2077 Question Solution

-PravinGupta

Subject :- Mathematics-I (BSc.CSIT 1st sem)

GUPTA TUTORIAL

Group A

1(a) If $f(x) = x^2$ then find $\frac{f(2+h)-f(2)}{h}$

Sol^n Given,

$$f(x) = x^2 \quad \text{---} \quad (i)$$

$$\frac{f(2+h)-f(2)}{h} = ? \quad \text{--} \quad (ii)$$

from eq^n (i)

$$\text{if, } f(x) = x^2$$

$$\text{if, } f(2+h) = (2+h)^2 = 4+4h+h^2$$

$$\text{if, } f(2) = (2)^2 = 4$$

then eq^n (ii) becomes

$$\frac{f(2+h)-f(2)}{h}$$

$$= \frac{4+4h+h^2-4}{h}$$

$$= \frac{4h+h^2}{h}$$

$$= \frac{\cancel{h}(4+h)}{\cancel{h}}$$

$$= h+4 \quad \text{Ans}$$

$$\therefore \frac{f(2+h)-f(2)}{h} = h+4$$

Ans

(b)(a) Dry air is moving upword. If the ground temperature is 20° and the temperature at a height of 1km is 10°C, express the temperature T in °C as a function of the height h (in kilometers), assuming that a linear model is appropriate.
(b) Draw the graph of the function in part(a). What does the slope represent? (c) What is the temperature at a height of 2km?

Sol

Given,

(a)

Ground temperature = 20°C
Temperature at height 1 km = 10°C
Height = 1Km

Consider the linear model for this, temperature and height are independent

~~T = mh +~~

$$T = mh + b \quad \cdots \quad (i)$$

Since we know, at ground level, $h=0$

so, $T = m \times 0 + b$

or, $20 = 0 + b$

$\therefore b = 20$

Therefore, ~~h = 1~~ the eq^n (i) becomes

$$T = mh + 20 \quad \cdots \quad (ii)$$

Also given $h = 1$ Km, $T = 10°C$ then eq^n (ii) becomes

$T = mh + 20$

or, $10 = m \times 1 + 20$

$\therefore m = -10$

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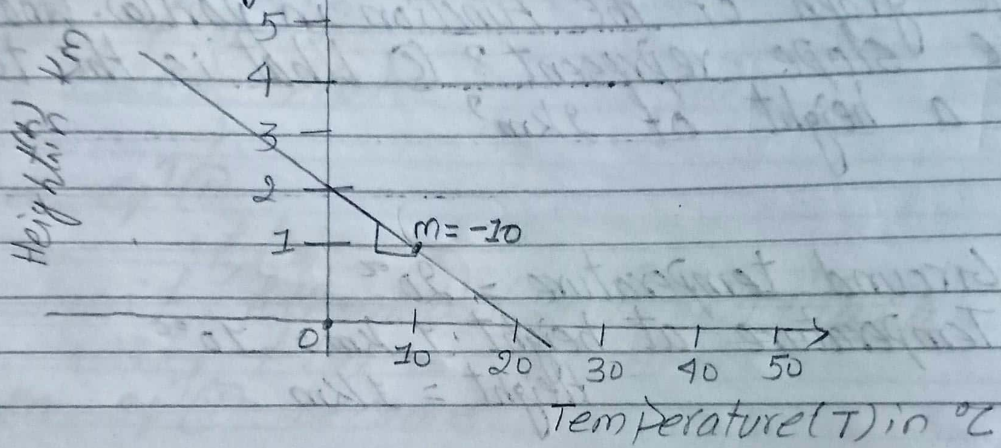
Therefore, appropriate linear model is
$T = -10h + 20$ --- (iii) is req^n model

(b) For the graph of the function



The slope ($m = -10$) represent, if height is increased by 1Km then the temperature is decreased by $-10^{\circ C}$

(c) Given, height = 2Km, T = ?
from eq^n (iii)

$$T = -10h + 20$$
$$\text{or, } T = -10 \times 2 + 20$$
$$\text{or, } T = -20 + 20$$
$$\therefore T = 0^{\circ C}$$

∴ Therefore at 2km temperature become $0^{\circ C}$.

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(c) Find the equation of the tangent to the parabola $y = x^2 + x + 1$ at $(0,1)$

Sol:

Given, eq^n of parabola is

$$y = x^2 + x + 1 \quad \text{--- (i)}$$

point $(x_1, y_1) = (0, 1)$

To find slope (m)

diff^n. eq^n (i) w.r. to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + x + 1)$$

$$\frac{dy}{dx} = 2x + 1 \quad \text{--- (ii)}$$

we know, slope $(m) = \frac{dy}{dx}$

so, eq^n (ii) becomes

$m = 2x + 1$ at $(0, 1)$

$m = 2 \times 0 + 1$

$\therefore m = 1$

Now,

eq^n of tangent to the parabola is,

$y - y_1 = m(x - x_1)$

or, $y - 1 = 1(x - 0)$

or, $y - 1 = x$

$\therefore x - y + 1 = 0$ which is req^d eq^n.

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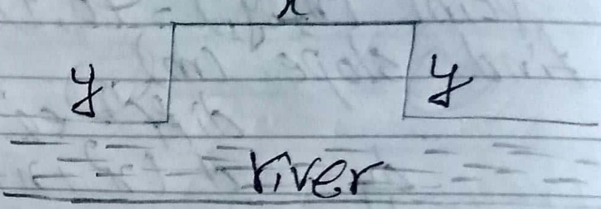
2(a) A farmer has 2000ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Sol Let us consider a river, x, y are dimension of Fence,



Given,

Total dimension covered by fence = 2000 ft

Length of rectangular field = x

Breadth " " " $= y$

From question, Total dimension of Fence (perimeter) is

$$x + y + y = 2000$$

$$\text{or,} \quad x + 2y = 2000$$

$$\therefore x = 2000 - 2y \quad \text{----} (1)$$

The area of rectangle,

$$A = L \times b = x \times y$$

$$A = y \times (2000 - 2y)$$

$$A = 2000y - 2y^2 \quad \text{----} (11)$$

Diffⁿ eqⁿ (11) w.r. to 'y'

$$\frac{dA}{dy} = \frac{d}{dy}(2000y - 2y^2)$$

$$= 2000 - 4y$$

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For largest area

$$\frac{dA}{dy} = 0$$

so, $2000 - 4y = 0$

or, $y = \frac{2000}{4}$

$\therefore y = 500 ft$

substuting the value of y in eq^n ①

$x = 2000 - 2y$

$x = 2000 - 2 \times 500$

$x = 1000 ft$

$\therefore$ The dimension of field which has largest area is, length (x) = 1000 ft, breadth (y) = 500 ft.

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2(b) Sketch the curve

$$y = \frac{1}{x-3}$$

Sol Given,

$$y = \frac{1}{x-3}$$

(I) Domain:-

for domain

$x - 3 \neq 0$

$\Rightarrow x \neq 3$

The function exists for all x except 3. so domain is

Domain $= (-\infty, \infty) - \{3\}$

(II) Intercept:

When $x = 0$, $y = -\frac{1}{3}$

It passes through $(0, -1/3)$

(III) Asymptote:-

As $x \to \infty$, $y \to 0$

and As $x \to 3$, $y \to \infty$

$\therefore$ $x = 3$, $y = 0$ are asymptote

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$(x-3)^{-1}$
$= -1(x-3)^{-1-1} * 1$

(IV) Critical point and increasing and decreasing

$$y = \frac{7}{(x-3)}$$

$$\therefore y' = \frac{7}{-(x-3)^2}$$

For critical point

$$y' = \frac{7}{0}$$

or, $\frac{7}{-(x-3)^2} = \frac{7}{0}$

or, $-(x-3)^2 = 0$

$\therefore x = 3$

Sign of y':

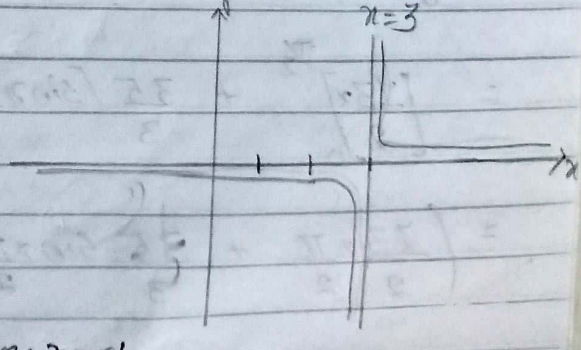
	$-\infty$ to 0	0 to 3	3 to $+\infty$
sign	$-$	$+$	$+$
	decreasing	increasing	increasing

(V) Concavity:-

$$y' = \frac{7}{-(x-3)^2}$$

$$\therefore y'' = \frac{7}{(x-3)^3}$$

$\therefore$ point of inflection is $x = 3$



Sign of y''

	$-\infty$ to 0	0 to 3	3 to $+\infty$
sign	$-$		$+$
	concave down	concave down	concave upward increasing

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3(a) Show that the $\int_1^\infty \frac{1}{x^2}$ converges and $\int_1^\infty \frac{1}{x}$ diverges

Sol Given,

$$\int_1^\infty \frac{1}{x^2}\,dx$$

It is improper integral so, we have to take limit for solving this

$$I = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx \Rightarrow \lim_{b\to\infty} \int_1^b x^{-2}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{x^{-2+1}}{-2+1}\right]_1^b$$

$$= \lim_{b\to\infty} \left[-x^{-1}\right]_1^b \Rightarrow \lim_{b\to\infty} \left[-b^{-1} - (-1)^{-1}\right]$$

$$= \lim_{b\to\infty} \left[-\frac{1}{b} + 1\right]$$

$$= \lim_{b\to\infty} \left(1 - \frac{1}{b}\right)$$

$$= 1 - \lim_{b\to\infty} \left(\frac{1}{b}\right)$$

$$= 1 - \frac{1}{\infty}$$

$$= 1 - 0$$

$$= 1$$

So, it is converges.

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Given,
$$\int_1^{\infty} \frac{1}{x}\,dx$$

It is improper integral. so, we have to take limit for solving this

$$I = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} [\log x]_1^b$$

$$= \lim_{b\to\infty} (\log b - \log 1)$$

$$= \lim_{b\to\infty} \log b \quad [\because \log 1 = 0)$$

$$= \infty$$

so, it is diverges.

3(b) IF $f(x,y) = xy/(x^2+y^2)$, does $f(x,y)$ exist, as $(x,y)\to(0,0)$?

Sol

Given function,
$$f(x,y) = \frac{xy}{(x^2+y^2)}$$

Given limit as $(x,y)\to(0,0)$

at $(x,y) = (0,0)$, we have $f(x,y) = 0/0$

Let us consider $y = mx$, where m is finite constant

Then,

$$f(x,y) = \frac{x \cdot mx}{x^2+(mx)^2} = \frac{mx^2}{x^2+m^2x^2}$$

Taking given limit $(x,y) \to (0,0)$ in $f(x,y)$

$$= \lim_{(x,y)\to(0,0)} \frac{mx^2}{x^2+m^2x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2 m}{x^2(1+m^2)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{m}{1+m^2}$$

$$= \frac{m}{1+m^2} \neq 0$$

Since, the limit of the given function gives same constant value, so the given function doesn't exist.

(c) A particle moves in a straight line and has acceleration given by $a(t) = 6t^2+1$. Its initial velocity is 4 m/sec and its initial displacement is $s(0) = 5$ cm. Find its position function $s(t)$.

Sol. Given,

Acceleration for a particle is,

$$a(t) = 6t^2+1$$

i.e. $s''(t) = 6t^2+1$

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Integrating we get, velocity function

$v(t) = s'(t) = 6\frac{t^3}{3} + t + c$

$s'(t) = 2t^3 + t + c$ ---- (i)

Again,

integrating to get position ~~position~~,

$s(t) = 2\frac{t^4}{4} + \frac{t^2}{2} + ct + d$ —— (iii)

$s(t) = \frac{t^4}{2} + \frac{t^2}{2} + ct + d$ --- (ii)

Given, initial velocity, $v(0) = 4m/sec$

i.e $s'(0) = 4$

or, $2t^3 + t + c = 4$ (i.e $s'(0)$, $t = 0$)

$\therefore c = 4$

Again,

$s(0) = 5$ (i.e $t = 0$)

or, $\frac{t^4}{2} + \frac{t^2}{2} + 4t + d = 5$ ($\because c = 4$)

$\therefore d = 5$ [$\because t = 0$]

Substitute the value of c and d in eq^n (ii) we get the position function $s(t)$

$\therefore s(t) = \frac{t^4}{2} + \frac{t^2}{2} + 4t + 5$ Ans

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4(a) Evaluate $\int_{-3}^{2}\int_{0}^{\pi/2}(y+y^2\cos x)\,dx\,dy$

Sol Given,

$\int_{-3}^{2}\int_{0}^{\pi/2}(y+y^2\cos x)\,dx\,dy$ $\left[\because \int_a^b\int_c^d f(x,y)\,dx\,dy = \int_c^d\int_a^b f(x,y)\,dy\,dx\right]$ Fubini's theorem

$= \int_0^{\pi/2}\left[\int_{-3}^{2}(y+y^2\cos x)\,dy\right]dx$

$= \int_0^{\pi/2}\left\{\left[\frac{y^2}{2}\right]_{-3}^{2} + \cos x\left[\frac{y^3}{3}\right]_{-3}^{2}\right\}dx$

$= \int_0^{\pi/2}\left\{\left(\frac{4}{2}-\frac{9}{2}\right) + \cos x\left(\left(\frac{8}{3}\right)-\frac{(-3)^3}{3}\right)\right\}dx$

$= \int_0^{\pi/2}\left\{-\frac{5}{2} + \cos x\left(\frac{8+27}{3}\right)\right\}dx$

$= \int_0^{\pi/2}\left(\cos x\cdot\frac{35}{3} - \frac{5}{2}\right)dx$

$= \frac{35}{3}\left[\sin x\right]_0^{\pi/2} - \left[\frac{5}{2}\cdot x\right]_0^{\pi/2}$

$= \frac{35}{3}\left(\sin\frac{\pi}{2} - \sin 0\right) - \frac{5}{2}\times\frac{\pi}{2}$

$= \frac{35}{3} - \frac{5\pi}{4}$ Ans

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(b) Find the Maclaurin's series for $\cos x$ and prove that it represents $\cos x$ for all x.

Sol. Given,

the given function is,

$f(x) = \cos x$ ---- (1)

we have to find Maclaurin's series generated by the function $f(x) = \cos x$ for this we need to find Taylor's series generated by the function.

$f(x) = \cos x$ about $x=0$ (i.e $x = a = 0$)

Diffⁿ eqⁿ (1) with respect to 'x' successively.

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{iv}(x) = \cos x$

$f^{v}(x) = -\sin x$

$f^{vi}(x) = -\cos x$

rearrangement all derivative as bellow

$f'(x) = -\sin x$

$f'''(x) = \sin x$

$f^{v}(x) = -\sin x$

⋮

$f^{2n+1}(x) = (-1)^n \sin x$

$f''(x) = -\cos x$

$f^{iv}(x) = \cos x$

$f^{vi}(x) = -\cos x$

⋮

$f^{2n}(x) = (-1)^n \cos x.$

at $x=0$,

$$f(0) = \cos(0) = 1$$

$f'(0) = -\sin 0 = 0$ | $f''(0) = -\cos 0 = -1$

$f'''(0) = \sin 0 = 0$ | $f^{iv}(0) = \cos 0 = 1$

$f^{v}(0) = -\sin 0 = 0$ | $f^{vi}(0) = -\cos 0 = -1$

$\vdots$

$f^{2n+1}(0) = (-1)^n \times 0 = 0$ | $f^{2n}(0) = (-1)^n$

We know that,

$$f(x) = f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \cdots + \frac{f^{n}(0)}{n!}x^n + \cdots$$

$$\cos x = 1 + 0 - \frac{1 \times x^2}{2!} + 0 + \frac{1 \times x^4}{4!} + \cdots + (-1)^n \times \frac{x^{2n}}{(2n)!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \times \frac{x^{2n}}{(2n)!}$$

Since, the cosine function (i.e $f(x) = \cos x$) and all the derivative of cosine function (i.e $f'(x), f''(x), f'''(x), \ldots$) have value less than or equal to 1 (i.e value ≥ 1)

So, by Taylor's inequality

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}$$

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$$|R_{2n}(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!}$$

Here, we have to take limit $n \to \infty$
then,

$$\lim_{n\to\infty} \frac{|x|^{2n+1}}{(2n+1)!}$$
$$= 0$$

i.e

$$\lim_{n\to\infty} R_{2n}(x) = 0 \quad \text{for all } x$$

Which shows that given series is converges to $\cos x$ for every value of x.

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5 If $f(x) = x^2 - 1$, $g(x) = 2x + 1$, find fog and gof domain of fog.

Sol Given,

$f(x) = x^2 - 1$,

$g(x) = 2x + 1$

$fog = ?$

$gof = ?$

For $fog = f(g(x))$

$= f(2x+1)$

$= (2x+1)^2 - 1$

$= 4x^2 + 4x + 1 - 1$

$= 4x^2 + 4x$

$= 4x(x+1)$

For $gof = g(f(x))$

$= g(x^2 - 1)$

$= 2(x^2 - 1) + 1$

$= 2x^2 - 2 + 1$

$= 2x^2 - 1$

For domain of fog,

since $fog(x) = 4x(x+1) = 4x^2 + 4x$

since this is polynomial function. so that it is continuous on every point. so, its domain is

Domain $(-\infty, \infty)$

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6. Define Continuity of a function at a point $x=a$. Show that the function $f(x)=\sqrt{1-x^2}$ is continuous on the interval $[-1,1]$

Sol: 1st part:-

A Function $f(x)$ is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$ (Functional value = limiting value)

Precisely, a function $f(x)$ is continuous at $x=a$ if

(a) $f(a)$ is defined (i.e a is in the domain of f)

(b) $\lim_{x\to a} f(x)$ exists.

(c) $\lim_{x\to a} f(x) = f(a)$

2nd part.

Given function,

$$f(x)=\sqrt{1-x^2} \text{ on } [-1,1] = [a,b]$$

(a) For $-1<c<1$ (i.e $a<c<b$)

$$\therefore \lim_{x\to c} f(x) = \lim_{x\to c} \left(\sqrt{1-x^2}\right)$$

$$= \sqrt{1-c^2} \text{ exist on } (-1,1)$$

Also, $f(c) = \sqrt{1-c^2}$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

$\therefore$ $f(x)$ is continuous at $c\in(-1,1)$

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(b) At left end point $a = -1$

$$\lim_{x \to (-1)^+} f(x)$$

$$= \lim_{x \to (-1)^+} \sqrt{1-x^2}$$

$$= \sqrt{1-(-1)^2}$$

$$= 0 \text{ exist on } [-1, 1]$$

$\therefore$ $f(x)$ is continuous at $a = -1$

(c) Right end point $b = 1$

$$\lim_{x \to 1^-} f(x)$$

$$= \lim_{x \to 1^-} \sqrt{1-x^2}$$

$$= \sqrt{1-1^2}$$

$$= 0 \text{ exist on } [-1, 1]$$

$\therefore$ $f(x)$ is continuous at point $b = 1$

From above condition (b) and (c), it shows that $f(x)$ is continuous on closed interval of $[-1, 1]$

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7. State Rolle's theorem and verify the Rolle's theorem for $f(x) = x^3 - x^2 - 6x + 2$ in $[0,3]$

Sol?

1st part

Let f be a function that satisfies the following three hypotheses:

(i) f is continuous on the closed interval $[a,b]$

(ii) f is differentiable on the open interval (a,b)

(iii) $f(a) = f(b)$

Then there is a number c in (a,b) such that $f'(c) = 0$

2nd part

Given,

$$f(x) = x^3 - x^2 - 6x + 2 \quad \text{in } [0,3]$$

(i) Since $f(x)$ is a polynomial function. so, it is continuous on the closed interval $[0,3]$

(ii) $f'(x) = 3x^2 - 2x - 6$ which exists for all $x \in (0,3)$

$\therefore$ $f(x)$ is differentiable in $(0,3)$

(iii) $f(0) = 2$

$f(3) = 3^3 - 3^2 - 6 \times 3 + 2 = 2$

$\therefore$ $f(0) = f(3)$

$\therefore$ all the condition of Rolle's theorem are satisfied

Hence, there exists at least a point $c \in (0,3)$ such that

$$f'(c) = 0$$

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or, $3c^2 - 2c - 6 = 0$ $\quad (\because f'(x) = 3x^2 - 2x - 6)$

comparing with $ax^2 + bx + c$

$$a = 3, \; b = -2, \; c = -6$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{4 + 72}}{2 \times 3}$$

$$x = \frac{2 \pm \sqrt{76}}{6}$$

$$x = \frac{2}{6} \pm \cancel{\frac{2\sqrt{19}}{6}} \quad \frac{2\sqrt{19}}{6}$$

$$x = \frac{1}{3} \pm \cancel{\frac{\sqrt{19}}{3}} \quad \frac{\sqrt{19}}{3}$$

Taking (+ve)

$$x = \frac{1}{3} + \frac{\sqrt{19}}{3} = 1.78 \in (0,3)$$

Taking (-ve)

$$x = \frac{1}{3} - \frac{\sqrt{19}}{3} = -1.11 \notin (0,3)$$

Hence, Rolle's theorem verified and $\frac{1 + \sqrt{19}}{3} \in (0,3)$ Ans

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8. Find the third approximation x_3 to the root of the equation $f(x) = x^3 - 2x - 7$, setting $x_1 = 2$

Sol. Given,

$$f(x) = x^3 - 2x - 7 \quad \text{--- (1)} \qquad x_1 = 2$$

diffⁿ eqⁿ (1) w.r. to 'x'

$$f'(x) = 3x^2 - 2$$

By Newton's method formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

when $x_1 = 2$,

$$x_{1+1} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2 - \frac{(2^3 - 2\times 2 - 7)}{(3\times 2^2 - 2)}$$

$$x_2 = 2 - \left(\frac{-3}{10}\right)$$

$$x_2 = 2.3$$

Again,

$$x_{2+1} = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 2.3 - \frac{(-7.573)}{13.87}$$

$$\therefore x_3 = 2.845$$

$\therefore$ The third approximation x_3 is 2.845 Ans

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9. Find the derivative of $r(t) = (1+t^2)i - te^{-t}j + \sin 2t\, k$ and find the unit tangent vector at $t=0$

Sol Given,

$$r(t) = (1+t^2)i - te^{-t}j + \sin 2t\, k \quad \text{--- (1)}$$

diffⁿ eqⁿ (1) w.r.to 't'

$$\frac{d(\vec{r}(t))}{d(t)} = \frac{d\{(1+t^2)i\}}{dt} - \frac{d(t.e^{-t})j}{dt} + \frac{d(\sin 2t)k}{dt}$$

$$= 2ti - (t.(-1)e^{-t} + e^{-t}.1)j + \frac{d(\sin(2t))}{d(2t)} \cdot \frac{d(2t)}{\cancel{d(t)}dt} k$$

$$= 2ti + (te^{-t} - e^{-t})j + \cos 2t \times 2k$$

$$= 2ti + e^{-t}(t-1)j + 2\cos 2t\, k$$

$$\therefore \frac{d\vec{r}}{dt} = 2ti + (t-1)e^{-t}j + 2\cos 2t\, k$$

At $t=0$,

$$r(0) = 1i - 0j + 0.k = 1i$$

$$\# \frac{d\vec{r}}{dt} = r'(0) = 0 + (0-1).1j + 2\times 1k = -1j + 2k$$

Since by using formula,

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

$$T(0) = \frac{-1j+2k}{\sqrt{(-1)^2+(2)^2}} = \frac{-1j+2k}{\sqrt{5}} = \frac{-1}{\sqrt{5}}j + \frac{2}{\sqrt{5}}k \quad \text{Ans}$$

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10. Find the volume of the solid obtained by rotating about the y-axis the region between $y=x$ and $y=x^2$

Sol: Here,

The given Parabola are $y=x$ and $y=x^2$

Solving these parabola to obtained the limiting value of x.

$$x^2 = x$$

or, $x - x^2 = 0$

or, $x(1-x) = 0$

$\therefore x = 0, 1$

Hence,

$$\text{Volume}(V) = \int_a^b 2\pi(x)\cdot f(x)\,dx$$

$$= \int_0^1 2\pi x\cdot(x - x^2)\,dx = 2\pi\int_0^1 (x^2 - x^3)\,dx = 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= 2\pi\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= 2\pi\left(\frac{4-3}{12}\right)$$

$$= \frac{2\pi\times 1}{12}$$

$$= \frac{\pi}{6}$$

$\therefore$ The volume of the solid obtained by rotating about y-axis is $\frac{\pi}{6}$ Ans

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11 Solve: $y''+y'=0$, $y(0)=5$, $y(\pi/4)=3$

Sol

Sol The given 2nd order homogenous linear eqn is

$$y''+y'=0 \quad \text{---} \quad (1)$$

Let

$y=e^{mx}$ be the reqn soln of eqn (1). We get an auxilarly eqn of (1) by replacing y'' by m^2 and y' by m

$$m^2+m=0$$

or, $m(m+1)=0$

$\therefore m=0,-1$

Let $m_1=0$, $m_2=-1$ which is real and unequal.

The reqn soln is

$$y=c_1e^{m_1x}+c_2e^{m_2x}$$

$$y=c_1e^{0\cdot x}+c_2e^{-1\cdot x}$$

$$y=c_1+c_2e^{-x} \quad \text{----} \quad (11)$$

Since, $y(0)=5$

if $x=0$, $y=5$ then eqn (11) becomes

$$5=c_1+c_2$$

$$c_1=5-c_2 \quad \text{---} \quad (111)$$

Again, $y(\pi/4)=3$

if $x=\frac{\pi}{4}$, $y=3$ then eqn (11) becomes

$$3=c_1+c_2e^{-\pi/4}$$

or, $3=5-c_2+c_2e^{-\pi/4}$

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$$3-5 = c_2 e^{-\pi/4} - c_2$$

or, $$-2 = c_2 (e^{-\pi/4} - 1)$$

$$\therefore c_2 = \frac{(-2)}{(e^{-\pi/4} - 1)}$$

Substuting the value of c_2 in eqⁿ (iii)

$$c_1 = 5 - \frac{(-2)}{e^{-\pi/4} - 1}$$

$$c_1 = \frac{5e^{-\pi/4} - 5 + 2}{e^{-\pi/4} - 1}$$

$$c_1 = \frac{5e^{-\pi/4} - 3}{e^{-\pi/4} - 1}$$

Now, replacing the value of c_1, c_2 in eqⁿ (ii)

$$y = \frac{5e^{-\pi/4} - 3}{e^{-\pi/4} - 1} + \left(\frac{-2}{e^{-\pi/4} - 1} \right) e^{-x}$$ Ans

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12. Show that the series $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ converges.

Sol: Given series is

$$\sum a_n = \sum_{n=0}^{\infty} \frac{1}{1+n^2}$$

$$\therefore a_n = \frac{1}{1+n^2}$$

$$a_n = \frac{1}{n^2(1/n^2+1)}$$

Take

$$b_n = \frac{1}{n^2}$$

clearly, this series is convergent by p-series test with $p=2$ i.e $p>1$

Being $\sum b_n$ is convergent, by limit comparison test the series $\sum a_n$ also convergent if

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = l \cdot a \text{ finite value.}$$

Here, $\lim_{n\to\infty} \frac{\left(\frac{1}{1+n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty}\left(\frac{\frac{1}{n^2(1/n^2+1)}}{1/n^2}\right) = \lim_{n\to\infty}\left(\frac{1}{\frac{1}{n^2}+1}\right)$

$$= \lim_{n\to\infty}\left(\frac{1}{1/n^2+1}\right) = \frac{1}{1/\infty+1} = \frac{1}{0+1}$$

$= 1$ which is a finite non-zero value.

$\therefore$ Then by limit comparison test, the given series $\sum a_n$ is converges.

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13. Find a vector perpendicular to the plane that passes through the points: $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$

Sol. Given points are
$R(1,4,6)$, $Q(-2,5,-1)$ and $R(1,-1,1)$

For perpendicular vector,

$$\vec{PQ} = (1,4,6)-(-2,5,-1)$$
$$= (3,-1,7)$$

$$\vec{PR} = (1,4,6) - (1,-1,1)$$
$$= (0,5,5)$$

The cross product of $\vec{PQ}$ and $\vec{PR}$ is

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 3 & -1 & 7 \\ 0 & 5 & 5 \end{vmatrix}$$

$$= (-5-35)i - (15-0)j + (15-0)k$$
$$= -40i - 15j + 15k$$
$$= (-40, -15, 15)$$

Since $(-40, 15, 15)$ is divisible by 5. so, $(-8,-3,3)$ is vector perpendicular to plane.

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14. Find the partial derivative of $f(x,y) = x^3 + 2x^2y^3 - 3y^2 + x + y$ at $(2,1)$

Sol. Given,

$$f(x,y) = x^3 + 2x^2y^3 - 3y^2 + x + y \quad \text{--- (1)}$$

we have to find $\frac{\delta f}{\delta x}$. For this keeping y as constant and differentiate $f(x,y)$ with respect to x then,

$$\frac{\delta f}{\delta x} = \frac{\delta}{\delta x}(x^3 + 2x^2y^3 - 3y^2 + x + y)$$

$$= 3x^2 + 4xy^3 - 0 + 1 + 0$$

$$= 3x^2 + 4xy^3 + 1$$

At point $(2,1)$

$$\left.\frac{\delta f}{\delta x}\right|_{(2,1)} = 3 \times 2^2 + 4 \times 2 \times 1^3 + 1$$

$$= 3 \times 4 + 8 + 1$$

$$= 12 + 8 + 1$$

$$= 21 \text{ Ans}$$

Again, we have to find $\frac{\delta f}{\delta y}$ for this keeping x as constant and differentiate $f(x,y)$ with respect to y. then,

$$\frac{\delta f}{\delta y} = \frac{\delta}{\delta y}(x^3 + 2x^2y^3 - 3y^2 + x + y)$$

$$= 0 + 6x^2y^2 - 6y + 0 + 1$$

$$= 6x^2y^2 - 6y + 1$$

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At point (2,1)

$$\left.\frac{\delta f}{\delta y}\right|_{(2,1)} = 6x^2y^2 - 6y + 1$$

$$= 6 \times 2^2 \times 1^2 - 6 \times 1 + 1$$

$$= 24 - 6 + 1$$

$$= 19$$ Ans

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For better understand of 2077 question solution I will suggest you to watch my video on youtube only in GUPTA TUTORIAL Channel

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15. Find the local maximum and minimum values, Saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$

Sol Given Function,

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

so,

$f_x = 4x^3 - 4y$ $\quad$ $f_y = 4y^3 - 4x$

$f_{xx} = 12x^2$ $\quad$ $f_{yy} = 12y^2$

$f_{xy} = -4$ $\quad$ $f_{yx} = -4$

For stationary point

$f_x = 0$ $\quad$ $f_y = 0$

$4x^3 - 4y = 0$ $\quad$ $4y^3 - 4x = 0$

$x^3 = y$ ---- (i) $\quad$ $y^3 = x$ ---- (ii)

from eq^n (i) and (ii)

$$(x^3)^3 = x \quad (\because y = x^3)$$

or, $x^9 = x$

or, $x^9 - x = 0$

or, $x(x^8 - 1) = 0$

$\therefore x = 0, x = 1$

substitute the point value of y in (i)

when $x = 0$, $y = 0$

" $x = 1$, $y = 1$

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At point (0,0)

$f_{xx} = 0$

$f_{yy} = 0$

Then,

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = 0-16 = -16 < 0$$

so, it has saddle point at (0,0)

At point (1,1)

$f_{xx} = 12$, $f_{yy} = 12$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yx} \end{vmatrix} = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 144-16 = 128 > 0$$

So, it has minimum value at ~~(1,1)~~ (1,1)

$\therefore$ Minimum value is

$$f(1,1) = 1^4 + 1^4 - 4\times1\times1 + 1$$
$$= 1+1-4+1$$
$$= -1$$

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