Chapter 15
周期性非正弦稳态电路

15.1 周期性函数的傅里叶级数
Trigonometric Fourier Series

15.2 平均功率和有效值
Average Power and rms Values

15.4 周期性非正弦电源激励下的稳态响应
Steady-state Response under Nonsinasoidal Input

目标：

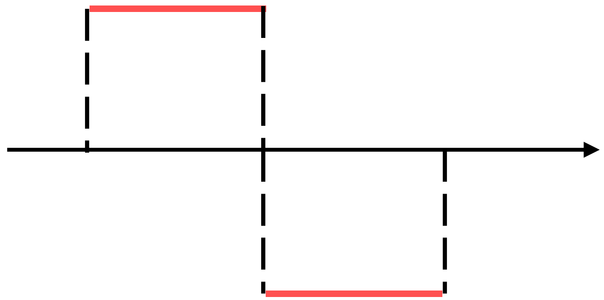
利用傅里叶级数和叠加原理计算周期电源下的稳态响应

15. 1 概述

周期性非正弦稳态响应：周期性非正弦电源激励下的稳态响应。

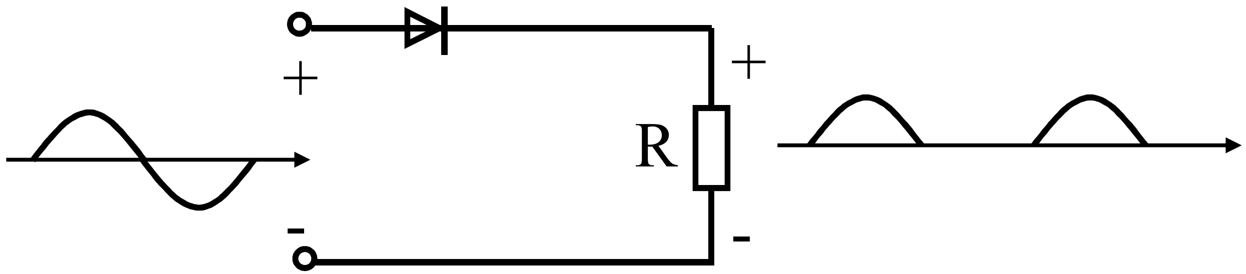
1 电路中产生周期性非正弦变化电压、电流的原因

- 电源提供的电压或电流是非正弦周期变化的



- 一个电路中有两个或两个以上不同频率的电源作用

- 电路中含有非线性元件



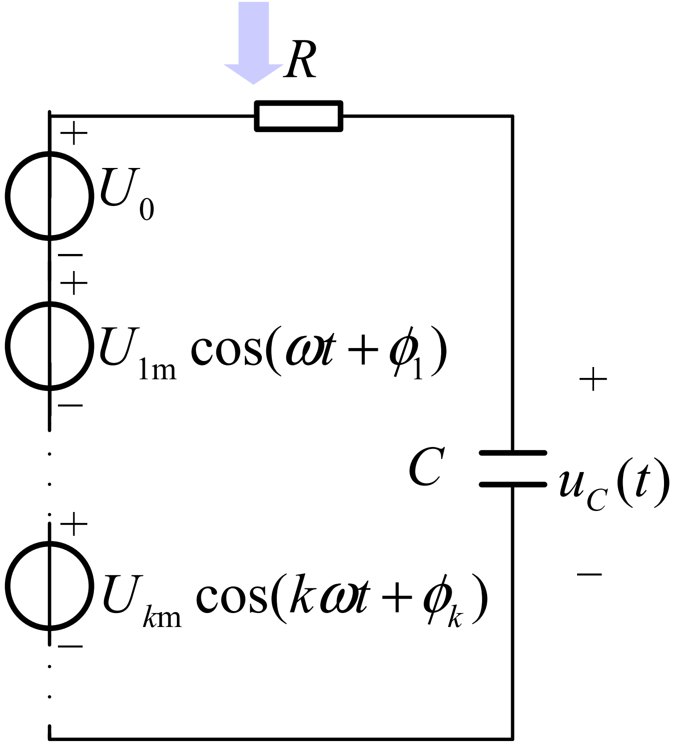
15. 1 概述

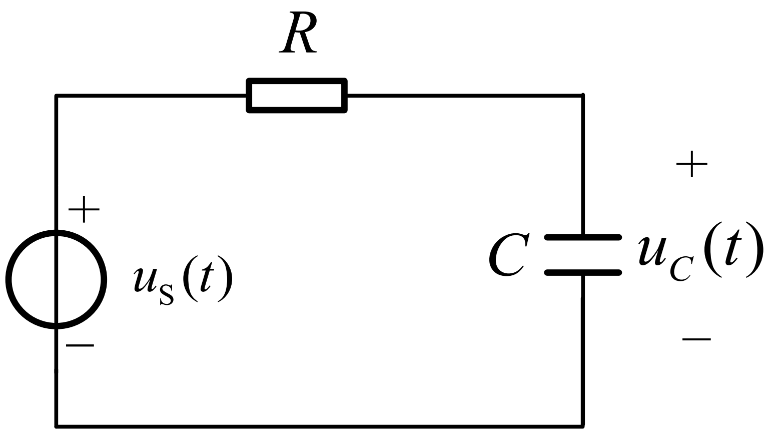
周期性非正弦稳态响应：周期性非正弦电源激励下的稳态响应。

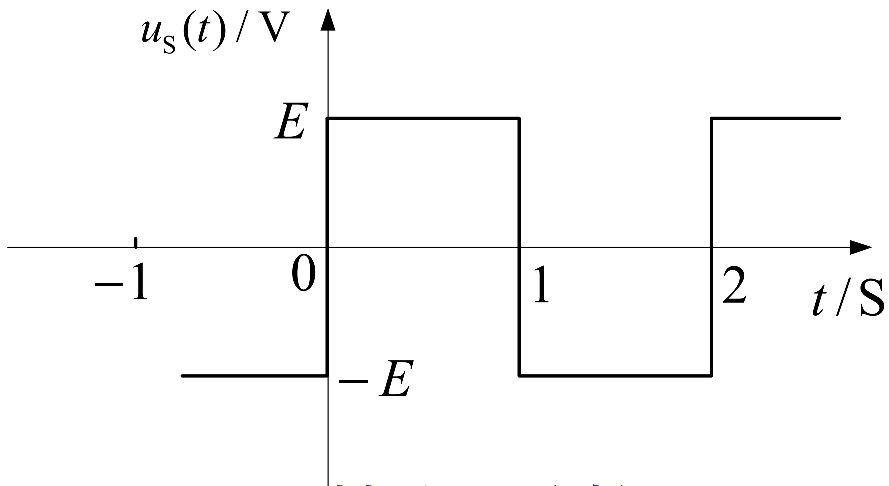
2 本章的讨论对象及处理问题的思路

➢线性时不变电路 —适用叠加定理

$$u_S(t) = U_0 + \sum_{k=1}^{\infty} U_{km}\cos(k\omega t + \phi_k)$$







周期性非正弦输入

15.2 周期性函数的傅里叶级数Fourier Series

1.周期函数 Periodic function

$$f(t)=f(t\pm T) \qquad \omega T=2\pi$$

$$f(t)=a_0+\sum_{k=1}^{\infty}(a_k\cos k\omega t+b_k\sin k\omega t) \quad =\underbrace{A_0}_{\text{dc}}+\sum_{k=1}^{\infty}\underbrace{A_{km}\cos(k\omega t+\phi_k)}_{\text{ac— harmonics}}$$

ω—基频 fundamental frequency

$k\omega$ —k次谐波频率 harmonic frequency

$$\begin{cases} a_0=\dfrac{1}{T}\displaystyle\int_0^T f(t)\mathrm{d}t \\ a_k=\dfrac{2}{T}\displaystyle\int_0^T f(t)\cos k\omega t\mathrm{d}t \\ b_k=\dfrac{2}{T}\displaystyle\int_0^T f(t)\sin k\omega t\mathrm{d}t \end{cases}$$

$$\boxed{A_{km}\angle\varphi_k=a_k-\mathrm{j}b_k}$$

$$\begin{cases} A_0=a_0 \\ A_{km}=\sqrt{a_k^2+b_k^2} \\ \phi_k=-\arctan\dfrac{b_k}{a_k} \end{cases}$$

15. 4 周期性非正弦稳态电路分析

15.4.1 有效值 u为周期性非正弦电压时，可展开为傅立叶级数：

$$u(t) = U_0 + \sum_{k=1}^{\infty} \sqrt{2} U_k \cos(k\omega t + \phi_k)$$

$$\boxed{U = \sqrt{\frac{1}{T}\int_0^T u^2 \mathrm{d}t}}$$

$$U = \sqrt{\frac{1}{T}\int_0^T [U_0 + \sum_{k=1}^{\infty} \sqrt{2} U_k \cos(k\omega t + \phi_k)]^2 \mathrm{d}t}$$

$$\frac{1}{T}\int_0^T U_0^2 \mathrm{d}t = U_0^2$$

$$\boxed{U = \sqrt{U_0^2 + \sum_1^{\infty} U_k^2}}$$

$$\frac{1}{T}\int_0^T [U_0 \cdot \sqrt{2} U_k \cos(k\omega t + \phi_k)] \mathrm{d}t = 0$$

$$\frac{1}{T}\int_0^T [\sqrt{2} U_k \cos(k\omega t + \phi_k)]^2 \mathrm{d}t = \frac{1}{T}\int_0^T 2U_k^2 \frac{1 - \cos 2(k\omega t + \phi_k)}{2} \mathrm{d}t = U_k^2$$

$$\frac{1}{T}\int_0^T [\sqrt{2} U_k \cos(k\omega t + \phi_k) \cdot \sqrt{2} U_q \cos(q\omega t + \phi_q)] \mathrm{d}t = 0 \qquad k \neq q$$

15. 4 周期性非正弦稳态电路分析

15.4.1 有效值

$$U = \sqrt{U_0^2 + \sum_{k=1}^{\infty} U_k^2}$$

周期性非正弦量的有效值

等于它的直流分量与各谐波分量有效值的平方之和的平方根。

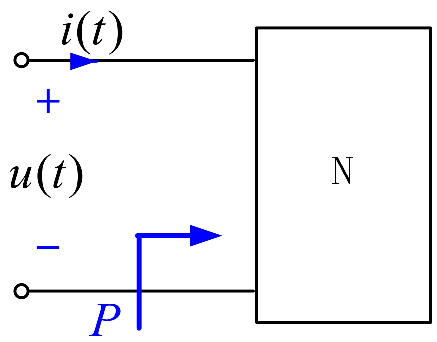
例：周期性矩形脉冲电流i(t)的傅立叶级数为下式，求其有效值。

$$i(t) = (\frac{\pi}{4} + \cos\omega_1 t - \frac{1}{3}\cos 3\omega_1 t + \frac{1}{5}\cos 5\omega_1 t - \frac{1}{7}\cos 7\omega_1 t)mA$$

解：$$I = \sqrt{I_0{}^2 + \sum_{n=1}^{\infty} I_n{}^2}$$

$$= \sqrt{\left(\frac{\pi}{4}\right)^2 + \frac{1^2}{2} + \frac{(1/3)^2}{2} + \frac{(1/5)^2}{2} + \frac{(1/7)^2}{2}} = 1.097mA$$

15. 4 周期性非正弦稳态电路分析



15.4.2. 平均功率

$$u(t)=U_0+\sum_{k=1}^{\infty}\sqrt{2}U_k\cos(k\omega t+\phi_{uk})$$

$$i(t)=I_0+\sum_{k=1}^{\infty}\sqrt{2}I_k\cos(k\omega t+\phi_{ik})$$

$$P=\frac{1}{T}\int_0^T u(t)i(t)\mathrm{d}t$$

$$=\frac{1}{T}\int_0^T\left[U_0+\sum_{k=1}^{\infty}\sqrt{2}U_k\cos(k\omega t+\phi_{uk})\right]\left[I_0+\sum_{k=1}^{\infty}\sqrt{2}I_k\cos(k\omega t+\phi_{ik})\right]dt$$

$$\frac{1}{T}\int_0^T U_0I_0\mathrm{d}t=U_0I_0=P_0 \qquad \frac{1}{T}\int_0^T U_0\sqrt{2}I_k\cos(k\omega t+\phi_{ik})\mathrm{d}t=0$$

$$\frac{1}{T}\int_0^T[\sqrt{2}U_k\cos(k\omega t+\phi_{uk})][\sqrt{2}I_k\cos(k\omega t+\phi_{ik})]\mathrm{d}t=U_kI_k\cos(\phi_{uk}-\phi_{ik})=P_k$$

$$\frac{1}{T}\int_0^T[\sqrt{2}U_k\cos(k\omega t+\phi_{uk})][\sqrt{2}I_q\cos(q\omega t+\phi_{iq})]\mathrm{d}t=0 \qquad (k\neq q)$$

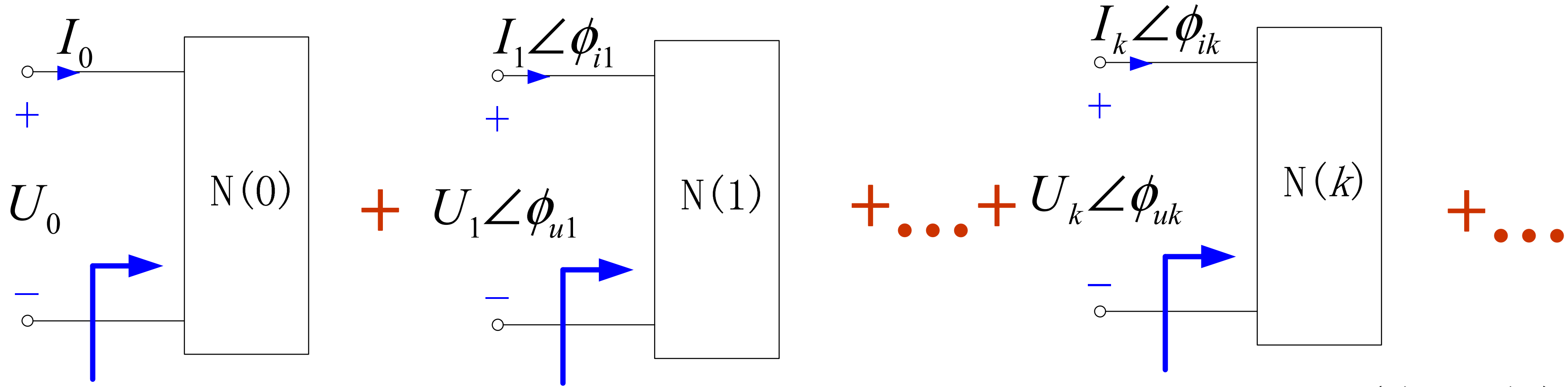
$$\boxed{P=U_0I_0+\sum_{1}^{\infty}U_kI_k\cos(\phi_{uk}-\phi_{ik})}\quad\boxed{=P_0+P_1+P_2+P_3+\cdots}$$

功率符合叠加原理

15. 4 周期性非正弦稳态电路分析

15.4.2. 平均功率

$$P=\frac{1}{T}\int_0^t u(t)i(t)dt=U_0I_0+\sum_{k=1}^{\infty}U_kI_k\cos(\phi_{uk}-\phi_{ik})=P_0+\sum_{k=1}^{\infty}P_k$$



$P_0=U_0I_0$ $\quad$ $P_1=U_1I_1\cos(\phi_{u1}-\phi_{i1})$ $\quad$ $P_k=U_kI_k\cos(\phi_{uk}-\phi_{ik})$

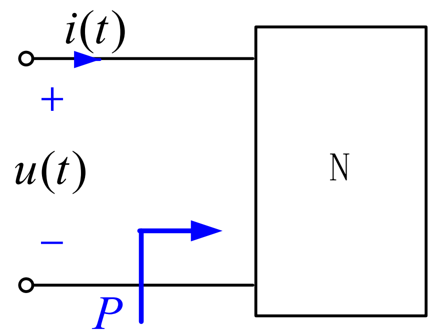
式中P_0为电压电流的直流分量产生的平均功率， P_k为电压电流的n次谐波产生的平均功率。

注意：只有相同频率的电压谐波和电流谐波才能构成平均功率，频率不同的电压谐波和电流谐波虽能构成瞬时功率，但在一周期内的平均值为0。

【例1】如图所示电路的周期电流和电压为：

$$u(t)=[50+85\sin(\omega_1 t-60°)+56.6\sin(2\omega_1 t-80°)]\text{V}$$

$$i(t)=[1-0.707\sin(\omega_1 t+70°)+0.424\sin(2\omega_1 t-40°)]\text{A}$$



解：此电路吸收的平均功率：

直流功率：$P_0=50\times1=50\text{W}$

基频功率：$P_1=\dfrac{85}{\sqrt{2}}\dfrac{0.707}{\sqrt{2}}\cos[-60°-(70°-180°)]=19.3\text{W}$

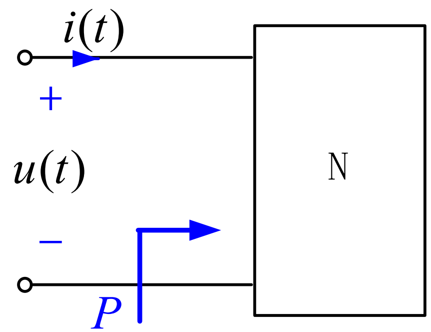
2次谐波功率：$P_2=\dfrac{56.6}{\sqrt{2}}\dfrac{0.404}{\sqrt{2}}\cos[-80°-(-40°)]=9.2\text{W}$

总功率：$P=50+19.3+9.2=78.5\text{W}$

【例2】 已知电路中某支路电压和电流分别为

$$u(t) = 20 + 100\sin 314t - 50\cos(628t + 30°) + 10\sin(1256t - 20°)\ \text{V}$$

$$i(t) = 0.1 + \cos(314t - 60°) + 0.2\cos(942t + 45°) + 0.1\cos(1256t + 10°)\ \text{A}$$



计算该支路的平均功率。

解：直流功率：$P_0 = 20 \times 0.1 = 2\text{W}$

基波功率：$P_1 = \dfrac{100}{\sqrt{2}} \times \dfrac{1}{\sqrt{2}} \times \cos(-90° + 60°) = 43.3\ \text{W}$

2次谐波功率：$P_2 = \dfrac{1}{\sqrt{2}} 50 \times 0 = 0\ \text{W}$

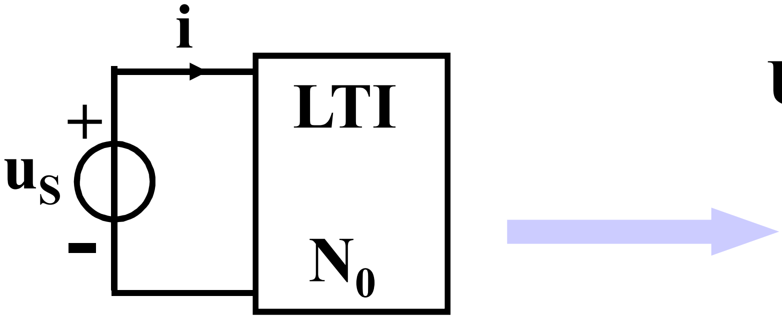
3次谐波功率：$P_3 = \dfrac{1}{\sqrt{2}} 0 \times 0.2 = 0\ \text{W}$

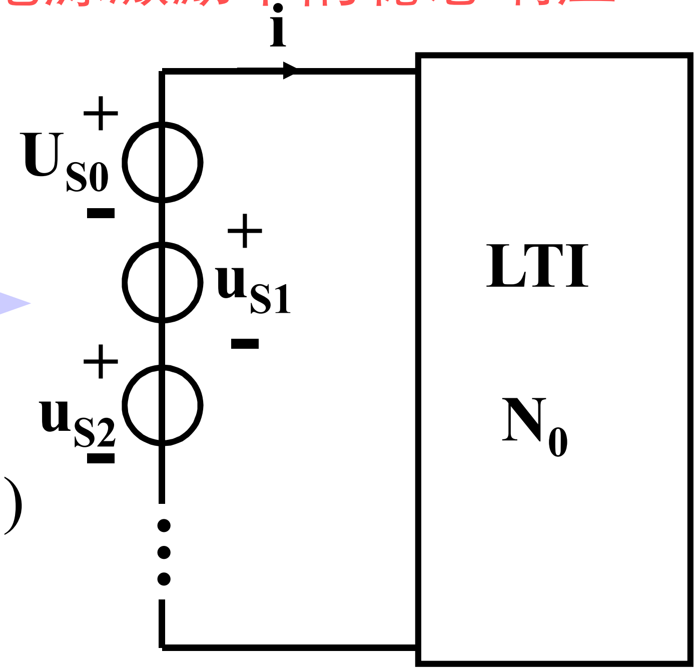
4次谐波功率：$P_4 = \dfrac{10}{\sqrt{2}} \times \dfrac{0.1}{\sqrt{2}} \times \cos(-20° - 90 - 10°) = -0.25\ \text{W}$

总功率：$P = P_0 + P_1 + P_2 + P_3 + P_4 = 2 + 43.3 - 0.25 = 45.05\ \text{W}$

15. 4 周期性非正弦稳态电路分析

15.4.3 线性电路在周期性非正弦电源激励下的稳态响应

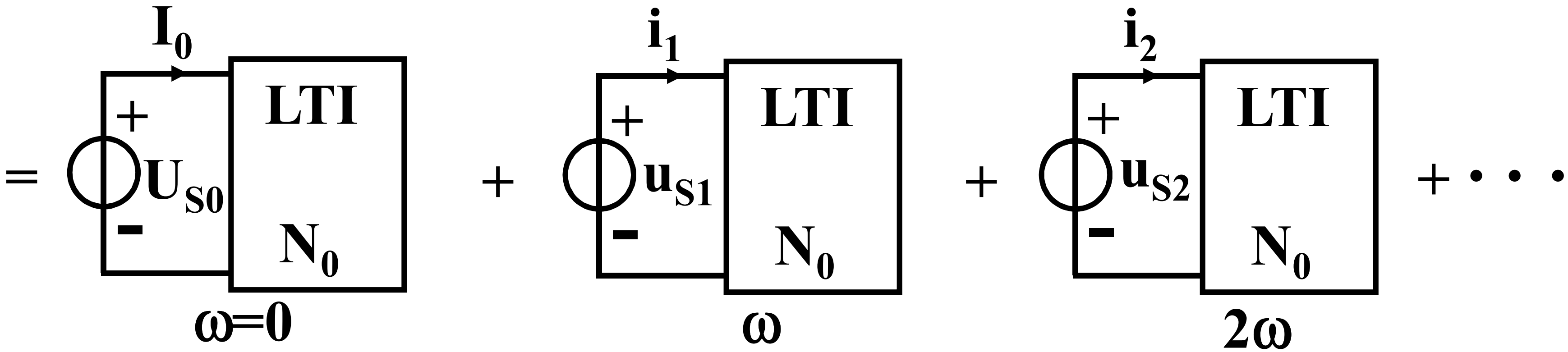




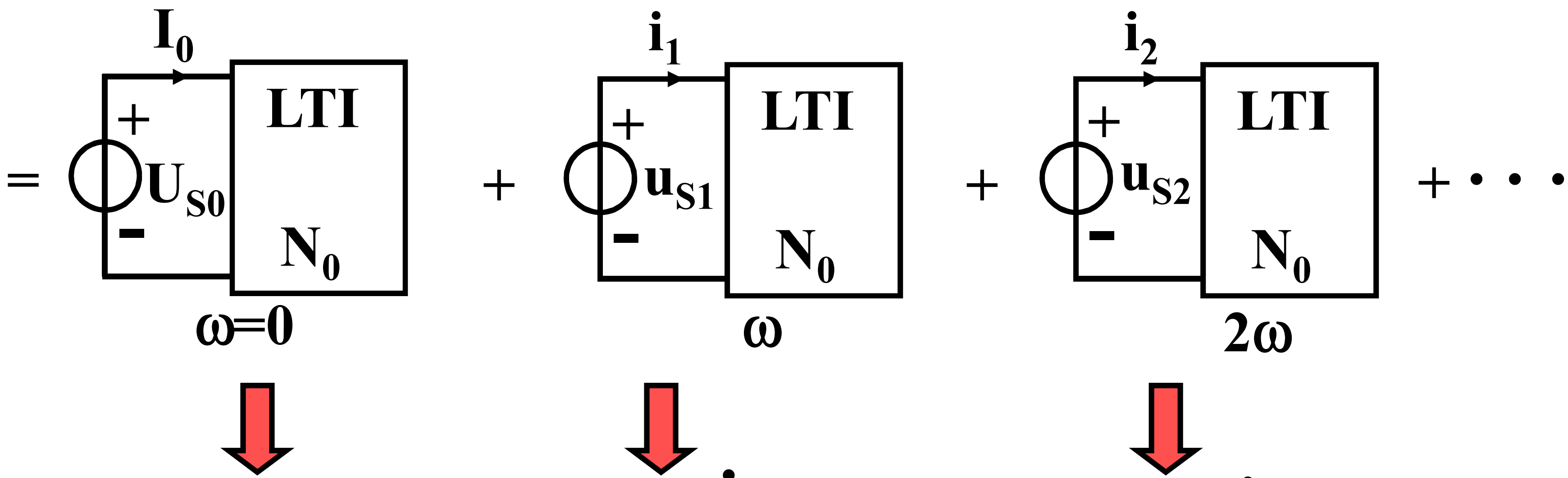
叠加定理

$$u_s(t) = U_{s0} + \sum_{k=1}^{\infty} \sqrt{2} u_{sk} \cos(k\omega t + \phi_k)$$

$$= U_{s0} + u_{s1} + u_{s2} + \cdots$$



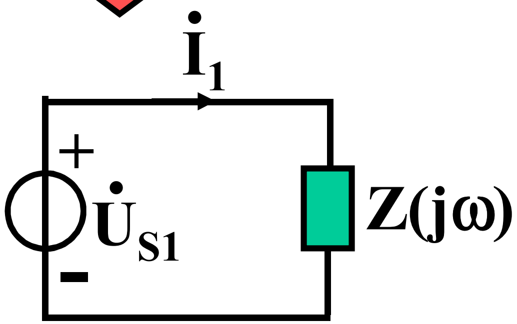
15.4.3 线性电路在周期性非正弦电源激励下的稳态响应

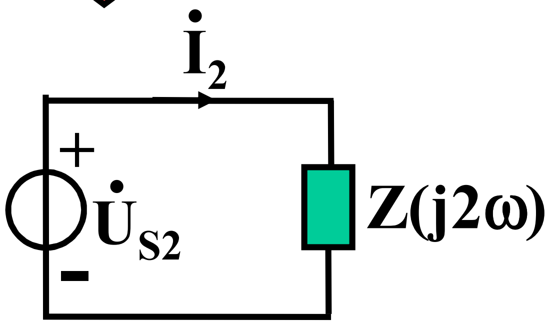


直流稳态电路

L ⇒短路

C ⇒开路



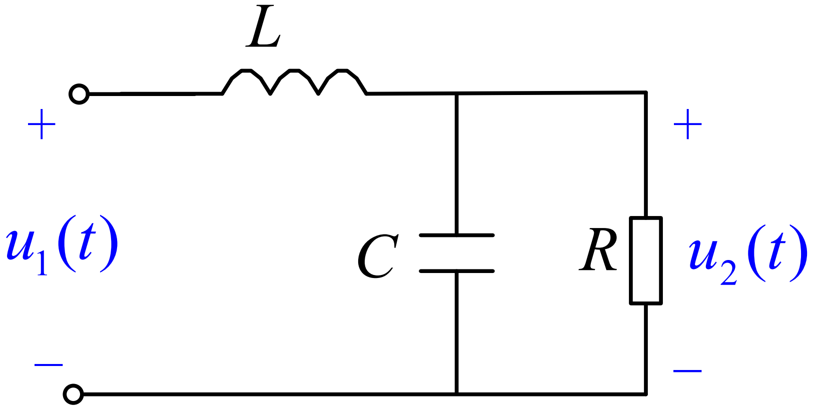


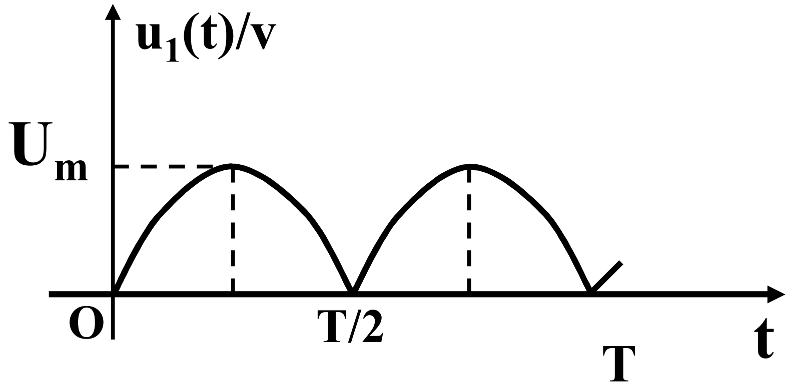
$$i(t)=I_0+i_1(t)+i_2(t)+\cdots$$

$$I=\sqrt{I_0^2+I_1^2+I_2^2+\cdots}$$

$$P=P_0+P_1+P_2+\cdots$$

【例1】 图示全波整流器的输出电压$u_1(t)$，U_m=157V，T=0.02s，通过LC滤波电路作用于负载R，L=5H，C=10μF，R= 2kΩ。求负载两端电压$u_2(t)$及其有效值。谐波电压考虑到4次谐波。





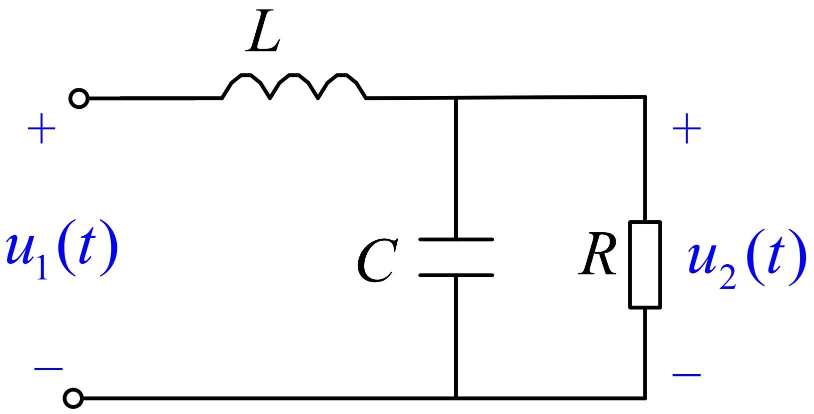
由表15-3-1，正弦全波整流波形傅里叶级数：

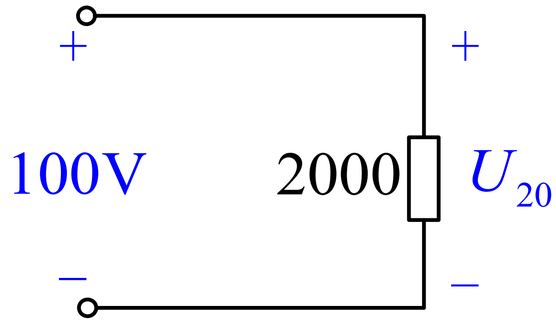
$$u_1(t)=\frac{2A}{\pi}-\frac{4A}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\cos 2k\omega t$$

$$=\frac{2\times157}{\pi}-\frac{4\times157}{\pi}\times\frac{1}{3}\cos 2\omega t-\frac{4\times157}{\pi}\times\frac{1}{15}\cos 4\omega t$$

$$\boxed{=100-66.7\cos 2\omega t-13.3\cos 4\omega t}$$

【例1】 图示全波整流器的输出电压$u_1(t)$，U_m=157V，T=0.02s，通过LC滤波电路作用于负载R，L=5H，C=10μF，R= 2kΩ。求负载两端电压$u_2(t)$及其有效值。谐波电压考虑到4次谐波。



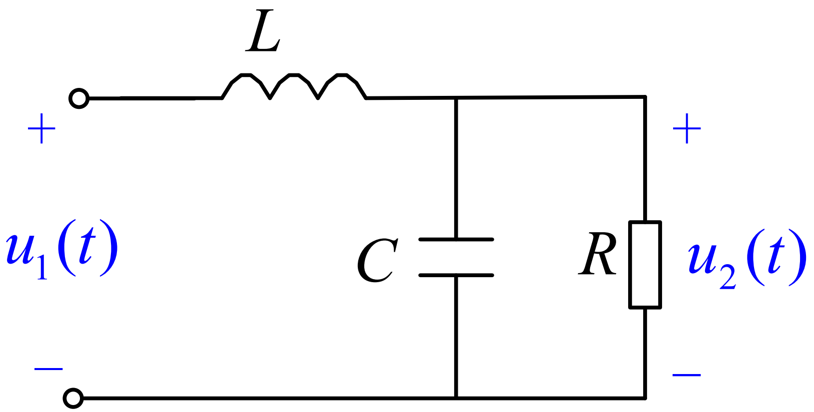


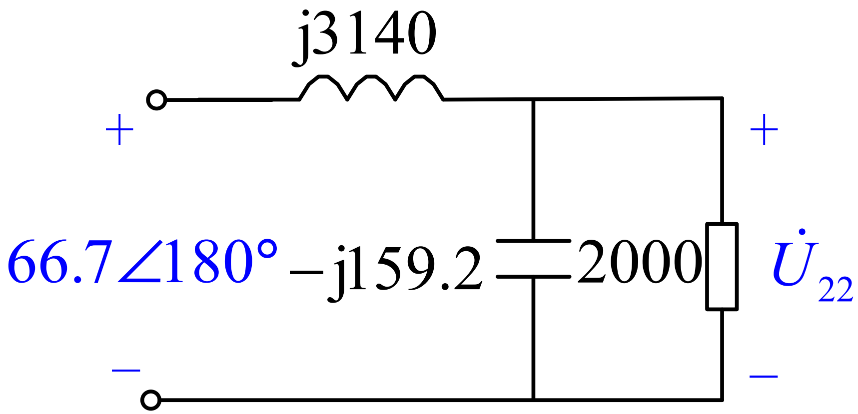
$$\boxed{u_1(t)=100-66.7\cos 2\omega t-13.3\cos 4\omega t}$$

➢直流分量单独作用：

$$U_{20}=100\text{V}$$

【例1】 图示全波整流器的输出电压$u_1(t)$，U_m=157V，T=0.02s，通过LC滤波电路作用于负载R，L=5H，C=10μF，R= 2kΩ。求负载两端电压$u_2(t)$及其有效值。谐波电压考虑到4次谐波。



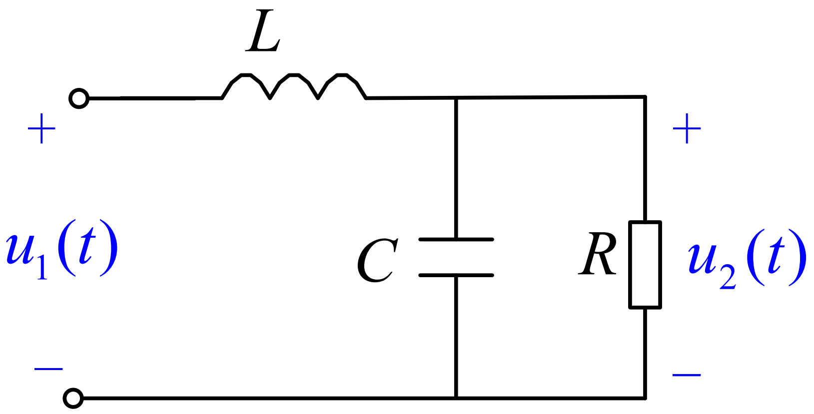


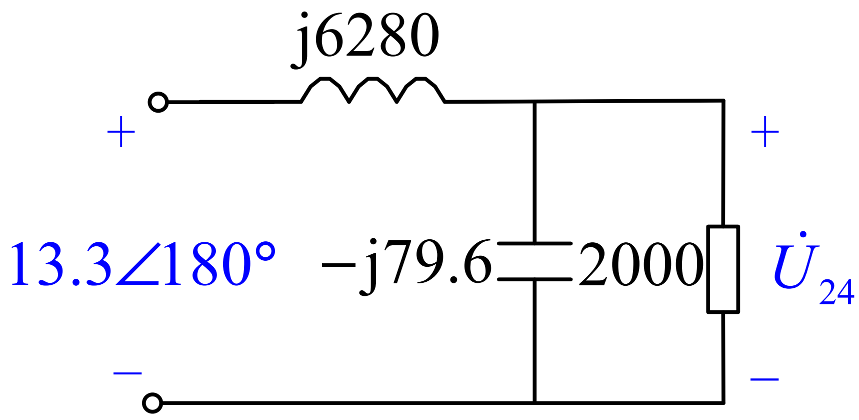
$$\boxed{u_1(t) = 100 - 66.7\cos 2\omega t - 13.3\cos 4\omega t}$$

➢二次谐波单独作用：

$$\dot{U}_{22} = \frac{66.7\angle 180°}{j3140 + (-j159.2)//2000} \times (-j159.2)//2000 = 3.55\angle 4.8°$$

【例1】 图示全波整流器的输出电压$u_1(t)$，U_m=157V，T=0.02s，通过LC滤波电路作用于负载R，L=5H，C=10μF，R= 2kΩ。求负载两端电压$u_2(t)$及其有效值。谐波电压考虑到4次谐波。



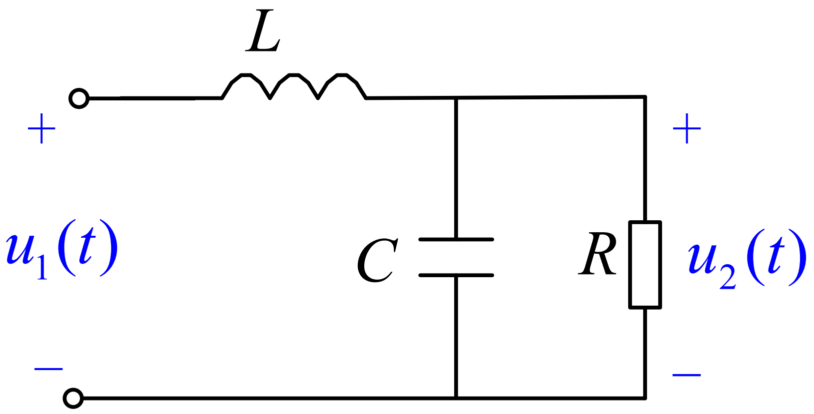


$$\boxed{u_1(t) = 100 - 66.7\cos 2\omega t - 13.3\cos 4\omega t}$$

➢四次谐波单独作用:

$$\dot{U}_{24} = \frac{13.3\angle 180°}{j6280 + (-j79.6)//2000} \times (-j79.6)//2000 \qquad \dot{U}_{24} = 0.17\angle 2.3°$$

【例1】 图示全波整流器的输出电压$u_1(t)$，U_m=157V，T=0.02s，通过LC滤波电路作用于负载R，L=5H，C=10μF，R= 2kΩ。求负载两端电压$u_2(t)$及其有效值。谐波电压考虑到4次谐波。



$$U_{20}=100\text{V}$$

$$\dot{U}_{22}=3.55\angle 4.8°$$

$$\dot{U}_{24}=0.17\angle 2.3°$$

$$u_1(t)=100-66.7\cos 2\omega t-13.3\cos 4\omega t$$

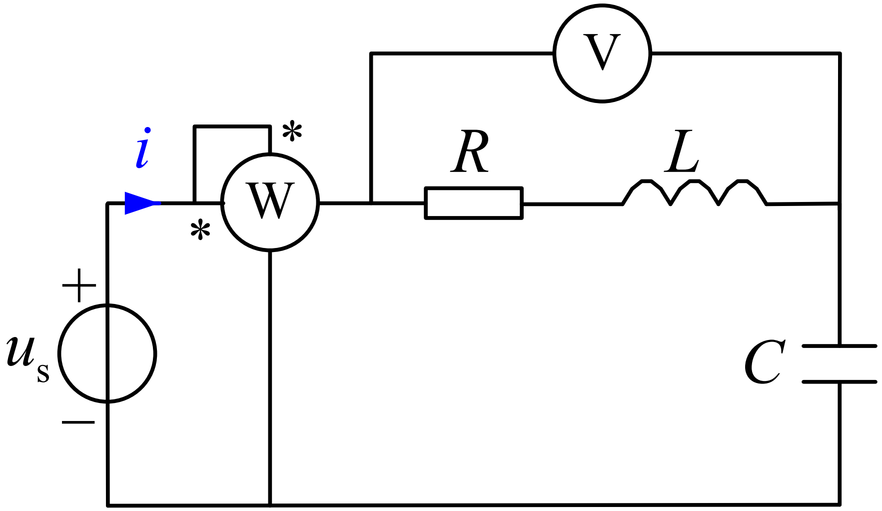
$$u_2(t)=100+3.55\cos(2\omega t+4.8°)+0.17\cos(4\omega t+2.3°)$$

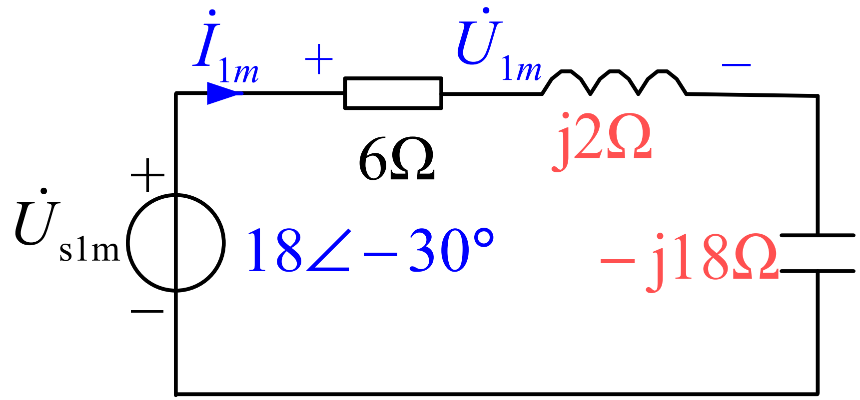
$$U=\sqrt{U_0^2+\sum_{k=1}^{\infty}U_k^2}=\sqrt{(100)^2+(\frac{3.55}{\sqrt{2}})^2+(\frac{0.17}{\sqrt{2}})^2}=100\text{V}$$

小结： 谐波阻抗 瞬时值叠加

【例2】已知 $u_s = 18\cos(\omega t - 30°) + 18\cos 3\omega t + 9\cos(5\omega t + 90°)$ V

R=6Ω， ωL=2Ω，1/ωC=18Ω，求电压表和功率表的读数。





➢基波电源单独作用:

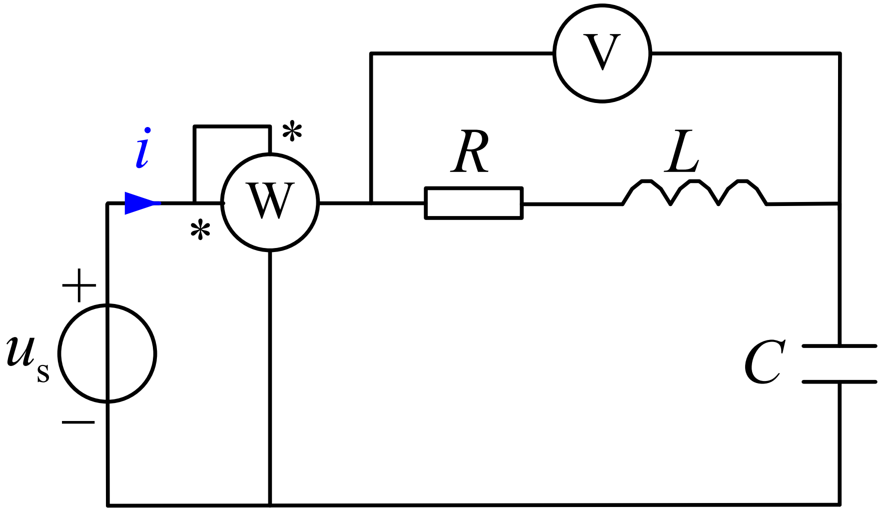
$$\dot{I}_{1m} = \frac{18\angle -30°}{6 + j2 - j18} = 1.05\angle 39.4°$$

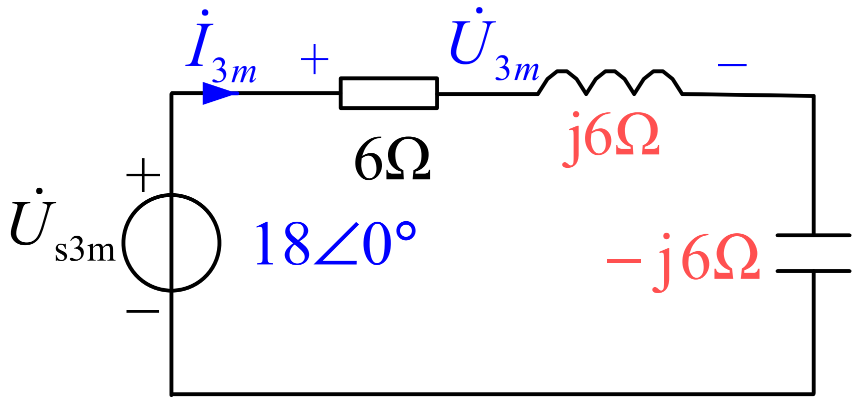
$$\dot{U}_{1m} = (6 + j2) \times 1.05\angle 39.4° = 6.64\angle 57.8°$$

$$P_1 = \frac{18}{\sqrt{2}} \times \frac{1.05}{\sqrt{2}} \cos(-30° - 39.4°) = 3.32\text{W} \qquad P_1 = I_1^2 R = 3.32\text{W}$$

【例2】已知 $u_s = 18\cos(\omega t - 30°) + 18\cos 3\omega t + 9\cos(5\omega t + 90°)$ V

R=6Ω， ωL=2Ω，1/ωC=18Ω，求电压表和功率表的读数。





➢三次谐波电源单独作用:

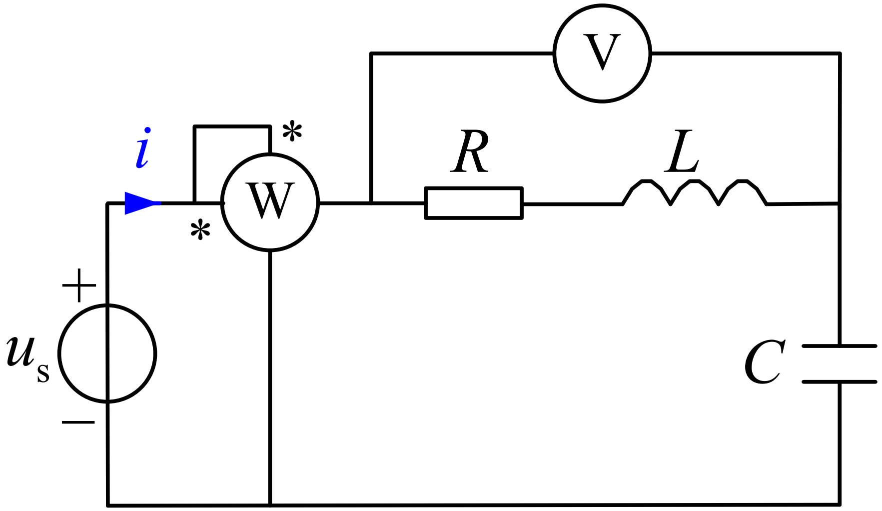
$$\dot{I}_{3m} = \frac{18\angle 0°}{6} = 3\angle 0°$$

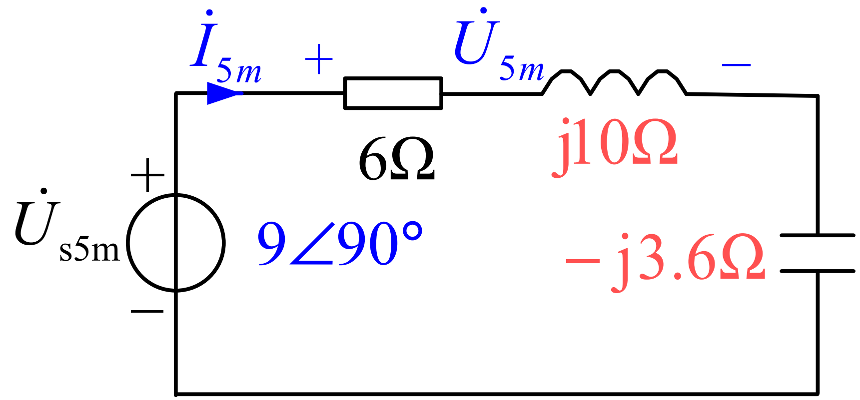
$$\dot{U}_{3m} = (6 + j6) \times 3\angle 0° = 25.5\angle 45°$$

$$P_3 = \frac{18}{\sqrt{2}} \times \frac{3}{\sqrt{2}} = 27\text{W}$$

【例2】已知 $u_s = 18\cos(\omega t - 30°) + 18\cos 3\omega t + 9\cos(5\omega t + 90°)$ V

R=6Ω， ωL=2Ω，1/ωC=18Ω，求电压表和功率表的读数。





➢五次谐波电源单独作用：

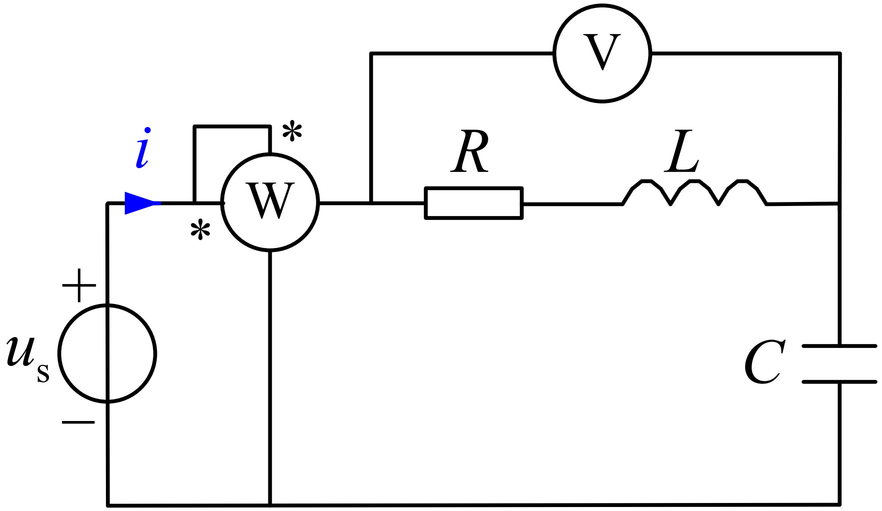
$$\dot{I}_{5m} = \frac{9\angle 90°}{6 + \mathrm{j}10 - \mathrm{j}3.6} = 1.03\angle 43.2°$$

$$\dot{U}_{5m} = (6 + \mathrm{j}10) \times 1.03\angle 43.2° = 12.1\angle 102.2°$$

$$P_5 = \frac{9}{\sqrt{2}} \times \frac{1.03}{\sqrt{2}} \cos(90° - 43.2°) = 3.17\mathrm{W}$$

【例2】已知 $u_s = 18\cos(\omega t - 30°) + 18\cos 3\omega t + 9\cos(5\omega t + 90°)$ V

R=6Ω， ωL=2Ω，1/ωC=18Ω，求电压表和功率表的读数。



$\dot{U}_{1m} = 6.64\angle 57.8°$ P_1=3.32W

$\dot{U}_{3m} = 25.5\angle 45°$ P_3=27W

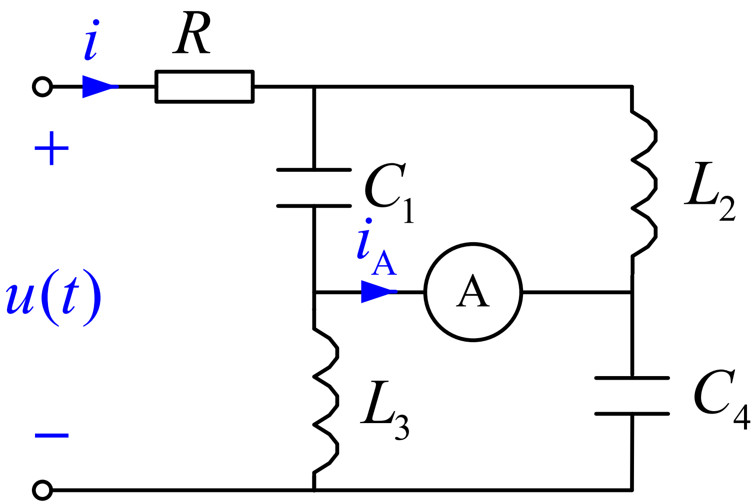
$\dot{U}_{5m} = 12.1\angle 102.2°$ P_5=3.17W

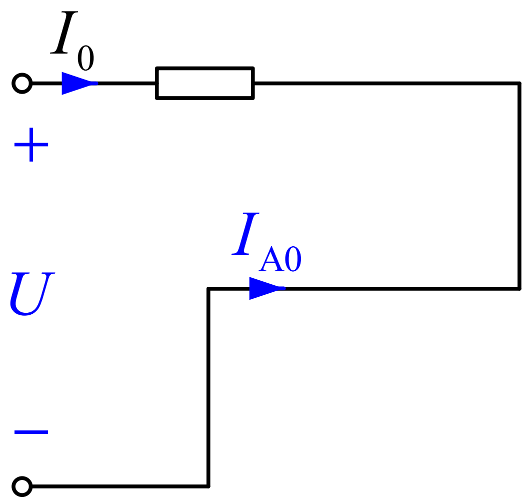
$$u(t) = 6.64\cos(\omega t + 57.8°) + 25.5\cos(3\omega t + 45°) + 12.1\cos(5\omega t + 102.2°)$$

$$U = \sqrt{U_0^2 + \sum_{k=1}^{\infty} U_k^2} = \sqrt{(\frac{6.64}{\sqrt{2}})^2 + (\frac{25.5}{\sqrt{2}})^2 + (\frac{12.1}{\sqrt{2}})^2} = 20.5\text{V}$$

$$P = P_0 + \sum_{k=1}^{\infty} P_k = P_1 + P_2 + P_3 = 3.32 + 27 + 3.17 = 33.49\text{W}$$

【例3】已知 $u(t)=60+282\sin\omega t+169\sin(2\omega t-22.5°)$ V R=10Ω，$1/\omega C_1=40\Omega$， $\omega L_2=\omega L_3=20\Omega$， $1/\omega C_4=20\Omega$，求电流表和电源提供的功率。



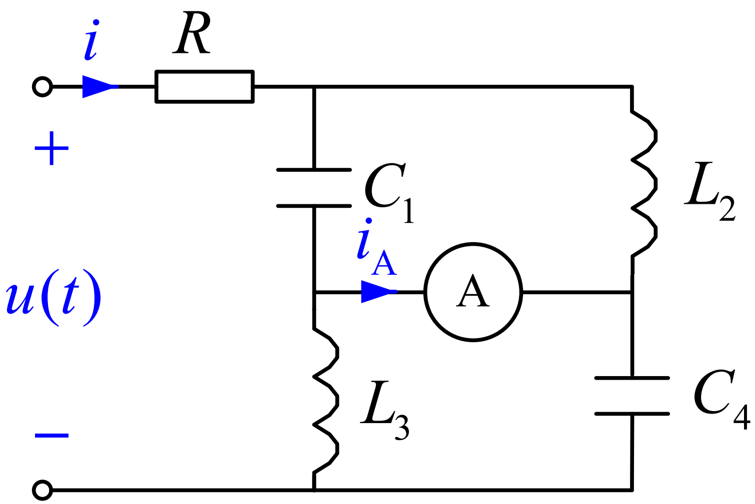


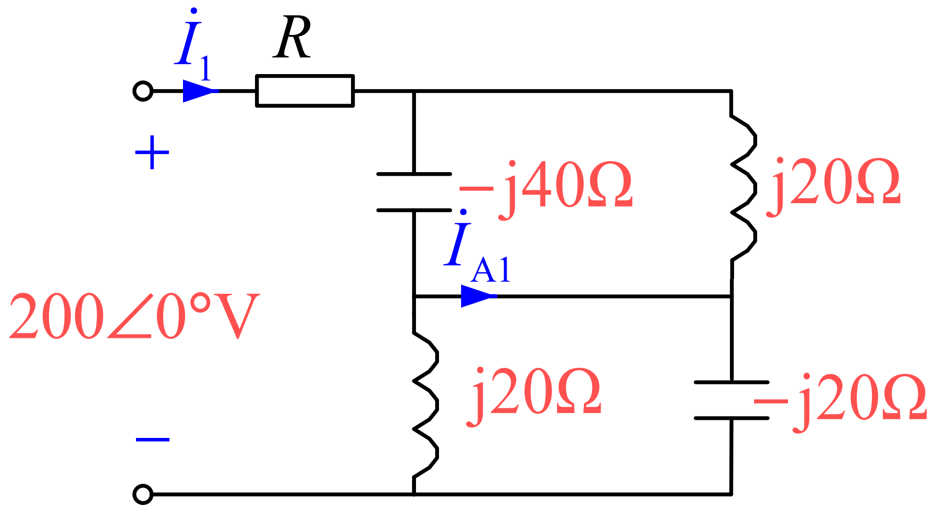
➢直流分量单独作用：

$$I_0=-I_{A0}=\frac{60}{10}=6\text{A}$$

$$P_0=U_0I_0=60\times 6=360\text{W}$$

【例3】已知 $u(t)=60+282\sin\omega t+169\sin(2\omega t-22.5°)\ \mathrm{V}$ R=10Ω，$1/\omega C_1=40\Omega$， $\omega L_2=\omega L_3=20\Omega$， $1/\omega C_4=20\Omega$，求电流表和电源提供的功率。



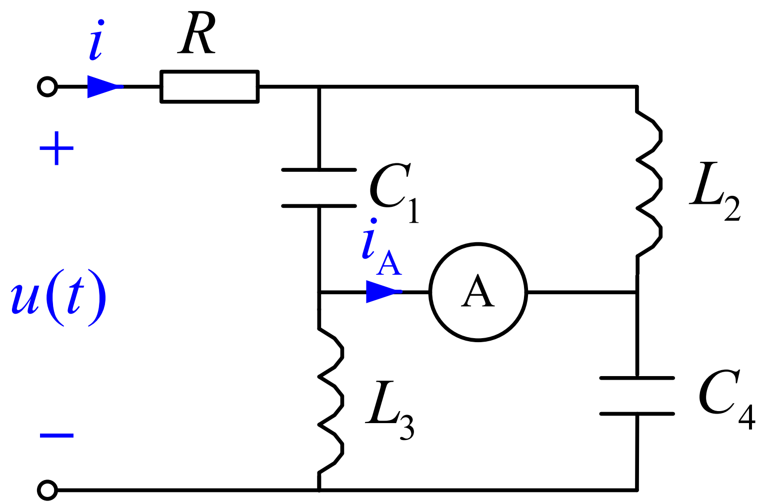


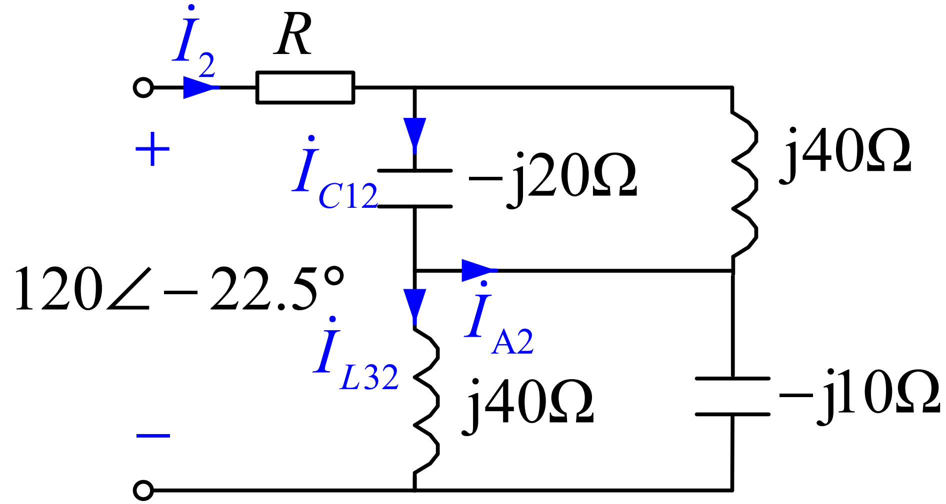
➢基波单独作用:

$$\dot{I}_1=0\mathrm{A}\qquad \dot{I}_{A1}=\frac{200\angle 0°}{-\mathrm{j}20}=10\angle 90°\mathrm{A}$$

$$P_1=0\mathrm{W}$$

【例3】已知 $u(t)=60+282\sin\omega t+169\sin(2\omega t-22.5°)$ V R=10Ω，1/ωC$_1$=40Ω， ωL$_2$=ωL$_3$=20Ω， 1/ωC$_4$=20Ω，求电流表和电源提供的功率。





➢二次谐波单独作用：

$$Z=10+\frac{-j20\times j40}{j20}+\frac{-j10\times j40}{j30}$$

$$=54\angle-79.4°$$

$$\dot{I}_2=\frac{120\angle-22.5°}{54\angle-79.4°}=2.22\angle56.9°\text{A}$$

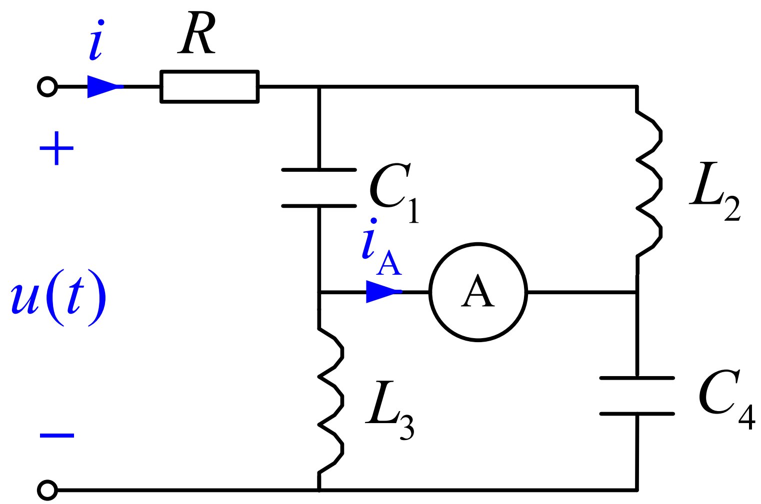
$$\dot{I}_{C12}=\frac{j40}{j20}\times\dot{I}_2=2.42+j3.71\text{A}$$

$$\dot{I}_{L32}=\frac{-j10}{j30}\times\dot{I}_2=-0.403-j0.618\text{A}$$

$$\dot{I}_{A2}=\dot{I}_{C12}-\dot{I}_{L32}=5.17\angle56.9°\text{A}$$

$$P_2=I_2{}^2R=49\text{W}$$

【例3】已知 $u(t) = 60 + 282\sin\omega t + 169\sin(2\omega t - 22.5°)$ V　R=10Ω，1/ωC$_1$=40Ω，　ωL$_2$=ωL$_3$=20Ω，　1/ωC$_4$=20Ω，求电流表和电源提供的功率。



$$I_0 = -I_{A0} = \frac{60}{10} = 6\text{A} \quad P_0 = 360\text{W}$$

$$\dot{I}_{A1} = 10\angle 90°\text{A} \quad P_1 = 0\text{W}$$

$$\dot{I}_{A2} = 5.17\angle 56.9°\text{A} \quad P_2 = 49\text{W}$$

$$\dot{I}_2 = 2.22\angle 56.9°\text{A}$$

$$I_A = \sqrt{6^2 + 10^2 + 5.17^2} = 12.8\text{A}$$

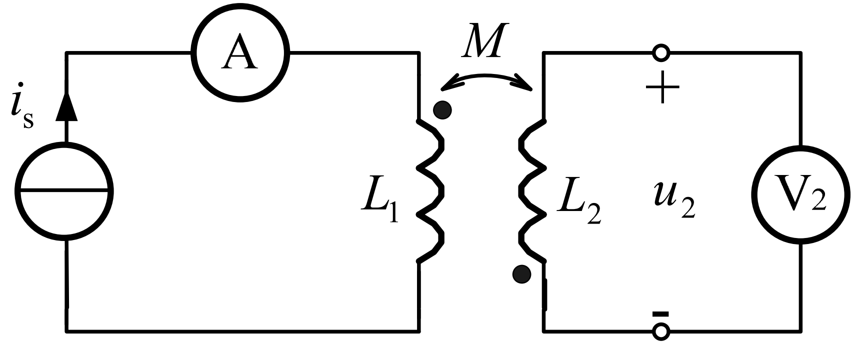
$$P = P_0 + \sum_{k=1}^{\infty} P_k = 360 + 49 = 409\text{W}$$

$$I = \sqrt{6^2 + 2.22^2} = 6.4\text{A}$$

$$P = I^2 R = 6.4^2 \times 10 = 409\text{W}$$

【练习】 $i_s = [5 + 10\cos(10t - 20°) - 5\sin(30t + 60°)]\text{A}$,

$L_1 = L_2 = 2\text{H}$，$M = 0.5\text{H}$。计算表的读数。



➢直流分量单独作用:

$i_{s(0)} = 5\text{A}$，$u_{2(0)} = 0$

➢基波单独作用:

$$\dot{I}_{s(1)} = 10\angle -20°\text{A}$$

$$\dot{U}_{2(1)} = -\text{j}\omega M\dot{I}_{s(1)} = -\text{j}10\times0.5\times10\angle -20° = 50\angle -110°\text{V}$$

➢三次谐波单独作用:

$$\dot{I}_{s(3)} = 5\angle 60°\text{A}$$

$$\dot{U}_{2(3)} = -\text{j}3\omega M\dot{I}_{s(3)} = -\text{j}30\times0.5\times5\angle 60° = 75\angle -30°\text{V}$$

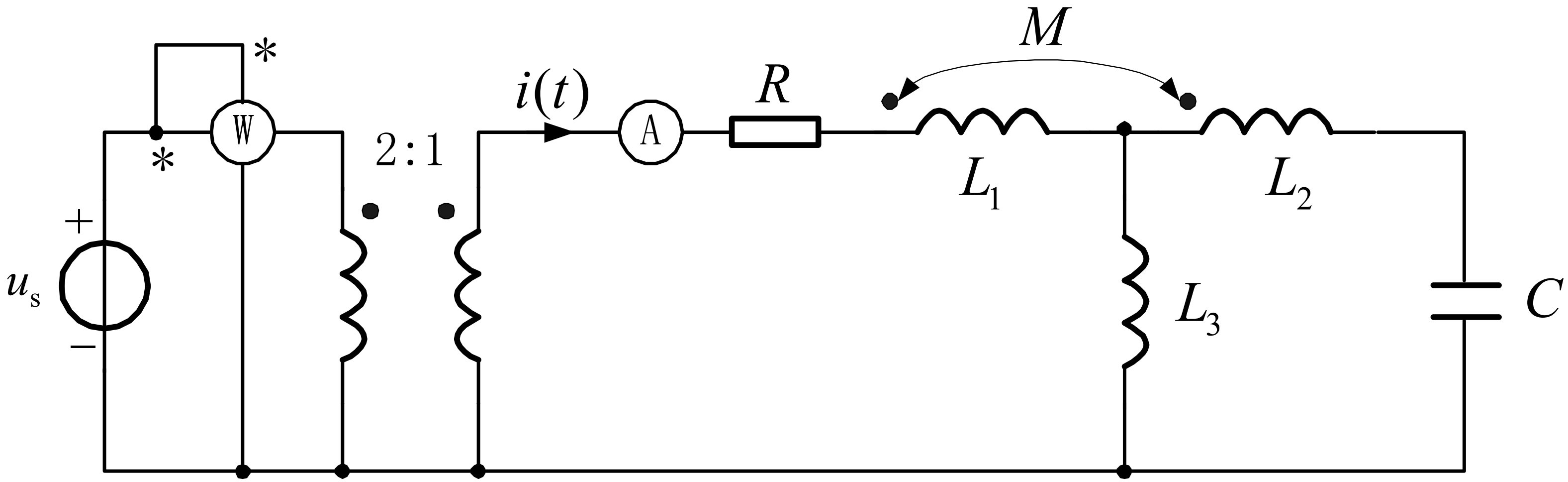
$$u_2 = [50\cos(10t - 110°) - 75\sin(30t - 30°)]\text{V}$$

$I_s = \sqrt{5^2 + 10^2/2 + 5^2/2} = 9.4\text{A}$，$U_2 = \sqrt{50^2/2 + 75^2/2} = 63.7\text{V}$

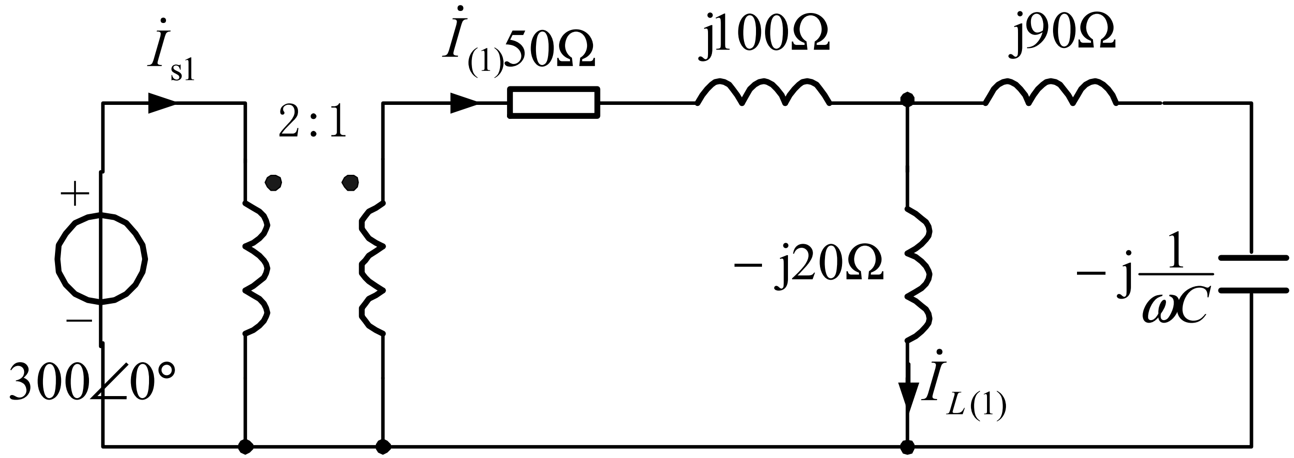
【课下练习】 $u_s(t) = (300\sqrt{2}\sin\omega t + 200\sqrt{2}\sin 3\omega t)$ $\quad R = 50\Omega$

$\omega L_1 = 60\Omega \qquad \omega L_2 = 50\Omega \qquad \omega M = 40\Omega \qquad \omega L_3 = 20\Omega$

L_3电流中没有基波分量。计算表的读数。例15-4-6



➢基波单独作用:



$\because \dot{I}_{L(1)} = 0$

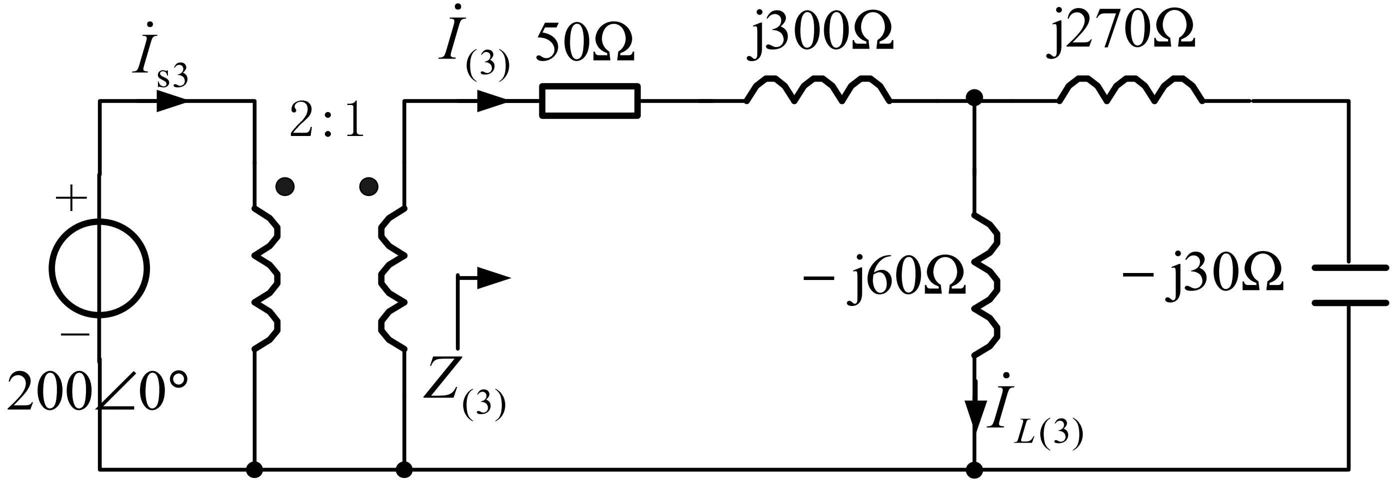
$\therefore \dfrac{1}{\omega C} = 90$

$\dot{I}_{(1)} = \dfrac{150}{50 + \mathrm{j}100}$

$= 1.34\angle -63.4°$

【课下练习】 $u_s(t)=(300\sqrt{2}\sin\omega t+200\sqrt{2}\sin 3\omega t)$ $R=50\Omega$

$\omega L_1=60\Omega$ $\omega L_2=50\Omega$ $\omega M=40\Omega$ $\omega L_3=20\Omega$



➢三次谐波单独作用：

$\dot{I}_{(1)}=1.34\angle-63.4°$

$$Z_{(3)}=50+\text{j}300+\frac{-\text{j}240\times\text{j}60}{\text{j}240-\text{j}60}=50+j220$$

$$\dot{I}_{(3)}=\frac{100}{Z_{(3)}}=0.44\angle-77.2°$$

$$I=\sqrt{I_{(1)}^2+I_{(3)}^2}=1.41\text{A} \qquad P=50I^2=99.5\text{W}$$

$$i(t)=1.34\sqrt{2}\sin(\omega t-63.4°)+0.44\sqrt{2}\sin(3\omega t-77.7°)\text{A}$$

计划学时：2学时；课后学习4学时

作业：

15-8，15-12 /周期非正弦电路分析
15-21/综合应用