⑤ Compatable relation:

If a relation is both reflexive and Symmetric then It is known as compatable relation.

Reflexive : $\forall x \ (x,x) \in R$

Symmetric : $\forall x, y \in R$, If $(x,y) \in R$ then $(y,x) \in R$

Transitive : $\forall x, y, z \in R$, If $(x,y) \in R$ and $(y,z) \in R$ then $(x,z) \in R$

Equivalence : Ref, Symmetric & Transitive

Compatable : Ref & Symmetric

ACE

Q. The number of equivalence relations on set $\{1, 2, 3, 4\}$ is

a) 15 ✓ b) 16

c) 24 d) 24

$A = \{1,2,3,4\}$ ✓

$A \times A = \{(1,1), (1,2), (1,3), (1,4),$
$(2,1), (2,2), (2,3), (2,4),$
$(3,1), (3,2), (3,3), (3,4),$
$(4,1), (4,2), (4,3), (4,4)\}$

I. $R_1 = \{(1,1), (2,2), (3,3), (4,4)\} \longrightarrow 1$

II. $R_2 = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1)\} \longrightarrow 6$

2+1+1

III. $R_3 = \{(1,1), (2,2), (3,3), (4,4), \underbrace{(1,2), (1,3), (2,1), (3,1), (2,3), (3,2)}_{1,2,3}\} \longrightarrow {}^4C_3 = 4 \longrightarrow 4$

1,2,4 ✓
2,3,4 ✓
1,3,4 ✓

IV. $R_4 = \{(1,1), (2,2), (3,3), (4,4), (1,2), (3,4), (2,1), (4,3)\} \longrightarrow 3$

V. $R_5 = A \times A \longrightarrow 1$

15

ACE

Method II

$A = \{1,2,3,4\}$

$$A \times A = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$$

$n(A) = |A| = 4$

4

$4 = 1+1+1+1 = 1$ ✓

$4 = \boxed{2+2} = \frac{4C_2}{2!} = 3$ ✓

$4 = 2+1+1 = 4C_2 = 6$ ✓

$4 = 3+1 = 4C_3 = 4$ ✓

$4 = 4+0 = 1$ ✓

$+$

15

No. of partitions

$R = \{(1,1), (2,2), (3,3), (4,4), \boxed{(1,2), (3,4)}\, (2,1), (4,3)\}$

$R = \{(1,1), (2,2), (3,3), (4,4), \boxed{(3,4), (1,2)}, (4,3), (2,1)\}$

Irreflexive: A relation R on set A is said to be irreflexive if $(x, x) \notin R$, $\forall\, x \in A$

$A = \{1, 2, 3, 4\}$ ✓

$R_1 = \{(1, 2), (1, 3), (2, 4), (4, 3)\}$. Irref ✓

$R_2 = \{\ \}$ Irref ✓

$\boxed{R_3 = (A \times A) - \Delta_A}$ Irref. ✓

$R_4 = A \times A$ Not Irref

$R_5 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3)\}$ Not Irref

$R_6 = \{(1, 1), (2, 2), (2, 3)\}$ Not Irref

R_6 is Neither reflexive Nor irreflexive = Non-reflexive

Diagonal Relation: Let A be any set, The diagonal relation on A consists of all ordered pairs (a, b) such that a = b

i.e., $\Delta_A = \{(a, a) / a \in A\}$

If $A = \{1, 2, 3\}$ then $\Delta_A = \{(1, 1), (2, 2), (3, 3)\}$

$R : A \to A = \{(1, 1), (2, 2), (3, 3), (1, 2), (3, 2)\}$ Reflexive but NOT diagonal

$\forall x \; (x, x) \in R$

Asymmetric Relation: A relation R on a set 'A' is said to be asymmetric, if

If $(x, y) \in R$ then $(y, x) \notin R \quad \forall\, x, y \in A$

$R_1 = \{(1, 2), (1, 3), (3, 2)\}$ Asymmetric.

$R_2 = \{(1, 2), (2, 2), (2, 1)\}$ Symmetric, But NOT Asymmetric.

$R_3 = \{(2, 2), (3, 3)\}$ Symmetric But NOT Asymmetric.

$R_4 = \{\ \}$ Symmetric & asymmetric.

$R_3 = \{(2,2), (3,3)\}$

$= \{\underset{a,\,b}{(2,2)}, \underset{b,\,a}{(2,2)}, \underline{(3,3), (3,3)}\}$

Anti-Symmetric: A relation R on a set 'A' is said to be anti symmetric

If (x R y) and (y R x) then x = y $\quad \forall x, y \in A$

If $\overset{(1,2)}{(x,y)}$ and $(y,x) \in R$ Then $x = y$

Let $\quad A = \{1, 2, 3, 4\}$

$R_1 = \{(1, 2), (1, 3)\}$ ✓ anti-Symmetric

× $R_2 = \{(\underset{a\ \ b}{1, 2}), (\underset{b\ \ a}{2, 1})\}$ = NOT anti-Symmetric

$R_3 = \{(2, 2), (3, 3)\}$ = $\{ \underset{a,\ b}{(2,2)}, \underset{b,\ a}{(2,2)}, \overset{x,\ y}{(3,3)}, \overset{y,\ x}{(3,3)} \}$ ✓ anti-Symmetric

$x = y$

$a = b$

$R_4 = \{ \}$ ✓ anti-Symmetric

Short Notes

1) Reflexive: $\forall x,\ (x,x) \in R$

2) Symmetric: $\forall x, y \in R,$

$$\begin{tikzcd} x \arrow[r, bend left, "\text{if}"] & y \arrow[l, bend left, "\text{then}"] \end{tikzcd}$$

3) Transitive: $\forall x, y, z,$

$$\begin{tikzcd} x \arrow[r, "\text{if}"] \arrow[rr, bend right, "\text{then}"'] & y \arrow[r, "\text{if}"] & z \end{tikzcd}$$

4) Compatable: R & S

5) Equivalence: R, S, T

6) Irref: $(x,x) \notin R,\ \forall x$

7) ASym:

$$\begin{tikzcd} x \arrow[r, bend left, "\text{if}"] & y \arrow[l, bend left, "\cancel{\text{then}}"] \end{tikzcd} \quad \forall x, y$$

8) Anti:

$$\begin{tikzcd} x \arrow[r, bend left, "\text{if}"] \arrow[r, phantom, "\text{then}"] & y \arrow[l, bend left, "\text{if}"] \end{tikzcd} \quad \forall x, y$$

9) Diagonal: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Partial-Order, Lattice and Boolean Algebra

Partial-Order

A relation R on set A is called a partial-order relation. If 'R' is

i) Reflexive ✓ ✓

ii) Anti-symmetric ✓ ✓

iii) Transitive ✓ ✓

Anti-Symmetric

If $\underbrace{\overbrace{(x, y) \in R}^{(a,b)\in R}}_{\text{B.Tech}}$ and $\underbrace{(y, x) \in R}_{\text{GATE}}$ then $\underbrace{x = y}_{\text{Master's}}$

×

Ex:- If A = {a, b, c} then

R : A → A = {(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)}

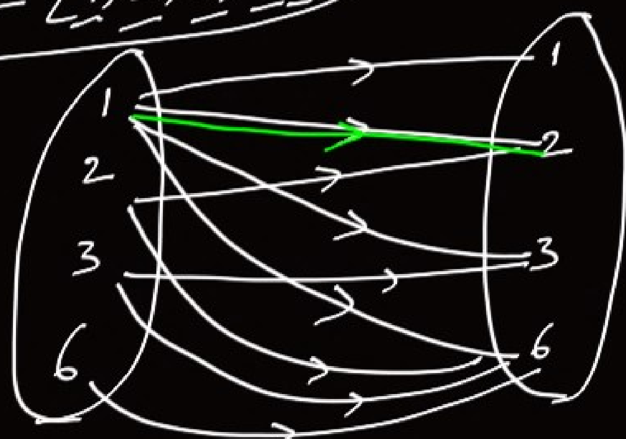
eg Consider the relation division on set D_6, (set of divisors of 6)

Sol Relation ($\preceq$) is division

x is related to y iff x divides y.

$R : D_6 \longrightarrow D_6$

$D_6 = \{1, 2, 3, 6\}$ ✓



$R = \{(1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), (6,6)\}$

R is partially-ordered

$R : D_6 \longrightarrow D_6$

↓

POSET

POSET diagram (or) Hasse diagram

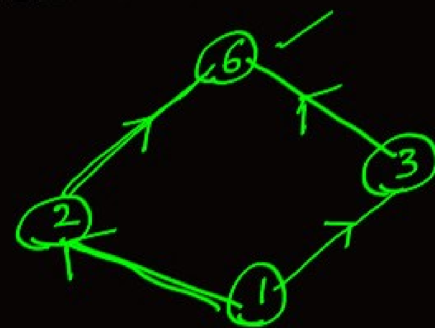
I. Every vertex in the graph represent an element in the POSET

II. If element x is related to y then their will be an edge from 'x' to 'y' in the POSET diagram.

III. Eliminate edges which represents reflexive and transitive property

$D_6 = \{1, 2, 3, 6\}$

$[D_6 : |]$



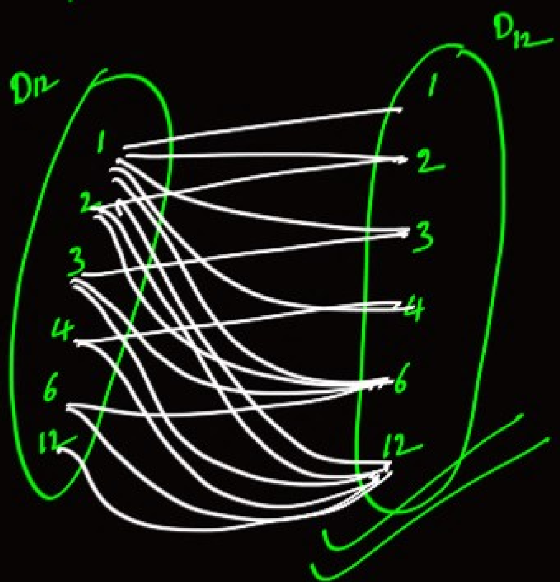
$$\left\{ \begin{array}{l} 2 \vee 3 = 6 \text{ and} \\ 2 \wedge 3 = 1 \end{array} \right.$$

2, 3 complements

$[D_{12} : |]$

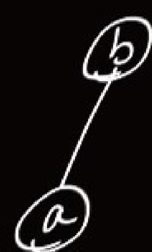
$D_{12} = \{1, 2, 3, 4, 6, 12\}$

$R : D_{12} \longrightarrow D_{12}$

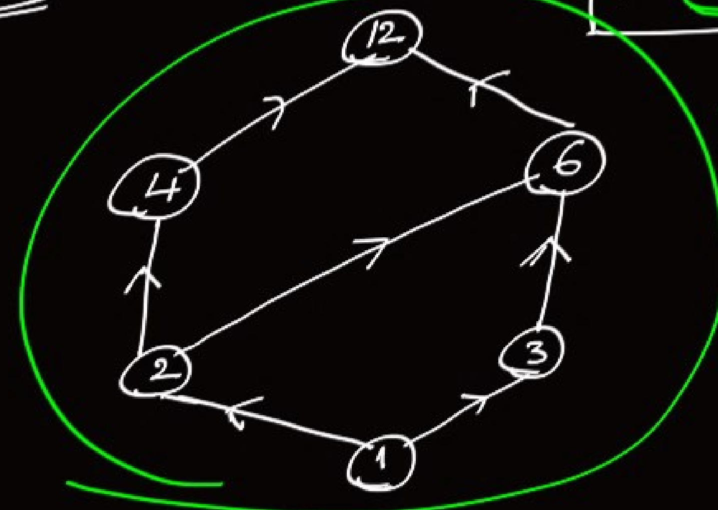


$R = \{ (1,1), (1,2), (1,3), (1,4), (1,6), (1,12),$
$(2,2), (2,4), (2,6), (2,12),$
$(3,3), (3,6), (3,12),$
$(4,4), (4,12), (6,6), (6,12), (12,12) \}$

POSET diagram



(a,b)



partially order

$\boxed{R : D_{12} \longrightarrow D_{12}}$

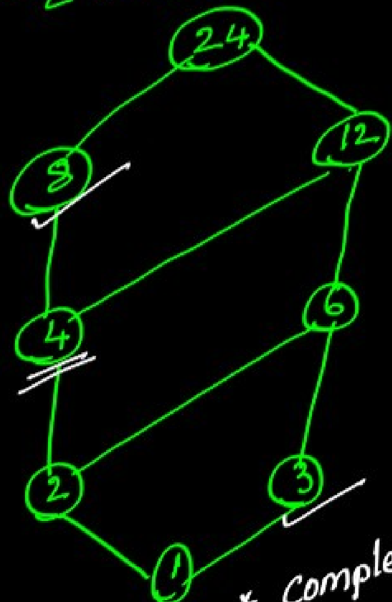
$\downarrow$

POSET

Q. Construct $[D_{24} : |]$

$[D_8 : |]$

$D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$



$D_8 = \{1, 2, 4, 8\}$

D_{24}

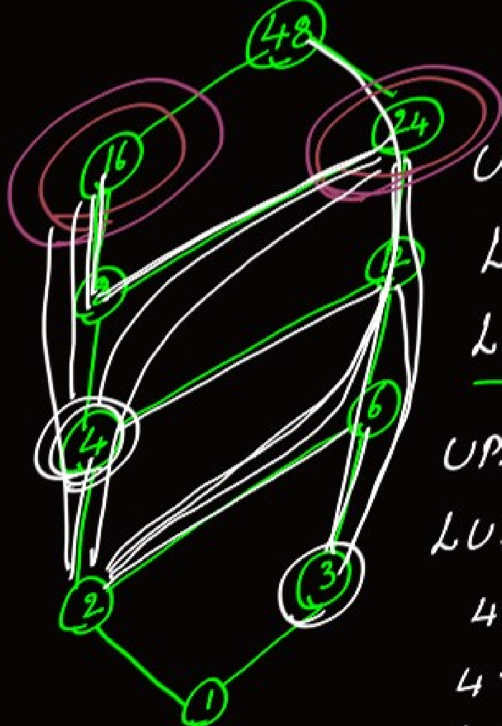
$\left\{\begin{array}{l} 3 \vee 8 = 24 \text{ and} \\ 3 \wedge 8 = 1 \end{array}\right.$

Complements

* Complement is not necessary to be unique

* NOT necessary "Every element should have complement"

$[D_{48} : |]$, $D_{48} = \{1, 2, 3, 4, 6, 8, 12, 16, 24, 48\}$



Upper Bounds

$UB\ (2,3) = \overset{*}{6}, 12, 24, 48$

Least upper Bound $(2,3) = 6$

$LUB(2,3) = \text{Join}(2,3) = 2 \vee 3 = 6$

Upper Bounds $(2,4) = \overset{*}{4}, 8, 12, 16, 24, 48$

$LUB(2,4) = 2 \vee 4 = 4$

$4 \vee 6 = 12$

$4 \vee 12 = 12$

$4 \vee 8 = 8$

$4 \vee 24 = 24$

$4 \vee 16 = 16$

$4 \vee 4 = 4$

$4 \vee 3 = 12$

$4 \vee 1 = 4$

Lower Bounds

Lower Bound$(16,24) = \overset{*}{8}, 4, 2, 1$

Greatest lower Bound $(16,24) = 8$

$GLB\ (16,24) = \text{Meet}\ (16,24) = 16 \wedge 24 = 8$

$8 \wedge 48 = 8$

$8 \wedge 16 = 8$

$8 \wedge 24 = 8$

$8 \wedge 12 = 4$

$8 \wedge 8 = 8$

$8 \wedge 4 = 4$

$8 \wedge 6 = 2$

$8 \wedge 2 = 2$

$8 \wedge 3 = 1$

$8 \wedge 1 = 1$

$\boxed{\underset{(\text{Join})}{LUB}\ (a,b)\ \underset{(\text{Meet})}{GLB}}$

Lattice

Lattice: A partial-order relation in which every pair of elements has LUB (Join) and GLB (Meet) is known as Lattice

$(L, \vee, \wedge)$

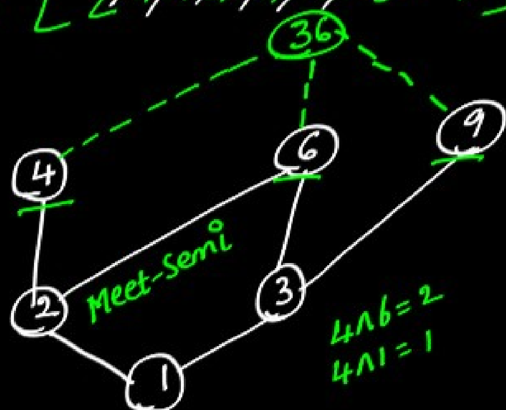
$(L, \vee, \wedge)$

* If a partial order relation has LUB (Join) for any pair of elements, Then it is known as Join-semi Lattice. $(L, \vee)$

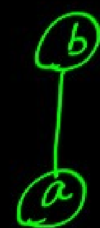
* If a partial-order relation has GLB (Meet) for any pair of elements, Then It is known as Meet-Semi Lattice. $(L, \wedge)$

Construct $[\{1, 2, 3, 4, 6, 9\} : |]$

$4 \vee 6 =$
$4 \vee 9 =$
$6 \vee 9 =$

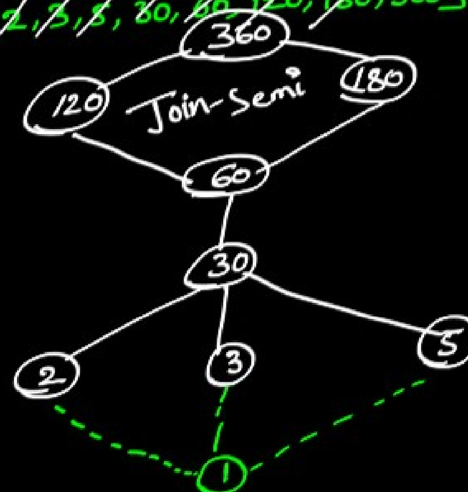


$4 \wedge 6 = 2$
$4 \wedge 1 = 1$



$a \vee b = b$
$a \wedge b = a$

$[\{2, 3, 5, 30, 60, 120, 180, 360\} : |]$



$30 \vee 60 = 60$
$2 \vee 5 = 30$
$2 \vee 360 = 360$

$2 \wedge 3 =$
$2 \wedge 5 =$
$3 \wedge 5 =$?

Complements: Two elements a, b are said to be complements if

i) $a \vee b = I$ and   I   1 → upper Bound

ii) $a \wedge b = 0$ 0   0 → Lower Bound

Complements Lattice:

A Lattice in which every element has a complement is known as complemented Lattice.

Complement of 'a'