Solution to GATE ST 2023.40

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Question: Let $X_1, X_2, ..., X_{10}$ be a random sample of size 10 from a population having $\mathcal{N}\left(0, \theta^2\right)$ distribution, where $\theta > 0$ is an unknown parameter. Let $T = \frac{1}{10}\sum_{i=1}^{10} X_i^2$. If the mean square error of cT $(c > 0)$ as an estimator of θ^2, is minimized at $c = c_0$, then the value of c_0 equals

(i) $\frac{5}{6}$
(ii) $\frac{2}{3}$
(iii) $\frac{3}{5}$
(iv) $\frac{1}{2}$

Solution:

$$X_i \sim \mathcal{N}\left(0, \theta^2\right) \tag{1}$$

$$\frac{X_i - 0}{\theta} \sim \mathcal{N}(0, 1) \tag{2}$$

$$\left(\frac{X_i - 0}{\theta}\right)^2 \sim \chi^2_{(1)} \tag{3}$$

$$\sum_{i=1}^{n}\left(\frac{X_i}{\theta}\right)^2 \sim \chi^2_{(n)} \tag{4}$$

For χ^2 distribution,

$$E\left(\chi^2_{(n)}\right) = n \tag{5}$$

$$V\left(\chi^2_{(n)}\right) = 2n \tag{6}$$

Now,

$$T = \frac{1}{10}\sum_{i=1}^{10} X_i^2 \tag{7}$$

$$= \frac{\theta^2}{10}\sum_{i=1}^{10}\frac{X_i^2}{\theta^2} \tag{8}$$

$$= \frac{\theta^2}{10}\chi^2_{(10)} \tag{9}$$

$$E(T) = E\left(\frac{\theta^2}{10}\chi^2_{(10)}\right) \tag{10}$$
$$= \frac{\theta^2}{10}(10) \tag{11}$$
$$= \theta^2 \tag{12}$$
$$V(T) = V\left(\frac{\theta^2}{10}\chi^2_{(10)}\right) \tag{13}$$
$$= \left(\frac{\theta^2}{10}\right)^2 (20) \tag{14}$$
$$= \frac{\theta^4}{5} \tag{15}$$

The mean square error of cT as an estimator of θ^2 is given by

$$f = E\left(\left(cT - \theta^2\right)^2\right) \tag{16}$$

Since

$$E\left(X^2\right) = V(X) + (E(X))^2 \tag{17}$$

Now,

$$f = V\left(cT - \theta^2\right) + \left(E\left(cT - \theta^2\right)^2\right) \tag{18}$$
$$= c^2 V(T) + \left(cE(T) - \theta^2\right)^2 \tag{19}$$
$$= \frac{c^2}{5}\theta^4 + \theta^4 (c-1)^2 \tag{20}$$

Minimizing the mean square error,

$$f' = 0 \tag{21}$$
$$f' = \frac{2c}{5}\theta^4 + 2\theta^4 (c-1) \tag{22}$$
$$\frac{c}{5} + c - 1 = 0 \tag{23}$$
$$6c = 5 \tag{24}$$
$$c = \frac{5}{6} \tag{25}$$

(GATE ST 2023)