Assignment - 3

ME-639 Intro to Robotics

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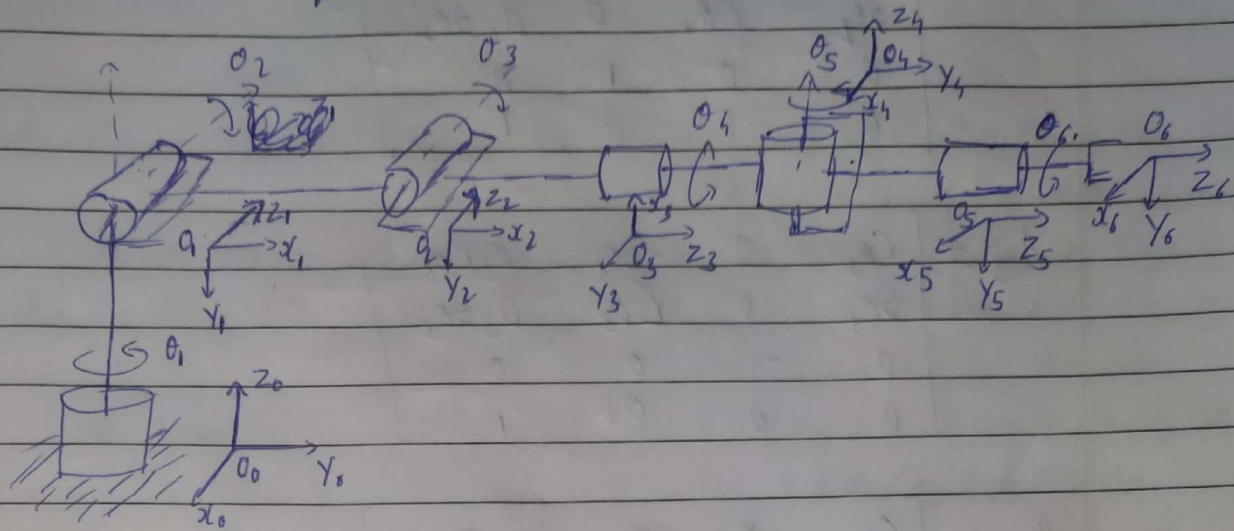
1 For a n-link manipulator, a configuration is said to be singular if the rank of the ~~Jacobian~~ Jacobian at that configuration is less than its maximum ~~value~~ possible value.

→ If a particular configuration is singular then the determinant of the Jacobian matrix must be zero.

→ ~~At singularities, bounded end effector may correspond to unbounded~~

→ If the determinant of the jacobian matrix tends to zero at some configuration then they are close to singularities.

5 Elbow Manipulator with spherical wrist.



Thus the DH parameters for the given manipulator ~~parameters~~ are

Link	a_i	α_i	d_i	θ_i
1	a_1	-90	$\cancel{d_1}$ 0	θ_1
2	a_2	0	$\cancel{d_2}$ 0	θ_2
3	a_3	-90	$\cancel{d_3}$ 0	θ_3
4	a_4	90	$\cancel{d_4}$ 0	θ_4
5	a_5	-90	$\cancel{d_5}$ 0	θ_5
6	a_6	0	$\cancel{d_6}$ 0	θ_6

Thus, we have

$$A_1 = \begin{bmatrix} C_{\theta_1} & 0 & -S_{\theta_1} & a_1 C_{\theta_1} \\ S_{\theta_1} & 0 & C_{\theta_1} & a_1 S_{\theta_2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & a_2C\theta_2 \\ S\theta_2 & C\theta_2 & 0 & a_2S\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} C\theta_3 & 0 & -S\theta_3 & a_3C\theta_3 \\ S\theta_3 & 0 & C\theta_3 & a_3S\theta_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} C\theta_4 & 0 & S\theta_4 & a_4C\theta_4 \\ S\theta_4 & 0 & -C\theta_4 & a_4S\theta_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} C\theta_5 & 0 & -S\theta_5 & a_5C\theta_5 \\ S\theta_5 & 0 & C\theta_5 & a_5S\theta_5 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} C\theta_6 & -S\theta_6 & 0 & a_6C\theta_6 \\ S\theta_6 & C\theta_6 & 0 & a_6S\theta_6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore T_0^6 = A_1 A_2 A_3 A_4 A_5 A_6$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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where

$\cancel{r_{11}}$

$$r_{11}=C\theta_1C\theta_2C\theta_3C\theta_4C\theta_5C\theta_6-C\theta_1C\theta_2C\theta_3S\theta_4S\theta_6-C\theta_1C\theta_4C\theta_5C\theta_6S\theta_2S\theta_3$$
$$+C\theta_1S\theta_2S\theta_3S\theta_4S\theta_6+S\theta_1C\theta_5C\theta_6C\theta_4$$
$$+C\theta_4S\theta_1S\theta_6-C\theta_1C\theta_2\cancel{C\theta_3}C\theta_6S\theta_5S\theta_3\cancel{C\theta_1C\theta_3S\theta_2}-C\theta_1C\theta_3C\theta_6S\theta_3S\theta_5$$

$$r_{12}=-C\theta_1C\theta_2C\theta_3C\theta_4C\theta_5C\theta_6-C\theta_1C\theta_2C\theta_3C\theta_6S\theta_4+C\theta_1C\theta_4C\theta_5C\theta_6S\theta_2S\theta_3$$
$$+C\theta_1C\theta_6S\theta_2S\theta_3S\theta_4S\theta_6$$
$$+-C\theta_5S\theta_1S\theta_4S\theta_6+C\theta_4C\theta_6S\theta_1+C\theta_1C\theta_2S\theta_3S\theta_5S\theta_6\cancel{+C\theta_1C\theta_3S\theta_2}$$
$$+C\theta_1C\theta_3S\theta_2S\theta_5S\theta_6$$

$$r_{13}=-C\theta_1C\theta_2C\theta_3C\theta_4S\theta_5+C\theta_1S\theta_2S\theta_3S\theta_5-S\theta_1S\theta_4S\theta_5$$
$$-C\theta_1C\theta_2C\theta_5S\theta_3-C\theta_1C\theta_3C\theta_5S\theta_2$$

$$dx=[C\theta_1C\theta_2C\theta_3-C\theta_1S\theta_2S\theta_3][a_6C\theta_4C\theta_5C\theta_6-a_6S\theta_4S\theta_6$$
$$+a_5C\theta_4C\theta_5+a_4C\theta_4]$$
$$+S\theta(a_6C\theta_5C\theta_6S\theta_4+a_6C\theta_4S\theta_6+a_5C\theta_5S\theta_4+a_4S\theta_4)$$
$$+(-C\theta_1C\theta_2S\theta_3-C\theta_1C\theta_3S\theta_2)(a_6C\theta_6S\theta_5+a_5S\theta_5)$$
$$+a_3C\theta_1C\theta_2C\theta_3\cancel{+}a_3C\theta_1S\theta_2S\theta_3+a_2C\theta_1C\theta_2+a_1C\theta_1$$

$$r_{21}=(C\theta_2C\theta_3S\theta_1-S\theta_1S\theta_2S\theta_3)(C\theta_4C\theta_5C\theta_6-S\theta_4S\theta_6)$$
$$-C\theta_1(C\theta_5C\theta_6S\theta_4+C\theta_4S\theta_6)$$
$$-(C\theta_2S\theta_1S\theta_3-C\theta_3S\theta_1S\theta_2)(C\theta_6S\theta_5)$$

$$r_{22}=(C\theta_2C\theta_3S\theta_1-S\theta_1S\theta_2S\theta_3)(-C\theta_4C\theta_5C\theta_6-S\theta_4C\theta_6)$$
$$-C\theta_1(C\theta_5S\theta_4S\theta_6+C\theta_4C\theta_6)$$
$$-(C\theta_2S\theta_1S\theta_3+C\theta_3S\theta_1S\theta_2)(-S\theta_5S\theta_6)$$

$$r_{23}=(C\theta_2C\theta_3S\theta_1-S\theta_1S\theta_2S\theta_3)(-C\theta_4S\theta_5)$$
$$+C\theta_1S\theta_4S\theta_5-C\theta_5(C\theta_2S\theta_1S\theta_3+C\theta_3S\theta_1S\theta_2)$$

$$d_y = (c_{\theta_2} c_{\theta_3} s_{\theta_1} - s_{\theta_1} s_{\theta_2} s_{\theta_3})(a_6 c_{\theta_4} c_{\theta_5} c_{\theta_6} - a_6 s_{\theta_4} s_{\theta_6} + a_5 c_{\theta_4} c_{\theta_5} + a_4 c_{\theta_4})$$
$$+ -c_{\theta_1}(a_6 c_{\theta_5} c_{\theta_6} s_{\theta_4} + a_6 c_{\theta_4} s_{\theta_6} + a_5 c_{\theta_5} s_{\theta_4} + a_4 s_{\theta_4})$$
$$- (c_{\theta_2} s_{\theta_1} s_{\theta_3} + c_{\theta_3} s_{\theta_1} s_{\theta_2})(a_6 c_{\theta_6} s_{\theta_5} + a_5 s_{\theta_5})$$
$$+ a_3 c_{\theta_2} c_{\theta_3} s_{\theta_1} - a_3 s_{\theta_1} s_{\theta_2} s_{\theta_3} + a_2 c_{\theta_2} s_{\theta_1} + a_1 s_{\theta_1}$$

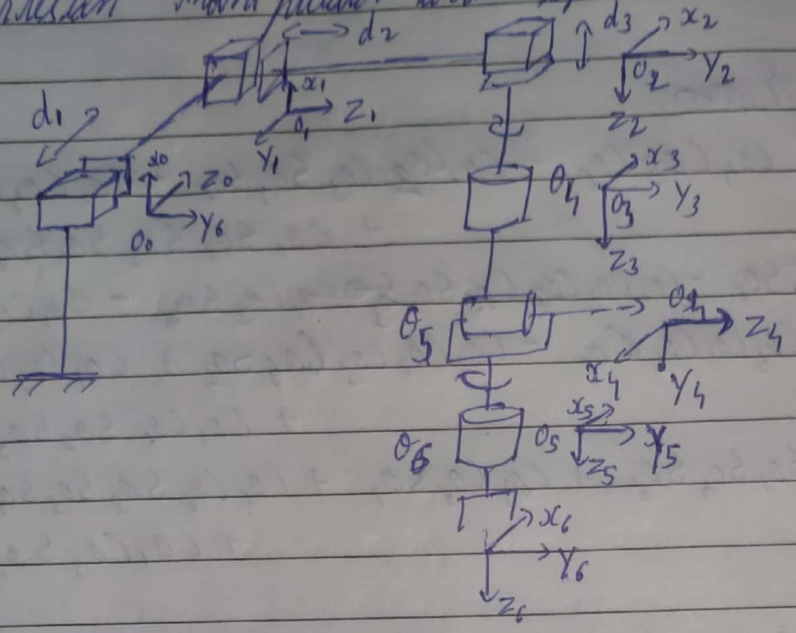
$$r_{31} = -(c_{\theta_3} s_{\theta_2} + c_{\theta_2} s_{\theta_3})(c_{\theta_4} c_{\theta_5} c_{\theta_6} - s_{\theta_4} s_{\theta_6})$$
$$+ (s_{\theta_2} s_{\theta_3} - c_{\theta_2} c_{\theta_3})(c_{\theta_6} s_{\theta_5})$$

$$r_{32} = -(c_{\theta_3} s_{\theta_2} + c_{\theta_2} s_{\theta_3})(\cancel{s_{\theta}} - c_{\theta_4} c_{\theta_5} c_{\theta_6} - s_{\theta_4} s_{\theta_6})$$
$$+ (s_{\theta_2} s_{\theta_3} - c_{\theta_2} c_{\theta_3})(-s_{\theta_5} s_{\theta_6})$$

$$r_{33} = -(c_{\theta_3} s_{\theta_2} + c_{\theta_2} s_{\theta_3})(\cancel{c_{\theta_4} c_{\theta_5}} - c_{\theta_4} s_{\theta_5})$$
$$+ (s_{\theta_2} s_{\theta_3} - c_{\theta_2} c_{\theta_3})(c_{\theta_5})$$

$$d_z = -(c_{\theta_3} s_{\theta_2} + c_{\theta_2} s_{\theta_3})(a_6 c_{\theta_4} c_{\theta_5} c_{\theta_6} - a_6 s_{\theta_4} s_{\theta_6} + a_5 c_{\theta_4} c_{\theta_5} + a_4 c_{\theta_4})$$
$$+ (s_{\theta_2} s_{\theta_3} - c_{\theta_2} c_{\theta_3})(a_6 c_{\theta_6} s_{\theta_5} + a_5 s_{\theta_5})$$
$$- a_3 c_{\theta_3} s_{\theta_2} - a_3 c_{\theta_2} s_{\theta_3} - a_2 s_{\theta_2}$$

6 Cartesian manipulator with spherical wrist



Thus the DH Parameters for the given manipulator are

Links	a_i	α_i	d_i	θ_i
1	0	-90	d_1	0
2	0	90	d_2	-90
3	0	0	d_3	0
4	a_4	90	0	θ_4
5	a_5	90	0	θ_5
6	a_6	0	0	θ_6

So, we have

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} C\theta_4 & 0 & S\theta_4 & a_4 C\theta_4 \\ S\theta_4 & 0 & -C\theta_4 & a_4 S\theta_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} C\theta_5 & 0 & S\theta_5 & a_5 C\theta_5 \\ S\theta_5 & 0 & -C\theta_5 & a_5 S\theta_5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} C\theta_6 & -S\theta_6 & 0 & a_6 C\theta_6 \\ S\theta_6 & C\theta_6 & 0 & a_6 S\theta_6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore T_0^6 = A_1 A_2 A_3 A_4 A_5 A_6$$

$$= \begin{bmatrix} h_{11} & h_{12} & h_{13} & d_x \\ h_{21} & h_{22} & h_{23} & d_y \\ h_{31} & h_{32} & h_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where,

$$r_{11} = -C\theta_6 S\theta_5$$

$$r_{12} = S\theta_5 S\theta_6$$

$$r_{13} = C\theta_5$$

$$d_x \ \cancel{r_{14}} = -a_6 C\theta_6 S\theta_5 - a_5 S\theta_5 - d_3$$

$$r_{21} = C\theta_5 C\theta_6 S\theta_4 - C\theta_4 S\theta_6$$

$$r_{22} = -C\theta_5 S\theta_4 S\theta_6 - C\theta_4 C\theta_6$$

$$r_{23} = S\theta_4 S\theta_5$$

$$d_y \ \cancel{r_{24}} = a_6 C\theta_5 C\theta_6 S\theta_4 - a_6 C\theta_4 S\theta_6 + a_5 S\theta_4 C\theta_5 + a_4 S\theta_4 + d_2$$

$$r_{31} = C\theta_4 C\theta_5 C\theta_6 + S\theta_4 S\theta_6$$

$$r_{32} = -C\theta_4 C\theta_5 S\theta_6 + C\theta_6 S\theta_4$$

$$r_{33} = C\theta_4 S\theta_5$$

$$d_z = a_6 C\theta_4 C\theta_5 C\theta_6 + a_6 S\theta_4 S\theta_6 + a_5 C\theta_4 C\theta_5 + a_4 C\theta_4 + d_1$$

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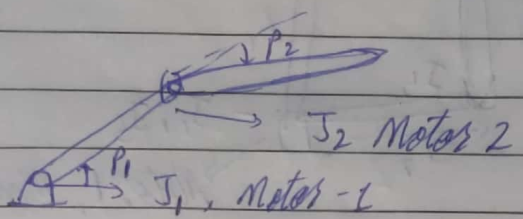
7 The different types of 2R manipulators are:

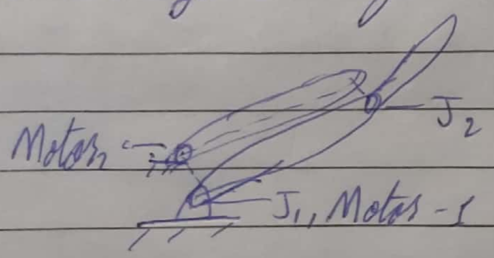
→ Direct drive :- In this configuration, the motors controlling the joint angles are attached at the joints.



~~Thus in reference to the base~~

Here, the driving angle P_2 is relative to the link 1. Because of its simple nature, many of the potential difficulties do not arise.
A drawback of this configuration is that an extra weight of motor 2 is also carried by motor 1.

→ Remotely driven :- In this configuration, the motors controlling the joint angles are attached at the base.



Here, the driving angle P_2 is independent of P_1. As the second joint is remotely driven from the base, the coriolis forces are eliminated.

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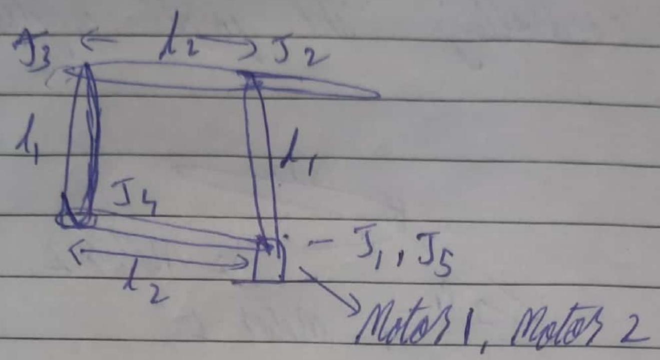
→ Five Bar linkage - In this configuration 4 links form a closed loop parallelogram configuration attached to the base.



Since it is a closed configuration the DoF of the system is 2. In this configuration it is possible to remove both coriollis and centrifugal terms by making the gen inertia matrix diagonal. Also it is possible to make the driving angles independently of each other adjustable. Also the potentials in the dynamic Equation are not coupled.
So in general this configuration can form a decoupled set of equations with proper mass and length values.

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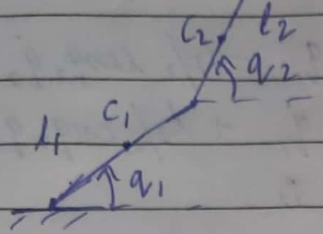
8 Equation of Motion for remotely driven 2R manipulator



C_1 & C_2 – Center of mass of links

Assuming the links to be uniform

i.e. $l_{c_1} = l_1/2 \qquad l_{c_2} = \frac{l_2}{2}$

∴ We have velocities of links as

$$V_{c_1} = \begin{bmatrix} -\frac{l_1}{2}\sin q_1 \\ l_1/2 \cos q_1 \\ 0 \end{bmatrix} \dot{q}_1 \qquad V_{c_2} = \begin{bmatrix} -l_1 \sin q_1 & -l_2/2 \sin q_2 \\ l_1 \cos q_1 & l_2/2 \cos q_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$w_1 = \dot{q}_1 \hat{k} \qquad w_2 = \dot{q}_2 \hat{k}$$

Kinetic Energy KE of the system is

$$K = \frac{1}{2} \sum_{i=1}^{n} m_i v_{c_i}^T v_{c_i} + \frac{1}{2} \sum_{i=1}^{n} w_i^T I_i w_i$$

$$= \frac{1}{2} \Big[m_1 \left[-\frac{l_1}{2}\sin q_1 \quad \frac{l_1}{2}\cos q_1 \quad 0 \right] \dot{q}_1 \begin{bmatrix} -l_1/2 \sin q_1 \\ l_1/2 \cos q_1 \\ 0 \end{bmatrix} \dot{q}_1^2$$

$$+ m_2 \left[-l_1 \sin q_1 \dot{q}_1 - \frac{l_2}{2} \sin q_2 \dot{q}_2 \quad l_1 \cos q_1 \dot{q}_1 + \frac{l_2}{2} \cos q_2 \dot{q}_2 \quad 0 \right] \times$$

$$\left[\begin{matrix} -l_1 \sin q_1 \dot{q}_1 - l_2/2 \sin q_2 \dot{q}_2 \\ l_1 \cos q_1 \dot{q}_1 + l_2/2 \cos q_2 \dot{q}_2 \\ 0 \end{matrix} \right] \Bigg]$$

$$+ \frac{1}{2} \left(I_1 w_1^2 + \frac{1}{2} I_2 w_2^2 \right)$$

$$= \frac{1}{2} m_1 \left(\frac{l_1^2}{4} \sin^2 q_1 + \frac{l_1^2}{4} \cos^2 q_1 \right) \dot{q}_1^2$$

$$+ \frac{1}{2} m_2 \Big(l_1^2 \sin^2 q_1 \dot{q}_1^2 + \frac{l_2^2}{4} \sin^2 q_2 \dot{q}_2^2 + l_1 l_2 \sin q_1 \sin q_2 \dot{q}_1 \dot{q}_2$$

$$+ l_1^2 \cos^2 q_1 \dot{q}_1^2 + \frac{l_2^2}{4} \cos^2 q_2 \dot{q}_2^2 + l_1 l_2 \cos q_1 \cos q_2 \dot{q}_1 \dot{q}_2 \Big)$$

$$+ \frac{1}{2} I_1 \dot{q}_1^2 + \frac{1}{2} I_2 \dot{q}_2^2$$

$$= \left(\frac{m_1 l_1^2 \sin^2 q_1}{8} + \frac{m_2 l_1^2 \sin^2 q_1}{2} + \frac{I_1}{2} \right) \dot{q}_1^2$$

$$+ \left(\frac{m_1}{4} l_1^2 \cos^2 q_1 + \frac{m_2}{2} l_1^2 \cos^2 q_2 \right) \dot{q}_1^2$$

$$+ \frac{m_2 l_2^2}{8} \sin^2 q_2 \dot{q}_2^2 + \frac{m_2}{8} l_2^2 \cos^2 q_2 \dot{q}_2^2 + \frac{I_2}{2} \dot{q}_2^2$$

$$+ \frac{m_2 l_1 l_2}{2} \cos(q_1 - q_2) \dot{q}_1 \dot{q}_2$$

$$= \left(\frac{m_1 l_1^2}{8} + \frac{m_2 l_1^2}{2} + \frac{I_1}{2} \right) \dot{q}_1^2 + \left(\frac{m_2 l_2^2}{8} + \frac{I_2}{2} \right) \dot{q}_2^2$$

$$+ \frac{m_2 l_1 l_2}{2} \cos(q_1 - q_2) \dot{q}_1 \dot{q}_2$$

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Thus we have

$$K = \frac{1}{2} \dot{q}^T D(q) \dot{q}$$

where

$$D(q) = \begin{bmatrix} m_1 l_1^2/4 + m_2 l_1^2 + I_1 & m_2 l_1 l_2 \cos(q_2 - q_1)/2 \\ m_2 l_1 l_2 \cos(q_2 - q_1)/2 & \frac{m_2 l_2^2}{4} + I_2 \end{bmatrix}$$

We have Potential Energy as

$$V(q) = \frac{m_1 g l_1}{2} \sin q_1 + m_2 g \left(l_1 \sin q_1 + \frac{l_2}{2} \sin q_2\right)$$

Thus the system of Equations of motion can be computed as

$$D(q)(\ddot{q}) + C(q, \dot{q}) \dot{q} + g(q) = \tau$$

where $C(q, \dot{q})$ - christoffel Matrix

& $$c_{kj} = \sum_{i=1}^{n} c_{ijk}(q) \dot{q}_i = \sum_{i=1}^{n} \frac{1}{2} \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \dot{q}_i$$

& $$g_i(q) = \frac{\partial V(q)}{\partial q_i}$$

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Computing christoffel symbols

$$C_{111} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0$$

$$C_{121} = C_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = 0$$

$$\cancel{C_{221} = \frac{\partial d_{12}}{\partial q_2} = -m_2 l_1 l_2}$$

$$C_{221} = \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1}$$

$$= -\frac{m_2 l_1 l_2}{2}\sin(q_2 - q_1)$$

$$C_{112} = \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} =$$

$$= \frac{m_2 l_1 l_2}{2}\sin(q_2 - q_1)$$

$$C_{212} = C_{122} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0$$

$$C_{222} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = 0$$

l

$$g_1 = \frac{\partial V(q)}{\partial q_1}$$

$$= \frac{m_1 g l_1}{2}\cos q_1 + m_2 g l_1 \cos q_1$$

$$g_2 = \frac{\partial V(q)}{\partial q_2}$$

$$= \frac{m_2 g l_2}{2}\cos q_2$$

$\therefore$ The Equation of motion are

$$d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{221}\dot{q}_2^2 + g_1 = \tau_1$$

$$\left(\frac{m_1 l_1^2}{4} + m_2 l_1^2 + I_1\right)\ddot{q}_1 + \frac{m_2 l_1 l_2}{2}\cos(q_2 - q_1)\ddot{q}_2 - \frac{m_2 l_1 l_2}{2}\sin(q_2 - q_1)\dot{q}_2^2$$

$$+ \frac{m_1 g l_1}{2}\cos q_1 + m_2 g l_1 \cos q_1 = \tau_1$$

$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + g_2 = \tau_2$$

$$\frac{m_2 l_1 l_2}{2}\cos(q_2 - q_1)\ddot{q}_1 + \left(\frac{m_2 l_2^2}{4} + I_2\right)\ddot{q}_2 + \frac{m_2 l_1 l_2}{2}\sin(q_2 - q_1)\dot{q}_1^2$$

$$+ \frac{m_2 g l_2}{2}\cos q_2 = \tau_2$$

We see that the Equations derived above as well as those in mini projects are same.

10 For the given robot we have $D(q)$ & $V(q)$. Thus to get the Equations of motion, we have

Kinetic Energy of the system as

$$K = \frac{1}{2} \dot{q}^T D(q) \dot{q} = \frac{1}{2} \sum_{i,j} d_{ij}(q) \dot{q}_i \dot{q}_j$$

Potential Energy of the system is $V(q)$

Now using the Euler Lagrangian method we can get the Equations of motion for the robot.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau$$

$$\text{where } L = K - P$$
$$= \frac{1}{2} \sum_{i,j} d_{ij}(q) \dot{q}_i \dot{q}_j - V(q)$$

The Partial derivative of the Lagrangian with k th joint velocity is

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj} \dot{q}_j$$

As $D(q)$ is symmetric the factor of 1/2 cancels out

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$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \sum_j d_{kj}\ddot{q}_j + \sum_j \frac{d}{dt}(d_{kj})\dot{q}_j$$

$$= \sum_j d_{kj}\ddot{q}_j + \sum_{i,j} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

Now the partial derivative of lagrangian with respect to k^{th} joint is q

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial V(q)}{\partial q_k}$$

Thus for each k the Euler lagrangian Equation is

$$\sum_j d_{kj}\ddot{q}_j + \sum_{i,j}\left(\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j - \frac{\partial V(q)}{\partial q_k}$$

$$= \tau_k$$

Due to the symmetry of $D(q)$ we can write

$$\sum_{i,j}\frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{i,j}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j}\right)\dot{q}_i\dot{q}_j$$

Thus $$\sum_{i,j}\left(\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j = \sum_{i,j}\frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j$$

$$= \sum_{i,j} c_{ijk}\dot{q}_i\dot{q}_j$$

where

$$C_{ijk} = \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}$$

Also

$$g_k = \frac{\partial P}{\partial q_k}$$

$\therefore$ We have our ~~sy~~ Euler Lagrangian Equations as

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k$$

where $k = 1, 2, \ldots n$