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Worksheet-4 (Laplace Transforms)

Find the Laplace transform of the following functions:

1. Sin(2t) Cos(2t) Ans: $\frac{2}{s^2+16}$

2. $\text{Cos}^2(3t)$ Ans: $\frac{1}{2s} + \frac{1}{2(s^2+36)}$

3. $t\,e^{2t}\,\text{Sin}(3t)$ Ans: $\frac{6(s-2)}{((s-2)^2+9)^2}$

4. $6\,e^{-5t} + e^{3t} + 5\,t^3 - 9$ Ans: $\frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$

5. 4 Cos (4t) − 9 Sin(4t) + 2 Cos(10t) Ans: $\frac{4(s-9)}{s^2+16} + \frac{2s}{s^2+100}$

6. 3Sinh(2t) + 3 Sin(2t) Ans: $\frac{6}{s^2-4} + \frac{6}{s^2+4}$

7. $e^{3t} + \text{Cos}(6t) - e^{3t}\text{Cos}(6t)$ Ans: $\frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$

8. $\frac{e^{bt}-e^{at}}{t}$ Ans: $\ln\left(\frac{s-a}{s-b}\right)$

9. $t^{-\frac{1}{2}}$ Ans: $\frac{\sqrt{\pi}}{s^{\frac{1}{2}}}$

10. $\frac{Sin\ 3t}{t}$ Ans: $\frac{\pi}{2} - \tan^{-1}\frac{s}{3}$

Find the inverse Laplace transform of the following functions:

1. $\frac{4}{s-2} - \frac{3}{s+5}$
2. $\frac{s+5}{s^2+9} = \frac{s}{s^2+9} + \frac{5}{s^2+9}$
3. $\frac{8(s+2)-4}{(s+2)^2+25} = \frac{8(s+2)}{(s+2)^2+25} - \frac{4}{(s+2)^2+25}$
4. $\frac{4}{s} - \frac{1}{s^2} + \frac{5}{s^3} + \frac{2}{s^4}$
5. $\frac{10}{(s-5)^2} + \frac{2}{(s-5)^3}$
6. $\frac{1}{s^2+6s+13}$ (start by completing the square)

Answers

1. $\mathcal{L}^{-1}\{\frac{4}{s-2} - \frac{3}{s+5}\} = 4e^{2t} - 3e^{-5t}$

2. $\mathcal{L}^{-1}\{\frac{s}{s^2+9} + \frac{5}{s^2+9}\} = \cos(3t) + \frac{5}{3}\sin(3t)$

3. $\mathcal{L}^{-1}\{\frac{8(s+2)}{(s+2)^2+25} - \frac{4}{(s+2)^2+25}\} = 8e^{-2t}\cos(5t) - \frac{4}{5}e^{-2t}\sin(5t)$

4. $\mathcal{L}^{-1}\{\frac{4}{s} - \frac{1}{s^2} + \frac{5}{s^3} + \frac{2}{s^4}\} = 4 - t + \frac{5}{2!}t^2 + \frac{2}{3!}t^3$

5. $\mathcal{L}^{-1}\{\frac{10}{(s-5)^2} + \frac{2}{(s-5)^3}\} = 10te^{5t} + \frac{2}{2!}t^2e^{5t} = 10te^{5t} + t^2e^{5t}$

6. $\mathcal{L}^{-1}\{\frac{1}{(s+3)^2+4}\} = \frac{1}{2}e^{-3t}\sin(2t)$

Que: Solve the initial value problem

$y'' - y' - 2y = 4,\ y(0) = 2,\ y'(0) = 1$ Ans: $y = -2 + 3e^{t} + e^{-2t}$

Que: Use Laplace transform to find the solution to

$y'' - 6y' + 5y = 3e^{2t},\ y(0) = 2,\ y'(0) = 3$ Ans: $-e^{2t} + \frac{e^{5t}}{2} + \frac{5e^{t}}{2}$

Que: Solve the IVP with variable coefficients

$ty'' - ty' + y = 2,\ y(0) = 2,\ y'(0) = -4$ Ans: $y(t) = 2 - 4t$

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Course: Mathematics-II (20MA103T)

<u>Worksheet-3 (Ordinary Differential Equations)</u>

1. Solve$(D^3 - 2D^2 - 4D + 8)y = 0$.

[**Ans:** $y = (C_1 + C_2x)e^{2x} + C_3e^{-2x}$]

2. Find the solution of $\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$, where $R^2C = 4L$ and R, C, L are constants.

Ans: $i = (C_1t + C_2)e^{-\left(\frac{R}{2L}\right)t}$, where C_1 and C_2 are arbitrary constants.

3. Solve $(D^4 + 2D^3 - 3D^2)y = 3e^{2x} + 4\sin(x)$

Ans: $y = C_1x + C_2 + C_3e^x + C_4e^{-3x} + \left(\frac{3}{20}\right)e^{2x} + \left(\frac{4}{5}\right)\sin(x) + \left(\frac{2}{5}\right)\cos(x)$

4. Solve $(D^2 - 4D + 1)y = e^{2x}\sin(x)$.

[**Ans**: $y = C_1e^{(2+\sqrt{3})x} + C_2e^{(2-\sqrt{3})x} - \frac{1}{4}e^{2x}\sin(x)$]

5. Solve $(D^2 - 4)y = x^2$. [**Ans**: $y = C_1e^{2x} + C_2e^{-2x} - \frac{1}{4}\left(x^2 + \frac{1}{2}\right)$]

6. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$

Ans: $y = c_1e^x + c_2e^{2x} + x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2e^x - 3xe^x$

7. $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 3x^2 + 4\sin x - 2\cos x$

Ans: $y = c_1 + c_2x + c_3\cos x + c_4\sin x + \frac{1}{4}x^4 - 3x^2 + x\sin x + 2x\cos x$

8. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x^2$

Ans: $y = c_1 e^{x} + c_2 e^{2x} + 2x^2 + 6x + 7$

9. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 6\sin 2x + 7\cos 2x$

Ans: $y = e^{-x}(c_1 \sin 2x + c_2 \cos 2x) + 2\sin 2x - \cos 2x$

10. Solve $x^2y'' - 4xy' + 6y = x$ [**Ans**: $y = C_1x^2 + C_2x^3 + \frac{x}{2}$]

11. Solve $x^2y'' - xy' + y = 2\log(x)$ [**Ans**: $y = (C_1\log(x) + C_2)x + 2\log(x) + 4.$]

12. Solve $(1+x)^2y'' + (1+x)y' + y = 4\cos\big(\log(1+x)\big)$

[**Ans**: $y = C_1\cos\big(\log(1+x)\big) + C_2\sin\big(\log(1+x)\big) + 2\log(1+x)\sin\big(\log(1+x)\big)$.]

13. Solve $(2x-1)^3y''' + (2x-1)y' - 2y = 0$

Ans: $y = C_1(2x-1) + C_2(2x-1)^{4+2\sqrt{3}} + C_3(2x-1)^{4-2\sqrt{3}}$

14. Solve the simultaneous ODEs $\frac{dx}{dt} - 4y = \cos(at)$, $\frac{dy}{dt} + 4x = \sin(at)$.

Ans: $x = C_1\cos(4t) + C_2\sin(4t) + \frac{1}{a+4}\sin(4t)$, $y = -C_1\sin(4t) + C_2\cos(4t) - \frac{1}{a+4}\cos(4t)$

15. $x' + y' + x = -e^{-t}$, $x' + 2y' + 2x + 2y = 0$ given that $x(0) = -1$, $y(0) = 1$.

Ans: $x(t) = -e^{-t}\big(\cos(t) + \sin(t)\big)$, $y(t) = e^{t}\big(1 + \sin(t)\big)$

16. The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u + kr^3 = 0$, where k is constant. Solve the equation under the conditions $u = 0$ when $r = 0$ and $r = a$.

Ans: $u = \frac{kr}{8}(a^2 - r^2)$

Applications of second-order linear ODE

17. A particle is executing simple harmonic motion with amplitude $20\ cm.$ and time $4\ seconds$. Find the time required by the particle in passing between

points which are at distances $15\ cm.$ and $5\ cm.$ from the centre of force and are on the same side of it.

Ans: Required time = $\frac{2}{\pi}\left(\cos^{-1}\frac{1}{4} - \cos^{-1}\frac{3}{4}\right)$ = 0.38 secs.

18. A circuit has in series an electromotive force given by $E = 100\sin 6\,0t$V, a resistor of 2Ω, an inductor of 0.1 H, and a capacitor of $\frac{1}{260}$farads. If the initial current and the initial charge on the capacitor are both zero, find the charge on the capacitor at any time $t > 0$.

Ans: $q = e^{-10t}(\frac{36}{61}\sin 5\,0t + \frac{30}{61}\cos 5\,0t) - \frac{25}{61}\sin 6\,0t - \frac{30}{61}\cos 6\,0t$

19. A 32-lb weight is attached to the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, thereby stretching 2ft. The weight is then pulled down 6 in. below its equilibrium position and released at $t = 0$. No external forces are present; but the resistance of the medium in pounds is numerically equal to $4(\frac{dx}{dt})$, where $\frac{dx}{dt}$is the instantaneous velocity in feet per second. Determine the resulting motion of the weight on the spring.

Ans: $x = e^{-2t}(\frac{\sqrt{3}}{6}\sin 2\sqrt{3}t + \frac{1}{2}\cos 2\sqrt{3}t)$

8

Ordinary Simultaneous Differential Equations

8.1 Introduction

In this chapter, we shall discuss differential equations is which there is one independent variable and two or more than two dependent variables. To solve such equations completely, there must be as many equations as there are dependent variables. Such equations are called its *ordinary simultaneous differential equations*.

8.2 Methods for solving ordinary simultaneous differential equations with constant coefficients

Let x and y be the dependent variables and t be the independent variable. Thus, in such equations there occur differential coefficients of x, y with respect to t. Let $D \equiv d/dt$. Then such equations can be put in the form

$$f_1(D)\, x + f_2(D)\, y = T_1 \qquad \text{... (1)}$$

and

$$g_1(D)\, x + g_2(D)\, y = T_2, \qquad \text{... (2)}$$

where T_1 and T_2 are functions of the independent variable t and $f_1(D)$, $f_2(D)$, $g_1(D)$ and $g_2(D)$ are all rational integral functions of D with constant coefficients. Such equations can be solved by the following two methods.

First method. Method of elimination (use of operator *D*).

In order to eliminate y between (1) and (2), operating on both sides of (1) by $g_2(D)$ and on both sides of (2) by $f_2(D)$ and subtracting, we have

$$\{f_1(D)\, g_2(D) - g_1(D)\, f_2(D)\}x = g_2(D)\, T_1 - f_2(D)\, T_2, \qquad \text{... (3)}$$

which is a linear differential equation with constant coefficients in x and t and can be solved to give the value of x in terms of t. Substituting this value of x in either (1) or (2), we get the value of y in terms of t. Equation (3) is solved by using methods of chapter 5.

Note 1. The above equations (1) and (2) can be also solved by first eliminating x between them and solving the resulting equation to get y in terms of t. Substituting this value of y in either (1) or (2), we get the value of x in terms of t.

Note 2. Since $f_2(D)$ and $g_2(D)$ are functions of D with constant coefficients, so

$$f_2(D)\, g_2(D) = g_2(D)\, f_2(D).$$

Note 3. In the general solutions of (1) and (2) the number of arbitrary constants is equal to the degree of D in the determinant $\Delta = \begin{vmatrix} f_1(D) & f_2(D) \\ g_1(D) & g_2(D) \end{vmatrix}$, provided $\Delta \neq 0$.

If $\Delta = 0$, then the system of equations (1) and (2) is dependent and such cases will not be considered.

Second method. Method of differentiation.

Sometimes, x or y can be eliminated easily if we differentiate (1) or (2). For example, assume that the given equations (1) and (2) connect four quantities x, y, dx/dt and dy/dt. Differentiating (1) and (2) with respect to t, we obtain four equations containing x, dx/dt, d^2x/dt^2, y, dy/dt and d^2y/dt^2. Eliminating three quantities y, dy/dt, d^2y/dt^2 from these four equations, y is eliminated and we get an equation of the second order with x as the dependent and t as the independent variable. Solving this equation we get value of x in terms of t. Substituting this value of x in either (1) or (2), we get value of y in terms of t.

In what follows we present solution of an ordinary simultaneous differential equations by above two methods. In future, we shall use first method or second method as per requirement of the problem.

AN ILLUSTRATIVE SOLVED EXAMPLE

Solve the simultaneous equations $(dx/dt) - 7x + y = 0$ *and* $(dy/dt) - 2x - 5y = 0$. **[Delhi Maths (Prog) 2007-09, 11; Lucknow 2001, 2000, Sagar 2000; Vikram 2003; Meerut 2007, 10]**

Sol. We shall solve the given system by two methods given in Art. 8.2.

First method. Method of elimination (use of operator *D*)

Step 1. Writing D for d/dt, the given equations can be rewritten in the symbolic form as follows:

$$(D-7)\,x + y = 0 \qquad \text{...(1)}$$

and

$$-2x + (D-5)\,y = 0. \qquad \text{...(2)}$$

Step 2. We now eliminate x (say) as follows. Multiplying (1) by 2 and operating (2) by $(D-7)$, we get

$$2(D-7)\,x + 2y = 0 \qquad \text{...(3)}$$

$$-2\,(D-7)\,x + (D-7)\,(D-5)\,y = 0 \qquad \text{...(4)}$$

Adding (3) and (4), $\quad [(D-7)\,(D-5)+2]\,y = 0 \quad$ or $\quad (D^2 - 12D + 37)\,y = 0,$

which is linear equation with constants coefficients.

Its. auxiliary equation is $\quad D^2 - 12D + 37 = 0 \quad$ so that $\quad D = 6 \pm i$

$\therefore \quad y = e^{6t}\,(c_1 \cos t + c_2 \sin t)$, c_1 and c_2 being arbitrary constants. ...(5)

Step 3. We now try to get x by using (5). In this connection remember that we must avoid integration to get x. Thus if we use (1) to get x, then after putting value of y we have to integrate for getting x. Hence we must use (2) because this will not involve any subsequent integration to obtain x. Now from (5), differentiating w.r.t. 't', we get

$$Dy = 6e^{6t}\,[(c_1 \cos t + c_2 \sin t) + e^{6t}(-\,c_1 \sin t + c_2 \cos t)$$

or

$$Dy = e^{6t}\,\{(6c_1 + c_2) \text{ cost } t + (6c_2 - c_1) \sin t] \qquad \text{...(6)}$$

Substituting the values of y and Dy given by (5) and (6) in (2), we have

$$2x = Dy - 5y = e^{6t}\,[6c_1 + c_2) \cos t + (6c_2 - c_1) \sin t - 5(c_1 \cos \mathrm{t} + c_2 \sin t)]$$

or

$$x = (1/2) \times e^{6t}\,[(c_1 + c_2) \cos t + (c_2 - c_1) \sin t] \qquad \text{...(7)}$$

Thus (5) and (7) together give the required solution.

Remark. We can also eliminate y first (as we did to eliminate x) and then obtain x. This value of x can be put in (1) to get the desired value of y.

Second method. Method of differentiation. Given that

$$(dx/dt) - 7x + y = 0 \qquad \text{...(1)}$$

and

$$(dy/dt) - 2x - 5y = 0. \qquad \text{...(2)}$$

To eliminate x, we differentiate (2) w.r.t. 't' and obtain

$$(d^2y/dt^2) - 2(dx/dt) - 5(dy/dt) = 0 \qquad \text{...(3)}$$

Now, from (2), we have

$$x = \frac{1}{2}\left(\frac{dy}{dt} - 5y\right). \qquad \text{...(4)}$$

Then, from (1), we get $$\frac{dx}{dt} = 7x - y = \frac{7}{2}\left(\frac{dy}{dt} - 5y\right) - y, \text{ using (4)}$$

$\therefore$ $$dx/dt = (7/2) \times (dy/dt) - (37y/2)$$

Substituting this value of dx/dt in (3), we have

$$(d^2y/dt^2) - 7\,(dy/dt) + 37y - 5\,(dy/dt) = 0 \quad \text{or} \quad (D^2 - 12D + 37)\,y = 0.$$

Now get y as done in first method. In fact repeat the whole method after this step. Thus we get the same values of x and y as in first method.

Note 1. Second method will be used when found very necessary. In almost all problems we shall use the first method.

Note 2. Generally t will be the independent variable and x and y will be dependent variables. In some problems any other variable, x say, will be given as the independent variable and y and z as the dependent variables. This point should be noted carefully while doing any problem.

8.3 Solved examples based on Art 8.2

Ex. 1. *Solve $dx/dt - y = t$, $dy/dt + x = 1$.* **[Agra 2000, Delhi Maths (G) 1998]**

Sol. Writing D for d/dt, the given equations become

$$Dx - y = t \qquad \dots (1)$$

and $$x + Dy = 1 \qquad \dots (2)$$

Differentiating (1) w.r.t. 't', $$D^2x - Dy = 1 \qquad \dots (3)$$

To eliminate y between (2) and (3), we add them and get

$$D^2x + x = 2 \quad \text{or} \quad (D^2 + 1)\,x = 2. \qquad \dots (4)$$

Now the auxiliary equation of (4) is $D^2 + 1 = 0$ so that $D = \pm\, i$.

$\therefore$ C.F. $= c_1 \cos t + c_2 \sin t$, c_1 and c_2 being arbitrary constants.

and $$P.I. = \frac{1}{1 + D^2}\,2 = (1 + D^2)^{-1}\,2 = (1 - D^2 + \dots)\,2 = 2$$

Hence the general solution of (4) is $$x = c_1 \cos t + c_2 \sin t + 2 \qquad \dots (5)$$

From (5), $$Dx = dx/dt = -\,c_1 \sin t + c_2 \cos t \qquad \dots (6)$$

$\therefore$ From (1), $$y = Dx - t = -\,c_1 \sin t + c_2 \cos t - t. \qquad \dots (7)$$

The required solution is given by (5) and (7).

Ex. 2. *Solve the simultaneous differential equations $dx/dt = 3x + 2y$, $dy/dt = 5x + 3y$.*

[Kanpur 2004, Lucknow 2001, 03]

Sol. Writing D for d/dt, the given equations become

$$(D - 3)\,x - 2y = 0 \qquad \dots (1)$$

and $$-\,5x + (D - 3)\,y = 0 \qquad \dots (2)$$

Operating on both sides of (1) by $(D - 3)$ and multiplying both sides of (2) by 2 and then adding, we have

$$\{(D - 3)^2 - 10\}\,x = 0 \quad \text{or} \quad (D^2 - 6D - 1)\,x = 0. \qquad \dots (3)$$

Now, auxiliary equation of (3) is $D^2 - 6D - 1 = 0$ so that $D = 3 \pm \sqrt{10}$.

$\therefore$ $$x = \text{C.F.} = e^{3t}[c_1 \cosh(t\sqrt{10}) + c_2 \sinh(t\sqrt{10})\}. \qquad \dots (4)$$

From (4), $$Dx = dx/dt = 3\,e^{3t}\,\{c_1 \cosh\,(t\sqrt{10}) + c_2 \sinh\,(t\sqrt{10})\}$$
$$+\, e^{3t}\{c_1\sqrt{10}\sinh(t\sqrt{10}) + c_2\sqrt{10}\cosh(t\sqrt{10})\}$$

or $$Dx = e^{3t}\{(3c_1 + c_2\sqrt{10})\cosh(t\sqrt{10}) + (3c_2 + c_1\sqrt{10})\sinh(t\sqrt{10})\} \qquad \dots (5)$$

Then, from (1), we have $\quad y = (1/2) \times (D - 3)\, x = (1/2) \times (Dx - 3x)$

i.e., $$y = (1/2) \times [e^{3t}\{(3c_1 + c_2\sqrt{10})\cosh(t\sqrt{10}) + (3c_2 + c_1\sqrt{10})\sinh(t\sqrt{10})\}$$
$$- 3e^{3t}\{c_1 \cosh(t\sqrt{10}) + c_2 \sinh(t\sqrt{10})\}], \text{ using (4) and (5)}$$

$$\therefore \qquad y = (\sqrt{10}/2) \times e^{3t}[c_2 \cosh(t\sqrt{10}) + c_1 \sinh(t\sqrt{10})] \qquad \dots (6)$$

The general solution is given by (4) and (6).

Ex. 3. *Solve the simultaneous differential equations* $(D - 17)\, y + (2D - 8)\, z = 0,$ $(13D - 53)\, y - 2z = 0,$ *where* $D \equiv d/dt$.

Sol. Given $$(D - 17)\, y + 2\,(D - 4)\, z = 0 \qquad \dots (1)$$

and $$(13D - 53)\, y - 2z = 0 \qquad \dots (2)$$

Operating on both sides of (2) by $(D - 4)$ and then adding to (1), we have

$$\{(D - 17) + (D - 4)(13D - 53)\}\, y = 0 \quad \text{or} \quad (D^2 - 8D - 15)\, y = 0 \qquad \dots (3)$$

Here auxiliary equation is $\quad D^2 - 8D - 15 = 0 \quad$ so that $\quad D = 3, 5.$

$$\therefore \qquad y = \text{C.F.} = c_1 e^{3x} + c_2 e^{5x},\ c_1 \text{ and } c_2 \text{ being arbitrary constants} \qquad \dots (4)$$

From (4), $$Dy = dy/dx = 3c_1 e^{3x} + 5c_2 e^{5x} \qquad \dots (5)$$

From (2), $$2z = 13Dy - 53y$$

or $$2z = 13\,(3c_1 e^{3x} + 5c_2 e^{5x}) - 53\,(c_1 e^{3x} + c_2 e^{5x}), \text{ by (4) and (5)}$$

$$\therefore \qquad z = 6C_2 e^{5x} - 7C_1 e^{3x} \qquad \dots (6)$$

The required general solution is given by (4) and (6).

Ex. 4(*a*). *Solve* $(dx/dt) + 5x + y = e^t,\ (dy/dt) - x + 3y = e^{2t}$. **[Kanpur 2005, Garhwal 2005, Delhi Maths (Hons.) 2000, 02, Delhi Maths (G) 2000]**

Sol. Given $$(D + 5)\, x + y = e^t \qquad \dots (1)$$

and $$-x + (D + 3)\, y = e^{2t} \qquad \dots (2)$$

Operating on both sides of (2) by $(D + 5)$, we get

$$-(D + 5)\, x + (D + 5)(D + 3)\, y = (D + 5)\, e^{2t} = 2e^{2t} + 5e^{2t}, \qquad \dots (3)$$

Adding (1) and (3), $$\{1 + (D + 5)(D + 3)\}\, y = e^t + 7e^{2t}$$

or $$(D + 4)^2\, y = e^t + 7e^{2t} \qquad \dots (4)$$

Its auxiliary equation is $\quad (D + 4)^2 = 0 \quad$ so that $\quad D = -4, -4.$

$$\therefore \qquad \text{C.F.} = (c_1 + c_2 t)\, e^{-4t}\ c_1 \text{ and } c_2, \text{ being arbitrary constants.}$$

$$P.I. = \frac{1}{(D+4)^2}(e^t + 7e^{2t}) = \frac{1}{(D+4)^2}e^t + 7\frac{1}{(D+4)^2}e^{2t} = \frac{1}{(1+4)^2}e^t + 7\frac{1}{(2+4)^2}e^{2t} = \frac{1}{25}e^t + \frac{7}{36}e^{2t}.$$

$\therefore$ Solution of (4) is $$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 t)\, e^{-4t} + (1/25)\, e^t + (7/36)\, e^{2t} \qquad \dots (5)$$

From (5), $$Dy = dy/dt = -4(c_1 + c_2 t)e^{-4t} + c_2 e^{-4t} + (1/25)\, e^t + (7/18)\, e^{2t} \qquad \dots (6)$$

$\therefore$ From (2), $\quad x = Dy + 3y - e^{2t}$, Using (5) and (6), this gives

$$x = -4\,(c_1 + c_2 t)\, e^{-4t} + c_2 e^{-4t} + (1/25)\, e^t + (7/18)\, e^{2t} + 3\,[(c_1 + c_2 t)\, e^{-4t} + (1/25)\, e^t + (7/36)\, e^{2t}] - e^{2t}$$

or $$x = -(c_1 + c_2 t)\, e^{-4t} + c_2 e^{-4t} + (4/25)\, e^t - (1/36)\, e^{2t}. \qquad \dots (7)$$

The required general solution is given by (5) and (7).

Ex. 4(*b*). *Solve* $dx/dt + 2y + x = e^t,\ dy/dt + 2x + y = 3e^t$. **[Delhi Maths (H) 2009]**

Sol. Writing D for d/dt, the given equations become

$$(D + 1)\, x + 2y = e^t \qquad \dots (1)$$

and $$2x + (D + 1)\, y = 3e^t \qquad \dots (2)$$

Operating on both sides of (1) by $(D + 1)$ and multiplying both sides of (2) by 2 and then subtracting, we get

$$[(D + 1)^2 - 4]\, x = (D + 1)\, e^t - 6e^t \quad \text{or} \quad (D^2 + 2D - 3)x = - 4e^t \qquad \text{... (3)}$$

The auxiliary equation is $D^2 + 2D - 3 = 0$ so that $D = 1, - 3.$

$\therefore$ C.F. $= c_1 e^t + c_2 e^{-3t}$, c_1 and c_2, being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{D^2 + 2D - 3}(-4e^t) = -4\frac{1}{(D-1)}\frac{1}{(D+3)}e^t$$

$$= -4\frac{1}{D-1}\frac{1}{1+3}e^t = -\frac{1}{(D-1)}e^t = -\frac{t}{1!}e^t, \text{ as } \frac{1}{(D-a)^n}e^{at} = \frac{t^n}{n!}e^{at}$$

$\therefore$ Solution of (3) is $\quad x = c_1 e^t + c_2 e^{-3t} - t\, e^t \qquad$ ... (4)

From (4), $\quad Dx = dx/dt = c_1 e^t - 3c_2 e^{-3t} - (e^t + t\, e^t) \qquad$ ... (5)

Now, $\quad 2y = e^t - Dx - x$, using (1)

or $\quad 2y = e^t - (c_1 e^t - 3c_2 e^{-3t} - e^t - t\, e^t) - (c_1 e^t + c_2 e^{-3t} - t\, e^t)$, using (4) and (5)

or $\quad y = e^t - c_1 e^t + c_2 e^{-3t} + t\, e^t \qquad$ ... (6)

The required general solution is given by (4) and (6).

Ex. 4(*c*). *Solve* $(dx/dt) + 2\, (dy/dt) - x + y = 0$ *and* $2\, (dx/dt) + (dy/dt) + 2x + y = 3e^{-t}$.

[Delhi Maths (Hons.) 1998]

Sol. Writing D for d/dt, the given equations become

$$(D - 1)\, x + (2D + 1)\, y = 0 \qquad \text{... (1)}$$

and
$$2(D + 1)\, x + (D + 1)\, y = 3e^{-t} \qquad \text{... (2)}$$

Operating on both sides of (1) by $(D + 1)$ and (2) by $(2D + 1)$ and then subtracting, we have

$$[(D + 1)\, (D - 1) - 2(2D + 1)\, (D + 1)]\, x = 0 - (2D + 1)\, (3e^{-t})$$

or
$$[D^2 - 1 - 2\, (2D^2 + 3D + 1)]\, x = - 6D\, e^{-t} - 3e^{-t} = 6e^{-t} - 3e^{-t} = 3e^{-t}$$

or
$$(- 3D^2 - 6D - 3)\, x = 3e^{-t} \quad \text{or} \quad (D + 1)^2\, x = - e^{-t} \qquad \text{... (3)}$$

Its auxiliary equation is $(D + 1)^2 = 0$ so that $D = - 1, - 1.$

$\therefore$ C.F. $= (c_1 + c_2 t)\, e^{-t}$, c_1 and c_2 being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{(D+1)^2}(-e^{-t}) = -\frac{t^2}{2!}e^{-t}, \quad \text{as } \frac{1}{(D-a)^n}e^{at} = \frac{t^n}{n!}e^{at}$$

$\therefore$ Solution of (3) is $\quad x = (c_1 + c_2 t) - (1/2) \times t^2 e^{-t}. \qquad$ ... (4)

From (4), $\quad Dx = - (c_1 + c_2 t)\, e^{-t} + c_2 e^{-t} - (1/2) \times (2t\, e^{-t} - t^2 e^{-t}) \qquad$ ... (5)

Multiplying both sides of (2) by 2, we have

$$(4D + 4)\, x + (2D + 2)\, y = 6e^{-t} \qquad \text{... (6)}$$

Subtracting (1) from (6), we have

$$(3D + 5)\, x + y = 6\, e^{-t} \quad \text{or} \quad y = 6\, e^{-t} - 3Dx - 5x$$

or $\quad y = 6\, e^{-t} - 3\, [- (c_1 + c_2 t)\, e^{-t} + c_2 e^{-t} - (1/2) \times (2t\, e^{-t} - t^2 e^{-t})] - 5[(c_1 + c_2 t)\, e^{-t} - (1/2) \times t^2 e^{-t}]$

or
$$y = 6\, e^{-t} - 2\, (c_1 + c_2 t)\, e^{-t} - 3c_2 e^{-t} + 3t\, e^{-t} + t^2 e^{-t}$$

or
$$y = - 2\, (c_1 + c_2 t)\, e^{-t} - 3c_2 e^{-t} + (t^2 + 3t + 6)\, e^{-t}. \qquad \text{... (7)}$$

The required general solution is given by (4) and (7).

Ex. 5. *Solve* $(dx/dt) - y = t^2$, $(dy/dt) + 4x = t$, *given* $x\, (0) = 0$ *and* $y\, (0) = 3/4$.

Sol. Writing D for d/dt, the given equations become

$$Dx - y = t^2 \qquad \text{... (1)}$$

and
$$4x + Dy = t \qquad \text{... (2)}$$

Operating on both sides of (1) by D and adding to (2), we get

$$D^2x + 4x = Dt^2 + t \quad \text{or} \quad (D^2 + 4)\, x = 2t + t = 3t \qquad \text{.... (3)}$$

Its auxiliary equation is $D^2 + 4 = 0$ so that $D = \pm 2i$

$\therefore$ C.F. $= c_1 \cos 2t + c_2 \sin 2t$, c_1 and c_2 being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{D^2+4} 3t = 3\frac{1}{4(1+D^2/4)} t = \frac{3}{4}\left(1+\frac{D^2}{4}\right)^{-1} = \frac{3}{4}\left(1-\frac{D^2}{4}+\dots\right)t = \frac{3t}{4}$$

$\therefore$ Solution of (3) is $x = c_1 \cos 2t + c_2 \sin 2t + (3t/4)$. ... (4)

From (4), $Dx = dx/dt = -2c_1 \sin 2t + 2c_2 \cos 2t + (3/4)$. ... (5)

From (1) and (5), $y = Dx - t^2 = -2c_1 \sin 2t + 2c_2 \cos 2t + (3/4) - t^2$ ... (6)

Putting $t = 0$ in (4) and using the fact that $x(0) = 0$, we get $c_1 = 0$. Again, putting $t = 0$ in (6) and using the fact that $y(0) = 3/4$, we get $3/4 = 2c_2 + 3/4$ so that $c_2 = 0$.

Hence, from (4) and (6), the required solution is $x = (3t/4)$, $y = (3/4) - t^2$.

Ex. 6. *Solve* $dydt = y$, $dx/dt = 2y + x$. **[Delhi Maths (G) 2000]**

Sol. Given that $dy/dt = y$ ... (1)

and $dx/dt = 2y + x$ ... (2)

From (1), $(1/y)\, dy = dt$.

Integrating, $\log y - \log c_1 = t$ or $y = c_1 e^t$ ... (3)

Substituting this value of y in (2), we have $(dx/dt) = 2c_1e^t + x$ or $(dx/dt) - x = 2c_1e^t$,

which is a linear equation. Its I.F. $= e^{\int(-1)dt} = e^{-t}$ and solution is

$$x \cdot e^{-t} = \int (2c_1e^t)\cdot e^{-t}\, dt + c_2 = 2c_1t + c_2, \quad \text{or} \quad x = (2c_1t + c_2)\, e^t$$

where c_1 and c_2 are arbitrary constants.

Hence the required solution is given by $x = (2c_1t + c_2)\, e^t$, $y = c_1e^t$.

Ex. 7(*a*). *Solve* $(dx/dt) + 4x + 3y = t$, $(dy/dt) + 2x + 5y = e^t$. **[Garhwal 2003; Lucknow 2003; Kerala 2001; Karnataka 2002; Vikram 2000; Osmania 2004, Meerut 2011; Delhi Maths (G) 1994, Delhi Maths (Hons.) 1999]**

Sol. Writing D for d/dt, the given equations become

$$(D + 4)\, x + 3y = t \qquad \text{... (1)}$$

and
$$2x + (D + 5)\, y = e^t \qquad \text{... (2)}$$

Operating on both sides of (1) by $(D + 5)$ and multiplying both sides of (2) by 3 and then subtracting, we get

$$\{(D + 5)(D + 4) - 6\}\, x = (D + 5)\, t - 3e^t \quad \text{or} \quad (D^2 + 9D + 14)\, x = 1 + 5t - 3e^t \quad \text{...(3)}$$

Its auxiliary equation is $D^2 + 9D + 14 = 0$ so that $D = -2, -7$.

$\therefore$ C.F. $= c_1 e^{-2t} + c_2 e^{-7t}$, c_1 and c_2 being arbitrary constants.

P.I. corresponding to $(1 + 5t)$

$$= \frac{1}{14+9D+D^2}(1+5t) = \frac{1}{14[1+(9/14)\,D+(1/14)D^2]}(1+5t) = \frac{1}{14}\left[1+\left(\frac{9}{14}D+\frac{1}{14}D^2\right)\right]^{-1}(1+5t)$$

$$= \frac{1}{14}\left[1-\left(\frac{9}{14}D+\frac{1}{14}D^2\right)+\dots\right](1+5t) = \frac{1}{14}\left[1+5t-\frac{9}{14}D(1+5t)\right] = \frac{1}{14}\left[1+5t-\frac{9}{14}\times 5\right] = \frac{5t}{14}-\frac{31}{196}.$$

P.I. corresponding to $(-3e^t) = \dfrac{1}{14+9D+D^2}(-3e^t) = -3\dfrac{1}{14+9\cdot 1+1^2}e^t = -\dfrac{3}{24}e^t = -\dfrac{e^t}{8}$.

$\therefore$ Solution of (3) is $\quad x = \text{C.F.} + \text{P.I.} = c_1e^{-2t} + c_2e^{-7t} + (5/14)\,t - (31/196) - (1/8)\,e^t \quad$ ...(4)

$\therefore \qquad Dx = dx/dt = -2c_1e^{-2t} - 7c_2e^{-7t} + (5/14) - (1/8)\,e^t. \qquad$ ...(5)

From (1), $\qquad 3y = t - Dx - 4x$. $\quad$ Using (4) and (5), this gives

$3y = t - [-2c_1e^{-2t} - 7c_2e^{-7t} + (5/14) - (1/8)\,e^t] - 4\,[c_1e^{-2t} + c_2e^{-7t} + (5/14)\,t - (31/196) - (1/8)\,e^t]$

or $\qquad y = (1/3)\,[-2c_1\,e^{-2t} + 3c_2\,e^{-7t} + (5/8)\,e^t + (27/98) - (3/7)\,t] \qquad$ ... (5)

The required general solution is given by (3) and (5).

Ex. 7(*b*). *Solve* $dx/dt + 2x - 3y = t,\ dy/dt - 3x + 2y = e^{2t}$.

[Ujjain 2003, Delhi Maths 2001; Delhi B.A. (Prog) II 2010]

Sol. Let $D \equiv d/dt$. Then the given equations become

$$(D+2)\,x - 3y = t \qquad \text{... (1)}$$

and
$$-3x + (D+2)\,y = e^{2t} \qquad \text{... (2)}$$

Eliminating y from (1) and (2), we have $\qquad (D+2)^2\,x - 9x = (D+2)\,t + 3\,e^{2t}$

or
$$(D^2 + 4D - 5)\,x = 2t + 1 + 3\,e^{2t} \qquad \text{... (3)}$$

Auxiliary equation for (3) is $\qquad D^2 + 4D - 5 = 0. \qquad$ Hence $\qquad D = 1, -5.$

$\therefore$ C.F. of (3) $= c_1\,e^t + c_2\,e^{-5t}$, $\ c_1$ and c_2 being arbitrary constants

P.I. corresponding to $(2t+1)$

$$= \frac{1}{D^2+4D-5}(2t+1) = -\frac{1}{5}\left[1-\left(\frac{4D}{5}+\frac{D^2}{5}\right)\right]^{-1}(2t+1) \quad = -\frac{1}{5}\left(1+\frac{4D}{5}+\ldots\right)(2t+1)$$

$$= -(1/5)\times(2t+1+8/5) = (10t+13)/25$$

P.I. corresponding to $3\,e^{2t} = 3\dfrac{1}{D^3+4D-5}e^{2t} = 3\dfrac{1}{4+8-5}e^{2t} = \dfrac{3}{7}e^{2t}$.

Hence the general solution of (3) is

$$x = c_1e^t + c_2\,e^{-5t} + (3/7)\,e^{2t} - (1/25)\,(10\,t + 13). \qquad \text{... (4)}$$

$\therefore$
$$Dx = c_1\,e^t - 5c_2\,e^{-5t} + (6/7)\,e^{2t} - (2/5). \qquad \text{... (5)}$$

From (1), $3y = Dx + 2x - t$. Using (4) and (5), it gives

$$3y = 3c_1\,e^t - 3c_2\,e^{-5t} + (12/7)\,e^{2t} - (9/5)\,t - (36/25)$$

$\therefore$
$$y = c_1\,e^t - c_2\,e^{-5t} + (4/7)\,e^{2t} - (3/5)\,t - (12/25) \qquad \text{... (6)}$$

The required solution is given by (4) and (6).

Ex. 7(c). *Solve* $dx/dt + dy/dt - 2y = 2\cos t - 7\sin t,\ dx/dt - dy/dt + 2x = 4\cos t - 3\sin t$.

[Lucknow 2005; Pune 2000; Delhi Maths (G) 2005; Agra 2002; Kanpur 1998]

Sol. Let $D \equiv d/dt$. Then the given equations become

$$Dx + (D-2)\,y = 2\cos t - 7\sin t \qquad \text{... (1)}$$

and
$$(D+2)\,x - Dy = 4\cos t - 3\sin t \qquad \text{... (2)}$$

Eliminating y from (1) and (2), we get

$$[D^2 + (D-2)\,(D+2)]x = D(2\cos t - \sin t) + (D-2)(4\cos t - 3\sin t)$$

or
$$(D^2 - 2)\,x = -9\cos t, \text{ on simplification} \qquad \text{... (3)}$$

Auxiliary equation is $\qquad D^2 - 2 = 0 \qquad$ giving $\qquad D = \pm\sqrt{2}$

$\therefore$ C.F. $= c_1 e^{\sqrt{2}\,t} + c_2 e^{-\sqrt{2}\,t}$, c_1 and c_2 being arbitrary constants.

Also, $$\text{P.I.} = -\frac{9}{D^2-2}\cos t = \frac{-9}{-1^2-2}\cos t = 3\cos t.$$

$\therefore$ Solution of (3) is $$x = c_1 e^{\sqrt{2}\,t} + c_2 e^{\sqrt{2}t} + 3\cos t. \qquad \text{... (4)}$$

From (4), $$Dx = c_1\sqrt{2}\, e^{\sqrt{2}\,t} - c_2\sqrt{2}\, e^{-\sqrt{2}\,t} - 3\sin t \qquad \text{... (5)}$$

Adding (1) and (2), $$2Dx + 2x - 2y = 6\cos t - 10\sin t$$

$\therefore$ $$y = Dx + x - 3\cos t + 5\sin t$$

or $y = c_1\sqrt{2}\, e^{\sqrt{2}\,t} - c_2\sqrt{2}\, e^{-\sqrt{2}\,t} - 3\sin t + c_1 e^{\sqrt{2}\,t} + c_2 e^{-\sqrt{2}\,t} + 3\cos t - 3\cos t + 5\sin t$, by (4) and (5)

Thus, $$y = (1+\sqrt{2})\, c_1 e^{\sqrt{2}\,t} + (1-\sqrt{2})\, c_2 e^{-\sqrt{2}\,t} + 2\sin t \qquad \text{... (6)}$$

The required solution is given by (4) and (6).

Ex. 7(*d*). *Solve the equations $4(dx/dt) + 9\,(dy/dt) + 11x + 31y = e^t$, $3\,(dx/dt) + 7\,(dy/dt) + 8x + 24y = e^{2t}$* **[Lucknow 1998, Meerut 1996]**

Sol. Writing D for d/dt, the given equations become

$$(4D + 11)\,x + (9D + 31)\,y = e^t \qquad \text{... (1)}$$

and $$(3D + 8)\,x + (7D + 24)\,y = e^{2t} \qquad \text{... (2)}$$

Operating on both sides of (1) by $(7D + 24)$ and (2) by $(9D + 31)$ and then subtracting, we have $$\{(7D + 24)\,(4D + 11) - (9D + 31)\,(3D + 8)\}\,x = (7D + 24)\,e^t - (9D + 31)\,e^{2t}$$

or $$(D + 4)^2\,x = 31\,e^t - 49\,e^{2t} \qquad \text{... (3)}$$

Its auxiliary equation is $(D + 4)^2 = 0$ so that $D = -4, -4$

$\therefore$ C.F. $= (c_1 + c_2\,t)\,e^{-4t}$, c_1 and c_2 being arbitrary constants

and $$\text{P.I.} = \frac{1}{(D+4)^4}(31e^t - 49e^{2t}) = 31\frac{1}{(D+4)^2}e^t - 49\frac{1}{(D+4)^4}e^{2t}$$

$$= 31\frac{1}{(1+4)^2}e^t - 49\frac{1}{(2+4)^2}e^{2t} = \frac{31}{25}e^t - \frac{49}{36}e^{2t}.$$

$\therefore$ Solution of (3) is $$x = \text{C.F.} + \text{P.I.} = (c_1 + c_2 t)\,e^{-4t} + (31/25)\,e^t - (49/36)\,e^{2t} \qquad \text{... (4)}$$

From (4), $$Dx = dx/dt = c_2 e^{-4t} - 4\,(c_1 + c_2 t)\,e^{-4t} + (31/25)\,e^t - (49/18)e^{2t} \qquad \text{... (5)}$$

Now, multiplying both sides of (1) by 7 and (2) by 9, we get

$$(28D + 77)\,x + (63D + 217)\,y = 7\,e^t \qquad \text{... (6)}$$

and $$(27D + 72)\,x + (63D + 216)\,y = 9\,e^{2t}. \qquad \text{... (7)}$$

Subtracting (7) from (6), $Dx + 5x + y = 7\,e^t - 9\,e^{2t}$ or $y = -Dx - 5x + 7e^t - 9e^{2t}$

or $y = -\,[c_2 e^{-4t} - 4\,(c_1 + c_2\,t)\,e^{-4t} + (31/25)\,e^t - (49/18)\,e^{2t}]$

$- 5\,[(c_1 + c_2\,t)\,e^{-4t} + (31/25)\,e^t - (49/36)\,e^{2t}] + 7\,e^t - 9\,e^{2t}$, by (4) and (5)

or $$y = -\,(c_2 + c_1 + c_2 t)\,e^{-4t} + (19/36)\,e^{2t} - (11/25)\,e^t \qquad \text{.... (8)}$$

The required general solution is given by (4) and (8).

Ex. 8. *Solve the following simultaneous equations :*

(i) $dy/dx + y = z + e^x$, $dz/dx + z = y + e^x$. **[Delhi Maths (P) 2005]**

(ii) $dx/dt + x = y + e^t$, $dy/dt + y = x + e^t$. **[Delhi Maths Hons. 2005]**

Sol. (*i*) Writing D for d/dx, the given equations become

$$(D + 1)\,y - z = e^x \qquad \text{... (1)}$$

and $$-\,y + (D + 1)\,z = e^x \qquad \text{... (2)}$$

Operating (1) by $(D + 1)$, we get $\quad (D + 1)^2 y - (D + 1) z = (D + 1) e^x \quad$ ... (3)

Adding (2) and (3), we get $\quad [(D + 1)^2 - 1] y = e^x + (e^x + e^x)$

or $\quad (D^2 + 2D) y = 3e^x \quad$ or $\quad D (D + 2) y = 3e^x \quad$ ...(4)

Auxiliary equation of (4) is $\quad D (D + 2) = 0 \quad$ giving $\quad D = 0, -2.$

$\therefore$ C.F. $= c_1 + c_2 e^{-2x}$ and P.I. $= 3\dfrac{1}{D(D+2)}e^x = 3\dfrac{1}{1\times(1+2)}e^x = e^x.$

$\therefore$ Solution of (4) is $\quad y = c_1 + c_2 e^{-2x} + e^x$, c_1, c_2 being arbitrary constants. ... (4)

From (4), $\quad Dy = dy/dx = -2c_2 e^{-2x} + e^x \quad$ ... (5)

$\therefore$ From (1), $\quad z = Dy + y - e^x = -2c_2 e^{-2x} + e^x + c_1 + c_2 e^{-2x} + e^x - e^x$, by (4) and (5)

or $\quad z = c_1 - c_2 e^{-2x} + e^x \quad$ ... (6)

The required solution is given by (4) and (6).

(*ii*) This is just the same as (*i*). Here we have t in place of x and x and y in place of y and z. You have to denote d/dt by D. **Ans.** $x = c_1 + c_2 e^{-2t} + e^t, y = c_1 - c_2 e^{-2t} + e^t.$

Ex. 9(*a*). *Solve* $dx/dt = ax + by$, $dy/dt = b x + ay$

[Punjab 2005; G.N.D.U. Amritsar 2000; Garhwal 1998, Lucknow 1999]

Sol. Writing D of d/dt, the given equations become

$$(D - a) x - by = 0 \qquad \text{... (1)}$$

and

$$-bx + (D - a) y = 0 \qquad \text{... (2)}$$

Operating both sides of (1) by $(D - a)$ and multiplying (2) by b, we get

$$(D - a)^2 x - b (D - a) y = 0$$

and

$$-b^2x + b (D - a) y = 0$$

Adding these, $\quad [(D - a)^2 - b^2] x = 0 \quad$ or $\quad (D - a - b) (D - a + b) x = 0 \quad$ ... (3)

Its auxiliary equation $\quad (D - a - b) (D - a + b) = 0$ yields $\quad D = a + b$ and $D = a - b$.

Hence, solution of (3) is $\quad x = c_1e^{(a+b)t} + c_2 e^{(a-b)t}$, c_1, c_2 being arbitrary constants ... (4)

From (4), $\quad dx/dt = c_1 (a + b) e^{(a+b)t} + c_2(a - b) e^{(a-b)t} \quad$ ... (5)

From the first given differential equatioin, we have

$y = (1/b) \times \{dx/dt - ax\}$

$\quad = (1/b) \times \{c_1(a + b) e^{(a+b)t} + c_2 (a - b) e^{(a-b)t} - ac_1e^{(a+b)t} - ac_2e^{(a-b)t}]$, using (4) and (5)

or $\quad y = c_1 e^{(a+b)t} - c_2 e^{(a-b)t}$ on simplification ... (6)

(4) and (6) together give the required solution.

Ex. 9(*b*). *Solve* $dx/dt = ax + by$, $dy/dt = a' x + b' y$.

[Garhwal 1999, G.N.D.U. Amritsar 2000, Lucknow 1999]

Sol. Writing D for d/dt, the given equations become

$$(D - a) x - by = 0 \qquad \text{... (1)}$$

and

$$-a'x + (D - b') y = 0 \qquad \text{... (2)}$$

Operating both sides of (1) by $(D - b')$ and multiplying (2) by b, we get

$$(D - b') (D - a) x - b (D - b') y = 0 \qquad \text{... (3)}$$

and

$$-a'bx + b (D - b) y = 0 \qquad \text{... (4)}$$

Adding (3) and (4), $\quad [(D - b') (D - a) - a'b] x = 0$

or $\quad [D^2 - D (a + b') + (ab' - a'b)] x = 0, \quad$ ... (5)

Its auxiliary equation is $\quad D^2 - D (a + b') + (ab' - a'b) = 0,$

giving

$$D = \frac{a+b' \pm \sqrt{\{(a+b')^2 - 4(ab'-a'b)\}}}{2} = \frac{a+b' \pm \sqrt{\{(a-b')^2 + 4a'b\}}}{2}$$

$\therefore$ $D = (1/2) \times [a + b' + \{(a - b')^2 + 4a'b\}^{1/2}] = \alpha_1$, say

and $D = (1/2) \times [a + b' - \{(a - b')^2 + 4a'b\}^{1/2}] = \alpha_2$, say

$\therefore$ Solution of (5) is $x = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t}$, c_1 and c_2 being arbitrary constants ... (6)

(1) $\Rightarrow by = (D - a)\, x$ or $y = (1/b) \times \{(dx/dt) - ax\}$

$\therefore$ $y = (1/b) \times [c_1 \alpha_1 e^{\alpha_1 t} + c_2 \alpha_2 e^{\alpha_2 t} - a(c_1 \alpha^{\alpha_1 t} + c_2 e^{\alpha_2 t})]$, by (6)

or $y = (1/b) \times [c_1 (\alpha_1 - a)\, e^{\alpha_1 t} + c_2\, (\alpha_2 - a)\, e^{\alpha_2 t}]$. ... (8)

(6) and (8) together give the required solution.

Ex. 9(*c*). *Solve $dx/dt = -wy$ and $dy/dt = wx$. Also show that the point (x, y) lies on a circle.*

[I.A.S. 2002, Meerut 2006; Nagpur 2007; Sagar 2001, 04]

Sol. Writing D for d/dt, the given equations become $Dx + wy = 0$ ... (1)

and $wx - Dy = 0$ ... (2)

Operating (1) by D and multiplying (2) by w, we get

$D^2 x + w\, Dy = 0$ and $w^2 x - w\, Dy = 0$.

Adding the above two equations, we get $(D^2 + w^2)\, x = 0$ ....(3)

Auxiliary equation for (3) is $D^2 + w^2 = 0$ giving $D = \pm\, iw$

Solution of (3) is $x = c_1 \cos wt + c_2 \sin wt$, c_1, c_2 being arbitrary constants ... (3)

(3) $\Rightarrow$ $dx/dt = Dx = -\, c_1 w \sin wt + c_2 w \cos wt$. ... (4)

$\therefore$ From (1), $y = -\,(1/w) \times Dx = -\,(1/w) \times (-\, c_1 w \sin wt + c_2 w \cos wt)$, by (4)

Thus, $y = c_1 \sin wt - c_2 \cos wt$ ... (5)

Thus (3) and (5) together give the required solution.

Squaring and adding (3) and (5), $x^2 + y^2 = (c_1 \cos wt + c_2 \sin wt)^2 + (c_1 \sin wt - c_2 \cos wt)^2$

Thus, $x^2 + y^2 = c_1{}^2 + c_2{}^2 = \{(c_1{}^2 + c_2{}^2)^{1/2}\}^2$, which is a circle.

Hence the point (x, y) lies on a circle.

Ex. 10(*a*). *Solve for x and y :$(dx/dt) + 2\,(dy/dt) - 2x + 2y = 3e^t$ and $3\,(dx/dt) + (dy/dt) + 2x + y = 4e^{2t}$.* **[Delhi B.Sc. (Prog) II 2010; Kanpur 2002, 07; Meerut 2007]**

Sol. Given $(dx/dt) + 2(dy/dt) - 2x + 2y = 3e^t$ ... (1)

and $3\,(dx/dt) + (dy/dt) + 2x + y = 4e^{2t}$ ... (2)

Multiplying both sides of (2) by 2, we have $6\,(dx/dt) + 2\,(dy/dt) + 4x + 2y = 8\, e^{2t}$ ... (3)

Subtracting (1) from (3), we have

$$5\frac{dx}{dt} + 6x = 8\, e^{2t} - 3\, e^t \quad \text{or} \quad \frac{dx}{dt} + \frac{6}{5}\, x = \frac{8}{5}\, e^{2t} - \frac{3}{5}\, e^t, \quad \text{... (4)}$$

which is a linear differential equation of order one.

I.F. of (4) $= e^{\int (6/5)\, dt} = e^{(6/5)t}$ and its solution is

$$x\, e^{(6/5)t} = \int \left(\frac{8}{5}\, e^{2t} - \frac{3}{5}\, e^t\right) e^{(6/5)t} dt + c_1 = \int \left[\frac{8}{5} e^{(16/5)t} - \frac{3}{5}\, e^{(11/5)t}\right] dt + c_1$$

or $$x\, e^{(6/5)t} = (8/5) \cdot (5/16)\, e^{(16/5)t} - (3/5) \cdot (5/11)\, e^{(11/5)t} + c_1$$

$x = (1/2)\, e^{2t} - (3/11)\, e^t + c_1\, e^{-(6/5)\,t}$, c_1 being an arbitrary constant ... (5)

Multiplying both sides of (1) by 3, $3\,(dx/dt) + 6\,(dy/dt) + 6x + 6y = 9e^t$ ... (6)

Subtracting (2) from (6), we have

$5\,(dy/dt) - 8x + 5y = 9e^t - 4e^{2t}$ or $5\,(dy/dt) + 5y = 8x + 9e^t - 4e^{2t}$

or $$5\frac{dy}{dt}+5y=8\left[\frac{1}{2}e^{2t}-\frac{3}{11}e^{t}+c_1 e^{-(6/5)t}\right]+9e^{t}-4e^{2t}, \text{ by (5)}$$

or $$5\frac{dy}{dt}+5y=\frac{75}{11}e^{t}+8c_1 e^{-(6/5)t} \qquad \text{or} \qquad \frac{dy}{dt}+y=\frac{15}{11}e^{t}+\frac{8c_1}{5}e^{-(6/5)t}$$

which is again a linear differential equation of order one.

Its integrating factor $= e^{\int dt}=e^{t}$ and its solution is

$$y\,e^{t}=\int\left[\frac{15}{11}e^{t}+\frac{8c_1}{5}e^{(-6/5)t}\right]e^{t}\,dt+c_2 \qquad \text{or} \qquad y\,e^{t}=\int\left[\frac{15}{11}e^{2t}+\frac{8c_1}{5}e^{-(1/5)t}\right]dt+c_2$$

or $$y\,e^{t}=(15/11)\cdot(1/2)\,e^{2t}+(8c_1/5)\cdot(-5)\cdot e^{-(1/5)t}+c_2$$

or $$y=c_2\,e^{-t}-8c_1\,e^{(-6/5)t}+(15/22)\,e^{t}.,\ c_2 \text{ being an arbitrary constant.} \qquad \text{... (7)}$$

(5) and (7) together give the required solution.

Ex. 10(*b*). *Solve $dx/dt + 2x + 3y = 0$, $dy/dt + 3x + 2y = 2e^{2t}$.* **[Delhi Maths 2002, 04]**

Sol. Writing D for d/dt, the given equations become

$$dx/dt + 2x + 3y = 0 \qquad \text{or} \qquad (D+2)\,x + 3y = 0 \qquad \text{... (1)}$$

and $$dy/dt + 3x + 2y = 2e^{2t} \qquad \text{or} \qquad 3x + (D+2)\,y = 2e^{2t} \qquad \text{... (2)}$$

Operating (2) by $(D + 2)$ and multiplying (1) by 3 and then subtracting, we have

$$[(D+2)^2-9]\,y=(D+2)\,2e^{2t} \qquad \text{or} \qquad (D^2+4D-5)\,y=8e^{2t} \qquad \text{... (3)}$$

Auxiliary equation of (3) is $D^2 + 4D - 5 = 0$ so that $D = 1, -5$

$\therefore$ C.F. of (3) $= C_1e^{t} + C_2\,e^{-5t}$, c_1 and c_2 being arbitrary constants

$$\text{P.I. of (3)} = \frac{1}{D^2+4D-5}8\,e^{2t}=8\,\frac{1}{2^2+4\cdot 2-5}e^{2t}=\frac{8}{7}e^{2t}$$

$\therefore$ solution of (3) is $$y = C_1e^{t} + C_2e^{-5t} + (8/7)\,e^{2t} \qquad \text{... (4)}$$

From (4), $$dy/dt = C_1e^{t} - 5C_2\,e^{-5t} + (16/7)\,e^{2t} \qquad \text{... (5)}$$

From (2), $$3x = 2e^{2t} - 2y - dy/dt$$

or $$3x=2e^{2t}-2\{C_1e^{t}+C_2e^{-5t}+(8/7)\,e^{2t}\}-\{C_1\,e^{t}-5C_2\,e^{-5t}+(16/7)\,e^{2t}\}$$

[On putting values of y and dy/dt from (4) and (5)]

or $$3x = -3C_1e^{t} + 3C_2e^{-5t} - (18/7)\,e^{2t} \qquad \text{or} \qquad x = -C_1e^{t} + C_2e^{-5t} - (6/7)\,e^{2t} \qquad \text{... (6)}$$

The required solution is given by (4) and (6).

Ex. 10(*c*). *Solve $(dx/dt) - (dy/dt) + 3x = \sin t$, $dx/dt + y = \cos t$, given that $x = 1$, $y = 0$ for $t = 0$.*

[Delhi Maths (H) 2001]

Sol. Writing D for d/dt, the given equations become

$$(dx/dt) - (dy/dt) + 3x = \sin t \qquad \text{or} \qquad (D+3)\,x - Dy = \sin t \qquad \text{... (1)}$$

and $$dx/dt + y = \cos t \qquad \text{or} \qquad Dx + y = \cos t \qquad \text{... (2)}$$

Operating (2) by D and adding it to (1), we get

$$[(D+3)+D^2]\,x = \sin t + D\cos t \qquad \text{or} \qquad (D + D + 3)\,x = 0 \qquad \text{... (3)}$$

Auxiliary equation of (3) is $D^2 + D + 3 = 0$, giving

$$D=\{-1\pm(1-12)^{1/2}\}/2=(-1/2)\pm i(\sqrt{11}/2)$$

So solution of (3) is $$x=e^{-t/2}\{C_1\cos(t\sqrt{11}/2)+C_2\sin(t\sqrt{11}/2)\} \qquad \text{... (4)}$$

Diff. (4) w.r.t 't', $\quad dx/dt = -(1/2)\, e^{-t/2}\{C_1 \cos(t\sqrt{11}/2) + C_2 \sin(t\sqrt{11}/2]$

$$+\, e^{-t/2}\{-(C_1\sqrt{11}/2)\sin(t\sqrt{11}/2) + (C_2\sqrt{11}/2)\cos(t\sqrt{11}/2)\} \quad \ldots (5)$$

From (2), $\quad y = \cos t - dx/dt$

or $\quad y = \cos t + (1/2)\, e^{-t/2}\ \{C_1 \cos\ (t\sqrt{11}/2)\ +\ C_2 \sin\ (t\sqrt{11}/2)\}$

$$-\, e^{-t/2}\{-(C_1\sqrt{11}/2)\sin(t\sqrt{11}/2) + (C_2\sqrt{11}/2)\cos(t\sqrt{11}/2)\}\text{, using (5)} \quad \ldots (6)$$

Given that $y = 0$ for $t = 0$. So the above equation gives

$$0 = 1 + (1/2)\,C_1 - (C_2\sqrt{11}/2) \quad \ldots (7)$$

Again, given that $x = 1$ for $t = 0$. So (4) gives $C_1 = 1$. With this value of C_1, (7) gives $C_2 =\ 3/\sqrt{11}$. Therefore, (4) and (6) give

$$x = e^{-t/2}\{\cos(t\sqrt{11}/2) + (3/\sqrt{11})\sin(t\sqrt{11}/2)\} \quad \ldots (7)$$

and $\quad y = \cos t + (1/2)\, e^{-t/2}[\cos(t\sqrt{11}/2) + (3/\sqrt{11})\sin(t\sqrt{11}/2)]$

$$-\, e^{-t/2}\{-(\sqrt{11}/2)\sin(t\sqrt{11}/2) + (3/2)\cos(t\sqrt{11}/2)\}$$

or $\quad y = \cos t - e^{-t/2}\cos(t\sqrt{11}/2) + e^{-t/2}(3/2\sqrt{11} + \sqrt{11}/2)\sin(t\sqrt{11}/2) \quad \ldots (8)$

The required solution is given by (7) and (8).

Ex. 10(*d*). *Solve $dx/dt - 3x + 4y = e^{-2t}$, $dy/dt - x + 2y = 3e^{-2t}$.* **[Delhi Maths (H) 2004, 06]**

Find also the particular solution, if $x = 12$, $y = 7$ when $t = 0$

Sol. Let $D \equiv d/dt$. Then, the given equation reduce to

$$(D - 3)\, x + 4y = e^{-2t} \quad \ldots (1)$$

and $\quad -\,x + (D + 2)\, y = 3e^{-2t}$

Eliminating y from (1) and (2), $\quad (D + 2)(D - 3)x + 4x = (D + 2)e^{-2t} - 12e^{-2t}$

or $\quad (D^2 - D - 2)\, x = -\ 12e^{-2t} \quad \ldots(3)$

Its auxiliary equation is $\quad D^2 - D - 2 = 0, \quad$ giving $\quad D = 2, -1.$

Its $\quad$ C.F. $= C_1e^{2t} + C_2e^{-t}$, C_1 and C_2 being arbitrary constants.

Its $\quad$ P.I. $= \dfrac{1}{D^2 - D - 2}(-12e^{-2t}) = -12\dfrac{1}{(-2)^2 + 2 - 2}e^{-2t} = -3\,e^{-2t}$

So solution of (3) is $\quad x = C_1e^{2t} + C_2e^{-t} - 3e^{-2t} \quad \ldots (4)$

From (1), $\quad 4y = e^{-2t} + 3x - (dx/dt)$

$$= e^{-2t} + 3\,(C_1e^{2t} + C_2e^{-t} - 3e^{-2t}) - (2C_1e^{2t} - C_2e^{-t} + 6e^{-2t})\text{, using (4)}$$

or $\quad y = (1/4) \times (C_1e^{2t} + 4C_2e^{-t} - 14e^{-2t}) \quad \ldots (5)$

(4) and (5) together give the required solution.

Second part Given that $x = 12$ and $y = 7$ when $t = 0$. So (4) and (5) reduce to

$C_1 + C_2 - 3 = 12 \quad$ giving $\quad C_1 + C_2 = 15 \quad \ldots(6)$

and $\quad (1/4) \times (C_1 + 4C_2 - 14) = 7 \quad$ giving $\quad C_1 + 4C_2 = 42 \quad \ldots (7)$

Solving (6) and (7), $C_1 = 6$ and $C_2 = 9$. Hence, the required solution is given by

$$x = 6\, e^{2t} + 9\, e^{-t} - 3e^{-2t}, \qquad y = (3/2)\, e^{2t} + 9e^{-t} - (7/2)\, e^{-2t}$$

Ex. 10(*e*). *Solve $dx/dt + dy/dt + 2x + y = e^t$, $dy/dt + 5x + 3y = t$.* **[Delhi Maths (G) 2004]**

Sol. Let $D \equiv d/dt$. Then, the given equations reduce to

$$(D+2)\,x + (D+1)\,y = e^t \qquad \text{... (1)}$$

$$5x + (D+3)\,y = t \qquad \text{... (2)}$$

Eliminating x from (1) and (2), $\{5(D+1) - (D+2)(D+3)\}y = 5e^t - (D+2)t$

or $(-D^2 - 1)\,y = 5e^t - 1 - 2t$ or $(D^2 + 1)\,y = 1 + 2t - 5e^t$ ... (3)

Its auxiliary equation is $D^2 + 1 = 0$, giving $D = \pm\, i$

$\therefore$ C.F. of (3) $= C_1 \cos t + C_2 \sin t$, C_1 and C_2 being arbitrary constants.

P.I. of (3) corresponding to $(1+2t) = \dfrac{1}{1+D^2}(1+2t)$

$$= (1+D^2)^{-1}(1+2t) = (1 - D^2 + \ldots)(1+2t) = 1 + 2t$$

and P.I. of (3) corresponding to $(-5e^t) = \dfrac{1}{D^2+1}(-5e^t) = -\dfrac{5}{2}e^t$

$\therefore$ Solution of (3) is $y = C_1 \cos t + C_2 \sin t + 1 + 2t - (5/2)\,e^t$ ... (4)

From (2), $5x = t - 3y - (dy/dt) = t - 3\,\{C_1 \cos t + C_2 \sin t + 1 + 2t - (5/2)\,e^t\}$

$-(-C_1 \sin t + C_2 \cos t + 2 - (5/2)\,e^t\}$, by (4)

or $x = \{(C_1 - 3C_2)/5\} \sin t - \{(3C_1 + C_2)/5\} \cos t - t - 1 + 2e^t$ ... (5)

(4) and (5) together give the required solution.

Ex. 10(*f*). *Solve* $dx/dt + dy/dt + 2x - y = 3\,(t^2 - e^{-t})$, $2\,(dx/dt) - (dy/dt) - x - y = 3\,(2t - e^{-t})$ **[I.A.S. 2003; Rajasthan 2007]**

Sol. Let $x_1 = dx/dt$, $x_2 = d^2x/dt^2$, $y_1 = dy/dt$ and $y_2 = d^2y/dt^2$

Then, re-writing the given equation, we have

$$x_1 + y_1 + 2x - y = 3\,(t^2 - e^{-t}) \qquad \text{... (1)}$$

and

$$2x_1 - y_1 - x - y = 3\,(2t - e^{-t}) \qquad \text{... (2)}$$

Differentiating (1) and (2) w.r.t. 't', we have $x_2 + y_2 + 2x_1 - y_1 = 3\,(2t + e^{-t})$ ... (3)

and $2x_2 - y_2 - x_1 - y_1 = 3\,(2 + e^{-t})$ ... (4)

Adding (3) and (4), $3x_2 + x_1 - 2y_1 = 6\,(t + 1 + e^{-t})$ ... (5)

Subtracting (2) from (1), $x_1 - 2y_1 - 3x = 3\,(2t - t^2)$ ... (6)

Subtracting (6) from (5), $3x_2 + 3x = 6 + 3t^2 + 6\,e^{-t}$

or $(D^2 + 1)\,x = 2 + t^2 + 2e^{-t}$, where $D \equiv d/dt$ ... (7)

Auxiliary equation of (7) is $D^2 + 1 = 0$ so that $D = \pm\, i$.

$\therefore$ C.F. of (7) $= c_1 \cos t + c_2 \sin t$, c_1 and c_2 being arbitrary constants

P.I. corresponding to $(2 + t^2)$

$$= \frac{1}{D^2+1}(2+t^2) = (1+D^2)^{-1}(2+t^2) = (1 - D^2 + \ldots)(2+t^2) = 2 + t^2 - 2 = t^2.$$

P.I. corresponding to $(2e^{-t}) = \dfrac{1}{D^2+1}2e^{-t} = 2\dfrac{1}{1+1}e^{-t} = e^{-t}$.

$\therefore$ Solution of (7) is $x = c_1 \cos t + c_2 \sin t + e^{-t} + t^2$ ... (8)

From (8), on differentiating, $x_1 = -\,c_1 \sin t + c_2 \cos t - e^{-t} + 2t$ ... (9)

From (6), $2y_1 = x_1 - 3x - 6t + 3t^2 = -\,c_1 \sin t + c_2 \cos t - e^{-t} + 2t$

$-\,3\,(c_1 \cos t + c_2 \sin t + e^{-t} + t^2) - 6t + 3t^2$, by (8) and (9)

$\therefore$ $y_1 = [(c_2 - 3c_1) \cos t - (c_1 + 3c_2) \sin t - 4t - 4e^{-t}]/2$ ... (10)

$\therefore$ From (2), $$y = 2x_1 - y_1 - x - 6t + 3e^{-t}$$

or $$y = 2\,(-c_1 \sin t + c_2 \cos t + 2t - e^{-t}) - (1/2)\,[(c_2 - 3c_1)\cos t$$
$$-(c_1 + 3c_2)\sin t - 4t - 4e^{-t}] - (c_1 \cos t + c_2 \sin t + e^{-t} + t^2) - 6t + 3e^{-t}, \text{ by (8) (9) and (10)}$$

or $$y = (1/2) \times (3c_2 + c_1)\cos t + (1/2) \times (c_2 - 3c_1)\sin t + 2e^{-t} - t^2 \qquad \text{... (11)}$$

(8) and (11) together give the desired solution.

Ex. 10(*g*). *Solve* $4x_1 + 9y_1 + 44x + 49y = t,\ \ 3x_1 + 7y_1 + 34x + 38y = e^t$ *where* $x_1 = dx/dt$ *and* $y_1 = dy/dt$. **[Kanpur 2005; Meerut 1997; Delhi Maths (Prog) 2007]**

Sol. Let $D \equiv d/dt$. Then the given equations can be re-written as

$$(4D + 44)\,x + (9D + 49)\,y = t \qquad \text{... (1)}$$

and $$(3D + 34)\,x + (7D + 38)\,y = e^t \qquad \text{... (2)}$$

Eliminating y from the above equations, we have

$$[(7D + 38)(4D + 44) - (9D + 49)(3D + 34)]\,x = (7D + 38)\,t - (9D + 49)\,e^t$$

or $$(D^2 + 7D + 6)\,x = 7 + 38t - 58t^2 \qquad \text{... (3)}$$

$\therefore$ C.F. of (3) $= c_1e^{-t} + c_2e^{-6t}$, c_1 and c_2 being arbitrary constants

P.I. corresponding to $(7 + 38\,t)$ is

$$= \frac{1}{D^2 + 7D + 6}(7 + 38t) = \frac{1}{6[1 + (D^2 + 7D)/6]}(7 + 38t) = \frac{1}{6}\left[1 + \frac{D^2 + 7D}{6}\right]^{-1}(7 + 38t)$$

$$= \frac{1}{6}\left(1 - \frac{D^2 + 7D}{6} + \ldots\right)(7 + 38t) = \frac{1}{6}\left[7 + 38t - \frac{7}{6} \times (38)\right] = \frac{19}{3}t - \frac{56}{9}.$$

P.I. corresponding to $(-58\,e^t) = -58\dfrac{1}{D^2 + 7D + 6}e^t = -\dfrac{29}{7}e^t.$

Hence, the solution of (3) is $\quad x = c_1e^{-t} + c_2e^{-6t} + (19/3)t - (29/77)\,e^t - (56/9) \qquad \text{... (4)}$

Now, $\quad (4) \Rightarrow x_1 = dx/dt = -c_1e^{-t} - 6c_2\,e^{-6t} + (19/3) - (29/7)\,e^t \qquad \text{... (5)}$

Eliminating y_1 from given equations, we have

$$x_1 + 2x + y = 7t - 9e^t \quad \text{so that} \quad y = 7t - 9e^t - x_1 - 2x$$

or $$y = 7t - 9e^t - \{-c_1e^{-t} - 6\,c_2e^{-6t} + (19/3) - (29/7)\,e^t\}$$
$$-2\{c_1e^{-t} + c_2e^{-6t} + (19/3)\,t - (29/7)\,e^t - (56/9)\}, \text{ using (4) and (5)}$$

or $$y = -c_1e^{-t} + 4c_2\,e^{-6t} - (17/3)\,t + (24/7)\,e^t + (55/9) \qquad \text{... (6)}$$

(4) and (6) together give the required solution.

Ex. 10(*h*). *Solve :* $dx/dt = ax + by + c,\quad dy/dt = a'\,x + b'\,y + c'$. **[Rajasthan 2004, 05]**

Sol. Given $$dx/dt - ax - by = c \qquad \text{... (1)}$$

and $$dy/dt - a'x - b'y = c' \qquad \text{... (2)}$$

Let $d/dt \equiv D$. Then (1) and (2) can be written as

$$(D - a)\,x - by = c \qquad \text{... (3)}$$

and $$-a'x + (D - b')\,y = c' \qquad \text{... (4)}$$

Eliminating y from (3) and (4), we have

$$[(D - b')(D - a) - a'b]\,x = (D - b')\,c + bc'$$

or $$[D^2 - (a + b')\,D + ab' - a'b]\,x = c'b - cb' \qquad \text{... (5)}$$

Here auxiliary equation of (5) is $\quad D^2 - (a + b')\,D + ab' - a'b = 0$

$$\Rightarrow D = \frac{a + b' \pm \sqrt{(a + b')^2 - 4(ab' - a'b)}}{2} = \frac{(a + b') \pm \sqrt{(a - b')^2 + 4a'b}}{2} \Rightarrow D = m_1, m_2 \text{ (say)}$$

$\therefore$ C.F. $= c_1 e^{m_1 t} + c_2 e^{m_2 t}$, c_1 and c_2 being arbitrary constants

$$\text{P.I.} = (c'b - cb')\frac{1}{D^2 - (a+b)\,D + ab' - a'b}\, e^{0.t} = \frac{c'b - cb'}{ab' - a'b}, \text{ provided } (ab' - a'b) \neq 0.$$

Hence, the general solution of (5) is $\quad x = c_1 e^{m_1 t} + c_2 e^{m_2 t} + \{(c'b - cb')/(ab' - a'b)\}$. ... (6)

Now, $\quad (6) \Rightarrow \quad dx/dt = c_1 m_1 e^{m_1 t} + c_2 m_2 e^{m_2 t}$. ... (7)

From (1), we have $\quad by = (dx/dt) - ax - c$

$\therefore \quad by = c_1 m_1 e^{m_1 t} + c_2 m_2 e^{m_2 t} - a[c_1 e^{m_1 t} + c_2 e^{m_2 t} + \{(c'b - cb')/(ab' - a'b)\}] - c$, by (6) and (7)

$$= (m_1 - a)\,c_1 e^{m_1 t} + (m_2 - a)\,c_2 e^{m_2 t} - \frac{a(c'b - cb') + c(ab' - a'b)}{ab' - a'b}$$

$$= (m_1 - a)\,c_1 e^{m_1 t} + (m_2 - a)\,c_2 e^{m_2 t} - \{b(ac' - ca')\}/(ab' - a'b)$$

$$\therefore \quad y = \frac{c_1}{b}(m_1 - a)\,e^{m_1 t} + \frac{c_2}{b}(m_2 - a)\,e^{m_2 t} - \frac{ac' - ca'}{ab' - a'b}. \quad \text{... (8)}$$

(6) and (8) together give the required solution.

Ex. 11(*a*). *Solve $d^2x/dt^2 - 3x - 4y = 0$, $d^2y/dt^2 + x + y = 0$.* **[Agra 2001, 04; Kanpur 2003; [Garhwal 2005, Gorakhpur 1999, Delhi Maths Hons. 1992, Meerut 2009]**

Sol. Writing D for d/dt, the given equations become

$$(D^2 - 3)\,x - 4y = 0 \quad \text{... (1)}$$

and

$$x + (D^2 + 1)\,y = 0 \quad \text{... (2)}$$

Eliminating y from (1) and (2), $\quad [(D^2 + 1)(D^2 - 3) + 4]\,x = 0 \quad$ or $\quad (D^2 - 1)^2 y = 0$ ... (3)

Auxiliary equation for (3) is, $\quad (D^2 - 1)^2 = 0 \quad$ so that $\quad D = 1, 1, -1, -1$.

Hence solution of (3) is $\quad x = (c_1 + c_2 t)\,e^t + (c_3 + c_4 t)\,e^{-t}$, ... (4)

where c_1, c_2, c_3 and c_4 are arbitrary constants.

(4) $\Rightarrow \quad Dx = c_2 e^t + (c_1 + c_2 t)\,e^t + c_4 e^{-t} - (c_3 + c_4 t)\,e^{-t} = (c_1 + c_2 + c_2 t)\,e^t + (c_4 - c_3 - c_4 t)\,e^{-t}$.

$$\therefore \quad D^2 x = c_2 e^t + (c_1 + c_2 + c_2 t)\,e^t - c_4 e^{-t} - (c_4 - c_3 - c_4 t)\,e^{-t}$$

or

$$D^2 x = (c_1 + 2c_2 + c_2 t)\,e^t - (2c_4 - c_3 - c_4 t)\,e^{-t} \quad \text{... (5)}$$

But from (1), $\quad 4y = D^2 x - 3x$ ... (6)

Hence using (4) and (5), (6) becomes

$$4y = (c_1 + c_2 + 2c_2 t)\,e^t - (2c_4 - c_3 - c_4 t)\,e^{-t} - 3[(c_1 + c_2 t)\,e^t + (c_3 + c_4 t)\,e^{-t}]$$

or

$$4y = 2(c_2 - c_1 - c_2 t)\,e^t - 2(c_4 - c_3 + c_4 t)\,e^{-t}$$

or

$$y = (1/2) \times (c_2 - c_1 - c_2 t)\,e^t - (1/2) \times (c_4 - c_3 + c_4 t)\,e^{-t} \quad \text{... (7)}$$

The required solution is given by (4) and (7).

Ex. 11(*b*). *Solve $D^2x + m^2y = 0$, $D^2y - m^2x = 0$, where $D \equiv d/dt$.*

[Gwaliar 2004; Rajasthan 1997; Rohilkhand 1995, Agra 1998, Poona 1994]

Sol. Given $\quad D^2 x + m^2 y = 0$ ... (1)

and $\quad -m^2 x + D^2 y = 0$. ... (2)

Eliminating y from (1) and (2), $\quad (D^4 + m^4)\,x = 0 \quad$ whose auxiliary equation is $\quad D^4 + m^4 \equiv 0$.

or $\quad (D^4 + 2m^2D^2 + m^4) - 2m^2D^2 = 0 \quad$ or $\quad (D^2 + m^2)^2 - (m\sqrt{2}D)^2 = 0$

or

$$(D^2 + \sqrt{2}mD + m^2)(D^2 - \sqrt{2}mD + m^2) = 0.$$

$\therefore \quad D = \{-m\sqrt{2} \pm (2m^2 - 4m^2)^{1/2}\}/2, \qquad \{m\sqrt{2} \pm (2m^2 - 4m^2)^{1/2}\}/2$

or $\qquad D = -(m/\sqrt{2}) \pm i\,(m/\sqrt{2}), \qquad (m/\sqrt{2}) \pm i(m/\sqrt{2})$

$$\therefore \; x = e^{-mt/\sqrt{2}}[c_1 \cos(mt/\sqrt{2}) + c_2 \sin(mt/\sqrt{2})] + e^{mt/\sqrt{2}}[c_3 \cos(mt/\sqrt{2}) + c_4 \sin(mt/\sqrt{2})]. \quad \ldots (3)$$

$$Dx = e^{-mt/\sqrt{2}}\left[-\frac{c_1 m}{\sqrt{2}}\sin\,(mt/\sqrt{2}) + \frac{c_2 m}{\sqrt{2}}\cos\,(mt/\sqrt{2})\right] - \frac{m}{\sqrt{2}}e^{-mt/\sqrt{2}}[c_1 \cos\,(mt/\sqrt{2}) + c_2 \sin\,(mt/\sqrt{2})]$$

$$+ e^{mt/\sqrt{2}}\left[-(1/\sqrt{2})c_3 m \sin(mt/\sqrt{2}) + (1/\sqrt{2})c_4 m \cos(mt/\sqrt{2})\right]$$

$$+ (m/\sqrt{2})\,e^{mt/\sqrt{2}}\left[c_3 \cos(mt/\sqrt{2}) + c_4 \sin(mt/\sqrt{2})\right]$$

$$= -\,(m/\sqrt{2})e^{-mt/\sqrt{2}}[(c_1 + c_2)\sin(mt/\sqrt{2}) + (c_1 - c_2)\cos(mt/\sqrt{2})]$$

$$+ (m/\sqrt{2})e^{mt/\sqrt{2}}[(c_3 + c_4)\cos(mt/\sqrt{2}) + (c_4 - c_3)\sin(mt/\sqrt{2})]$$

$$\therefore \quad D^2 x = -\frac{m}{\sqrt{2}}e^{-mt/\sqrt{2}}\left[\frac{m(c_1 + c_2)}{\sqrt{2}}\cos\left(\frac{mt}{\sqrt{2}}\right) - \frac{m(c_1 - c_2)}{\sqrt{2}}\sin\left(\frac{mt}{\sqrt{2}}\right)\right]$$

$$+ \frac{m^2}{2}e^{-mt/\sqrt{2}}\left[(c_1 + c_2)\sin\left(\frac{mt}{\sqrt{2}}\right) + (c_1 - c_2)\cos\left(\frac{mt}{\sqrt{2}}\right)\right]$$

$$+ \frac{m}{\sqrt{2}}e^{mt/\sqrt{2}}\left[-\frac{m(c_3 + c_4)}{\sqrt{2}}\sin\left(\frac{mt}{\sqrt{2}}\right) + \frac{m(c_4 - c_3)}{\sqrt{2}}\cos\left(\frac{mt}{\sqrt{2}}\right)\right]$$

$$+ \frac{m^2}{2}e^{mt/\sqrt{2}}\left[(c_3 + c_4)\cos\left(\frac{mt}{\sqrt{2}}\right) + (c_4 - c_3)\sin\left(\frac{mt}{\sqrt{2}}\right)\right]$$

$$= m^2 e^{-mt/\sqrt{2}}[c_1 \sin\,(mt/\sqrt{2}) - c_2 \cos(mt/\sqrt{2})] + m^2 e^{mt/\sqrt{2}}[c_4 \cos(mt/\sqrt{2}) - c_3 \sin(mt/\sqrt{2})]. \quad \ldots (4)$$

Now (1), $\qquad y = -\,(1/m^2) \times D^2 x.$

or $\quad y = e^{-mt/\sqrt{2}}[c_2 \cos(mt/\sqrt{2}) - c_1 \sin(mt/\sqrt{2})] + e^{mt/\sqrt{2}}[c_3 \sin(mt/\sqrt{2}) - c_4 \cos(mt/\sqrt{2})]$, by (4) $\quad \ldots (5)$

The required solution is given by (3) and (6).

Ex. 11(*c*). *Solve* $\quad d^2x/dt^2 - 3x - 4y + 3 = 0, \quad d^2y/dt^2 + y + x + 5 = 0.$

[Delhi Maths (G) 1999]

Sol. Let $D \equiv d/dt$. Then the given equations become

$d^2x/dt^2 - 3x - 4y + 3 = 0 \qquad$ or $\qquad (D^2 - 3)\,x - 4y = -\,3 \qquad \ldots (1)$

and $\quad d^2y/dt^2 + y + x + 5 = 0 \qquad$ or $\qquad x + (D^2 + 1)\,y = -\,5 \qquad \ldots (2)$

Operate (1) by $(D^2 + 1)$ and multiply (2) by 4 and then add. Thus, we get

$\{(D^2 + 1)\,(D^2 - 3) + 4]\,x = -\,(D^2 + 1)3 - 20 \qquad$ or $\qquad (D^4 - 2D^2 + 1)\,x = -23 \qquad \ldots (3)$

The auxiliary equation of (3) is $\qquad (D^2 - 1)^2 = 0 \qquad$ gives $\qquad D = 1, 1, -1, -1$

$\therefore$ C.F. $= (C_1 + C_2 t)\,e^t + (C_3 + C_4 t)\,e^{-t}$, C_1, C_2, C_3 and C_4 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^4 - 2D^2 + 1}(-23)\,e^{0.t} = \frac{1}{0^2 - (2 \times 0^2) + 1}(-23)\,e^{0.t} = -\,23$$

$\therefore$ Solution of (3) is $\qquad x = (C_1 + C_2 t)\,e^t + (C_3 + C_4 t)\,e^{-t} - 23 \qquad \ldots (4)$

From (4), $dx/dt = (C_1 + C_2 t)\, e^t + C_2 e^t - (C_3 + C_4 t)\, e^{-t} + C_4 e^{-t}$ ... (5)

From (5), $d^2x/dt^2 = (C_1 + C_2 t)\, e^t + 2C_2 e^t + (C_3 + C_4 t)\, e^{-t} - 2C_4 e^{-t}$ ... (6)

From (1), $4y = d^2x/dt^2 - 3x + 3$

or $4y = (C_1 + 2C_2 + C_2 t)\, e^t + (C_3 - 2C_4 + C_4 t)\, e^{-t} - 3(C_1 + C_2 t)\, e^t$

$+ (C_3 + C_4 t)\, e^{-t} - 23\} + 3$, using (4) and (6)

or $4y = (2C_2 - 2C_1 - 2C_2 t)\, e^t - (2C_4 + 2C_3 + 2C_4 t)\, e^{-t} + 72$

or $y = (1/2) \times (C_2 - C_1 - C_2 t)\, e^t - (1/2) \times (C_4 + C_3 + C_4 t)\, e^{-t} + 18$ ... (7)

The required solution is given by (4) and (7).

Ex. 11(*d*). *Solve the simultaneous equations* $(d^2x/dt^2) + 4x + y = t\, e^{3t}$ and $(d^2y/dt^2) + y - 2x = \cos^2 t$. **[Meerut 1997]**

Sol. Writing D for d/dt, the given equations become $(D^2 + 4)\, x + y = t\, e^{3t}$ ... (1)

and $-2x + (D^2 + 1)\, y = \cos^2 t.$ ... (2)

Operating both sides of (1) by $(D^2 + 1)$, we get

$$(D^2 + 1)\,(D^2 + 4)\, x + (D^2 + 1)\, y = (D^2 + 1)\,(t\, e^{3t})$$

or $(D^4 + 5D^2 + 4)\, x + (D^2 + 1)\, y = D\,\{D\,(te^{3t})\} + t\, e^{3t}$

or $(D^4 + 5D^2 + 4)\, x + (D^2 + 1)\, y = D\,(e^{3t} + 3t\, e^{3t}) + t\, e^{3t}$

or $(D^4 + 5D^2 + 4)\, x + (D^2 + 1)\, y = 3\, e^{3t} + 3\,(e^{3t} + 3t\, e^{3t}) + t\, e^{3t}.$

or $(D^4 + 5D^2 + 4)\, x + (D^2 + 1)\, y = 6\, e^{3t} + 10\, t\, e^{3t}.$ ... (3)

Subtracting (2) from (3), $(D^4 + 5D^2 + 6)\, x = 6e^{3t} + 10t\, e^{3t} - \cos^2 t$

or $(D^4 + 5D^2 + 6)\, x = 6\, e^{3t} + 10\, t\, e^{3t} - (1/2) \times (1 + \cos 2t),$ ... (4)

whose auxiliary equation is $D^2 + 5D^2 + 6 = 0$ so that $D = \pm i\sqrt{3}, \pm i\sqrt{2}$

$\therefore$ C.F. $= c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t + c_3 \cos\sqrt{2}t + c_4 \sin\sqrt{2}t$, c_1, c_2, c_3 and c_4 being arbitrary constants.

P.I. corresponding to $6e^{3t} = 6\dfrac{1}{D^4 + 5D^2 + 6}e^{3t} = 6\dfrac{1}{3^4 + (5 \times 3^2) + 6}e^{3t} = \dfrac{e^{3t}}{22}.$

P.I. corresponding to $10\, te^{3t} = 10\dfrac{1}{D^4 + 5D^2 + 6}t\, e^{3t} = 10\, e^{3t}\dfrac{1}{(D+3)^4 + 5(D+3)^2 + 6}t$

$$= 10\, e^{3t}\frac{1}{132 + 138D + \ldots}t = 10\, e^{3t}\frac{1}{132\,\{1 + (23/22)\, D + \ldots\}}t$$

$$= \frac{5\, e^{3t}}{66}\left(1 + \frac{23}{22}D + \ldots\right)^{-1} t = \frac{5\, e^{3t}}{66}\left(1 - \frac{23}{22}D + \ldots\right)t = \frac{5\, e^{3t}}{66}\left(t - \frac{23}{22}\right).$$

P.I. corresponding to $\left(-\dfrac{1}{2}\right) = \dfrac{1}{D^4 + 5D^2 + 6}\left(-\dfrac{1}{2}\right) = -\dfrac{1}{2}\dfrac{1}{D^4 + 5D^2 + 6}e^{0.t} = -\dfrac{1}{2}\cdot\dfrac{1}{6} = -\dfrac{1}{12}.$

P.I. corresponding to $\left(-\dfrac{1}{2}\cos 2t\right) = -\dfrac{1}{2}\dfrac{1}{D^4 + 5D^2 + 6}\cos 2t = -\dfrac{1}{2}\dfrac{1}{(D^2)^2 + 5D^2 + 6}\cos 2t$

$$= -\frac{1}{2}\frac{1}{\times(-2^2)^2 + 5\times(-2^2) + 6}\cos 2t = -\frac{1}{4}\cos 2t.$$

$\therefore$ Solution of (4) is $x = c_1 \cos\sqrt{3}\, t + c_2 \sin\sqrt{3}\, t + c_3 \cos\sqrt{2}\, t + c_4 \sin\sqrt{2}\, t + (1/22)\, e^{3t}$

$+ (5/66)\, e^{3t}\,(t - 23/22) - (1/12) - (1/4)\cos 2t.$ ... (5)

Differentiating both sides of (5) w.r.t. 't' twice, we get

$$dx/dt = -c_1\sqrt{3}\sin\sqrt{3}\,t + c_2\sqrt{3}\cos\sqrt{3} - c_3\sqrt{2}\,\sin\sqrt{2}\,t + c_4\sqrt{2}\,\cos\sqrt{2}\,t$$
$$+ (3/22)\ e^{3t} + (5/22)\ e^{3t}\ (t - 23/22) + (5/66)\ e^{3t} + (1/2)\sin 2t$$

and $$d^2x/dt^2 = -3c_1\cos\sqrt{3}\,t - 3c_2\sin\sqrt{3}\,t - 2\,c_3\cos\sqrt{2}\,t - 2\,c_4\sin\sqrt{2}\,t$$
$$+ (9/22)\ e^{3t} + (15/22)\ e^{3t}\ (t - 23/22) + (5/22)\ e^{3t} + (5/22)\ e^{3t} + \cos 2t. \quad ...(6)$$

Now, (1) $\Rightarrow$ $y = -(d^2x/dt^2) - 4x + t\,e^{3t}$

$\therefore$ $y = 3c_1\cos\sqrt{3}\,t + 3c_2\sin\sqrt{3}\,t + 2c_3\cos\sqrt{2}t + 2c_4\sin\sqrt{2}\,t - (9/22)e^{3t} - (15/22)\ e^{3t}\ (t - 23/22)$
$+ (5/11)\ e^{3t} - \cos 2t - 4[c_1\cos\sqrt{3}\,t + c_2\sin\sqrt{3}\,t + c_3\cos\sqrt{2}\,t + c_4\sin\sqrt{2}\,t + (1/22)\ e^{3t}$
$+ (5/66)\ e^{3t}\ (t - 23/22) - (1/12) - (1/4)\cos 2t] + t\,e^{3t}$, using (5) and (6)

$$\therefore\ y = -\ c_1\cos\sqrt{3}t - c_2\sin\sqrt{3}t - 2c_3\cos\sqrt{2}t - 2c_4\sin\sqrt{2}t + (1/66)te^{3t} - (23/2452)e^{3t} + (1/3). \quad ...(7)$$

(5) and (7) together give the required solution.

Ex. 11(*e*). *Solve $(d^2x/dt^2) - (dy/dt) = 2x + 2t$, $(dx/dt) + 4(dy/dt) = 3y$.* **[G.N.D.U. 1997]**

Sol. Given $$(d^2x/dt^2) - (dy/dt) - 2x = 2t \quad ...(1)$$

and $$(dx/dt) + 4\,(dy/dt) - 3y = 0. \quad ...(2)$$

Writing D for d/dt, the given equations (1) and (2) become $(D^2 - 2)\,x - Dy = 2t$ ... (3)

and $$Dx + (4D - 3)\,y = 0. \quad ...(4)$$

Operating both sides of (3) and (4) by $(4D - 3)$ and D respectively, we get

$$(4D - 3)\,(D^2 - 2)\,x - (4D - 3)\,Dy = 2\,(4D - 3)\,t \quad ...(5)$$

and $$D^2x + (4D - 3)\,Dy = 0. \quad ...(6)$$

Adding (5) and (6), $$\{(4D - 3)\,(D^2 - 2) + D^2\}\,x = 8\,Dt - 6t$$

or $$(2D^3 - D^2 - 4D + 3)\,x = 4 - 3\text{t}. \quad ...(7)$$

Its auxiliary equation is $2D^3 - D^2 - 4D + 3 = 0$, giving $D = 1, 1, -3/2$.

$\therefore$ C.F. of (7) $= (c_1 + c_2t)\,e^t + c_3\,e^{-3t/2}$, c_1 and c_2 being arbitrary constants.

$$\text{P.I. of (7)} = \frac{1}{2D^3 - D^2 - 4D + 3}(4 - 3t) = \frac{1}{3[1 - (4D/3 + D^2/3 - 2D^3/3)]}(4 - 3t)$$
$$= (1/3) \times [1 - (4D/3 + D^2/3 - 2D^3/3)]^{-1}\,(4 - 3t)$$
$$= (1/3) \times \{1 + 4D/3 + ...\}\,(4 - 3t) = (1/3)\,\{4 - 3t + (4/3)\cdot(-3)\} = -t$$

$\therefore$ Solution of (7) is, $$x = (c_1 + c_2t)\,e^t + c_3\,e^{-3t/2} - t. \quad ...(8)$$

where c_1, c_2 and c_3 are arbitrary constants.

From (8), $$dx/dt = (c_1 + c_2t)\,e^t + c_2\,e^t - (3/2)\,c_3\,e^{-3t/2} - 1$$

or $$dx/dt = (c_1 + c_2 + c_2t)\,e^t - (3/2)\,c_3\,e^{-3t/2} - 1. \quad ...(9)$$

From (9), $$d^2x/dt^2 = (c_1 + c_2 + c_2t)\,e^t + c_2\,e^t + (9/4)\,c_3\,e^{-3t/2}$$

or $$d^2x/dt^2 = (c_1 + 2c_2 + c_2t)\,e^t + (9/4)\,c_3\,e^{-3t/2}. \quad ...(10)$$

(1) $\Rightarrow$ $dy/dt = (d^2x/dt^2) - 2x - 2t$

$= (c_1 + 2c_2 + c_2t)\,e^t + (9/4)\,c_3\,e^{-3t/2} - 2\,[(c_1 + c_2t)\,e^t + c_3\,e^{-3t/2} - t] - 2t$, by (8) and (10)

$\therefore$ $$dy/dt = (2c_2 - c_1 - c_2t) + (1/4)\,c_3\,e^{-3t/2} \quad ...(11)$$

(2) $\Rightarrow 3y = (dx/dt) + 4\,(dy/dt)$

$= (c_1 + c_2 + c_2t)\,e^t - (3/2)\,c_3\,e^{-3t/2} - 1 + 4\,[(2c_2 - c_1 - c_2t)\,e^t + (1/4)\,c_3\,e^{-3t/2}]$, by (9) and (11)

Thus, $$3y = (9c_2 - 3c_1 - 3c_2\,t)\,e^t - (1/2)\,c_3\,e^{-3t/2} - 1.$$

$\therefore$ $$y = (3c_2 - c_1 - c_2t)\,e^t - (1/6)\,c_3\,e^{-3t/2} - (1/3). \quad ...(12)$$

(8) and (12) together give the required solution.

Ex. 11(*f*). *Solve* $d^2x/dt^2 - 2(dy/dt) - x = e^t \cos t,\ d^2y/dt^2 + 2(dx/dt) - y = e^t \sin t.$

Sol. Given $$d^2x/dt^2 - x - 2(dy/dt) = e^t \cos t \qquad \text{... (1)}$$

and $$2\,(dx/dt) + d^2y/dt^2 - y = e^t \sin t. \qquad \text{... (2)}$$

Writing D for d/dt, (1) and (2) become

$$(D^2 - 1)\,x - 2\,Dy = e^t \cos t \qquad \text{... (3)}$$

and $$2Dx + (D^2 - 1)\,y = e^t \sin t. \qquad \text{... (4)}$$

Operating both sides of (3) and (4) by $(D^2 - 1)$ and $2D$ respectively, we get

$$(D^2 - 1)^2 x - 2D\,(D^2 - 1)\,y = (D^2 - 1)\,(e^t \cos t) \qquad \text{... (5)}$$

and $$4D^2x + 2D\,(D^2 - 1)\,y = 2D\,(e^t \sin t). \qquad \text{... (6)}$$

Adding (5) and (6), we have

$$[(D^2 - 1)^2 + 4D^2]\,x = D^2\,(e^t \cos t) - e^t \cos t + 2D\,(e^t \sin t)$$

or $$(D^2 + 1)^2\,x = D\,[e^t \cos t - e^t \sin t] - e^t \cos t + 2\,[e^t \sin t + e^t \cos t]$$

or $$(D^2 + 1)^2\,x = e^t \cos t - e^t \sin t - (e^t \sin t + e^t \cos t) - e^t \cos t + 2(e^t \sin t + e^t \cos t)$$

or $$(D^2 + 1)^2\,x = e^t \cos t. \qquad \text{... (7)}$$

whose auxiliary equation is $(D^2 + 1)^2 = 0$ so that $D = \pm\, i, \pm\, i.$

$\therefore$ C.F. of (7) $= (c_1 + c_2t) \cos t + (c_3 + c_4t) \sin t$, c_1, c_2, c_3, c_4 being arbitrary constants.

$$\text{P.I.} = \frac{1}{(D^2+1)^2} e^t \cos t = e^t \frac{1}{[(D+1)^2+1]^2} \cos t$$

$$= e^t \frac{1}{(D^2+2D+2)^2} \cos t = e^t \frac{1}{D^4+4D^2+4+4D^3+8D+4D^2} \cos t$$

$$= e^t \frac{1}{(-1)^2+4(-1)+4+4D(-1)+8D+4(-1)} \cos t$$

$$= e^t \frac{1}{4D-3} \cos t = e^t \frac{4D+3}{(4D-3)\,(4D+3)} \cos t$$

$$= e^t\,(4D+3) \frac{1}{16D^2-9} \cos t = e^t\,(4D+3) \frac{1}{-16-9} \cos t$$

$$= -\left(1/25\right) \times e^t\,(4\,D \cos t + 3 \cos t) = -\left(1/25\right) \times e^t\,(-4 \sin t + 3 \cos t).$$

$\therefore$ Solution of (7) is

$$x = (c_1 + c_2\,t) \cos t + (c_3 + c_4t) \sin t + (1/25)\,e^t\,(4 \sin t - 3 \cos t) \qquad \text{... (8)}$$

$$(8) \Rightarrow dx/dt = c_2 \cos t - (c_1 + c_2t) \sin t + c_4 \sin t + (c_3 + c_4\,t) \cos t + (1/25)\,\{e^t\,(4 \sin t - 3 \cos t) + e^t\,(4 \cos t + 3 \sin t)\}$$

or $$dx/dt = (c_2 + c_3 + c_4t) \cos t + (c_4 - c_1 - c_2t) \sin t + (1/25) \times e^t\,(7 \sin t + \cos t) \qquad \text{... (9)}$$

$$(9) \Rightarrow d^2x/dt^2 = c_4 \cos t - (c_2 + c_3 + c_4t) \sin t - c_2 \sin t + (c_4 - c_1 - c_2t) \cos t + (1/25) \times \{e^t\,(7 \sin t + \cos t) + e^t\,(7 \cos t - \sin t)\}$$

or $$d^2x/dt^2 = (2c_4 - c_1 - c_2t) \cos t - (2c_2 + c_3 + c_4t) \sin t + (1/25) \times e^t\,(6 \sin t + 8 \cos t) \qquad \text{... (10)}$$

From (1), we have $$2\,(dy/dt) = (d^2x/dt^2) - x - e^t \cos t$$

$\therefore$ $$2(dydt) = (2c_4 - c_1 - c_2t) \cos t - (2\,c_2 + c_3 + c_4t) \sin t + (1/25)\,e^t\,(6 \sin t + 8 \cos t) - (c_1 + c_2\,t) \cos t - (c_3 + c_4t) \sin t - (1/25) \times e^t\,(4 \sin t - 3 \cos t) - e^t \cos t, \text{ by (8) and (10)}$$

$$= (2c_4 - 2c_1 - 2c_2t) \cos t - (2c_2 + 2c_3 + 2c_4t) \sin t + (1/25) \times e^t\,(2 \sin t - 14 \cos t).$$

or $$dy/dt = (c_4 - c_1 - c_2t) \cos t - (c_2 + c_3 + c_4t) \sin t + (1/25) \times e^t\,(\sin t - 7 \cos t). \qquad \text{... (11)}$$

(11) $\Rightarrow d^2y/dt^2 = -c_2 \cos t - (c_4 - c_1 - c_2 t) \sin t - c_4 \sin t - (c_2 + c_3 + c_4) \cos t$
$+(1/25) \times [e^t (\sin t - 7 \cos t) + e^t (\cos t + 7 \sin t)]$

or $d^2y/dt^2 = -(2c_2 + c_3 + c_4 t) \cos t - (2c_4 - c_1 - c_2 t) \sin t + (1/25) \times e^t (8 \sin t - 6 \cos t)$ ... (12)

Now, from (2), we have $y = 2\,(dx/dt) + (d^2y/dt^2) - e^t \sin t$

or $y = 2\,(c_2 + c_3 + c_4 t) \cos t + 2\,(c_4 - c_1 - c_2 t) \sin t + (2/25) \times e^t (7 \sin t + \cos t) - (2c_2 + c_3 + c_4 t) \cos t$

$-(2c_4 - c_1 - c_2 t) \sin t + (1/25) \times e^t (8 \sin t - 6 \cos t) - e^t \sin t$, using (9) and (12)

or $y = (c_3 + c_4 t) \cos t - (c_1 + c_2 t) \sin t + (1/25) \times e^t (4 \cos 3t - 3 \sin t)$ ... (13)

(8) and (13) give the required solution.

Exercise 8

Solve the following simultaneous differential equations :

1. (a) $dx/dt = x - 2y,\ dy/dt = 5x + 3y$ **[Andhra 2003]**

Ans. $x = e^{2t} (c_1 \cos 3t + c_2 \sin 3t),\ y = \{(3c_1 - c_2) \sin 3t - (c_1 + 3c_2) \cos 3t\}/2$

(b) $dx/dt = 3x + 2y,\ dy/dt = -5x - 3y$ **[Lucknow 2002; Meerut 2000]**

Ans. $x = c_1 \cos t + c_2 \sin t,\ y = (1/2) \times (c_2 - 3c_1) \cos t - (1/2) \times (c_1 + 3c_2) \sin t$

2. $dz/dx = x + y,\ dy/dx = x + z$ **Ans.** $y = \{c_1 e^{-x} + c_2 e^x - 2(x + 1)\}/2,\ z = \{c_2 e^x - c_1 e^{-x} - 2(x + 1)\}/2$

3. $dx/dt + 7x - y = 0,\ dy/dt + 2x + 5y = 0$ **[Agra 2006; Kanpur 2003; Lucknow 2005]**

Ans. $x = e^{-6t} (c_1 \cos t + c_2 \sin t),\ y = e^{-6t} \{(c_1 + c_2) \cos t - (c_1 - c_2) \sin t\}$

4. (a) $dx/dt = 3x + 2y,\ dy/dt + 5x + 3y = 0$ **Ans.** $x = c_1 \cos t + c_2 \sin t,\ y = \{(c_2 - 3c_1) \cos t - (c_1 + 3c_2) \sin t\}/2$

(b) $dx/dt + dy/dt + 2x + y = 0,\ dy/dt + 5x + 3y = 0$

[Gujrat 2005, Indore 2003; Karnatak 2000; Meerut 1998; Vikram 1998)

Ans. $x = c_1 \cos t + c_2 \sin t,\ y = (1/2) \times (c_2 - 3c_1) \cos t - (1/2) \times (c_1 + 3c_2) \sin t$

5. $dx/dt + x - y = e^t,\ dy/dt + y - x = 0$ **Ans.** $x = c_1 + c_2 e^{-2t} + (2t/3),\ y = c_1 - c_2 e^{-2t} + (1/3)\, e^t$

6. $dx/dt - y = e^{-t},\ dy/dt + x = e^t$ **Ans.** $x = c_1 \cos t + c_2 \sin t + (e^t - e^{-t})/2,\ y = c_2 \cos t - c_1 \sin t + (e^t - e^{-t})/2$

7. $(dx/dt) - y = t^2,\ (dy/dt) + 4x = t$, given $x(0) = 0,\ y(0) = 3/4$. **Ans.** $x = 3t/4,\ y = (3/4) - t^2$

8. $(5D + 4)\, y - (2D + 1)\, z = e^{-x},\ (D + 8)\, y - 3z = 5\, e^{-x}$, where $D \equiv d/dx$ **[Pune 2000; Kolkata 2003]**

Ans. $y = c_1 e^{-2x} + c_2 e^x + 2\, e^{-x},\ z = 2\, c_1 e^{-2x} + 3c_2 e^x + 3\, e^{-x}$

9. $\left(\dfrac{d}{dt} + 2\right) x + 3y = 0,\ 3x + \left(\dfrac{d}{dt} + 2\right) y = 2e^{3t}$ **Ans.** $x = c_1 e^t - c_2 e^{-5t} - (3/8)\, e^{3t},\ y = c_2 e^{-5t} - c_1 e^t + (5/8)\, e^{3t}$

10. $dx/dt + 2x + 4y = 1 + 4t,\ dy/dt + x - y = 3\, t^2/2$ **[I.A.S. 1987]**

Ans. $x = c_1 e^{2t} + c_2 e^{-3t} + t^2 + (t/3) - (5/6),\ y = -4\, c_1 e^{2t} + c_2 e^{-3t} - 2\, t^2 + (4t/3) + (7/3)$

11. $dx/dt - dy/dt - y = e^t,\ dy/dt + x - y = e^{2t}$ **[Agra 1994]**

Ans. $x = c_1 + \cos t\, c_2 \sin t + (3/5)\, e^{2t},\ y = (1/2) \times (c_1 + c_2) \cos t - (1/2)\,(c_1 - c_2) \sin t + (1/2)\, e^t + (2/5)\, e^{2t}$

12. $4(dx/dt) - (dy/dt) + 3x = \sin t,\ (dx/dt) + y = \cos t$, given that $x = 1,\ y = 1$ for $t = 0$. **Ans.** $x =$ $2\, e^{-t} - e^{-3t},\ y = 2\, e^{-t} - 3\, e^{-3t} + \cos t$

13. $(dx/dt) + 2\,(dy/dt) + x + 7y = e^t - 3,\ (dy/dt) - 2x = 3y = 12 - 3\, e^t$

Ans. $x = (1/2) \times e^{-4t} \{(c_2 - c_1) \cos t - (c_1 + c_2) \sin t\} + (31/26)\, e^t - (3/17)$

$y = e^{-4t} (c_1 \text{cost } t + c_2 \sin t) + (6/17) - (2/13)\, e^t$

14. $3\,(dx/dt) + 2\,(dy/dt) - 4x + 3y = 8\, e^{-3t},\ 4\,(dx/dt) + (dy/dt) + 3x + 4y = 8e^{-3t}$, given that $x = 1/5,\ y = 0$ when $t = 0$.

Ans. $x = e^{-t} \{\cos 2t - (1/18) \times \sin 2t\} - (4/5)\, e^{-3t},\ y = e^{-t} \{(21/10) \sin 2t - (4/5) \cos 2t\} + (4/5)\, e^{-3t}$

15. Solve $4x_1 + 9y_1 + 2x + 31y = e^t,\ 3x_1 + 7y_1 + x + 24y = 3$, where, $x_1 = dx/dt$ and $y_1 = dy/dt$.

Ans. $x = e^{-4t} (c_1 \cos t + c_2 \sin t) + (31/26)\, e^t - (93/17)$.

$y = e^{-4t} [(c_2 - c_1) \sin t - (c_1 + c_2) \cos t] - (2/13)\, e^t + (6/17)$

16. $d^2x/dt^2 + 16x - 6(dy/dt) = 0,\ 6(dx/dt) + d^2y/dt^2 + 164 = 0$. **[Agra 1994]**

Ans. $x = c_1 \cos 2t + c_2 \sin 2t + c_3 \cos 8t + c_4 \sin 8t,\ y = c_1 \sin 2t + c_2 \cos 2t + c_3 \sin 8t + c_4 \cos 8t$

17. $d^2x/dt^2 - 4(dx/dt) + 4x = y,\ d^2y/dt^2 + 4(dy/dt) + 4y = 25 + 16\, e^t$.

Ans. $x = c_1 e^{3t} + c_2 e^{-3t} + c_3 \cos t + c_4 \sin t - e^t$, $y = c_1 e^{3t} + 25c_2 e^{-3t} + 7\, c_3 \cos t - c_4 \sin t - e^t$

18. $d^2x/dt^2 + 4x + y = t\, e^t,\ d^2y/dt^2 + y - 2x = \sin^2 t$.

Ans. $x = c_1 \cos \sqrt{2}\,t + c_2 \sin \sqrt{2}\,t + c_3 \cos \sqrt{3}\,t + c_4 \sin \sqrt{3}\,t + (1/6)\, e^t (6t-1) - (1/12) + (1/4) \cos 2t$;

$y = -2\, c_1 \cos \sqrt{2}\,t - 2\, c_2 \sin \sqrt{2}\,t - c_3 \cos \sqrt{3}\,t - c_4 \sin \sqrt{3}\,t + (1/36)\, e^t (6t-7) + (1/3)$

19. Solve $d^2x/dt^2 - 2(dy/dt) - x = e^t \cos t,\ d^2y/dt^2 + 2(dx/dt) - y = e^t \sin t$.

Ans. $x = (c_1 + c_2 t) \cos t + (c_3 + c_4 t) \sin t - (1/25)\, e^t (3 \cos t - 4 \sin t)$;

$y = -(c_1 + c_2 t) \sin t + (c_3 + c_4 t) \cos t - (1/25)\, e^t (3 \sin t + 4 \cos t)$

8.4 Solution of simultaneous differential equations involving operators *x* (*d/dx*) or *t* (*d/dt*)

In such problems we begin with use of methods of chapter 6. We transform the given equat ions into ordinary simultaneous differential equations and then proceed as explained in Art. 8.3.

8.5 Solved Examples based on Art. 8.4

Ex. 1. *Solve* $t\,(dx/dt) + y = 0,\ t\,(dy/dt) + x = 0$.

[G.N.D.U. Amritsar 2004; Rajasthan 2010, Meerut 2001, 08, Lucknow 2001, 06]

Sol. Let $t = e^z$. Let $D_1 \equiv d/dz \equiv t(d/dt)$. Then given equations become

$$D_1 x + y = 0 \qquad \ldots (1)$$

and
$$x + D_1 y = 0. \qquad \ldots (2)$$

Eliminating y from (1) and (2), $\quad D_1^2 x - x = 0 \quad$ or $\quad (D_1^2 - 1)\, x = 0. \quad \ldots (3)$

Its auxiliary equation is $\quad D_1^2 - 1 = 0 \quad$ so that $\quad D_1 = 1, -1$.

$\therefore$ Solution of (3) is $\quad x = c_1 e^z + c_2 e^{-z} \quad$ and so $\quad D_1 x = c_1 e^z - c_2 e^{-z}$

$\therefore$ From (1), $\quad y = -D_1 x = c_2 e^{-z} - c_1 e^z$

Since $t = e^z$, the required solution is $\quad x = c_1 t + c_2 t^{-1}, \quad y = c_2 t^{-1} - c_1 t$.

Ex. 2. *Solve :* $t^2 (d^2x/dt^2) + t\,(dx/dt) + 2y = 0,\ t^2 (d^2y/dt^2) + t\,(dy/dt) - 2x = 0$.

Sol. Let $t = e^z$ so that $z = \log t$. Let $D_1 \equiv d/dz = t\,(d/dt)$. Then $t^2 (d^2/dt^2) = D_1 (D_1 - 1)$.

Using the above values, given equations become

$$[D_1(D_1 - 1) + D_1]\, x + 2y = 0 \qquad \text{and} \qquad [D_1(D_1 - 1) + D_1]\, y - 2x = 0.$$

i.e.,
$$D_1^2 x + 2y = 0 \qquad \ldots (1)$$

and
$$-2x + D_1^2 y = 0 \qquad \ldots (2)$$

Eliminating y from (1) and (2), $\quad (D_1^4 + 4)\, x = 0 \quad \ldots (3)$

Its auxiliary equation is $\quad D_1^4 + 4 = 0 \quad$ or $\quad (D_1^2 + 2)^2 - (2D_1)^2 = 0$.

$(D_1^2 - 2D_1 + 2)(D_1^2 + 2D_1 + 2) \quad$ so that $\quad D = 1 \pm i, -1, \pm i$.

$\therefore$ Solution of (3) is $\quad x = e^z (c_1 \cos z + c_2 \sin z) + e^{-z} (c_3 \cos z + c_4 \sin z). \quad \ldots (4)$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

$$\therefore\ D_1 x = e^z (c_1 \cos z + c_2 \sin z) - e^{-z} (c_3 \cos z + c_4 \sin z) + e^z (-c_1 \sin z + c_2 \cos z) + e^{-z} (-c_3 \sin z + c_4 \cos z)$$

$$= e^z [(c_1 + c_2) \cos z + (c_2 - c_1) \sin z] + e^{-z} [(c_4 - c_3) \cos z - (c_3 + c_4) \sin z].$$

$$\therefore\ D_1^2 x = e^z [(c_1 + c_2) \cos z + (c_2 - c_1) \sin z] + e^z [-(c_1 + c_2) \sin z + (c_2 - c_1) \cos z] - e^{-z} [(c_4 - c_3) \cos z - (c_3 + c_4) \sin z] + e^{-z} [-(c_4 - c_3) \sin z - (c_3 + c_4) \cos z]$$

or
$$D_1^2 x = 2e^z (c_2 \cos z - c_1 \sin z) + 2e^{-z} (c_3 \sin z - c_4 \cos z).$$

Using this value of $D_1^2 x$ in (1), we get

$$y = e^z (c_1 \sin z - c_2 \cos z) + e^{-z} (c_4 \cos z - c_3 \sin z). \qquad \ldots (5)$$

Since $t = e^z$ and $\log t = z$, from (4) and (5) the required solution is given by

$$x = t\,(c_1 \cos \log t + c_2 \sin \log t) + t^{-1} (c_3 \cos \log t + c_4 \sin \log t)$$

and
$$y = t\,(c_1 \sin \log t - c_2 \cos \log t) + t^{-1} (c_4 \cos \log t - c_3 \sin \log t).$$

Ex. 3. $t\,Dx + 2\,(x - y) = t,\ t\,Dy + x + 5y = t^2$, *where* $D \equiv d/dt$. **[Lucknow 2002]**

Sol. Let $t = e^z$ and $D_1 \equiv d/dz \equiv t\,(d/dt)$. Given equations become

$$(D_1 + 2)\, x - 2y = e^z \qquad \ldots (1)$$

and
$$x + (D_1 + 5)\, y = e^{2z}. \qquad \ldots (2)$$

Eliminating y from (1) and (2), $\quad (D_1 + 5)(D_1 + 2)\, x + 2x = (D_1 + 5) e^z + 2e^{2z}$

or
$$(D_1^2 + 7D_1 + 12)\, x = 6\, e^z + 2\, e^{2z}. \qquad \ldots (3)$$

Its auxiliary equation is $\quad D_1^2 + 7D_1 + 12 = 0 \quad$ giving $\quad D_1 = -3, -4$, C.F. of (3) $= c_1 e^{-3z} + c_2 e^{-4z}$, where c_1 and c_2, are arbitrary constants.

$$\text{P.I. corresponding to } 6e^z = 6\frac{1}{D_1^2 + 7D_1 + 12} e^z = \frac{3}{10} e^z.$$

$$\text{P.I. corresponding to } 2e^{2z} = 2\frac{1}{D_1^2 + 7D_1 + 12} e^{2z} = \frac{1}{15} e^{2z}.$$

$\therefore$ Solution of (3) is $\qquad x = c_1 e^{-3z} + c_2 e^{-4z} + (3/10) \times e^z + (1/15) \times e^{2z}$ ... (4)

$\therefore \qquad D_1 x = -3c_1 e^{-3z} - 4c_2 e^{-4z} + (3/10)\, e^z + (2/15) \times e^{2z}$ ... (5)

From (1) and (5), $\qquad y = -(1/2)\, c_1 e^{-3z} - c_2 e^{-4z} - (1/20)\, e^z + (2/15)\, e^{2z}$ ... (6)

Putting $t = e^z$ in (4) and (6), the required general solution is

$$x = c_1 t^{-3} + c_2 t^{-4} + 3t/10 + t^2/15, \qquad y = -(1/2)\, c_1 t^{-3} - c_2 t^{-4} + 2t^2/15 - t/20$$

8.6 Miscellaneous examples on chapter 8

Ex. 1. *Solve* $t\,dx = (t - 2x)\,dt$ *and* $t\,dy = (tx + ty + 2x - t)\,dt$.

Sol. Given $\qquad t\,dx = (t - 2x)\,dt.$ ... (1)

and $\qquad t\,dy = (tx + ty + 2x - t)\,dt.$ ... (2)

From (1), $\qquad \dfrac{dx}{dt} = 1 - \dfrac{2x}{t} \qquad$ or $\qquad \dfrac{dx}{dt} + \dfrac{2}{t}x = 1,$ which is a linear equation.

Its $\qquad$ I.F. $= e^{\int (2/t)\,dt} = e^{2\log t} = t^2 \qquad$ and so its solution is

$$xt^2 = c_1 + \int t^2\,dt = c_1 + (t^2/3) \qquad \text{or} \qquad x = c_1 t^{-2} + (t/3) \qquad \text{... (3)}$$

Now from (2), $\quad t\,dy = t\,(x + y)\,dt - (t - 2x)\,dt \quad$ or $\quad t\,dy = t\,(x + y)\,dt - t\,dx$, using (1)

or $\qquad dx + dy = (x + y)\,dt \qquad$ or $\qquad (dx + dy)/(x + y) = dt.$

Integrating, $\qquad \log(x + y) - \log c_2 = t \qquad$ or $\qquad x + y = c_2 e^t \qquad$ or $\qquad y = c_2 e^t - x.$

or $\qquad y = c_2 e^t - c_1 t^{-2} - (1/3)\, t$, using (3) ... (4)

The required solution is given by (3) and (4).

Ex. 2. *Solve* $\quad dx/dt = ny - mz, \quad dy/dt = lz - nx, \quad dz/dt = mx - ly.$ **[Meertut 1996]**

Sol. Given $\qquad dx/dt = ny - mz.$ ... (1)

$\qquad dy/dt = lz - nx$ ... (2)

and $\qquad dz/dt = mx - ly.$ ... (3)

Multiplying (1), (2) and (3) by $2x$, $2y$ and $2z$ respectively and adding,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} = 0 \qquad \text{or} \qquad \frac{d}{dt}(x^2 + y^2 + z^2) = 0.$$

Integrating, $\qquad x^2 + y^2 + z^2 = c_1$, c_1 being an arbitrary constant ... (4)

Again multiplying (1), (2), (3) by $2lx$, $2my$, $2nz$ respectively and then adding, we have

$$2lx\frac{dx}{dt} + 2my\frac{dy}{dt} + 2nz\frac{dz}{dt} = 0 \qquad \text{or} \qquad \frac{d}{dt}(lx^2 + my^2 + nz^2) = 0.$$

Integrating, $\qquad lx^2 + my^2 + nz^2 = c_2$, c_2 being an arbitrary constant ... (5)

Now multiplying (1), (2) and (3) by l, m and n respectively and adding,

$$l\frac{dx}{dt} + m\frac{dy}{dt} + n\frac{dz}{dt} = 0 \qquad \text{or} \qquad \frac{d}{dt}(lx + my + nz) = 0.$$

Integrating, $\qquad lx + my + nz = c_3$, c_3 being an arbitrary constant ... (6)

The required solution is given by (4), (5) and (6).

Ex. 3. *Solve :* $\quad lt\dfrac{dx}{dt} = mn(y - z), \qquad mt\dfrac{dy}{dt} = nl(z - x), \qquad nt\dfrac{dz}{dt} = lm(x - y).$

Sol. Re-writing the given equations, we have

$$\frac{l\,dx}{(1/t)\,dt} = mn(y - z), \qquad \frac{m\,dy}{(1/t)\,dt} = nl(z - x), \qquad \frac{n\,dz}{(1/t)\,dt} = lm(x - y). \qquad \text{... (1)}$$

Putting $\qquad l\,dx = dX, \qquad m\,dy = dY, \qquad n\,dz = dZ \qquad$ and $\qquad (1/t)\,dt = dT,$ ... (2)

$\qquad (1) \Rightarrow dX/dT = nY - mZ, \qquad dY/dT = lZ - nX, \qquad dZ/dT = mX - lY.$

Now proceed as in Ex. 3 and finally replace X, Y, Z and T by their values given by (2), namely $X = lx$, $Y = my$, $Z == nz$ and $T = \log t$.

Ex. 4. *$dx/dt = 2y$, $dy/dt = 2z$ and $dz/dt = 2x$.*

Sol. Let as an exercise. **Ans.** $x = c_1 e^{2t} + c_2 e^{-t} \cos(\sqrt{3}\,t + c_3)$,

$$y = c_1 e^{2t} + c_2 e^{-t}\cos(\sqrt{3}\,t + c_3 + 2\pi/3), \qquad z = c_1 e^{2t} + c_2 e^{-t}\cos(\sqrt{3}\,t + c_3 + 4\pi/3)$$

Ex. 5. *Solve* $$(D + 1)\,x + (D - 1)\,y = e^t. \qquad \ldots (1)$$
$$(D^2 + D + 1)\,x + (D^2 - D + 1)\,y = t^2 \qquad \ldots (2)$$
where $D \equiv d/dt$. **[Kurukshetra 2005; 07; Mysore 2001, 03]**

Sol. Here the determinant Δ formed by operator 'coefficients' is given by

$$\Delta = \begin{vmatrix} D+1 & D-1 \\ D^2+D+1 & D^2-D+1 \end{vmatrix} = (D+1)(D^2-D+1) - (D-1)(D^2+D+1)$$

or $$\Delta = (D^3 + 1) - (D^3 - 1) = 2.$$

Since the degree of D in Δ is zero, the general solution of the given system should not contain any arbitrary constant (refer note of Art 8.2 for understanding).

Now operating (1) by $(D^2 - D + 1)$, (2) by $(D - 1)$ and then subtracting the equations thus obtained, we get

$$[(D^2 - D + 1)(D + 1) - (D - 1)(D^2 + D + 1)]\,x = (D^2 - D + 1)\,e^t - (D - 1)\,t^2$$

or $$[(D^3 + 1) - (D^3 - 1)]\,x = e^t - e^t + e^t - 2t + t^2$$

or $2x = e^t - 2t + t^2$ or $x = (1/2)\,(e^t - 2t + t^2)$. ... (3)

Similarly, on eliminating x, (1) and (2) give

$$[(D^2 + D + 1)(D - 1) - (D + 1)(D^2 - D + 1)]\,y = (D^2 + D + 1)\,e^t - (D + 1)\,t^2$$

or $$[D^3 - 1 - (D^3 + 1)]\,y = e^t + e^t + e^t - 2t - t^2$$

or $-2y = 3e^t - 2t - t^2$ or $y = (2t + t^2 - 3e^t)/2$ ... (4)

The required solution is given by (3) and (4).

Ex. 6. Solve $dx/dt = x^2 + xy$, $dy/dt = y^2 + xy$, satisfying the initial condition $x = 1$, $y = 2$ when $t = 0$.

Sol. Given $$dx/dt = x\,(x + y) \qquad \ldots(1)$$
$$dy/dt = y\,(x + y) \qquad \ldots(2)$$

Given initial condition are $x = 1$, $y = 2$ when $t = 0$ ...(3)

Dividing (2) by (1), we get $dy/dx = y/x$ or $(1/y)dy = (1/x)\,dx$

Integrating it, we get $\log y = \log x + \log c$ or $y = cx$, ...(4)

where c in an arbitrary constant.

Putting $x = 1$ and $y = 2$ in (4), we get $c = 2$. Hence (4) reduce to

$$y = 2x \qquad \ldots(5)$$

Using (5), (1) gives $dx/dt = x\,(x + 2x)$ or $(1/x^2)dx = 3dt$

Integrating it, $-(1/x) = 3t + c_1$, c_1 being an arbitrary constant ...(6)

Putting $x = 1$ and $t = 0$ in (6) we get $c_1 = -1$

$\therefore$ (6) reduces to $-(1/x) = 3t - 1$ or $x = 1/(1-3t)$ ...(7)

From (5) and (7), we get $y = 2/(1-3t)$ ...(8)

Hence the required solution is given by (7) and (8).

Objective problems on chapter 8

Ex. 1. *The general solution of the system of equations $y + (dz/dx) = 0$, $dy/dx - z = 0$ is given by*

(a) $y = \alpha e^x + \beta e^{-x}$, $z = \alpha e^x - \beta e^{-x}$ *(b)* $y = \alpha \cos x + \beta \sin x$, $z = \alpha \sin x - \beta \cos x$

(c) $y = \alpha \sin x - \beta \cos x$, $z = \alpha \cos x + \beta \sin x$ *(d)* $y = \alpha e^x - \beta e^{-x}$, $z = \alpha e^x + \beta e^{-x}$

[GATE 2005]

Sol. Ans. (*c*). Use Art 8.2 and Art. 8.3.

Ex. 2. The general solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ *of the system $x' = -x + 2y$, $y' = 4x + y$ is given by* *(a)* $\begin{pmatrix} c_1 e^{3t} - c_2 e^{-3t} \\ 2c_1 e^{3t} + c_2 e^{-3t} \end{pmatrix}$ *(b)* $\begin{pmatrix} c_1 e^{3t} \\ c_2 e^{-3t} \end{pmatrix}$ *(c)* $\begin{pmatrix} c_1 e^{3t} + c_2 e^{-3t} \\ 2c_1 e^{3t} + c_2 e^{-3t} \end{pmatrix}$ *(d)* $\begin{pmatrix} c_1 e^{3t} - c_2 e^{-3t} \\ -2c_1 e^{3t} + c_2 e^{-3t} \end{pmatrix}$

[GATE 2004]

Sol. Ans. (*a*) Writing D for d/dt, the given equations become

$$(D + 1)\, x - 2y = 0 \qquad \text{... (1)}$$

and
$$-4x + (D - 1)\, y = 0 \qquad \text{... (2)}$$

Operating (1) by $(D - 1)$, multiplying (2) by 2 and then adding the resulting equations, we get

$$(D - 1)\,(D + 1)\, x - 8x = 0 \qquad \text{or} \qquad (D^2 - 9)x = 0$$

whose solution is
$$x = c_1' e^{3t} + c_2' e^{-3t} \qquad \text{... (3)}$$

From (1), $\quad 2y = dx/dt + x = 3c_1' e^{3t} - 3c_2' e^{-3t} + c_1' e^{3t} + c_2' e^{-3t}$, using (3)

Thus,
$$y = 2c_1' e^{3t} - c_2' e^{-3t} \qquad \text{...(4)}$$

Setting $c_1' = c_1$ and $c_2' = -c_2$, (3) and (4) yield, $x = c_1 e^{3t} - c_2 e^{-3t}$, $y = 2c_1 e^{3t} + c_2 e^{-3t}$ ...(5)

Putting equations of (5) in matrix form, we get $\quad \begin{pmatrix} x(t) \\ 3(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{3t} - c_2 e^{-3t} \\ 2c_1 e^{3t} + c_2 e^{-3t} \end{pmatrix}$

Miscellaneous Examples on Chapter 8

Ex. 1. *Solve the simultaneous differential equations $2(D - 2)x + (D + 1)y = e^{2t}$, $(D + 2)\, x + (D - 3)\, y = 0$, $D = d/dt$* **[Guwahati 2007]**

Sol. Given
$$2(D - 2)x + (D + 1)y = e^{2t} \qquad \text{... (1)}$$

$$(D+2)x + (D-3)y = 0 \qquad \text{... (2)}$$

Operating (1) by $(D - 3)$ and (2) by $(D + 1)$ and then subtracting, we have

$$\{2(D-3)\,(D-2) - (D+1)\,(D+2)\}x = (D-3)e^{2t} \qquad \text{or} \qquad (D^2 - 13D + 10)x = -e^{2t} \qquad \text{... (3)}$$

Auxiliary equation of (3) is $\quad D^2 - 13D + 10 = 0 \quad$ giving $\quad D = (13 \pm \sqrt{129})/2$

$\therefore$ C.F. of (3) = $c_1\, e^{x(13+\sqrt{129})/2} + c_2\, e^{x(13-\sqrt{129})/2}$, c_1, c_2 being arbitrary constants

$$\text{P.I of (3)} = \frac{1}{D^2 - 13D + 10}\,(-e^{2t}) = -\frac{1}{2^2 + (13 \times 2) + 10} e^{2t} = -\frac{1}{40} e^{2t}$$

$\therefore$ General solution of (3) is given by

$$x = c_1\, e^{x(13+\sqrt{129})/2} + c_2\, e^{x(13-\sqrt{129})/2} - (1/40) \times e^{2t} \qquad \text{... (4)}$$

From (2),
$$Dy = 3y - Dx - 2x \qquad \text{... (5)}$$

From (1),
$$y = e^{2t} - Dy - 2Dx + 4x$$

or
$$y = e^{2t} - (3y - Dx - 2x) - 2Dx + 4x, \text{ using (5)}$$

or
$$4y = e^{2t} - Dx + 6x \qquad \text{... (6)}$$

From (4), we have

$$Dx = (c_1/2) \times (13 + \sqrt{129})e^{x(13+\sqrt{129})/2} + (c_2/2) \times (13 - \sqrt{129})\, e^{x(13-\sqrt{129}/2} - (1/20) \times e^{2t} \quad \text{..... (7)}$$

Substituting the valvue of x and Dx given by (4) and (7) respectively in (6), we have

$$4y = e^{2t} - (c_1/2) \times (13 + \sqrt{129})\, e^{x(13+\sqrt{129})/2} - (c_2/2) \times (13 - \sqrt{129})\, e^{x(12-\sqrt{129})/2}$$
$$+ (1/20) \times e^{2t} + 6c_1 e^{x(13+\sqrt{129})/2} + 6c_2 e^{x(13-\sqrt{129})/2} - (3/20)\, e^{2t}$$

or
$$4y = (9/10) \times e^{2t} - (c_1/2) \times (1 + \sqrt{129})\, e^{x(13+\sqrt{129})/2} - (c_2/2) \times (1 - \sqrt{129})\, e^{x(12-\sqrt{129})/2}$$

or
$$y = (9/40) \times e^{2t} - (c_1/8) \times (1 + \sqrt{129})\, e^{x(13+\sqrt{129})/2} - (c_2/8) \times (1 - \sqrt{129})\, e^{x(12-\sqrt{129})/2} \quad \text{... (8)}$$

The required solution is given by (4) and (8).

UNIT-1 COMPLEX ANALYSIS

* Basics Of Complex Numbers

- $z = a + ib$, $\bar{z} = a - ib$
- $Re(z) = \frac{z+\bar{z}}{2}$, $Im(z) = \frac{z-\bar{z}}{2}$
- $z\bar{z} = |z|^2$, $\overline{(\bar{z})} = z$, $|z| = |\bar{z}|$
- Polar Form $\Rightarrow$ $z = re^{i\theta}$, $r = \sqrt{a^2+b^2} = |z|$

  $\Rightarrow Arg(z) = \theta = \tan^{-1}(b/a)$

  $\Rightarrow arg(z) = Arg(z) + 2n\pi$

- ★ De Moivre's Thm: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

  $z^n = r^n e^{in\theta}$

- $z + z^{-1} = 2\cos\theta$, $z - z^{-1} = 2i\sin\theta$
- Euler's Formula : $e^{i\theta} = \cos\theta + i\sin\theta$

  $e^z = e^x . e^{iy} = e^x(\cos y + i\sin y)$

- $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, $\sin x = \frac{e^{ix}-e^{-ix}}{2}$
- $\cosh x = \frac{e^x+e^{-x}}{2}$, $\sinh x = \frac{e^x-e^{-x}}{2}$, $\tanh x = \frac{e^x-e^{-x}}{e^x+e^{-x}}$

★ $\sin z = \sin x \cosh y + i \cos x \sinh y$

$\cos z = \cos x \cosh y - i \sin x \sinh y$

$\sin ix = i \sinh x$

$\cos ix = \cosh x$

- $\sinh(ix) = i\sin x$

  $\cosh(ix) = \cos x$

  $\tanh(ix) = i\tan x$

  $cosech(ix) = -i\, cosec\, x$

  $sech(ix) = \sec x$

  $\coth(ix) = -i\cot x$

* CR Equations

→ Cartesian → $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

→ Polar → $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

* Harmonic Functions

If $f(z) = u + iv$, and

$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

& $\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$,

Then u & v are Harmonic Conjugates of each other.

* Complex Integration

If $f(z)$ is analytic → Then integration is independent of PATH taken.

* Cauchy's Integral Formula

★ $\oint \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$

$\oint \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0)$

* Taylor's Series

$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) \ldots$

* Maclauren's Series

Taylor's Series @ ORIGIN

$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) \ldots\ldots$

* Laurent's Series

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Regular Part}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{Principal Part}}$$

* Pole & It's Order - Theory

* Cauchy's Residue Thm

$\oint_c f(z)\, dz = 2\pi i\, (\underbrace{R_1 + R_2 + R_3 + \ldots\ldots + R_n}_{\text{Residues}})$

* Other Imp Concepts

- Conformal Mapping
- Bilinear Transformation (Mobius Transformation)
- Orthogonal Trajectories
- Evaluation of Real Integrals

UNIT - 2 ORDINARY DIFFRENTIAL EQUATIONS

* CONCEPTS:

- ODE v/s PDE
- Order of DE
- Degree of DE
- Linear & Non Linear DE
- Wronskian
- Linearly Dependent & Independent
  - Dependent → $W = 0$
  - Independent → $W \neq 0$
- Formation of D.E.

*** 1st order D.E.**
- Variable Seperable
- Homogeneous
- Linear + Bernoulli's
- Exact

① **Variable Seperable** - DE can be written as

$$f(y)\,dy = g(x)\,dx$$

Then Integrate Both Sides

② **Homogeneous** - Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

③ **Linear** - of the form $\frac{dy}{dx} + Py = Q$

- IF $= e^{\int P dx}$
- Solution : $y \times (IF) = \int Q.(IF)\,dx + c$

④ **Exact** - Form : $M dx + N dy = 0$

Condition : $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Solution : $\int M dx + \int N dy = c$ ↳ Only involving y terms

* RULES for CF & PI

C.F.

	Types of Roots	C.F.
1)	Distinct & Real $m_1, m_2, \ldots$	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots$
2)	Real & Equal $m_1 = m_2, m_3 \ldots$ $m_1 = m_2 = m_3, m_4 \ldots$	$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} \ldots$ $y = (c_1 + x c_2 + x^2 c_3)e^{m_1 x} + c_4 e^{m_4 x} \ldots$
3)	Complex Conjugate $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, $m_3 \ldots$	$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} \ldots$
4)	Pair of Complex $m_1 = m_3 = \alpha + i\beta$, $m_2 = m_4 = \alpha - i\beta$, m_5	$y = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$

P.I.

1) $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$

2) $\frac{1}{f(D)} x^n = [f(D)]^{-1} x^n$ ↳ Expand & Then operate

3) $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$ | $\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$

* Case of failure — If $f(-a^2) = 0$

$\frac{1}{D^2 + a^2} \cos ax = \frac{x}{2a} \sin ax$ | $\frac{1}{D^2 + a^2} \sin ax = \frac{-x}{2a} \cos ax$

4) $\frac{1}{f(D)} e^{ax}.\phi(x) = e^{ax}.\frac{1}{f(D+a)} \phi(x)$

5) $\frac{1}{D+a} \phi(x) = e^{-ax} \int e^{ax}.\phi(x)\,dx$

* VARIATION OF PARAMETERS

→ From CF, we get $c_1 y_1 + c_2 y_2$

Let P.I. be $u_1 y_1 + u_2 y_2$

→ Now, task is to find u_1 & u_2

Given DE → $y'' + p y' + q y = b(x)$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$u_1 = -\int \frac{y_2\, b(x)}{W}$$

$$u_2 = \int \frac{y_1\, b(x)}{W}$$

∴ P.I. $= u_1 y_1 + u_2 y_2$

And $y = CF + PI$

* Other Methods

- Reduction of Order
- Method of Undetermined Coeff.

* IMP CONCEPTS

- Euler's Eqn
- Cauchy's Homogeneous Linear Eqn
- Legendre's " " "
- Simultaneous D.E.

UNIT - 3 PARTIAL DIFFRENTIAL EQUATIONS

* PDE are those which involves partial derivatives with respect to two or more independent variables.

* General Notation $\Rightarrow$ $p = \frac{\partial z}{\partial x}$ | $q = \frac{\partial z}{\partial y}$ | $r = \frac{\partial^2 z}{\partial x^2}$ | $s = \frac{\partial^2 z}{\partial x \partial y}$ | $t = \frac{\partial^2 z}{\partial y^2}$ |

* FORMATION OF P.D.E :

① Elimination of Arbitrary Constants

② Elimination of Arbitrary Functions

$$\boxed{\frac{\partial(u,v)}{\partial(y,z)} p + \frac{\partial(u,v)}{\partial(z,x)} q = \frac{\partial(u,v)}{\partial(x,y)}}$$

* SOLUTION by DIRECT INTEGRATION :

This method is applicable to those problems, where direct integration is possible.

Eg $\frac{\partial^2 z}{\partial x \partial y} = \cos x \cos y$, $\frac{\partial^2 z}{\partial x^2} + z = 0$

* LAGRANGE'S EQUATION :

Form : $Pp + Qq = R$

Solution : $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

* CHARPIT'S METHOD :

FORM : $f(x,y,z,p,q) = 0$

$f(x,y,z,p,q,a) = 0$, $\Big\rangle$ Compatible

Solution : $\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$

* SPECIAL TYPES OF I ORDER PDE

Form 1 : $f(p,q) = 0$

Solution : Replace $p \to a$, $q \to b$

Find $b = F(a)$

$z = ax + F(a)y + c$

Form 2 : $f(z,p,q) = 0$

Solution : Assume $q = ap$

Find $p = \phi(z,a)$

$\frac{dz}{\phi(z,a)} = dx + a\,dy$

$\int \frac{dz}{\phi(z,a)} = x + ay + b$

Form 3 : Seperable

$f_1(x,p) = f_2(y,q)$

Solution : $f_1(x,p) = f_2(y,q) = a$

Solving , $p = \phi_1(x,a)$

$q = \phi_2(y,a)$

$z = \int p\,dx + \int q\,dy + b$

Form 4 : Clairaut's Form

$z = px + qy + f(p,q)$

Solution : Replace $p \to a$, $q \to b$

$z = ax + by + f(a,b)$

* RULES FOR FINDING C.F. & P.I.

C.F.

	Types of Root	CF
1)	Real & Distinct $m_1, m_2 \ldots$	$f_1(y+m_1x) + f_2(y+m_2x) \ldots$
2)	Two Equal $m_1 = m_2, m_3 \ldots$	$f_1(y+m_1x) + x f_2(y+m_1x) + f_3(y+m_3x) \ldots$
	Three Equal $m_1 = m_2 = m_3, m_4 \ldots$	$f_1(y+m_1x) + x f_2(y+m_1x) + x^2 f_3(y+m_1x) + f_4(y+m_4x) \ldots$

General Method

$\frac{1}{D - mD'} F(x,y) = \int F(x, c - mx)\,dx$

Then replace c by $y + mx$

$[\, y + mx = c \,]$

P.I.

1) If $F(x,y) = e^{ax+by}$

$PI = \frac{1}{f(D,D')} e^{ax+by}$

$= \frac{1}{f(a,b)} e^{ax+by}$

if $f(a,b) \neq 0$

2) $F(x,y) = \sin(ax+by)$

$P.I. = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by)$

$P.I. = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by)$

Same for $\cos(ax+by)$

3) $F(x,y) = e^{ax+by} \phi(x,y)$

$P.I. = \frac{1}{f(D,D')} e^{ax+by} \phi(x,y)$

$= e^{ax+by} \frac{1}{f(D+a, D'+b)} \cdot \phi(x,y)$

4) $F(x,y) = x^m y^n$

$P.I. = \frac{1}{f(D,D')} x^m y^n$

$= [f(D,D')]^{-1} x^m y^n$

$L[af(t) + bg(t)] = a\,L[f(t)] + b\,L[g(t)]$

LAPLACE TRANSFORM OF FUNCTIONS :

$f(t)$	$\bar{f}(s)$
1	$1/s$
e^{at}	$1/s-a$
$\sin at$	$1/s^2+a^2$
$\cos at$	s/s^2+a^2
$\sinh at$	a/s^2-a^2
$\cosh at$	s/s^2-a^2
t^n	$\frac{\overline{\lvert n+1}}{s^{n+1}}$ or $\frac{n!}{s^{n+1}}$

* I Shifting Thm :

If $L[f(t)] = \bar{f}(s)$

Then $\boxed{L[e^{at}f(t)] = \bar{f}(s-a)}$

$f(t)$	$\bar{f}(s)$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sinh bt$	$\frac{b}{(s-a)^2-b^2}$
$e^{at}\cosh bt$	$\frac{s-a}{(s-a)^2-b^2}$
$e^{at}\,t^n$	$\frac{\overline{\lvert n+1}}{(s-a)^{n+1}}$ or $\frac{n!}{(s-a)^{n+1}}$

If $L[f(t)] = \bar{f}(s)$

Then $L^{-1}\{\bar{f}(s)\} = f(t)$

$\bar{f}(s)$	$f(t)$
$L^{-1}\left(\frac{1}{s}\right)$	1
$L^{-1}\left(\frac{1}{s-a}\right)$	e^{at}
$L^{-1}\left(\frac{1}{s^2+a^2}\right)$	$\frac{1}{a}\sin at$
$L^{-1}\left(\frac{s}{s^2+a^2}\right)$	$\cos at$
$L^{-1}\left(\frac{1}{s^2-a^2}\right)$	$\frac{1}{a}\sinh at$
$L^{-1}\left(\frac{s}{s^2-a^2}\right)$	$\cosh at$
$L^{-1}\left(\frac{1}{s^n}\right)$	$\frac{t^{n-1}}{\overline{\lvert n}}$ or $\frac{t^{n-1}}{(n-1)!}$
$L^{-1}\left(\frac{1}{(s-a)^2+b^2}\right)$	$\frac{1}{b}e^{at}\sin bt$
$L^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right)$	$e^{at}\cos bt$
$L^{-1}\left(\frac{1}{(s-a)^2-b^2}\right)$	$\frac{1}{b}e^{at}\sinh bt$
$L^{-1}\left(\frac{s-a}{(s-a)^2-b^2}\right)$	$e^{at}\cosh bt$
$L^{-1}\left(\frac{1}{(s-a)^n}\right)$	$\frac{e^{at}\,t^{n-1}}{(n-1)!}$
* $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$	$\frac{1}{2a}t\sin at$
* $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$	$\frac{1}{2a^3}[\sin at - at\cos at]$

$L[f''(t)] = s^2\bar{f}(s) - sf(0) - f'(0)$

*** Laplace Transform of Integral :** $L\left[\int_0^t f(u)\,du\right] = \frac{\bar{f}(s)}{s}$

*** Differentiation of Laplace Transform :**

$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$

$L[t\,f(t)] = -1\frac{d}{ds}\bar{f}(s)$

$f(t) = \frac{-1}{t}L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\}$

*** Integration of Laplace Transform :**

$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$

see limits carefully

*** CONVOLUTION THM :** If $L^{-1}[\bar{f}(s)] = f(t)$ & $L^{-1}[\bar{g}(s)] = g(t)$

Then $\boxed{L^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = \int_0^t f(u)\,g(t-u)\,du = f*g}$

*** Laplace Transform of Periodic Function :**

If $f(t+P) = f(t)$

Here $P \rightarrow$ Period of $f(t)$

$L[f(t)] = \frac{1}{1-e^{-Ps}}\int_0^P e^{-st}f(t)\,dt$

*** Second Shifting Theorem**

If $L[f(t)] = \bar{f}(s)$

Then $L[f(t-a)\,u(t-a)] = e^{-as}\bar{f}(s)$

OR $L^{-1}[e^{-as}\bar{f}(s)] = f(t-a)\,u(t-a)$

*** Laplace Transform of Dirac Delta Function :**

* $L\{\delta(t-a)\} = e^{-as}$

OR $L^{-1}[e^{-as}] = \delta(t-a)$