### Algebraic Structure

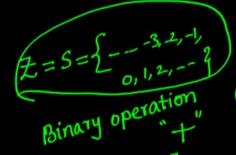


Algebraic Structure: A non-empty set which is equipped with some operations and some properties is known as algebraic structure.

(S, \*)

- Groupoid
- Semi-group
- Monoid
- Group /
- Abelian Group

#### **Some Properties:**



I. Closure: Let 'S' be the given algebraic structure, '\*' is the binary operation and a, b are any two elements in S,

(If a \* b  $\in$  S then we can say (S, \*) follows closure property.

$$\forall a, b \in S, a * b \in S$$

$$2+3=5 \in \Xi$$
 $-2+5=3 \in \Xi$ 
 $0+(-1)=-1 \in \Xi$ 

IL Associative:

$$\forall a, b, c \in S,$$

$$a * (b * c) = (a * b) * c$$

III. Identity:

$$2 + (3+14) = (2+3)+4$$

$$2 + (3+14) = (5)+4$$

$$2 + (7) = (5)+4$$

$$9 = 9$$

$$\forall a \in S, \exists e \in S, \ni For every$$

$$a * e = e * a = a$$
Such that

a\*e=e\*a=a such that a\*e=a 2+0=2

### ACE

#### IV. Inverse:

$$\frac{\forall \ a \in S, \quad \exists \ b \in S, \quad \ni}{a * b = b * a = e}$$

$$a \times b = e$$
 $-2 + (2) = 0$ 
 $-5 + (5) = 0$ 
 $9 + (-9) = 0$ 
 $0 + 0 = 0$ 

### V. Commutative:

$$\forall a, b \in S$$

$$\boxed{a * b = b * a}$$

$$2+3=3+2$$

$$(Z,+) \subseteq (R,+)$$

$$Z \subseteq R$$

$$(H,*) \subseteq (Q,*)$$

### **Classification of Algebraic Structure**



Semi-group (2)	Monoid (3)	Group *	Sub-Group	Abelian (5)
1) Closure 🖊	1) Closure	1) Closure	1) Closure	1) Closure
2) Associative	2) Associative	2) Associative	2) Associative	2) Associative
	3) Identity	3) Identity	3) Identity	3) Identity
		4) Inverse	4) Inverse	4) Inverse
				5) Commutative
	1) Closure / 2) Associative/	1) Closure / 2) Associative / 3) Identity	1) Closure / 1) Closure / 2) Associative / 2) Associative / 3) Identity / 3) Identity	1) Closure 1) Closure 2) Associative 2) Associative 3) Identity 3) Identity 4) Inverse 4) Inverse

### **Sub-Group:**



Let (G, \*) be a group, H is a subset of 'G' and (H, \*) is also group then we can say (H, \*) is a subgroup of (G, \*)

$$(H, *) \subseteq (G, *)$$

$$\not\exists \subseteq R$$

$$(\not\exists, +) \subseteq (R, +)$$

Q. Check the properties of commutative and associative on binary operation '\*' is defined by a \*  $b = a^b$ ,  $\forall$  a,  $b \in N$ 



even binary operation 
$$*$$
 defined by

Associative:

Take  $2,3,4$ 

Commutative:

 $2 \times 3 = 2 = 3$ 

consider  $2 \times 3 = 2 = 3$ 
 $2 \times 3 \times 2 = 3 = 3$ 
 $2 \times 3 \times 2 = 3 \times 2 = 9$ 
 $2 \times 3 \times 2 = 3 \times 2 = 3$ 
 $2 \times 3 \times 2 = 3 \times 2 = 3$ 
 $2 \times 3 \times 2 = 3 \times 2 = 3$ 
 $2 \times 3 \times 2 = 3 \times 2 = 3$ 
 $2 \times 3 \times 2 = 3 \times 2 = 3 \times 3 \times 4 = 3 \times 4 =$ 

Q. Show that the set of all rational number 
$$Q - \{0\}$$
 forms an abelian group under composition '\*' defined by a \* b =  $\frac{ab}{2}$ 



Sol civen set = 
$$0 - 203 = 5$$
 (say)

Binary operation  $a \times b = \frac{ab}{2}$ 

$$= \frac{\chi(\frac{43}{2})}{2} = \frac{\chi y }{4}$$

$$(\chi * y) * 3 = (\frac{\chi y}{2}) * 3$$

$$x + y = \frac{x \cdot y}{2} \in S$$
  
(S, \*) is closed.

$$=\frac{xy^{2}}{4}$$

$$(x*y)*s = (xy)*s$$

$$= xys$$

$$: (S,*) \text{ is associative}$$

$$III. Identity: Let  $e \in S$  such that
$$a*e = a$$

$$\underline{ae} = a$$

$$\underline{ce} = 2$$$$

$$\int C = 2$$

# Inverse: Let us suppose there is some b ES

such that

$$a + b = e$$

$$ab = 2$$

$$b = 4$$

$$c = 4$$

$$c = 4$$

Commutative: 工

Take 
$$x \times y = \frac{xy}{2}$$

$$=y*x$$

$$= \frac{g\pi}{2}$$

$$= y + \pi$$

$$(s, *) is commutative$$

$$(s, *) is an abelian group$$

$$(s, *) is an abelian group$$



## S.T. Set of trational numbers undsome conditions with binary operation \*X' defined by [a+b=a+b-ab] is an abelian group



### closure:

Associative:

Identity: eES

a \* e = ag(1-a) = 0 e(1-a) = 0

le = 0

Inverse:

$$a+b-ab=0$$

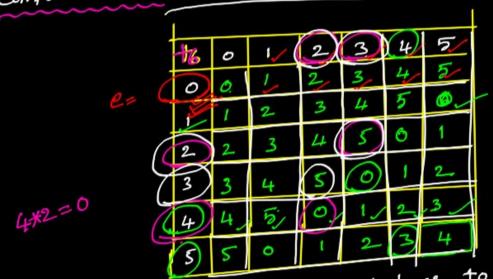
$$b = \frac{a}{1-a} = \frac{a}{a-1}$$

commutative: axb = a+b-ab

#### Show that $[G, +_6]$ is a group where $G = \{0, 1, 2, 3, 4, 5\}$ Q.



### Composition Table



$$2 + \frac{1}{6}(3 + \frac{1}{6}4) = 2 + \frac{1}{6}(1)$$

$$= 3$$

$$(2 + \frac{1}{6}3) + \frac{1}{6}4 = 5 + \frac{1}{6}4$$

$$= 3$$

$$(2+3)$$
  $\frac{1}{6}4 = 5 + 64$ 

All the entries of composition table belongs to a : (a, +6) is closed.

Identity
$$0+60=0$$

$$1+60=1$$

$$2+0=2$$

$$8+60=3$$

$$4+60=4$$

$$5+60=5$$

$$3+60=5$$

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$$5+$$



### Order of Group (Vs) Order of element:



\* The number of the elements (Cardinality) of a given group is known as order of group, O(G)

\* Let (G, \*) be a group and an element  $a \in G$ ,

If  $a^n = e$ , where n is a least positive integer). Then 'n' is called order of the

element 'a'

element 2:  $2^{2} = 2 \times 2 = 2 + 2 = 4$   $2^{3} = 2^{2} + 2 = 4 + 2 = 0 = e$  $a = 2^{3} = 2^{2} + 2 = 4 + 2 = 0 = e$ 

el	ement 4:	
~	4'=4	
4	,2= 4 +64 = 2	1=0=e
4	$3 = 4^2 + 74 = 2 + 76$	,
	1/10 = 3	9 4

1							_
1	46	0	1/(	2)	3	(4)	5
þ	0	0	1	2	3	4	5
_			2	3	4	5	<b>Ø</b> ~
				11	5	0	1
	2			4	=		
7	3	3	4	(5)	0		2
7		1.	5	0	1/	2	3
Q	<u> </u>	7,			2	3	4
ł	5	5	0	1		9	

element 
$$0! = 0$$
 (e)  
element  $1! = 1!$   
 $1^2 = 1 + 6! = 2 + 6! = 3$   
 $1^3 = 1^2 + 6! = 2 + 6! = 3$ 

$$|^{3} = |^{2} + 6| = 2 + 6| = 3$$

$$|^{4} = |^{3} + 6| = 3 + 6| = 4$$

$$|^{5} = |^{4} + 6| = 4 + 6| = 5$$

$$|^{6} = |^{5} + 6| = 5 + 6| = 0 \quad (e)$$

$$|^{6} = |^{5} + 6| = 5 + 6| = 0 \quad (e)$$

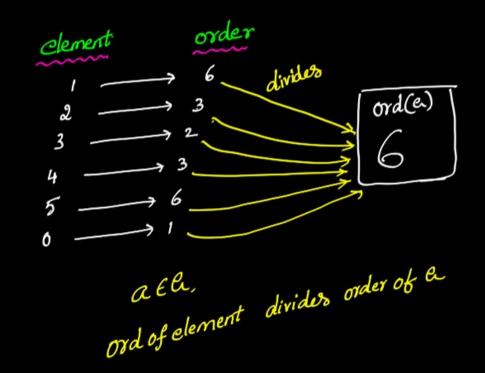
element 3: 3'=3  $3^2=3+6^3=o(e)$ ord(3)=2

element 
$$5'$$
:

 $5'=5$ 
 $5^2=5+5=4$ 
 $5^3=4+5=3$ 
 $5^4=3+6^5=2$ 
 $5^5=2+6=1$ 
 $5^6=1+5=0$  (e).

 $ord(5)=6$ 



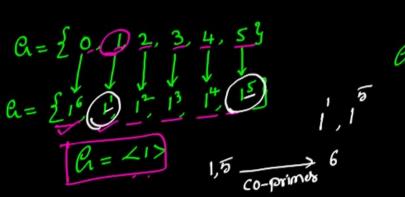


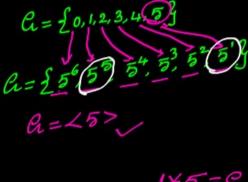


### **Generators & Cyclic Group:**



Let (G, \*) be a group and  $a \in G$ , If every element of (G, \*) can be expressed as integral power of 'a' then 'a' is called generator of 'G' and group (G, \*) is known as cyclic group.





HX	5	=	و
+6	5	=	0

	16	0	1/(	(2)	3	(4)	5
	0		1	2	3	4	2
			2	3	4	5	<b>Ø</b> _
S	2	2	3	4(	5	Ó	1
Ç	2	_		$\overline{a}$		1	2_
(	3	3	4	(3)			,
1	4)	4	5	(0)	1/	2	3 /
	3	5	0	t	2	3	4
4						' '	

\* Every cyclic group is an abelian group



\* If 'a' is the generator of group (G, \*) then a-1 is also be generator.

\* Every group of order ≤ 6 is an abelian.

\* Every group of prime order 1s cyclic

Q. Analyse (G, x) where 
$$G = \{1, -1, i, -i\}$$



0 -1	meition
Con	position
~	~

	×	11	-1/	را	-2/
		11	-1,2	رنا	-i
_	-1	-1	ı	- L	î
	°	9	-L	-1	1_
	-i	-î	i	1	-1

we can prove 
$$1 \times (-1 \times 1) = (1 \times -1) \times 1$$

Inverse: 
$$| \rightarrow |$$
 $| \rightarrow |$ 
 $| \rightarrow |$ 

$$(l_i, x)$$
 is commutative

## a=21,-1,2,-i3, (a,x)

				9
×	11			
	11	-1)	i_	-i_
-1	-1	ı	- L	Ĺ
ů	î	-L	-1	1_
-i	-i	i	1	-1
	1 -1	1) 1/ -1 -1 2 2	1) 1/ -1/ -1 -1 1 2 2 -2	1 -1 1 -i  i i -i -1

ord(a) = 4 
$$^{\circ}$$
  
Divisors of 4 = 1, 2, 4  
and of ele = 1, 2, 4

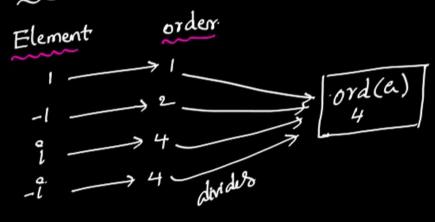
element 1: 
$$1 = 1$$
 ord  $(1) = 1$ 

element  $-1$ :  $(-1)^{i} = -1$ 
 $(-1)^{2} = -1 \times -1 = -1 \times -1 = 1$  (e)

ord  $(-1) = 2$ 

element  $i$ :  $i' = i$ 
 $i'' = -1$ 
 $i''$ 

### Lagrange 18 theorem

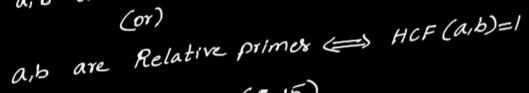




Cyclic & Generators:

$$e_{i} = \underbrace{\sum_{i=1}^{i} \frac{1}{i!} \frac{1}{i$$

a, b are Said to Co-primer (=> gcd(a,b)=1 Co-primes: Two numbers (or)



(8,15), examples Co-primes (9,16)

(3,7)

(5,6)

(4,9)



order. How many generators are there for a cyclic group of '8'? Q.



ord of group 
$$a = o(a) = 8$$

Let us consider generator of en = a.

$$C_1 = \{a', a^2, a^3, a^4, a^5, a^6, a^7, a^8\}$$

a', a3, a5, at are 4 generators

2: How many generators are there for a Cyclic group of order 9'?

 $o\gamma d(a) = 8$ 



#### Lagrange's Theorem:



Let (G, \*) be a group and (H, \*) be a subgroup of G then order of subgroup (H, \*) is always divides of group (G, \*)

Let (G, \*) be a group and an element  $a \in G$ , then the order of element 'a' always divides order of group

$$O(a) \mid O(G)$$



Q. The set {1, 2, 3, 5, 6, 7, 8, 9} under multiplication modulo 10 is not a group. Given below are four possible reasons. Which one of them is false?

[2006:1 Mark]

a) It is not closed

b) 2 does not have an inverse

c) 3 does not have an inverse

d) 8 does not have an inverse

element's: 
$$a * b = e$$
  
 $3*7 = 3*10^7 = 1(e)$   
Triverse of element (3) = 7

(a) closure; 
$$\forall a,b \in S$$
.  $a * b \in S$ 

$$2 * 5 = 2 \times_{10} 5 = 0 \notin C$$

$$(C, X_{10}) \text{ is not closed } (TRUE reason)$$

$$=) (C, X_{10}) \text{ is NOT group}$$

b Inverse: 
$$0 \times b = \mathcal{C}$$
  $2 \times 6 = 2$   
 $2 \times 1 = 2$   $2 \times 7 = 4$   
 $2 \times 3 = 2 \times 10^3 = 6$   $2 \times 8 = 6$   
 $2 \times 5 = 0$   $2 \times 9 = 8$   
2 does not have inverse (TRUE)  $2 \times 8 = 4$ 

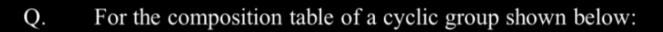


Q. Which one of the following is NOT necessarily a property of a Group?

[2009:1 Mark]



- b) Associativity
- c) Existence of inverse for every element
- d) Existence of identity





element C:

$$c' = c$$
 $c^2 = c * c = b$ 
 $c^3 = c^2 * c = b * c = d$ 
 $c^4 = c^3 * c = d * c = a$ 

Which one of the following choices is correct?

[2009 : 2 Marks]

$$a'=a$$

$$a^2 = a * a = a$$

a) a, b are generators
b) b, c are generators
c) c, d are generators
d) d, a are generators
$$a' = a$$

$$a^2 = a * a = a$$

$$b' = b * b = a$$

$$b^2 = b * b = a$$

$$b^3 = b^2 * b = a * b = b$$

$$b^{2} = b^{2} + b = a + b = b$$



A binary operation  $\oplus$  on a set of integers is defined as  $x \oplus y = x^2 + y^2$ . Q. Which one of the following statements is TRUE about  $\oplus$ ?

[2013:1 Mark]

- a) Commutative but not associative
- b) Both commutative and associative
- c) Associative but not commutative
- d) Neither commutative nor associative

$$2xy = x^2 + y^2$$

$$= y^2 + x^2 = y \times x$$
commutative

Take 
$$\chi_{+}(y+3) = \chi_{+}(y^{2}+3^{2})$$

$$= \chi_{+}^{2}(y^{2}+3^{2})^{2}$$

ative Take 
$$x \neq (y \neq 3) = x \Rightarrow (y^2 + 3^2)$$

$$= x^2 + (y^2 + 3^2)^2$$

$$= (x \Rightarrow y) + 3 = (x^2 + y^2) \times 3$$

$$= (x^2 + y^2) \times 4 \times 3$$

Q. Let G be a group of 15 elements. Let L be a subgroup of G. It is known that  $L \neq G$  and that the size of L is at least 4. The size of L is



order (l.) = 15 (GATE-14-Set3)

L' is subgroup of la

order of subgroup of can be 1, 3, 5, 15

possibilities |L| = o(L) = 1, 2/5 15  $L \neq ll$ ,  $|L| \geq 4$ Size of L = 5

Q. The set {1, 2, 4, 7, 8, 11, 13, 14} is a group under multiplication modulo 15. The inverse of 4 and 7 are respectively.



**x**b) 2 and 11

c) 4 and 13  $(4)^{-1} = ?$ 

 $\chi$ <sup>d) 8 and 14</sup>

Binary operation = 
$$X_{15}$$
  
 $C = \{1, 2, 4, 7, 8, 11, 13, 14\}$ 

$$(4)^{-1} = ?$$

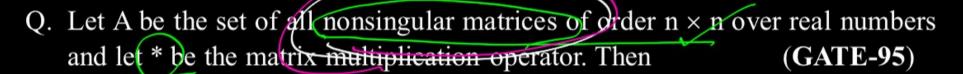
$$(7)^{-1} = ?$$

$$(7)^{-1} = ?$$
option-a:  $4*3=e$ 
 $7*13=e$ 
 $check$ 
 $4*3=4*_{15}3=12 \neq e$ 

(a) 
$$4 \times 2 = 4 \times_{15} 2 = 8 \neq 0$$
  
(c)  $4 \times 4 = 4 \times_{15} 4 = 1 = 0$   
 $7 \times 13 = 7 \times_{15} 13 = 1 = 0$ 

Q. Let G be a group of 35 elements. Then the largest possible size of a subgroup of G other than G itself is \_\_\_\_\_. (GATE-20)







a) A is closed under \* but  $\langle A, * \rangle$  is not a semigroup

b) 
$$\langle A, * \rangle$$
 is a semigroup but not a monoid

c) 
$$\langle A, * \rangle$$
 is a monoid but not a group

Non-singular = 
$$\det A \neq 0$$
  
=> Inverse exists  
A:  $A^{-1} = \frac{adjA}{dtA}$   
A:  $A^{-1} =$ 

Associative:

$$A_1(A_2A_3) = (A_1A_2)A_3$$

Commutative AXB 7 BXA



Identity: Haes, axe= exa=e

$$A \times I = I \times A = A$$

Inxn. Identity

Inverse: 4aes, a\*b=b\*a=e  $A \times B = B \times A = I$   $A \times A^{-1} = A^{-1} \times A = I$ 





Q. The following is the incomplete operation table of a 4-element group

(GATE-04)

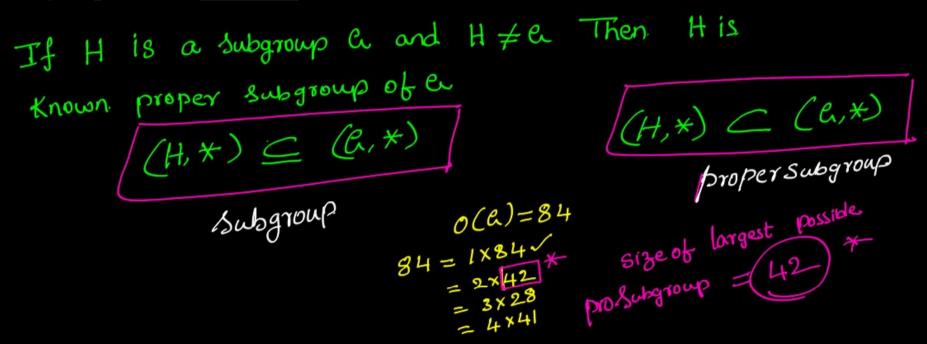
*	е	a	Ь	С
е	e	a	b	С
a	a	b	С	e
b	b	C	e	a
С	(c	e	a	6

The last row of the table is

ord(a)=
$$4 \le 6$$
  
: a is an abelian group  
 $2 \times y = y \times x$ ,  $y = y \times x$ 

Q. Let G be a finite group on 84 elements. The size of a largest possible proper subgroup of G is \_\_\_\_\_. (GATE-18-CSIT)





# Q. Let G be a group of order 6, and H be a subgroup of G such that 1 < |H| < 6. (GATE-21-Set1)



a) Both G and H are always cyclic

d) Both G and H may not be cyclic

lic 
$$ord(a) = 6$$
  
y not be cyclic  $ord(a) = 6$   
always cyclic  $ord(a) = 6$   
clic  $ord(a) = 6$   
clic  $ord(a) = 6$   
 $ord(a) = 6$   

Q. Let  $S = \{0,1,2,3,4,5,6,7\}$  and  $\otimes$  denote multiplication modulo 8, that is,  $x \otimes y = (xy) \mod 8$ .



a) Prove that  $(\{0,1\}, \otimes)$  is not a group

(GATE-2000)

b) Write 3 distinct groups  $(G, \otimes)$  where  $G \subset S$  and G has 2 elements.