Tribhuvan University

Institute of Science and Technology

2078

Full Marks: 80
Pass Marks: 32
Time 3 Hrs.

BScCSIT Level/First Semester
Mathematics[MTH 112]
Calculus

Candidates are required to give their answers in their own words as far as practicable.

Group A (10 × 3 = 30)

Attempt any **THREE** questions.

1. (a) If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{3 - x}$ then find fog and its domain and range. [5]

   (b) A rectangular storage container with an open top has a volume of $20m^3$. The length of its base is twice its width. Material for the base costs $Rs10$ per square meter; material for the sides costs $Rs4$ per square meter. Express the cost of materials as a function of the width of the base. [5]

2. (a) Using rectangles, estimate the area under the parabola $y = x^2$ from 0 to 1. [5]

   (b) A particle moves along a line so that its velocity v at time t is [5]

$$v = t^2 - t + 6$$

   (i) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

   (ii) Find the distance traveled during this time period.

3. (a) Find the area of the region bounded by $y = x^2$ and $y = 2x - x^2$. [5]

   (b) Using trapezoidal rule, approximate $\int_1^2 1/x dx$ with $n = 5$. [5]

4. (a) Solve: $y' = x^2/y^2, y(0) = 2$. [5]

   (b) Solve the initial value problem: $y'' + y' - 6y = 0, y(0) = 1, y'(0) = 0$. [5]

Group B (10 × 5 = 50)

Attempt any **TEN** questions.

5. Recent studies indicate that the average surface temperature of the earth has been rising steadily. Some scientists have modeled the temperature by the linear function $T = 0.03t + 8.50$, where T is temperature in degree centigrade and t represents years since 1900. (a) What do the slope and T-intercept represent? (b) Use the equation to predict the average global surface temperature in 2100.

2-1 a=1 b=2

MTH112-2078

6. Find the equation of tangent at $(1, 2)$ to the curve $y = 2x^2$.

7. State Rolle's theorem and verify the Rolle's theorem for $f(x) = x^2 - 3x + 2$ in $[0, 3]$.

8. Use Newton's method to find $\sqrt[6]{2}$ correct to five decimal places.

9. Find the derivative of $\mathbf{r}(t) = (1 - t^2)\mathbf{i} - te^{-t}\mathbf{j} + \sin 2t\mathbf{k}$ and find the unit tangent vector at $t = 0$.

10. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

11. Solve: $y' + 2xy - 1 = 0$

12. What is a sequence? Is the sequence

$$a_n = \frac{n}{\sqrt{5 + n}}$$

convergent? 40

13. Find a vector perpendicular to the plane that passes through the points: $P(1, 4, 6)$, $Q(-2, 5, -1)$ and $R(1, 1, -1)$

14. Find the partial derivativesof $f(x, y) = x^2 + 2x^3y^2 - 3y^2 + x + y$, at $(1, 2)$.

15. Find the local maximum and minimum values, saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

2078 Question Solution
BSC.CSIT 1st sem

- Pravin Gupta

Subject:- Mathematics-I / calculus

Group 'A'

1(a) If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{3-x}$ then find fog and its domain and range.

Sol

Given,

$$f(x) = \sqrt{x}$$
$$g(x) = \sqrt{3-x}$$
$$fog = ?$$

domain and range of fog = ?

Now,

$$fog$$
$$= fog(x)$$
$$= f(\sqrt{3-x})$$
$$= \sqrt{\sqrt{3-x}} \quad \text{or} \quad (3-x)^{1/4}$$

For Domain,

first we have to find domain of ~~f~~ g(x) = ?

$$g(x) = \sqrt{3-x}$$
$$3-x \geq 0$$
$$x \leq 3$$
$$\text{i.e } A = (-\infty, 3]$$

then

$$\text{domain of } f(g(x)) = (3-x)^{1/4} \text{ or } \sqrt{\sqrt{3-x}}$$
$$3-x \geq 0$$
$$\therefore x \leq 3$$
$$\text{i.e } B = (-\infty, 3]$$

$$\therefore \text{Domain of fog} = A \cap B$$
$$= (-\infty, 3] \cap (-\infty, 3]$$
$$= (-\infty, 3]$$

For Range,

$$Fog(x) = \sqrt{\sqrt{(3-x)}} \text{ or } (3-x)^{1/4}$$

Let,

$$y = \sqrt{\sqrt{(3-x)}}$$

s.b.s

$$y^2 = \sqrt{(3-x)}$$

Again,

s.b.s

$$y^4 = (3-x)$$
$$y^4 = 3-x$$
$$x = 3-y^4$$

$\therefore$ Range of fog $= 3-y^4$
$= [0, \infty)$

Hence,

$fog = \sqrt{\sqrt{3-x}}$
Domain of fog $= (-\infty, 3]$
Range of fog $= [0, \infty)$
Ans

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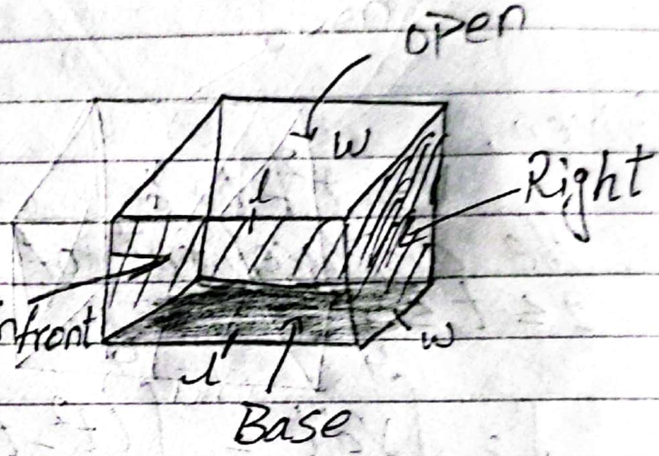
P.N: 3

1 (b) A rectangular storage container with an open top has a volume of $20m^3$. The length of its base is twice its width. Material for the base costs Rs 10 per square meter, material for the sides costs Rs 4 per square meter. Express the cost of materials as a function of the width of the base.

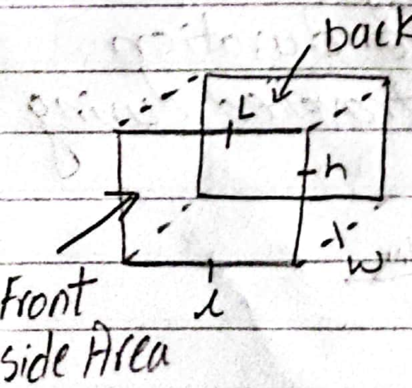
Sol

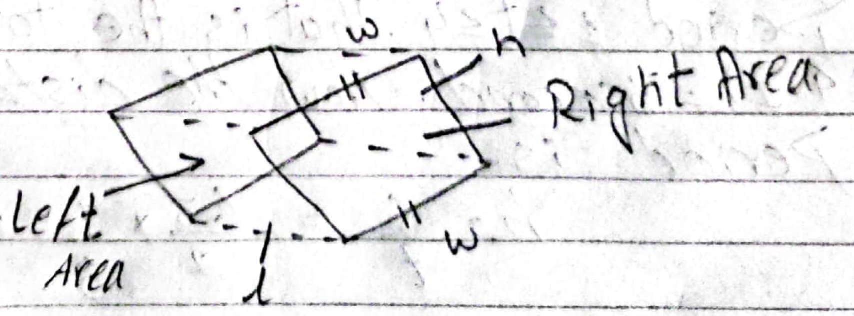
Given,
Volume (V) = $20m^3$
width (w) = x (Let)
length (l) = $2x$.
cost for base area = $10m^{-2}$
cost of side area = $4m^{-2}$



There are total 6 sides in a rectangular box (total side = 4 if we count upside and base then there are 6).
But in question, box is open so that are are only 5 side (Area)

which are,
Right Area = Left Area
Front Area = Back Area
Base Area ≠ UP (open)





Then, calculate heigth.

$$V = l \times w \times h$$

or, $20 = 2x \times x \times h$

or, $20 = 2x^2 \times h$

$\therefore h = \frac{10}{x^2}$

Base Area $= l \times w = 2x \times x = 2x^2$

As we know that, there are 4 sides in rectangulor box, so that,

(Right) one side Area $= h \times w$

$= \frac{10}{x^2} \times x = \frac{10}{x}$

There are Two of right so, $= 2 \times \frac{10}{x} = 20/x$

Infront Area $= l \times h = 2x \times \frac{10}{x^2} = \frac{20}{x}$

(Infront + Back) Area $= \frac{20}{x} + \frac{20}{x} = \frac{40}{x}$

Total side Area of box $= \frac{20}{x} + \frac{40}{x} = \frac{60}{x}$

Calculate, cost for base Area $= 2x^2 \times 10$ (i.e Base area × cost)

$= 20x^2$

Calculate, Cost for side area, = Total side area × cost

$= \frac{60}{x} \times 4 = \frac{240}{x}$

Total cost $= 20x^2 + \frac{240}{x}$

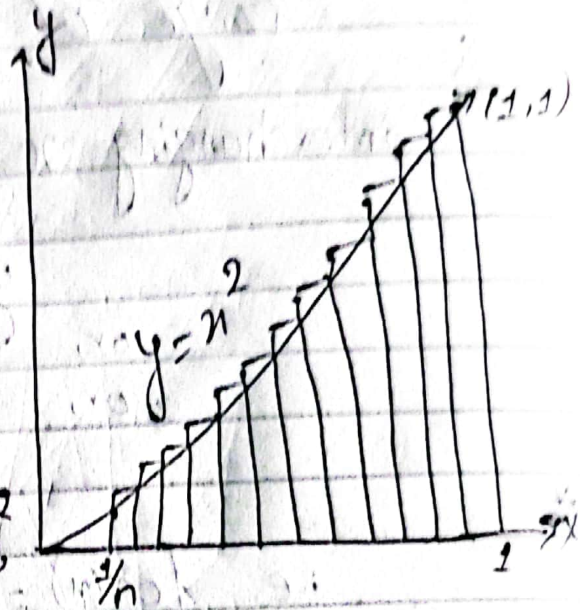
$\therefore f(x) = 20x^2 + \frac{240}{x}$ Ans

P:N 124

2(a) Using rectangles, estimate the area under the parabola $y = x^2$ from 0 to 1.

Sol

Let R_n be the Sum of the area of the n rectangles as shown in figure. Each rectangle has width $1/n$ and the heights are the values of the function $f(x) = x^2$ at the Points $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots \frac{n}{n}$



i.e the heights are $\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2, \ldots \left(\frac{n}{n}\right)^2$

Thus,

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n^3}\left(1 + 2^2 + 3^2 + \cdots + n^2\right)$$

$$= \frac{1}{n^3} \times \frac{n(n+1)(2n+1)}{6}$$

$$R_n = \frac{(n+1)(2n+1)}{6n^2}$$

Thus, we have,

$$\text{Area}(A) = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \quad (\text{i.e } 1/\infty = 0)$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(1 + 1/n\right)\left(2 + 1/n\right) = \frac{1}{6} \times 1 \times 2 = 1/3$$

1(b) A particle moves along a line so that its velocity v at time t is
$$v = t^2 - t + 6$$
(i) Find the displacement of the particle during the time period $1 \le t \le 4$.
(ii) Find the distance travel during this time period.

Sol:

(i) Given,
$$v = t^2 - t + 6$$
Now, the displacement of particle during time period $1 \le t \le 4$ is

$$S = \int_1^4 v\,dt = \int_1^4 (t^2 - t + 6) = \left[\frac{t^3}{3} - \frac{t^2}{2} + 6t\right]_1^4$$
$$= \left(\frac{64}{3} - \frac{16}{2} + 24\right) - \left(\frac{1}{3} - \frac{1}{2} + 6\right)$$
$$= \left(\frac{128 - 48 + 144}{6}\right) - \left(\frac{2 - 3 + 36}{6}\right)$$
$$= \frac{189}{6} = \frac{63}{2} \quad \text{Ans}$$

(ii) Given that the particle moves along a line for the time period $1 \le t \le 4$. That is the total traveled time duration is $(4-1) = 3$ second. Now the distance travelled during this period is
$$h = \frac{s}{t} = \frac{63}{2} \times \frac{1}{3} = \frac{21}{2} \quad \text{Ans}$$

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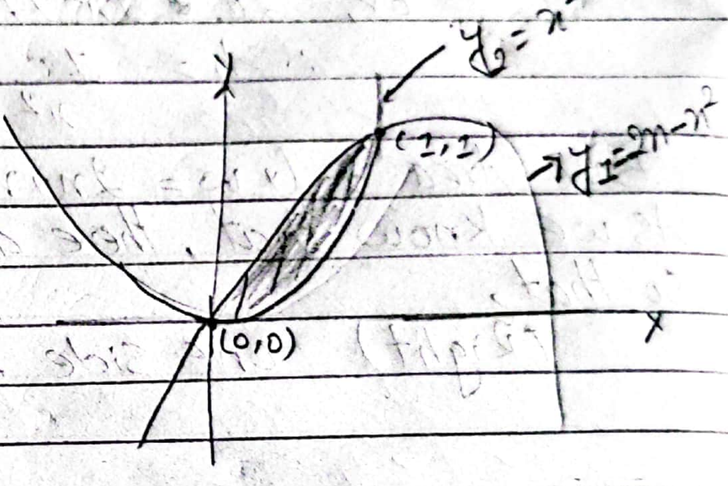
3(a) Find the area of the ~~graph~~ region bounded by $y=x^2$ and $y=2x-x^2$.

sol Given,

$$y=x^2 \quad \text{--- (i)}$$
$$y=2x-x^2 \quad \text{--- (ii)}$$



Solving eqn (i) and (ii)

$$x^2=2x-x^2$$
$$2x^2-2x=0$$
$$\therefore 2x(x-1)=0$$

Either, or,

$$x=0, \quad x=1$$

then,

if $x=0$, $y=0$

if $x=1$, $y=1$.

$\therefore (x,y)=(0,0)$ and $(1,1)$ is the Point of intersection.

The Area is

$$A=\int_0^1 (y_1-y_2)\,dx = \int_0^1 (2x-x^2-x^2)\,dx$$
$$=\int_0^1 (2x-2x^2)\,dx = 2\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1$$
$$=2\left(\frac{1}{2}-\frac{1}{3}\right)$$
$$=2\times\frac{(3-2)}{6}$$
$$=1/3 \text{ sq. unit}$$

Ans

(b) Using trapezoidal rule, approximate $\int_1^2 \frac{1}{x}\,dx$ with $n=5$

Sol Given,

$$f(x) = \frac{1}{x} \quad \text{i.e.} \int_1^2 \frac{1}{x}\,dx$$

$n=5$, $a=1$, $b=2$

$$\Delta x = \frac{2-1}{5} \quad \left(\text{i.e. } \frac{b-a}{n}\right)$$

$$= 0.2$$

So, 1, 1.2, 1.4, 1.6, 1.8, 2

$$\int_1^2 \frac{1}{x}\,dx = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right]$$

$$= \cancel{\frac{0.2}{2}[(f(x_1)} -$$

$$= \frac{0.2}{2}\left[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)\right]$$

$$= 0.1\left[\frac{1}{1} + 2\times\frac{1}{1.2} + 2\times\frac{1}{1.4} + 2\times\frac{1}{1.6} + 2\times\frac{1}{1.8} + \frac{1}{2}\right]$$

$$= 0.1\left[1 + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right]$$

$$= 0.1\left[1 + 1.666667 + 1.428571 + 1.250000 + 1.111111 + 0.5\right]$$

$$= 0.1 \times 6.956349$$

$$= 0.695635 \text{ Ans}$$

4(a) Solve: $y' = \frac{x^2}{y^2}$, $y(0) = 2$

Sol Given,

$$y' = \frac{x^2}{y^2} \quad ; \quad y(0) = 2$$

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

or, $y^2\,dy = x^2\,dx$

On integrating bothsides.

$$\int y^2 dy = \int x^2 dx$$

or, $\frac{y^3}{3} = \frac{x^3}{3} + c$ ------ (*)

or, $\frac{y^3}{3} = \frac{x^3}{3} + c/3$

$\therefore y^3 = x^3 + c$ ---------- (*)

On, $y(0) = 2$

if $x = 0$, $y = 2$ then eqn (*) becomes.

$\frac{y^3}{3} = \frac{x^3}{3} + c$

$\frac{8}{3} = 0 + c$

$c = 8/3$

substituting ~~Substitu~~ the value of c.

$\frac{y^3}{3} = \frac{x^3}{3} + \frac{8}{3}$

$\therefore y^3 = x^3 + 8$ Ans

or

$y^3 = x^3 + c$

$2^3 = 0^3 + c$

$\therefore c = 8$

then eqn (*) becomes

$y^3 = x^3 + 8$ Ans

4(b) Solve the initial value problem:

$$y''+y'-6y=0, \quad y(0)=1, \quad y'(0)=0$$

Sol^n The given 2nd order homogenous linear eq^n is

$$y''+y'-6y=0 \quad \text{---} (1)$$

Let

$y=e^{mx}$ be the req^n sol^n of eq^n(1)

Now,

we get an auxilarly eq^n of (1) by replacing y'' by m^2, y' by m and y by 1.

$$m^2+m-6=0$$

or, $m^2+3m-2m-6=0$

or, $m(m+3)-2(m+3)=0$

$(m-2)(m+3)=0$

Either or

$m=2$, $m=3$

$\therefore m_1=2, m_2=-3$ which is real and unequal

The req^n sol^n is;

$$y=c_1e^{m_1x}+c_2e^{m_2x}$$

$$y=c_1e^{2x}+c_2e^{-3x} \quad \text{---} (ii)$$

Since, $y(0)=1$, i.e $x=0, y=1$

from eq^n(ii)

$$1=c_1e^0+c_2e^0$$

$$1=c_1+c_2 \quad \text{---} (iii)$$

Again,

diffⁿ eqⁿ (ii) with respect to x

$$y' = 2c_1 e^{2x} - 3c_2 e^{-3x} \quad \text{---} \text{(iv)}$$

since,

$$y'(0) = 0 \quad \text{i.e } y' = 0,\ x = 0$$

from eqⁿ (iv)

$$0 = 2c_1 - 3c_2 \quad \text{---} \text{(v)}$$

from eqⁿ (iii) and (v)

$2 \times$ eqⁿ (iii) − (v)

$$\begin{array}{r} \cancel{2c_1} + 2c_2 = 2 \\ -\cancel{2c_1} \mp 3c_2 = 0 \\ \hline 5c_2 = 2 \\ c_2 = 2/5 \end{array}$$

if $c_2 = 2/5$ then,

$$c_1 + c_2 = 1$$

$$c_1 = 1 - c_2$$

$$c_1 = 1 - \frac{2}{5} = 3/5$$

Substuting the value of c_1 and c_2 in eqⁿ (ii)

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

$$y = \frac{3}{5} e^{2x} + \frac{2}{5} e^{-3x} \quad \underline{\text{Ans}}$$

Group 'B'

5. Recent studies indicate that the average Surface temperature of the earth has been rising steadily. Some Scientists have modeled the temperature by the linear function $T = 0.03t + 8.50$, where T is temperature in degree centigrade and t represents years since 1900

(a) What do the slope and T-intercept represent?

(b) Use the equation to predict the average global Surface temperature in 2100

Sol

Given, linear function

$T = 0.03t + 8.50$ ---(i) where T = temperature in degree

t = years

We know, linear model is in the form of

$y = mx + c$. so for given variable and condition it will be;

$T = mt + c$ ---(ii)

comparing eqn (i) with (ii)

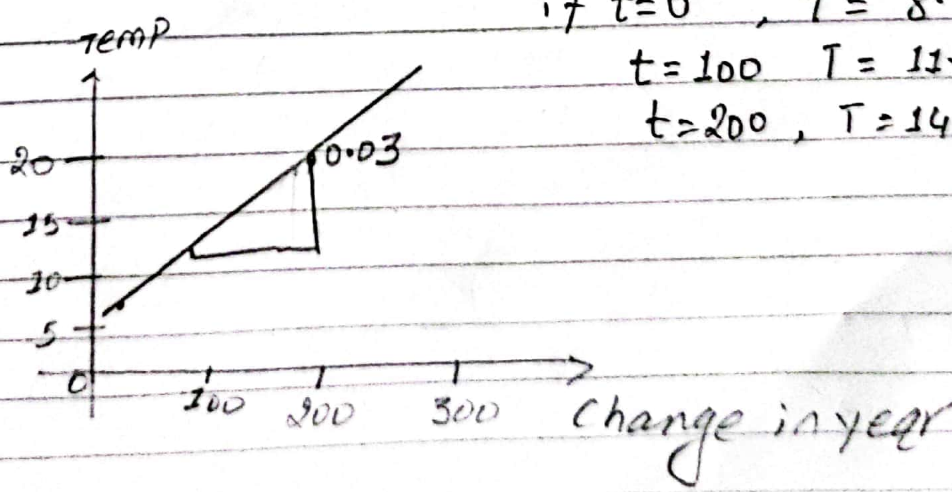
$\therefore m = 0.03, \quad c = 8.50$

$T = 0.03t + 8.50$

if $t = 0$, $T = 8.50$

$t = 100$, $T = 11.5$

$t = 200$, $T = 14.5$



(a) Slope represent change of temperature with respect to year is 0.03. It means when there is change in year with 1, then temperature is also increased by 0.03.

And

T-intercept means in a initial stage or $t = 0$ Temperature is 8.50

(5) Average global Temperature in 2100, using linear model,

$$T = 0.03t + 8.50$$

Since there is 2100 we have to convert into year by

$$2100 - 1900 = 200$$

$$\therefore \quad T = 0.03 \times 200 + 8.50$$
$$= 14.5^\circ C$$

Thus in 2100, temperature of earth will be 14.5°C

Formula eq^n of tangent $\Rightarrow y-y_1 = m(x-x_1)$

P.N; 71

6. Find the equation of tangent at (1,2) to the curve $y = 2x^2$

Sol Given,

The curve is, $y = 2x^2$ ....(1)

Given Point is (1,2)

diff^n eq^n (1) w.r.to x.

$$y' = 4x$$

At Point (1,2), $y' = 4x = 4\times1 = 4$

This show the curve has slope 4 at the Point (1,2). Since the point (1,2) is the common point of the curve and the tangent line $y = 2x^2$. Therefore the slope of the tangent line is 4.

Thus, the eq^n of tangent at (1,2) to the curve $y = 2x^2$ is

$$y - y_1 = m(x - x_1)$$

or, $y - 2 = 4(x-1)$

or, $y - 2 = 4x - 4$

or, $y = 4x - 4 + 2$

$\therefore y = 4x - 2$ Ans

$\therefore$ The eq^n of tangent is $y = 4x - 2$ Ans

7 State Rolle's theorem and verify the Rolle's theorem for $f(x) = x^2 - 3x + 2$ in $[0, 3]$.

Sol

1^{st} Part,

Let f be a function that satisfies the following three conditions;

(i) f is continuous on the closed interval $[a,b]$

(ii) f is differentiable on the open interval (a,b)

(iii) $f(a) = f(b)$

Then there is a number c in (a,b) such that $f'(c) = 0$

2^{nd} Part,

Given,

$$f(x) = x^2 - 3x + 2 \quad \text{in } [0,3]$$

(i) Since $f(x)$ is a polynomial function. so, it's continuous on the closed interval $[0,3]$

(ii) $f'(x) = 2x - 3$ which exists for all $x \in (0,3)$

$\therefore f(x)$ is differentiable in $(0,3)$

(iii) $f(0) = 0 - 0 + 2 = 2$

$f(3) = 3^2 - 3\times3 + 2 = 9 - 9 + 2 = 2$

$\therefore f(0) = f(3)$

$\therefore$ All the conditions of Rolle's theorem are satisfied then there exists at least a point $c \in (0,3)$ such that $f'(c) = 0$

CamScanner

$f'(x) = 2x-3$

i.e

$f'(c) = 2c-3$

so,

$f'(c) = 0$

or, $2c-3=0$

or, $2c=3$

$\therefore c = 3/2 = 1.5 \in (0,3)$ Ans

Hence, Rolle's theorem verified and $1.5 \in (0,3)$ Ans

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P.N; 114

8. Use Newton's method to find $\sqrt[6]{2}$ correct to five decimal places.

Sol^n

Let,

$$x = \sqrt[6]{2}$$

So,

$$x^6 = 2$$

$$x^6 - 2 = 0$$

Hence, $f(x) = x^6 - 2$

$$f'(x) = 6x^5$$

By Newton's formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

i.e $$x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5}$$

Let $n = 1, 2, \ldots$

let $x_1 = 1$

$n=1 \Rightarrow x_{1+1} = x_2 \Rightarrow x_1 - \frac{x_1^6 - 2}{6x_1^5} = 1 - \frac{(1^6 - 2)}{6 \times 1^5} = 1.166667$

$n=2 \Rightarrow x_{2+1} = x_3 \Rightarrow x_2 - \frac{x_2^6 - 2}{6x_2^5} = 1.166667 - \frac{(1.166667)^6 - 2}{6 \times (1.166667)^5}$

$= 1.166667 - \frac{0.521626}{12.968383}$

$x_3 = 1.126444$

$x_4 = 1.126444 - \frac{(1.126444)^6 - 2}{6 \times (1.126444)^5} = 1.126444 - \frac{(0.042951)}{(10.881763)}$

$= 1.122497$

$$x_5 = 1.122497 - \frac{(1.122497)^6 - 2}{6 \times (1.122497)^5}$$

$$= 1.122497 - \frac{0.000374}{10.692449}$$

$$x_5 = 1.122462$$

$$x_6 = 1.122462 - \frac{(1.122462)^6 - 2}{6 \times (1.122462)^5}$$

$$x_6 = 1.122462 - \frac{0}{10.690782}$$

$$\therefore x_6 = 1.122462$$

Here if we compare with x_5 and x_6 the value are same upto 6 decimal place.

$\therefore$ the root of $f(x)$ is given by 1.12246 at 6^{th} iteration.

Same as 2077

9. Find the derivative of $r(t) = (1-t^2)i - te^{-t}j + \sin 2t k$ and find the unit tangent vector at $t=0$

Sol

Given,

$$r(t) = (1-t^2)i - te^{-t}j + \sin 2t k \quad \text{----} (1)$$

diff^n eq^n(1) w.r. to 't'

$$\frac{d(\vec{r}(t))}{d(t)} = \frac{d\{(1-t^2)i\}}{dt} - \frac{d(t.e^{-t})}{dt}j + \frac{d(\sin 2t)}{dt}k$$

$$= -2ti - (t.(-1).e^{-t} + e^{-t}.1)j + \frac{d(\sin 2t)}{d(2t)} \cdot \frac{d(2t)}{dt}.k$$

$$= -2ti + (te^{-t} - e^{-t})j + \cos 2t * 2k$$

$$= -2ti + e^{-t}(t-1)j + 2\cos 2t k$$

$$\therefore \frac{d\vec{r}}{dt} = -2ti + (t-1)e^{-t}j + 2\cos 2t k$$

At $t=0$,

$$r(0) = 1i - 0j + 0.k = 1i$$

$$\frac{d\vec{r}}{dt} = r'(0) = -0 + (0-1).1.j + 2.\cos(0)k$$

$$= -1j + 2k$$

Since, by using formula,

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

$$T(0) = \frac{-1j+2k}{\sqrt{(-1)^2+2^2}} = \frac{-1j+2k}{\sqrt{5}} = -\frac{1}{\sqrt{5}}j + \frac{2}{\sqrt{5}}k \quad \text{Ans}$$

Sameas 2077

10. Find the volume of the Solid obtained by rotating about the y-axis the region between $y=x$ and $y=x^2$

Sol: Here,

the given Parabola are
$y=x$ and $y=x^2$

Solving these parabola to obtained the limiting values of x

$$x^2 = x$$
$$\text{or, } x - x^2 = 0$$
$$\text{or, } x(1-x) = 0$$
$$\therefore x = 0, 1$$

Hence,

$$\text{Volume}(v) = \int_a^b 2\pi(x).f(x)\,dx.$$

$$= \int_0^1 2\pi x\,(x-x^2)\,dx = 2\pi \int_0^1 (x^2-x^3)\,dx = 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= 2\pi\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= 2\pi\left(\frac{4-3}{12}\right)$$

$$= \frac{2\pi}{12}$$

$$= \pi/6 \text{ Ans}$$

$\therefore$ The volume of the solid obtained by rotating about y-axis is $\frac{\pi}{6}$ Ans

(2071.4 q.N.7)

11. Solve : $y' + 2xy - 1 = 0$

Sol:

Given,

$$y' + 2xy - 1 = 0$$

$$\frac{dy}{dx} + 2xy = 1 \text{ ------ (1)}$$ comparing with the standard linear ordinary differential equation

$$\frac{dy}{dx} + Py = Q$$

where, $P = 2x$, $Q = 1$

Now,

the integrating factor (I.F) is

$$I.F = e^{\int P dx} = e^{\int 2x} = e^{\frac{2x^2}{2}} = e^{x^2}$$

Multiplying eq^n (1) by I.F

$$e^{x^2}\left[\frac{dy}{dx} + 2xy\right] = e^{x^2}$$

$$d(e^{x^2} \cdot y) = e^{x^2}$$

on integrating bothside

$$\int d(e^{x^2} \cdot y) = \int e^{x^2} dx$$

or, $$e^{x^2} \cdot y = \int e^{x^2} dx + C$$

$$y = \frac{1}{e^{x^2}}\left(\int e^{x^2} dx + C\right)$$

$$y = e^{-x^2}\int e^{x^2} dx + e^{-x^2} C$$ Ans

converges → finite value (i.e 1, 2, 3 ....)
diverges → doesn't gives finite value (i.e ∞)

12. What is a Sequence? Is the Sequence $a_n = \frac{n}{\sqrt{5+n}}$ $\left(a_n = \frac{n}{\sqrt{5+n^2}} \right)$ convergent?

Sol. 1st part,

A Sequence is an ordered list of things. Such thinks may finite or infinite. In a sequence, the terms are separated by commas (,)

2nd part,

Given sequence is

$$\{a_n\} = \left\{ \frac{n}{\sqrt{5+n^2}} \right\}$$

To check convergent,

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(\frac{n}{\sqrt{5+n^2}} \right) = \lim_{n \to \infty} \left(\frac{n}{\sqrt{n^2\left(\frac{5}{n^2}+1\right)}} \right)$$

$$= \lim_{n \to \infty} \left(\frac{n}{n\sqrt{\left(5/n^2 + 1\right)}} \right)$$

$$= \lim_{n \to \infty} \left(\frac{1}{\sqrt{\frac{5}{n^2}+1}} \right)$$

$$= \frac{1}{\sqrt{\frac{5}{\infty}+1}} = \frac{1}{\sqrt{0+1}} = 1$$

Hence, the given series $\{a_n\}$ is 1 i.e convergent

Same as 2077 question

13. Find a vector perpendicular to the plane that passes through the points: $P(1,4,6)$, $Q(-2,5,-1)$ and $R(1,1,-1)$

Sol^n

Given points are;
$P(1,4,6)$, $Q(-2,5,-1)$, $R(1,1,-1)$

For perpendicular vector,

$$\vec{PQ} = (1,4,6) - (-2,5,-1)$$
$$= (1-(-2), 4-5, 6-(-1))$$
$$= (3, -1, 7)$$

$$\vec{PR} = (1,4,6) - (1,1,-1)$$
$$= (1-1, 4-1, 6-(-1))$$
$$= (0, 3, 7)$$

The cross product of $\vec{PQ}$ and $\vec{PR}$ is

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 3 & -1 & 7 \\ 0 & 3 & 7 \end{vmatrix}$$

$$= i(-7-21) - j(21-0) + k(9+0)$$
$$= -28i - 21j + 9k \quad \text{Ans}$$

Thus, the vector $(-28, -21, 9)$ is perpendicular to the given plane.

14. Find the partial derivatives of $f(x,y) = x^2+2x^3y^2-3y^2+x+y$ at $(1,2)$

Sol Given, ~~function~~

$$f(x,y) = x^2+2x^3y^2-3y^2+x+y \quad \text{--- (1)}$$

we have to find $\frac{\delta f}{\delta x}$. for this keeping y as constant and differentiate $f(x,y)$ with respect to x then

$$\frac{\delta f}{\delta x} = \frac{\delta}{\delta x}\left(x^2+2x^3y^2-3y^2+x+y\right)$$

$$= 2x+6x^2y^2-0+1+0$$

$$= 2x+6x^2y^2+1$$

At point $(1,2)$

$$\left.\frac{\delta f}{\delta x}\right|_{(1,2)} = 2\times1+6\times1^2\times2^2+1$$

$$= 2+24+1$$

$$= 27 \text{ Ans}$$

Again,

we have to find $\frac{\delta f}{\delta y}$ for this keeping x as constant and differentiate $f(x,y)$ with keeping to y then

$$\frac{\delta f}{\delta y} = \frac{\delta}{\delta y}\left(x^2+2x^3y^2-3y^2+x+y\right)$$

$$= (0+4x^3y-6y+0+1)$$

$$= 4x^3y-6y+1$$

At point $(1,2)$

$$\left.\frac{\delta f}{\delta x}\right|_{(1,2)} = 4\times1^3\times2-6\times2+1 = 8-12+1 = -3 \text{ Ans}$$

15. Find the local maximum and minimum values, saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$

Sol^n Given Function,

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

so,

$f_x = 4x^3 - 4y$ $\qquad$ $f_y = 4y^3 - 4x$

$f_{xx} = 12x^2$ $\qquad$ $f_{yy} = 12y^2$

$f_{xy} = -4$ $\qquad$ $f_{yx} = -4$

For stationary Point

$f_x = 0$ $\qquad$ $f_y = 0$

$4x^3 - 4y = 0$ $\qquad$ $4y^3 - 4x = 0$

$x^3 = y$ ---- (i) $\qquad$ $y^3 = x$ ---- (ii)

from eq^n (i) and (ii)

$$(x^3)^3 = x \qquad (\because y = x^3)$$

or, $x^9 = x$

or, $x^9 - x = 0$

or, $x(x^8 - 1) = 0$

$\therefore x = 0, \; x = 1$

Substitute the point value of y in (i)

when $x = 0, \; y = 0$

" $x = 1, \; y = 1$

At point $(0,0)$

$$f_{xx} = 0$$
$$f_{yy} = 0$$

Then,

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = 0 - 16 = -16 < 0$$

so, it has saddle point at $(0,0)$

At point $(1,1)$

$$f_{xx} = 12 \quad , \quad f_{yy} = 12$$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yx} \end{vmatrix} = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 144 - 16$$
$$= 128 > 0$$

so, it has minimum value at ~~(1,1)~~ $(1,1)$

$\therefore$ Minimum value is

$$f(1,1) = 1^4 + 1^4 - 4 \times 1 \times 1 + 1$$
$$= 1 + 1 - 4 + 1$$
$$= -1$$