Generating Functions

Transforming problems about sequences into problems about functions is known as generating functions.

$< a_0\ a_1\ a_2\ a_3\ \ldots\ldots\ a_i\ \ldots\ldots > \leftrightarrow a_0x^0 + a_1x^1 + a_2x^2 + \ldots\ldots\ldots\ldots + a_ix^i + \ldots\ldots\ldots + a_nx^n$

Rule is a_i is acting as co-efficient of $\cancel{x_i}$ x^i

$< 1\ 2\ 3\ 4 \ldots > \mapsto 1.x^0 + 2.x^1 + 3.x^2 + 4.x^3$

$x^0\ x^1\ x^2\ x^3$

(where indices are 0, 1, 2, 3, ……)

* $\langle 1\ 1\ 1\ 1 \ldots \rangle \longmapsto 1 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + 1 \cdot x^3 + \cdots$

$= 1 + x + x^2 + x^3 + \cdots$ G.P $= S_\infty = \frac{a}{1-r}$

$= \frac{1}{1-x}$

$= (1-x)^{-1}$

$\langle 1\ {-1}\ 1\ {-1} \ldots \rangle \longmapsto 1 - x + x^2 - x^3 + \cdots$

$\langle 1\ 0\ 1\ 0 \ldots \rangle \longmapsto 1 + x^2 + x^4 + \cdots$

$\langle 2\ 0\ 2\ 0 \ldots \rangle \longmapsto$

$\{a_1\}$

n	Elements	No. of ways
$n=0$	—	1 ✓
$n=1$	a_1	1 ✓
$n=2$	Not possible	0 ✓
$n=3$	a.	0 ✓

$$\therefore \text{Sequence} = \langle 1 \; 1 \; 0 \; 0 \; \ldots \rangle \longmapsto 1\cdot x^0 + 1\cdot x^1 + 0\cdot x^2 + 0\cdot x^3 - - -$$

$$= (1+x)$$

Q. The generating function for choosing n-elements from

$\{a_1\} = (1+x)$

$\{a_2\} = (1+x)$

$\{a_1, a_2\} = (1+x)(1+x)$

$\{a_1, a_2, a_3\} = (1+x)(1+x)(1+x)$

$\{a_1, a_2\}$

n	Elements	No. of ways
$n=0$	—	1
$n=1$	a_1, a_2	$\longrightarrow 2C_1 = 2$
$n=2$	$a_1 a_2$	$\longrightarrow 2C_2 = 1$
$n=3$	NOT possible.	0
$n=4$	"	0
		⋮

Sequence is

$\langle 1\ 2\ 1\ 0\ 0 \ldots \rangle \longmapsto 1.x^0 + 2.x^1 + 1.x^2 + 0.x^3 + \cdots$

$= 1 + 2x + x^2$

$= (1+x)^2$

$= (1+x)(1+x)$

= Generating function for seq choosing-n-element from $\{a_1\} * \{a_2\}$

$\{a_1, a_2, a_3\}$ $\qquad (1+x)^3$

n	Elements	No.of ways
$n=0$	—	1
$n=1$	$a_1, a_2, a_3 \longrightarrow 3C_1$	$= 3$
$n=2$	$a_1a_2, a_1a_3, a_2a_3 \longrightarrow 3C_2$	$= 3$
$n=3$	$a_1a_2a_3 \longrightarrow 3C_3$	$= 1$
$n=4$	Not possible	0

Sequence is

$\langle 1\ 3\ 3\ 1\ 0\ 0 \ldots \rangle$

$\longmapsto 1.x^0 + 3.x^1 + 3.x^2 + 1.x^3 + 0.x^4 + \ldots$

$= 1 + 3x + \underline{3}x^2 + x^3$ $\qquad \underline{3}.x^{2}$

$= (1+x)^3$

$= (1+x)(1+x)(1+x)$

Generating function for choosing n-elements from $\{a_1\} \times \{a_2\} \times \{a_3\}$

Q. Generating function for choosing n-elements from

$\{a_1, a_2, a_3, a_4\} = (1+x)(1+x)(1+x)(1+x)$

$= (1+x)^4$

$= 1+4x+6x^2+4x^3+x^4$

$= 1+4.x^1+6.x^2+4.x^3+1.x^4$

$4.x^1$

$6.x^2 =$ No. of ways choosing 2-elements from $(1+x)^4$ is 6

$4x^3 =$ No. of ways choosing 3-elements $= x^3$ co-eff $= 4$

Generating function for choosing n-elements of $\{a_1, a_2, a_3, \ldots.., a_k\}$

$$= (1 + x)(1 + x)(1 + x) \ldots\ldots (1 + x) \text{ [k – times]}$$

$$= (1 + x)^k$$

* The number of ways of choosing 3-objects of $\{a_1, a_2, \ldots.., a_k\}$

$$= \text{Co-efficient of } x^3 \text{ in } (1 + x)^k$$

* The number of ways of choosing n-objects of $\{a_1, a_2, \ldots.., a_k\}$

$$= \text{Co-efficient of } x^n \text{ in } (1 + x)^k$$

$$= {}^kC_n \checkmark$$

* $< 1\ 1\ 1\ 1\ \ldots\ldots > \leftrightarrow 1 + x + x^2 + x^3 + \ldots\ldots$

$$\leftrightarrow \frac{1}{1-x} = (1-x)^{-1}$$

* The generating function for choosing n-objects of $\{a_1\}$ with repetitions

$$= 1.x^0 + 1.x^1 + 1.x^2 + 1.x^3 + \ldots\ldots\ldots\ldots$$

$$= 1 + x + x^2 + x^3 + \ldots\ldots$$

$$= (1-x)^{-1}$$

n	$\{a_1\}$ Elements	No. of ways
$n=0$	—	1
$n=1$	a_1	1
$n=2$	$a_1 a_1$	1
$n=3$	$a_1 a_1 a_1$	1

$< 1\ 1\ 1\ 1 ---- >$

* The generating function for choosing n-objects of $\{a_2\}$ with repetition = $(1 - x)^{-1}$

* The generating function for choosing n-objects of

$\{a_1, a_2, a_3, \ldots\ldots, a_k\}$ with repetitions $= (1-x)^{-1}(1-x)^{-1} \ldots\ldots (1-x)^{-1}$ [k times]

$$= (1-x)^{-k}$$

* The number of ways of choosing n-objects of $\{a_1, a_2, a_3, \ldots\ldots, a_k\}$

With repetitions = Co-efficient of x^n in $(1-x)^{-k}$

$$\sum_{n=0}^{\infty} {}^{n+k-1}C_n \cdot x^n$$

$$= \frac{(n+k-1)!}{(k-1)!n!} = {}^{(n+k-1)}C_n$$

$${}^{n+k-1}C_n$$

$*(1-x)^{-k} = \Sigma_{n=0}^{\infty}\ {}^{(n+k-1)}C_n \cdot x^n$ $= \sum_{n=0}^{\infty} {}^{n+k-1}C_n \cdot x^n$

$*\ (1-a\cancel{x})^{-k} = \Sigma_{n=0}^{\infty}\ {}^{(n+k-1)}C_n \cdot a^n \cdot x^n$ where ‘k’ is positive integer

$(1-ax)^{-k} = \sum_{n=0}^{\infty} {}^{n+k-1}C_n \cdot a^n \cdot x^n$

Q. $(1-3x)^{-1} = ?$

$a=3, \quad k=1$

$$(1-ax)^{-k} = \sum_{n=0}^{\infty} {}^{n+k-1}C_n \cdot a^n \cdot x^n$$

$$= \sum_{n=0}^{\infty} {}^{n+1-1}C_n \cdot 3^n \cdot x^n$$

$$= \sum_{n=0}^{\infty} 3^n \cdot x^n$$

Q. The generating function of the sequence

$\{1, -2, 4, -8, 16, -32, \ldots\ldots\}$

$x^0 \quad x^1 \quad x^2 \quad x^3 \quad x^4$

$$\{1, -2, 4, -8, 16, -32, \ldots\} \longmapsto 1\cdot x^0 - 2\cdot x^1 + 4\cdot x^2 - 8\cdot x^3 + 16\cdot x^4 - 32\cdot x^5 + \ldots$$

$$= 1 - 2x + 4x^2 - 8x^3 + 16x^4 - 32x^5 + \ldots$$

$$= \frac{1}{1+2x}$$

$$= (1+2x)^{-1}$$

$a = 1, \quad r = -2x$

$S_\infty = \frac{a}{1-r}$

$(1-2x)^{-1}$

$(1-ax)^{-k}$

Q. The generating function of the sequence

$\{0, 1, 3, 9, 27, \ldots\ldots\} \longmapsto 0\cdot x^0 + 1\cdot x^1 + 3\cdot x^2 + 9\cdot x^3 + 27\cdot x^4 + \cdots$

$= x + 3x^2 + 9x^3 + 27x^4$

$a = x, \quad r = 3x$

$= \dfrac{x}{1-3x}$

$= x(1-3x)^{-1}$

Q. The Co-efficient of x^{27} in the expansion of

$(x^4 + 2x^5 + 3x^6 + 4x^7 + \ldots\ldots\ldots + \infty)^5$ is

$$= \left[x^4(1+2x+3x^2+4x^3+\cdots+\infty)\right]^5$$

$$= x^{20}(1+2x+3x^2+4x^3+\cdots+\infty)^5$$

$$= x^{20}\left[(1-x)^{-2}\right]^5$$

$$= x^{20}(1-x)^{-10}$$

$16C_7$

$$x^{20}(1-x)^{-10}$$

$$= x^{20}\sum_{n=0}^{\infty} {}^{n+k-1}C_n \cdot x^n$$

$$= x^{20}\sum_{n=0}^{\infty} {}^{n+10-1}C_n \cdot x^n$$

$$= \boxed{x^{20}\sum_{n=0}^{\infty} {}^{n+9}C_n \cdot x^n}$$

Put $n=7$, $16C_7$

$\langle 1\ 2\ 3\ 4 \cdots \rangle$

$$(1-x)^{-k} = \sum_{n=0}^{\infty} {}^{n+k-1}C_n \cdot x^n$$

$n=0,\ k=2$

$${}^{n+k-1}C_n = {}^{0+2-1}C_0 = {}^1C_0 = 1$$

$${}^{1+2-1}C_1 = {}^2C_1 = 2$$

$${}^{2+2-1}C_2 = {}^3C_2 = 3$$

Q. Consider the polynomial

$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$,

Where $a_i \neq 0, \forall i$. The minimum number of multiplications needed to evaluate p on an input x is **(GATE-CS-06)**

a) 3

b) 4

c) 6

d) 9

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$
$$= a_0 + x\left[a_1 + a_2x + a_3x^2\right]$$
$$= a_0 + \underline{x}\left[a_1 + \underline{x}\left(a_2 + \underline{a_3x}\right)\right]$$

Q. $\sum_{x=1}^{99} \frac{1}{x(x+1)} =$ ______ (GATE-15-Set1)

$$\sum_{x=1}^{99} \frac{1}{x(x+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \cdots$$

$$= \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{5}}\right) + \cdots \left(\cancel{\tfrac{1}{99}} - \tfrac{1}{100}\right)$$

$$= 1 - \frac{1}{100}$$

$$= \frac{99}{100} = 0.99$$

Q. Let $G(x) = 1/(1-x)^2 = \sum_{i=0}^{\infty} g(i)x^i$,

Where $|x| < 1$ what is $g(i)$? **(GATE-CS-05)**

a) 1

b) $i + 1$

c) $2\ i$

d) $2i$

$$G(x) = \frac{1}{(1-x)^2} = \sum_{i=0}^{\infty} g(i)\, x^i$$

Rules for finding (C.F) are given below.

1. Characteristics roots are real and distinct say $t_1, t_2 \ldots.. t_k$

$$C.F = c_1.t_1^n + c_2.t_2^n + \ldots\ldots + c_k.t_k^n \quad = c_1.t_1^n + c_2.t_2^n + c_3.t_3^n + \ldots$$

2. Roots are real and two roots are equal say $t_1, t_1, t_3, t_4 \ldots., t_k$ $\quad t_1, t_1, t_3$

$$C.F = (c_1 + c_2.n)t_1^n + c_3.t_3^n + \ldots\ldots + c_k.t_k^n \quad =$$

3. Roots are real and 3 roots are equal say $t_1, t_1, t_1, t_4 \ldots.. t_k$

$$C.F = (c_1 + c_2.n + c_3.n^2).t_1^n + c4.t_4^n + \ldots\ldots + c_k.t_k^n$$

4. Suppose if all the roots are equal say $t_1, t_1, t_1, \ldots.. t_1$

$$C.F = (c_1 + c_2.n + c_3.n^2 + \ldots\ldots + c_k.n^{k-1})t_1^n$$

5. A pair of roots are complex say $(\alpha \pm i\beta)$

$$C.F = r^n(c_1.\cos(n\theta) + c_2.\sin(n\theta)\}$$

Where

$$r = \sqrt{\alpha^2 + \beta^2} \text{ and}$$

$$\theta = tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

Particular Solution (P.S):-

$$\phi(E)\, a_n = F(n)$$
$$a_n = \frac{F(n)}{\phi(E)}$$

From equation (2),

$$P.S = \frac{1}{\phi(E)}\{F(n)\}$$

The solution of equation (1) is

$$\boxed{a_n = C.F + P.S}$$

Rule to find P.S:-

When $F(n) = b^n$ we have

$$P.S = \frac{b^n}{\phi(E)} = \frac{b^n}{\phi(b)} \text{ provided } \phi(b) \neq 0$$

$F(n) = b^n \cdot n^k$

$$P.S = \frac{F(n)}{\phi(E)}$$

$E^2 - 2E + 2$

Case of failure:-

When $\phi(b) = 0$, use the formula

$$\frac{b^n}{(E-b)^k} = C(n.k).\,b^{n-k} \quad (k = 1, 2, 3, \ldots.)$$

$\phi(E) = E^2 - 2E + 1$

$= (E-1)^2$

Method of undetermined Co-efficient:

$$\phi(E)\, a_n = F(n)$$

$$F(n) = b^n . n^k$$

$b \neq t$ $\quad k=2$

i) If 'b' is not characteristic root then

$$P.S = b^n(A_0 n^k + A_1 n^{k-1} + \ldots\ldots + A_k) = b^n[cn^2 + dn + e]$$

ii) If 'b' is the characteristic root with multiplicity 'm' then $\quad b = t$

$(t-5)^2 = 0$

$t = 5, 5$

$m = 2$

$$P.S = b^n(A_0 n^k + A_1 n^{k-1} + \ldots\ldots + A_k)n^m$$

Particular Solution

- $F(n) \neq 0$
  - $F(n) = b^n$
    - $\phi(b) \neq 0$: $P.S = \dfrac{F(n)}{\phi(b)}$
    - $\phi(b) = 0$: $P.S = nc_k \cdot b^{n-k}$
  - $F(n) = b^n n^k$
    - $b \neq t$: $P.S = b^n \left[A_0 n^k + A_1 n^{k-1} + A_2 n^{k-2} + \cdots A_k\right]$
    - $b = t$: $P.S = b^n \left[A_0 n^k + A_1 n^{k-1} + A_2 n^{k-2} + \cdots + A_k\right] n^m$

Q. Solve the following recurrence relation (GATE-CS-98)

$$X_n = 2X_{n-1} - 1 \quad n > 1$$

$$\boxed{X_1 = 2}$$

Given Recurrence Relation is

$$x_n = 2x_{n-1} - 1$$

$$x_n - 2x_{n-1} = -1$$

put $n = n+1$

$$x_{n+1} - 2x_n = -1$$

$$E^1(x_n) - 2E^0(x_n) = -1$$

$$(E-2)x_n = -1$$

This is in form $\phi(E)a_n = F(n)$

$$\boxed{\phi(E) = E-2}$$

char. eq. $\phi(t) = t-2 = 0 \Rightarrow t = 2$

$$\boxed{C.F = c_1 t_1^n = c_1 2^n} \checkmark$$

Particular solution:

$$F(n) = -1 = (-1)(1)^n$$

$b = 1$ ✓

$$P.S = \frac{F(n)}{\phi(b)} = -1\left[\frac{(1)^n}{b-2}\right] = -1\left[\frac{1^n}{1-2}\right] = 1 \checkmark$$

Solution to RR is

$$x_n = C.F + P.S$$

$$\boxed{x_n = c_1 2^n + 1}$$

Initial condition $x_1 = 2$

$$x_1 = c_1 2^1 + 1 = 2$$

$$2c_1 = 1$$

$$c_1 = \frac{1}{2}$$

$$\therefore \quad x_n = \frac{1}{2} \cdot 2^n + 1$$

$$\boxed{x_n = 2^{n-1} + 1}$$

Q. Solve the recurrence equations (GATE-CS-87)

$T(n) = T(n-1) + n$

$T(1) = 1$ ✓ $a_1 = 1$

Given Recurrence relation

$T(n) = T(n-1) + n$ ✓

$a_n = a_{n-1} + n$ ✓

$a_n - a_{n-1} = n$ ✓

Put $n = n+1$

$a_{n+1} - a_n = n+1$

$E^1(a_n) - E^0(a_n) = n+1$

$(E-1)\,a_n = n+1$

$\phi(E)\,a_n = F(n)$

$\phi(E) = E - 1$

char. eq. $\phi(t) = (t-1)^1 = 0 \Rightarrow \boxed{t=1}$ ✓ $m = 1$

$\boxed{C.F = C_1 t_1^n = C_1 1^n = C_1}$ ✓

$F(n) = n+1 = 1^n (n^1 + 1) = b^n \cdot n^k$

$\boxed{b=1}$, $k = \underline{1}$

Here $b = t$

$P.S = b^n [A_0 n^k + A_1 n^{k-1} + \dots] n^m = 1^n [cn + d] n^1$

$$\boxed{P.S = cn^2 + dn}$$

$\therefore$ Solution to RR,

$$a_n = C.F + P.S$$

$$\boxed{a_n = c_1 + cn^2 + dn}$$

Let us Substitute P.S in R.R.

$$a_n - a_{n-1} = n$$

$$(cn^2 + dn) - [c(n-1)^2 + d(n-1)] = n$$

$$(cn^2 + dn) - [c(n-1)^2 + d(n-1)] = n$$

put $n=0$

$$0 - c + d = 0$$

$$-c + d = 0 \longrightarrow (1)$$

put $n=1$

$$c + d = 1 \longrightarrow (2)$$

$$c = \frac{1}{2} ; \quad d = \frac{1}{2}$$

$$\therefore \boxed{a_n = c_1 + \frac{1}{2}n^2 + \frac{1}{2}n}$$

initial $a_1 = 1$

$$a_1 = c_1 + \frac{1}{2}(1)^2 + \frac{1}{2}(1) = 1$$

$$\Rightarrow c_1 = 0$$

$$\therefore \boxed{a_n = \frac{1}{2}n^2 + \frac{1n}{2}}$$

$$\boxed{a_n = \frac{n(n+1)}{2}}$$

Method - II :

Given Recurrence Relation is

$$T(n) = T(n-1) + n$$

Initial $T(1) = 1$

$$T(2) = T(1) + 2 = 1 + 2 = \cancel{x}.3$$

$$T(3) = T(2) + 3 = (1+2) + 3$$

$$T(4) = T(3) + 4 = (1+2+3) + 4$$

$$T(5) = T(4) + 5 = (1+2+3+4) + 5$$

$$\vdots$$

$$T(n) = 1+2+3+4+5+ \cdots + n = \frac{n(n+1)}{2}$$

Method of undetermined Co-efficient:

$$\phi(E)\, a_n = F(n)$$

$$\boxed{F(n) = b^n.n^k}$$

i) If ‘b’ is not characteristic root then

$$P.S = b^n(\underline{A_0}n^k + \underline{A_1}n^{k-1} + \ldots.. + \underline{A_k}) \checkmark$$

ii) If ‘b’ is the characteristic root with multiplicity ‘m’ then

$$P.S = b^n(\underline{A_0}n^k + \underline{A_1}n^{k-1} + \ldots.. + \underline{A_k})n^m \checkmark$$

Q. Solve the recurrence equations:

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

$$T(1) = 1$$

(GATE-CS-88)

Given Recurrence Relation

$T(n) = T\left(\frac{n}{2}\right) + 1$

$T(n) = aT\left(\frac{n}{b}\right) + \theta\left(n^k \cdot \log_b^p n\right)$

$a = 1,\ b = 2,\ k = 0,\ p = 0$

$a = 1,\ b^k = 2^0 = 1$

clearly $a = b^k$

and also $p > -1$

$T(n) = \theta\left[n^{\log_b^a} \cdot \log_b^{p+1} n\right]$

$T(n) = \theta\left[n^{\log_2^1} \log_2^{0+1} n\right]$

$= \theta(\log n)$

Method-II :

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

let $n = 2^x$

$$\therefore T(2^x) = T\left(\frac{2^x}{2}\right) + 1$$

$$T(2^x) = T(2^{x-1}) + 1$$

$$a_x = a_{x-1} + 1$$

$$a_x - a_{x-1} = 1$$

put $x = x+1$

$$a_{x+1} - a_x = 1$$

$$E^1(a_x) - E^0(a_x) = 1$$

$$(E-1)a_x = 1$$

$$\phi(E)\, a_n = F(n)$$

$$\boxed{\phi(E) = E-1} = (E-1)^1 = (E-1)^{1=K}$$

char. eq. $\phi(t) = t-1 = 0 \Longrightarrow t = 1$

$$\boxed{C.F = c_1 \cdot t_1^{\,n} = c_1 . 1^n = c_1}$$

$$F(n) = 1 = 1^n = b^n$$

$$\phi(b) = 0$$

$$P.S = \frac{b^n}{(E-b)^K} = {}^nC_K \cdot b^{n-K}$$

$$= {}^xC_K \cdot b^{x-K}$$

$$= {}^xC_1 \cdot (1)^{x-1}$$

$$= x$$

$$n = 2^x$$

$$\log_2 n = x$$

Solution to R.R.

$$a_x = C.F + P.S$$

$$\boxed{a_x = c_1 + x}$$ ✓

initial condition $T(1) = 1$

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

$$n = 2^x = 2^0$$

$$T(1)$$

$$T(1) = T(n) = T(2^x) = T(2^0)$$

$$= a_0 = c_1 + 0 = 1$$

$$c_1 = 1$$

$$\therefore \; a_x = c_1 + x$$

$$\boxed{a_x = 1 + x}$$

$$a_x = T(2^x) = 1 + x$$

$$= \boxed{T(n) = 1 + \log_2 n}$$

$$\theta(1 + \log n)$$

Q. Find the particular solution of

$$an - 2a_{n-1} + a_{n-2} = n^2 + n + 1$$

$$a_n - 2a_{n-1} + a_{n-2} = n^2 + n + 1$$

$$\boxed{p.s = cn^4 + dn^3 + en^2}$$

Q. The recurrence equation (GATE-CS-04)

$T(1) = 1$

$T(n) = 2T(n-1) + n, n \geq 2$ evaluates to

a) $2^{n+1} - n - 2$

b) $2^{n} - n$

c) $2^{n+1} - 2n - 2$

d) $2^{n} + n$

Q. The solution to the recurrence equation $T(2^k) = 3T(2^{k-1})+1$, $T(1) = 1$ is

(GATE-CS-02)

a) 2^k

b) $(3^{k+1} - 1)/2$

c) $3^{\log 2^k}$

d) $2^{\log 3^k}$