Ball-Wippe-System: Aufstellen des nichtlinearen Zustandsraummodells aus den Bewegungsgleichungen

Bewegungsgleichungen

$$\underbrace{\left(m + \frac{I_b}{r^2}\right)}_{a_1} \ddot{x}' + \underbrace{(m r^2 + I_b)\frac{1}{r}}_{a_2} \ddot{\alpha} - m x' \dot{\alpha}^2 = \underbrace{m g}_{a_3} \sin(\alpha)$$

$$\Rightarrow a_1 \ddot{x}'(t) + a_2 \ddot{\alpha}(t) - m x'(t) \left(\dot{\alpha}(t)\right)^2 = a_3 \sin(\alpha(t)) \qquad (I)$$

$$(m x'^2 + \underbrace{I_b + I_w}_{b_1})\ddot{\alpha} + (\underbrace{2m}_{b_2} \dot{x}' x' + \underbrace{b \ell^2}_{b_3})\dot{\alpha} + \underbrace{k \ell^2}_{b_4} \alpha + \underbrace{(m r^2 + I_b)\frac{1}{r}}_{b_5} \ddot{x}' - \underbrace{m g}_{b_6} x' \cos(\alpha) = u \ell \cos(\alpha)$$

$$\Rightarrow \left(m (x'(t))^2 + b_1\right) \ddot{\alpha}(t) + \left(b_2 \dot{x}'(t) x'(t) + b_3\right) \dot{\alpha}(t) + b_4 \alpha(t) + b_5 \ddot{x}'(t) - b_6 x'(t) \cos(\alpha(t))$$

$$= u(t)\, \ell \cos(\alpha(t)) \qquad (II)$$

Zustandsraummodell (nili)

Wahl ZVs lt. Aufgabe

$x_1 = x'$ Ballposition

$x_2 = \dot{x}'$ Ballgeschw.

$x_3 = \alpha$ Wippenwinkel

$x_4 = \dot{\alpha}$ Wippenwinkelgeschw.

Stellgröße u: Antriebskraft

Regelgröße y: Ballposition

$$\dot{x}_1(t) = \dot{x}'(t) = x_2(t) \qquad f_1$$

$$\dot{x}_2(t) = \ddot{x}'(t) \overset{(I)}{=} \frac{-a_2 \dot{x}_4(t) + m x_1(t) (x_4(t))^2 + a_3 \sin(x_3(t))}{a_1} \qquad (III)$$

$$\dot{x}_3(t) = \dot{\alpha}(t) = x_4(t) \qquad f_3$$

$$\dot{x}_4(t) = \ddot{\alpha}(t) \overset{(II)}{=} \frac{-(b_2 x_2(t) x_1(t) + b_3) x_4(t) - (b_4 x_3(t) + b_5 \dot{x}_2(t))}{m (x_1(t))^2 + b_1}$$

$$+ \frac{b_6 x_1(t) \cos(x_3(t)) + u(t)\, \ell \cos(x_3(t))}{m (x_1(t))^2 + b_1} \qquad (IV)$$

(IV) in (III) einsetzen und nach $\dot{x}_2(t)$ auflösen:

$$\dot{x}_2 = \frac{m x_1 x_4^2 + a_3 \sin(x_3)}{a_1} + \frac{a_2 (b_2 x_2 x_1 + b_3) x_4}{a_1 (m x_1^2 + b_1)} + \frac{(b_4 x_3 + \overbrace{b_5 \dot{x}_2}^{\text{rüberziehen}}) a_2}{a_1 (m x_1^2 + b_1)}$$

$$- \frac{a_2 (b_6 x_1 \cos(x_3) + u \ell \cos(x_3))}{a_1 (m x_1^2 + b_1)}$$

$$\dot{x}_2 - \frac{b_5 a_2 \dot{x}_2}{a_1 (m x_1^2 + b_1)} = \dot{x}_2 \left(1 - \frac{b_5 a_2}{a_1 (m x_1^2 + b_1)}\right) = \frac{m x_1 x_4^2 + a_3 \sin(x_3)}{a_1} + \frac{a_2 (b_2 x_2 x_1 + b_3) x_4}{a_1 (m x_1^2 + b_1)}$$

$$+ \frac{b_4 x_3 a_2}{a_1 (m x_1^2 + b_1)} - \frac{a_2 (b_6 x_1 \cos(x_3) + u \ell \cos(x_3))}{a_1 (m x_1^2 + b_1)} \qquad \Big| : \left(1 - \frac{b_5 a_2}{a_1 (m x_1^2 + b_1)}\right)$$

$$\Rightarrow \dot{x}_2 = \frac{m x_1 x_4^2 + a_3 \sin(x_3)}{a_1 \left(1 - \frac{b_5 a_2}{a_1(m x_1^2 + b_1)}\right)} + \frac{a_2 \left(b_2 x_1 x_2 + b_3\right) x_4 + b_4 x_3}{a_1 (m x_1^2 + b_1)\left(1 - \frac{b_5 a_2}{a_1(m x_1^2 + b_1)}\right)} \Big\} \; NR := a_1(m x_1^2 + b_1) - b_5 a_2$$

$$- \frac{a_2 \left(b_6 x_1 \cos(x_3) + u l \cos(x_3)\right)}{a_1 (m x_1^2 + b_1) - b_5 a_2}$$

$$\Rightarrow \dot{x}_2 = \frac{(m x_1^2 + b_1)\left[m x_1 x_4^2 + a_3 \sin(x_3)\right] + a_2 \left[(b_2 x_1 x_2 + b_3) x_4 + b_4 x_3 - b_6 x_1 \cos(x_3) - u l \cos(x_3)\right]}{a_1 (m x_1^2 + b_1) - b_5 a_2} \quad (\mathrm{V})$$

f_2

(V) in (IV) einsetzen und umformen:

$$\dot{x}_4 = \frac{-(b_2 x_2 x_1 + b_3) x_4 - b_4 x_3 + b_6 x_1 \cos(x_3) + u l \cos(x_3)}{m x_1^2 + b_1}$$

$$- \frac{b_5 a_2 \left[(b_2 x_1 x_2 + b_3) x_4 + b_4 x_3 - b_6 x_1 \cos(x_3)\right]}{(m x_1^2 + b_1)\left(a_1 (m x_1^2 + b_1) - b_5 a_2\right)}$$

$$- \frac{b_5 \left[m x_1 x_4^2 + a_3 \sin(x_3)\right]}{a_1 (m x_1^2 + b_1) - b_5 a_2}$$

zusammenfassen

$$+ \frac{b_5 a_2 u l \cos(x_3)}{(m x_1^2 + b_1)\left(a_1 (m x_1^2 + b_1) - b_5 a_2\right)}$$

in RL

$$\Rightarrow \dot{x}_4 = \underbrace{\frac{-(b_2 x_2 x_1 + b_3) x_4 - b_4 x_3 + b_6 x_1 \cos(x_3)}{m x_1^2 + b_1}}_{(1)} - \underbrace{\frac{b_5 a_2 \left[(b_2 x_1 x_2 + b_3) x_4 + b_4 x_3 - b_6 x_1 \cos(x_3)\right]}{(m x_1^2 + b_1)\left(a_1 (m x_1^2 + b_1) - b_5 a_2\right)}}_{(2)}$$

$$- \underbrace{\frac{b_5 \left[m x_1 x_4^2 + a_3 \sin(x_3)\right]}{a_1 (m x_1^2 + b_1) - b_5 a_2}}_{(3)} + \underbrace{\left(1 + \frac{b_5 a_2}{a_1 (m x_1^2 + b_1) - b_5 a_2}\right) \frac{u l \cos(x_3)}{m x_1^2 + b_1}}_{(4)}$$

f_4

$$y(t) = x'(t) = x_1(t)$$

g

Ball-Wippe-System: Linearisierung der nichtlinearen ZRMs in Ruhelage $\underline{x}_0 = (x_{10}, 0, 0, 0)^T, u_0$

Es gilt: $x_{20} = x_{30} = x_{40} = 0$; $\sin(x_3) \approx x_3$, $\cos(x_3) \approx 1$ für kleine Winkel

A-Matix

$$\left.\frac{\partial f_1}{\partial \underline{x}}\right|_{RL} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

erste Zeile von $\underline{A}$

$$\left.\frac{\partial f_3}{\partial \underline{x}}\right|_{RL} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

dritte Zeile von $\underline{A}$

im RL

$$\dot{x}_2 = \frac{(m x_1 \overset{0}{x_4}{}^2 + a_3 \overbrace{\sin(\overset{0}{x_3})}^{0})(m x_1^2 + b_1)}{\underset{(1)}{a_1 (m x_1^2 + b_1) - b_5 a_2}} + \frac{a_2 [(b_2 x_1 \overset{0}{x_2} + b_3)\overset{0}{x_4} + b_4 \overset{0}{x_3} - b_6 x_1 \overbrace{\cos(\overset{0}{x_3})}^{1}]}{\underset{(2)}{a_1 (m x_1^2 + b_1) - b_5 a_2}}$$

$$- \frac{a_2 u \ell \overbrace{\cos(\overset{0}{x_3})}^{1}}{\underset{(3)}{a_1 (m x_1^2 + b_1) - b_5 a_2}}$$

f_2 von oben anders dargestellt

$$\left.\frac{\partial f_2}{\partial x_1}\right|_{RL} \overset{\text{Quotientenregel}}{=} - \frac{a_2 b_6 (a_1 (m x_{10}^2 + b_1) - b_5 a_2) - a_2 b_6 x_{10} \cdot 2 a_1 m x_{10}}{(a_1 (m x_{10}^2 + b_1) - b_5 a_2)^2} \qquad \left.\frac{\partial (2)}{\partial x_1}\right|_{RL}$$

$\left.\frac{\partial (1)}{\partial x_1}\right|_{RL} = 0$

$$- \frac{0 - a_2 u_0 \ell \cdot 2 a_1 m x_{10}}{(a_1 (m x_{10}^2 + b_1) - b_5 a_2)^2} \qquad \left.\frac{\partial (3)}{\partial x_1}\right|_{RL}$$

$$= \frac{-a_2 b_6 (a_1 (m x_{10}^2 + b_1) - b_5 a_2) + 2 a_1 a_2 m x_{10} (b_6 x_{10} + u_0 \ell)}{(a_1 (m x_{10}^2 + b_1) - b_5 a_2)^2} \qquad \boxed{A_{21}}$$

$$\left.\frac{\partial f_2}{\partial x_2}\right|_{RL} = \frac{a_2 (b_2 x_{10} + b_3) \overset{0}{x_{40}}}{a_1 (m x_{10}^2 + b_1) - b_5 a_2} = 0$$

$\left.\frac{\partial (1)}{\partial x_2}\right|_{RL} = 0$, $\left.\frac{\partial (3)}{\partial x_2}\right|_{RL} = 0$ da kein x_2

$$\left.\frac{\partial f_2}{\partial x_3}\right|_{RL} = \frac{a_3 \overbrace{\cos(x_{30})}^{1} (m x_{10}^2 + b_1) + a_2 b_4}{a_1 (m x_{10}^2 + b_1) - b_5 a_2} = \frac{a_3 (m x_{10}^2 + b_1) + a_2 b_4}{a_1 (m x_{10}^2 + b_1) - b_5 a_2} \qquad \boxed{A_{23}}$$

$\left.\frac{\partial (2)}{\partial x_3}\right|_{RL} = 0$, $\left.\frac{\partial (3)}{\partial x_3}\right|_{RL} = 0$ da $\sin(x_{30}) = 0$
$x_{30} = 0$

$$\left.\frac{\partial f_2}{\partial x_4}\right|_{RL} = \frac{2 m x_{10} \overset{0}{x_{40}} (m x_{10}^2 + b_1)}{a_1 (m x_{10}^2 + b_1) - b_5 a_2} + \frac{a_2 b_2 x_{10} \overset{0}{x_{20}} \overset{0}{x_{40}} + a_2 b_3}{a_1 (m x_{10}^2 + b_1) - b_5 a_2} = \frac{a_2 b_3}{a_1 (m x_{10}^2 + b_1) - b_5 a_2}$$

$$\boxed{A_{24}}$$

$\left.\frac{\partial (3)}{\partial x_4}\right|_{RL} = 0$ da kein x_4

$$\left.\frac{\partial f_4}{\partial x_1}\right|_{RL} \overset{Q\text{-Regel}}{=} \frac{b_6 \overbrace{\cos(x_{30})}^{\wedge}\,(m x_{10}^2+b_1) - 2m x_{10} b_6 x_{10} \overbrace{\cos(x_{30})}^{1}}{(m x_{10}^2+b_1)^2} \qquad \left.\frac{\partial(1)}{\partial x_1}\right|_{RL}$$

$$- b_5 a_2 \frac{-b_6 (m x_{10}^2 + b_1)(a_1(m x_{10}^2+b_1)-b_5a_2) + b_6 x_{10} \overbrace{2m x_{10}(2a_1(m x_{10}^2+b_1)-b_5a_2)}^{NR1}}{[(m x_{10}^2+b_1)(a_1(m x_{10}^2+b_1)-b_5a_2)]^2} \qquad \left.\frac{\partial(2)}{\partial x_1}\right|_{RL}$$

$$-0 \qquad \left.\frac{\partial(3)}{\partial x_4}\right|_{RL} = 0 \quad \text{da } x_{40}=0$$

$$+\frac{0 - u_0 l \overbrace{\cos(x_{30})}^{1} 2m x_{10}}{(m x_{10}^2+b_1)^2} - \frac{0 - b_5 a_2 u_0 l \overbrace{\cos(x_{30})}^{1} \overbrace{2m x_{10}(2a_1(m x_{10}^2+b_1)-b_5a_2)}^{NR1}}{[(m x_{10}^2+b_1)(a_1(m x_{10}^2+b_1)-b_5a_2)]^2} \qquad \left.\frac{\partial(4)}{\partial x_1}\right|_{RL}$$

$$= \frac{b_6(-m x_{10}^2+b_1)}{(m x_{10}^2+b_1)^2} - b_5 a_2 b_6 \frac{\overbrace{m x_{10}^2(3a_1 m x_0^2 + 2a_1b_1 - b_5a_2) + b_1(b_5a_2 - a_1b_1)}^{NR2}}{[(m x_{10}^2+b_1)(a_1(m x_{10}^2+b_1)-b_5a_2)]^2}$$

$$-\left(1 + b_5 a_2 \frac{2a_1 m x_{10}^2 + 2a_1 b_1 - b_5a_2}{(a_1(m x_{10}^2+b_1)-b_5a_2)^2}\right)\frac{u_0 l\, 2m x_{10}}{(m x_{10}^2+b_1)^2} \qquad \boxed{A_{41}}$$

$$\left.\frac{\partial f_4}{\partial x_2}\right|_{RL} = 0 \qquad \left.\frac{\partial(1)}{\partial x_2}\right|_{RL} = \left.\frac{\partial(2)}{\partial x_2}\right|_{RL} = 0 \text{ da } x_{40}=0, \quad \left.\frac{\partial(3)}{\partial x_2}\right|_{RL} = \left.\frac{\partial(4)}{\partial x_2}\right|_{RL} = 0 \text{ da kein } x_2$$

$$\left.\frac{\partial f_4}{\partial x_3}\right|_{RL} = \frac{\overset{\sin(x_{30})=0}{-b_4}}{(m x_{10}^2+b_1)} - \frac{\overset{\sin(x_{30})=0}{b_5 a_2 b_4}}{(m x_{10}^2+b_1)(a_1(m x_{10}^2+b_1)-b_5a_2)}$$

$$- \frac{b_5 a_3 \overbrace{\cos(x_{30})}^{1}}{a_1(m x_{10}^2+b_1)-b_5a_2} + (\quad)\cdot 0 \text{ — wg. } \sin(x_{30})=0 \qquad \boxed{A_{43}}$$

$$\left.\frac{\partial f_4}{\partial x_4}\right|_{RL} = \frac{-b_3}{m x_{10}^2+b_1} - \frac{b_5 a_2 b_3}{(m x_{10}^2+b_1)(a_1(m x_{10}^2+b_1)-b_5a_2)} \qquad \boxed{A_{44}}$$

$$\frac{\partial(3)}{\partial x_4} = 0 \quad \text{da } x_{40}=0, \qquad \frac{\partial(4)}{\partial x_4} = 0 \quad \text{da kein } x_4$$

$$NR1: \frac{\partial}{\partial x_1}\left((m x_1^2+b_1)(a_1(m x_1^2+b_1)-b_5a_2)\right) \overset{\text{Produkt-regel}}{=} (2m x_1)(a_1(m x_1^2+b_1)-b_5a_2) + (m x_1^2+b_1)(2a_1 m x_1)$$

$$= 2m x_1(2a_1(m x_1^2+b_1)-b_5a_2)$$

$$NR\ 2:\ -b_5 a_2 b_6 \cdot (-1)(m x_{10}^2 + b_1)(a_1 m x_{10}^2 + a_1 b_1 - b_5 a_2) + x_{10}\, 2m x_{10} \left((2a_1 m x_{10}^2 + 2a_1 b_1) - b_5 a_2\right)$$

$$= -b_5 a_2 b_6 \left[-a_1 m^2 x_{10}^4 - a_1 b_1 m x_{10}^2 + m x_{10}^2 b_5 a_2 - a_1 b_1 m x_{10}^2 - a_1 b_1^2 + b_5 a_2 b_1 + x_{10}^2\, 2m\, 2a_1 m x_{10}^2 + 2a_1 b_1\, 2m x_{10}^2 - b_5 a_2\, 2m x_{10}^2\right]$$

$$= -b_5 a_2 b_6 \left[m x_{10}^2 \left(-a_1 m x_{10}^2 - a_1 b_1 + b_5 a_2 - a_1 b_1 + 4a_1 m x_{10}^2 + 4a_1 b_1 - 2b_5 a_2\right) + b_1(-a_1 b_1 + b_5 a_2)\right]$$

$$= -b_5 a_2 b_6 \left[m x_{10}^2 \left(3a_1 m x_{10}^2 + 2a_1 b_1 - b_5 a_2\right) + b_1 (b_5 a_2 - a_1 b_1)\right]$$

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b-Vektor

$$\left.\frac{\partial f_1}{\partial u}\right|_{RL} = \left.\frac{\partial f_3}{\partial u}\right|_{RL} = 0 \qquad \text{(da kein } u\text{)}$$

$$\left.\frac{\partial f_2}{\partial u}\right|_{RL} = \frac{-a_2\, l\, \overbrace{\cos(x_{30})}^{\wedge}}{a_1 (m x_{10}^2 + b_1) - b_5 a_2} \quad \boxed{B_2} \quad ; \left.\frac{\partial f_4}{\partial u}\right|_{RL} = \left(1 + \frac{b_5 a_2}{a_1 (m x_{10}^2 + b_1) - b_5 a_2}\right) \frac{l\, \overbrace{\cos(x_{30})}^{\wedge}}{m x_{10}^2 + b_1} \quad \boxed{B_4}$$

C^T-Vektor

$$\left.\frac{\partial g}{\partial \underline{x}}\right|_{RL} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

Durchgriff d

$$\left.\frac{\partial g}{\partial u}\right|_{RL} = 0 \qquad \text{(da kein } u\text{)}$$

lin. ZRM in Matrix-Form

$$\dot{\underline{x}}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ A_{21} & 0 & A_{23} & A_{24} \\ 0 & 0 & 0 & 1 \\ A_{41} & 0 & A_{43} & A_{44} \end{pmatrix} \underline{x}(t) + \begin{pmatrix} 0 \\ B_2 \\ 0 \\ B_4 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \underline{x}(t)$$