Q. What is the minimum possible weight of a path P from vertex 1 to vertex 2 in this graph such that P contains at least 3 edges?

a) 7 ✓

b) 8

c) 9

d) 10

path from vertex '1' to vertex '2' suchthat

path length $(P^L) \geq 3$

path = 1 − 0 − 3 − 4 − 2 = 1+1+2+3 = 7 units

Q. Let G be connected undirected graph of 100 vertices and 300 edges. The weight of a minimum spanning tree of G is 500. When the weight of each edge of G is increased by five, the weight of a minimum spanning tree becomes ______ **(GATE)**

$n = 100, \quad e = 300, \quad cost(MST) = 500$

Each edge cost increased by '5'

MST has 100 vertices with 99 edges

$99 \times 5 = 495$

$cost(MST)_{new} = 500 + 495 = 995$

Planar Graphs: A simple graph without any edge crossing. *A simple graph which immersed in a two-dimensional plane without cross edges is known as a plane graph*

Ex: K_3, K_4, $K_{2,3}$

For Plane Graph:

$r + v = e + 2$

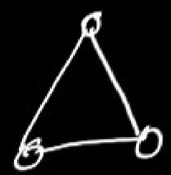
$r = e - v + 2$

$\boxed{e \leq 3v - 6}$

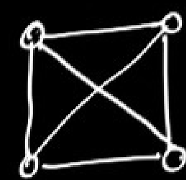
$r \rightarrow$ No. of regions

$v \rightarrow$ No. of vertices

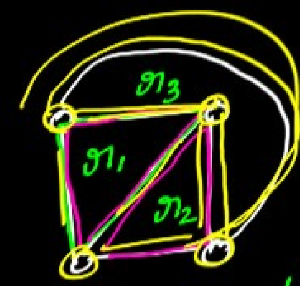
$e \rightarrow$ No. of edges

① 

K_3

② 

K_4

$\Longrightarrow$ 

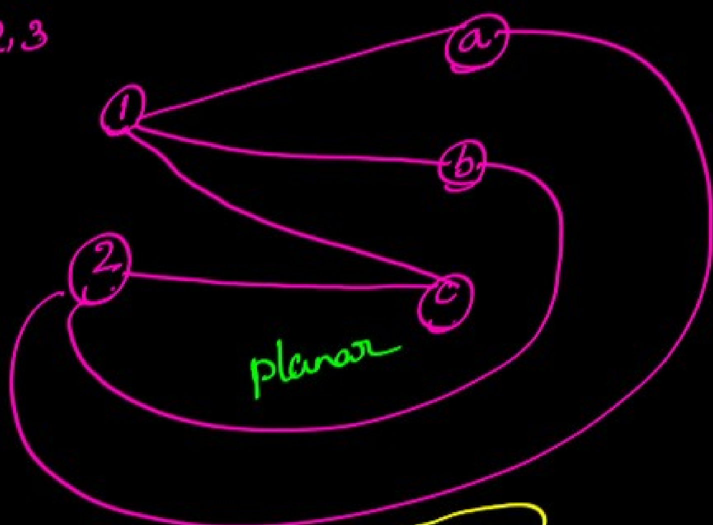
Region	Degree
$r_1 \longrightarrow$	3
$r_2 \longrightarrow$	3
$r_3 \longrightarrow$	3
$r_4 \longrightarrow$	∞

4 regions
3 are bound,
1 unbounded

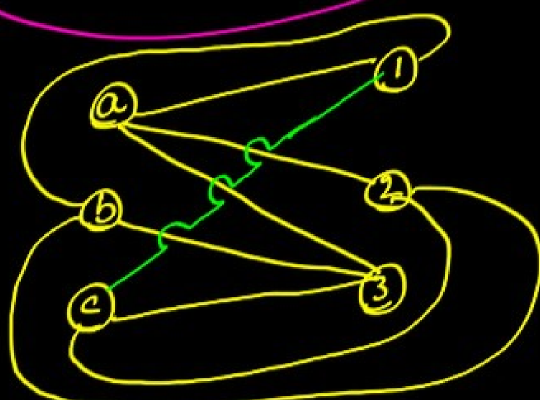
$r + v = e + 2$

$4 + 4 = 6 + 2$

$K_{2,3}$

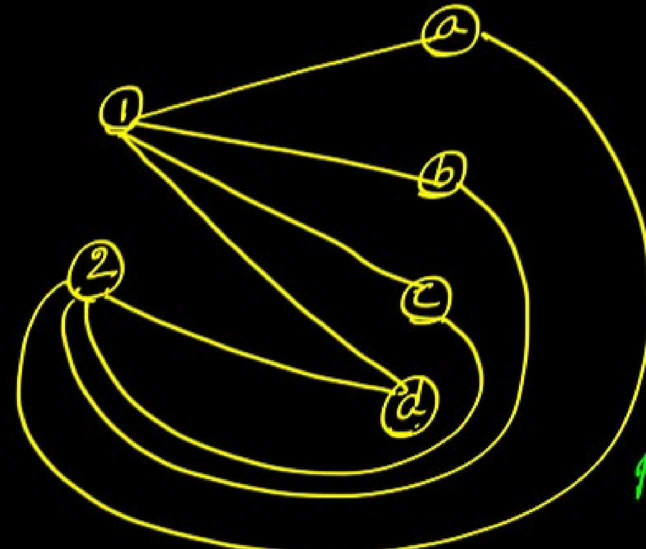


$K_{3,3}$



Non-planar

$K_{2,4}$ (or) $K_{4,2}$



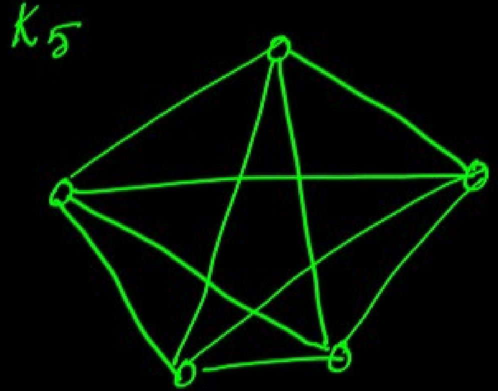
planar

K_5

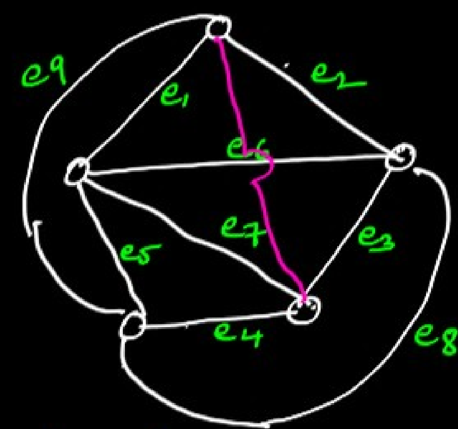
$K_{2,10}$

$K_{2,100}$

$K_{n,2}$



$\Longrightarrow$



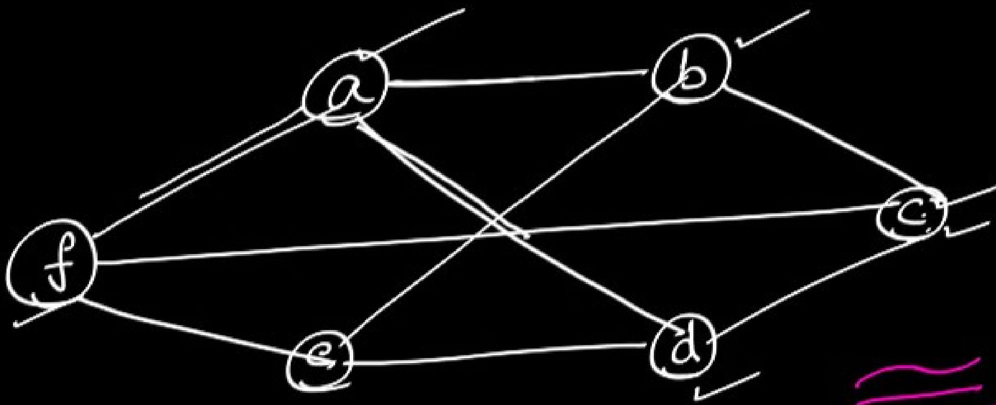
$e = nC_2 = \frac{n(n-1)}{2} = \frac{5\times4}{2} = 10$

K_5 is Non-planar.

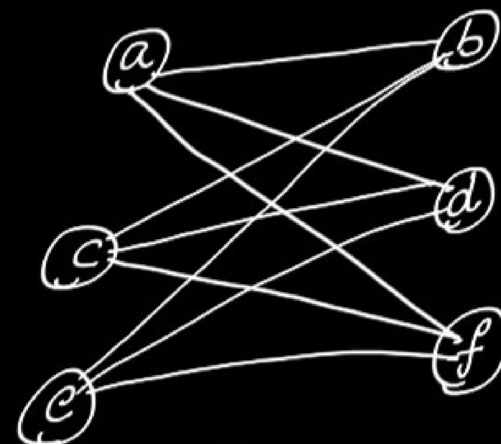
$e \leq 3v - 6$

$e \leq 3(5) - 6$

$\boxed{e \leq 9}$



$\cong$



$K_{3,3}$

Isomorphic

Kuratowski's

K_5 (or) $K_{3,3}$ Non-planar

Homeomorphic to K_5 (or) $K_{3,3}$

$\boxed{e \leq 3v-6}$

$\therefore$ $K_{3,3}$ is Non-planar.

v

M N

Four color Theorem: Every plane graph is four-colorable. Maximum Number of colors required for proper coloring is 4

i.e., $\chi(G) \leq 4$

Note: This statement is true for both vertex coloring and map coloring.

$\chi(G) \leq 4$

chi

Q. The minimum number of colours that is sufficient to vertex colour any planar graph is ___4___. **(GATE)**

Q. In an undirected connected planar graph G, there are eight vertices and five faces. The number of edges in G is ________. **(GATE)**

$v = 8, \quad r = 5, \quad e = ?$

for plane graph,

$$r + v = e + 2$$

$$5 + 8 = e + 2$$

$$\boxed{e = 11}$$

faces $= r = 5$

Bounded faces $= r - 1 = 4$

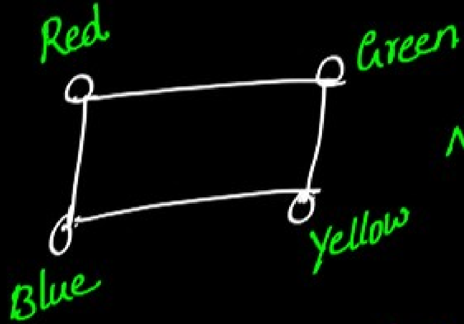
Graph Coloring: A coloring of a simple graph is the assignment of colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Chromatic Number: The minimum number of colors needed for vertex colouring of a graph G is called the chromatic number of G, denoted by $\chi(G)$.

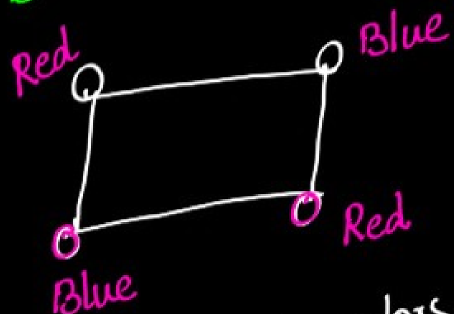
Adjacent Regions: In a planar graph two regions are said to be **adjacent** if they have a common edge.

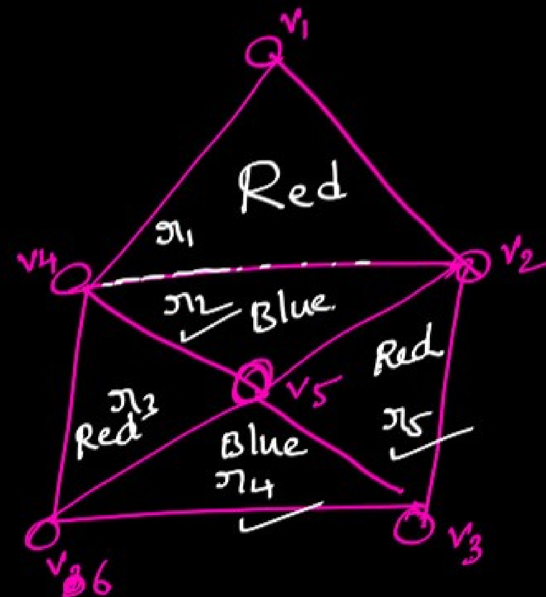
Region colouring (map colouring): An assignment of colors to the regions of a map such that adjacent regions have different colors.

A map 'M' is **n-colorable** if there exists a coloring of M which uses atmost n colors.



No. of colors = 4





Min no. of colors for proper coloring graph $= \chi(G) = 2$

Welch-powell's algorithm:

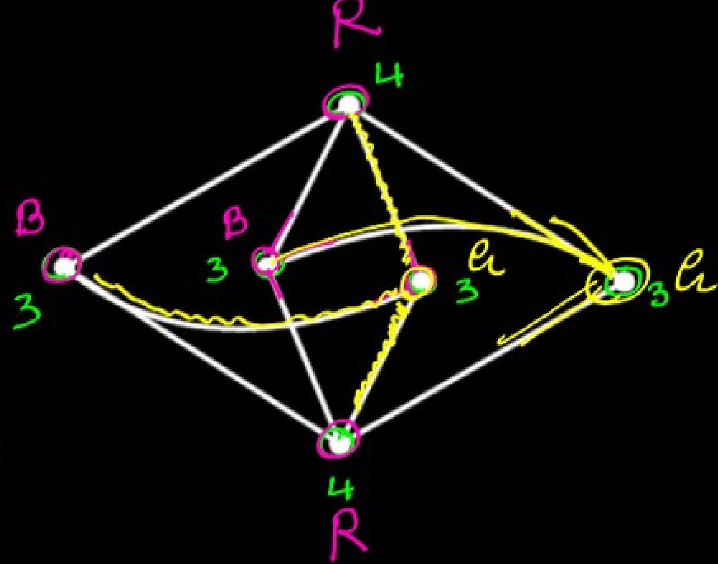
(for vertex colouring)

Degree Seq.

Step 1: Arrange the vertices of the graph G in the descending order of their degrees.

Step 2: Assign colours to the vertices in the above order so that no two adjacent vertices have same colour.

Q. For the graph shown below, the chromatic number is ________.



$n = 6$

Degree sequence:

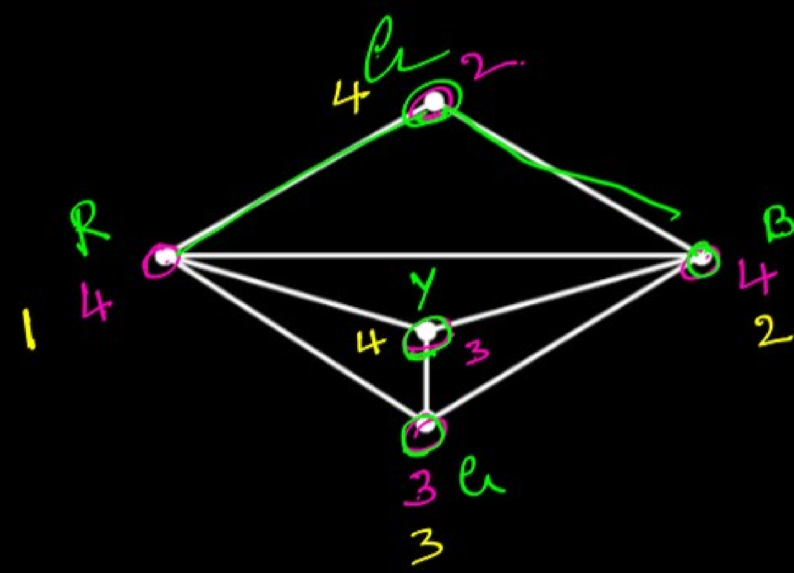
4, 4, 3, 3, 3, 3

*

Welch-Powell's

$\chi(G) = 3$

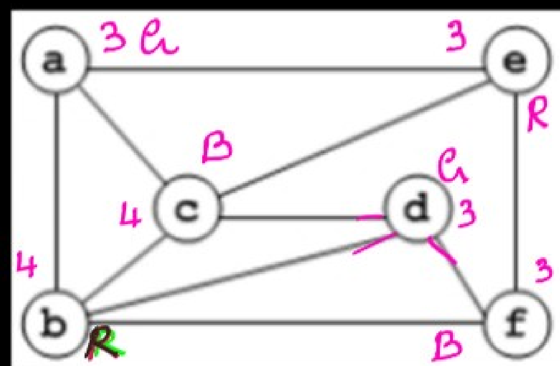
Q. For the graph shown below, the chromatic number is ________.



$n=5$

$\chi(G)=4$

Q. The chromatic number of the following graph is _________. (GATE)



$\chi(G) = 3$

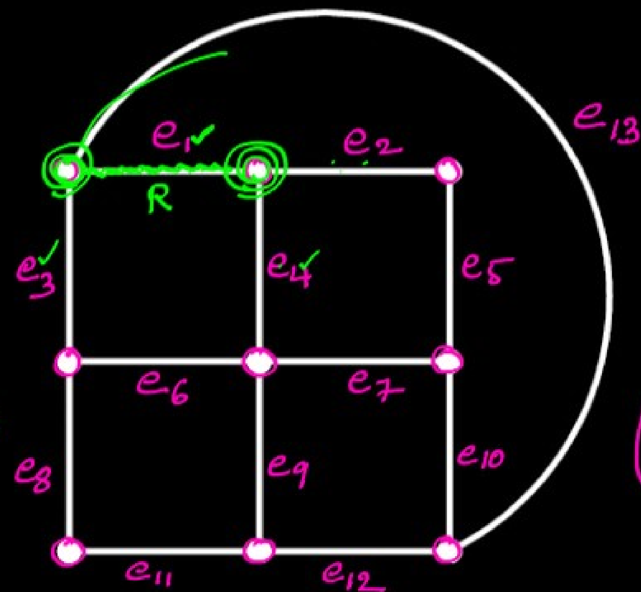
Chromatic Number

i) Vertex Coloring

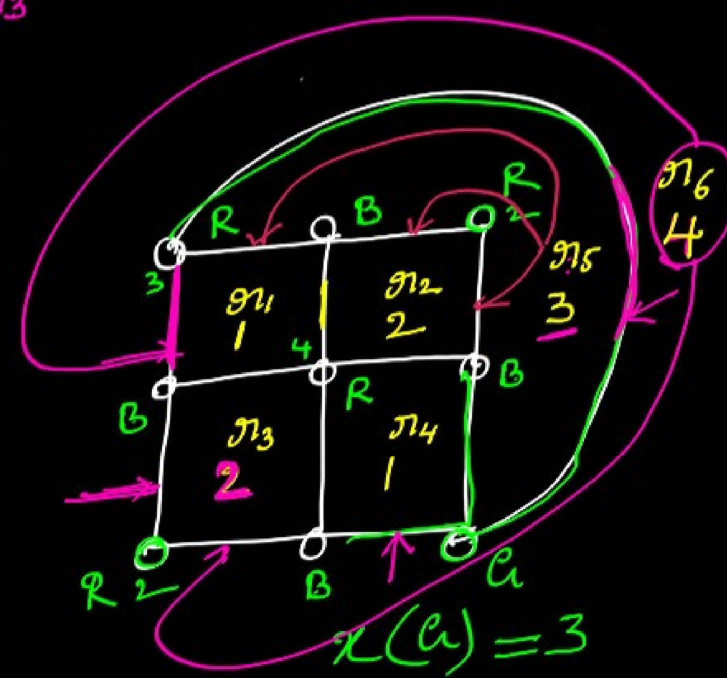
ii) Region Coloring

iii) Edge Coloring

e_1 adjacent to e_2, e_3, e_{13}, e_4



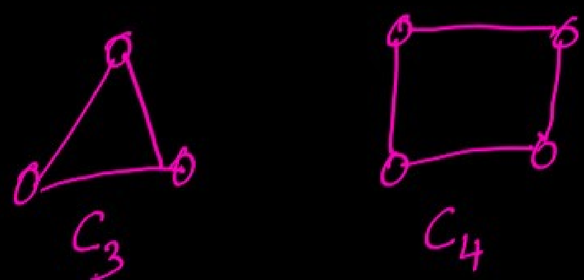
Map coloring chrom = 4

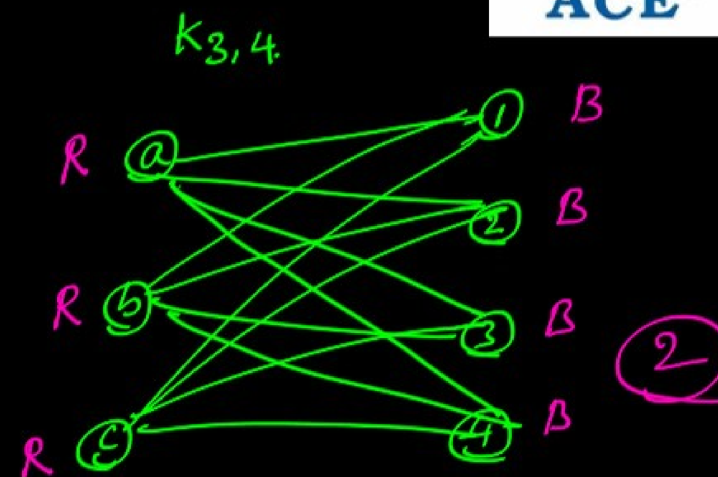


* Chromatic Number of $C_n = 2$ (n is even)
$= 3$ (n is odd)

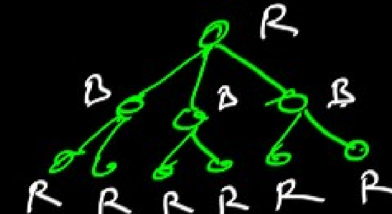
* Chromatic Number of $K_n = n$

* Chromatic Number of $K_{m,n} = 2$





* Chromatic Number of Tree $= 2$



Q. Graph G is obtained by adding vertex s to $K_{3,4}$ and making s adjacent to every vertex of $K_{3,4}$. The minimum number of colours required to edge colour G is ________. (GATE)

Matchings, Coverings, Independent Sets:

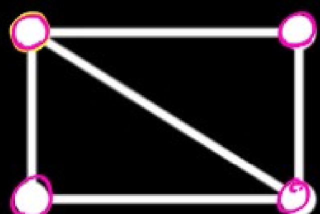
Matching: A matching M, of a graph 'G' is a subgraph such that every vertex of 'G' is incident with atmost one edge in M

$$\deg(v_i) \le 1 \ \forall \ v_i \in G$$

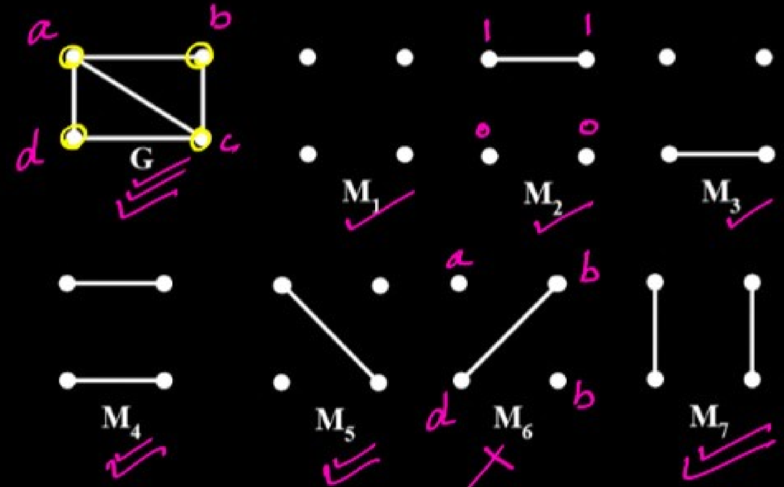
$\boxed{deg(v_i) \le 1}$

degree of any vertex is atmost one.

Ex:



G



M_6 is NOT Subgraph of G

$\therefore$ M_6 is not Matching

Maximal = cannot add

Minimal = cannot remove

* A Matching M is called Maximal if THERE IS NO POSSIBILITY to add any edge to it.

In the example M_4, M_5, M_7 are maximal matchings.

* A maximal matching containing maximum no. of edges is called Largest Maximal Matching. $|M_4| = 2$, $|M_5| = 1$ $|M_7| = 2$

* The no. of edges in a Largest Maximal Matching is called Matching number $M = 2$

$|E(M_4)| = 2$

Note:

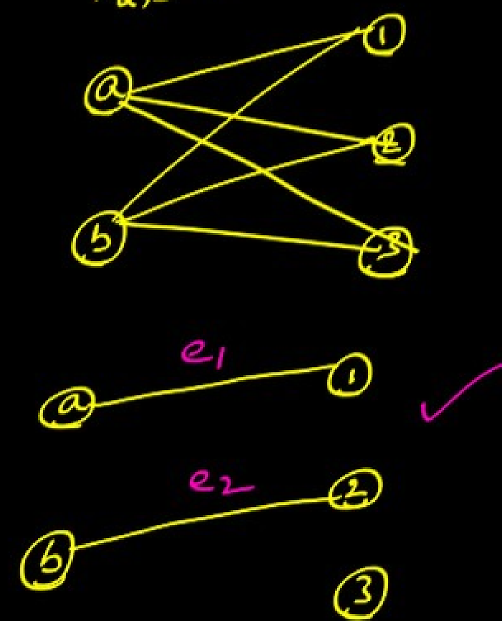
* Matching number of a complete Bi-patriate graph $K_{m,n} = \min\{m, n\}$

$K_{2,3} = \min\{2,3\} = 2$

* Matching number of a complete graph $K_n = \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$

* Matching number of a cycle graph $C_n = \left\lfloor \frac{n}{2} \right\rfloor$

* Matching number of a wheel graph $W_n = \left\lfloor \frac{n}{2} \right\rfloor$



* **Perfect Matching:** A matching is said to be perfect if

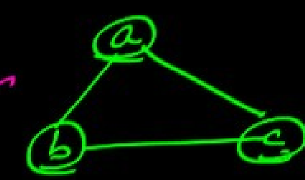
$$\text{if } \deg(v_i) = 1 \quad \forall\, v_i \in G$$

$\deg(v_i) = 1$

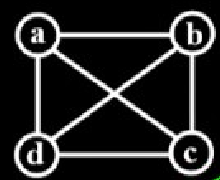
Complete ~~Common~~ graph K_n has perfect matching iff n is even.

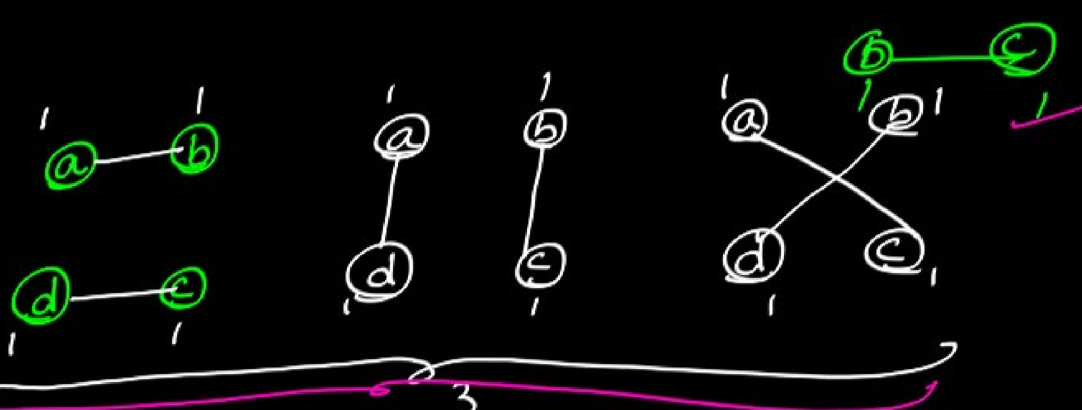
$\therefore$ No. of perfect matching in $K_{2n} = \frac{(2n)!}{2^n * n!} = \frac{2n!}{2^n * n!}$

K_3



Ex:

$K_4 =$ 



3

* $K_{m,n}$ has perfect matching iff m = n

Number of perfect matching $K_{n,n}$ = n!

98853 27372

No. of perfect Matchings for $K_{2n} = \dfrac{(2n)!}{2^n * n!}$

$$K_4 = K_{2(2)} = \frac{4!}{2^2 * 2!} = \frac{4*3*2*1}{4*2*1} = 3$$

* The Number of perfect Matchings on K_6 =

No. of perfect Matchings for $K_{2n} = \dfrac{(2n)!}{2^n * n!}$

$$K_6 = K_{2(3)} = \frac{6!}{2^3 * 3!} = \frac{\cancel{6}\times5\times\cancel{4}\times3\times\cancel{2}\times1}{\cancel{8}\times\cancel{6}} = 15$$

$K_{2n} = K_{2x} = K_{2K}$

No. of vertices

Coverings:

Line Covering: The ~~number~~ Set of edges, which covers all vertices of a graph.

Cannot Remove

Minimal Line Covering: A line covering is called minimal if there is NO possibility to remove any edge from it.

Smallest minimal Line Covering: A minimal line covering containing minimum no. of edges.

- Number of edges in a smallest minimal line covering is called

LINE COVERING NUMBER (α_1)

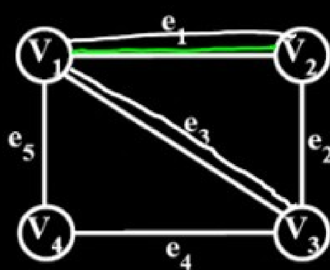
Vertexset = $\{v_1, v_2, v_3, v_4\}$

Edges set = $\{e_1, e_2, e_3, e_4, e_5\}$

$\{e_1, e_3, \cancel{e_5}\} \longrightarrow v_4$ Missing

$\{e_1, \cancel{e_3}, e_5\} \longrightarrow v_3$ "

$\{\cancel{e_1}, e_3, e_5\} \longrightarrow v_2$ "



$\boxed{\alpha_1 = 2}$

$\{e_1, e_2, e_3, e_4, e_5\}$ = Line covering

$\{e_1, e_3, e_5\}$ = Minimal line covering

$\{e_2, e_3, e_4\}$ = Minimal line covering.

$\{e_1, e_4\}$ = Smallest Minimal L.C.

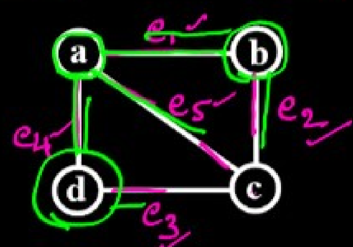
$\{e_2, e_5\}$ = Smallest Minimal L.C.

* **Vertex Covering:** The set of vertices, which covers all edges

* **Minimal Vertex Covering:** A vertex covering is said to be minimal if the no possibility to remove any vertex from the set.

* **Smallest Minimal Vertex Covering:** A minimal vertex covering containing minimum no. of vertices is called smallest minimal vertex covering.

* The number of vertices in a smallest minimal vertex covering is called VERTEX COVERING NUMBER (α_2)



$\{a, b, d\}$

$a \to e_1, e_5, e_4$

$b \to e_1, e_2$

$d \to e_4, e_3$

{a, b, c, d} vertex covering

{a, b, d} minimal vertex covering $\{a, b, \cancel{d}\}$

{b, d, c} minimal vertex covering

{a, c} smallest minimal vertex covering

$\alpha_2 = 2$

Independent Sets:

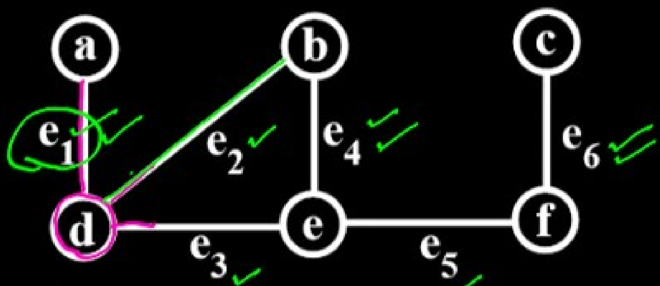
* **Line Independent Set:** Set of edges which are NOT adjacent.

* **Maximal Line Independent Set:** A Line Independent set is said to be maximal if it is not possible to add any edge to it.

* **Largest Maximal Line Independent Set:** A maximal Line Independent Set containing maximum no.of edges

* **No. of edges:** In a largest maximal line independent set is called LINE INDEPENDENT NUMBER (β_1).

G:



$e_1 \xrightarrow{\text{adjacent}} e_2, e_3$

$e_2 \longrightarrow e_1, e_3, e_4$

$\{e_2, e_5, \cancel{e_1}, \cancel{e_3}, \cancel{e_4}, \cancel{e_6}$

$\{e_1, e_4\}$ Line Independent Set

$\{e_1, e_5\}$ Maximal Line Independent

$\{e_1, e_6\}$ Line Independent Set

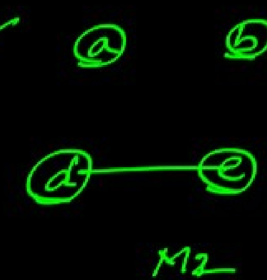
* $\{e_2, e_5\}$ Maximal Line Independent Set

* $\{e_2, e_6\}$ Maximal Line Independent Set

$\{e_4, e_6\}$ Line Independent Set

* $\{e_1, e_4, e_6\}$ Largest Maximal Line Independent

$\beta_1 = 3$



Vertex Independent Set: The vertex independent set of a graph 'G' is the vertices of 'G' which are not adjacent with each other.

Maximal Vertex Independent Set: A vertex independent set is said to be maximal if there is no possibility to add any vertex to it.

Largest Maximal Vertex Independent Set: A maximal vertex independent containing maximum number of vertices is called Largest Maximal Vertex Independent Set.

Number of vertices in it is called VERTEX INDEPENDENT NUMBER (β_2).

β_2

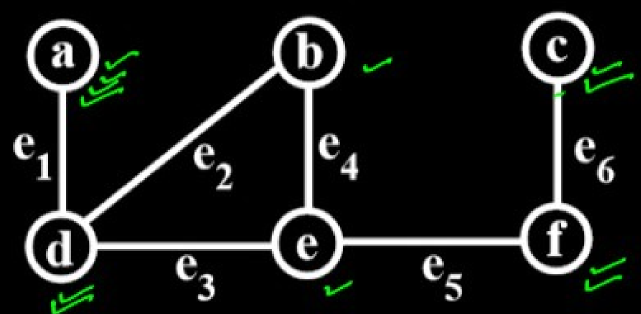
$\{a, b, c\}$

$\{a, b\}$ Vertex Independent Set

$\{a, e\}$ Vertex Independent Set

$\{a, f\}$ Vertex Independent Set

$\{a, c\}$ Vertex Independent Set



* $\{d, c\}$ Maximal Vertex Independent Set $\{d, c\}$

* $\{d, f\}$ Maximal Vertex Independent Set

* $\{a, b, c\}$ Largest Maximal Vertex Independent Set

$\beta_2 = 3$

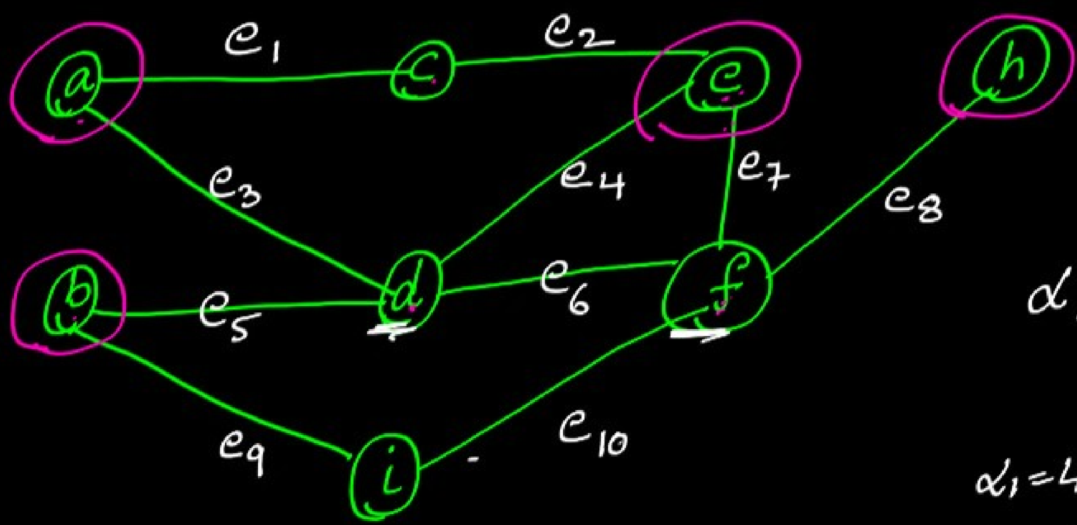
$a \xrightarrow{\text{Neigh}} d$

$d \longrightarrow a, b, e$

$c \longrightarrow f$

Matchings, Coverings, Independent Sets

G:



$\alpha_1 = ?$ $\alpha_2 = ?$ $\beta_1 = ?$ $\beta_2 = ?$

4

$\alpha_1 = 4$ line cov $= \{e_2, e_3, e_8, e_9\}$

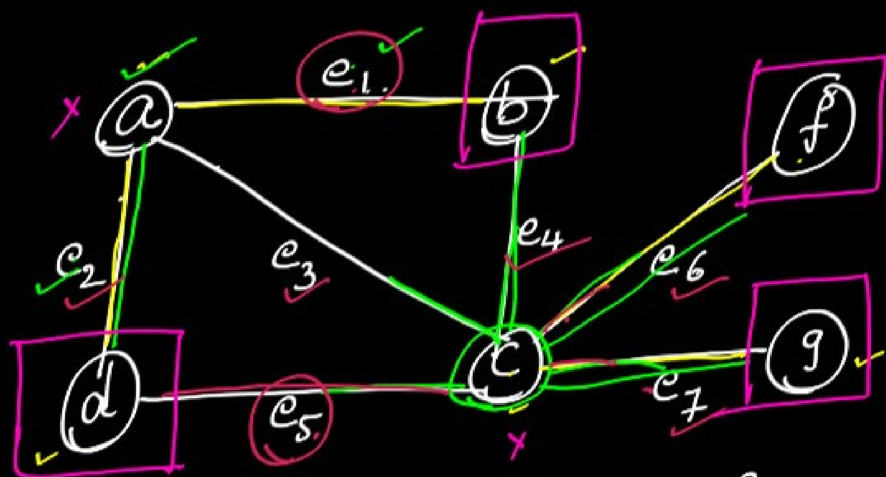
$\alpha_2 = 4$ verte cov $= \cancel{\{a, e, h, b\}}$

$\beta_1 = 4$ line Ind $= \{e_1, e_4, e_8, e_9\}$

$\beta_2 = 4$ vert Ind $= \{a, b, e, \cancel{f}\}$ h

$\{d, f, c, b\}$ ✓

G



$\alpha_1 = ?$ $\alpha_2 = ?$ $\beta_1 = ?$ $\beta_2 = ?$

$\alpha_1 = 4$ $\{e_6, e_7, e_1, e_2\}$ ✓

$\{e_2, e_4, e_6, e_7\}$ ✓

$\alpha_2 = 2$ $\{c, a\}$

$\beta_1 = 2$ $\{e_1, e_5\}$

$\beta_2 = 4$ $\{b, d, f, g\}$

Min. coverings + Max independset

= order of graph

$\boxed{\alpha_2 + \beta_2 = 2 + 4 = 6 = O(G)}$

$\boxed{\alpha_1 + \beta_1 = \alpha_2 + \beta_2}$

$$\alpha_1 + \beta_1 = \alpha_2 + \beta_2$$

Q. What is the size of the smallest MIS (Maximal Independent Set) of a chain of nine nodes? **(GATE)**

a) 5

b) 4

c) 3

d) 2

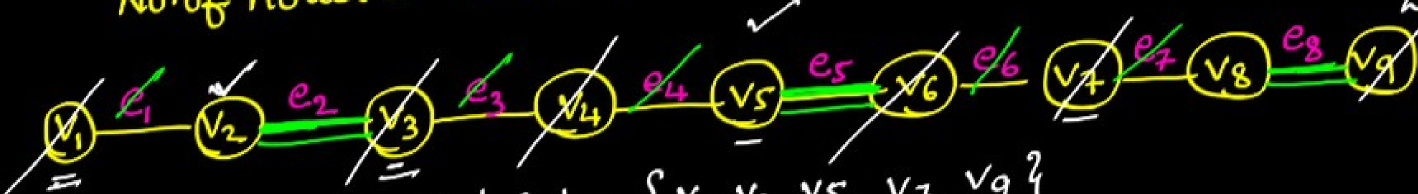
Smallest line independent set $= \{e_1, e_5, e_8\}$

Line independent set

Maximal line indpendent set

Largest Maximal line indp set (β_1)

No. of nodes $= n = v = 9$



Here Smallest Maximal Ind Set

Maximal vertex Ind. set $= \{v_1, v_3, v_5, v_7, v_9\}$

Smallest Maximal vertex Ind. set $= \{v_2, v_5, v_8\}$

Vertex Ind set $= \{v_1, v_3\}$ but not Maximal

Isomorphic:

G_1, G_2 two graph are said to Isomorphic iff $f: G_1 \to G_2$ is a Bijective functions.

[one-one & onto]

Conditions to follow:

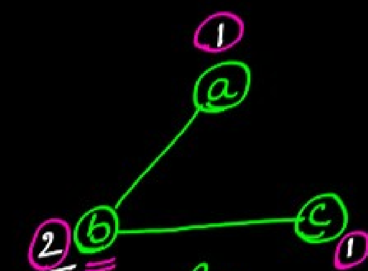
i) Number of Vertices $O(G_1) = O(G_2)$

ii) Number of Edges $|E(G_1)| = |E(G_2)|$

iii) Degree Sequence

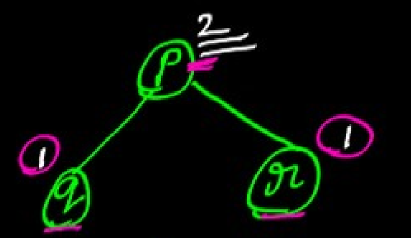
iv) Number of Cycles → even → Length; → odd → length

v) Adjacency Preserved



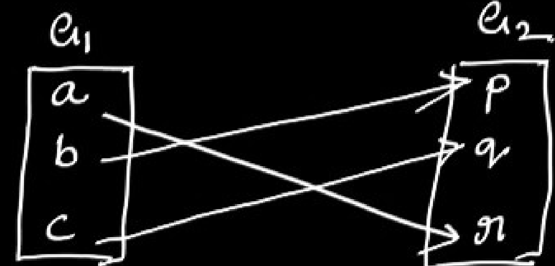
G_1

Deg seq : 2, 1, 1 ✓



G_2

Deg seq : 2, 1, 1 ✓



$f: G_1 \to G_2$ Bijective

$\boxed{G_1 \equiv G_2}$

Self Complementary Graph: A graph is Isomorphic to it's complement graph then the graph is known as self complementary.

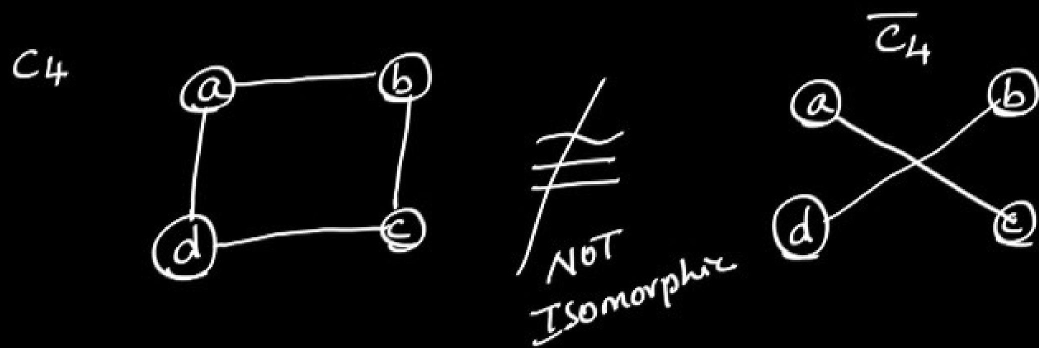
$G \cong \overline{G} \rightarrow$ No. of vertices 4K (or) 4K + 1

* If G is ismorphic to $\overline{G}^{-}$ Then the order of graph G can be 4K (or) 4K+1

8 9 $^{8}C_2$

16 17 2

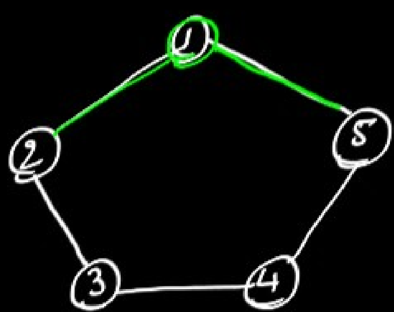
Ex Is C_4 self-Complementary?



Is C_5 Self-complementary?

Sol

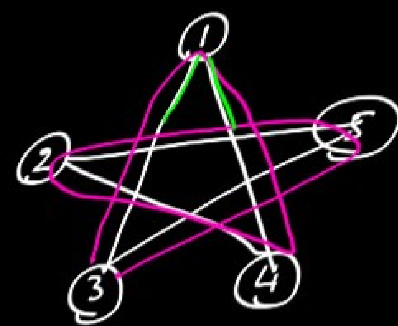
C_5



$V = 5$
$e = 5$
2-reg.
cycle of length '5'

$\cong$

$\overline{C_5}$



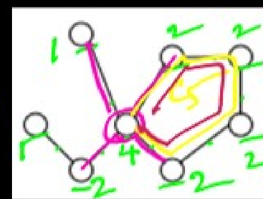
$V = 5$
$e = 5$
2, 2, 2, 2, 2
3-1-4-2-5-3
Cycle 5

$C_1 \cong \overline{C_5}$

So C_1 is self-complementary

Q. Which of the following graphs is isomorphic to **(GATE)**

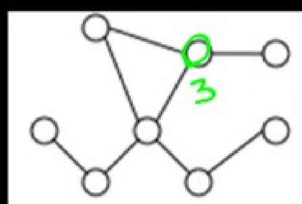
Q.
Question:



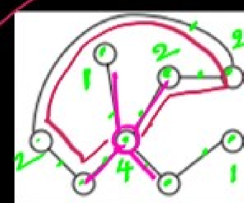
$o(G) = 8$ ✓
size = 8 ✓
4, 2, 2, 2, 2, 2, 1, 1 ✓

(b)

✗ a)

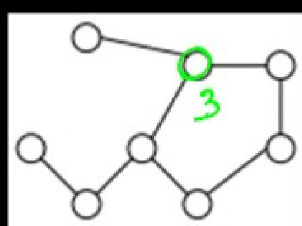


✓ b)

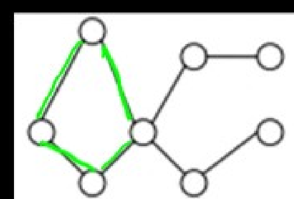


(b) No.of vertices = $o(G) = 8$ ✓
size = 8 ✓
Deg seq. : 4, 2, 2, 2, 2, 2, 1, 1 ✓
C_5

✗ c)



✗ d)

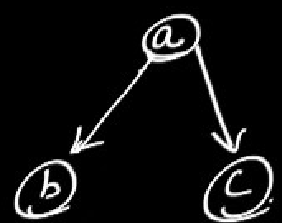


C_4

Topological Sorting:

A sorting of nodes of given simple directed graph, in which, If there an edge from (u,v) then "u comes before v" in ordering, is known as Topological Sorting (or) Topological ordering.

example



a b c ✓

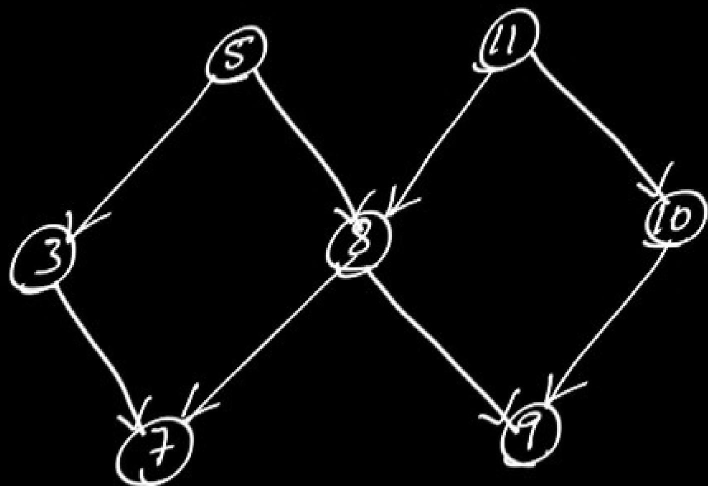
b a c ✗

a c b ✓

c a b ✗

(a, b)

(a, c)



orderings :

a) 5, 11, 3, 8, 10, 7, 9 ✓ Topological ordering

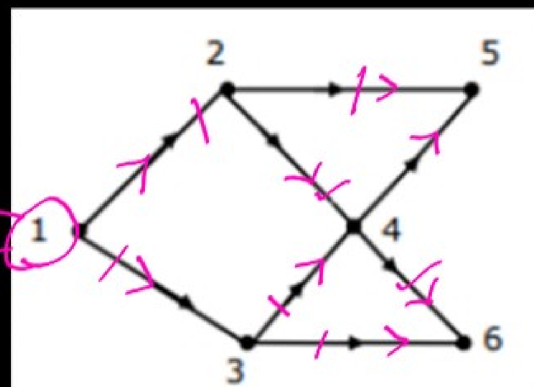
b) 5, 3, $\boxed{7, 8,}$ 11, 10, 9 ✗

c) 5, 3, 11, 8 7, 10, 9 ✓ Topological ordering

d) 11, 5, 3, 10, 8, 9, 7 ✓ Topological sorting.

e) 5, 3, $\boxed{8, 11,}$ 7, 10, 9 ✗

Q. Consider the DAG with V = {1, 2, 3, 4, 5, 6}, shown below: **(GATE)**



Which of the following is NOT a topological ordering?

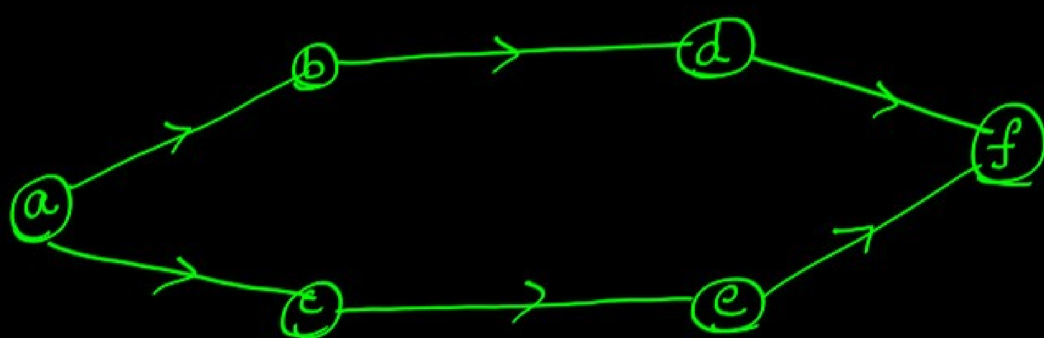
Top a) 1 2 3 4 5 6 ✓

Top b) 1 3 2 4 6 5

Top c) 1 3 2 4 5 6

d) 3 2 4 1 6 5 ✓ NOT

Number of Topological orderings of the following DAG



Topological sortings

1) a, b, c, d, e, f
2) a, c, b, d, e, f
3) a, b, c, e, d, f
4) a, c, b, e, d, f
5) a, b, d, c, e, f
6) a, c, e, b, d, f

$\therefore$ No. of Topological orderings = 6

Q. G is an undirected graph with n vertices and 25 edges such that each vertex of G has degree at least 3. Then the maximum possible value of n is ________. (GATE)

No. of vertices $= n$

No. of edges $= e = 25$

Min. deg of $e = k = 3$.

Sum of degrees theorem

$$\boxed{k \cdot n \leq 2e}$$

$$3 \cdot n \leq 2(25)$$

$$3n \leq 50$$

Max. possible value $n = 16$

$3 \times 16 = 48$

$3 \times 17 = 51$

Q. Let T be a tree with 10 vertices. The sum of the degree of all the vertices in T is ________. **(GATE)**

Graph = Tree (n-nodes with (n-1) edges)

$n = 10$

$e = n-1 = 9$

According to sum of deg.

$$\sum_{i=1}^{n} \deg(v_i) = 2e$$

$$= 2(9)$$

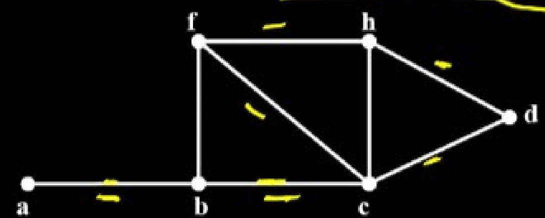
$$= 18$$

Eccentricity

Distance: Distance between two vertices u and v is denoted by d(u, v)

d(u, v) = number of edges in a shortest path between 'u' and 'v'.

d (u, v)



path (a,h) = a-b-c-d-h = 4
path (a,h) = a-b-c-f-h = 4
path (a,h) = a-b-c-h = 3
path (a,h) = a-b-f-h = 3 ✓

Ex: In the above graph

* d(a, h) = 3 ✓

d(b, a) = 1 ✓

d(b, d) = 2 ✓

Distance from 'a' to all vertices:

$d(a,b) = 1$

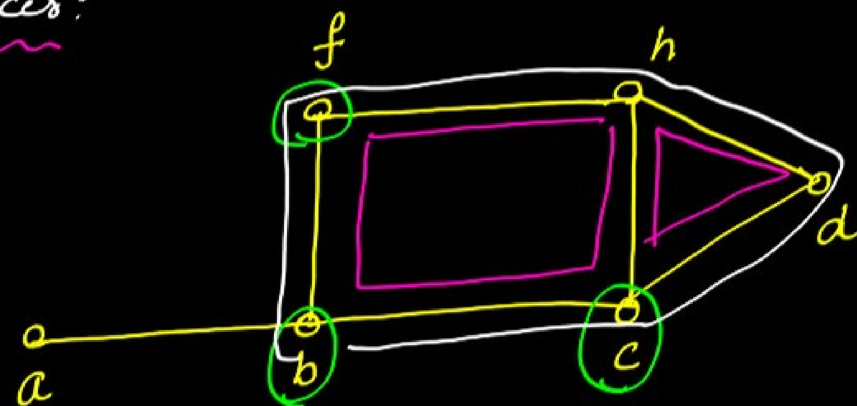
$d(a,c) = 2$

$d(a,d) = 3$

$d(a,f) = 2$

$d(a,h) = 3$

$\{1, 2, \overset{*}{3}, 2, 3\}$



Distance from 'b' to all vertices $= \{1, 1, \overset{*}{2}, 1, 2\}$

Distance from 'c' to all vertices $= \{2, 1, 1, \overset{*}{2}, 1\}$

Distance from 'd' to all vertices $= \{\overset{*}{3}, 2, 1, 2, 1\}$

Distance from 'f' to all vertices $= \{\overset{*}{2}, 1, 2, 2, 1\}$

Distance from 'h' to all vertices $= \{\overset{*}{3}, 2, 1, 1, 1\}$

Eccentricity: eccentricity of a vertex v is denoted by e(v). is

$e(a)$ ~~e(a)~~ = 3, e(b) = 2, e(c) = 2,

e(d) = 3, e(h) = 3, e(f) = 2

$$e(v_i) = \text{Maximum}\{d(v_i, v_j),\ \forall v_j \in V(G)\}$$

$d(v_i, v_1) = *$

$d(v_i, v_2) =$

$d(v_i, v_3) =$

Radius: Radius of a connected graph G is denoted by r(G).

r(G) = minimum of the eccentricities of all vertices in G.

For the graph given above

$\boxed{r(G) = 2}$

$e(a) = 3$
$e(b) = 2*$
$e(c) = 2$
$e(d) = 3*$
$e(h) = 3$
$e(f) = 2 = r(G)$

Diameter: Diameter of a connected graph G is denoted by d(G).

d(G) = Maximum of the ecentricies of all vertices in G.

For the graph given above

d(G) = 3

$d(G)$
$d(u, v)$

Central Point: If $e(v) = r(G)$ then v is called a central point of G.

For the graph given above,

b, c and f are central points of G.

Center: Set of all central points of G is called center of G and denoted by C(G).

For the graph given above

$C(G) = \{b, c, f\}$ $\{b, c, f\} = C(G)$

Circumference of 'G': The number of edges in a longest cycle of G is called circumference of G.

For the graph given above

Circumference of G is 5.

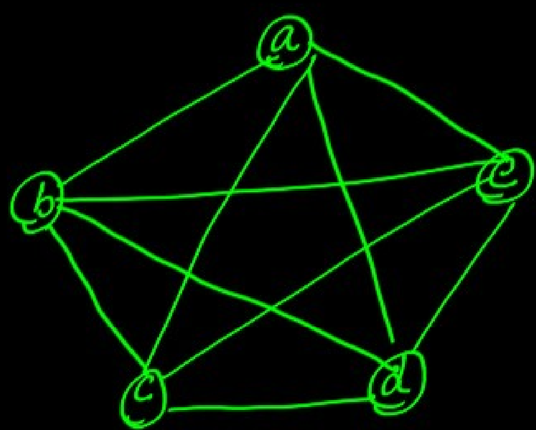
Girth: The number of edges in a shortest cycle of G is called girth of G is denoted by g(G).

For the graph given above

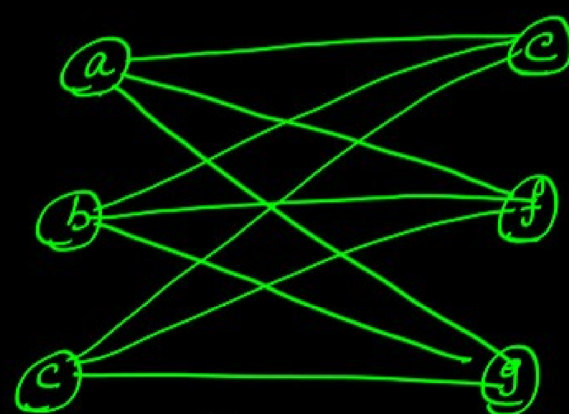
$g(G) = \cancel{5}$. $g(G) = 3$

Kurarowski's Theorem:-

It states that "A graph is planar if and only if it does not contain subgraph which is homeomorphic to K_5 (or) $K_{3,3}$



K_5



$K_{3,3}$

Homeomorphic Graphs:

In a Graph G, if another graph G* taken by dividing edge of G with additional vertices (or) we can say that a Graph G* is obtained by introducing the vertices of degree 2 in any edge of Graph G, then G & G* are said to be Homeomorphic.

Ex: