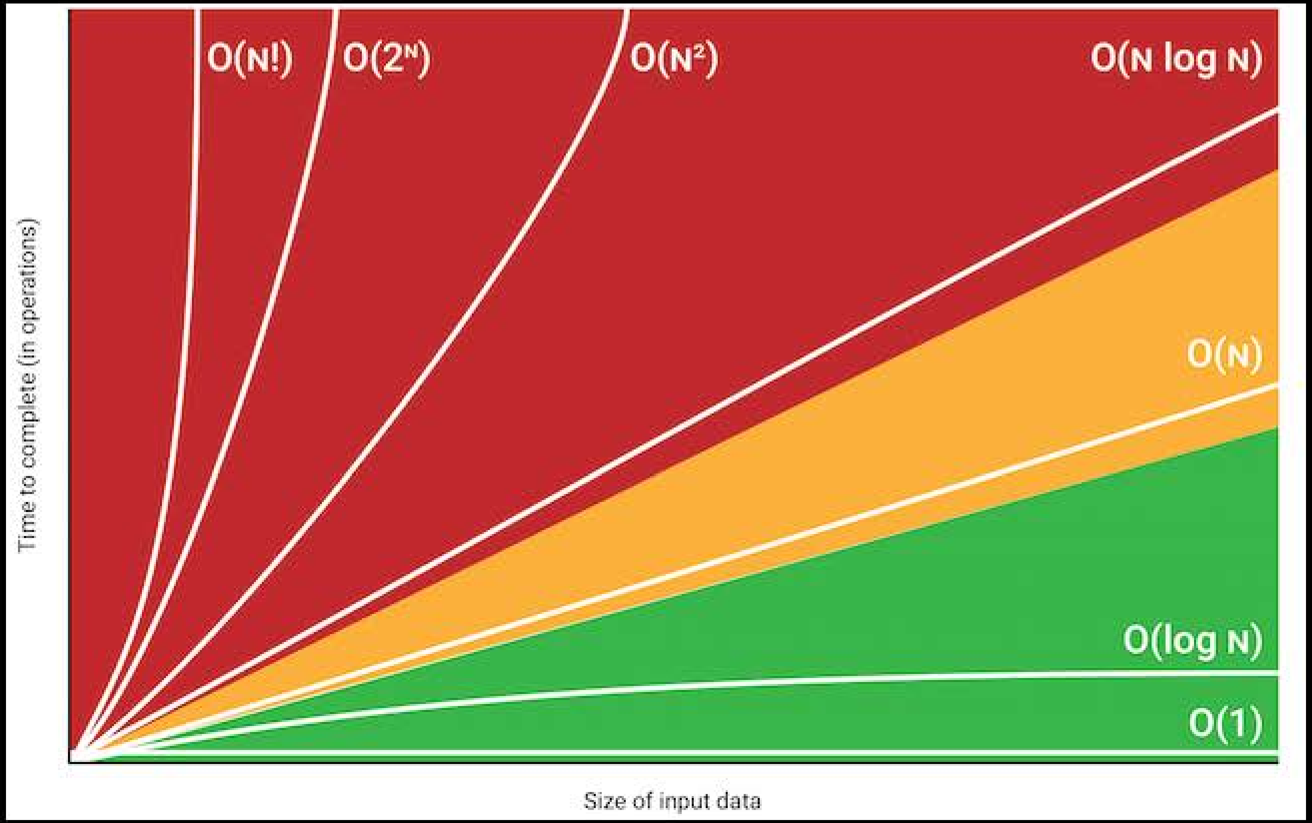


Asymptotic Notation

1. Running time Algorithm is a function of Input size
2. Compare the function

$f(n) \leq c \cdot g(n)$ $n > n_0$

O - Notation

n^2, n^3, n^4

$f(n) \geq c \cdot g(n)$ n^2

Ω Notation

$c_2 \cdot g(n) \leq f(n) \leq c_1 g(n)$ - Θ

Compare the No

15 18

o Notation

$O \quad \leq$

$\underline{o} \quad \underline{<}$

$f(n)$ is $o(g(n))$ iff

$f(n) < c \cdot g(n)$ $\quad \underline{n > n_0}$

$$\lim_{\underline{n \to \infty}} \frac{f(n)}{g(n)} = 0$$

$\uparrow$

o Notation

- [little "oh"] The function $f(n) = o(g(n))$ (read as f of n is little oh of g of n) iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

$n < n^2$

'o' $f(n) < c \cdot g(n)$

'O' $f(n) \leq c \cdot g(n)$

$\log < n$

$1 < \log n$

ω Notation

$f(n)$ is $\omega(g(n))$ $\Rightarrow f(n) > c \cdot g(n)$

$n > n_0$

$$\lim_{n \to \infty} \frac{g(n)}{f(n)} = 0$$

ω Notation

- [little "omega"] The function $f(n) = \omega(g(n))$ (read as f of n is little omega of g of n) iff

$$\lim_{n \to \infty} \frac{g(n)}{f(n)} = 0$$

f_1 f_2

Big 'O' – $\leq$

Big 'Ω' – $\geq$

Thetha Θ – $==$

Small 'o' $<$

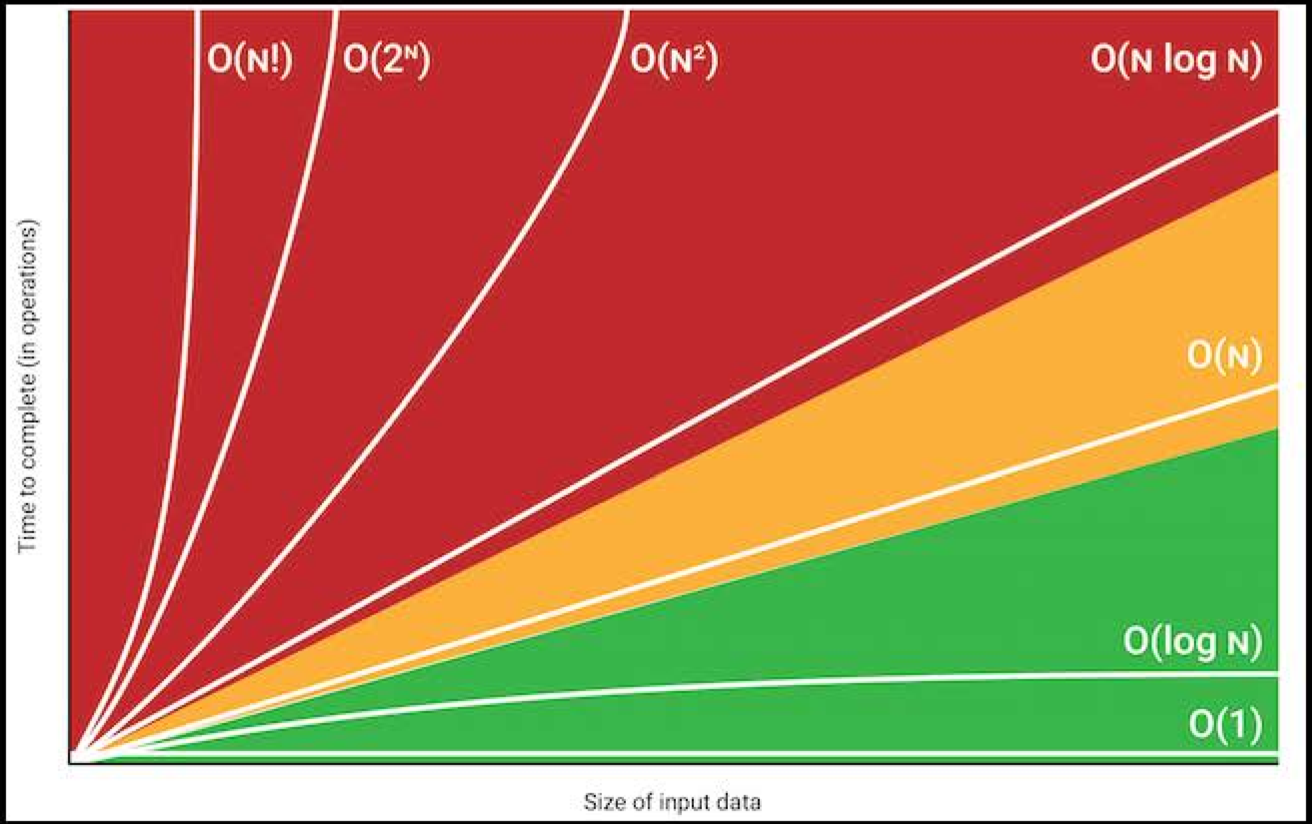
Small ω $>$

Symbols Compare the function

$\log n \leq (n^2)$ are true

$\log \leq O(n^2)$

11th



Properties of Asymptotic Notation

Properties of Asymptotic Notations

- Multiply by constant ✓
- Addition of function ✓
- Multiplication of function ✓
- Power of logarithmic function $\boxed{\log_2 n}$
- Polynomial function always upper bound for logarithmic function
- Exponential function are upper bound for polynomial function

Natural order

Properties of Asymptotic Notations

- Multiply by constant :

$f(n)$ is $O(g(n)) \Rightarrow f(n) \leq c_2 \cdot g(n)$

$c \cdot f(n)$ is $O(g(n))$

c. constant

- If two function are Related with Asymptotic Notation then multiplying by constant does not change the Relation.

Properties of Asymptotic Notations

$f(n)$ *is same as* $c \times f(n)$

$f(n)$ *is* $O(f(n))$ then $c \times f(n)$ is $O(f(n))$

$f(n)$ is $O(g(n))$ then $c \cdot f(n)$ is $O(g(n))$

Properties of Asymptotic Notations

- Addition of function

$\underset{n}{f(n)}$ is $O(\overset{n^2}{g(n)})$

$\underset{n^2}{h(n)}$ is $O(\underset{n^3}{k(n)})$

$n^3 + n^2$

$f(n) + h(n)$ is $O(g(n) + k(n))$

$(n + n^2)$ is lesser function $(\max(g(n), k(n)))$

$\boxed{n+n^2 \text{ is } O(n^3)}$ → true/False

Properties of Asymptotic Notations

- Addition of function

log n ≤ n   1 ≤ n

f(n) is lesser g(n)

$f(n)$ is $O(g(n))$ and $h(n)$ is $O(k(n))$

$f(n) + h(n) = O(max(g(n), k(n)))$

log n is O (n)

Log ≤ c. n . true/False

Properties of Asymptotic Notations

- Multiplication of function

$$\overset{n}{f}(n) \text{ is } O(\overset{n^2}{g}(n))$$

$$\underset{n^2}{\underline{h(n)}} \text{ is } O(k(n))$$

$$f(n).\,h(n) \text{ is } O(\overset{n^2}{g}(n) * k(n))$$

Properties of Asymptotic Notations

- Multiplication of function

$$f(n) \ \ is \ \ O(g(n)) \ and \, h(n) is \, O(k(n))$$

$$\underline{f(n)} \times \underline{h(n)} = O((\underline{g(n)} \times \underline{k(n)}))$$

Properties of Asymptotic Notations

- Power of logarithmic function (x is a constant)

Properties of Asymptotic Notations

- Power of logarithmic function (x is a constant)

$$\log_2 n^x = O(\log_2 n)$$

Properties of Asymptotic Notations

- Polynomial function always upper bound for logarithmic function

Properties of Asymptotic Notations

- Polynomial function always upper bound for logarithmic function

$$\log_2^x n = (\log_2 n)^x \; is \; O(n^y) \;\; for \, (x > 0, y > 0)$$

Properties of Asymptotic Notations

- Exponential function are upper bound for polynomial function

Properties of Asymptotic Notations

- Exponential function are upper bound for polynomial function

$n \to \infty$

$$n^x \text{ is } O(a^n) (a > 1, x > 1)$$

n^{1000} is $O(1.0001^n)$

Relation : $aRb \Rightarrow a \geq b$] Reflexive Relation

1.
2.
3.

$\{(1R2)$ (1R1) (3R2)

① ② ③

No yes $3 \geq 2$

$1 \not\geq 2$ $1 \geq 1$ yes

③ Natural

$aRb \Rightarrow$ Reflexive

a divide b

2 R 4

aRa - a divide a

Question

Any Natural will be Related with it self

① aRb - $a \geq b$ (1, 1) (2, 2) (3, 3)

Reflexive (4, 4) $\boxed{a < b}$

② aRb - $a > b$ Not Reflexive

will Natural No. will be Related with itself or Not

$\underline{aRb}$

Natural No

Reflexive aRa $\forall a$

Symmetric $aRb \Rightarrow bRa$ (all pure)

①

e.g. (I) aRb — $a \le b$ — $2 \le 3 \not\Rightarrow 3 \le 2$

(II) aRb — $a \ge b$ $3 \ge 2 \not\Rightarrow 2 \ge 3$

$\boxed{1R1}$ Symmetric ← (III) aRb — $a == b$

Symmetric ← (IV) aRb — $a+b$ even No.

$2R4$ — $2+4=6$ Even No

$4R2$ — $4+2$ — Even No

Transitive Relationship aRb ; bRc

$\Rightarrow aRc$

$\frac{1R2}{2R3} \Rightarrow 1 \leq 3$

(I) aRb – $a \leq b$ transitive yes

(II) aRb – $a \geq b$ transitive ~~2R3~~ $\frac{3R2 \quad 2R1}{3R1}$

(III) aRb – $a == b$ transitive

(IV) aRb – a divides b — yes

$\frac{2R4}{\ }$

$4R8 \Rightarrow 2R8$

Properties of Asymptotic Notations

	≤ **Big – O**	≥ **Big-Ω**	== **Θ**	< **Little-oh**	> **Little-ω**
Reflexive properties	Yes	Yes	Yes	No	No
Symmetric Properties	No	No	Yes	No	No
Transitive Properties	Yes	Yes	Yes	Yes	Yes
Transpose Symmetry	~~f(n) = O(g(n)) if and only if g(n) = Ω(f(n))~~			~~f(n) = o(g(n)) if and only if g(n) = ω(f(n))~~	

$\leq$ O

$\geq$ Ω

$==$ Θ

$<$ o

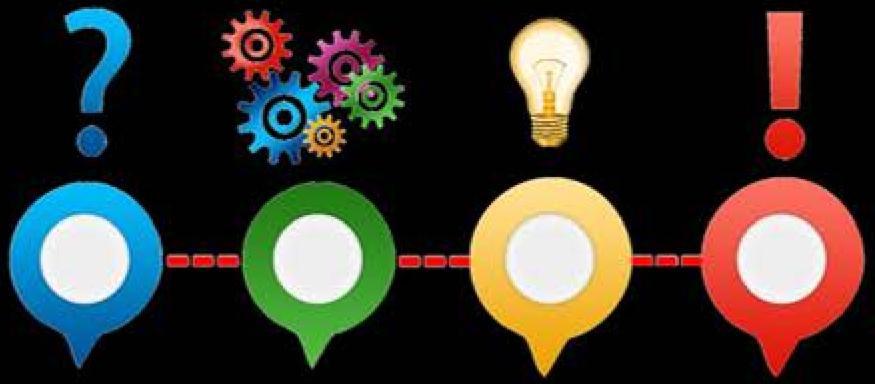
$>$ ω

$\boxed{\text{DM}}$

$n^2 \; \Theta(n^2)$

$n^2 \leq n^2$

$1 < 1$



Problem Solving
Asymptotic Notation

Relation Between Best, Worst & Average Case

for any Algorithm $\quad A(n), B(n), W(n)$

$$B(n) \leq A(n) \leq W(n)$$

GATE 2012, Question Number 18, 1-Mark, Question Category-MCQ

Subject: *Algorithms,* Topic: *Asymptotic Analysis*

Let W(n) and A(n) denote respectively, the worst case and average case running time of ~~an~~ 2021 Merge Sort algorithm executed on an input of size n. Which of the following is ALWAYS TRUE?

a) $A(n)=\Omega(W(n))$ — $n\log n \geq n\log n$ ✓

b) $A(n)=\Theta(W(n))$ — $n\log n == n\log n$

c) $A(n)=O(W(n))$ — $n\log n \leq n\log n$

d) $A(n)=o(W(n))$

$A(n) = n\log n$

$W(n) - n\log(n)$

$A(n) < c \cdot W(n)$

$A(n) = W(n) \leftarrow$

$A(n) \boxed{\leq} W(n)$ $\boxed{\text{yes}}$

$\boxed{A(n) \quad O(W(n))}$

Question

Let $f(n) = \Omega(n)$, $g(n) = O(n)$ and $h(n) = \Theta(n)$. Then $[f(n).g(n)]+h(n)$ is:

(A) Ω(n)

(B) O(n)

(C) Θ(n)

(D) None of these

Question

Let $f(n) = \Omega(n)$, $g(n) = O(n^2)$. Then $[f(n)+g(n)]$ is:

(A) $\Omega(n)$ True

(B) $O(n)$ False

(C) $\Theta(n)$ False

(D) $\Omega(n^2)$ ✗

$f(n)+g(n) - n^2$

n^2 n $1/\log n$ n^2 f n

n

$\boxed{[A]}$

$n+n$ — $g(n) = n$

$n+1$ $g(n) - 1$

$n+\log n$ $g(n)\ \log n$

$f(n) - n^2$

n^2+n

n^2+1

$n^2+\log n$

n is $\Omega(n^2)$

Every situation / Every choice

Question

Let $f(n) = \Omega(n)$, $g(n) = O(n)$ and $h(n) = \Theta(n)$. Then $[f(n).g(n)]+h(n)$ is:

(A) $\Omega(n)$

(B) $O(n)$

(C) $\Theta(n)$

(D) ω (n)

GATE 2004-IT, Question Number 55

Let f(n), g(n) and h(n) be functions defined for positive integers such that

$f(n)=O(g(n))$, $g(n)\neq O(f(n))$, $g(n)=O(h(n))$, and $h(n)=O(g(n))$.

H.W

Which one of the following statements is FALSE?

logically disqualified

a) $f(n)+g(n)=O(h(n)+h(n))$

1 Ans

b) $f(n)=O(h(n))$

c) $h(n)\neq O(f(n))$

d) $f(n)h(n)\neq O(g(n)h(n))$

O

Ω

Θ

Mathematical Background

Logarithm Simplification

- $\log_b a + \log_b c = \log_b ac$
- $\log_b a - \log_b c = \log_b \frac{a}{c}$
- $\log_b a = \frac{\log_c a}{\log_c b}$
- $b^{\log_c a} = a^{\log_c b}$
- $b^{\log_b a} = \boxed{a}$

Logarithm Simplification

- $\log_b a + \log_b c = \log_b ac$
- $\log_b a - \log_b c = \log \frac{a}{c}$
- $\log_b a = \frac{\log_c a}{\log_c b} \leftarrow$

$f(n)$ $\log_2 n$ is θ $(\log_4 n)$

- $b^{\log_c a} = a^{\log_c b}$
- $b^{\log_b a} = a$

Logarithm Simplification

- $\log_b a^n =$ $n \log_b a \Rightarrow \log_b a^n$
- $\log_b \frac{1}{a} =$ $= \log_b 1 - \log_b a$

  $= -\log_b a$ Both are equal
- $\log_b a =$ $\frac{1}{\log_a b}$

$f(n) = 2^{2\log_2 n} = n^2$

$g(n) = n^2$

$f(n) = 2^{\log_2 n^2}$

$f(n) = n^2$

Logarithm Simplification

- $\log_b a^n = n\log_b a$
- $\log_b \frac{1}{a} = -\log_b a$
- $\log_b a = \frac{1}{\log_a b}$

Logarithm Simplification

- $\log_2 2 = 1$
- $\log_2 3 = 1.58496$
- $\log_2 4 = 2$
- $\log_2 5 = 2.32192$
- $\log_2 6 = 2.584962$

Memorize

useful

Natural Log

Solving

Some Complex

Recurrence

Relation

Logarithm Simplification

Powers of $\log n$, such as $(\log n)^7$. We will usually write this as $\log^7 n$.

Summations

For loop

Summations naturally arise in the analysis of iterative algorithms. Also, more complex forms of analysis, such as recurrences, are often solved by reducing them to summations.

Constant Series

$$\sum_{i=a}^{b} 1 =$$

for (i = 69 ; i < 96 ; i++)
a++ ; ← 27 times

How many times a++ will be executed?

i = 69
i 95

$10 - 4 + 1$

$6 + 1 = 7$

4
5
6
7
8
9 10

UB − LB + 1

$95 - 69 + 1$

$26 + 1 = 27$

Constant Series

$$\sum_{i=a}^{b} 1 = b - a + 1$$ ← In case Array Address

6 to 60 Calculation - Data structure

1 + 1 + 1 + 1 + 1

6 7 8 9 10

10 - 6 + 1 = 5

Arithmetic Series: For n $\geq$ 0

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 \ldots\ldots\ldots\ldots + n =$$

Arithmetic Series: For n $\geq$ 0

$$\sum_{i=1}^{n} i^{1} = 1 + 2 + 3 + 4 \ldots\ldots\ldots\ldots\ldots + n = \frac{n(n+1)}{2} = \Theta(n^2)$$

$$1 + 2 + 3 + \ldots + (n-1) = \frac{n(n-1)}{2}$$

Quadratic Series: For n ≥ 0

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 \ldots\ldots\ldots\ldots\ldots + n^2 =$$

Quadratic Series: For n ≥ 0

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 \ldots\ldots\ldots\ldots\ldots + n^2 == \frac{n(n+1)(2n+1)}{6}$$

$\Theta(n^3)$

Cubic Series

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + \dots\dots\dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Cubic Series

$10 - \cancel{1} + \cancel{1}$

10

$1\ 2 \dots 10$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + \dots\dots\dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 = (n^4 + \dots\dots)$$

$$\sum_{i=1}^{n} i^k = \qquad \text{is } \Theta(\_\_\_)$$

Largerer polynomical

$$\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$$

Approximate using Integrals:Riemann sum

- Integration and summation are closely related. (Integration is in some sense a continuous form of summation.) Here is a handy formula. Let $f(x)$ be any monotonically increasing function (the function increases as x increases).

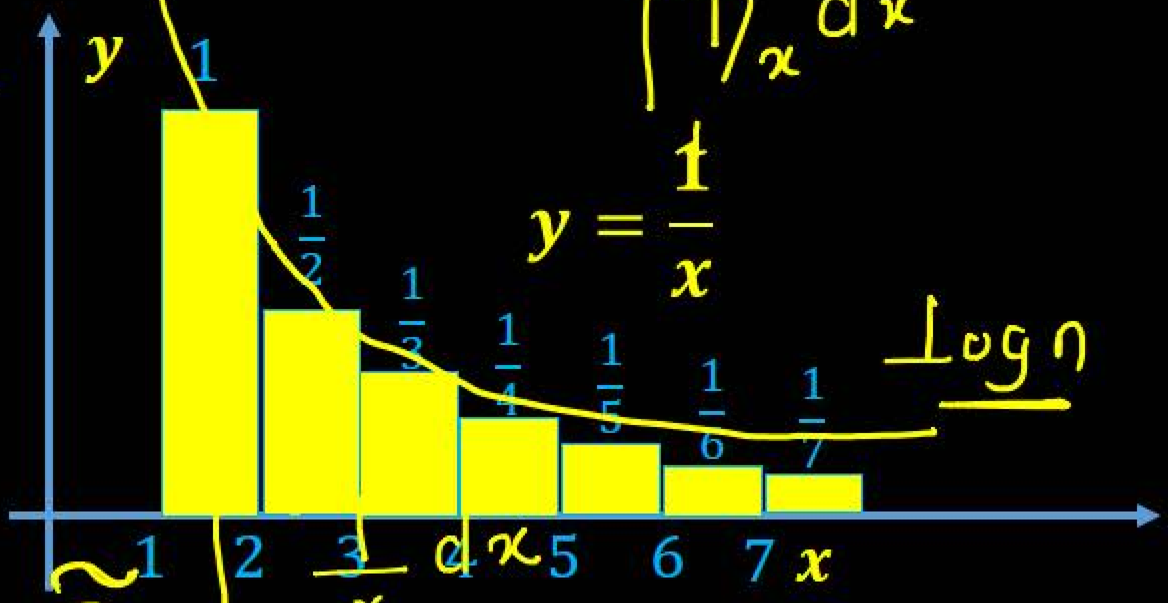
Approximate using Integrals:Riemann sum

$$\int_{i=a-1}^{b} f(x)dx \le \sum_{i=a}^{b} f(i) \le \int_{i=a}^{b+1} f(x)dx$$

Harmonic Series

- This arises often in probabilistic analyses of algorithms. It does not have an exact closed form solution, but it can be closely approximated.

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} =$$



$\int^n 1/x \, dx$

$\log n$

$\approx \int_1^n \frac{1}{x} dx$

$= \log x \Big|_1^n$

$\approx \log n$

$= O(\log n)$

$n > n_0$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \frac{1}{n} =$

approximated $\log n$

Harmonic Series:

- This arises often in probabilistic analyses of algorithms. It does not have an exact closed form solution, but it can be closely approximated.

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} =$$

Harmonic Series

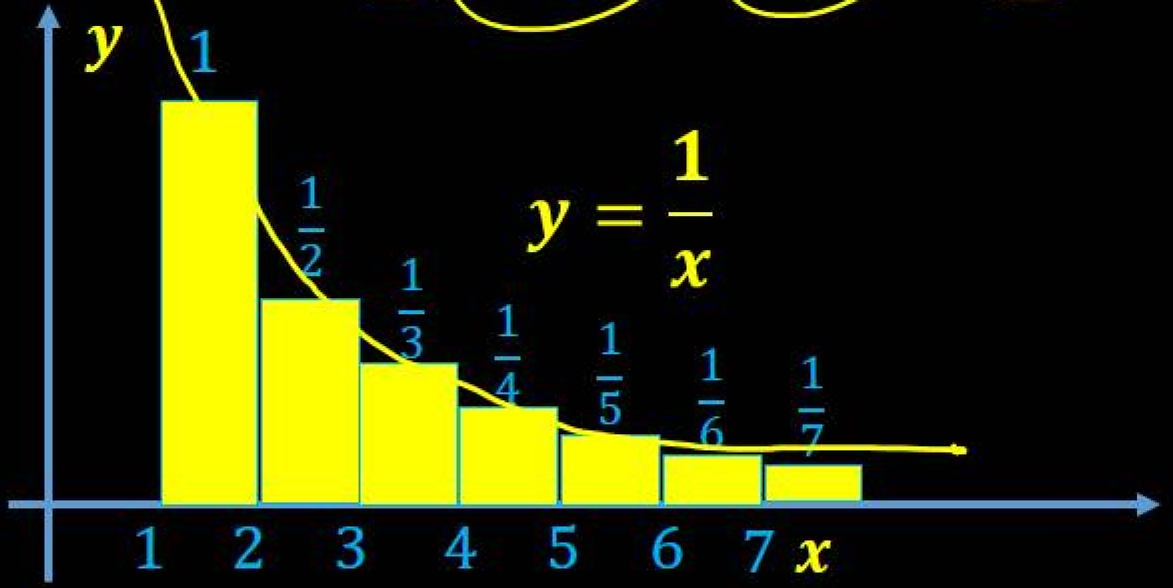
$$H_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} = \log n$$

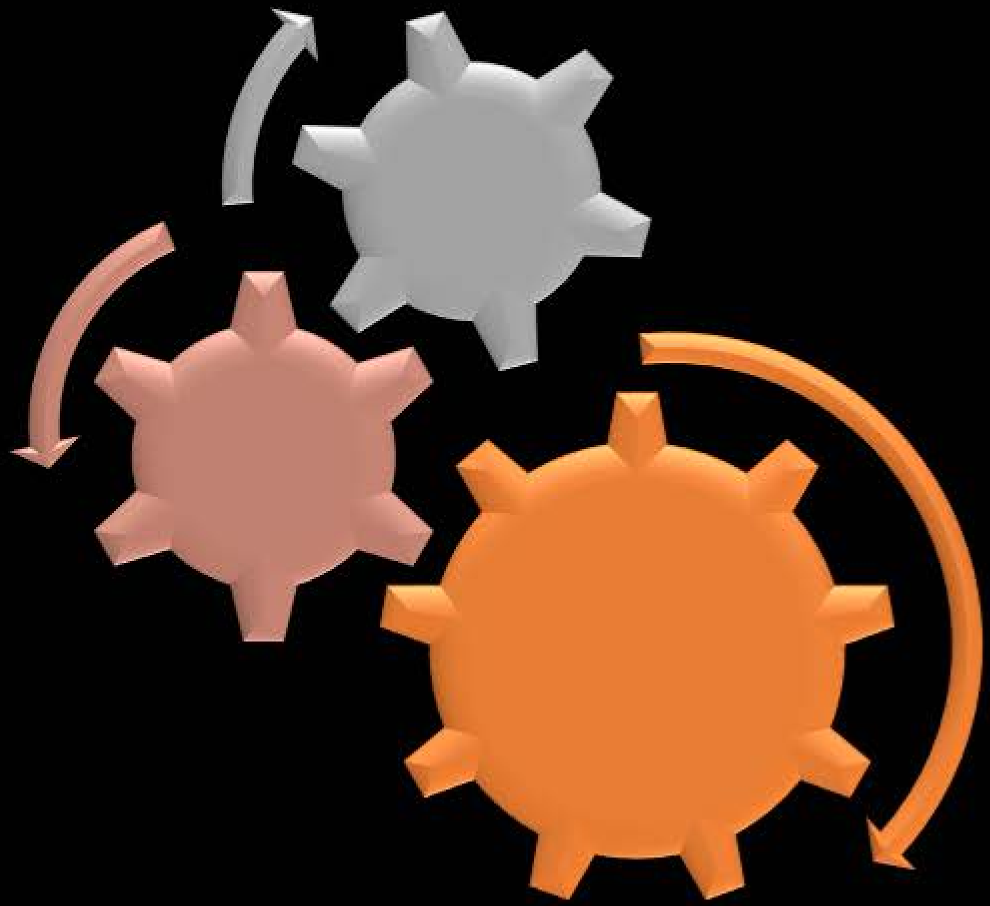
$$\approx \int_1^n 1/x \, dx$$

$\log n$

Recurrence

Series –





Problem Solving

Question

(a) Is $2^{n+1} = O(2^n)$? – True $2^{n+1} \leq 2^n$

false ← (b) Is $2^{2n} = O(2^n)$? $2^{2n} \leq 2^n$ No

x^3 x^2

Exponential function

For a exponential function additive constant can be Ignored but Not multiplicative

we Never compare con1

2^{n+1} 2^n

$2 \cdot 2^n$ 2^n (compare of function)

Constant of value

$$\boxed{2^{2n} \leq 3 \cdot 2^n}$$ $c = 3$

Question

For each of the following pairs of functions, either

$n^6 > n^2$ polyno

1 $f(n)$ *is in* $O(g(n))$ ✓

$\log n^6$ $\log n^2$

$6 \log n == 2 \log n$

$f(n)$ *is in* $\Omega(g(n))$ ✓ or

$\boxed{f(n) \text{ is in } \Theta(g(n))}$ → Both. Determine which relationship is correct.

$$\underbrace{f(n) = \overset{\log n}{\log n^2}}; \; \underbrace{g(n) = \overset{\log n}{\log n + 5}}$$

$\log n^2$

$2 \log n$

$g(n) = \log n + 5$

Question

5, (6=2), 7, 4, 1, ~~2~~, 3

Consider the following function

$4n^2$, $\log_3 n$, 3^n, $20n$, 2, $\log_2 n$, $n^{2/3}$

1 2 3 4 5 6 7

Arrange the expressions by asymptotic

growth rate from slowest to fastest.

$f(n)$ $g(n)$

$\log_2 n$ $\log_4 n$

$\Rightarrow \dfrac{\log_2 n}{\log_2 4} = \dfrac{\log_2 n}{\text{constant } (2)}$

~~Arra~~ Arrange them

$f(n)$ is $\Theta(g(n))$

$f(n) = \log_2 n$

$g(n) = \log_4 n$

(I) $f(n)$ is $O(g(n))$

(II) $g(n)$ is $O(f(n))$

(III) or both correct

Question

(a) Is $2^{n+1} = O(2^n)$?

(b) Is $2^{2n} = O(2^n)$?

Question

For each of the following pairs of functions, either

$f(n)\ is\ in\ O(g(n))$

$f(n)\ is\ in\ \Omega(g(n))$ or

$f(n)\ is\ in\ \Theta(g(n))$. Determine which relationship is correct.

$$f(n) = \log n^2;\ g(n) = \log n + 5$$

Question

For each of the following pairs of functions,

either $f(n)$ $is\ in\ O(g(n))$

$\boxed{f(n)\ is\ in\ \Omega(g(n))}$ or $f(n)\ is\ in\ \Theta(g(n))$.

Determine which relationship is correct and

briey explain why.

$$f(n) = \log^2 n;\ g(n) = \log n$$

$\log n = x$

$n \to \infty$

$(\log n)^2 \quad \log n$

$x^2 > x$

polylogarithmic function

* $(\log n)^7$ as $\log^7 n =$

* $\log\log n$ to $(\log_2 8)^7$

$(8)^7$

$\log\log_2 8$

$\log_2 8$

3