

# Free monads

## Haskell and Cryptocurrencies

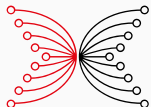
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INPUT | OUTPUT

# Goals

- Case study monad transformers
- Free monads

## Case study: Evolving an interpreter

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## A simple language for expressions

```
data Expr =  
    Lit Int  
  | Add Expr Expr
```

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data Expr =  
    Lit Int  
  | Add Expr Expr
```

```
eval :: Expr -> Int  
eval (Lit n)      = n  
eval (Add e1 e2) = eval e1 + eval e2
```

## Adding division

```
data Expr =  
  Lit Int  
| Add Expr Expr  
| Div Expr Expr
```

## Adding division

```
data Expr =  
    Lit Int  
  | Add Expr Expr  
  | Div Expr Expr
```

```
eval :: Expr -> Either String Int  
eval (Lit n)      = pure n  
eval (Add e1 e2) = (+) <$> eval e1 <*> eval e2  
eval (Div e1 e2) = do  
    v1 <- eval e1  
    v2 <- eval e2  
    if v2 == 0  
    then throwError "division by 0"  
    else return (v1 `div` v2)
```

## Two evaluator versions

```
eval :: Expr -> Int
eval (Lit n)      = n
eval (Add e1 e2) = eval e1 + eval e2
```

```
eval :: Expr -> Either String Int
eval (Lit n)      = pure n
eval (Add e1 e2) = (+) <$> eval e1 <*> eval e2
eval (Div e1 e2) = do
  v1 <- eval e1
  v2 <- eval e2
  if v2 == 0
    then throwError "division by 0"
    else return (v1 `div` v2)
```

Rewrite affects every line.



## Rewrite to identity monad

```
eval :: Expr -> Identity Int
eval (Lit n)      = pure n
eval (Add e1 e2) = (+) <$> eval e1 <*> eval e2
```

```
eval :: Expr -> ExceptT String Identity Int
eval (Lit n)      = pure n
eval (Add e1 e2) = (+) <$> eval e1 <*> eval e2
eval (Div e1 e2) = do
  v1 <- eval e1
  v2 <- eval e2
  if v2 == 0
    then throwError "division by 0"
    else return (v1 `div` v2)
```

Rewrite now affects only the new case.

## Intermediate summary and plan

- If we accept applicative / monadic style, we seem to be able to add new effects to our evaluator with minimal impact on existing cases.
- Let's define an “abstract” monad `Eval` that we gradually equip with new features.
- We will use monad transformers to implement the interface dictated by `Eval`.
- Extensions typically require adapting only the interesting cases of the evaluator, and possibly the monad.

# The `Eval` monad

Required interface:

```
data Eval  -- abstract
divByZeroError :: Eval a
runEval       :: Eval a -> Either String a
```

## Evaluator with division, again

```
eval :: Expr -> Eval Int
eval (Lit n)      = pure n
eval (Add e1 e2) = (+) <$> eval e1 <*> eval e2
eval (Div e1 e2) = do
  v1 <- eval e1
  v2 <- eval e2
  if v2 == 0
    then divByZeroError
    else return (v1 `div` v2)
```

## Implementing the interface

```
newtype Eval a =  
  Eval (ExceptT String Identity a)  
deriving  
  ( Functor, Applicative  
  , Monad, MonadError String  
  )
```

- A **newtype** makes it easier to still treat **Eval** as abstract for most of the code.
- The **GeneralizedNewtypeDeriving** extension makes it possible to lift class instances from the inner type to the wrapper type in the obvious way.

## Implementing the interface (contd.)

```
divByZeroError :: Eval a  
divByZeroError = throwError "division by 0"
```

```
runEval :: Eval a -> Either String a  
runEval (Eval m) =  
    runIdentity (runExceptT m)
```

## Adding variables

```
data Expr =  
    Lit Int  
  | Add Expr Expr  
  | Div Expr Expr  
  | Var String
```

## Adding variables

```
data Expr =  
    Lit Int  
  | Add Expr Expr  
  | Div Expr Expr  
  | Var String
```

`Eval` interface extension / modification:

```
varLookup :: String -> Eval Int  
runEval   :: Eval a -> Env -> Either String a
```



## Adapting the evaluator

```
eval :: Expr -> Eval Int
eval (Lit n)      = pure n
eval (Add e1 e2) = (+) <$> eval e1 <*> eval e2
eval (Div e1 e2) = do
  v1 <- eval e1
  v2 <- eval e2
  if v2 == 0
    then divByZeroError
    else return (v1 `div` v2)
eval (Var x)      = varLookup x
```

Only the last case is new; all others unchanged.

## Implementing the interface extension

```
newtype Eval a =  
  Eval (ReaderT Env (ExceptT String Identity) a)  
deriving  
  ( Functor, Applicative  
    , Monad, MonadError String, MonadReader Env )
```

Monad changes. Now we include a reader.

## Implementing the interface extension (contd.)

```
varLookup :: String -> Eval Int
varLookup x = do
  env <- ask
  case M.lookup x env of
    Nothing -> unknownVar x
    Just n   -> return n
```

```
unknownVar :: String -> Eval a
unknownVar x =
  throwError $ "unknown: " ++ show x
```

## Implementing the interface extension (contd.)

```
runEval :: Eval a -> Env -> Either String a
runEval (Eval m) env =
  runIdentity (runExceptT
    (runReaderT m env))
```

The code for `divByZeroError` needs no changes.

## Examples

```
GHCi> runEval (eval (Div (Var "x") (Var "x")))
      Map.singleton "x" 2
```

```
Right 1
```

```
GHCi> runEval (eval (Div (Var "x") (Var "x")))
      Map.singleton "x" 0
```

```
Left "division by 0"
```

```
GHCi> runEval (eval (Div (Var "x") (Var "x")))
      Map.empty
```

```
Left "unknown: \"x\""
```

## Adding mutation

```
data Expr =  
    Lit Int  
  | Add Expr Expr  
  | Div Expr Expr  
  | Var String  
-- these are new:  
  | Seq Expr Expr  
  | Assign String Expr
```

## Adding mutation

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data Expr =  
    Lit Int  
  | Add Expr Expr  
  | Div Expr Expr  
  | Var String  
  -- these are new:  
  | Seq Expr Expr  
  | Assign String Expr
```

`Eval` interface extension:

```
varSet :: String -> Int -> Eval ()
```

## Adapting the evaluator

Again, all the old cases are unchanged:

```
eval :: Expr -> Eval Int
eval (Seq e1 e2) = eval e1 >> eval e2
eval (Assign x e) = do
  v <- eval e
  varSet x v
  return v
...
```



## Implementing the interface extension

```
newtype Eval a =  
  Eval (StateT Env (ExceptT String Identity) a)  
deriving  
  ( Functor, Applicative  
    , Monad, MonadError String, MonadState Env )
```

We are changing `MonadReader` to `MonadState`.

## Implementing the interface extension (contd.)

```
varSet :: String -> Int -> Eval ()  
varSet x v =  
    modify (M.insert x v)
```

Recall:

```
modify :: MonadState s m => (s -> s) -> m ()  
modify f = do  
    s <- get  
    put (f s)
```

## Implementing the interface extension (contd.)

Minor changes in `varLookup` and `runEval`:

```
varLookup :: String -> Eval Int
varLookup x = do
  env <- get
  case M.lookup x env of
    Nothing -> throwError $ "unknown: " ++ show x
    Just n   -> return n
```

```
runEval :: Eval a -> Env -> Either String a
runEval (Eval m) env =
  runIdentity (runExceptT
    (evalStateT m env))
```

## Examples

```
program :: Expr
program =
  Assign "x" (Lit 16)
  `Seq` Assign "x" (Div (Var "x") (Lit 2))
  `Seq` Add (Var "x") (Lit 1)
```

```
GHCi> runEval (eval program) Map.empty
Right 9
```

# Summary

- Using an abstract monad, we can easily extend code with new kinds of effects, usually without affecting existing code.
- Monad transformers are a useful way to quickly implement such interfaces.
- The notion of sequencing changes with changing the underlying monad. We have a “programmable semicolon”.

## Revisiting the interpreter

---

## (Extended) Expression language

```
data Expr =  
  Lit Int  
| Add Expr Expr  
| Div Expr Expr  
| Var String  
| Seq Expr Expr  
| Assign String Expr
```

```
eval :: Expr -> Eval Int
eval (Div e1 e2) = do
  v1 <- eval e1
  v2 <- eval e2
  if v2 == 0
    then divByZeroError
    else return (v1 `div` v2)
...
```



# Evaluation monad

```
data Eval  -- abstract
instance Monad Eval

divByZeroError :: Eval a
unknownVar     :: String -> Eval a
varLookup      :: String -> Eval Int
varSet         :: String -> Int -> Eval ()
runEval        :: Eval a -> Env -> Either String a
```

# Evaluation monad

```
data Eval  -- abstract
instance Monad Eval
divByZeroError :: Eval a
unknownVar     :: String -> Eval a
varLookup      :: String -> Eval Int
varSet         :: String -> Int -> Eval ()
runEval        :: Eval a -> Env -> Either String a
```

- Had an implementation using monad transformers.
- Can we implement in a way that abstracts from the interface?

## Abstracting from the interface

- Provide several different implementations of the interface.
- Have programs in the expression language available as data.
- Analyze programs written in the expression language and possibly transform or even compile them.

# Abstracting from the interface

- Provide several different implementations of the interface.
- Have programs in the expression language available as data.
- Analyze programs written in the expression language and possibly transform or even compile them.

Basic idea: turn functions into constructors.

## A first attempt

```
data Eval a =  
    DivByZeroError  
  | UnknownVar String  
  | VarLookup String  
  | VarSet String Int
```

## A first attempt

```
data Eval a =  
    DivByZeroError  
  | UnknownVar String  
  | VarLookup String  
  | VarSet String Int
```

This loses info about the result types:

```
VarLookup :: String -> Eval a
```

vs.

```
varLookup :: String -> Eval Int
```

Also, it's unclear how to define a monad instance.

# Generalized Algebraic Datatypes (GADTs)

---

## Using a GADT for `Eval`

```
data Eval :: * -> * where
  DivByZeroError :: Eval a
  UnknownVar      :: String -> Eval a
  VarLookup       :: String -> Eval Int
  VarSet          :: String -> Int -> Eval ()
```

- Constructors are listed by type signature.
- We can restrict the result type.



## Using a GADT for `Eval`

```
data Eval :: * -> * where
  DivByZeroError :: Eval a
  UnknownVar      :: String -> Eval a
  VarLookup       :: String -> Eval Int
  VarSet          :: String -> Int -> Eval ()
  Return          :: a -> Eval a
  Bind            :: Eval a -> (a -> Eval b)
                  -> Eval b
```

- Constructors are listed by type signature.
- We can restrict the result type.
- We can do the same for the monad operations.

```
instance Monad Eval where
    return = Return
    (>=)    = Bind

instance Functor Eval where
    fmap    = liftM

instance Applicative Eval where
    pure    = return
    (<*>)   = ap
```

To literally implement the interface, we need wrappers:

```
divByZeroError = DivByZeroError  
unknownVar     = UnknownVar  
varLookup      = VarLookup  
varSet         = VarSet
```

## Interpreting the GADT

Let `MT` be the module defining the evaluation monad using monad transformers.

```
fromEval :: Eval a -> MT.Eval a
fromEval DivByZeroError = MT.divByZeroError
fromEval (UnknownVar x) = MT.unknownVar x
fromEval (VarLookup x)  = MT.varLookup x
fromEval (VarSet x v)   = MT.varSet x v
fromEval (Return x)     = return x
fromEval (Bind m f)     =
    fromEval m >>= fromEval . f
```

## Monad laws

```
return x >>= f = f x  
a >>= return  = a  
(a >>= f) >>= g = a >>= (\ x -> f x >>= g)
```

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```
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```

Does the `Eval` GADT adhere to the monad laws?

# Monad laws

```
return x >>= f = f x  
a >>= return    = a  
(a >>= f) >>= g = a >>= (\ x -> f x >>= g)
```

Does the `Eval` GADT adhere to the monad laws?

Strictly speaking, it quite obviously does not.

## How bad is this?

- One can argue that violating the monad laws to a certain extent is ok if the violation is ultimately not observable.
- For example, if `Eval` is always interpreted into `MT.Eval` and `MT.Eval` adheres to the monad laws, that seems ok.
- But for every function we define that operates on `Eval`, we inherit a proof obligation that it must be compatible with the monad laws.
- Wouldn't it be nice if that wasn't necessary, and we could just define a variant of `Eval` that *does* adhere to the monad laws?



## Free monads

---

## Observation

In essence, the monad laws say that every monadic computation has a normal form:

**do**

```
x1 <- step1  
x2 <- step2  
...  
xn <- stepn  
return something
```

# Observation

In essence, the monad laws say that every monadic computation has a normal form:

**do**

```
x1 <- step1  
x2 <- step2  
...  
xn <- stepn  
return something
```

- Every individual step is followed by a bind.
- We don't need **Bind** as a constructor if we pair it with steps.

## Normalising steps

```
data Eval :: * -> * where
  DivByZeroError :: Eval a
  UnknownVar      :: String -> Eval a
  VarLookup       :: String -> Eval Int
  VarSet          :: String -> Int -> Eval ()
  Return          :: a -> Eval a
  Bind            :: Eval a -> (a -> Eval b)
                  -> Eval b
```

## Normalising steps

```
data Eval :: * -> * where
  DivByZeroError :: Eval a
  UnknownVar      :: String -> Eval a
  VarLookup       :: String -> Eval Int
  VarSet          :: String -> Int -> Eval ()
  Return          :: a -> Eval a
  Bind            :: Eval a -> (a -> Eval b)
                  -> Eval b

varLookup :: String -> (Int -> Eval a) -> Eval a
varLookup x = Bind (VarLookup x)

varSet :: String -> Int -> Eval a -> Eval a
varSet x v k = Bind (VarSet x v) (const k)
```

## Normalising steps

```
data Eval :: * -> * where
  DivByZeroError :: Eval a
  UnknownVar      :: String -> Eval a

  Return          :: a -> Eval a

  VarLookup       :: String -> (Int -> Eval a)
                  -> Eval a

  VarSet          :: String -> Int -> Eval a
                  -> Eval a
```

## Normalising steps

```
data Eval :: * -> * where
  DivByZeroError :: Eval a
  UnknownVar      :: String -> Eval a
  VarLookup       :: String -> (Int -> Eval a)
                  -> Eval a
  VarSet          :: String -> Int -> Eval a
                  -> Eval a
  Return          :: a -> Eval a
```

- No longer a proper GADT.
- We could (but don't need to) add continuations to the error cases, because they “never return” and ignore their continuations.

## Normalising steps

```
data Eval a =  
    DivByZeroError  
  | UnknownVar String  
  | VarLookup String (Int -> Eval a)  
  | VarSet String Int (Eval a)  
  | Return a
```



## Monad instance

```
instance Monad Eval where
  return = Return

  Return x          >>= f = f x
  DivByZeroError >>= _ = DivByZeroError
  UnknownVar x      >>= _ = UnknownVar x
  VarLookup x k     >>= f = VarLookup x
                        (\ v -> k v >>= f)
  VarSet x v k       >>= f = VarSet x v (k >>= f)
```

Instances for `Functor` and `Applicative` as usual.

## Bind is substitution

If we look at terms of type `Eval a` as trees with `Return` at the leaves, then `(>>=)` implements substitution.

## Still implementing the interface?

```
divByZeroError :: Eval a
divByZeroError = DivByZeroError

unknownVar :: String -> Eval a
unknownVar = UnknownVar

varLookup :: String -> Eval Int
varLookup x = VarLookup x Return

varSet :: String -> Int -> Eval ()
varSet x v = VarSet x v (Return ())
```

## Still possible to write an interpreter?

```
fromEval :: Eval a -> MT.Eval a
fromEval DivByZeroError = MT.divByZeroError
fromEval (UnknownVar x) = MT.unknownVar x
fromEval (VarLookup x k) =
    MT.varLookup x >=> fromEval . k
fromEval (VarSet x v k) =
    MT.varSet x v >> fromEval k
fromEval (Return x)      = return x
```

## And the monad laws hold as well!

For example,

```
return x >>= f = f x
```

holds by construction.

## And the monad laws hold as well!

For example,

```
return x >>= f = f x
```

holds by construction.

The other two laws can be proved via a simple induction.

## Another similar example

Recall the exercises:

```
data GP a =  
  End a  
| Get (Int -> GP a)  
| Put Int (GP a)
```

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Recall the exercises:

```
data GP a =  
    End a  
  | Get (Int -> GP a)  
  | Put Int (GP a)
```

```
instance Monad GP where  
    return = End  
  
    End a    >>= f = f a  
    Get k    >>= f = Get (\ n -> k n >>= f)  
    Put n k >>= f = Put n (k >>= f)
```

Again, `(>>=)` implements substitution.



# Free monad generalization

---

## Abstracting from the steps

```
data Eval a =  
    DivByZeroError  
  | UnknownVar String  
  | VarLookup String (Int -> Eval a)  
  | VarSet String Int (Eval a)  
  | Return a
```

## Abstracting from the steps

```
data EvalOp a =  
    DivByZeroError  
  | UnknownVar String  
  | VarLookup String (Int -> Eval a)  
  | VarSet String Int (Eval a)  
  
data Eval a =  
    Return a  
  | Wrap (EvalOp a)
```

## Abstracting from the steps

```
data EvalOp b =  
    DivByZeroError  
  | UnknownVar String  
  | VarLookup String (Int -> b)  
  | VarSet String Int b  
data Eval a =  
    Return a  
  | Wrap (EvalOp (Eval a))
```

## Abstracting from the steps

```
data EvalOp b =  
    DivByZeroError  
  | UnknownVar String  
  | VarLookup String (Int -> b)  
  | VarSet String Int b  
data Free f a =  
    Return a  
  | Wrap (f (Free f a))  
type Eval = Free EvalOp
```

## Free vs. Fix

```
data Fix f =  
  In (f (Fix f))
```

```
data Free f a =  
  Return a  
  | Wrap (f (Free f a))
```

The `Free` type is like a type-level fixed point with a built-in possibility to stop via `Return`.

## The same construction for **GP**

```
data GP a =  
    End a  
  | Get (Int -> GP a)  
  | Put Int (GP a)
```

```
data GPOp b =  
    Get (Int -> b)  
  | Put Int b  
type GP = Free GPOp
```

## Monad instance for **Free**

```
data Free f a =  
    Return a  
  | Wrap (f (Free f a))
```

```
instance Functor f => Monad (Free f) where  
    return :: a -> Free f a  
    return = Return  
  
    (>=) :: Free f a -> (a -> Free f b) -> Free f b  
    Return x >= f = f x  
    Wrap c    >= f = Wrap (fmap (>= f) c)
```

For **every** functor, we get a monad instance **for free**.



## Free structures, categorically

In category theory, a structure is called **free** if it makes no more assumptions as necessary.

# Free structures, categorically

In category theory, a structure is called **free** if it makes no more assumptions as necessary.

If we start with a functor and make no further assumptions than that the monad laws should hold, we obtain the free monad of that functor.

## Are `EvalOp` and `GPOp` functors?

```
data EvalOp b =  
    DivByZeroError  
  | UnknownVar String  
  | VarLookup String (Int -> b)  
  | VarSet String Int b  
deriving Functor
```

```
data GPOp b =  
    Get (Int -> b)  
  | Put Int b  
deriving Functor
```

## More examples

---

# The constant functor

Defined in `Data.Functor.Const`:

```
newtype Const a b = Const {getConst :: a}  
instance Functor (Const a) where  
    fmap _ (Const a) = Const a
```

## Question

What is `Free (Const ())`?

## Question

What is `Free (Const ())`?

```
Return :: a -> Free (Const ()) a
```

```
Wrap   :: Const () (Free (Const ()) a)  
        -> Free (Const ()) a  
≈ () -> Free (Const ()) a  
≈ Free (Const ()) a
```

## Question

What is `Free (Const ())`?

```
Return :: a -> Free (Const ()) a
Wrap   :: Const () (Free (Const ()) a)
        -> Free (Const ()) a
        ≈ () -> Free (Const ()) a
        ≈ Free (Const ()) a
```

This is `Maybe`!



## Question

What is `Free (Const Void)`?

Where

```
data Void
```

is an uninhabited type.

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What is `Free (Const Void)`?

Where

```
data Void
```

is an uninhabited type.

```
Return :: a -> Free (Const Void) a
```

But `Wrap` cannot be applied at all.

## Question

What is `Free (Const Void)`?

Where

```
data Void
```

is an uninhabited type.

```
Return :: a -> Free (Const Void) a
```

But `Wrap` cannot be applied at all.

This is `Identity`!

## Intermediate summary

- For every functor, we can obtain a monad.
- The functor describes what operations we can perform in every step.
- The values are just terms.
- Bind is substitution.

## Intermediate summary

- For every functor, we can obtain a monad.
- The functor describes what operations we can perform in every step.
- The values are just terms.
- Bind is substitution.
- The primary advantage of using free monads is that we get to analyze the terms and provide possibly several interpretations of them.
- One disadvantage of free monads is that in the default definition, `(>>=)` is potentially very inefficient (can you see why)?

# Several interpretations

Again, recall the exercise for **GP**:

- We had one interpretation as an **IO** action.
- We had another interpretation as a simulation, taking predefined inputs to predefined outputs.

# Several interpretations

Again, recall the exercise for **GP**:

- We had one interpretation as an **IO** action.
- We had another interpretation as a simulation, taking predefined inputs to predefined outputs.

For **Eval**, we can still provide multiple interpretations:

- as a monad transformer stack,
- as a monolithic monad providing the correct functionality,
- as a combined monad using some other way to combine effects.

Another example: simulating  
concurrency

---



# Cooperative concurrency

This idea goes back to Koen Claessen:

```
data ProcessOp :: * -> * where
  Atomically :: IO a -> (a -> r) -> ProcessOp r
  Fork       :: Process () -> r -> ProcessOp r
```

Another GADT, although just a special case: the type of the action is hidden from the result – this is often called an **existentially quantified** type. **IO**

# Wrappers

```
type Process = Free ProcessOp
```

```
atomically :: IO a -> Process a  
atomically m = Wrap (Atomically m Return)  
fork :: Process () -> Process ()  
fork p = Wrap (Fork p (Return ()))
```

## Scheduling concurrent operations

```
schedule :: [Process ()] -> IO ()
schedule [] =
  return ()
schedule (Return _ : ps) =
  schedule ps
schedule (Wrap (Atomically m k) : ps) =
  do
    x <- m
    schedule (ps ++ [k x])
schedule (Wrap (Fork p1 p2) : ps) =
  schedule (ps ++ [p2, p1])
```

## Example

```
example :: Process ()
example = do
  fork (replicateM_ 5
        (atomically (putStrLn "Haskell")))
  fork (replicateM_ 6
        (atomically (putStrLn "cryptocurrencies")))
  atomically (putStrLn "2023")
```

## Example (contd.)

```
GHCi> schedule [example]
```

```
Haskell
```

```
2023
```

```
cryptocurrencies
```

```
Haskell
```

```
cryptocurrencies
```

```
Haskell
```

```
cryptocurrencies
```

```
Haskell
```

```
cryptocurrencies
```

```
cryptocurrencies
```

## Using free monads yourself

- The `Free` type (with slightly different constructor names) is provided by the `free` package.
- This also provides a `MonadFree` class and a `FreeT` monad transformer version.

## Fixing the performance problem

There is a trick commonly used, corresponding to a continuation passing style transformation, which avoids the inefficient definition of `(>>=)`.

This is also implemented in the **free** package.