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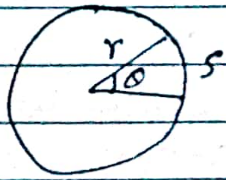
Unit 1: Mechanics

Chapter: 1 Rotational Dynamics

- Equation of motion

We know



$$\boxed{\theta = \frac{s}{r}}$$

$$\boxed{s = r\theta}$$

differentiating

$$\frac{ds}{dt} = r \cdot \frac{d\theta}{dt}$$

$$\boxed{v = r\omega} \quad — \text{(1)}$$

θ : Angular ~~velocity~~ displacement (radi
ω : Angular velocity

- unit radian s^{-1}

α : Angular acceleration

$$\boxed{\alpha = \frac{d\omega}{dt}}$$

$$\boxed{\begin{aligned}\alpha &= \frac{d\omega}{dt}\frac{d\theta}{d\theta} \\ &= \omega \frac{d\omega}{dt}\end{aligned}}$$

We know

$|v = r\omega$

differentiating

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$$\frac{dv}{dt} = r \cdot \frac{d\omega}{dt}$$

$$\boxed{a = r\alpha}$$

Relation between linear and angular eqn

	Linear	Angular
i	displacement (S)	Angular displacement (θ)
ii)	Velocity (v)	" Velocity ω
iii)	Acceleration	" Acceleration α
iv)	$v = u + at$	$\omega_2 = \omega_1 + \alpha t$
v)	$s = ut + \frac{1}{2}at^2$	$\theta = \omega_1 t + \frac{1}{2}\alpha t^2$
vi	$v^2 = u^2 + 2as$	$\omega_2^2 = \omega_1^2 + 2\alpha\theta$

Q.1) If $\theta = t^3 + t^2 + 1$ and $R = 2m$ then find ω, α, a, v, a_c $t = 2s$

sol^n:

$$\omega = \frac{d\theta}{dt} = \frac{d(t^3+t^2+1)}{dt} = 3t^2 + 2t = 3\times2^2 + 2\times2 = 16\ rad\,s^{-1}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d(3t^2+2t)}{dt} = 12t + 2 = 12\times2 + 2 = 14\ rad\,s^{-2}$$

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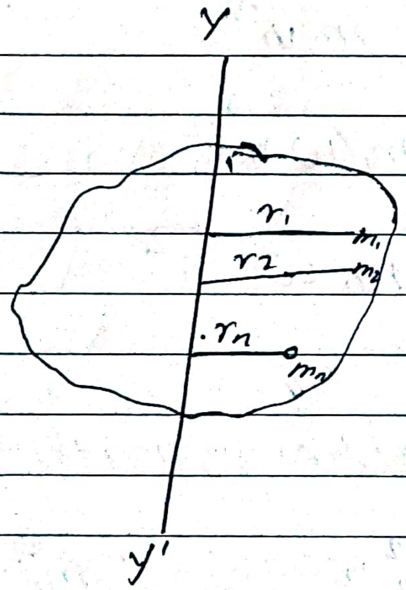
$a = r\alpha$ $= 2 \times 14$ $= 28\ ms^{-2}$	$v = r\omega$ $= 2 \times 16$ $= 32$	$a_c = \frac{v^2}{r}$ $= \frac{(32)^2}{2}$ $= 512\ ms^{-2}$

* kinetic energy of rotation of rigid body



consider a rigid body of mass 'm' rotating about an axis YY' with angular velocity 'ω'

Let us suppose this body consist of 'n' particle of masses $m_1, m_2, m_3 - - - m_n$ which are distance of $r_1, r_2 - - - r_n$ from the axis of rotation.

All particle has same angular velocity

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Let v_1 v_2 - - - - v_n represents the linear velocity of m_1, m_2 - - m_n

we know,

$v_1 = \omega r_1$

$v_2 = \omega r_2$

$v_n = \omega r_n$

Now, rotational K.E of mass (m_1) is

$$(K.E_1) = \frac{1}{2} m_1 v_1^2$$

$$= \frac{1}{2} m_1 (\omega r_1)^2$$

$$= \frac{1}{2} m_1 \omega^2 r_1^2$$

similarly for other particle K.E will be

$$\frac{1}{2} m_2 r_2^2 \omega^2 ; \quad \frac{1}{2} m_3 r_3^2 \omega^2$$

$$; \quad \frac{1}{2} m_n r_n^2 \omega^2$$

Total rotational K.E of rigid body

$$K.E = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 - - - + \frac{1}{2} m_n r_n^2 \omega^2$$

$$K.E = \frac{1}{2}(m_1 r_1^2 + m_2 r_2^2 \cdots m_3 r_3^2)\omega^2$$

$$K.E = \frac{1}{2}\left(\sum_{i}^{n} m_i r_i^2\right)\omega^2$$

$$KE = \frac{1}{2} I\omega^2$$

I is momento of inertia.

* kinetic energy Rolling body

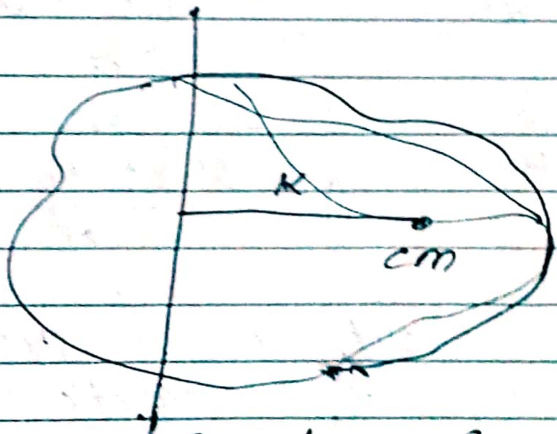
Moment of inertia (I)

The rotational inertia is called moment of inertia.

$$I = MR^2$$

M = mass
R = Radius

Radius of Gyration (K)



axis of rotation

The distance from axis of rotation to a point where whole mass of the body supposed to be concentrated

We know

$$I = \sum_{i=1} m_i r_i^2$$

Also : $I = Mk^2$

$$MK^2 = \sum_{i=1}^{n} m_i r_i^2$$

$$MK^2 = m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2$$

If all particle have same mass

$$MK^2 = m(r_1^2 + r_2^2 \cdots r_n^2)$$

$$MK^2 = \frac{mn(r_1^2 + r_2^2 \cdots r_n^2)}{n}$$

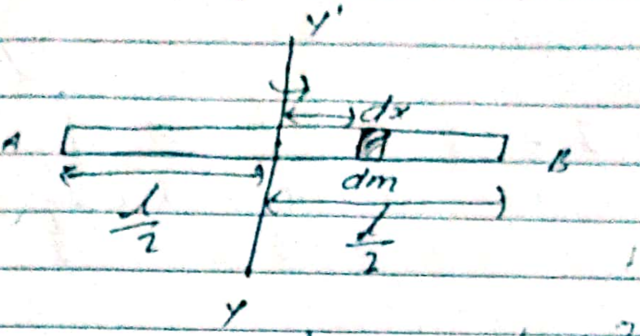
$$\cancel{M}K^2 = \frac{\cancel{M}(r_1^2 + r_2^2 \cdots + r_n^2)}{n}$$

$$K = \sqrt{\frac{(r_1^2 + r_2^2 \cdots + r_n^2)}{n}}$$

i.e root mean square distance of all the particle of the body is called radium of gyration

Moment of inertia of thin rod

a) About an axis passing through it's center of mass.



$dm\,x^2$

Let us consider thin uniform rod of mass 'M' and length 'l' let YY' be an axis passing through the ~~com~~ center of rod.
Let us conside small element of the length 'dx' which is at distance 'x' from the mass of rotation YY'

~~$dm = \frac{M\,dx}{l}$~~

Then mass per unit length of rod
$= \frac{M}{l}$ mass of im :

The mass of small element (dm) is $\frac{M\,dx}{l}$

then $dm = \frac{M}{l}dx$

The moment of inertial of small element $dI = dm.x^2$

$$\int dI = \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{M}{l}\,dx\,x^2$$

$$I = \frac{M}{l}\int_{-\frac{l}{2}}^{l/2} dx . x^2$$

$$I = \frac{M}{l}\left[\frac{x^3}{3}\right]_{-\frac{l}{2}}^{l/2}$$

$$\cancel{I = \frac{M}{l}\left(\frac{l}{2x}\right)}$$

$$I = \frac{M}{l}\left[\frac{l^3}{8\times 3} - \left(\frac{-l^3}{8\times 3}\right)\right]$$

$$I = \frac{M}{l}\,\frac{l^3 + l^3}{24}$$

$$I = \frac{M}{\cancel{l}}\,\frac{\cancel{2}l^{\cancel{3}}}{\cancel{24}_{12}}$$

$$\boxed{I = \frac{Ml^2}{12}}$$ Required.

Torque: The turning effect of a body is called torque.

Torque τ : Force $\times$ perpendicular distance of force from the axis of rotation.

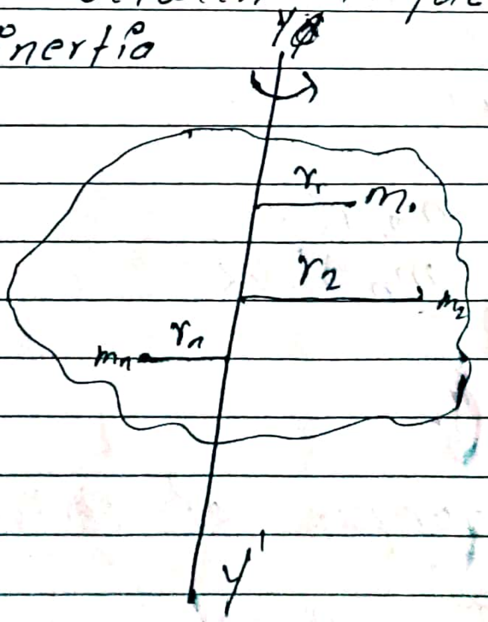
τ (τ) = $F \times r$

unit Nm

τ is vector quantity.

$$\boxed{\tau = \vec{r} \times \vec{F}}$$

Relation between Torque and moment of inertia



Let us conside a rigid body consisting n particle of masses

m_1 m_2 - - - m_n which are at distance r_1 r_2 - - - r_n from axis of rotation YY'

Let α represents the the angular acc. of each particle.

The acceleration of particle of mass (m_1) $a_1 = r_1\alpha$

Then force on 'm_1' $F_1 = m_1 a_1$

$= m_1 r_1 \alpha$

The Torque produced on particle of mass 'm_1' is

$\tau_1 = F_1 r_1$

$= m_1 r_1 \alpha r_1$

$= m_1 r_1^2 \alpha$

similarly

$\tau_2 = m_2 r_2^2 \alpha$

$\tau_n = m_n r_n^2 \alpha$

Total torque of rigid body

$\tau = \tau_1 + \tau_2 + \cdots \tau_n$

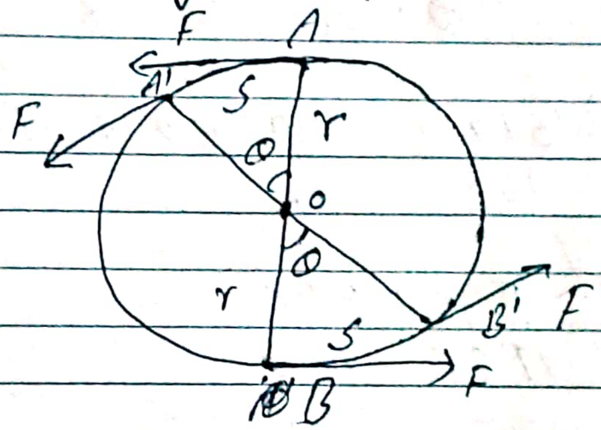
$$\tau = m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + \cdots m_n r_n^2 \alpha$$

$$\tau = (m_1 r_1^2 + m_2 r_2^2 + \cdots m_n r_n^2)\alpha$$

$$\tau = \left[\sum_{i=1}^{n} m_i r_i^2\right]\alpha$$

$\boxed{\tau = I\alpha}$ which is required relation.

Work done by couple



$\tau = I\alpha$.

Two equal and opposite forces applied at two different points in rigid body due to which there is rotational effect on the body is called couple. The work done by the pair of forces is called work done by couple.

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Let two equal ~~fo~~ force (F) is applied on a rigid body at A & B. A moves to A' in the same times B moves to B'.

The work done by force at A is

$= F \cdot S$
$= F\theta r$

Similarly the work done by force B is

$= F\theta r$

Now total work done in the body is

$W = F\theta r + F\theta r$
$= 2F\theta r$

$W = F 2r\,\theta$

$$\boxed{W = \tau\theta}$$

Also power $P = \frac{dW}{dt}$

$(P) = \frac{d}{dt}(\tau\theta)$

$P = \tau \frac{d\theta}{dt}$

$$\boxed{P = \tau\omega}$$

Angular momentum (L)

The moment of linear momentum is called angular momentum
i.e $L = p \cdot r$
r = distance from axis of rotation

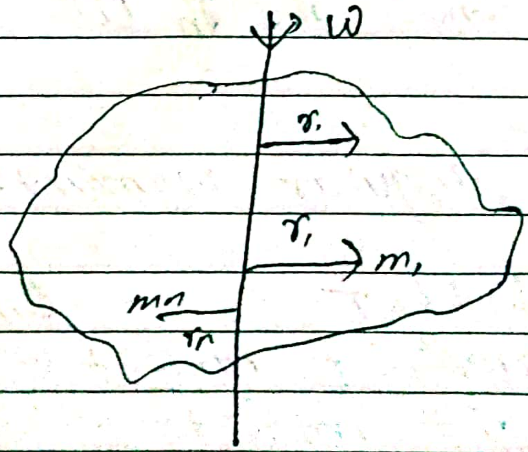
$\boxed{L = mvr}$

In vector form $\boxed{\vec{L} = \vec{r} \times \vec{p}}$

$\boxed{L = mr^2\omega}$

unit of L $kg\,m^2 s^{-1}$

Relation between angular momentum and moment of inertia.



Let (ω) be the angular velocity of each particle, then

Linear velocity of particle (m_1) is

$$v_1 = \omega r_1$$

~~Linear~~

Angular momentum of particle (m_1) is

$$L_1 = m_1 v_1 r_1$$

$$L_1 = m_1 \omega r_1 r_1$$

$$L_1 = m_1 r_1^2 \omega$$

Similarly

$$L_2 = m_2 r_2^2 \omega$$

$$L_3 = m_3 r_3^2 \omega$$

$$L_n = m_n r_n^2 \omega$$

Total angular momentum

$$L = L_1 + L_2 + \dots L_n$$

$$L = m_1 r_1^2 \omega + m_2 r_2^2 + \dots m_n r_n^2 \omega$$

$$L = (m_1 r_1^2 + m_2 r_2^2 + \dots m_n r_n^2)\omega$$

$$L -$$

$$L \quad \sum_{i=1}^{n} m_i r_i$$

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$\boxed{L = I\omega}$

where $I = \sum_{i=1}^{n} m_i r_i^2$

which is required relation.

Relation between Angular moment & Torque

we know

$$L = I\omega$$

Diffrentiating on both side w.r.t 't'

$$\frac{dL}{dt} = \frac{d(I\omega)}{t}$$

$$\frac{dL}{dt} = I\frac{d\omega}{dt}$$

$$\frac{dL}{dt} = I\alpha \quad \text{--- (i)}$$

we also know,

$$\tau = I\alpha \quad \text{--- (ii)}$$

comparing (i) & (ii)

$$\tau = \frac{dL}{dt}$$

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i.e ~~O~~ Torque is equal to rate of change of angular momentum

[Note : Which is equavalent to

$$F = \frac{dp}{dt}$$]

Imp

Principle of conservation of Angular momentum.

In absence of external torque the total angular momentum of the system remains constant.

We know

$$\frac{dL}{dt} = \tau$$

In absence of external Torq

$$\frac{dL}{dt} = 0$$

$\Rightarrow$ $L =$ constant

$$\boxed{I\omega = \text{constant}}$$

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For numerical

$\boxed{I_1 \omega_1 = I_2 \omega_2}$ $\boxed{\omega = 2\pi f}$

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Chapter - 2 Periodic Motion

The motion which repeats itself after a fixed interval of time is called periodic motion.

Eg. Motion of pendulum clock, Motion of second arm. etc.

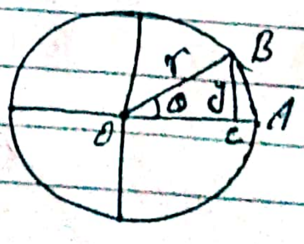
Simple Harmonic Motion (SHM)

A to and fro motion is called SHM. In this motion there is constant amplitude and single frequency.
In SHM, the acceleration produced on a body is directly proportional to displacement from mean position
i.e $a \propto x$

$a = -kx$

The negetive sign shows that acceleration is in a direction opposite to displacement from mean position.

Characteristics of SHM
(i) Displacement (y)



Let particle under SHM moves from point A to B covering angular displacement θ.

In ΔOBC

$$\sin\theta = \frac{y}{r}$$

$$y = r\sin\theta$$

$$y = r\sin(\omega t)$$

where 'r' is maximum displacement from mean position. It is called amplitude.

(ii) Velocity (v)

We know, $v = \frac{dy}{dt}$

$$v = \frac{d\, r\sin(\omega t)}{dt}$$

$$v = r\cos\omega t \cdot \omega$$

$$v = r\omega\cos(\omega t)$$

$$v = \omega r\sqrt{1-\sin^2\omega t}$$

$$v = \omega\sqrt{r^2 - r^2\sin\omega t}$$

$$v = \omega\sqrt{r^2-y^2}$$

Special case

(i) A mean position

$y = 0$

$v = \omega\sqrt{r^2 - 0^2}$

$v = \omega r$

i.e the velocity will be maximum at mean position.

(ii) At extreme position

$y = r$

$v = \omega\sqrt{r^2 - r^2}$

$v = 0$

The velocity will be minimum at extreme position.

(iii) Acceleration (a)

We know $a = \frac{dv}{dt}$

$= \frac{d\, r\omega\cos\omega t}{dt}$

$= r\omega(-\sin\omega t)\omega$

$a = -\omega^2 r\sin(\omega t)$

$a = -\omega^2 y$

Special cases

(1) At mean position $y = 0$

$$a = -\omega^2 0$$

$$\boxed{a = 0 \quad \text{minimum}}$$

(2) At Extreme position

$$y = r$$

$$\boxed{a = -\omega r \quad \text{maximum}}$$

(iv) Time period (T)

It is the time taken to complete one oscillation or one cycle.

we know,

$$a = -\omega^2 y$$

Neglecting -ve sign

$$a = \omega^2 y$$

$$\omega^2 = \frac{a}{y}$$

$$\omega = \sqrt{\frac{a}{y}}$$

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$$\frac{2\pi}{T} = \sqrt{\frac{a}{y}}$$

$$T = 2\pi\sqrt{\frac{y}{a}}$$

(v) Frequency : No. of oscillation completed per second

$$f = \frac{1}{T}$$

(vi) Amplitude (r) :

Maximum displacement of the particle from the mean position is called amplitude

unit :- meter

(vii) phase difference (ϕ) : It helps to us to locate position of particle in term of angle

Eg $y = r\sin(\omega t \pm \phi)$

where ϕ represents phase angle.

$y = 5\sin(2\pi t - 0.5)$

Find r, f, ϕ

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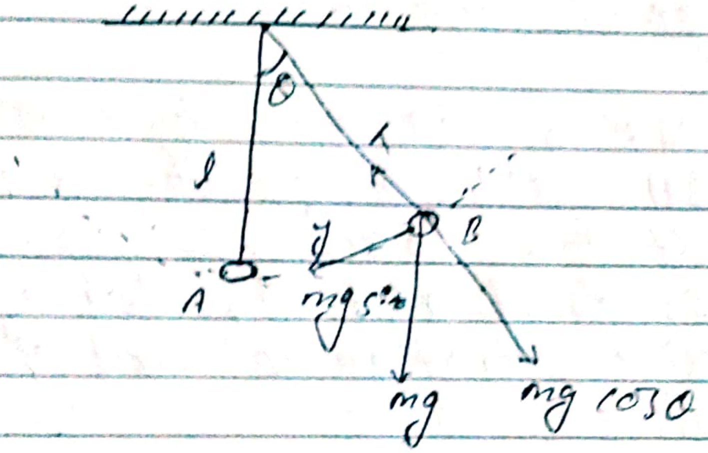
$y = r\sin(\omega t - \phi)$

$y = r\sin(2\pi f t - \phi)$

Simple ~~pen~~ pendulum



A heavy point mass object attached to free end of a string where other end of the string is fixed to rigid support so that the bob will oscillate in a vertical plane is called simple pendulum.

Let us consider a metal bob of mass(m) is attached to free end of string (l). The upper end of the string is fixed to a rigid support. Let the bob is displaced from mean position through an angle (θ) after the bob is released it will oscillatate.

In fig

mg = weight of metal bob

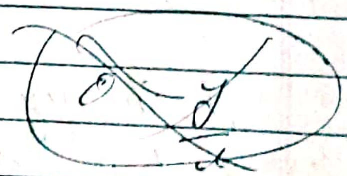
y = displacement from mean position

T = Tension in the string

Let us resolve mg into two componets i.e $mg\cos\theta$, $mg\sin\theta$.

Here, $mg\cos\theta$ is balanced by tension(T) and $mg\sin\theta$ act as restoring force thus, restoring force(F) = $-mg\,\cancel{\cos}\sin\theta$.
-ve shows that restoring force is in opposite direction to that of displacement.

$$ma = -mg\sin\theta$$



$$a = -g\sin\theta$$

For small angle $\sin\theta \simeq \theta$

$$\Rightarrow \quad a = -g\theta$$

From fig

$$\theta = \frac{\text{arc lengh}}{\text{radius}} = \frac{y}{l}$$

by putting value of θ

$$a = -g\,\frac{y}{l}$$

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$$a = -\left(\frac{g}{l}\right)y \quad \text{---} \quad (1)$$

$a \propto y$

which is necessary condition for SHM
Thus, motion of simple pendulum is SHM.

For $a = -\omega^2 y$ — (11)

Comparing eqn (1) & (2)

$$\omega^2 = \frac{g}{l}$$

$$\omega = \sqrt{\frac{g}{l}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{g}{l}}$$

$$\Rightarrow T = 2\pi\sqrt{\frac{l}{g}}$$

which is required expression for time period of simple pendulum.

Note (1) $T \propto \sqrt{l}$
$T \propto \frac{1}{\sqrt{g}}$

(iii) $l_2 = l_1(1 + \alpha\Delta\theta)$

If temp. increase, length increase so time period also increase.

4) second pendulum → A simple pendulum having time period of 2 sec.

Limitation of simple pendulum

1) A heavy point mass is not possible.
2) A weighless and inextensible string is also not possible.
3) The angular displacement of about 4° is difficult to achive. ~~practially.~~ practically.

3) The angular displacement of about 4° is difficult to achive practically.

Energy In SHM :

When an objects is in SHM, it posess both K.E & P.E

$$K.E = \frac{1}{2} m v^2$$

$$= \frac{1}{2} m \left(\omega \sqrt{r^2 - y^2} \right)^2$$

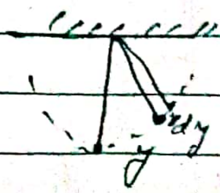
$$= \frac{1}{2} m \omega^2 (r^2 - y^2)$$

where r. amplitude
y : displacement from mean position
m : mass
ω : angular velocity

For potential energy
Restoring force (F) = $-ky$

$F = ma$
$F = m(-\omega^2 y)$
$F = -m\omega^2 y$
$F = -ky$

Let the object is further displaced by dy
The work done has to be done against restoring force - then



$$dW = -F dy$$

$$dW = -(-ky) dy$$

$$dW = ky\,dy$$

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Total workdone for displacement y is

$$\int dW = \int_0^y ky\,dy$$

$$W = k\left[\frac{y^2}{2}\right]_0^y$$

$$W = \frac{1}{2}k(y^2 - 0)$$

Putting value of $k = m\omega^2$

$$\therefore W = \frac{1}{2} m\omega^2 y^2$$

The workdone stored the P.E

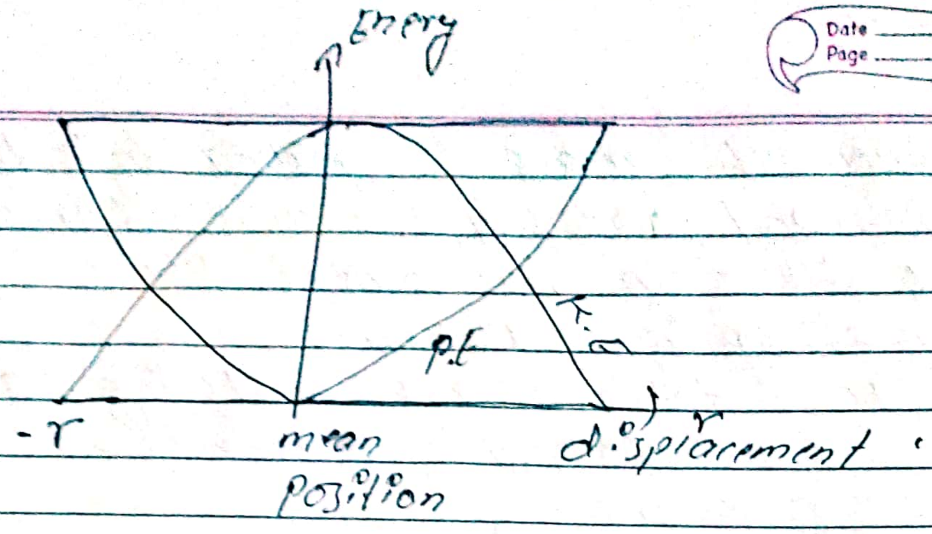
$$P.E = \frac{1}{2} m\omega^2 y^2$$

$\therefore$ The total energy = K.E + P.E

$$E = \frac{1}{2} m\omega^2 (r^2 - y^2) + \frac{1}{2} m\omega^2 y^2$$

$$E = \frac{1}{2} m\omega^2 r^2 - \cancel{\frac{1}{2} m\omega^2 y^2} + \cancel{\frac{1}{2} m\omega^2 y^2}$$

$$E = \frac{1}{2} m\omega^2 r^2$$



Mass-spring system

① Horizontal mass spring system

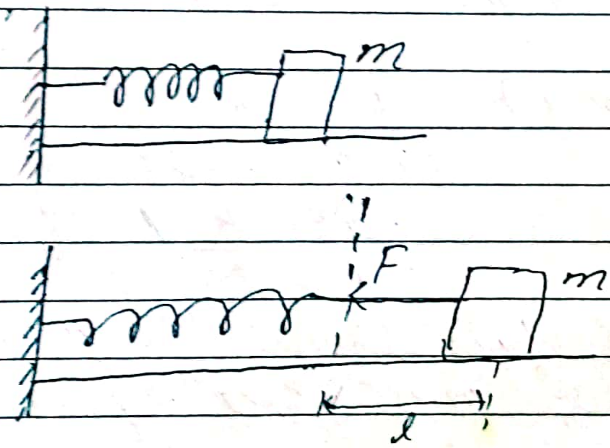


fig: mass-spring system laid in horizontal plane.

Let use consider a spring whose one end is fixed to a rigid support and mass (m) is attached to another end of system is kept in horizontal plane as shown in fig.

When the mass is pulled by distance (l) and released, the mass-spring will oscillate in horizontal plane, the mass-spring D-plane

In this case Restoring Force

$F \propto$ ~~con~~ displacement (l)

$F \propto l$

$F = -kl$ --- (i)

Where, K → sprin constant

-ve sign show that restoring Force and displacement are in opposite direction.

From eqn (1) $F = -kl$

$ma = -kl$

$a = \frac{-k}{M} l$ — (2)

thus $a \propto l$ this shows the motion of mass-spring in SHM

For SHM $a = -\omega^2 l$ — (iii)

comparing (2) & (3)

$\omega^2 = \sqrt{\frac{K}{M}}$

Time period $(T) = \frac{2\pi}{\omega}$

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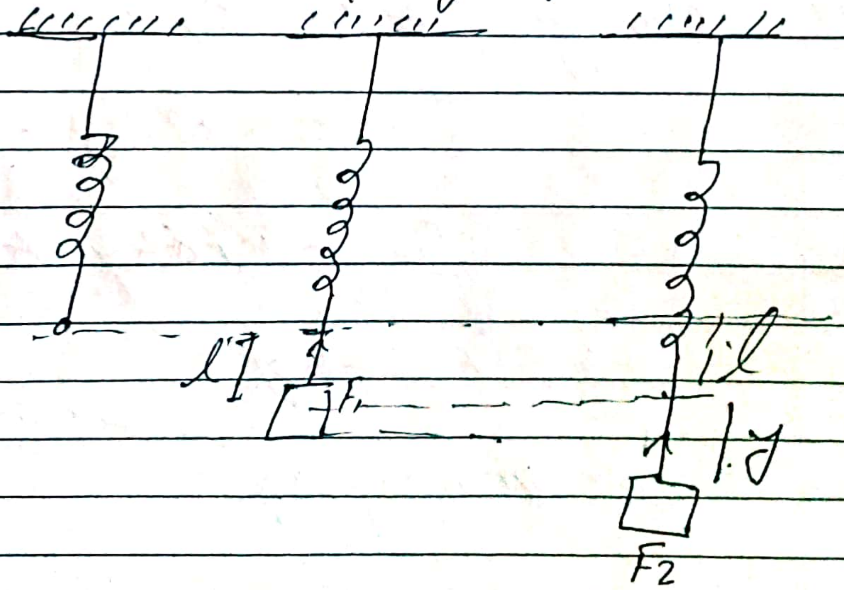
$$T = \frac{2\pi}{\sqrt{\frac{K}{RM}}}$$

$$\boxed{T = 2\pi\sqrt{\frac{RM}{K}}}$$

(2) Vertical mass-spring System



let use consider spring whose upper end is fixed to a rigid support and at the free end a mass (m) is hanged due to which it extend to

by length (l).
Let us extend the spring further by distance (y) as shown in fig.
After the mass is released from displaced position it will oscillate in a verticle plane

Here, restoring force $(F_1) \propto l$

$$F_1 = -kl$$

Also, $F_2 \propto (l+y)$

$$F_2 = -k(l+y)$$

net restoring fore $F = F_2 - F_1$

$$F = -k(l+y) + kl$$

$$F = -\cancel{kl} - ky + \cancel{kl}$$

$$F = -ky$$

$$ma = -ky$$

$$a = \frac{-k}{m} y \quad \text{--- (1)}$$

$$\Rightarrow \quad a \propto y$$

Thus, motion in SHM.

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For SHM (a)

We know,

$$a = -\omega^2 y \quad \text{---} (ii)$$

Comparing (i) & (ii)

$$\omega^2 = \frac{K}{M}$$

$$\omega = \sqrt{\frac{K}{M}}$$

$$T = \frac{2\pi}{\omega}$$

$$T = \frac{2\pi}{\sqrt{\frac{K}{M}}}$$

$$\boxed{T = 2\pi\sqrt{\frac{M}{K}}}$$ Required.

Angular simple Harmonic motion (Angular SHM)

In this motion Torque (τ) is

proportional to angular displacement θ

$\tau \propto \theta$

$\tau = -k\theta$

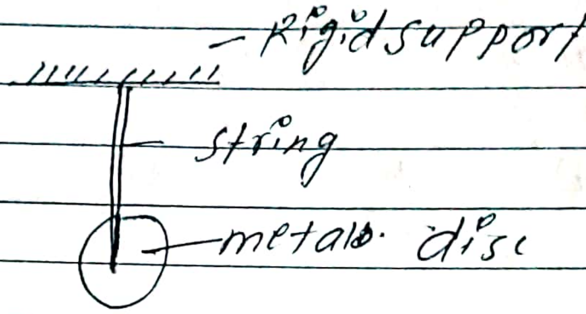

Fig: Tension Pendulum.

The restoring torque is directly proportional to angular displacement

$\tau \propto \theta$

$\tau = -k\theta$

k is proportionility constan called torsion constant

$I\alpha = -k\theta$

$\alpha = \frac{-k}{I}\theta$

which is equivalent to $a = -\omega^2 x$

$\omega^2 = \frac{k}{I}$

$\omega = \sqrt{\frac{k}{I}}$

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$$\frac{2\pi}{T} = \sqrt{\frac{K}{I}}$$

$$\boxed{T = 2\pi\sqrt{\frac{I}{K}}}$$

Types of oscillation

(i) Free oscillation :- The oscillation which is free from the opposing forces. Due to which amplitude of oscillition.

(ii) Damped oscillation:- The oscillation where there is presence of opposing forces like air resistance, friction.
Here, amplitude of oscillation decreases continuously.

(iii) Force oscillation:- The oscillation in which external forced applied so that the oscillation remains in uniform motion.