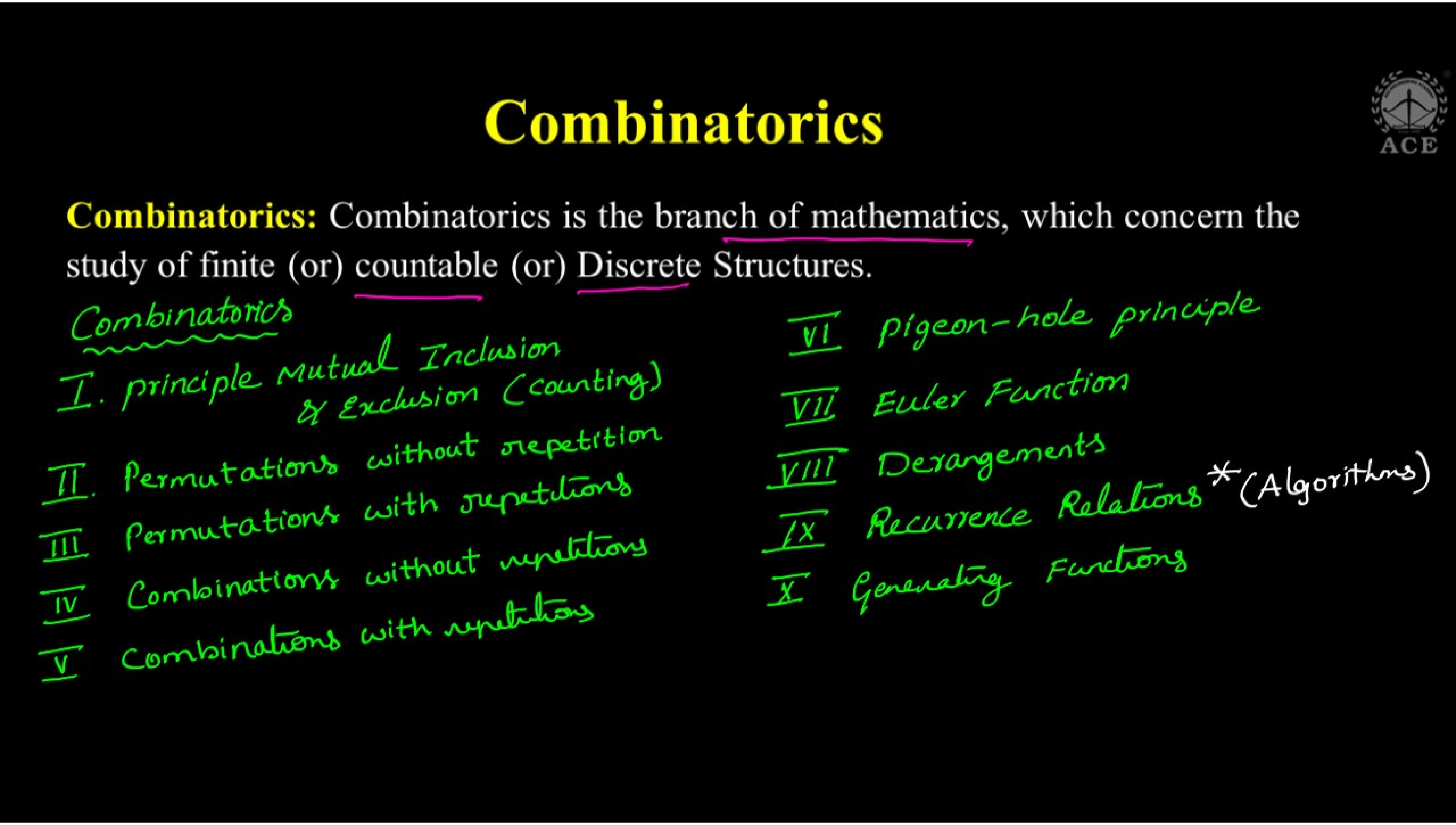


Combinatorics

Combinatorics: Combinatorics is the branch of mathematics, which concern the study of finite (or) countable (or) Discrete Structures.

Combinatorics

I. principle Mutual Inclusion & Exclusion (counting)

II. Permutations without repetition

III Permutations with repetitions

IV Combinations without repetitions

V Combinations with repetitions

VI pigeon-hole principle

VII Euler Function

VIII Derangements

IX Recurrence Relations * (Algorithms)

X Generating Functions

Principle of Mutual Inclusion & Exclusion:

$A = \{1,2,3,4\}$ ✓ $B = \{3,4,5,6\}$

$A \cup B = \{1,2,3,4,5,6\}$

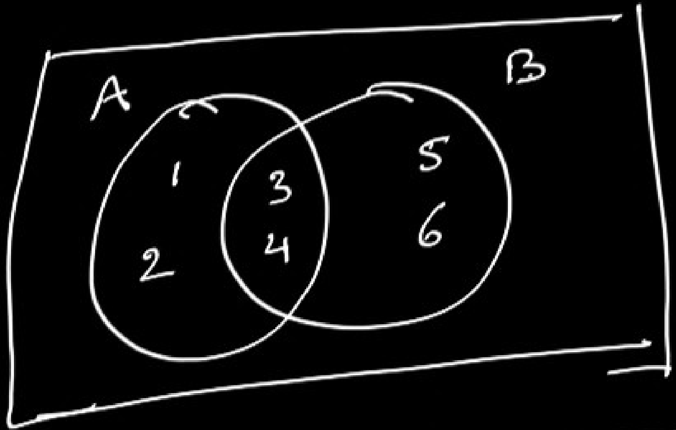
$A \cap B = \{3,4\}$

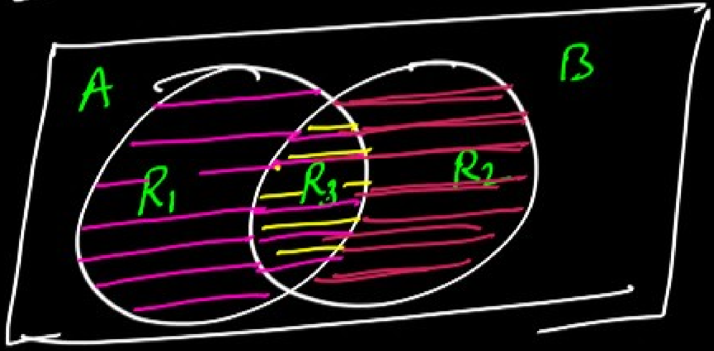
$n(A) = 4$ ✓ $n(B) = 4$ ✓

$n(A \cup B) = 6$ $n(A \cap B) = 2$

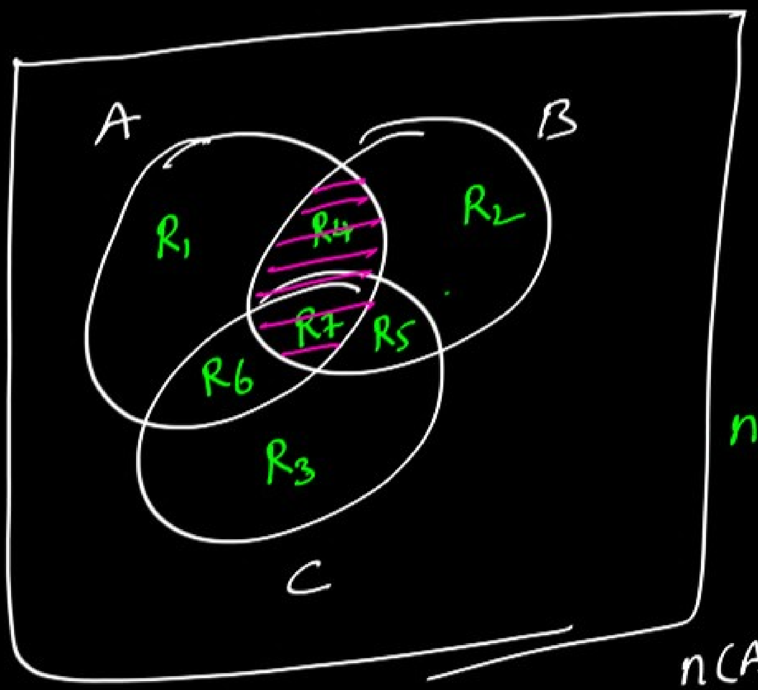
$n(A \cup B) = 4+4-2$

$* \; n(A \cup B) = n(A)+n(B)-n(A \cap B)$





$R_1 + R_2 - R_3$



$$n(A\cup B^{\cup C}) = R_1 + \overset{\frown}{R_2} + \overset{\frown}{R_3} - R_4 - R_5$$
$$- R_6 + R_7$$

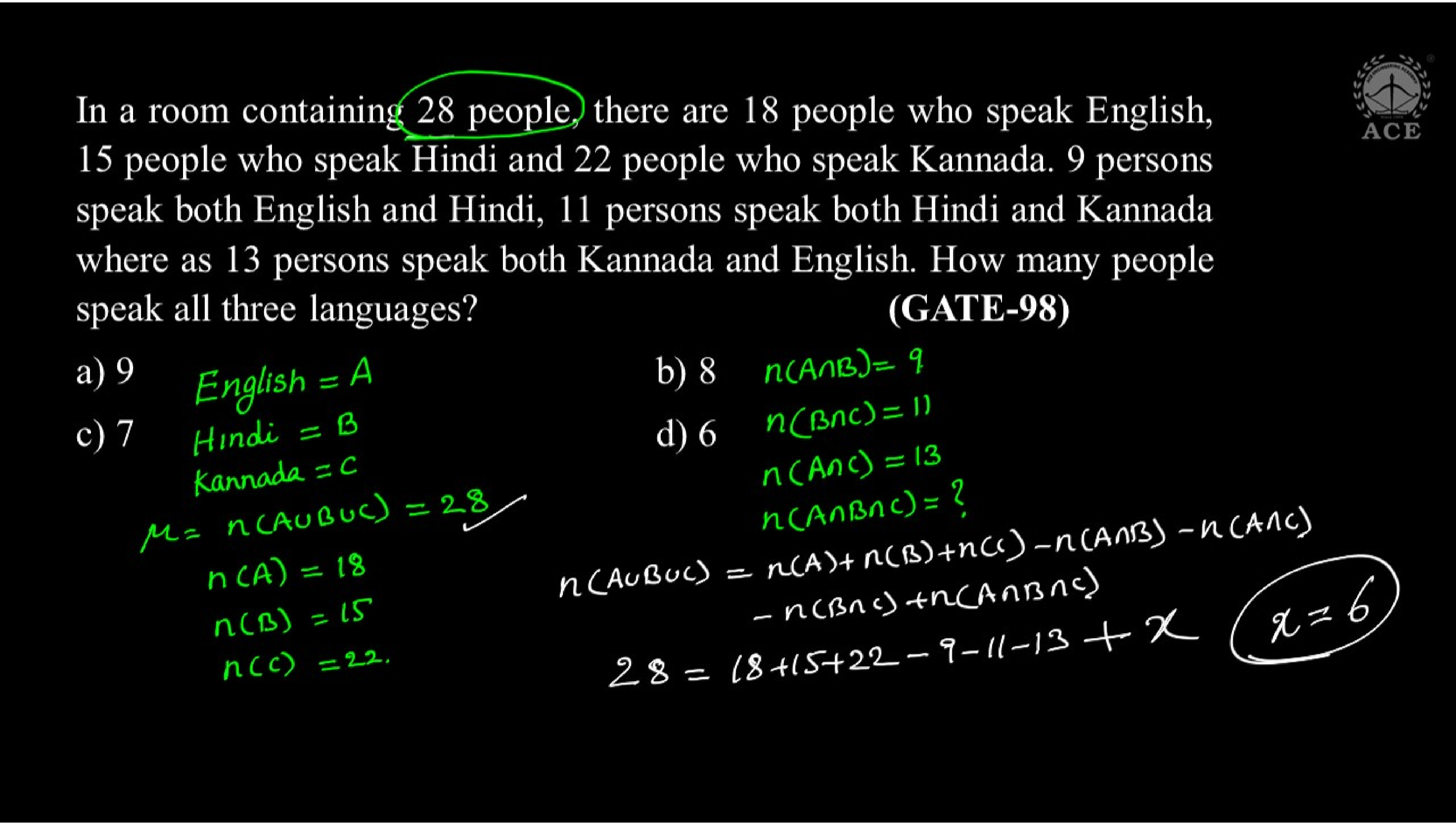
$$n(A\cup B\cup C) = n(A) + n(B) + n(C) - n(A\cap B)$$
$$- n(B\cap C) - n(A\cap C) + n(A\cap B\cap C)$$

$$n(A\cup B\cup C\cup D) = n(A) + n(B) + n(C) + n(D) - n(A\cap B)$$
$$- n(A\cap C) - n(A\cap D) - n(B\cap C) - n(B\cap D)$$
$$- n(C\cap D) + n(A\cap B\cap C) + n(A\cap B\cap D)$$
$$+ n(B\cap C\cap D) + n(A\cap C\cap D) -$$
$$- n(A\cap B\cap C\cap D)$$

* $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

* $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) +$
$n(A \cap B \cap C)$

*$n(A \cup B \cup C \cup D) = n(A) + n(B) + n(C) + n(D) - n(A \cap B) - n(A \cap C) -$
$n(A \cap D) - n(B \cap C) - n(B \cap D) - n\,(C \cap D) +$
$n(A \cap B \cap C) + n(A \cap B \cap D) + n(A \cap C \cap D) +$
$n(B \cap C \cap D) - n(A \cap B \cap C \cap D)$



In a room containing 28 people, there are 18 people who speak English, 15 people who speak Hindi and 22 people who speak Kannada. 9 persons speak both English and Hindi, 11 persons speak both Hindi and Kannada where as 13 persons speak both Kannada and English. How many people speak all three languages? **(GATE-98)**

a) 9
b) 8
c) 7
d) 6

English = A
Hindi = B
Kannada = C

$\mu = n(A \cup B \cup C) = 28$ ✓

$n(A) = 18$

$n(B) = 15$

$n(C) = 22.$

$n(A \cap B) = 9$

$n(B \cap C) = 11$

$n(A \cap C) = 13$

$n(A \cap B \cap C) = ?$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$28 = 18 + 15 + 22 - 9 - 11 - 13 + x$$

$$\boxed{x = 6}$$

Q. In a class of 200 students, 125 students have taken Programming Language course, 85 students have taken Data Structures course, 65 students have taken Computer Organization course; 50 students have both Programming Language and Data Structures, 35 students have taken both Programming Languages and Computer Organization; 30 students have taken both Data Structures and Computer Organization; 15 students have taken all the three courses.

How many students have not taken any of the three courses?

$n(A \cup B \cup C)$ $\cancel{200}$ $= 125+85+65-50-35-30+15$

$= 175$

$n(A \cup B \cup C)^c = M - n(A \cup B \cup C)$

$= 200-175$

$= 25.$

(GATE-IT-04)

a) 15 b) 20

c) 25 ✓ d) 35

Q. What is the cardinality of the set if integers X defined below?

(GATE-IT-06)

X = {n | 1≤ n ≤ 123, n is not divisible by either 2, 3 or 5}

a) 28

b) 33

c) 37

d) 44

Let $n(2)$ = no. of integers divisible by (from 1 to 123) $= \frac{123}{2} = 61$

$$\overline{X} = n(2 \vee 3 \vee 5) = n(2)+n(3)+n(5)-n(2\wedge 3)-n(3\wedge 5)-n(2\wedge 5)+n(2\wedge 3\wedge 5)$$

$$= \left\lfloor\frac{123}{2}\right\rfloor+\left\lfloor\frac{123}{3}\right\rfloor+\left\lfloor\frac{123}{5}\right\rfloor-\left\lfloor\frac{123}{6}\right\rfloor-\left\lfloor\frac{123}{15}\right\rfloor-\left\lfloor\frac{123}{10}\right\rfloor+\left\lfloor\frac{123}{30}\right\rfloor$$

$$= 61+41+24-20-8-12+4$$

$$= 130-40 = 90$$

$$X = 123-90$$
$$= 33$$

Q. The number of integers between 1 and 500 (both inclusive) that are divisible by 3 or 5 or 7 is ______. **(GATE-17-Set1)**

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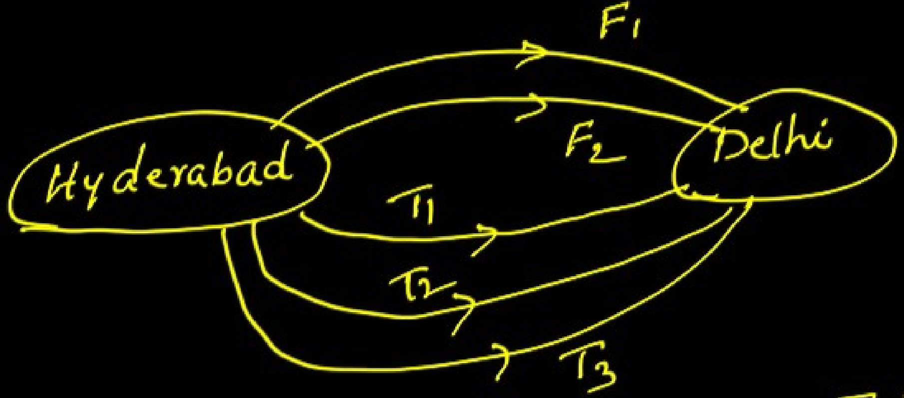
Sum Rule:

If E_1, E_2, E_3 are Mutually exclusive events, and E_1 can happen in e_1 ways, E_2 can happen in e_2 ways, …….., E_n can happen in e_n ways, then (E_1 or E_2 or …. Or E_n) can happen in $(e_1 + e_2 + ….. + e_n)$ ways.



F_1, F_2 = Flights = 2

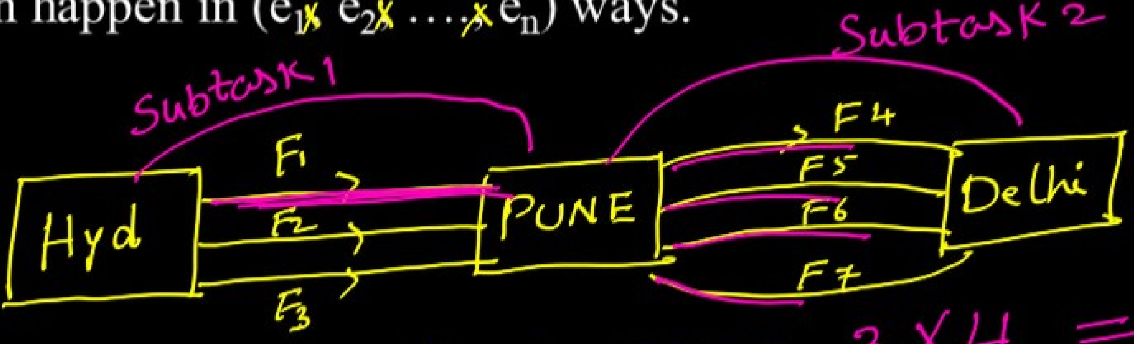
T_1, T_2, T_3 = Trains = 3

Total ways = 2 + 3

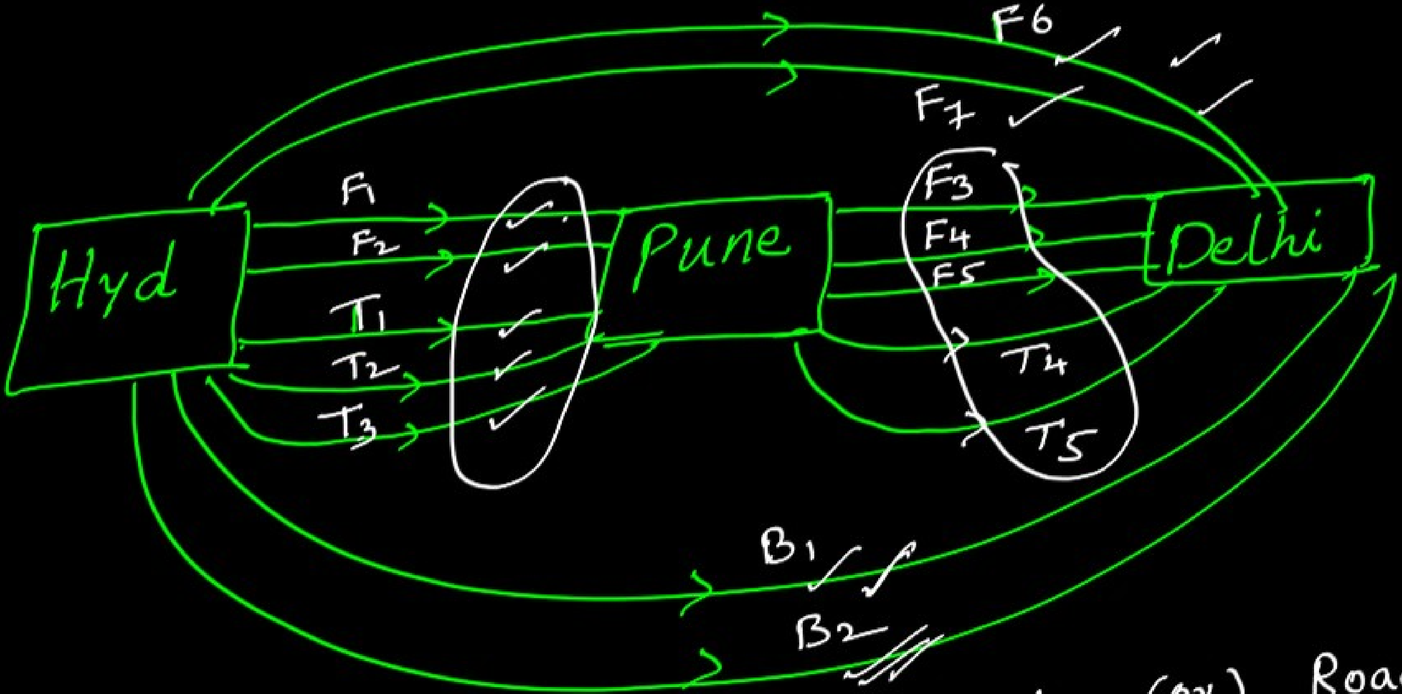
= 5

Product Rule:

If the independent events $E_1, E_2, \ldots\ldots, E_n$ can happen in $e_1, e_2, \ldots\ldots, e_n$ ways respectively, then the sequence of events E_1 first, followed by $E_2, \ldots\ldots$, followed by E_n can happen in $(e_1 \times e_2 \times \ldots \times e_n)$ ways.



From Hyd to Delhi via pune = $3 \times 4 = 12$ $(F_1F_4, F_1F_5, F_1F_6, F_1F_7$
$F_2F_4, F_2F_5, F_2F_6, F_2F_7$
$F_3F_4, F_3F_5, F_3F_6, F_3F_7)$



$2+(5\times5)+2$

$= 29$

Hyd to Delhi ~~via pune~~ = Air route (or) Road Route [Train or Bus]

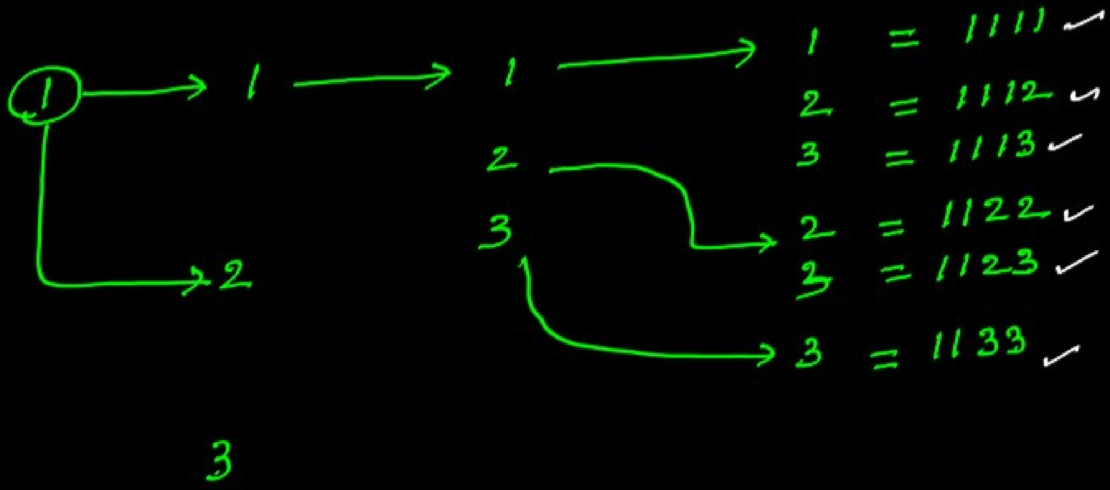
=

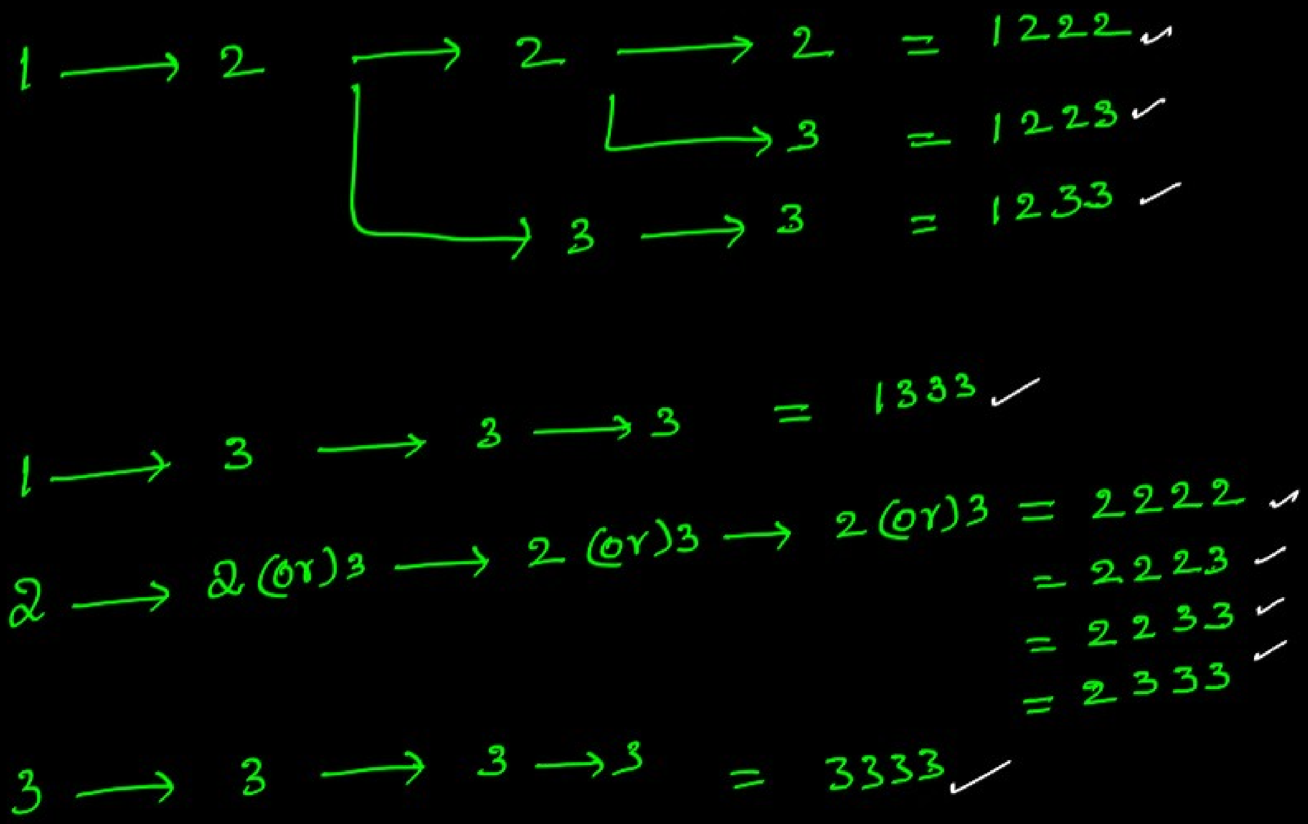
$= 2 + (2\times3) + (3\times2) + 2$

$= 16$

Q. The number of 4 digit numbers having their digits is non decreasing order (from left to right) constructed by using the digits belonging to the set {1, 2, 3} is _____ **(GATE-15-Set3)**

$\{1,2,3\}$





1 → 2 → 2 → 2 = 1222 ✓

3 = 1223 ✓

3 → 3 = 1233 ✓

1 → 3 → 3 → 3 = 1333 ✓

2 → 2 (or) 3 → 2 (or) 3 → 2 (or) 3 = 2222 ✓

= 2223 ✓

= 2233 ✓

= 2333 ✓

3 → 3 → 3 → 3 = 3333 ✓

⑮

Permutation Without Repetition:

A permutation of n-objects taken ‘r’ at a time is an “ordered selection” (or) “arrangement” of ‘r’ of the objects.

$$P(n, r) = {}^{n}P_{r} = \frac{n!}{(n-r)!}$$

Permutation With Repetition:

The number of r-permutations of n-objects with unlimited rep$\overset{e}{\text{i}}$titions is denoted by U(n, r)

$$U(n, r) = n^{\cancel{r}} = n^{r}$$

The number of permutations of n objects of which 'n_1' are alike n_2 are alike, …… n_r are alike is

$$\frac{n!}{n_1!n_2!\ldots\ldots n_r!}$$

Combination Without Repetition: A combination of ‘n’ objects taken ‘r’ at a time is an unordered selection of objects.

$$\begin{bmatrix} n \\ r \end{bmatrix} = C(n, r) = \frac{n!}{r!\,(n-r)!} = {}^{n}C_{r} \qquad (r \leq n)$$

Combination With Repetition:

The number of ways of selecting r-objects from available n-objects with unlimited repetition is known as combination with repetition.

$$\underline{V(n, r)} = C(n + r - 1, r) = {}^{n+r-1}C_r \quad \checkmark$$

Permutation = arrangement = 39, 93

Combination = Selecting

example let us Take 4-objects $\{a,b,c,d\}$

Combination (2)	Comb(2) with rep	pemut (2)	perm(2) with rep
ab ✓ ac ✓ ad ✓ bc ✓ bd ✓ cd ✓ ⑥	ab, ac, ad bc, bd, cd aa, bb, c,c dd, ⑩ ✓	ab, ba. ac, ca bc, cb bd, db cd, dc ⑫ ✓ ad,da	ab, ba - ac, ca ad,da bc, cb - bd, db - cd, dc - aa, bb, cc dd ⑯ ✓
$4C_2 = \frac{4\times3}{2} = 6$	$^{n+r-1}C_r = {}^{4+2-1}C_2 = 5C_2 = 10$ ✓	$4P_2 = 4\times3 = 12$ ✓	$c(n,r) = n^r = 4\times4 = 16$ ✓

Q. Let A be a sequence of 8 distinct integers sorted in ascending order. How many distinct pairs of sequences, B and C are there such that (i) each is sorted in ascending order. (ii) B has 5 and C has 3 elements, and (iii) the result of merging B and C gives A? **(GATE-CS-03)**

a) 2

b) 30

c) 56

d) 256

$$\boxed{nC_r = nC_{n-r}}$$

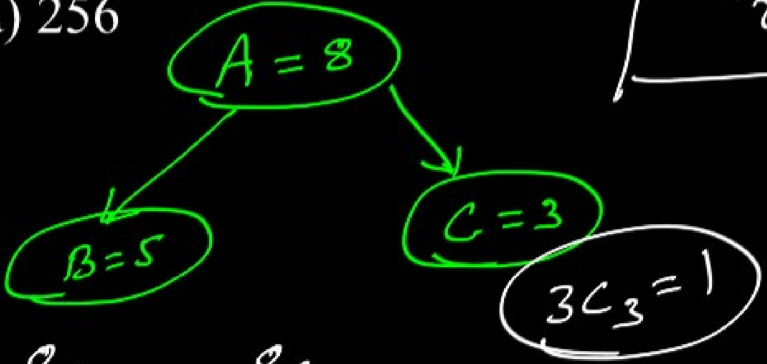
8 integers

$A = \{ 1, 2, 5, 6, 8, 9, 14, 17 \}$

Seq B : 1, 2, 5, 6, 8

Seq C : 9, 14, 17

$$8C_5 = 8C_3 = \frac{8\times\cancel{7}\times\cancel{6}}{\cancel{3\times2\times1}} = 56$$



$$8C_5 = 8C_3$$

Q. The number of binary strings of n zeros and k ones such that no two ones are adjacent is: **(GATE-CS-99)**

a) $^{n+1}C_k$

b) $^{n}C_k$

c) $^{n}C_{k+1}$

d) None of these

n-zero and k-ones

11 ✗

let $n=4$, $k=2$

(0000) (11)

$\underset{\cdot}{\_}\,0\,\underset{\cdot}{\_}\,0\,\underset{\cdot}{\_}\,0\,\underset{\cdot}{\_}\,0\,\underset{\cdot}{\_}$

5gaps with 2-ones

$^{5}C_2 = {}^{4+1}C_2 = {}^{n+1}C_K$

Statement for linked Answer questions Q33 & Q34: **(GATE-CS-07)**

Suppose that a robot is placed on the Cartesian plane. At each step it is allowed to move either one unit up or unit right i.e if it is at(i, j) then it can move to either (i + 1, j) or (i, j + 1).

Q. How many distinct paths are there for the robot to reach the point (10,10) starting from the initial position (0,0)?

$^{20}P_{10}$ ✓ a) $\binom{20}{10} = \binom{n}{r} = {}^{n}C_{r}$.

b) 2^{20}

c) 2^{10}

d) none of these

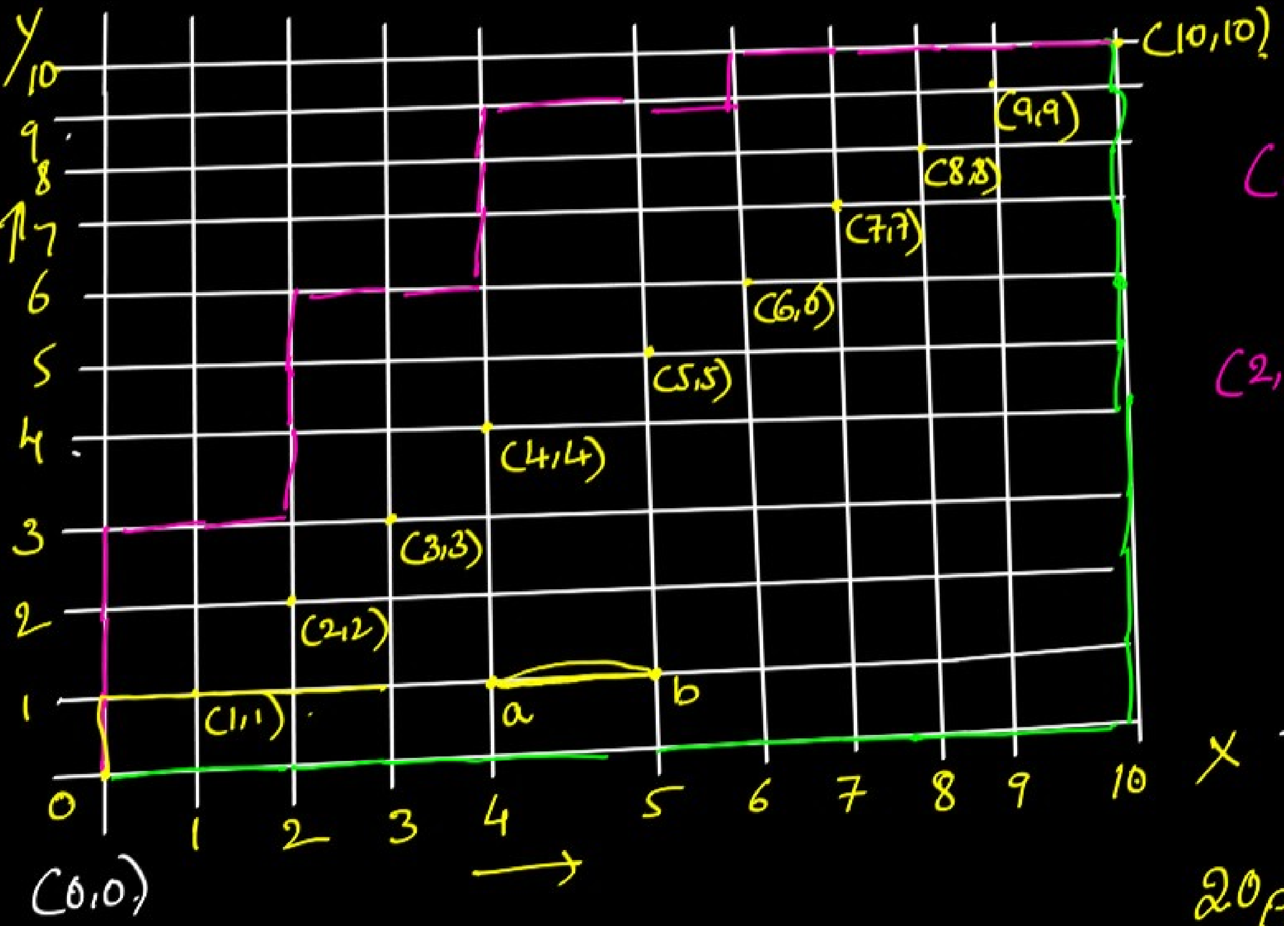
$\text{MANJU} = 5! = 120$

$M_1ADAM_2 = \dfrac{5!}{2!\,2!}$

M_2ADAM_1

$= 30$

UANJM



$(i,j) \rightarrow (i+1, j)$

$(i,j) \rightarrow (i, j+1)$

$(2,4) \rightarrow (3,4)$

$(2,4) \rightarrow (2,5)$

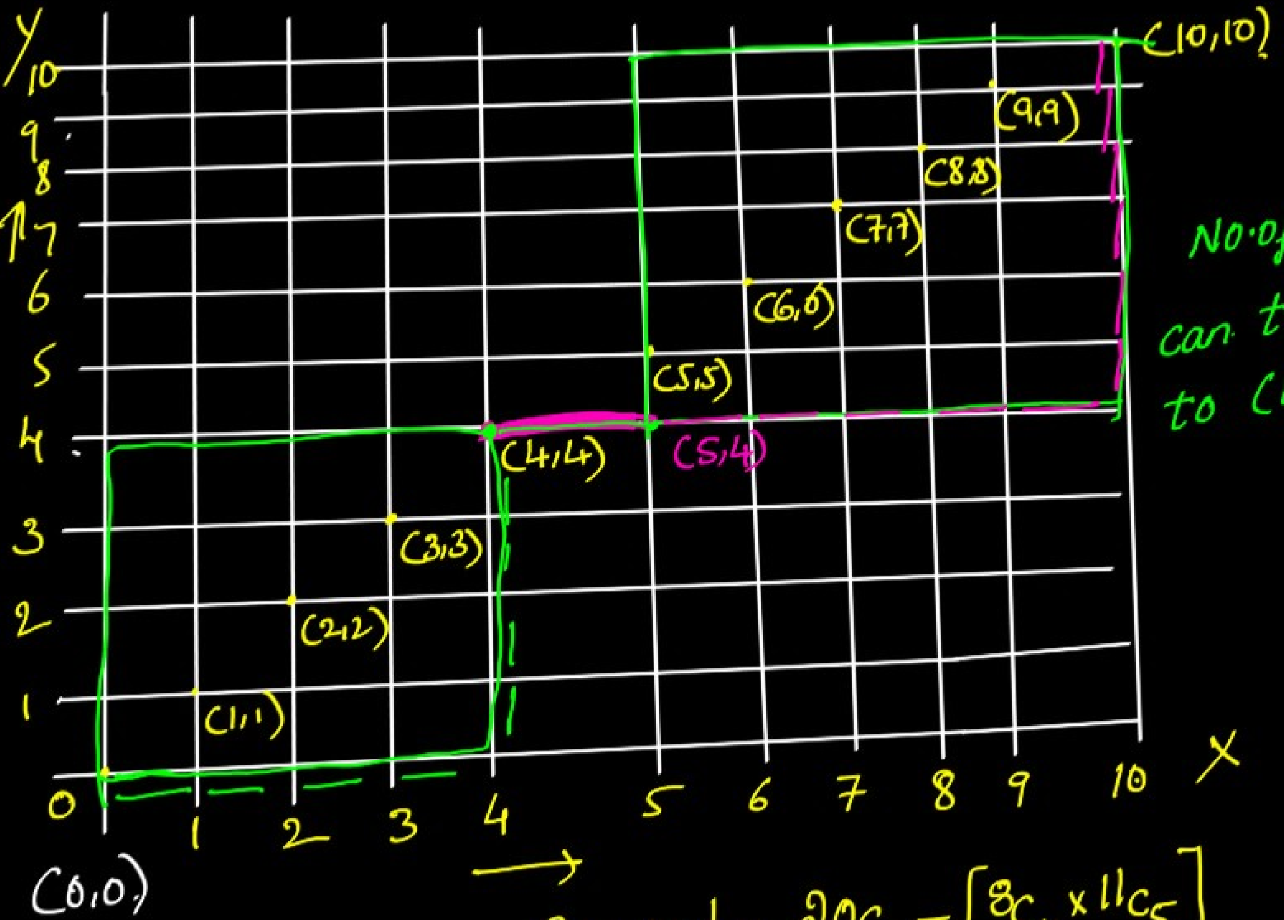
Right Move = R ✓

UP Move = U ✓

$\boxed{(0,0)} \rightarrow \boxed{(10,10)} = 10R \;\&\; 10U$

X $= \underline{RRRRRRRRRR}\ \underline{UUUUU\ UUUUU}$

$${}^{20}P_{10} = \frac{20!}{10!\,10!} = {}^{20}C_{10} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$



No.of ways th Robot can traverse from (0,0) to (10,10) through (4,4) and (5,4)

$= \underbrace{(0,0) \text{ to } (4,4)}_{\text{Subtask1}} \ \& \ \underbrace{(5,4) \text{ to } (10,10)}_{\text{Subtask2}}$

$= [4R \ \& \ 4U] \times [5R \ \& \ 6U]$

$= \frac{8!}{4!4!} \times \left[\frac{11!}{5!6!}\right.$

$= {}^{8}C_{4} \times {}^{11}C_{5}$

$\text{Required} = {}^{20}C_{10} - [{}^{8}C_{4} \times {}^{11}C_{5}]$

Q. Suppose that the robot is not allowed to traverse the line segment from (4,4) to (5,4). With this constraint, how many distinct paths are there for the robot to reach (10, 10) starting from (0, 0)?

a) 2^9

b) 2^{19}

c) $\binom{8}{4} \times \binom{11}{5}$

[✓ d) $\binom{20}{10} - \binom{8}{4} \times \binom{11}{5}$]

Q. How many non empty sub strings can be formed from a character string of length n?

(GATE-CS-89)

abc

a
ab
bc

length $n=8$,

abcdefgh

No.of substrings canbe formed of length $1 = n$ 8
" " " " " " $2 = n-1$ 7
" " " " " " $3 = n-2$
⋮
$n = 1$

Total No.of Substrings $= n+(n-1)+(n-2)+ ------ +1$

$$= \Sigma n = \frac{n(n+1)}{2}$$

Q. The number of non empty substrings (of all lengths inclusive) that can be formed from a character string of length n is: **(GATE-CS-94)**

a) n

b) n^2

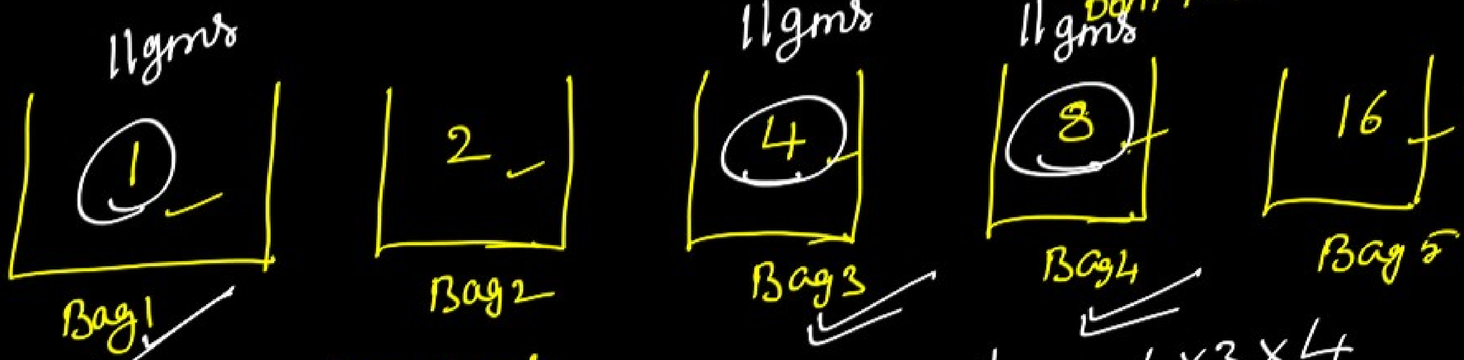
c) $\frac{n(n-1)}{2}$

[✓ d) $\frac{n(n+1)}{2}$]

Q. There are 5 bags labeled 1 to 5. All the coins in given bag have the same weight. Some bags have coins of weight 10 gm, others have coins of weight 11 gm. I pick 1, 2, 4, 8, 16 coins respectively from bags 1 to 5. Their total weight comes out to 323 gm. Then the product of the labels of the bags having 11 gm coin is ______ **(GATE-14-Set1)**

~~1+3+4~~
~~=13~~
13

31 coins = 323g



10coins
31 × 10 gms = 310 gms
↓ +13
323gms

10gm (or) 11gms

Required = 1 × 3 × 4
= 12

Q. A pennant is a sequence of numbers, each number being 1 or 2. An n-pennant is a sequence of numbers with sum equal to n. For example, (1, 1, 2) is a 4-pennat. The set of all possible 1-pennants is {(1)}, the set of all possible 2-pennants is {(2), (1, 1)} and the set of all 3-pennats is {(2, 1), (1, 1, 1), (1, 2)}. Note that the pennant (1, 2) is not the same as the pennant (2, 1). The number of 10-pennats is ________.

(GATE-14-Set1)

$(1,1,2) = 1+1+2 =$ 4-pennant

$10 = 1+1+1+1+1+1+1+1+1+1$ ✓

$= 1+1+1+1+1+1+1+1+2$ ✓ $= \frac{9!}{8!}$

2's	1's
0	10
1	8
2	6
3	4
4	2
5	0

$$= \frac{10!}{10} + \frac{9!}{8!} + \frac{8!}{2!6!}$$

$$+ \frac{7!}{3!4!} + \frac{6!}{4!2!} + \frac{5!}{5!}$$

$$= 89$$

$(\underline{2}, 2, 2, 2, 2)$

Q. How many 4 digit even numbers have all 4 digits distinct?

(GATE-CS-01)

a) 2240

~~b)~~ ✓ 2296

c) 2620

d) 4536

No. of 4 digit even numbers = _ _ _ _

0 ✓
2 ✓
4 ✓
6 ✓
8 ✓

{ ~~0~~, 1, 2, 3, 4, 5, 6, 7, 8, 9 }

$7 \times 8 \times 9$

$= 9 \times 8 \times 7 \; \frac{1}{0}$ (or) $8 \times 8 \times 7 \times 4$

1 2 3 ~~8~~ 6 7 8 9

0 1 2 ~~6~~ 7 8 9

2 ✓ ~~4~~ ✓ 6 ✓ 8 ✓

even '0'

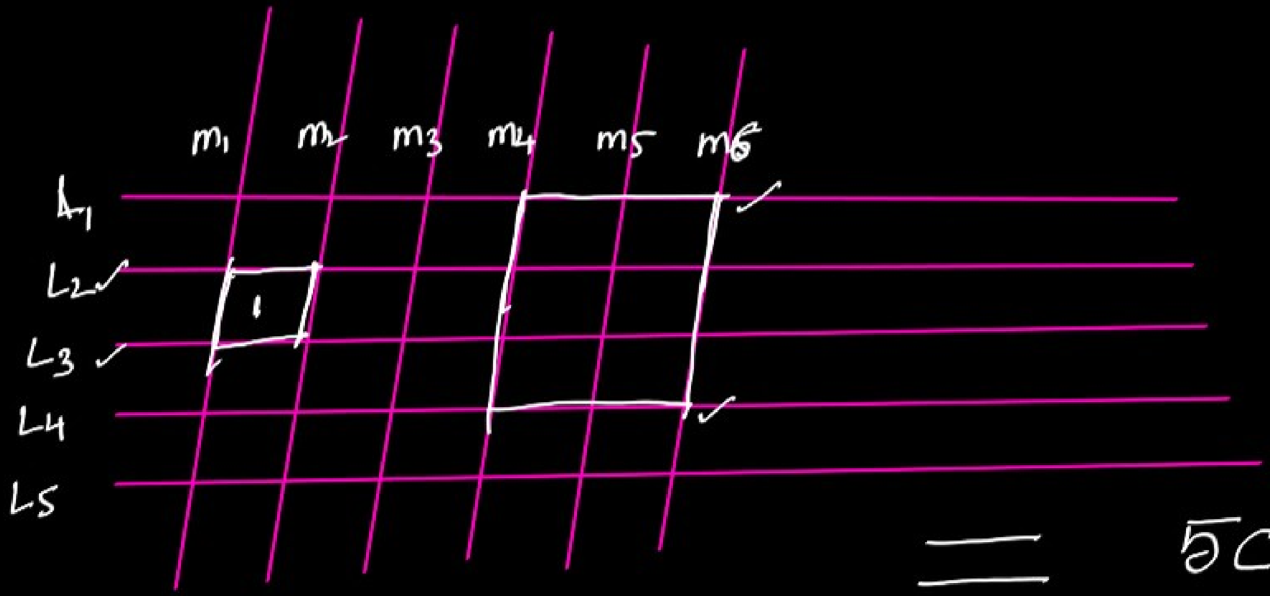
$= \boxed{504} + \boxed{1792}$

$= 2296$ ✓

$8 \times 8 \times 7 \times \frac{4}{2}$
4
6
8

64, 2
×7
448

Q. A set of 5 parallel lines intersect with another set of 6 parallel lines, Find the possible number of parallelograms in this setup



$=$ $^5C_2$

$\parallel$ $^6C_2$

$\# \ ^5C_2 * {^6C_2}$

$= 10 \times 15$

$= 150$