# Minimum Spanning Tree

Data Structures and Algorithms


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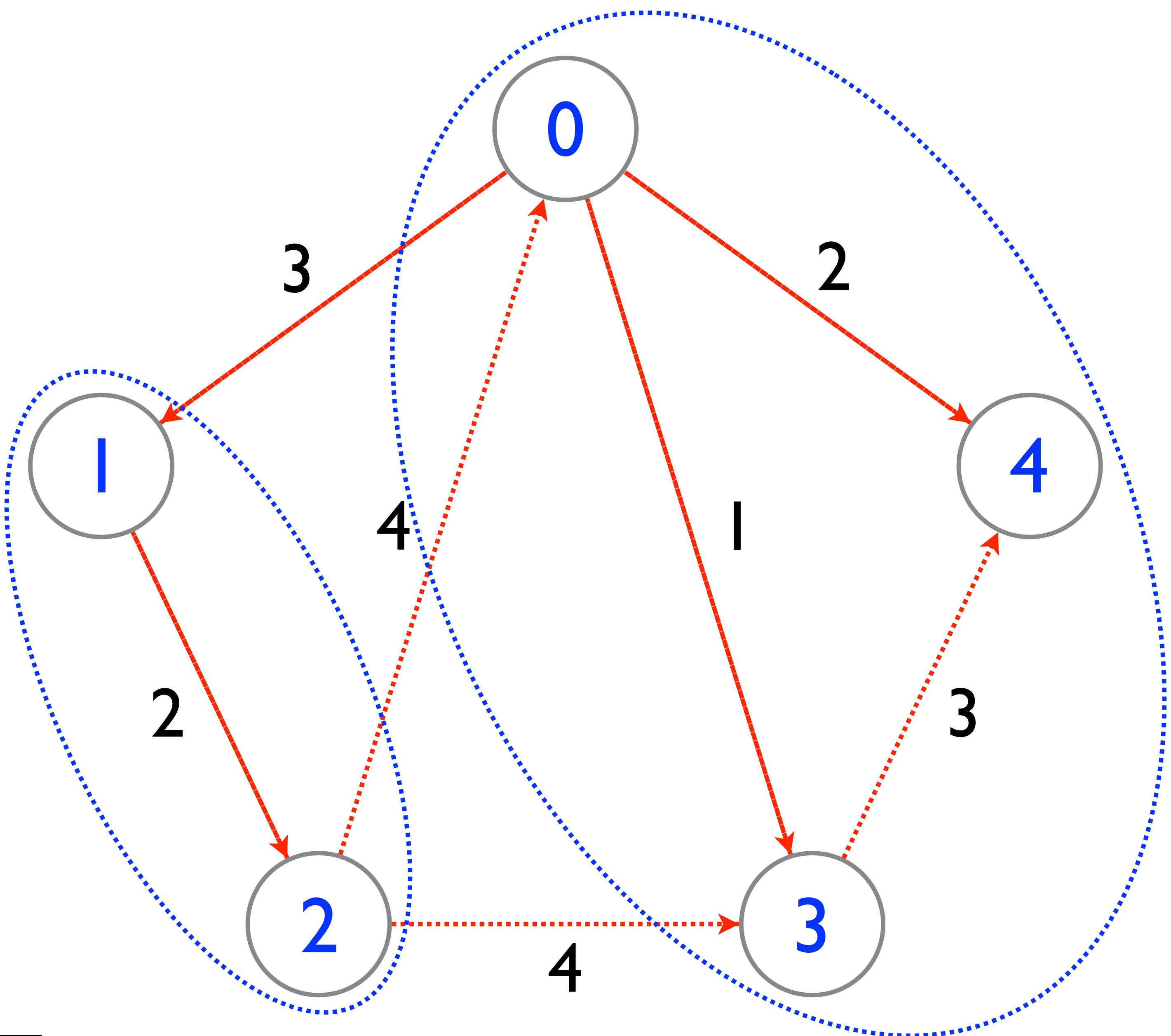
# Types of Graphs

G = ( V , E )




Undirected

Weighted

Directed

Acyclic

Tree

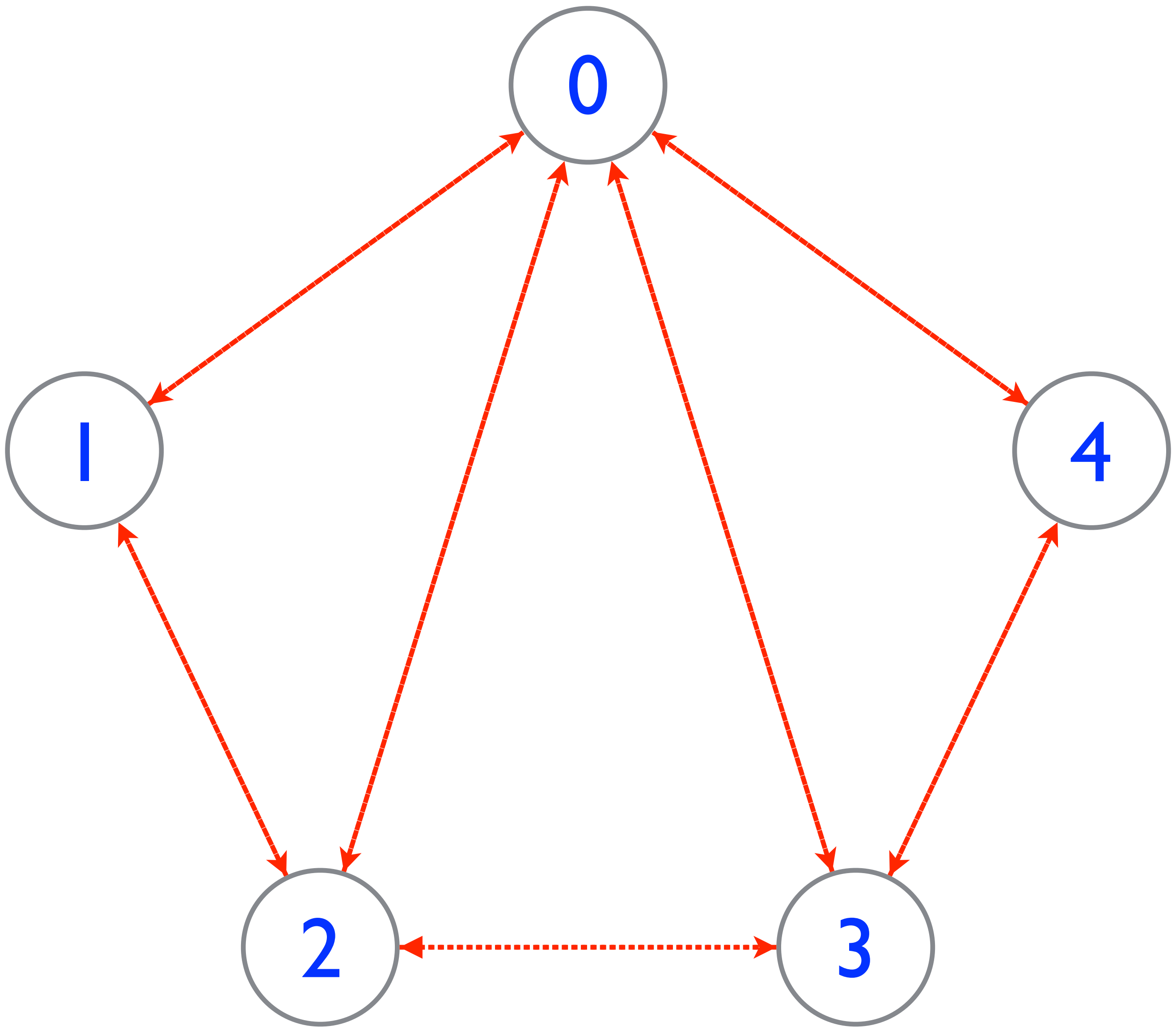
Forest

Every node except for the root must have exactly one incoming edge.

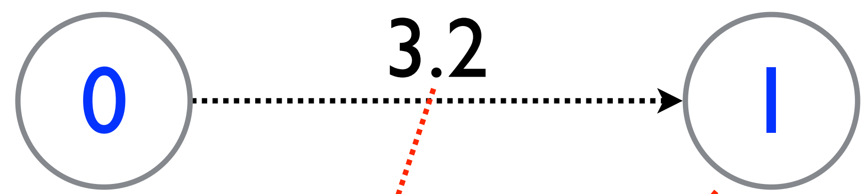
Every node must be reachable from the root.

# Undirected vs. Directed Graph

Can we represent undirected graphs using directed graphs?



# Edge




```
public class Edge implements Comparable<Edge>
{
  private int    source;
  private int    target;
  private double weight;

  public Edge(int source, int target, double weight)
  {
    setSource(source);
    setTarget(target);
    setWeight(weight);
  }
}
```

by weight

```
public int compareTo(Edge edge)
{
  double diff = weight - edge.weight;
  if      (diff > 0)  return  1;
  else if (diff < 0)  return -1;
  else                return  0;
}
```



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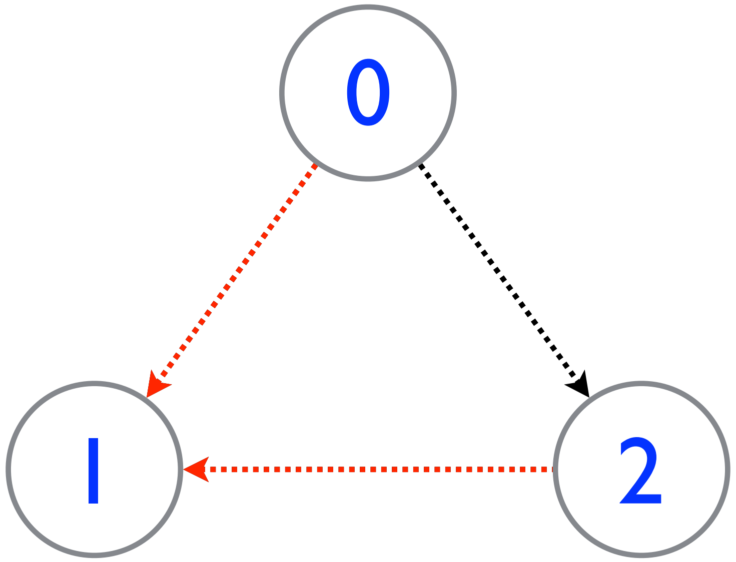
# Graph

```
incoming_edges[0] = {};
incoming_edges[1] = {1 <- 0, 1 <- 2};
incoming_edges[2] = {2 <- 0};
```



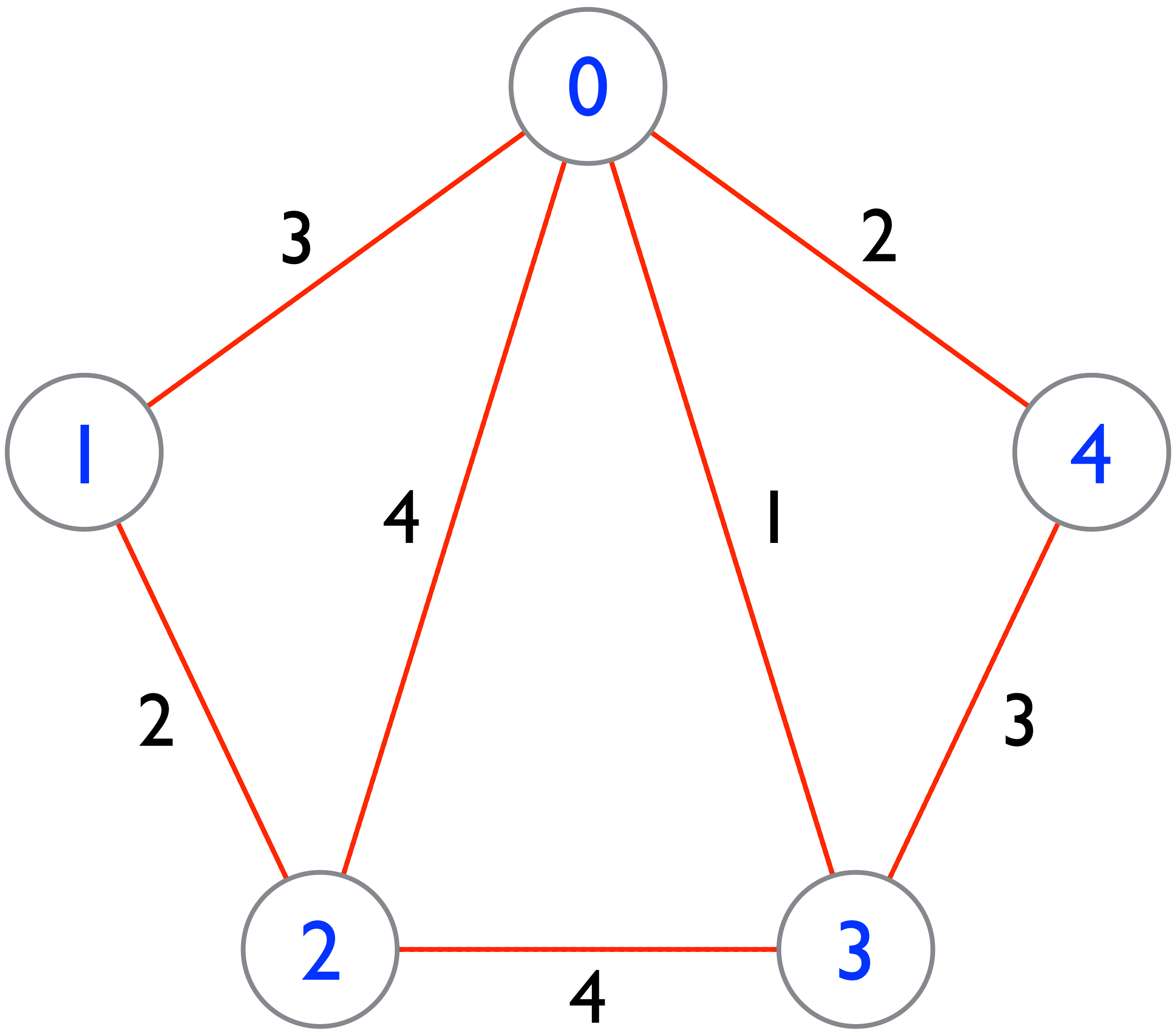

```
getIncomingEdges(1);
```

```
public class Graph {
  private List<Edge>[] incoming_edges;

  public Graph(int size) {
    incoming_edges = (List<Edge>[])Stream.generate(ArrayList<Edge>::new)
                     .limit(size).toArray(List<?>[]::new);
  }

  public List<Edge> getIncomingEdges(int target) {
    return incoming_edges[target];
  }
}
```

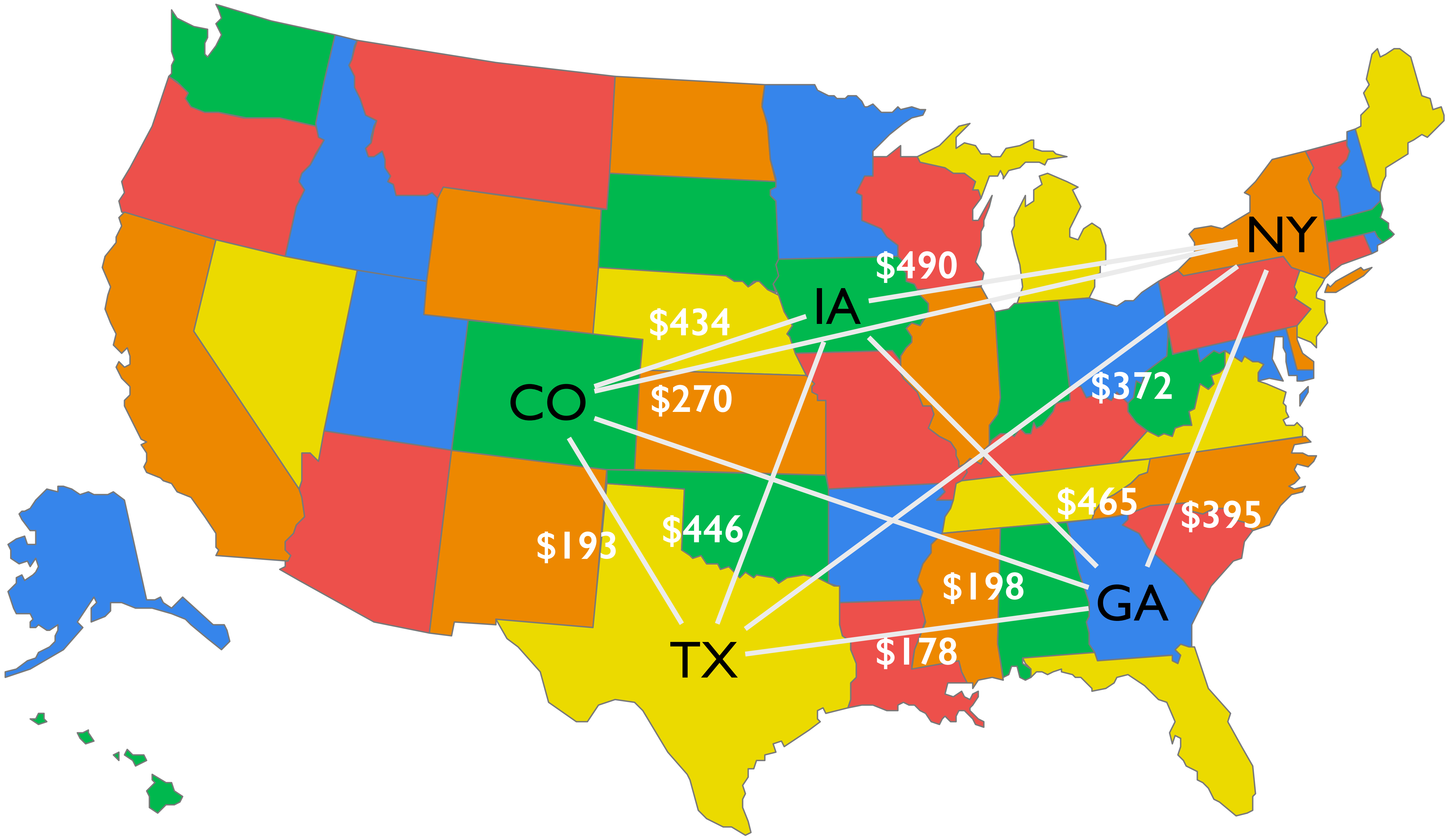
# Graph


```
public void setDirectedEdge(int source, int target, double weight)
{
  List<Edge> edges = getIncomingEdges(target);
  edges.add(new Edge(source, target, weight));
}

public void setUndirectedEdge(int source, int target, double weight)
{
  setDirectedEdge(source, target, weight);
  setDirectedEdge(target, source, weight);
}
```

# Minimum Spanning Tree



# Minimum Spanning Tree



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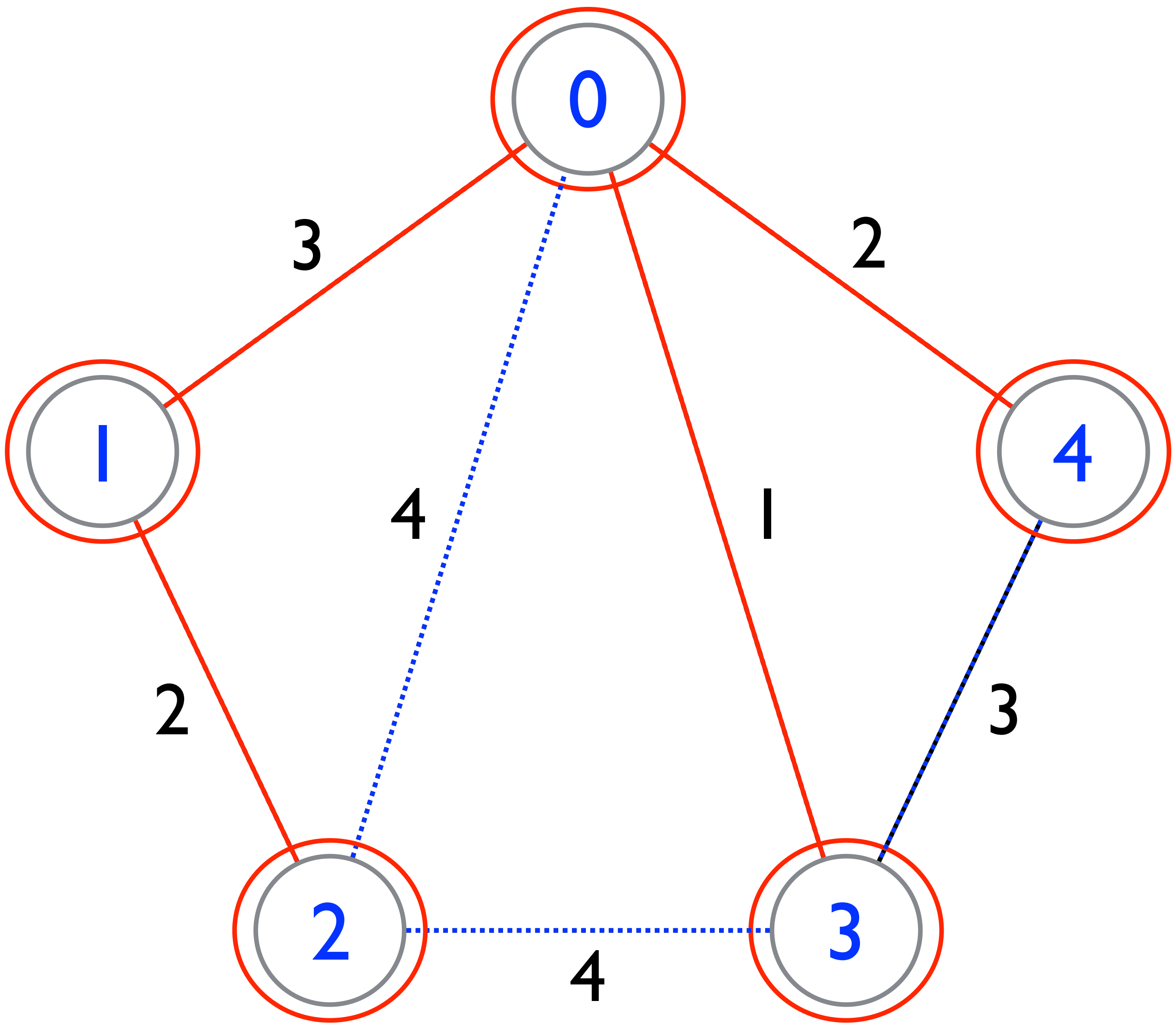
# Spanning Tree

```java
public class SpanningTree implements Comparable<SpanningTree>
{
  private List<Edge> edges;
  private double total_weight;

  public SpanningTree()
  {
    edges = new ArrayList<>();
  }

  public void addEdge(Edge edge)
  {
    edges.add(edge);
    total_weight += edge.getWeight();
  }
}
```

```java
public int size()
{
  return edges.size();
}

public double getTotalWeight()
{
  return total_weight;
}

public int compareTo(SpanningTree tree)
{
  double diff = total_weight
              - tree.total_weight;
  if      (diff > 0) return 1;
  else if (diff < 0) return -1;
  else               return 0;
}
```

# Prim's Algorithm




```
0 <- 3
0 <- 4
0 <- 1
1 <- 2
```

A spanning tree is found!

# Prim's Algorithm

```
public SpanningTree getMinimumSpanningTree(Graph graph)
{
  PriorityQueue<Edge> queue = new PriorityQueue<>();
  SpanningTree tree = new SpanningTree();
  Set<Integer> set = new HashSet<>();
  Edge edge;

  add(queue, set, graph, 0);

  while (!queue.isEmpty())
  {
    edge = queue.poll();

    if (!set.contains(edge.getSource()))
    {
      tree.addEdge(edge);
      if (tree.size()+1 == graph.size()) break;
      add(queue, set, graph, edge.getSource());
    }
  }

  return tree;
}
```

Complexity?

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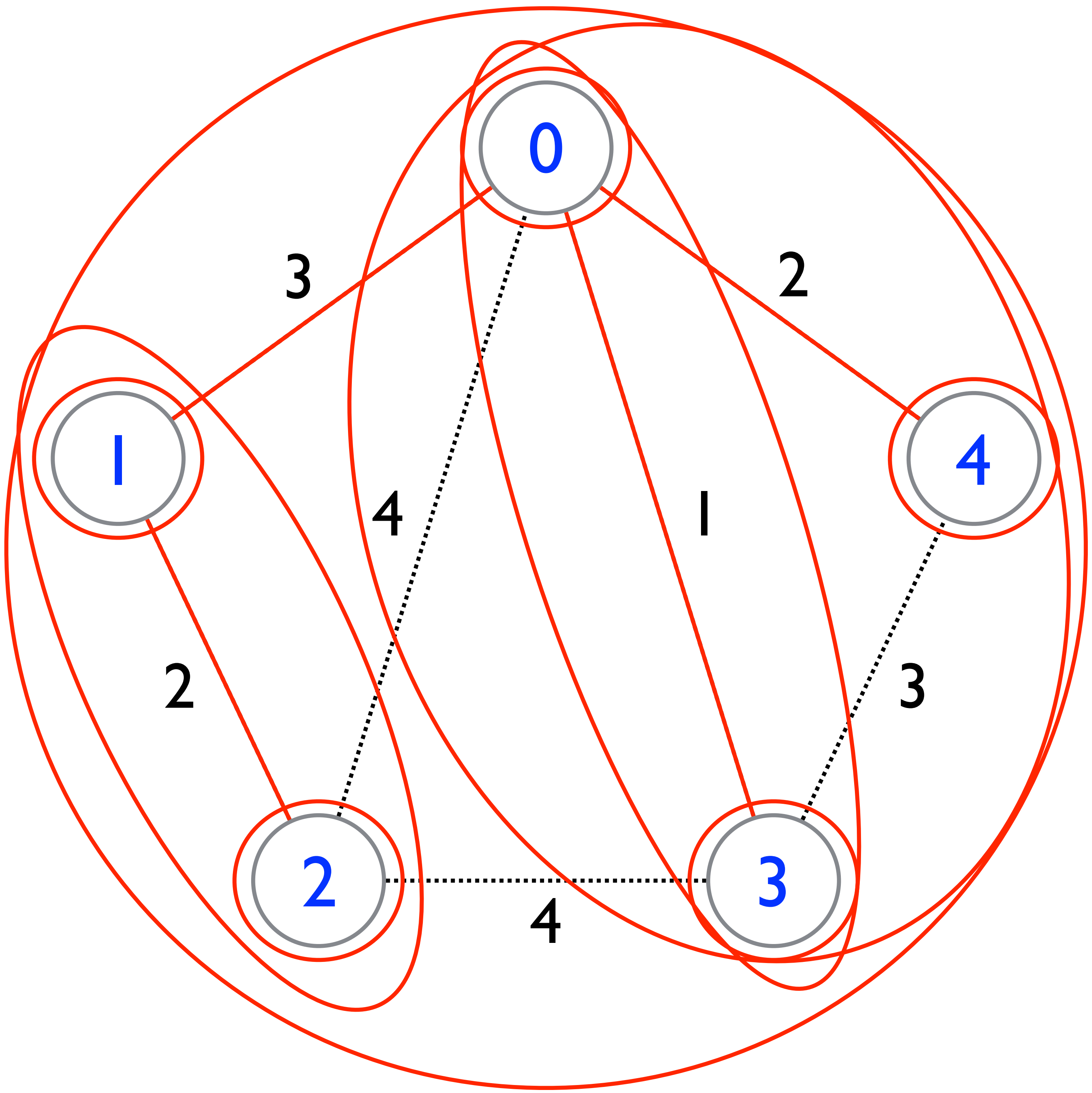
# Kruskal's Algorithm




```
0 <- 3
1 <- 2
0 <- 4
0 <- 1
```

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# Kruskal's Algorithm

```
public SpanningTree getMinimumSpanningTree(Graph graph) {
  PriorityQueue<Edge> queue = new PriorityQueue<>(graph.getAllEdges());
  DisjointSet forest = new DisjointSet(graph.size());
  SpanningTree tree = new SpanningTree();
  Edge edge;

  while (!queue.isEmpty()) {
    edge = queue.poll();

    if (!forest.inSameSet(edge.getTarget(), edge.getSource())) {
      tree.addEdge(edge);
      if (tree.size()+1 == graph.size()) break;
      forest.union(edge.getTarget(), edge.getSource());
    }
  }

  return tree;
}
```

# Proof by Induction

$$\sum_{n=0}^{k} = \frac{k(k+1)}{2}$$

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# Proof by Induction

**Base**

$$\sum_{n=0}^{0} = \frac{0 \cdot 1}{2} = 0$$

**Induction**

Assume $$\sum_{n=0}^{k} = \frac{k(k+1)}{2}$$ Then,

$$\sum_{n=0}^{k} + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{K^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2}$$

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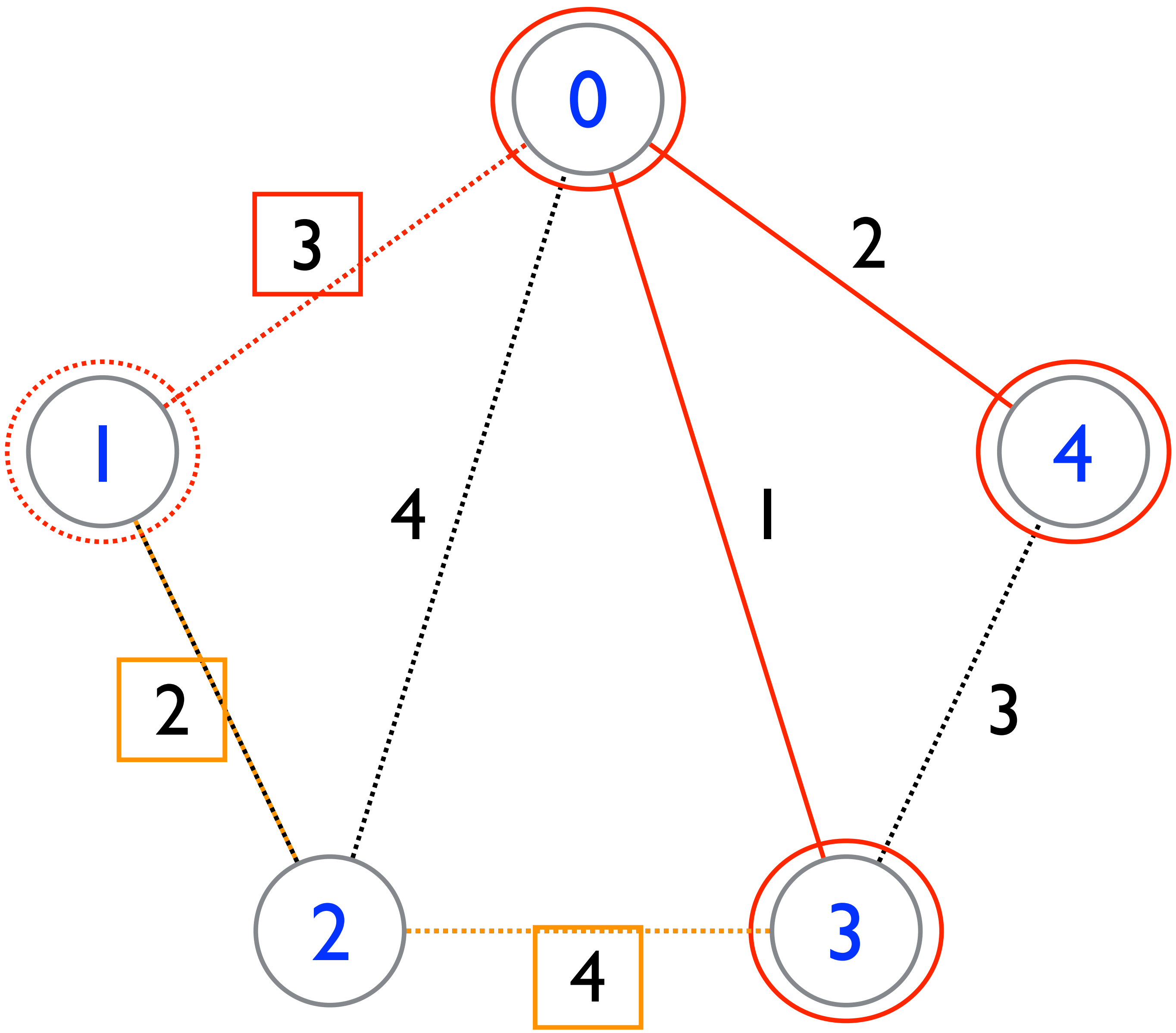
# Proof by Contradiction

$$n^2 = n \cdot n = 2k \cdot 2k = \boxed{2(2 \cdot 2k^2)}$$

Contradiction

# Correctness of Prim's Algorithm

# Correctness of Prim's Algorithm




Prim's algorithm finds $e$.

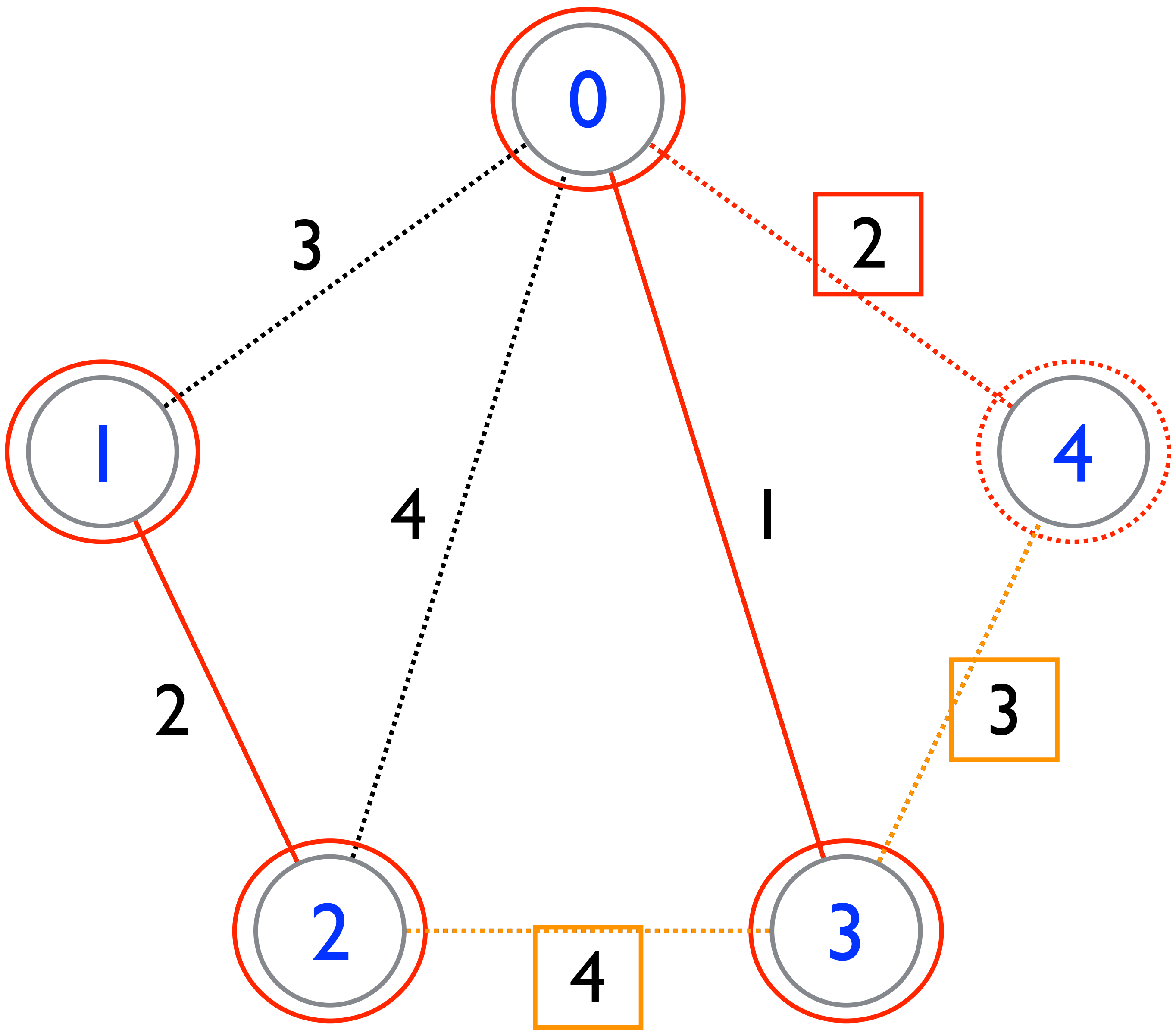
Suppose that there exists another path $p$ from $T$ to $1$ whose total weight < $e$.

Then, all edges in $p$ must have a weight < $e$.

Proof by contradiction!

# Correctness of Kruskal's Algorithm

# Correctness of Kruskal's Algorithm




Kruskal's algorithm finds $e$.

Suppose that there exists another path $p$ from $T_s$ to $T_t$ whose total weight < $e$.

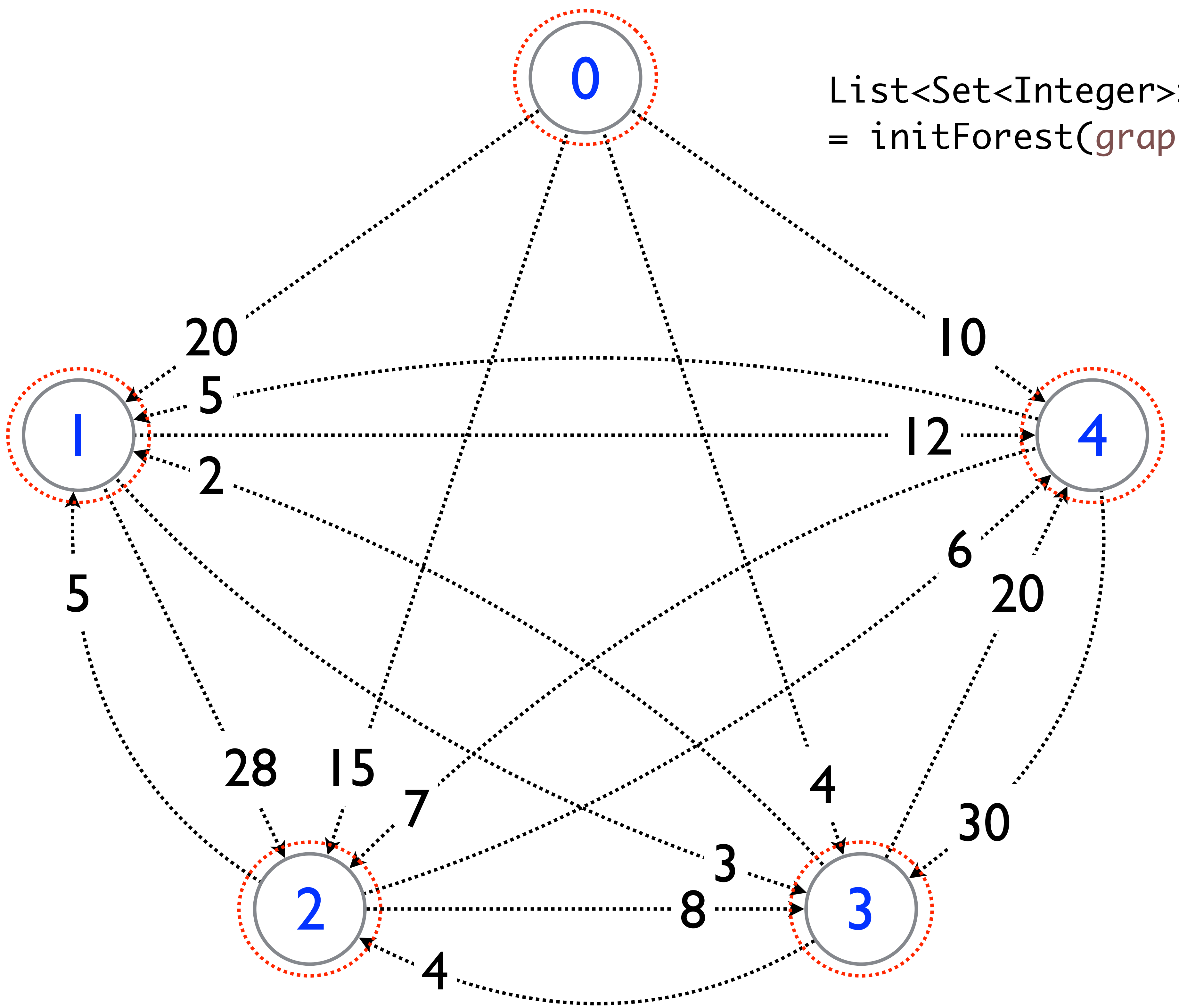
Then, all edges in $p$ must have a weight < $e$.

Proof by contradiction!

# Chu–Liu-Edmonds' Algorithm

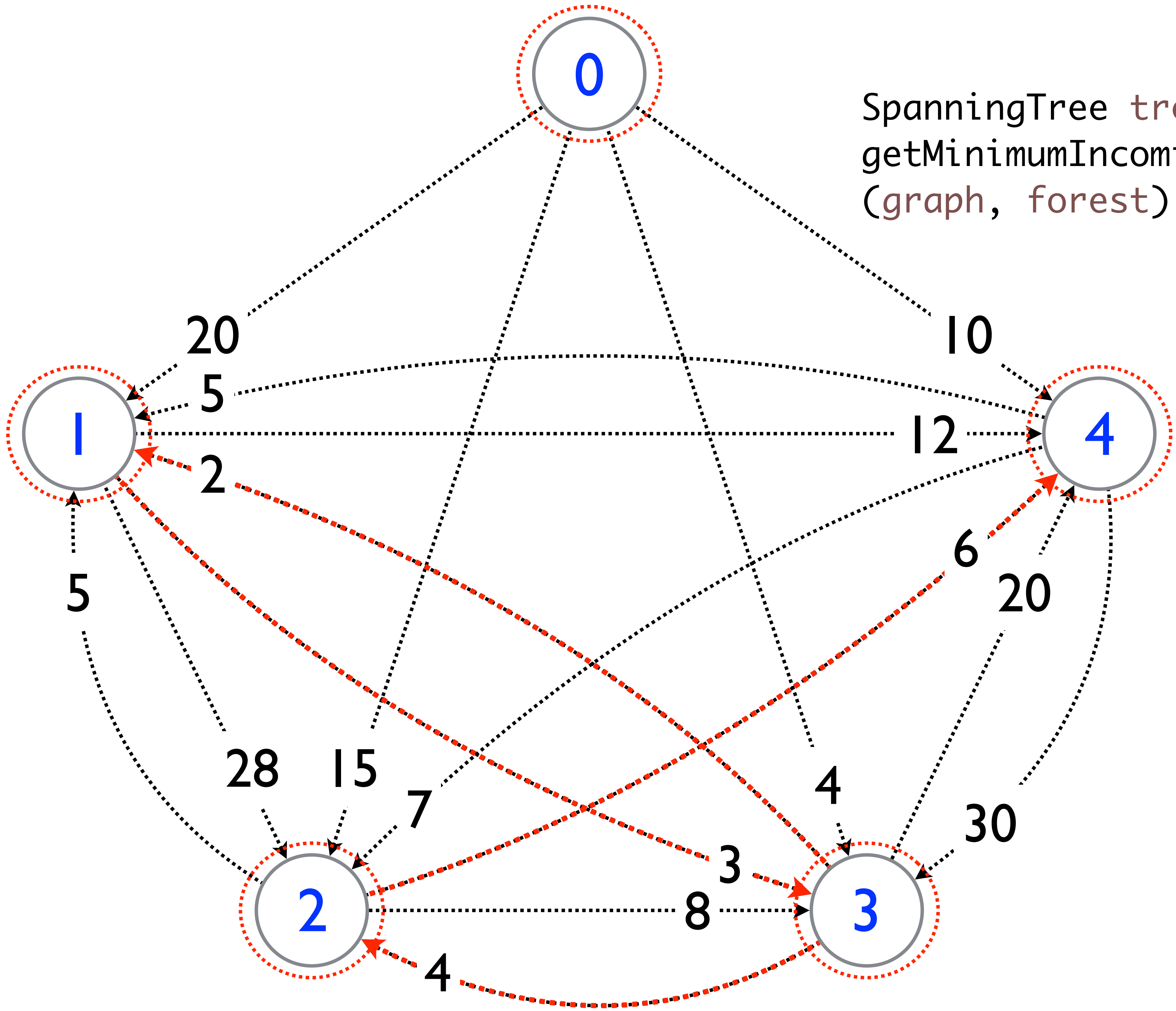
- Chu–Liu-Edmonds' algorithm
  - Finding minimum spanning trees in directed graphs.
- Algorithm
  1. Initially, every vertex is considered a tree.
  2. For each tree, keep 1 incoming edge with the minimum weight.
  3. If there is no cycle, go to #5.
  4. If there is a cycle, merge trees with the cycle into one and update scores for all incoming edges to this tree, and goto #2.
     - For each vertex in the tree, add the weight of its outgoing edge chain to its incoming edges not in the tree.
  5. Break all cycles by removing edges that cause multiple parents.

# Chu–Liu-Edmonds' Algorithm



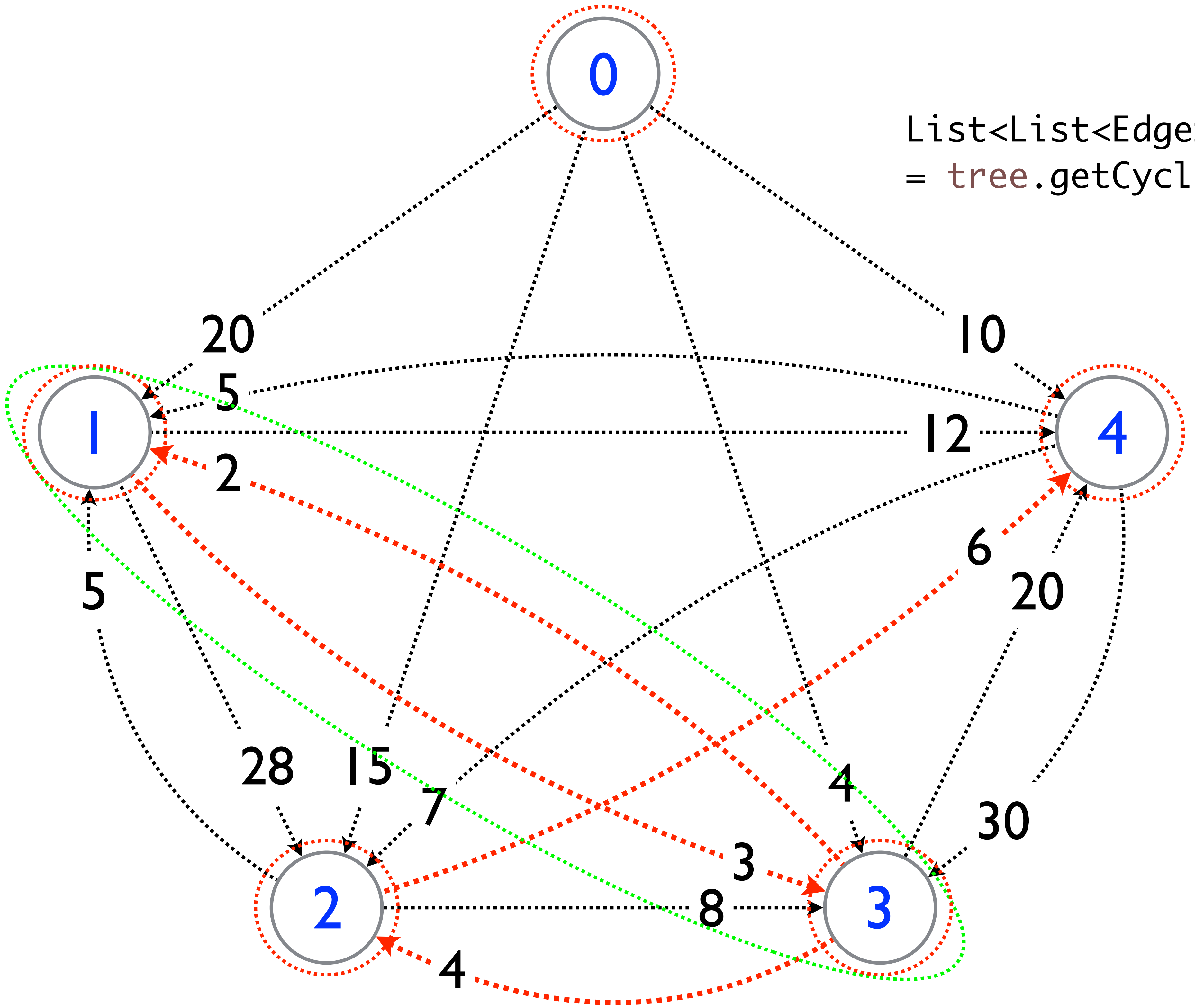

```
List<Set<Integer>> forest
= initForest(graph.size());
```

# Chu–Liu-Edmonds' Algorithm




```
SpanningTree tree =
getMinimumIncomingEdges
(graph, forest);
```

# Chu–Liu-Edmonds' Algorithm




```
List<List<Edge>> cycles
= tree.getCycles();
```

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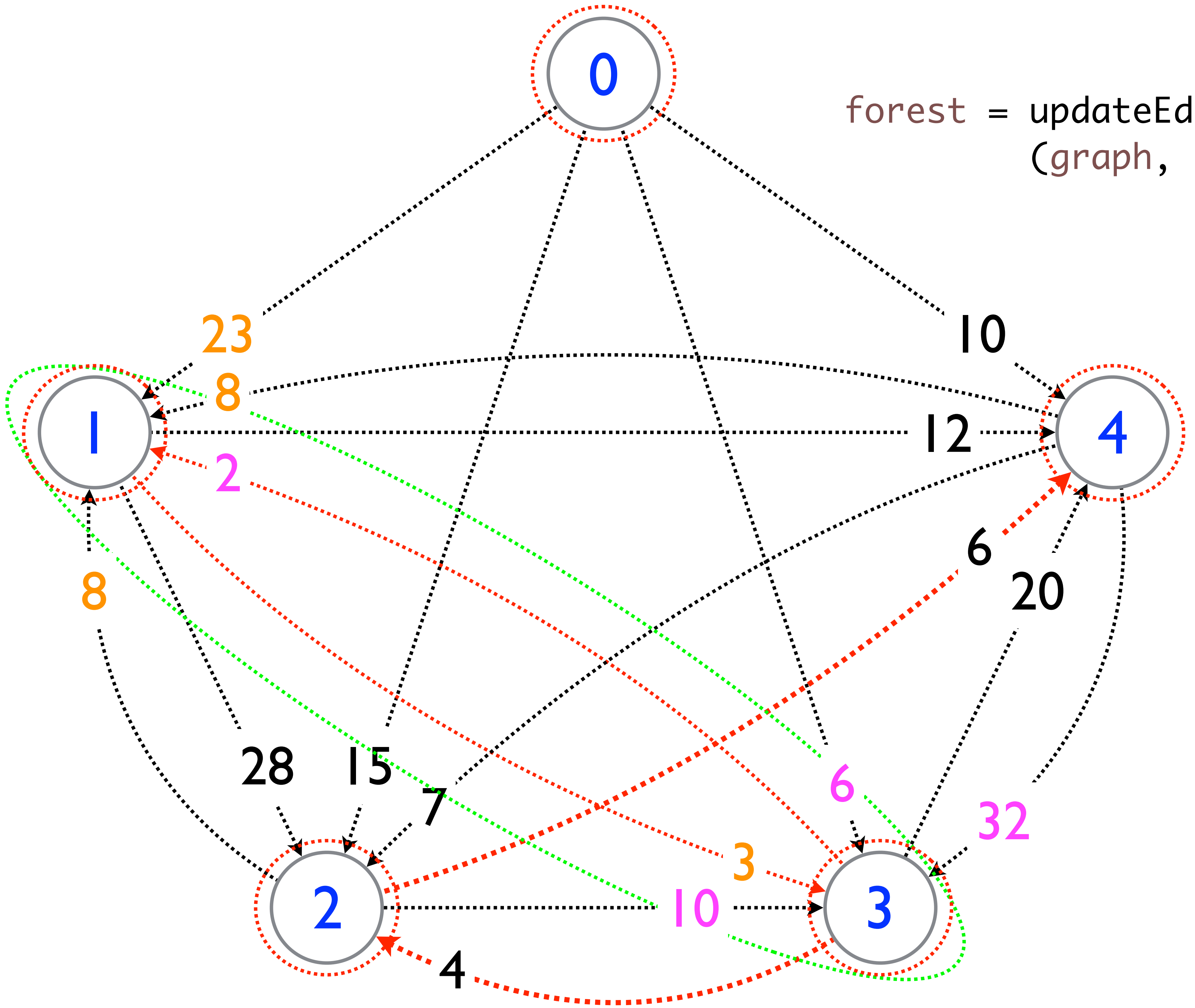
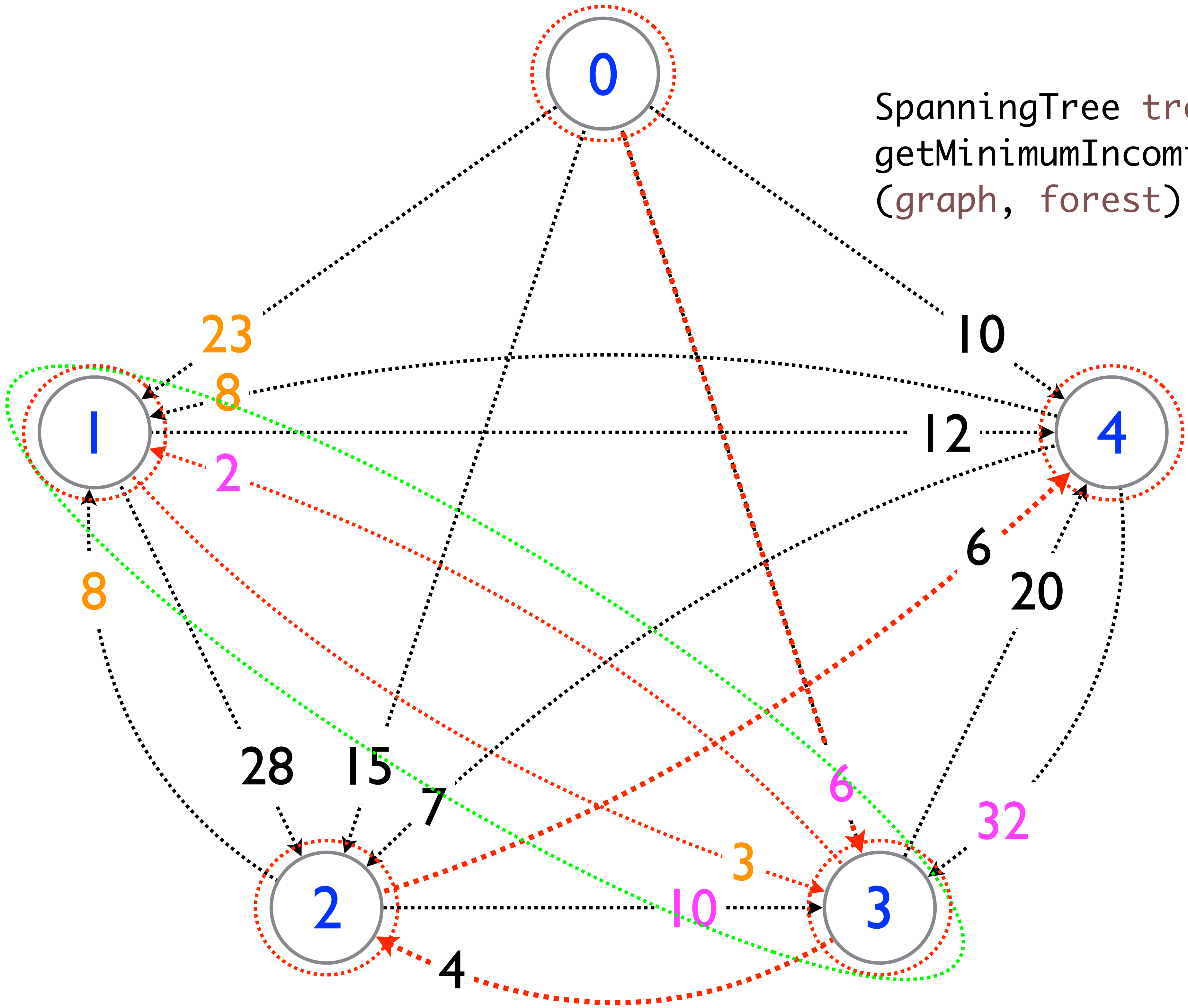
# Chu–Liu-Edmonds' Algorithm

```
forest = updateEdgeWeights
         (graph, cycles);
```



# Chu–Liu-Edmonds' Algorithm

```
SpanningTree tree =
getMinimumIncomingEdges
(graph, forest);
```

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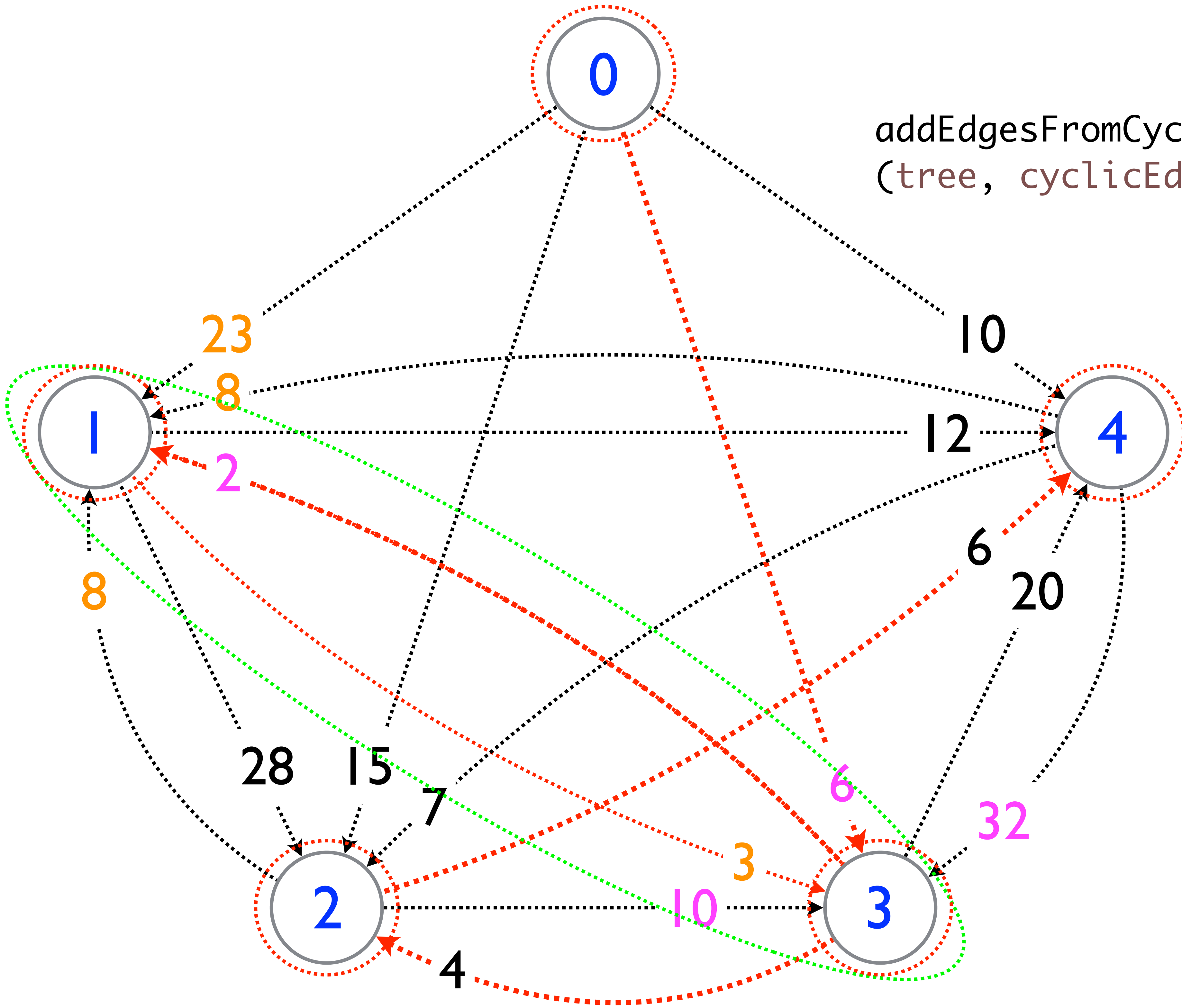
# Chu–Liu-Edmonds' Algorithm

```
addEdgesFromCycles
(tree, cyclicEdges);
```



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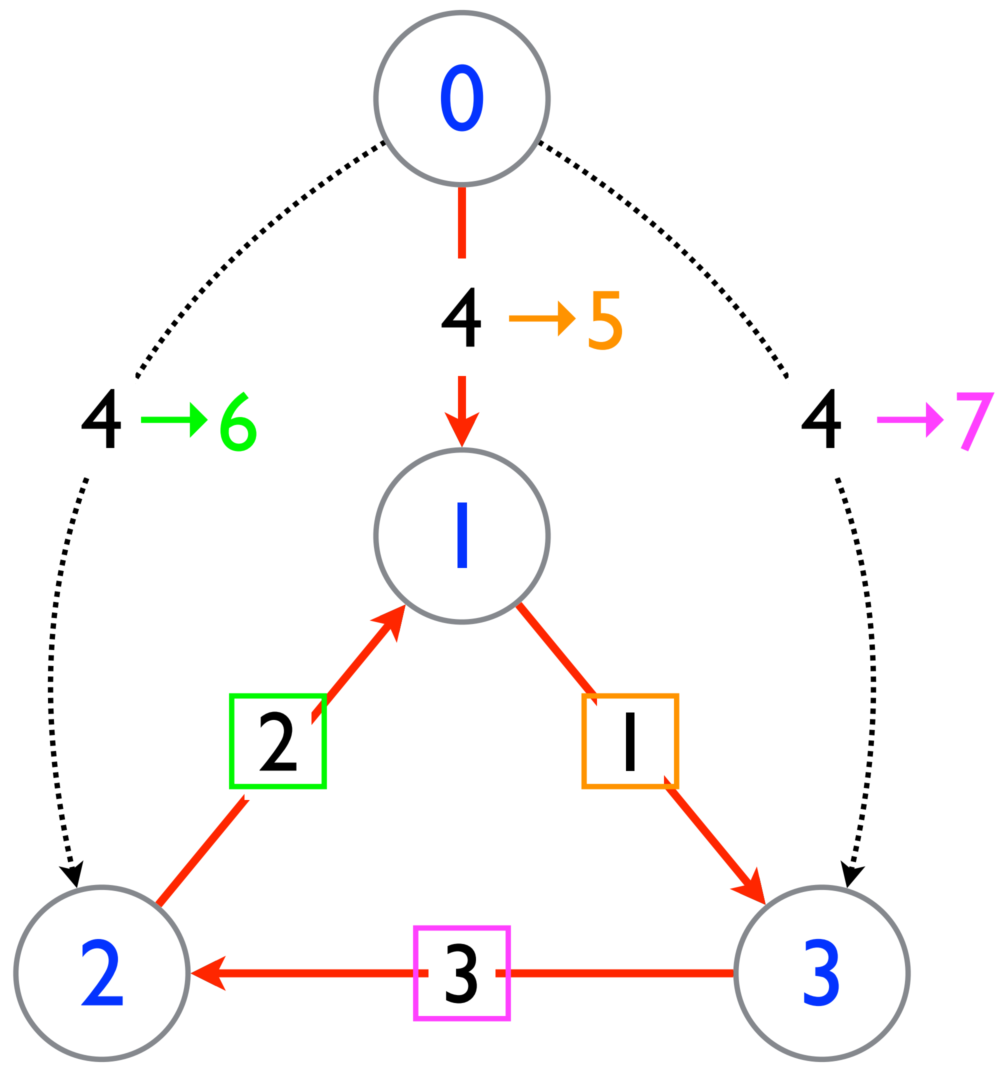
# Chu–Liu-Edmonds' Algorithm




Minimum Spanning Tree?

# Chu–Liu-Edmonds' Algorithm