Assignment 0

(Q1) (Elegance) This is just for practice, related to Linear Algebra but not Quantum Mechanics. Look at both the *Oddtown* and *Eventown* problems (hopefully not the solutions) and solve them. (You might need to go over the field $\mathbb{F}_2$ before this)

Let m be the number of clubs, n be the population of town, $\mathcal{C} = \{C_1, \ldots, C_m\} \in 2^{[n]}$ be such that $|C_i \cap C_j|$ is even for every $i, j \in [m]$ and thus C_i be any club vector of size n.
Let $v_1, \ldots, v_m$ denote the incidence vectors of the clubs. That is, $v_i \in \mathbb{F}_2^n$ has entry $j = 1$ if $j \in C_i$ and 0 otherwise. Since the intersection of any two clubs is even, $v_i \cdot v_j = 0$ for all $i, j \in [m]$, where the dot product is taken over $\mathbb{F}_2$. Consider the span V of $\{v_1, \ldots, v_m\}$. We note that V must be a subspace of $V^\perp := \{w \in \mathbb{F}_2^n : v \cdot w = 0$ for all $v \in V\}$ for if $u = a_1v_1 + \ldots + a_mv_m$ and $v = b_1v_1 + \ldots + b_mv_m$ both lie in V, then each $a_ib_jv_iv_j = 0$, and thus $u \cdot v = 0$.
Next, when V is a subspace of W, then dim$V$+dim$V^\perp$ =dim$W$. Since $V \subset \mathbb{F}_2^n$ of dimension n and $V \subset V^\perp$, we have that

$$2dimV \leq dimV + dimV^\perp = n$$

in which case

$$dimV \leq n/2$$

which yields the result that

$$|V| \leq 2^{n/2}$$

Thus, there can be no more than $2^{n/2}$ clubs in Eventown.

Let's enumerate the inahbitants by $1, \ldots, n$, as well as the clubs by $1, \ldots, N$.
Let $v_1, \ldots, v_N \in \mathbb{Z}_2^n$ be the 'club vectors' where for each club one puts a 1 in the j^{th} entry if j is a member, and 0 for non-members. We claim that $v_1, \ldots, v_N$ are *linearly independent*. Thus, assume

$$a_1v_1 + a_2v_2 + \ldots + a_Nv_N = 0,$$

with scalars $a_1, \ldots, a_N \in \mathbb{Z}_2$. For fixed j, the dot product $v_i \cdot v_j$ equals the modulo 2 of common members of clubs i and j. It is thus 1 if $i = j$, and 0 if $i \neq j$. Hence, taking the dot product of both sides of above equation with v_j, we obtain

$$a_j = 0$$

This shows $v_1, \ldots, v_N$ are linearly independent.
Since $dim(\mathbb{Z}_2^n) = n$, it follows that $N \leq n$.
Thus, there can be no more than n clubs in Oddtown.

(Q2) (Weaker suffices!) Suppose A is any linear operator on Hilbert space, V. A is said to be Hermitian iff $\langle x, Ay \rangle$=$\langle Ax, y \rangle$ for all vectors $x, y \in V$.

Now suppose V is finite dimensional. Show that a necessary and sufficient condition for an operator A in V to be Hermitian is:

$$\langle x, Ax \rangle = \langle Ax, x \rangle$$

for all vectors $x \in V$.

We can probably make use of the Adjoint operator. The Adjoint operator A^* of A is the unique operator such that

$$\langle Ax, y \rangle = \langle x, A^*y \rangle$$

for all $x, y \in V$.
Now, suppose that A is Hermitian. Then for any $x, y \in V$, we have:

$$\langle x, Ay \rangle = \langle Ax, x \rangle$$

Taking the complex conjugate on both sides, we get:

$$\langle Ax, y\rangle = \langle x, Ay\rangle$$

Applying the definition of Adjoint operator, we have:

$$\langle A^*y, x\rangle = \langle y, Ax\rangle$$

Since this holds for all $x, y \in V$, we have that $A^*y = Ax$ for all $y \in V$.
Now, for all $x \in V$, we have:

$$\langle x, Ax\rangle = \langle Ax, x\rangle$$

Applying the definition of Adjoint operator, we have:

$$\langle x, Ax\rangle = \langle x, A^*x\rangle$$

Taking the complex conjugate of both sides, we get:

$$\langle Ax, x\rangle = \langle A^*x, x\rangle$$

Since $A^*x = Ax$ for all $x \in V$, we have:

$$\langle x, Ax\rangle = \langle Ax, x\rangle = \langle A^*x, x\rangle$$

Therefore, A is Hermitian iff $\langle x, Ax\rangle = \langle Ax, x\rangle$ for all $x \in V$.

(Q3) (Higher Dimensions!) We're going to give you some matrices, and your job is to find their eigenvalues (no calculator, and write down how you find them, because that is what we want to know)

(a)

$$\begin{bmatrix} 0 & 5 & 0 & 4 \\ 5 & 0 & 4 & 0 \\ 0 & 3 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{bmatrix}$$

Next, applying the definition: determinant of $A - \lambda I$ to be equated to 0.
Thus,

$$det\begin{bmatrix} -\lambda & 5 & 0 & 4 \\ 5 & -\lambda & 4 & 0 \\ 0 & 3 & -\lambda & 2 \\ 3 & 0 & 2 & -\lambda \end{bmatrix} = 0$$

$$-\lambda\begin{vmatrix} -\lambda & 4 & 0 \\ 3 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} - 5\begin{vmatrix} 5 & 4 & 0 \\ 0 & -\lambda & 2 \\ 3 & 2 & -\lambda \end{vmatrix} - 4\begin{vmatrix} 5 & -\lambda & 4 \\ 0 & 3 & -\lambda \\ 3 & 0 & 2 \end{vmatrix} = 0$$

$$\lambda(-\lambda(\lambda^2 - 4) - 4(-3\lambda)) + 5(5(\lambda^2 - 4) - 4(0 - 6)) + 4(5(6) + \lambda(3\lambda) + 4(-9)) = 0$$

$$\lambda(-\lambda^3 + 16\lambda) + 5(5\lambda^2 + 4) + 4(3\lambda^2 - 6) = 0$$

$$\lambda^4 - 16\lambda^2 - 25\lambda^2 - 20 - 12\lambda^2 + 24 = 0$$

$$\lambda^4 - 53\lambda^2 + 4 = 0$$

Solving these, we get approximate solutions as:
λ = -7.2749, 7.2749, -0.2749, 0.2749 as the *eigenvalues*.

(b)

$$\begin{bmatrix} 0 & 0 & 5 & 4 \\ 0 & 0 & 3 & 2 \\ 5 & 4 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{bmatrix}$$

Following the similar steps as above:

$$det\begin{bmatrix} -\lambda & 0 & 5 & 4 \\ 0 & -\lambda & 3 & 2 \\ 5 & 4 & -\lambda & 0 \\ 3 & 2 & 0 & -\lambda \end{bmatrix} = 0$$

$$-\lambda\begin{vmatrix} -\lambda & 3 & 2 \\ 4 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} + 5\begin{vmatrix} 0 & -\lambda & 2 \\ 5 & 4 & 0 \\ 3 & 2 & -\lambda \end{vmatrix} - 4\begin{vmatrix} 0 & -\lambda & 3 \\ 5 & 4 & -\lambda \\ 3 & 2 & 0 \end{vmatrix} = 0$$

$$-\lambda(-\lambda(\lambda^2) - 3(-4\lambda) + 2(2\lambda)) + 5(\lambda(-5\lambda) + 2(10-12)) - 4(\lambda(3\lambda) + 3(10-12)) = 0$$

$$-\lambda(-\lambda^3 + 12\lambda + 4\lambda) + 5(-5\lambda^2 - 4) - 4(3\lambda^2 - 6) = 0$$

$$\lambda^4 - 16\lambda^2 - 25\lambda^2 - 20 - 12\lambda^2 + 24 = 0$$

$$\lambda^4 - 53\lambda^2 + 4 = 0$$

Having the same characteristic equation as the previous question, we have:
λ = -7.2749, 7.2749, -0.2749, 0.2749 as the *eigenvalues*.

(c)

$$\begin{bmatrix} 25 & 20 & 20 & 16 \\ 15 & 10 & 12 & 8 \\ 15 & 10 & 12 & 8 \\ 9 & 6 & 6 & 4 \end{bmatrix}$$

Following the norms, we have:

$$det\begin{bmatrix} 25-\lambda & 20 & 20 & 16 \\ 15 & 10-\lambda & 12 & 8 \\ 15 & 10 & 12-\lambda & 8 \\ 9 & 6 & 6 & 4-\lambda \end{bmatrix} = 0$$

{Solving this would be no fun;)}

$$(25-\lambda)\begin{vmatrix} 10-\lambda & 12 & 8 \\ 10 & 12-\lambda & 8 \\ 6 & 6 & 4-\lambda \end{vmatrix} - 20\begin{vmatrix} 15 & 12 & 8 \\ 15 & 12-\lambda & 8 \\ 9 & 6 & 4-\lambda \end{vmatrix} + 20\begin{vmatrix} 15 & 10-\lambda & 8 \\ 15 & 10 & 8 \\ 9 & 6 & 4-\lambda \end{vmatrix}$$

$$-16\begin{vmatrix} 15 & 10-\lambda & 12 \\ 15 & 10 & 12-\lambda \\ 9 & 6 & 6 \end{vmatrix} = 0$$

By no chance can I proceed to solve it in normal procedure and no other idea is striking!

(Q4) (Algebra and Technicalities) Look up the definitions of a norm and a metric. Show that $|x|$ = $\sqrt{\langle x|x\rangle}$ is a valid norm and d(x,y) = $|x-y|$ is a valid metric.

A Hilbert space is just a *complete* inner product space. This means that every Cauchy sequence in the vector space converges to some element in the vector space.

Consider the set of all real continuous functions on [-1,1]. Confirm that this is a vector space over $\mathbb{R}$ with the normal operations: $(f+g)(t) = f(t) + g(t)$, $(\alpha f)(t) = \alpha f(t)$.

Define the inner product as $\langle f|g\rangle$=$\int_{-1}^{1} f(t)g(t)dt$. Now take the sequence of functions:

$$f_n(t) = \begin{cases} 1 & \text{t} \in \text{[-1,0]} \\ 1 - nt & \text{t} \in [0, \frac{1}{n}] \\ 0 & \text{otherwise} \end{cases}$$

Compute $\langle f_n | f_m \rangle$ and show that f_n is Cauchy. Find where this sequence converges and conclude that this inner product space is not complete.

Show the following remarkable result: *Finite dimensional vector space V under field $\mathbb{F}$ is a Hilbert Space under any valid inner product on V for $\mathbb{F}$= $\mathbb{R}/\mathbb{C}$.*

Hint: You might be familiar with the result for $V = \mathbb{R}^n$. Try to show the same for $\mathbb{C}^n$. Next, note that any finite vector space V over field $\mathbb{F}$ is isomorphic (look up the definition) to $\mathbb{F}^n$ where $n = dim(V)$. Use this to reduce the general problem to that of $\mathbb{F}^n$.

Norm is defined on vector space endowed with an inner product, or in other words, in Hilbert Space. Now, some of the properties of Inner Product are:
(a) $\langle x|\alpha y\rangle$ = $\alpha\langle x|y\rangle$
(b) $\langle \alpha x|y\rangle$ = $\alpha^*\langle x|y\rangle$
These simply leads to the result that $\langle x|x\rangle$ will always be positive. Also, in $\mathbb{R}^2$ and $\mathbb{R}^3$, graphical representation can be given, claiming the norm as the length of the vector justifying $|x|$ = $\sqrt{\langle x|x\rangle}$ as an appropriate definition for norm.
Similarly, $|x-y|$ defines a vaid metric since $x-y$ denotes the vector showing the difference between two vectors **x** and **y** and the norm of it shows its length and hence the associated metric.

$$\langle f_n|f_m\rangle = \int_{-1}^{1} f_n f_m dt$$

Assuming n>m, we can write down the integral as:

$$= \int_{-1}^{0} 1 * 1dt + \int_{0}^{1/n} (1-nt)(1-mt)dt + \int_{1/n}^{1/m} 0 * (1-mt)dt + \int_{1/m}^{1} 0 * 0dt$$

$$= 1 + (\frac{1}{n} - \frac{(n+m)}{2n^2} + \frac{nm}{3n^3})$$

$$\langle f_n|f_m\rangle = 1 + \frac{1}{2n} - \frac{m}{6n^2}$$

We can show that f_n is Cauchy since there exists an N after which $|t_{n+1} - t_n|$ converges to 0. However, the inner product has value as mentioned above, which can be shown to be greater than 1, which doesn't lie in the space. Hence, the inner product is not complete.

An n-dimensional real inner product space is not just topologically isomorphic to $\mathbb{R}^n$ (because that would indeed not impy completeness, since it is not a topological property). It is isometric to it.
Let V be such a space and let $v_1, \ldots, v_n$ be an orthonormal basis in V. Let $e_1, \ldots, e_n$ be the standard basis in $\mathbb{R}^n$. Defining a linear transformation $T: V \to \mathbb{R}^n$ by declaring $T(v_i) = e_i$ for $i = 1, \ldots, n$ and extending linearly.
Then T is clearly a bijection and $\langle T(v_i), T(v_j)\rangle = \langle e_i, e_j\rangle = \delta_{ij} = (v_i, v_j)$, so T respects the inner product structures. That is, T is an isomorphism of inner product spaces and therefore any property of V (as an inner product space) is implied by the corresponding property in $\mathbb{R}^n$.
In particular, it is Hilbert!

(Q5) (Hilbert-Schmidt and Vectorization) Verify that $\mathcal{L}(A, B)$ is a vector space. Consider the following function T: $\mathcal{L}(A, B) \to A \otimes B$ defined by

$$\mathcal{U} = \sum_{i,j} \alpha_{ji} |w_j\rangle\langle v_i| \longmapsto u = \sum_{i,j} \alpha_{ij} |v_i\rangle \otimes |w_j\rangle$$

where $\mathcal{B} = (\{v_i\}, \{w_i\})$ is a fixed input-output basis pair. Show that T is a bijection. This establishes $A \otimes B \cong \mathcal{L}(A, B)$. Consider the inner product $\langle \mathcal{U}|V\rangle_{HS} := \langle T\mathcal{U}|TV\rangle = \langle u|v\rangle$ on $\mathcal{L}(A, B)$ (Note that the latter inner product refers to that of the tensor product space). Show that $\langle \mathcal{U}|V\rangle_{HS}$ reduces to $tr(\mathcal{U}^\dagger V)$. This is called the Hilbert-Schimdt inner product and is quite useful once we see density operators later on.

One more connection between the two spaces: define the vectorization operator *vec*: $\mathcal{M}_{m\times n}(\mathbb{F}) \to \mathcal{M}_{mn\times 1}(\mathbb{F})$ by:

$$vec(A) := [a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{mn}]^T$$

that is, *vec* flattens the matrix A in row-major form into one long vector. This is exactly how C-style 2D arrays are stored in memory - as their vectorized forms. Vectorization operators are used extensively in Quantum Operations. Here's a neat fact: Consider $\mathcal{U} \in \mathcal{L}(A, B)$. Show that

$$(T\mathcal{U})_{\mathcal{B}} = vec(\mathcal{U}_{\mathcal{B}})$$

Here, $\mathcal{U}_{\mathcal{B}}$ is the matrix form of $\mathcal{U}$ under bases $\mathcal{B}$, and $u_{\mathcal{B}}$ is the matrix form of $u \in A \otimes B$ under basis $\{v_i \otimes w_j\}_{i,j}$ (wait a minute, doesn't u also count as a linear operator? Yes! Then where are the two bases for matrix representation? Why does one suffice? Think about it.) This equality means that *vec* is a sort of 'matrix version' of the bijection T.

Did gave it a shot but couldn't go much ahead!