Complements Lattice:

A Lattice in which every element has a complement is known as complemented Lattice.

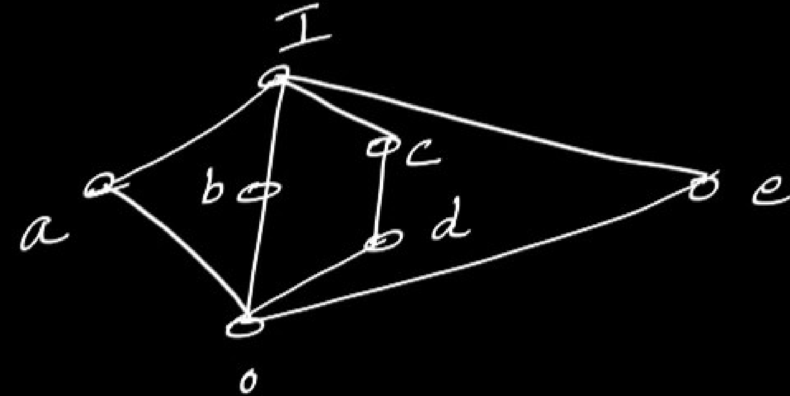
Complement of 'a'

Element		Complement
a	$\longrightarrow$	b, c, d, e
b	$\longrightarrow$	a, c, d, e
c	$\longrightarrow$	a, b, e
d	$\longrightarrow$	a, b, e
e	$\longrightarrow$	a, b, c, d
I	$\longrightarrow$	0
0	$\longrightarrow$	I

$$\left.\begin{aligned} a \vee b &= I \\ a \wedge b &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} a \vee c &= I \\ a \wedge c &= 0 \end{aligned}\right\}$$

Complemented Lattice



Distributive Lattice:

A Lattice which follows distributive law is known as Distributive Lattice.

Distributive Law: For any three elements a, b, c

i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $\forall a, b, c$

ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

5 elements

$5C_3 = \frac{5 \times \cancel{4}^{2} \times \cancel{3}}{\cancel{3} \times \cancel{2} \times 1} = 10$

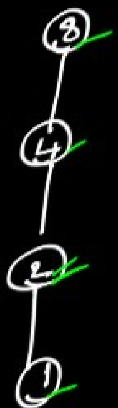
Q. Construct

i) $[D_8 : |]$ ✓

ii) $[D_{16} : |]$ ✓

iii) $[D_{32} : |]$ ✓

$D_8 = \{1, 2, 4, 8\}$



$D_{16} = \{1, 2, 4, 8, 16, 32\}$

Distributive

$2 \vee (4 \wedge 8)$	$(2 \vee 4) \wedge (2 \vee 8)$
$2 \vee (4)$	$4 \wedge 8$
$4 =$	4

$D_{32} = \{1, 2, 4, 8, 16, 32\}$

Can you show complements?

Any pair of elements related

8, 4

$(8,4) \in R$ (or) $(4,8) \in R$

$8 R 4$ (or) $4 R 8$

Total Order:

A partial-order relation in which every pair of elements are COMPARABLE, is known as Totally-ordered relation (or) Linearly-ordered (or) chain

* Every chain is Distributive

COMPARABLE: In a relation 'R',

Two elements x, y are comparable $\Leftrightarrow$ x R y (or) y R x

$(x,y) \in R$ (or) $(y,x) \in R$

Boolean Algebra Structure:

A Lattice which is both distributive and complemented, is known as Boolean Algebra structure

relation

Ex: Construct $[P(A) : \subseteq]$

Where $A = \{a, b, c\}$ and $P(A)$ is a powerset of 'A'

$A = \{a, b, c\}$

power set $A = P(A)$

$= \{ \cancel{\phi}, \cancel{\{a\}}, \cancel{\{b\}}, \cancel{\{c\}},$

$\cancel{\{a,b\}}, \cancel{\{a,c\}}, \cancel{\{b,c\}}$

$\cancel{\{a,b,c\}} \}$

$[P(A) ; \subseteq]$

Element	complement
$\{a\} \longrightarrow$	$\{b,c\}$, $\cancel{\{b\}}$
$\{b\} \longrightarrow$	$\{a,c\}$
$\{c\} \longrightarrow$	$\{a,b\}$
$\phi \longrightarrow$	$\{a,b,c\}$

Every element has complement

So It is complemented Lattice

every element has only one complement $\therefore$ Distributive

Take

$\{a\}, \{b\}, \{a,c\}$

$LHS = \{a\} \vee [\{b\} \wedge \{a,c\}]$

$= \{a\} \vee \phi$

$= \{a\}$

$RHS = [\{a\} \vee \{b\}] \wedge [\{a\} \vee \{a,c\}]$

$= [\{a,b\}] \wedge [\{a,c\}]$

$= \{a\}$

Given structure Distributive

$\therefore [P(A) : \subseteq]$ is Distributive & Complemented

$\therefore [P(A) : \subseteq]$ is Boolean algebra structure.

Transitive Closure:

If ‘R’ is any relation on a set ‘A’ then transitive closure of ‘R’ denoted by R* is the smallest transitive relation on A which contains R.

Ex: Let A = {a, b, c} and

$R = \{(a, b), (b, c)\}$ Here R is NOT Transitive

$R^* = \{\underbrace{(a, b), (b, c)}_{R}, (a, c)\}$

$\underset{\text{NOT Trans}}{R} \cup \text{something} = \underset{\text{Transitive}}{R^*}$

Reflexive Closure:

If 'R' is any relation on a set A then reflexive closure of R denoted by $R^{\#}$ is the smallest reflexive relation on A which contains R

$$R^{\#} = R \cup \Delta_A$$

Ex: If $A = \{a, b, c\}$

$R = \{(a, b), (b, c)\}$

$R^{\#} = \{\underbrace{\boxed{(a, b), (b, c)}}_{R}, (a, a), (b, b), (c, c)\}$

$\underset{\text{NOT ref}}{R} \cup \text{Some} = \underset{\text{Reflexive}}{R^{\#}}$

Symmetric Closure:

If 'R' is any relation on a set 'A' then symmetric closure of R denoted by R^+ is

$$R^+ = \text{The smallest symmetric relation on A which contains R} = R \cup R^{-1}$$

Ex: Let $A = \{a, b, c\}$ and

$R = \{(a, b), (b, c)\}$ then

$R^+ = \{\underbrace{(a, b), (b, c)}_{R}, (b, a), (c, b)\}$

$\underset{\text{NOT Symmetric}}{R} \cup \text{Something} = \text{Symmetric}$

$= R^+.$

Q. The Transitive closure of the relation

$\{(1, 2), (2, 3), (3, 4), (5, 4)\}$ on the set $A = \{1, 2, 3, 4, 5\}$ is __________

$R = \{ (1,2), (2,3), (3,4), (5,4) \}$

$1 \longrightarrow 2 \longrightarrow 3 = (1,3) \notin R$

$2 \longrightarrow 3 \longrightarrow 4 = (2,4) \notin R$

$3 \longrightarrow 4$

$5 \longrightarrow 4$

$R' = \{ (1,2), (2,3), (3,4), (5,4) \} \cup \{ (1,3), (2,4) \}$

$= \{ (1,2), \underbrace{(1,3)}_{x\,y}, (2,3), (2,4), \underbrace{(3,4)}_{y\,z}, (5,4) \}$

$1 \longrightarrow 3 \longrightarrow 4 = (1,4) \notin R$

$R^* = \{ (1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (5,4) \}$

$\longrightarrow 5 \longrightarrow 4 \longrightarrow$

$\left.\begin{matrix} (1,1) \\ (2,2) \\ (3,3) \\ (4,4) \end{matrix}\right]$ Don't transitive

Formula:

If A is a set with n elements then

i) Number of reflexive relations on A = $2^{n(n-1)}$

ii) Number of Irreflexive relations on A = $2^{n(n-1)}$

iii) Number of symmetric relations on A = $2^{\frac{n(n+1)}{2}}$

iv) Number of anti-symmetric relations on A = $2^n . 3^{\frac{n(n-1)}{2}}$

v) Number of asymmetric relations on A = $3^{\frac{n(n-1)}{2}}$

Q. Find the possible number of reflexive relations on 5-elements set?

$n = 5$

No of Reflexive relations $= 2^{n(n-1)}$

$= 2^{5\times 4}$

$= 2^{20}$

$= (2^{10})^2$

$= \underline{\underline{10,48,576}}$

$$\begin{array}{r} 1024 \\ 1024 \\ \hline 10,48,576 \\ \hline \end{array}$$

Q. Let A be a finite set of size n. the number of elements in the power set of $A \times A$ is (GATE-93)

a) 2^{2^n}

b) 2^{n^2} ✓

c) 2^n

d) n^2

$n(A) = |A| = n$

$|A \times A| = n \times n = n^2$

$|P(A \times A)| = 2^{n^2}$

Q. Let X = {2,3,6,12,24}. Let $\leq$ be the partial order defined by $x \leq y$ if x divides y. the number of edges in the Hasse diagram of $(X, \leq)$ is **(GATE-96)**

a) 3

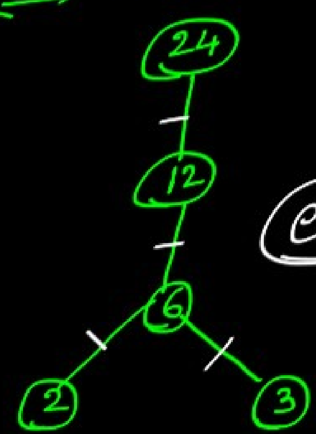
b) 4 ✓

c) 9

d) None of these

$x \leq y$ if x divides y

$x R y \iff x$ divides y



$e = 4$

Join-semi Lattice

Q. Suppose A is finite set with n elements. The number of elements in the largest equivalence relation of A is (GATE-98)

$A = \{a,b,c\}$

RST

$|A| = n = 3$

No. of relations $= 2^{n^2} = 2^9 = 512$

a) n

b) n^2 ✓

$R_1 = \{(a,a), (b,b), (c,c)\} \longrightarrow$ Min $= 3 = n$.

c) 1

$R_2 = A \times A = \{(a,a), (a,b), (a,c)$
$(b,a), (b,b), (b,c)$
$(c,a), (c,b), (c,c)\}$ RST = Eq

d) n+1

Max.
$n \times n = n^2$

Q. Let R_1 & R_2 be two equivalence relations on a set. Consider the following assertions: **(GATE-98)**

i) $R_1 \cup R_2$ is an equivalence relation

ii) $R_1 \cap R_2$ is an equivalence relation

Which of the following is correct?

a) Both assertions are true

b) Assertion (i) is true but assertion (ii) is not true

c) Assertion (ii) is true but assertion (i) is not true ✓

d) Neither (i) nor (ii) is true

$A = \{a, b, c, d\}$

$R_1 = \{(a,a), (b,b), (c,c), (d,d), \underline{(a,b)}, (b,a)\}$

$R_2 = \{(a,a), (b,b), (c,c), (d,d), \underline{(b,c)}, (c,b)\}$

(a,c)

$R_1 \cap R_2 = eq$

$R_1 \cup R_2 =$ NOT Trans

$\Rightarrow$ NOT equivalence

Q. The number of binary relations on a set with n elements is: **(GATE-99)**

a) n^2

b) 2^n

c) 2^{n^2}

d) None of these

Q. A relation R is defined on ordered pairs of integers as follows: **(GATE-06)**

$(x, y)\ R\ (u, v)$ if $\boxed{x < u \ \& \ y > v}$. then R is

a) neither a partial order nor an equivalence relation ✓

b) A partial order but not a total order

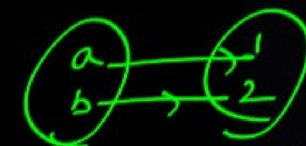
c) a total order

d) An equivalence relation

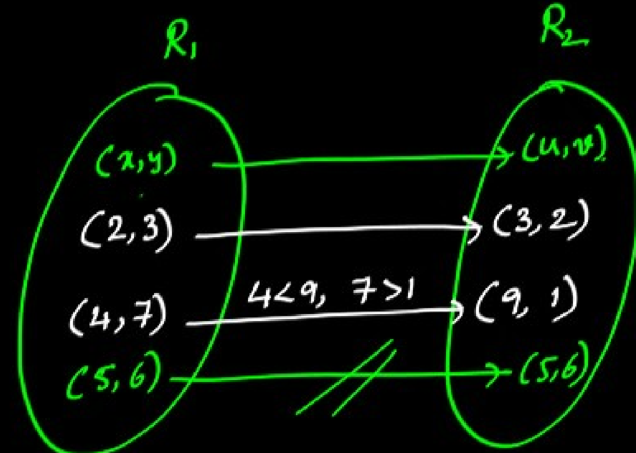
R is defined on sets

$R : A \longrightarrow A$

$R : R_1 \longrightarrow R_2$



$\{(a,1),(b,2)\}$

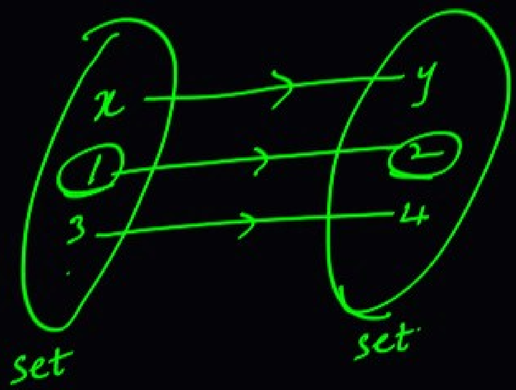


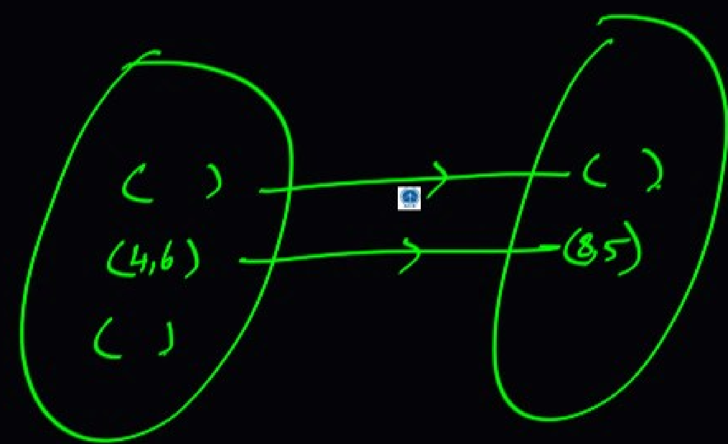
$\{((x,y),(u,v)),$

NOT reflexive.

$\Rightarrow$ NOT equivalence

$x R y$





$R = \{(1,2),(3,4),\}$

$\{((4,6),(8,5))$

Q. Let S be a set of n elements. The number of ordered pairs in the largest and the smallest equivalence relations on S are (GATE-07)

a) n and n

* b) n^2 and n ✓

c) n^2 and 0

d) n and 1

Q. Consider the binary relation R = {(x, y), $\boxed{(x, z), (z, x)}$, (z, y)} on the set {x, y, z}. which one of the following is TRUE? (GATE-09)

a) R is symmetric but NOT anti-symmetric

b) R is NOT symmetric but anti-symmetric

c) R is both symmetric and anti-symmetric

d) R is neither symmetric nor anti-symmetric

NOT Symmetric. $(x,y) \in R$ But $(y,x) \notin R$

NOT anti-Symm $(x,z),(z,x) \in R$ But $x \neq z$

Q. What is the possible number of reflexive relations on a set 5 elements?

(GATE-10)

a) 2^{10}

b) 2^{15}

c) 2^{20} ✓

d) 2^{25}

$2^{n(n-1)}$

$= 2^{5\times4}$

$= 2^{20}$

Q. Consider the set X = {a, b, c, d, e} under the partial ordering

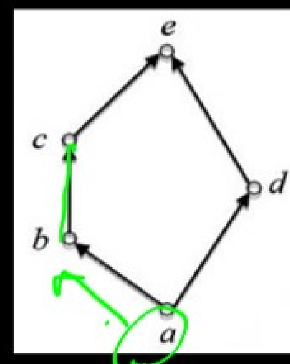
R= {(a, a), (a, b), (a, c), (a, d), (a, e), (b, b), (b, c), (b, e), (c, c), (c, e), (d, d), (d, e), (e, e)}. **(GATE-17-Set2)**

The Hasse diagram of the partial order (X, R) is shown below.

for every pair
there is a Join ($\vee$) and Meet ($\wedge$)

$$\left\{\begin{array}{l} b \vee d = e \\ b \wedge d = a \end{array}\right.$$

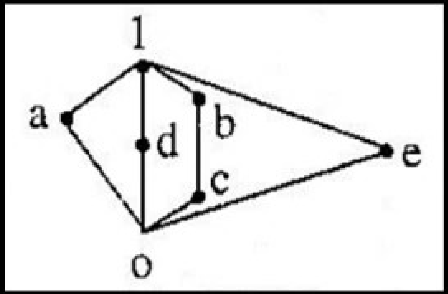
$$\left.\begin{array}{l} b \vee c = c \\ b \wedge c = b \end{array}\right\}$$



The minimum number of ordered pairs that need to be added to R to make (X, R) a lattice is ________. 0

Q. The complement(s) of the element 'a' in the lattice shown in Fig. is (are)……….

Complements (a) = d, b, c, e



(GATE-88)

Q. The transitive closure of the relation $\{(1, 2), (2, 3), (3, 4), (5, 4)\}$

On the set $A = \{1, 2, 3, 4, 5\}$ is__________ **(GATE-89)**

$\{(1,3),(2,4),(1,4)\}$ ✓

$$R^* = \{\underbrace{\boxed{(1,2),(2,3),(3,4),(5,4)}}_{R},\underbrace{(1,3),(2,4),(1,4)}\}$$

Q. Let R be a non-empty relation on a collection of sets defined by $A^R B$ if and only if $A \cap B = \phi$. Then, (pick the true statement) **(GATE-96)**

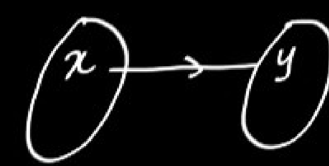
a) R is reflexive and transitive

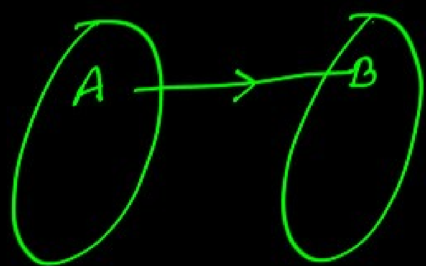
b) R is symmetric and not transitive

c) R is an equivalence relation

d) R is not reflexive and not symmetric

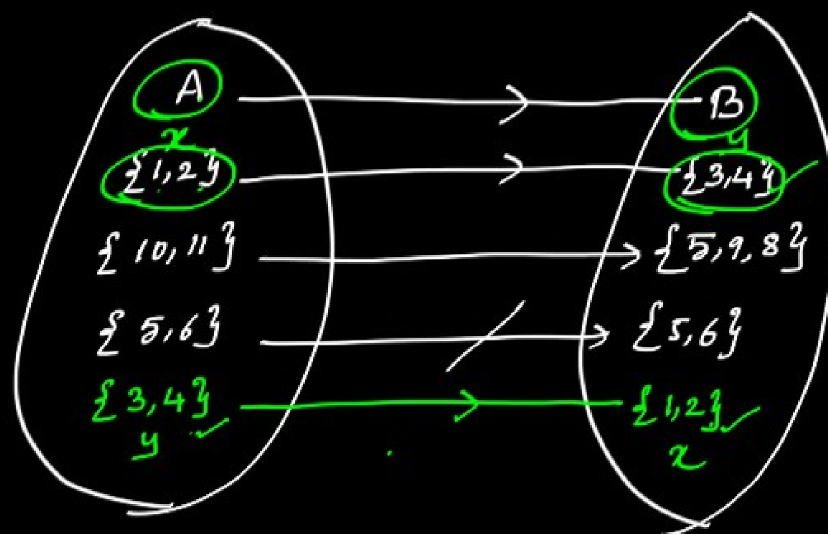
$xRy \iff$





$\{ (A,B)$

$\{$



$A R B \Longleftrightarrow A \cap B = \phi$ ✓

Ref: $A R A \Longleftrightarrow A \cap A = \phi$ (False)

(NOT)

But $A \cap A = A$

$A \not R A$

NOT Reflexive.

Symmetric: If aRb then bRa $\forall a, b$

Her. $\underline{ARB}$ then $\underline{BRA}$

$ARB \Longrightarrow A \cap B = \phi$

$\Longrightarrow B \cap A = \phi$

$\Longrightarrow BRA$ Symmetric.

Transitive:

If aRb, bRc then aRc, $\forall a, b, c$

If $A \cap B = \phi$, $B \cap C = \phi$ then $A \cap C = \phi$

$A = \{1, 2\}$ $B = \{3, 4\}$ $C = \{1, 5\}$

$A \cap B = \phi$, $B \cap C = \phi$, But $A \cap C \neq \phi$

NOT Transitive.

Q. The binary relation R = {(1, 1), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)} on the set A = {1, 2, 3, 4} is **(GATE-98)**

a) Reflexive, symmetric and transitive.

b) Neither reflexive nor irreflexive but transitive ✓

c) Irreflexive, symmetric and transitive

d) irreflexive and antisymmetric

$2 \to 1$, $2 \to 2$, $2 \to 3$

$3 \to 1$ (2,1)

$3 \to 2$ (2,2)

$3 \to 3$ (2,3)

$3 \to 4$ (2,4)

$1 \to 1$

$2 \to 1 \to 1$

$2 \to 2$

$2 \to 3$

$(4,4) \notin R$

NOT reflexive

$(1,1) \in R$

NOT irreflexive

Q. a) Mr. X claims the following: **(GATE-99)**

If a relation R is both symmetric and transitive, then R is reflexive. For this, Mr. X offers the following proof.

"from xRy, using symmetry we get yRx. Now because R is transitive, xRy and yRx together imply xRx. Therefore, R is reflexive"

Trans: aRb, bRc then aRc

xRy, yRx then xRx

Briefly point out the flaw in Mr. X's proof.

ϕ

Empty $= \phi =$ Sym, trans, = NOT ref

a=c

b) Give an example of a relation R which is symmetric and transitive but not reflexive.

$A = \boxed{\{1,2,3,4\}}$

$R = \{ \underbrace{(1,2)}_{a\,b}, \underbrace{(2,1)}_{b\,c}, (1,1) \}$

R is Symmetric

R is Transitive

Q. A relation R is defined on the set of integers as xRy iff (x + y) is even. Which of the following statements is true? **(GATE-00)**

a) R is not an equivalence relation

b) R is an equivalence relation having 1 equivalence class

c) R is an equivalence relation having 2 equivalence classes

d) R is an equivalence relation having 3 equivalences classes

xRy iff $(x+y)$ even

$2+4 = 6 \longrightarrow$ even

$3+5 = 8 \longrightarrow$ even

$7+8 = 15 \longrightarrow$ odd

$\{2, 4, 6, 8, --$

$\{1, 3, 5, 7, --\}$