Recurrence Relations

Let $\{a_0, a_1, a_2, \ldots\ldots, a_n\}$ be a sequence of real numbers, A formula that relates 'a_n' with one (or) more of the previous term is called a recurrence relation

S_n = Sum of first n-terms

$$= 1 + 2 + 3 + \ldots\ldots\ldots + (n-2) + (n-1) + n$$

$$S_n = S_{n-1} + n$$

$$S_{10} = S_9 + 10$$

A.P. : a, a + d, a + 2d, a + 3d, a + 4d, - - - - - - - - -, a + (n – 1)d

$t_1 = a$

$t_2 = t_1 + d$

$t_3 = t_2 + d$

$\vdots$

$t_n = t_{n-1} + d$

$t_n = t_{n-1} + t_{n-2} + t_{n-3}$

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$

G.P. : a, a r, ar^2, ar^3, - - - - - - - - -, ar^{n-1}

$t_1 = a$

$t_2 = t_1 * r$

$t_3 = t_2 * r$

$\boxed{t_n = t_{n-1} * r}$

Q. Obtain the Recurrence relation to following sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, - - - - - - - -

F_0

$F_0 = 0, \quad F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \; ---$

$F_2 = F_1 + F_0$

$F_3 = F_2 + F_1$

$F_4 = F_3 + F_2$

$\vdots$

$\boxed{F_n = F_{n-1} + F_{n-2},}$ with $F_0 = 0, \; F_1 = 1$

Q. Let a_n be the number of n-bit strings that do NOT contain two consecutive 1's. Which one of the following is the recurrence relation for a_n?

(GATE-16-Set1)

a) $a_n = a_{n-1} + 2a_{n-2}$

b) $\boxed{a_n = a_{n-1} + a_{n-2}}$ ✓

c) $a_n = 2a_{n-1} + a_{n-2}$

d) $a_n = 2a_{n-1} + 2a_{n-2}$

a_n = Number of n-bit strings that do not contain two consecutive one's

$n=4$,
1010 ✓
0000 ✓
0100 ✓
1101 ✗
1011 ✗

Here string can start with either 0 (or) 1

i) $0 \underbrace{- - - - - - - - - - -}_{n-1} = a_{n-1}$

ii) $1\ 0 \underbrace{- - - - - - - -}_{n-2} = a_{n-2}$

Let x_n denote the number of binary strings of length n that contain no consecutive 0s.

(GATE-CS-08)

Q. Which of the following recurrences does x_n satisfy?

a) $x_n = 2x_{n-1}$

b) $x_n = x_{[n/2]} + 1$

c) $x_n = x_{[n/2]} + n$

d) $\boxed{x_n = x_{n-1} + x_{n-2}}$

X_n = No. of binary strings of length 'n' that do not contain successive 0's.

1010 ✓
0110 ✓
0010 ✗

$$\boxed{X_n = X_{n-1} + X_{n-2}}$$

i) Begin '1'

$1\underbrace{\text{-----------}}_{n-1} = X_{n-1}$

ii) Begin '0'

$0\,1\underbrace{\text{---------}}_{n-2} = X_{n-2}$

$$X_n = X_{n-1} + X_{n-2}$$

$X_1 = 2$

$X_2 = 3$

$X_3 = 5$

$X_4 = 8$

$X_5 = 13$

$X_6 = 21$

Length $n = 1$

string = 0 (or) 1

$X_1 = 0$ $\quad$ $X_1 = 2$

$n = 2$

strings =
0 0	×
0 1	✓
1 0	✓
1 1	✓

$X_2 = 3$

$n = 3$

$X_3 = 5$

$n = 3$

string =
0 0 0	×
0 0 1	×
0 1 0	✓
0 1 1	✓
1 0 0	×
1 0 1	✓
1 1 0	✓
1 1 1	✓

Q2. The value of x_5 is

a) 5

b) 7

c) 8

d) 13

$x_1 = 2$

$x_2 = 3$

$x_3 = 5$

$x_4 = 8$

$x_5 = 13$

Q. The solution to the recurrence relation $a_n = 2a_{n-1} - 1$ with $a_1 = 3$ is

a) 3^n b) $2^n + 1$

c) 5^n d) None

$x + y = 7$

$x =$ $y =$

Given Recurrence Relation is

$\boxed{a_n = 2a_{n-1} - 1}$; with $a_1 = 3$

option (a)

$a_n = 3^n$ $a_9 = 3^9$

$a_{n-1} = 3^{n-1}$ $a_{100} = 3^{100}$

Initial condition $a_1 = 3^1 = 3$

Initial condition is satisfied

Given Recurrence Relation is

$a_n = 2a_{n-1} - 1$

$RHS = 2a_{n-1} - 1$

$= 2[3^{n-1}] - 1 \neq LHS$

This is not a solution.

option (b)

Solution is $\boxed{a_n = 2^n + 1}$ ✓

$a_{n-1} = 2^{n-1} + 1$ ✓

Initial condition $a_1 = 3$

$a_1 = 2^1 + 1 = 3$

Initial condition is satisfied

R.R. $a_n = 2a_{n-1} - 1$

$RHS = 2a_{n-1} - 1$

$= 2[2^{n-1} + 1] - 1$

$= 2 \cdot 2^{n-1} + 2 - 1$

$= 2^n + 1$

$= a_n$

$= LHS$

$LHS = RHS$

$\therefore$ This is solution R.R.

Solution to Recurrence Relation:

1) Substitution Method

2) Master's Method

3) Method of characteristic roots

4) Method of undetermined co-efficient

Substitution Method

Q. Solve the recurrence relation $a_n = a_{n-1} + n$ with $a_0 = 4$

Sol Given R.R $a_n = a_{n-1} + n$ with $a_0 = 4$

$$a_1 = a_0 + n = 4 + 1 = 5 \quad 1$$
$$a_2 = a_1 + 2 = 5 + 2 = 7 \quad 2$$
$$a_3 = a_2 + 3 = 7 + 3 = 10 \quad 3$$
$$a_4 = a_3 + 4 = 10 + 4 = 14 \quad 4$$
$$\vdots$$
$$a_n = a_{n-1} + n$$

Here we have

$$\cancel{a_1} - a_0 = 1$$
$$\cancel{a_2} - \cancel{a_1} = 2$$
$$\cancel{a_3} - \cancel{a_2} = 3$$
$$\cancel{a_4} - \cancel{a_3} = 4$$
$$\vdots$$
$$+ \quad a_n - \cancel{a_{n-1}} = n$$

$$a_n - a_0 = 1 + 2 + 3 + \cdots + n$$

$$a_n - a_0 = \frac{n(n+1)}{2}$$

$$a_n - 4 = \frac{n(n+1)}{2}$$

$$\boxed{a_n = \frac{n(n+1)}{2} + 4}$$

Q. Solve the recurrence relation $x_n = 2x_{n-1} - 1$ where $n > 1$ and $x_1 = 2$

$x_n = 2x_{n-1} - 1, \quad x_1 = 2$

$x_2 = 2x_1 - 1 = 2(2) - 1 = 3$

$x_3 = 2x_2 - 1 = 2(3) - 1 = 5$

$x_4 = 2x_3 - 1 = 2(5) - 1 = 9$

$x_5 = 2x_4 - 1 = 2(9) - 1 = 17$

$\vdots$

$x_n = 2x_{n-1} - 1$

$1 = 2^0$

$2 = 2^1$

$4 = 2^2$

$8 = 2^3$

$\cancel{x_2} - x_1 = 1 = 2^0$

$\cancel{x_3} - \cancel{x_2} = 2 = 2^1$

$\cancel{x_4} - \cancel{x_3} = 4 = 2^2$

$\cancel{x_5} - \cancel{x_4} = 8 = 2^3$

$\vdots$

$x_n - \cancel{x_{n-1}} = 2^{n-2}$

$\oplus$

$x_n - x_1 = 2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{n-2}$

[n-1] terms

$$x_n - x_1 = 2^0 + 2^1 + 2^2 + \cdots + 2^{n-2} \quad ((n-1) \text{ terms}$$

$$s_n = \frac{a(r^n - 1)}{r-1}$$

$$= \frac{a[r^{n-1} - 1]}{r-1}$$

$$= \frac{1(2^{n-1} - 1)}{2-1}$$

$$x_n - x_1 = 2^{n-1} - 1$$

$$x_n - 2 = 2^{n-1} - 1$$

$$\boxed{x_n = 2^{n-1} + 1}$$

Q. Consider the recurrence relation $a_1 = 8$,

$a_n = 6n^2 + 2n + a_{n-1}$. Let $a_{99} = K \times 10^4$. The value of K is ______.

(GATE-16-Set1)

Initial condition $a_1 = 8 = \underline{6(1)^2} + 2(1)$

R.R. : $a_n = 6n^2 + 2n + a_{n-1}$

$a_2 = \underline{6(2)^2} + 2(2) + a_1$

$a_3 = \underline{6(3)^2} + 2(3) + a_2$

$a_4 = \underline{6(4)^2} + 2(4) + a_3$

$a_5 = \underline{6(5)^2} + 2(5) + a_4$

⋮

$a_{99} = 6(99)^2 + 2(99) + a_{98}$

$= 6(99)^2 + 2(99) + [6(98)^2 + 2(98) + a_{97}]$

$= \underline{6(99)^2} + 2(99) + \underline{6(98)^2} + 2(98) + \underline{6(97)^2} + 2(97) + a_{96}$

$= 6[99^2 + 98^2 + 97^2 + \cdots + 1^2] + 2[99 + 98 + 97 + \cdots + 1]$

$= \cancel{6}\left[\frac{99(99+1)(2.99+1)}{\cancel{6}}\right] + \cancel{2}\ \frac{99(99+1)}{\cancel{2}}$

$$= (99 \times 100 \times 199) + (99 \times 100)$$

$$= 99 \times 100 \, (199 + 1)$$

$$= 99 \times 100 \times 200$$

$$= 1980000$$

$$= 198 \times 10^4$$

$$\therefore K = 198$$

$$\sum n = \frac{n(n+1)}{2}$$

$$\sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

Note:

$T(n) = \theta(n)$
- $\rightarrow O(n), O(n \log n), O(n^2), O(n^3)$
- $\rightarrow \theta(n)$
- $\rightarrow \Omega(n), \Omega(\log n), \Omega(\sqrt{n}), \Omega(n/2)$

$A \xrightarrow{40} B \quad C$

place 'A' → place B

20min, 30min, 40min → $O(n)$

20min → $\theta(n)$

20min
15 min
10 min
$\boxed{5 \text{ min}}$ → $\Omega(n)$

$\theta(n) = \theta(c.n)$

$n \log n = \theta(n \log n)$

Reflexive : $\theta[f(n)] = f(n)$

Asymptotic Notations:

Asymptotic Notations are used to describe the running time of an algorithm.

how much time an algorithm takes with a given input 'n'

we have three different Notations

i) big (O) → $f(n) \leq c.g(n)$ Upper Bound

ii) Theta (θ) → $f(n) = c.g(n)$ Exact (or) Tignbound

iii) big omega (Ω) → $f(n) \geq c.gn$ Lower Bound

Master's Theorem :

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

* Specific recurrence relations

$T(n)$ = Given task

Sub task

Each subtask $T\left(\frac{n}{b}\right)$

a = Number of subtasks.

$$T(n) = 5\,T\left(\frac{n}{3}\right) + \theta(n \log n)$$

Each subtask. $\frac{n}{3}$

How Many 5

$a : b^K \rightarrow P$

Master Theorem by Division:

If $T(n) = a\,T\left(\frac{n}{b}\right) + \theta(n^k \,.\, \log_b^p n)$ ✓

Where $a > 0$, $b > 1$, $k \geq 0$ and p is real

Case – 1: If $a > b^k$ then $T(n) = \theta\left[n^{\log_b^a}\right]$ ✓

$T(n) = a\,T(\frac{n}{b}) + f(n)$

$T(n) = a\,T(\frac{n}{b}) + \theta\left[n^k . \log_b^p n\right]$

$(\log_b n)^p = \log_b^p n$

Case – 2: If $a = b^k$ and

i) $P > -1$ then $T(n) = \theta\left[n^{log_b^a}.log_b^{p+1}n\right]$ $= \theta\left[n^{log_b^a} * log_b^{n^{p+1}}\right]$

ii) $P = -1$ then $T(n) = \theta\left[n^{log_b^a}.log_b^{log_b^n}\right]$ $= \theta\left[n^{log_b^a} * log_b log_b^n\right]$

iii) $P < -1$ then $T(n) = \theta\left[n^{log_b^a}\right]$

Case-3: If $a < b^k$ and

i) $P \geq 0$ then $T(n) = \theta[n^k . log_b^p n]$

ii) $P < 0$ then $T(n) = \theta(n^k)$

Master Theorem by Division

Case 1: If $a > b^k$

- $T(n) = \theta\left[n^{log_b^a}\right]$

Case – 2: If $a = b^k$ and

- $P > -1$ then $T(n) = \theta\left[n^{log_b^a}.log_b^{p+1}n\right]$
- $P = -1$ then $T(n) = \theta\left[n^{log_b^a}.log_b^{log_b^n}\right]$
- $P < -1$ then $T(n) = \theta\left[n^{log_b^a}\right]$

Case-3: If $a < b^k$ and

- * $P \geq 0$ then $T(n) = \theta\left[n^k.log_b^p\, n\right]$
- $P < 0$ then $T(n) = \theta(n^k)$

ACE

Q. The recurrence relation $T(1) = 2$

$T(n) = 3T\left(\frac{n}{4}\right) + n$ has the solution T(n) equal to **(GATE-CS-96)**

a) O(n)

b) O(log n)

c) O($n^{3/4}$)

d) none of these

Given Recurrence Relation is

$T(n) = 3T(\frac{n}{4}) + n = 3T(\frac{n}{4}) + \theta(n^1)$

Compare $T(n) = a\, T(\frac{n}{b}) + \theta(n^K \cdot \log_b^P n)$

Here

$a=3, \quad b=4, \quad K=1, \quad P=0$

Here $a=3, \quad b^K = 4^1 = 4$

clearly $a < b^K, \quad P \geq 0$

$\therefore T(n) = \theta[n^K \cdot \log_b^P n]$

$= \theta[n^1 \cdot \log_b^0 n]$

$= \theta(n) \longleftrightarrow O(n)$

Master Theorem for Subtraction:

If $T(n) = a\ T(n - b) + \theta(n^k)$ where $a > 0$, $b \geq 1$, $k \geq 0$

i) If $a > 1$, $T(n) = \theta(n^k \,.a^{n/b})$

ii) If $a = 1$, $T(n) = \theta(n^{k+1})$

iii) If $a < 1$, $T(n) = \theta(n^k)$

Q. $T(1) = 1, T(n) = 2T\left(\frac{n}{2}\right) + \sqrt{n}$ for $n \geq 2$

a) $T(n) = O(\sqrt{n})$

b) $T(n) = O(n)$

c) $T(n) = O(\log n)$

d) None

Given Recurrence Relation is

$T(n) = 2T\left(\frac{n}{2}\right) + \sqrt{n}$ for $n \geq 2$

$T(n) = aT\left(\frac{n}{b}\right) + \theta\left(n^k . \log^p_b n\right)$

$a = 2, b = 2, k = \frac{1}{2}, p = 0$

$a = 2, b^k = 2^{1/2} = \sqrt{2}$

clearly $a > b^k$.

$T(n) = \theta\left[n^{\log_b a}\right]$

$= \theta\left[n^{\log_2 2}\right]$

$= \theta\left[n^1\right] \longleftrightarrow O(n)$

Shift Operation E:

The shift operator E is defined as

$$E(a_n) = a_{n+1}$$

$$E^2(a_n) = a_{n+2}$$

$$E^3(a_n) = a_{n+3}$$

$$\dots \dots \qquad \dots$$

$$E^k(a_n) = a_{n+k}$$

III) Methods of characteristic Roots:-

Consider the linear recurrence relation $l_0 a_n + l_1 . a_{n-1} + \ldots\ldots + l_k . a_{n-k} = f(n)$ ✓ ………….. (1)

Replacing n by n + k, we have

$\Rightarrow l_0.a_{n+k} + l_1.a_{n+k-1} + \ldots\ldots + l_k.a_n = F(n)$ ✓

$\Rightarrow l_0.E^k(a_n) + l_1.E^{k-1}(a_n) + \ldots\ldots + l_k.a_n = F(n)$

$\Rightarrow (l_0.E^k + l_1.E^{k-1} + \ldots\ldots + l_k)a_n = F(n)$

$\Rightarrow \boxed{\phi(E)\ a_n = F(n)}$ ………… (2)

Where $\phi(E) = l_0.E^k + l_1.E^{k-1} + \ldots\ldots + l_k$

$l_0 a_n \quad + l_1 a_{n-1} \quad + l_2 a_{n-2}$

$a_n + a_{n-1} + a_{n-2} = n^2 + n.$

put $n = n+2$

$a_{n+2} + a_{n+1} + a_n = (n+2)^2 + (n+2)$

$r^{n+2} \quad r^{n+1} \quad r^{n}$

$E^2(a_n) \quad E^1(a_n) \quad E^0(a_n)$

The characteristics equation is

$$\phi(t) = 0$$

The roots of this equation are called **characteristic roots.**

Let $t = t_1, t_2 \ldots\ldots t_k$ be the characteristic roots

$\phi(E)\, a_n = F(n)$

$\phi(E)$

$E^2 + 2E + 1$

$\phi(t) = 0$

$t^2 + 2t + 1 = 0$

$a_n = 2a_{n-1} + 3a_{n-2} + n^2$

$a_n - 2a_{n-1} \mp 3a_{n-2} = n^2$ put $n = n+2$

$a_{n+2} - 2a_{n+1} \mp 3a_n = (n+2)^2$

$E^2(a_n) - 2E^1(a_n) \mp 3E^0(a_n) = (n+2)^2$

$(E^2 - 2E \mp 3)\, a_n = F(n)$

$\phi(E)\, a_n = F(n)$

$\phi(E) = E^2 - 2E \mp 3$

$\phi(t) = t^2 - 2t \mp 3 = 0$

$t = t_1, t_2$

$t = t_1, t_2, t_3$

Complimentary Function (C.F):-

This is solution of equation (1)

When f(n) = 0,

i.e., the solution of homogenous part eq of equation (1).

$$a_n + a_{n-1} + a_{n-2} = n^2+n+1$$

$$a_n + a_{n-1} + a_{n-2} = 0.$$ Homogenous eq.

Complete Solution $a_n = C.F + P.S$

Rules for finding (C.F) are given below.

1. Characteristics roots are real and distinct say $t_1, t_2 \ldots\ldots t_k$

$$C.F = c_1.t_1^n + c_2.t_2^n + \ldots\ldots + c_k.t_k^n \quad = c_1.t_1^n + c_2.t_2^n + c_3.t_3^n + \cdots$$

2. Roots are real and two roots are equal say $t_1, t_1, t_3, t_4 \ldots, t_k$ $\quad t_1, t_1, t_3$

$$C.F = (c_1 + c_2.n)t_1^n + c_3.t_3^n + \ldots\ldots + c_k.t_k^n \quad =$$

3. Roots are real and 3 roots are equal say $t_1, t_1, t_1, t_4 \ldots\ldots t_k$

$$C.F = (c_1 + c_2.n + c_3.n^2).t_1^n + c4.t_4^n + \ldots\ldots + c_k.t_k^n$$

4. Suppose if all the roots are equal say $t_1, t_1, t_1, \ldots\ldots t_1$

$$C.F = (c_1 + c_2.n + c_3.n^2 + \ldots\ldots + c_k.n^{k-1})t_1^n$$

5. A pair of roots are complex say $(\alpha \pm i\beta)$

$$C.F = r^n(c_1.\cos(n\theta) + c_2.\sin(n\theta)\}$$

Where

$$r = \sqrt{\alpha^2 + \beta^2} \text{ and}$$

$$\theta = tan^{-1}\left(\frac{\beta}{\alpha}\right)$$