MIT 18.06SC - Problem Set 1.4#5

Problem 4.1)

$$A=\begin{bmatrix}1&3&0\\2&4&0\\2&0&1\end{bmatrix}\xrightarrow[E_{21}]{\text{Row}2+(-2\times\text{Row}1)}\begin{bmatrix}1&3&0\\0&-2&0\\2&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&3&0\\0&-2&0\\2&0&1\end{bmatrix}\xrightarrow[E_{31}]{\text{Row}3+(-2\times\text{Row}1)}\begin{bmatrix}1&3&0\\0&-2&0\\0&-6&1\end{bmatrix}$$

$$\begin{bmatrix}1&3&0\\0&-2&0\\0&-6&1\end{bmatrix}\xrightarrow[E_{32}]{\text{Row}3+(-3\times\text{Row}2)}\begin{bmatrix}1&3&0\\0&-2&0\\0&0&1\end{bmatrix}$$

$E_{32}E_{31}E_{21}A = U$ i.e $E_{32}E_{31}E_{21} = E$ such that $EA = U$

$$E_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}\quad E_{31}=\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}$$

$$E_{21}=\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}$$

$$E_{32}E_{31}E_{21}=\begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}$$

~~$E = LU \quad A = LU$~~

$EA = U$ so $A = E^{-1}U$, and since $E^{-1} = L$, then $A = LU$.

$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Problem 4.2)

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

$$\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} \xrightarrow{E_1} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

$$\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix} \xrightarrow{E_2} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

$$\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix} \xrightarrow{E_3} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-b-c \end{bmatrix}$$

$$U = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} \quad \begin{matrix} a \neq 0 \\ b \neq a \\ c \neq b \\ d \neq c \end{matrix} \} \text{ Conditions}$$

$EA = U$ and $E_3E_2E_1 = E$, so $E_3E_2E_1A = U$.
Since $EA = E^{-1}A$, and $E^{-1} = L$ when $A = LU$.
Since $E^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}$, $L = E_1^{-1}E_2^{-1}E_3^{-1}$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \Rightarrow E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \Rightarrow E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$