MTH112-2074✡

Tribhuvan University

Institute of Science and Technology

2074

✡

Bachelor Level / First Year/ First Semester/ Science
Computer Science and Information Technology (MTH. 112)
(Mathematics I)

Full Marks: 80
Pass Marks: 32
Time: 3 hours.

Candidates are required to give their answers in their own words as for as practicable.
The figures in the margin indicate full marks.

Attempt any three questions: $(3\times10=30)$

1. (a) A function is defined by $f(x)=\begin{cases}x+2 & if \quad x<0\\ 1-x & if \quad x>0\end{cases}$, calculate f(-1), f(3), and sketch the graph. (5)

   (b) Prove that the $\lim_{x\to0}\frac{|x|}{x}$ does not exist. (5)

2. (a) Find the derivative of $f(x)=\sqrt{x}$ and to state the domain of f'. (3+2)

   (b) Estimate the area between the curve $y^2=x$ and the lines $x=0$ and $x=2$. (5)

3. (a) Find the Maclaurin series for e^x and prove that it represents e^x for all x. (4)

   (b) Define initial value problem. Solve that initial value problem of $y'+5y=1, y(0)=2$. (4)

   (c) Find the volume of a sphere of radius r. (2)

4. (a) For what values of x does the series $\sum_{n=1}^{\infty}\frac{(x-3)^n}{x}$ converge? (5)

   (b) Calculate $\iint_R f(x,y)dA$ for $f(x,y)=100-6x^2y$ and $R: 0\le x\le2, -1\le y\le1$. (5)

Attempt any ten questions: $(10\times5=50)$

5. If $f(x)=\sqrt{x}$ and $g(x)=\sqrt{3-x}$, find gof and gog. (5)

6. Use continuity to evaluate the limit, $\lim_{x\to4}\frac{5+\sqrt{x}}{\sqrt{5+x}}$. (5)

7. Verify Mean value theorem of $f(x)=x^3-3x+3$ for [-1, 2]. (5)

8. Sketch the curve $y=x^3+x$. (5)

9. Determine whether the integral $\int_1^{\infty}\frac{1}{x}dx$ is convergent or divergent. (5)

10. Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points (1, 1) and (4, 8). (5)

11. Find the solution of $y'' + 6y' + 9 = 0$, $y(0) = 2$, $y'(0) = 1$. (5)

12. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$. (5)

13. Define cross product of two vectors. If $a = i + 3j + 4k$ and $b = 2i + 7j - 5k$, find the vector $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$. (1+2+2)

14. Define limit of a function. Find $\lim_{x\to\infty}\left(x - \sqrt{x}\right)$. (1+4)

15. Find the extreme values of $f(x, y) = y^2 - x^2$. (5)

2074 Question Solution

BSc.CSIT 1^{st} sem

Subject: Mathematics-I

-Pravin Gupta.

Group 'A'

1(a) A function is defined by $f(x) = \begin{cases} x+2 & \text{if } x<0 \\ 1-x & \text{if } x>0 \end{cases}$

Calculate $f(-1)$, $f(3)$ and sketch the graph

Soln Given,

$$f(x) = \begin{cases} x+2, & \text{if } x<0 \\ 1-x, & \text{if } x>0 \end{cases}$$

then,

$$f(-1) = x+2 \quad , \text{ being } x<0 \text{ i.e } -1<0$$
$$= -1+2$$
$$= 1$$

$$f(3) = 1-x \quad , \text{ being } x>0 \text{ i.e } 3>0$$
$$= 1-3$$
$$= -2$$

For graph, $x<0$

~~at~~ Let $y = x+2$ ---- (1)

at $x=-1$, $y = -1+2 = 1$

at $x=-3$, $y = -3+2 = -1$

$\therefore$ eqn (1) passes through the point $(-1,1)$ $(-3,-1)$

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Again,

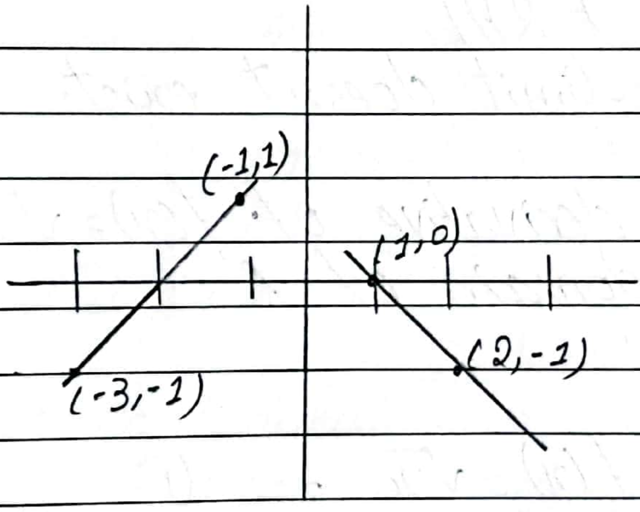
$f(x) = 1 - x \cdots (ii) \; x > 0$

let $y = 1 - x$

if $x = 1$, $y = 1 - 1 = 0$

if $x = 2$, $y = 1 - 2 = -1$

$\therefore$ eq^n (ii) passes through the point $(1, 0)$ $(2, -1)$



(b) Prove that the $\lim\limits_{x \to 0} \frac{|x|}{x}$ does not exist

Sol Here,

Given function

~~$f(x) = \lim\limits_{x\to 0} \frac{|x|}{x}$~~ $\lim\limits_{x \to 0} \frac{|x|}{x}$

we know that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Then

$$LHL = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0} \frac{-x}{x} = -1$$

RHL

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0} \frac{x}{x} = 1$$

$\therefore LHL \neq RHL$

Hence, the limit doesn't exist.

2(a) Find the derivative of $f(x) = \sqrt{x}$ and to state the domain of f'

Sol^n

Given,

$$f(x) = \sqrt{x} \quad \ldots\ldots (1)$$

$$f'(x) = ?$$

domain of $f'(x) = ?$

diff^n eq^n (1) w.r.to x.

$$f(x) = \sqrt{x} \text{ or } (x)^{1/2}$$

$$f'(x) = \frac{1}{2}(x)^{1/2 - 1}$$

$$= \frac{1}{2}(x)^{-1/2}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

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Now,
domain of $f'(x)$

For that we have,

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Since, $\sqrt{x}$ is a root function, which is continuous only for $x \geq 0$

Also, $\frac{1}{2\sqrt{x}}$ is a rational function that is continuous except $2\sqrt{x} = 0 \Rightarrow x = 0$

$\therefore$ $f'(x)$ is continuous for $x > 0$

So, the ~~fo~~ domain of $f'(x)$ is $(0, \infty)$

$\therefore$ The derivative of $f(x)$ is $\frac{1}{2\sqrt{x}}$ and its domain is $(0, \infty)$

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2(b) Estimate the area between the curve $y^2 = x$ and the line $x=0$ and $x=2$.

Sol^n Given curve

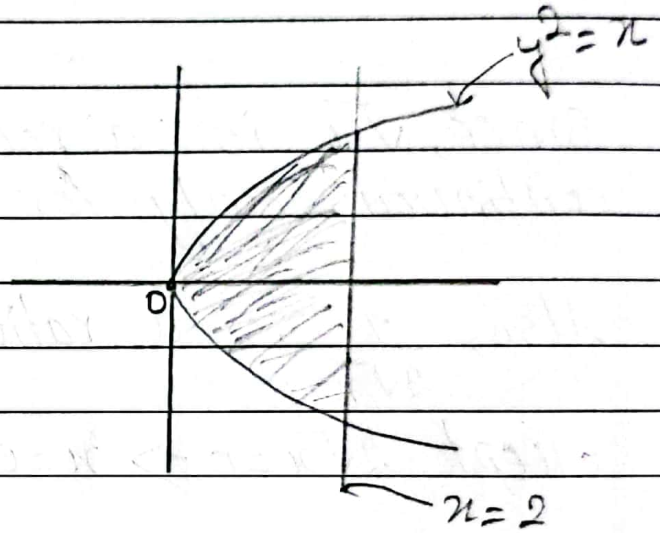
$$y^2 = x \quad \text{....(1)}$$

and the lines are

$$x = 0$$
$$x = 2$$

suppose

x	1	2
y	1	$\sqrt{2}$



From figure,

$$A = 2\int_0^2 y\,dx$$

$$= 2\int_0^2 \sqrt{x}\,dx$$

$$= 2\int_0^2 x^{1/2}\,dx$$

$$= 2\left[\frac{x^{1/2+1}}{1/2+1}\right]_0^2$$

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^2$$

$$= 2 \times \frac{2}{3}\left[x^{3/2}\right]_0^2$$

$$= \frac{4}{3}\left[(2)^{3/2} - 0\right]$$

$$= \frac{4}{3} \times (2)^{3/2}$$

$$= \frac{4}{3} \times (2^3)^{1/2}$$

$$= \frac{4}{3} \times 8^{1/2}$$

$$= \frac{4}{3} \times 2\sqrt{2}$$

$$= \frac{8\sqrt{2}}{3} \text{ sq. unit}$$

i.e. $8^{1/2} = \sqrt{8} = \sqrt{2\times2\times2}$
$= \sqrt{2^2 \times 2}$
$= 2\sqrt{2}$

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3(a) Find the Maclaurin series for e^x and prove that it represents e^x for all x

Sol

The given function is

$$f(x) = e^x \quad \text{----} \; (1)$$

we have to find a Maclaurin series generated by the function $f(x) = e^x$ for this we need to find Taylor's series generated by the function

$$f(x) = e^x \text{ about } x = 0 \quad (\text{i.e } x = a = 0)$$

diff^n eq^n (1) successively,

$$\begin{aligned} f(x) &= e^x \\ f'(x) &= e^x \\ f''(x) &= e^x \\ f'''(x) &= e^x \\ f^{iv}(x) &= e^x \\ &\vdots \\ f^n(x) &= e^n \\ &\vdots \end{aligned} \quad \xrightarrow{\text{at } x=0} \quad \begin{aligned} f(0) &= e^0 = 1 \\ f'(0) &= e^0 = 1 \\ f''(0) &= e^0 = 1 \\ f'''(0) &= e^0 = 1 \\ f^{iv}(0) &= e^0 = 1 \\ &\vdots \\ f^n(0) &= e^0 = 1 \end{aligned}$$

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we know that,

the reqⁿ Maclaurian series is

$$f(x) = f(0) + \frac{f'(0)(x-0)^1}{1!} + \frac{f''(0)(x-0)^2}{2!} + \dots + \frac{f^n(0)(x-0)^n}{n!}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots$$

Hence, it is Maclaurian series of e^x

Also,

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + R_n(x)$$

where, $R_n(x)$ is the remainder of the series

since,

$$|R_n(x)| \leq \frac{x^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty$$

This shows that e^x Converges to its Maclaurian's series.

3(b) Define initial value Problem. Solve that initial value Problem of $y' + 5y = 1$, $y(0) = 2$

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Sol) 1st part

A differential equation together with initial condition(s) is called the initial value problem. For example:-

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0, \; y(0) = -1, \; y'(0) = 1$$

Here, $y(0) = -1$ and $y'(0) = 1$ is an initial condition

2nd Part

Given

$$y' + 5y = 1 \qquad , \; y(0) = 2$$

$$\frac{dy}{dx} + 5y = 1 \text{ ---- (1)}$$

Comparing eqn (1) with $\frac{dy}{dx} + Py = Q$

where, $P = 5$, $Q = 1$

Now,

Integrating factor (I.F) $= e^{\int P dx} = e^{\int 5 dx} = e^{5x}$

Now, multiplying by I.F in eqn (1) on both sides

$$e^{5x}\left(\frac{dy}{dx} + 5y\right) = 1 \times e^{5x}$$

or, $d(e^{5x} \cdot y) = e^{5x}$

On integrating,

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$$\int d(e^{5x}\cdot y) = \int e^{5x}\,dx$$

or, $$y\times e^{5x} = \frac{e^{5x}}{5} + C$$

$$\therefore\ y\times e^{5x} = \frac{e^{5x}}{5} + C \quad \text{-----} \ (ii)$$

Now,

$y(0) = 2$

if $x=0$, $y=2$ then eq^n (ii) becomes,

or, $$2\times e^{0} = \frac{e^{0}}{5} + C$$

or, $$2 = \frac{1}{5} + C$$

or, $$2 - \frac{1}{5} = C$$

$$\therefore\ C = 9/5$$

Substuting the value of C in eq^n (ii)

$$y\times e^{5x} = \frac{e^{5x}}{5} + \frac{9}{5}$$

or, $$y\times e^{5x} = \cancel{1}\ \frac{e^{5x}}{5} + \frac{9}{5}$$

or, $$y = \frac{\cancel{e^{5x}}}{5\times \cancel{e^{5x}}} + \frac{9}{5\times e^{5x}}$$

$$\therefore\ y = \frac{1}{5}\left(1 + 9\times e^{-5x}\right)$$ Ans

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3(c) Find the volume of sphere of radius r

Sol

Since, we know that intersection of sphere of radius r and the plane is of a circle of radius r

Then circle is

$$x^2+y^2=r^2 \quad \cdots \text{(1)}$$

clearly, the circle has ends $x=-r$ to $x=r$

Now, the volume of the solid that is generated by revolving the circle (1) about x-axis is (i.e $y=0$)

$$\text{volume} = \pi\int_{-r}^{r} y^2 dx$$

$$= \pi\int_{-r}^{r}(r^2-x^2)dx$$

$$= \pi\left[r^2x-\frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi\left\{\left(r^2\times r-\frac{r^3}{3}\right)-\left(r^2\times(-r)+\frac{r^3}{3}\right)\right\}$$

$$= \pi\left\{\frac{3r^3-r^3}{3}+r^3-\frac{r^3}{3}\right\}$$

$$= \pi\left\{\frac{2r^3}{3}+\frac{2r^3}{3}\right\}$$

$$= \frac{4\pi r^3}{3}$$

$\therefore$ The volume of the sphere whose radius r is $\frac{4\pi r^3}{3}$

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4(a) For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{x}$ converge?

Sol. Given,

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{x} \text{ - - - - (i)}$$

Comparing it with $\sum_{n=1}^{\infty} a_n$ then

$$a_n = \frac{(x-3)^n}{x} \text{ - - - (ii)}$$

Here,

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left[\frac{(x-3)^{n+1}}{\cancel{x}} \times \frac{\cancel{x}}{(x-3)^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{\cancel{(x-3)^n}\cdot(x-3)}{\cancel{(x-3)^n}}\right]$$

$$= \lim_{n\to\infty}(x-3)$$

$$= x-3$$

By D'Alembert ratio test the given series is convergent for

$|x-3| < 1$ i.e $-1 < (x-3) < 1$

$\Rightarrow 2 < x < 4$

and is divergent for $|x-3| > 1$.

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And further test is needed for $|x-3|=1$
i.e at $x=2$, $x=4$

At $x=2$, the series in eqn (1) becomes

$$\sum_{n=1}^{\infty}(a_n) = \sum_{n=1}^{\infty}\frac{(-1)^n}{2} \qquad \left[\text{i.e } \frac{(x-3)^n}{2} = \frac{(2-3)^n}{2}\right]$$

This is an alternative series. Comparing it with $\sum_{n=1}^{\infty}(-1)^n V_n$ then

$$V_n = \frac{1}{2}$$

$$\lim_{n\to\infty} V_n = \lim_{n\to\infty}\left(\frac{1}{2}\right) = \frac{1}{2} \neq 0$$

So, the given series is divergent at $x=2$ by Leibnitz test

And at $x=4$, the series eqn (1) becomes

$$\sum_{n=1}^{\infty}(a_n) = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right) = \frac{1}{2}\sum_{n=1}^{\infty}(1) = \frac{1}{2}(\infty) = \infty$$

This means the given series is divergent at $x=4$

Thus, the given series is convergent only for
$2 < x < 4$

4(b) Calculate $\iint_R f(x,y)\,dA$ for $f(x,y) = 100 - 6x^2y$ and

$R: 0 \le x \le 2,\ -1 \le y \le 1$

Sol^n Given,

$f(x,y) = 100 - 6x^2y \quad ; R: 0 \le x \le 2,\ -1 \le y \le 1$

Here,

$$= \int_{-1}^{1}\int_{0}^{2} (100 - 6x^2y)\,dx\,dy$$

$$= \int_{-1}^{1} \left[100x - \frac{6x^3y}{3}\right]_0^2 dy$$

$$= \int_{-1}^{1} \left[100x - 2x^3y\right]_0^2 dy$$

$$= \int_{-1}^{1} \left[100 \times 2 - 2 \times (2)^3 y\right] dy$$

$$= \int_{-1}^{1} (200 - 16y)\,dy$$

$$= \left[200y - \frac{16y^2}{2}\right]_{-1}^{1}$$

$$= \left[200y - 8y^2\right]_{-1}^{1}$$

$$= (200 \times (1) - 8 \times (1)^2) - (200 \times (-1) - 8 \times (-1)^2)$$

$$= (200 - 8) - (-200 - 8)$$

$$= 200 - 8 + 200 + 8$$

$$= 400 \text{ Ans}$$

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Group 'B'

5. If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{3-x}$, find gof and gog

Sol^n Given,

$f(x) = \sqrt{x}$
$g(x) = \sqrt{3-x}$

then, $gof = ?$
$gog = ?$

For gof

gof
$= g(\sqrt{x})$
$= \sqrt{3-\sqrt{x}}$

And for gog

gog
$= g(\sqrt{3-x})$
$= \sqrt{3-(\sqrt{3-x})}$

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6. Use continuity to evaluate the limit.

$$\lim_{x \to 4} \frac{5+\sqrt{x}}{\sqrt{5+x}}$$

Sol^n Given,

$$\lim_{x \to 4} \frac{5+\sqrt{x}}{\sqrt{5+x}}$$

Since the function $f(x) = \frac{g(x)}{h(x)}$ is being a quotient of two continuous function $g(x) = 5+\sqrt{x}$ and $h(x) = \sqrt{5+x}$ everywhere in their domain In Particular at $x=4$ and hence the quotient function $f(x)$ is also Continuous at $x=4$

$$\lim_{x \to 4} f(x) = f(4) = \frac{5+\sqrt{4}}{\sqrt{5+4}}$$

$$= \frac{5+2}{\sqrt{9}}$$

$$= \frac{7}{3}$$

$$\therefore \lim_{x \to 4} \frac{5+\sqrt{x}}{\sqrt{5+x}} = \frac{7}{3}$$

7 Verify Mean value Theorem of $f(x) = x^3 - 3x + 3$ for $[-1, 2]$

Sol

Statement of Mean Value Theorem

Let f be a function that satisfies the following hypothesis:

(i) f is continuous on the closed interval $[a,b]$

(ii) f is differentiable on the open interval (a,b)

Then there is a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

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Given function is,

$$f(x) = x^3 - 3x + 3 \quad \text{for } [-1, 2]$$

(i) Since $f(x)$ is a Polynomial function. So, it is continuous on the closed interval $[-1, 2]$

(ii) $f'(x) = 3x^2 - 3$ which exists for all $x \in (-1, 2)$

$\therefore$ $f(x)$ is differentiable on the open interval $(-1, 2)$

all the condition of Mean value theorem are Satisfied Hence, there exists a number c in $(-1, 2)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$f'(x) = 3x^2 - 3$

$f'(c) = 3c^2 - 3$

$f(x) = x^3 - 3x + 3$

$f(a) = f(-1) = (-1)^3 - 3\times(-1) + 3 = -1 + 3 + 3 = 5$

$f(b) = f(2) = (2)^3 - 3\times 2 + 3 = 8 - 6 + 3 = 5$

then,

$$3c^2 - 3 = \frac{5-5}{2-(-1)}$$

$$\text{or, } 3(c^2 - 1) = \frac{5-5}{3}$$

$$\text{or, } 3(c^2 - 1) = 0$$

$$c^2 - 1 = 0$$

$$\therefore c^2 = 1$$

$$c \Rightarrow \pm 1$$

Hence, $c = 1 \in (-1, 2)$. Mean value theorem verify.

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8. Sketch the curve $y = x^3 + x$

Sol

Given,

$$y = x^3 + x$$

(A) Domain:

For domain,

$$y = x^3 + x$$

clearly, this function exists for all value of x i.e $(-\infty, \infty)$

So, domain of $y = (-\infty, \infty)$

(B) Intercept:-

For x-intercept put $y = 0$,

$$y = x^3 + x$$

or, $0 = x^3 + x$

or, $x(x^2 + 1) = 0$

$$x^2 = -1$$

this gives no real value.

For y-intercept, put $x = 0$,

$$y = x^3 + x$$

$$y = 0 + 0 = 0$$

$\therefore (x, y) = (0, 0)$

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(c) Symmetry :-

Curve symmetric about y-axis $f(x) = f(-x)$ and for curve symmetric about origin $f(-x) = -f(x)$

then,

$$f(x) = x^3 + x$$

if $x = -x$, $f(-x) = (-x)^3 + (-x) = -x^3 - x = -(x^3 + x) = -f(x)$

$$\therefore f(-x) = -f(x)$$

This means y i.e $f(x)$ is Symmetric about origin.

(D) Asymptote

Horizontal asymptote :-

If either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$ ~~is~~ then $y = L$ is horizontal asymptote of curve $y = f(x)$

So,

$$\lim_{x \to \infty} x^3 + x = (\infty)^3 + (\infty) = \infty$$

Vertical asymptote:- A line $x = a$ is vertical asymptote of $y = f(x)$ if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$

So, $\lim_{x \to a} (x^3 + x) = a^3 + a \neq \infty$ for any finite value of a

$\therefore$ This means y i.e $f(x)$ has no ~~asy~~ asymptote

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(E) Interval of increasing and decreasing:-

Here,

$$f(x) = x^3 + x$$

so, $$f'(x) = 3x^2 + 1$$

For critical point,

$$f'(x) = 0$$

or, $3x^2 + 1 = 0$

or, $3x^2 = -1$

$$x^2 = \frac{-1}{3}$$

$\therefore$ x has no real value.

This means y has no critical point.

So,

Interval	sign of $f'(x)$	Nature of $f(x)$
$(-\infty, \infty)$	+ve	Increasing

(F) Concavity

$$f(x) = x^3 + x$$

$$f'(x) = 3x^2 + x$$

$$f''(x) = 6x$$

For concavity

$f''(x) = 0$ and $f''(x) = \infty$ find the interval of concave up and concave down and find the point of inflection

$$f''(x) = 0$$

$$6x = 0$$

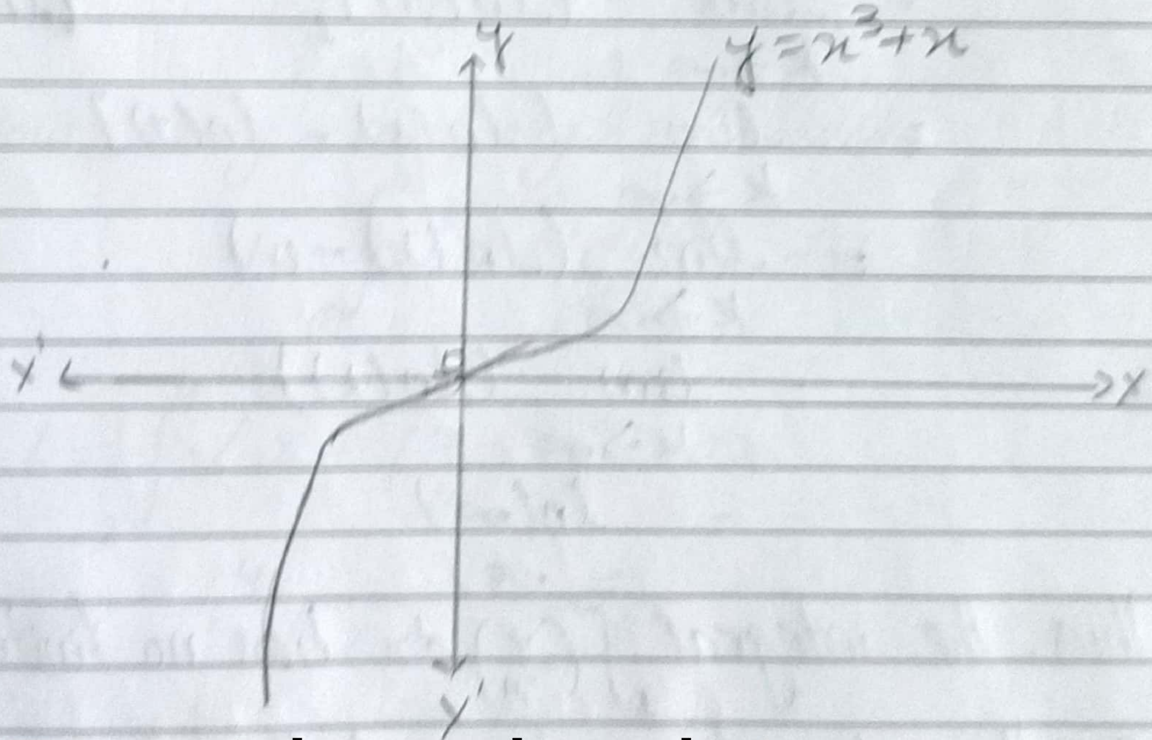
$$\therefore x = 0$$

Interval	sign of $f''(x)$	Nature of $f(x)$
$(-\infty, 0)$	$-ve$	Concave down
$(0, \infty)$	$+ve$	Concave up

(G) Summary

Interval	Nature of $f(x)$	
$(-\infty, 0)$	Increasing	Increasing
$(0, \infty)$	Concave down	Concave up

With the help of these information, the sketch of the curve is as;



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9. Determine whether the integral $\int_1^\infty \frac{7}{x}\,dx$ is convergent or divergent.

Sol

Given,

$$\int_1^\infty \left(\frac{7}{x}\right) dx$$

$$= \lim_{K\to\infty} \int_1^K \left(\frac{7}{x}\right) dx$$

$$= \lim_{K\to\infty} [\ln(x)]_1^K \qquad [\ln(1)=0]$$

$$= \lim_{K\to\infty} [\ln(K) - \ln(1)]$$

$$= \lim_{K\to\infty} (\ln(K) - 0)$$

$$= \lim_{K\to\infty} (\ln(K))$$

$$= \ln(\infty)$$

$$= \infty$$

Thus, the integral $\int_1^\infty \left(\frac{7}{x}\right) dx$ has no finite value.

So, the integral is divergent. Then, by integral test the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ is divergent.

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10. Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1,1)$ and $(4,8)$

Sol: Given

$$y^2 = x^3$$

taking root on bothsides

$$\sqrt{y^2} = \sqrt{x^3}$$

or, $$y = \sqrt{x^3}$$

$$\therefore y = x^{3/2} \quad \text{---} \quad (1)$$

diffⁿ eqⁿ (1) w.r.to x,

$$\frac{dy}{dx} = \frac{3}{2} x^{3/2 - 1}$$

$$\therefore \frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

Now, using arc length formula,

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$L = \int_1^4 \sqrt{1 + \left(\frac{3}{2} x^{1/2}\right)^2} \, dx$$

$$L = \int_1^4 \sqrt{1 + \frac{9}{4} x^{1/2 \times 2}} \, dx$$

$$\int_1^4 \sqrt{1 + \frac{9}{4} x} \, dx.$$

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Let

$$Z = 1 + \frac{9}{4}x$$

$$dZ = \frac{9}{4}\,dx \Rightarrow \frac{4}{9}dZ = dx$$

changing limit

$\Rightarrow$ if $x = 1$, $Z = 1 + \frac{9}{4} = \frac{13}{4}$

if $x = 4$, $Z = 1 + \frac{9 \times 4}{4} = 10$

Now,

$$L = \int_{\frac{13}{4}}^{10} \sqrt{Z} \times \frac{4}{9}\,dZ = \int_{\frac{13}{4}}^{10} (Z)^{\frac{1}{2}} \times \frac{4}{9}\,dZ$$

$$= \frac{4}{9}\left[\frac{Z^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_{\frac{13}{4}}^{10}$$

$$= \frac{4}{9} \times \frac{2}{3}\left[Z^{3/2}\right]_{13/4}^{10}$$

$$= \frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right)$$

$$= \frac{1}{27}\left(8 \times \sqrt{10^3} - 8 \times \sqrt{\left(\frac{13}{4}\right)^3}\right)$$

$$= \frac{1}{27}\left(8 \times 10\sqrt{10} - \cancel{8}^{2} \times \frac{13}{\cancel{4}}\sqrt{\frac{13}{4}}\right)$$

$$= \frac{1}{27}\left(80\sqrt{10} - 26\sqrt{\frac{13}{4}}\right)$$ ~~Ans~~

or,

$$= \frac{1}{27}\left(80\sqrt{10} - \cancel{26}^{13} \times \frac{1}{\cancel{2}}\sqrt{13}\right) \Rightarrow \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)$$ Ans

11. Find the solution of $y'' + 6y' + 9 = 0$, $y(0) = 2$, $y'(0) = 1$

Sol

The given 2^{nd} order homogenous linear eqn is

$$y'' + 6y' + 9 = 0 \quad \dots (1)$$

let

$y = e^{mx}$ be the reqn soln of eqn (1)

Now, we get an auxilarly eqn of (1) by replacing y'' by m^2 and y' by m

$$m^2 + 6m + 9 = 0$$

$$m^2 + m(3+3) + 9 = 0$$

or, $m^2 + 3m + 3m + 9 = 0$

or, $m(m+3) + 3(m+3) = 0$

or, $(m+3)(m+3) = 0$

either   or,

$m + 3 = 0$ $m + 3 = 0$

$m = -3$ $m = -3$

which is real and real

The reqn soln is

$$y = e^{mx}(C_1 + C_2 x)$$

$$y = e^{-3x}(C_1 + C_2 x) \quad \dots (11)$$

Since, $y(0) = 2$, if $x = 0$, $y = 2$ then eqn (11) becomes

$$2 = e^0 (C_1 + C_2 \times 0)$$

$$2 = C_1$$

$$\therefore C_1 = 2$$

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Diff^n eq^n (ii) w.r. to 'x'

$$y = e^{-3x}(C_1 + C_2 x)$$

$$y = e^{-3x}C_1 + e^{-3x}C_2 x$$ (using formula u.v)

diff^n both sides w.r. to 'x'

$$\frac{dy}{dx} \text{ or } y' = (-3e^{-3x}C_1 + e^{-3x} \times 0) + (e^{-3x} \times C_2 + xC_2 \times (-3) \times e^{-3x})$$

$$y' = (-3e^{-3x}C_1 + e^{-3x}C_2 - xC_2 \times 3e^{-3x})$$

$$y' = e^{-3x}C_2 - 3e^{-3x}C_1 - xC_2 \times 3e^{-3x}$$

$$y' = e^{-3x}C_2 - 3e^{-3x}(C_1 + xC_2)$$

$$\therefore y' = C_2 \times e^{-3x} - 3(C_1 + xC_2)e^{-3x} \quad \text{--- (iii)}$$

if $y'(0) = 1$ $\quad x = 0, y' = 1$ then eq^n (iii) becomes

$$1 = C_2 \times e^0 - 3(C_1 + 0)e^0$$

or, $1 = C_2 - 3C_1$

or, $1 = C_2 - 3 \times 2$ (using $C_1 = 2$)

or, $1 + 6 = C_2$

$\therefore C_2 = 7$

Substuting the value of C_1 and C_2 in eq^n (ii)

$$y = e^{-3x}(2 + 7x)$$ Ans

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12. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Sol. Given series is

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

comparing it with $\sum_{n=1}^{\infty} a_n$

where, $a_n = \frac{n^n}{n!}$

then,

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{(n+1)^n}{(n+1)!} \times \frac{n!}{n^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\cancel{(n+1)}(n+1)^n}{\cancel{(n+1)}\,\cancel{n!}} \times \frac{\cancel{n!}}{n^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(n+1)^n}{n^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(\frac{\cancel{n}}{\cancel{n}} + \frac{1}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \quad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

$$= e$$

$$= 2.71 > 1$$

So, the given series is divergent by D'Alembert ratio test

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13. Define Cross product of two vectors. If $a = i + 3j + 4k$ and $b = 2i + 7j - 5k$, find the vector $a \times b$ and $b \times a$

Sol

1st part

let $\vec{a}$ and $\vec{b}$ are two non-zero vectors. Then the cross product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ and is defined as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \hat{n}$$

where θ be the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is unit vector along $(\vec{a} \times \vec{b})$

2nd Part

Given,

$$\vec{a} = \vec{i} + 3\vec{j} + 4\vec{k}$$

$$\vec{b} = 2\vec{i} + 7\vec{j} - 5\vec{k}$$

Then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \vec{i}(-15 - 28) - \vec{j}(-5 - 8) + \vec{k}(7 - 6)$$

$$= (-43)\vec{i} + 13\vec{j} + 1\vec{k} \quad \text{Ans}$$

and we know,

$$(\vec{a} \times \vec{b}) = -(\vec{b} \times \vec{a})$$

So,

$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$$

$$= -\{-43\vec{i} + 13\vec{j} + 1\vec{k}\}$$

$$= (43\vec{i} - 13\vec{j} - 1\vec{k}) \quad \text{Ans}$$

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14 Define limit of a function. find $\lim_{x \to \infty} (x - \sqrt{x})$

Sol. 1st part,

let f(x) be a function defined on some open interval that contains the number a except possibly at itself, then the limit of f(x) on x approaches to a is l (finite value) and we write as

$$\lim_{x \to a} f(x) = l$$

2nd Part,

Given,

$$\lim_{x \to \infty} (x - \sqrt{x}) \qquad (x = \sqrt{x} \times \sqrt{x})$$

$$= \lim_{x \to \infty} (\sqrt{x} \times \sqrt{x} - \sqrt{x})$$

$$= \lim_{x \to \infty} \sqrt{x} (\sqrt{x} - 1) \qquad (\infty \times \infty)$$

$$= \infty (\infty - 1)$$

$$= \infty$$

Thus,

$$\lim_{x \to \infty} (x - \sqrt{x}) = \infty \quad \text{Ans}$$

15. Find the extreme value of $f(x,y) = y^2 - x^2$

Sol

Given,

$$f(x,y) = y^2 - x^2 \quad \text{---- (i)}$$

diff^n eq^n (i) w.r. to 'x'

$$f_x = -2x \quad \text{---- (ii)}$$

Again, diff^n eq^n (i) w.r. to 'y'

$$f_y = 2y \quad \text{--- (iii)}$$

For critical point,

$f_x = 0$

$-2x = 0$

$\therefore x = 0$

$f_y = 0$

$2y = 0$

$\therefore y = 0$

at point $(x,y) = (0,0)$.

Again, diff^n eq^n (ii) w.r. to 'x'

$$f_{xx} = -2$$

Again, diff^n eq^n (iii) w.r. to 'y'

$$f_{yy} = 2$$

Again, diff^n eq^n (ii) w.r. to 'y'

$$f_{xy} = 0$$

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Again, diffⁿ eqⁿ (iii) w.r. to 'x'

$$f_{yx} = 0$$

Here, at point (0,0)

$$f_{xx} = -2 > 0 \text{ (Maximum}$$

then,

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4 < 0$$

$$\therefore f(x,y) = y^2 - x^2$$

$$\text{at}(x,y) = f(0,0) = 0 - 0 = 0$$

Thus, the function $f(x,y)$ attains its maximum at (0,0) and maximum value is 0 Ans

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