Q. What is the minimum number of ordered pairs of non-negative numbers that should be chosen to ensure that there are two pairs(a,b) and (c,d) in the chosen set such that $a \equiv c \bmod 3$ and $b \equiv d \bmod 5$

0,1,2,3,4

(GATE-CS-05)

$a \equiv c \bmod 3$

a) 4 b) 6

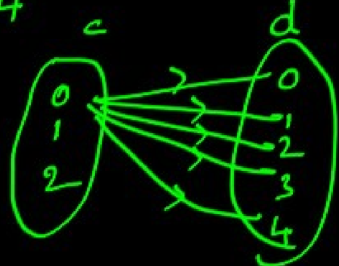
$3 \,)\, a \,($

c) 16 d) 24

$c \rightarrow 0,1,2$

$c = 0, 1, 2$

$d = 0, 1, 2, 3, 4$



$(c, d) = \{\boxed{(0,0)}, (0,1), (0,2), (0,3), (0,4), (1,0), (1,1), (1,2), (1,3), (1,4), (2,0), \boxed{(2,1)}, (2,2), (2,3), (2,4)\}$

16

$(a, b) = ($

$$\begin{matrix} a & b \\ (5, & 6)\end{matrix} \longrightarrow \begin{matrix}(c, & d) \\ (2, & 1)\end{matrix}$$

$$\begin{matrix} a & b \\ (24, & 15)\end{matrix} \longrightarrow \begin{matrix}(c, & d) \\ (0, & 0)\end{matrix}$$

04. The minimum number of colors required to color the vertices of a cycle with n nodes in such a way that no two adjacent nodes have the same color is:

(GATE-CS-02)

a) 2

b) 3

c) 4

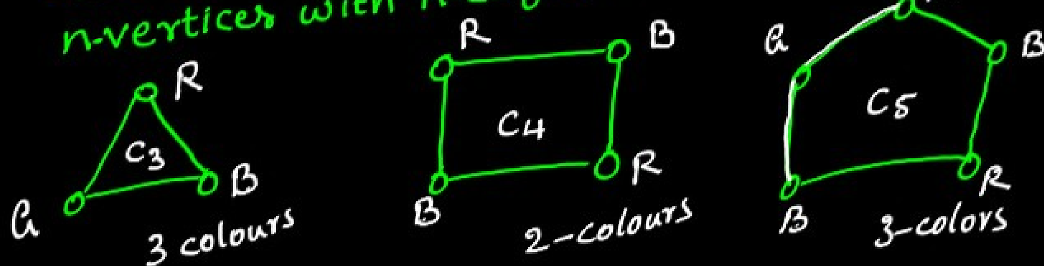
d) $n-2\lfloor n/2\rfloor+2$ $= 5-2\left\lfloor \frac{5}{2}\right\rfloor+2$

$= 7-2(2)$

$= 3$

cycle $[C_n]$

n-vertices with n-edges with exactly one cycle



(d) $n-2\left\lfloor \frac{n}{2}\right\rfloor+2$

$= 4-2\left\lfloor \frac{4}{2}\right\rfloor+2 = 2$

Euler Function:

If ‘n’ is a positive integer then $\phi(n)$ = The number of integers x such that $1 \le x \le n$ and ‘n’ and ‘x’ are relatively prime (co prime)

$1 \le x \le n$

$\phi(n)$ = Number of positive integers, which are less than ‘n’ and co-primes to ‘n’.

$$\phi(n) = n \times \left(1 - \frac{1}{P_1}\right) \times \left(1 - \frac{1}{P_2}\right) \times \left(1 - \frac{1}{P_3}\right) \times \cdots \ldots \ldots ..$$

Number $\boxed{n = P_1^{\alpha_1} \times P_2^{\alpha_2} \times P_3^{\alpha_3} \times \ldots\ldots\ldots}$

P_i = Prime

$\alpha_i \in N$

Number of divisors of n =

$$\tau(n) = (\alpha_1+1)(\alpha_2+1)(\alpha_3+1)\ldots\ldots$$

Co-primes

a, b are co-primes $\Longleftrightarrow$ $\gcd(a,b) = 1$

examples $(8,15)$
$(4,9)$
$(7,13)$
$(8,19)$

example

Take 15

$15 \rightarrow 1, 2, \cancel{3}, 4, \cancel{5}, \cancel{6}, 7, 8, \cancel{9}, \cancel{10},$
$11, \cancel{12}, 13, 14, \cancel{15}$

$\gcd(3,15) = 3$

$x = \phi(15) = $ No. of Co-primes $= 8$

$15 = 3^1 \times 5^1$

$p_1 = 3, \quad p_2 = 5$

$$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\left(1-\frac{1}{p_3}\right)\cdots$$

$$\phi(15) = 15 \times \left(1-\frac{1}{3}\right) \times \left(1-\frac{1}{5}\right)$$

$$= \cancel{15} \times \frac{2}{\cancel{3}} \times \frac{4}{\cancel{5}} = 8$$

$\phi(24)$

$\phi(8)$

$\phi(16)$ ✓

$\phi(36)$ ✓

$\phi(4) = 2$ ✓

$\phi(9) = 6$ ✓

$\phi(21) =$

$\phi(17) = 16$

$\phi(24)$

$24 = 2^3 \times 3^1$

$P_1 = 2, \quad P_2 = 3$

$\phi(24) = \cancel{24}^{8} \times \frac{1}{\cancel{2}} \times \frac{2}{\cancel{3}}$

$= 8$

Properties of Euler Function:

I. $\phi(P) = P - 1$, where P is prime

Eg: $\phi(23) = 22$

$\phi(17) = 17 - 1 = 16$

II. $\phi(m \times n) = \phi(m) \times \phi(n)$, where $\gcd(m, n) = 1$

Eg: $\phi(21) = \phi(3 \times 7)$

$= \phi(3) \times \phi(7) = 2 \times 6 = 12$

$\phi(36) = \phi(4 \times 9)$

$= \phi(4) \times \phi(9)$

$= 2 \times 6 = 12$

III. $\phi(P^n) = P^n - P^{n-1}$, where P is prime

Eg: $\phi(8) = \phi(2^3) = 2^3 - 2^2$

$= 8 - 4 = 4$

$\phi(16) = \phi(2^4) = 2^4 - 2^3$

$= 16 - 8$

$= 8$

Q. Number of positive integers which are less than 1368 and co-prime to 1368 is ______

$$\phi(1368) = ?$$

$$\begin{array}{r|l} 2 & 1368 \\ \hline 2 & 684 \\ \hline 2 & 342 \\ \hline 19 & 171 \\ \hline & 9 = 3^2 \end{array}$$

$$1368 = 2^3 \times 3^2 \times 19^1$$

$$P_1 = 2, \quad P_2 = 3, \quad P_3 = 19$$

$$\phi(1368) = 1368 \times \frac{1}{2} \times \frac{2}{3} \times \frac{18}{19}$$

$$= \overset{72}{\cancel{1368}} \times \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \times \frac{\overset{\cancel{9}\ 3}{\cancel{18}}}{\cancel{19}}$$

$$= 432$$

Q. The formula for number of positive integers 'm' which are less than P^k and relatively prime to P^k, where 'P' is a prime and k is a positive integer is ______

a) $P^k(P-1)$

b) $P^{k-2}(P-1)$

c) $P^k(P-2)$

d) $P^{k-1}(P-1)$ ✓

$x = m$

$n = p^k$

$$\phi(n) = n \times \left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right) \cdots$$

$$\phi(p^k) = p^k * \left(1-\frac{1}{p}\right)$$

$$= p^k\left(\frac{p-1}{p}\right)$$

$$= p^{k-1}(p-1)$$

$n = 2^3 \times 5^2$

$n = p^k$

Q. Let $n = p^2q$, where p and q are distinct prime numbers. How many numbers m satisfy $1 \leq m \leq n$ and $\gcd(m,n) = 1$? Note that $\gcd(m, n)$ is the greatest common divisor of m and n. **(GATE-IT-05)**

a) $p(q-1)$

b) pq

c) $(p^2-1)(q-1)$

d) $p(p-1)(q-1)$ ✓

$\boxed{n = p^2 \times q^1}$, p, q are primes

$\phi(n) = n \times (1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots$

$= n \times (1 - \frac{1}{p}) \times (1 - \frac{1}{q})$

$= p^2q * \frac{p-1}{p} * \frac{q-1}{q} = p(p-1)(q-1)$

Q. The exponent of 11 in the prime factorization of 300! is

(GATE-IT-08)

a) 27

b) 28

c) 29 ✓

d) 30

300!

$$\begin{array}{r|l} 11 & 300 \\ \hline 11 & 27 \\ \hline 11 & 2 \\ \hline & 0 \\ \hline \end{array} \quad \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} 27+2+0 = 29$$

Q. The number of divisors of 2100 is ______ (GATE-15-Set2)

ACE

$$2100 = 2^2 \times 3^1 \times 5^2 \times 7^1$$

$$\alpha_1 = 2, \ \alpha_2 = 1, \ \alpha_3 = 2, \ \alpha_4 = 1$$

$$\tau(2100) = (\alpha_1+1)(\alpha_2+1)(\alpha_3+1)(\alpha_4+1)$$

$$= 3 \times 2 \times 3 \times 2$$

$$= 36$$

Q. The number of distinct positive integral factors of 2014 is ________.

(GATE-14-Set2)

$2014 = 2^1 \times 19^1 \times 53^1$

$\tau(2014) = 2 \times 2 \times 2 = 8$

* According Euler's,

If a, n are +ve integers such that gcd (a, n) = 1

Then $\boxed{a^{\phi(n)} \equiv 1 \pmod{n}}$

$a \equiv b \bmod c$

'c' divides $(a-b)$

* According to Fermatt's little theorem,

If 'p' is a prime number and 'a' is some positive integer such that gcd $(a, p) = 1$ Then

$$\boxed{a^{p-1} \equiv 1 \pmod{p}}$$

$$5\overline{)16}(3$$
$$\underline{15}$$
$$1$$

$16 \equiv 1 \pmod{5}$

3) 17 (5
15
2

$17 \equiv 2 \pmod 3$

$3 \mid (17-2)$

4) 17 (4
16
1

$17 \equiv 1 \pmod 4$

4 divides $(17-1)$

$4 \mid (17-1)$

Q. The value of the expression 13^{99} (mod 17), in the range 0 to 16, is ____.

(GATE-16-Set2)

$13^{99} \pmod{17} = ?$

According to Fermatt's

$$\boxed{a^{p-1} = 1 \pmod{p},}$$

p = prime, $\gcd(a, p) = 1$

$$a^{16} = 1 \pmod{17}$$

$$\boxed{13^{16} = 1 \pmod{17}}$$

$= 13^{99} \pmod{17}$

$= 13^{96} \times 13^{3} \pmod{17}$

$= (13^{16})^{6} * 13^{3} \pmod{17}$

$= (13^{16} \bmod 17)^{6} * [13^{3} \pmod{17}]$

$= 1 * 4 = 4$

$13^3 = 2197$

$17)2197($

Derangements:

Among the permutations of {1, 2, ….., n} there are some, called **derangements,** in which none of the n integers appears in its natural place (correct place).

D_n = The number of derangements of n distinct objects.

$$D_n = n!\left\{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots\ldots + \frac{(-1)^n}{n!}\right\}$$

$$\approx n!.e^{-1}$$

$D_1 = 0$ $\quad D_4 = 9$

$D_2 = 1$ $\quad D_5 = 44$

$D_3 = 2$ $\quad D_6 = 265$

$$D_3 = 3! * \left(\frac{1}{2!} - \frac{1}{3!}\right) = 6\left(\frac{1}{2} - \frac{1}{6}\right) = \cancel{6}\left(\frac{3-1}{\cancel{6}}\right) = 2$$

$$D_4 = 4! * \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 24 * \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right) = \cancel{24} * \left(\frac{12 - 4 + 1}{\cancel{24}}\right) = 9$$

$$D_5 = 5! * \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 120 * \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right) = 120 * \left(\frac{60-20+5-1}{120}\right) = 44$$

Take $\{a,b,c\}$

Permutations : $\boxed{a}$ $\boxed{b}$ $\boxed{c}$

1^{st} 2^{nd} 3^{rd} → original places

$3! - D_3$

a	b	c
$\boxed{a}$	c	b
b	a	$\boxed{c}$
b	c	a
c	a	b
c	$\boxed{b}$	a

→ No object is at original place

Derangements $= D_3 = 2$

Take $\{a,b\}$

a b

$\boxed{b \quad a}$ Derange

$D_2 = 1$

$D_1 = 0$

$D_2 = 1$

$D_3 = 2$

In particular, $D_5 = 5!\left\{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right\} = 44$

$D_6 = 265,$

$D_4 = 9,$

$D_3 = 2,$

$D_2 = 1, D_1 = 0$

$D_1 = 0$

$D_2 = 1$

$D_3 = 2 (D_2$

Note: (1) $D_n = nD_{n-1} + (-1)^n \quad , (n \geq 2)$

$D_n = n\, D_{n-1} + (-1)^n$

$$\boxed{(2)\ D_n = (n-1)\,\{D_{n-1} + D_{n-2}\},\ (n \geq 3)}$$

$D_3 = 2(D_2 + D_1) = 2(1+0) = 2$

$D_4 = 3(D_3 + D_2) = 3(2+1) = 9$

$D_5 = 4(D_4 + D_3) = 4(9+2) = 44$

$D_6 = 5(D_5 + D_4) = 5(44+9) = 5(53) = 265$

$D_7 = 1854 \quad D_8 = 14833$

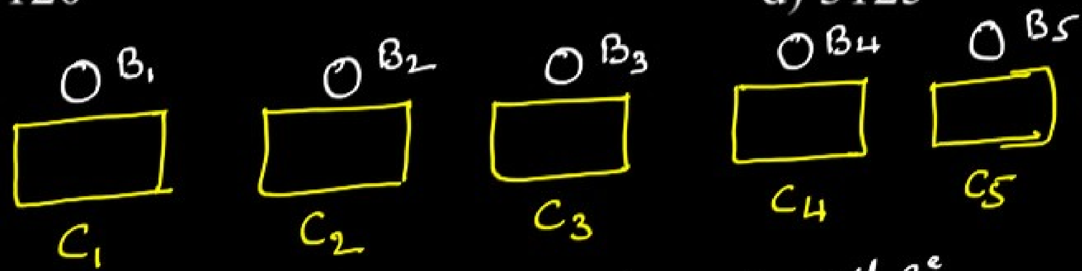
Q. In how many ways can we distribute 5 distinct balls, $B_1, B_2, \ldots, B_5$ in 5 distinct cells, $C_1, C_2, \ldots, C_5$ such that Ball B_i is not in cell C_i, $\forall\ i = 1, 2, \ldots, 5$ and each cell contains exactly one ball? **(GATE-04)**

a) 44 ✓

b) 96

c) 120

d) 3125



B_i Ball should not place in cell C_i

$\Rightarrow$ No Ball will be in its Natural place

$= D_5 = 44$

Q. How many ways we can put 5 letters L_1, L_2, L_3, L_4, L_5 in 5 envelops e_1, e_2, e_3, e_4 and e_5 (at one letter per envelope) so that

i) No letter is correctly place is ___ $D_5 = 44$

ii) At least one letter correctly place is ___ $5! - D_5 = 76$

iii) Exactly two letters are correctly place is ___ $5C_2 * D_3 = 20$

iv) At most one letter is correctly placed is ___ $(5C_1 * D_4) + D_5 = 89$

Max

v) At least one letter is wrongly placed is ___ $5! - 1 = 119 = 119$

Min

vi) Exactly one letter is wrongly placed is ___ $0 = 0$

$e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5$

correct place : $L_1 \quad L_2 \quad L_3 \quad L_4 \quad L_5$ → 1 case

(iii) Exactly two correct :

L_1	L_2	—	—	—	✓
L_1	—	L_3	—	—	✓
L_1	—	—	L_4	—	✓
L_1	—	—	—	L_5	✓
—	L_2	L_3	—	—	✓
—	L_2	—	L_4	—	✓
—	L_2	—	—	L_5	✓
—	—	L_3	L_4	—	✓
—	—	L_3	—	L_5	✓
—	—	—	L_4	L_5	✓

$5C_2 = 10$

$\boxed{5C_2 * D_3}$

(iv) Atmost one letter - correct

= one letter correct (or) No letter correct

$= [5C_1 * D_4] + [D_5]$

(v) At least one letter wrong

= All — all letters correct

$= 5! - 1$

(vi) exactly one letter wrong

$L_1 \quad L_2 \quad L_3 \quad L_4 \quad L_5$

L_1

Q. There are 8 letters to different people to be placed in 8 different addressed envelopes. Find the number of ways of doing this so that at least 1 letter goes to the right person.

$8! - D_8$

Q. Each of the n children in a class is given a book by the teacher; the books are all distinct. The students are required to return the books after 1 week. The same n books are again distributed for another week. In how many distributions does nobody get the same book twice?