Pandit Deendayal Energy University Gandhinagar
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Worksheet-4 (Laplace Transforms)

Find the Laplace transform of the following functions:

1. $\sin(2t)\cos(2t)$ Ans: $\frac{2}{s^2+16}$

2. $\cos^2(3t)$ Ans: $\frac{1}{2s} + \frac{1}{2(s^2+36)}$

3. $t\,e^{2t}\sin(3t)$ Ans: $\frac{6(s-2)}{((s-2)^2+9)^2}$

4. $6e^{-5t} + e^{3t} + 5t^3 - 9$ Ans: $\frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$

5. $4\cos(4t) - 9\sin(4t) + 2\cos(10t)$ Ans: $\frac{4(s-9)}{s^2+16} + \frac{2s}{s^2+100}$

6. $3\sinh(2t) + 3\sin(2t)$ Ans: $\frac{6}{s^2-4} + \frac{6}{s^2+4}$

7. $e^{3t} + \cos(6t) - e^{3t}\cos(6t)$ Ans: $\frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$

8. $\frac{e^{bt}-e^{at}}{t}$ Ans: $\ln\left(\frac{s-a}{s-b}\right)$

9. $t^{-\frac{1}{2}}$ Ans: $\frac{\sqrt{\pi}}{s^{\frac{1}{2}}}$

10. $\frac{\sin 3t}{t}$ Ans: $\frac{\pi}{2} - \tan^{-1}\frac{s}{3}$

Find the inverse Laplace transform of the following functions:

1. $\frac{4}{s-2} - \frac{3}{s+5}$
2. $\frac{s+5}{s^2+9} = \frac{s}{s^2+9} + \frac{5}{s^2+9}$
3. $\frac{8(s+2)-4}{(s+2)^2+25} = \frac{8(s+2)}{(s+2)^2+25} - \frac{4}{(s+2)^2+25}$
4. $\frac{4}{s} - \frac{1}{s^2} + \frac{5}{s^3} + \frac{2}{s^4}$
5. $\frac{10}{(s-5)^2} + \frac{2}{(s-5)^3}$
6. $\frac{1}{s^2+6s+13}$ (start by completing the square)

Answers

1. $\mathcal{L}^{-1}\{\frac{4}{s-2} - \frac{3}{s+5}\} = 4e^{2t} - 3e^{-5t}$
2. $\mathcal{L}^{-1}\{\frac{s}{s^2+9} + \frac{5}{s^2+9}\} = \cos(3t) + \frac{5}{3}\sin(3t)$
3. $\mathcal{L}^{-1}\{\frac{8(s+2)}{(s+2)^2+25} - \frac{4}{(s+2)^2+25}\} = 8e^{-2t}\cos(5t) - \frac{4}{5}e^{-2t}\sin(5t)$
4. $\mathcal{L}^{-1}\{\frac{4}{s} - \frac{1}{s^2} + \frac{5}{s^3} + \frac{2}{s^4}\} = 4 - t + \frac{5}{2!}t^2 + \frac{2}{3!}t^3$
5. $\mathcal{L}^{-1}\{\frac{10}{(s-5)^2} + \frac{2}{(s-5)^3}\} = 10te^{5t} + \frac{2}{2!}t^2e^{5t} = 10te^{5t} + t^2e^{5t}$
6. $\mathcal{L}^{-1}\{\frac{1}{(s+3)^2+4}\} = \frac{1}{2}e^{-3t}\sin(2t)$

Que: Solve the initial value problem

$y'' - y' - 2y = 4,\ y(0) = 2,\ y'(0) = 1$ Ans: $y = -2 + 3e^{t} + e^{-2t}$

Que: Use Laplace transform to find the solution to

$y'' - 6y' + 5y = 3e^{2t},\ y(0) = 2,\ y'(0) = 3$ Ans: $-e^{2t} + \frac{e^{5t}}{2} + \frac{5e^{t}}{2}$

Que: Solve the IVP with variable coefficients

$ty'' - ty' + y = 2,\ y(0) = 2,\ y'(0) = -4$ Ans: $y(t)=2-4t$

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Course: Mathematics-II (20MA103T)

Worksheet-3 (Ordinary Differential Equations)

1. Solve$(D^3 - 2D^2 - 4D + 8)y = 0$.

[**Ans:** $y = (C_1 + C_2 x)e^{2x} + C_3 e^{-2x}$]

2. Find the solution of $\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$, where $R^2 C = 4L$ and R, C, L are constants.

Ans: $i = (C_1 t + C_2)e^{-\left(\frac{R}{2L}\right)t}$, where C_1 and C_2 are arbitrary constants.

3. Solve $(D^4 + 2D^3 - 3D^2)y = 3e^{2x} + 4\sin(x)$

Ans: $y = C_1 x + C_2 + C_3 e^x + C_4 e^{-3x} + \left(\frac{3}{20}\right)e^{2x} + \left(\frac{4}{5}\right)\sin(x) + \left(\frac{2}{5}\right)\cos(x)$

4. Solve $(D^2 - 4D + 1)y = e^{2x}\sin(x)$.

[**Ans**: $y = C_1 e^{(2+\sqrt{3})x} + C_2 e^{(2-\sqrt{3})x} - \frac{1}{4}e^{2x}\sin(x)$]

5. Solve $(D^2 - 4)y = x^2$. [**Ans**: $y = C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{4}\left(x^2 + \frac{1}{2}\right)$]

6. $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$

Ans: $y = c_1 e^x + c_2 e^{2x} + x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2 e^x - 3xe^x$

7. $\frac{d^4 y}{dx^4} + \frac{d^2 y}{dx^2} = 3x^2 + 4\sin x - 2\cos x$

Ans: $y = c_1 + c_2 x + c_3 \cos x + c_4 \sin x + \frac{1}{4}x^4 - 3x^2 + x\sin x + 2x\cos x$

8. $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x^2$

Ans: $y = c_1 e^x + c_2 e^{2x} + 2x^2 + 6x + 7$

9. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 6\sin 2x + 7\cos 2x$

Ans: $y = e^{-x}(c_1 \sin 2x + c_2 \cos 2x) + 2\sin 2x - \cos 2x$

10. Solve $x^2y'' - 4xy' + 6y = x$ [**Ans**: $y = C_1x^2 + C_2x^3 + \frac{x}{2}$]

11. Solve $x^2y'' - xy' + y = 2\log(x)$ [**Ans**: $y = (C_1\log(x) + C_2)x + 2\log(x) + 4.$]

12. Solve $(1+x)^2y'' + (1+x)y' + y = 4\cos\big(\log(1+x)\big)$

[**Ans**: $y = C_1\cos\big(\log(1+x)\big) + C_2\sin\big(\log(1+x)\big) + 2\log(1+x)\sin\big(\log(1+x)\big)$.]

13. Solve $(2x-1)^3y''' + (2x-1)y' - 2y = 0$

Ans: $y = C_1(2x-1) + C_2(2x-1)^{4+2\sqrt{3}} + C_3(2x-1)^{4-2\sqrt{3}}$

14. Solve the simultaneous ODEs $\frac{dx}{dt} - 4y = \cos(at)$, $\frac{dy}{dt} + 4x = \sin(at)$.

Ans: $x = C_1\cos(4t) + C_2\sin(4t) + \frac{1}{a+4}\sin(4t)$, $y = -C_1\sin(4t) + C_2\cos(4t) - \frac{1}{a+4}\cos(4t)$

15. $x' + y' + x = -e^{-t}$, $x' + 2y' + 2x + 2y = 0$ given that $x(0) = -1$, $y(0) = 1$.

Ans: $x(t) = -e^{-t}\big(\cos(t) + \sin(t)\big)$, $y(t) = e^t\big(1 + \sin(t)\big)$

16. The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u + kr^3 = 0$, where k is constant. Solve the equation under the conditions $u = 0$ when $r = 0$ and $r = a$.

Ans: $u = \frac{kr}{8}(a^2 - r^2)$

Applications of second-order linear ODE

17. A particle is executing simple harmonic motion with amplitude $20\ cm.$ and time $4\ seconds$. Find the time required by the particle in passing between

points which are at distances $15\ cm.$ and $5\ cm.$ from the centre of force and are on the same side of it.

Ans: Required time = $\frac{2}{\pi}\left(\cos^{-1}\frac{1}{4} - \cos^{-1}\frac{3}{4}\right)$ = 0.38 secs.

18. A circuit has in series an electromotive force given by $E = 100\sin 6\,0t$V, a resistor of 2Ω, an inductor of 0.1 H, and a capacitor of $\frac{1}{260}$farads. If the initial current and the initial charge on the capacitor are both zero, find the charge on the capacitor at any time $t > 0$.

Ans: $q = e^{-10t}(\frac{36}{61}\sin 5\,0t + \frac{30}{61}\cos 5\,0t) - \frac{25}{61}\sin 6\,0t - \frac{30}{61}\cos 6\,0t$

19. A 32-lb weight is attached to the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, thereby stretching 2ft. The weight is then pulled down 6 in. below its equilibrium position and released at $t = 0$. No external forces are present; but the resistance of the medium in pounds is numerically equal to $4(\frac{dx}{dt})$, where $\frac{dx}{dt}$is the instantaneous velocity in feet per second. Determine the resulting motion of the weight on the spring.

Ans: $x = e^{-2t}(\frac{\sqrt{3}}{6}\sin 2\sqrt{3}t + \frac{1}{2}\cos 2\sqrt{3}t)$