1. 按定义计算下列行列式：

(1) $$\begin{vmatrix} 1 & x & y & z \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1-x^2 & x & y & z \\ 0 & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1-x^2-y^2-z^2 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$=1-x^2-y^2-z^2$$

(2) $$\begin{vmatrix} x+y & z+y & z+x \\ y+z & z+x & x+y \\ z+x & x+y & y+z \end{vmatrix} = \begin{vmatrix} x+y+z+y-(z+x) & z+y & z+x \\ y+z+z+x-(x+y) & z+x & x+y \\ z+x+x+y-(y+z) & x+y & z+x \end{vmatrix}$$

$$=2\begin{vmatrix} y & y+z & z+x \\ z & z+x & x+y \\ x & x+y & y+z \end{vmatrix} = 2\begin{vmatrix} y & z & x \\ z & x & y \\ x & y & z \end{vmatrix} = 2(xyz+xyz+xyz-x^3-y^3-z^3)$$
$$=6xyz-2x^3-2y^3-2z^3$$

2. 证明：$$\begin{vmatrix} 1+x_1^2 & x_1x_2 & x_1x_3 & x_1x_4 \\ x_2x_1 & 1+x_2^2 & x_2x_3 & x_2x_4 \\ x_3x_1 & x_3x_2 & 1+x_3^2 & x_3x_4 \\ x_4x_1 & x_4x_2 & x_4x_3 & 1+x_4^2 \end{vmatrix} = 1+\sum_{i=1}^{4} x_i^2$$

proof：左边 $=\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ x_1 & 1+x_1^2 & x_1x_2 & x_1x_3 & x_1x_4 \\ x_2 & x_2x_1 & 1+x_2^2 & x_2x_3 & x_2x_4 \\ x_3 & x_3x_1 & x_3x_2 & 1+x_3^2 & x_3x_4 \\ x_4 & x_4x_1 & x_4x_2 & x_4x_3 & 1+x_4^2 \end{vmatrix} = \begin{vmatrix} 1 & -x_1 & -x_2 & -x_3 & -x_4 \\ x_1 & 1 & 0 & 0 & 0 \\ x_2 & 0 & 1 & 0 & 0 \\ x_3 & 0 & 0 & 1 & 0 \\ x_4 & 0 & 0 & 0 & 1 \end{vmatrix}$

$=1+\sum_{i=1}^{4} x_i^2 =$ 右边

3. 计算下列行列式：

(1) $$\begin{vmatrix} x_1-m & x_2 & \cdots & x_n \\ x_1 & x_2-m & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_n-m \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^n x_i-m & x_2 & \cdots & x_n \\ \sum_{i=1}^n x_i-m & x_2-m & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^n x_i-m & x_2 & \cdots & x_n-m \end{vmatrix}$$

$$=\begin{vmatrix} \sum_{i=1}^n x_i-m & x_2 & \cdots & x_n \\ 0 & -m & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -m \end{vmatrix} = (-1)^{n-1}(m)^{n-1}\left(\sum_{i=1}^n x_i-m\right)$$

(2) $\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix}, a_i \neq 0, i=1,2,\cdots,n$

$$= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1+a_1 & 1 & \cdots & 1 \\ 0 & 1 & 1+a_2 & & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 1 & & & 1+a_n \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & & 1 \\ -1 & a_1 & & & \\ -1 & 0 & a_2 & & \\ \vdots & \vdots & & \ddots & \\ -1 & 0 & & & a_n \end{vmatrix} = \begin{vmatrix} 1+\frac{1}{a_1}+\cdots+\frac{1}{a_n} & 0 & 0 & \cdots & 0 \\ -1 & a_1 & 0 & \cdots & 0 \\ -1 & 0 & a_2 & & \\ \vdots & & & \ddots & \\ -1 & & & & a_n \end{vmatrix}$$

$$= \left(1+\sum_{i=1}^{n} \frac{1}{a_i}\right) a_1 a_2 \cdots a_n$$

4. 用 Cramer 法则解下列方程组：

(1) $\begin{cases} 2x_1 - x_2 - x_3 = 4 \\ 3x_1 + 4x_2 - 2x_3 = 11 \\ 3x_1 - 2x_2 + 4x_3 = 11 \end{cases}$

$$D = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 4 & -2 \\ 3 & -2 & 4 \end{vmatrix} = 60 \quad D_1 = \begin{vmatrix} 4 & -1 & -1 \\ 11 & 4 & -2 \\ 11 & -2 & 4 \end{vmatrix} = 180 \quad D_2 = \begin{vmatrix} 2 & 4 & -1 \\ 3 & 11 & -2 \\ 3 & 11 & 4 \end{vmatrix} = 60$$

$$D_3 = \begin{vmatrix} 2 & -1 & 4 \\ 3 & 4 & 11 \\ 3 & 2 & 11 \end{vmatrix} = 60 \qquad x_1 = \frac{D_1}{D} = 3, \ x_2 = \frac{D_2}{D} = 1, \ x_3 = \frac{D_3}{D} = 1$$

(2) $\begin{cases} x+y+z=a+b+c \\ ax+by+cz=a^2+b^2+c^2 \\ bcx+cay+abz=3abc \end{cases}$ （其中 a,b,c 互不相等）

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a) \neq 0$$

$$D_1 = \begin{vmatrix} a+b+c & 1 & 1 \\ a^2+b^2+c^2 & b & c \\ 3abc & ca & ab \end{vmatrix} = a \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = aD$$

$$D_2 = \begin{vmatrix} 1 & a+b+c & 1 \\ a & a^2+b^2+c^2 & c \\ b & 3abc & ab \end{vmatrix} = b \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b & ac & ab \end{vmatrix} = bD$$

$$D_3 = cD$$

$$x_1 = \frac{D_1}{D} = a \qquad x_2 = \frac{D_2}{D} = b \qquad x_3 = \frac{D_3}{D} = c$$

3. 求下列矩阵的幂：

(1) $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^n$ $\underline{\underline{定义为}}$ $B+\lambda E$ ，$B=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

则 $A^n=(B+\lambda E)^n=(\lambda E)^n+C_n^1(\lambda E)^{n-1}\cdot B+\cdots+B^n$

又 $B^2=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad B^3=B^2\cdot B=0$

故 $A^n=(B+\lambda E)^n=(\lambda E)^n+C_n^1(\lambda E)^{n-1}B+C_n^2(\lambda E)^{n-2}\cdot B^2$

$=\lambda^n E+n\lambda^{n-1}B+\frac{n(n-1)}{2}\lambda^{n-2}\cdots B^2$

$=\begin{bmatrix} \lambda^n & n\cdot\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\cdot\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix}$

(2) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^n$ $\underline{\underline{定义为}}$ $B+E$，$B=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$A^n=(B+E)^n=E^n+C_n^1E^{n-1}\cdot B+\cdots+B^n$

由于 $B^2=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=0$

故 $B^2=B^3=\cdots B^n=0$

故 $A^n=E^n+C_n^1E^{n-1}B=E+nB=\begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$

4. 求与矩阵 $\boldsymbol{A}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ 乘积可交换的矩阵。

A+

3,1

设 $B=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$，则

$AB=\begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix}$ $\quad BA=\begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix}$

$AB=BA$，得 $g=h=d=0$，$f=b$，$e=a=i$，

即 $B=\begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$

附：2. $\begin{vmatrix} 1+x_1^2 & x_1x_2 & x_1x_3 & x_1x_4 \\ x_2x_1 & 1+x_2^2 & x_2x_3 & x_2x_4 \\ x_3x_1 & x_3x_2 & 1+x_3^2 & x_3x_4 \\ x_4x_1 & x_4x_2 & x_4x_3 & 1+x_4^2 \end{vmatrix} = 1+\sum_{i=1}^{4} x_i^2$

证明：x_1, ~~x_2, x_3, x_4~~ $\neq 0$ 时.

$$左边 = \begin{vmatrix} 1+x_1^2 & x_1x_2 & x_1x_3 & x_1x_4 \\ -\frac{x_2}{x_1} & 1 & 0 & 0 \\ -\frac{x_3}{x_1} & 0 & 1 & 0 \\ -\frac{x_4}{x_1} & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1+\sum_{i=1}^{4} x_i^2 & x_1x_2 & x_1x_3 & x_1x_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= 1+\sum_{i=1}^{4} x_i^4 = 右边.$$

$x_1=0$ 时

$$左边 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+x_2^2 & x_2x_3 & x_2x_4 \\ 0 & x_3x_2 & 1+x_3^2 & x_3x_4 \\ 0 & x_4x_2 & x_4x_3 & 1+x_4^2 \end{vmatrix} = \begin{vmatrix} 1+x_2^2 & x_2x_3 & x_2x_4 \\ x_3x_2 & 1+x_3^2 & x_3x_4 \\ x_4x_2 & x_4x_3 & 1+x_4^2 \end{vmatrix} = 1+\sum_{i=2}^{3} x_i^2 = 1+\sum_{i=1}^{4} x_i^2$$

$=$ 右边（得证）

3. 求 $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^n$

可归纳猜测其为 $\begin{pmatrix} \lambda^n & n\cdot\lambda^{n-1} & \frac{n^2-n}{2}\cdot\lambda^{n-2} \\ 0 & \lambda^n & n\cdot\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}$

下面用数学归纳法证明之.