Network Flow

Data Structures and Algorithms

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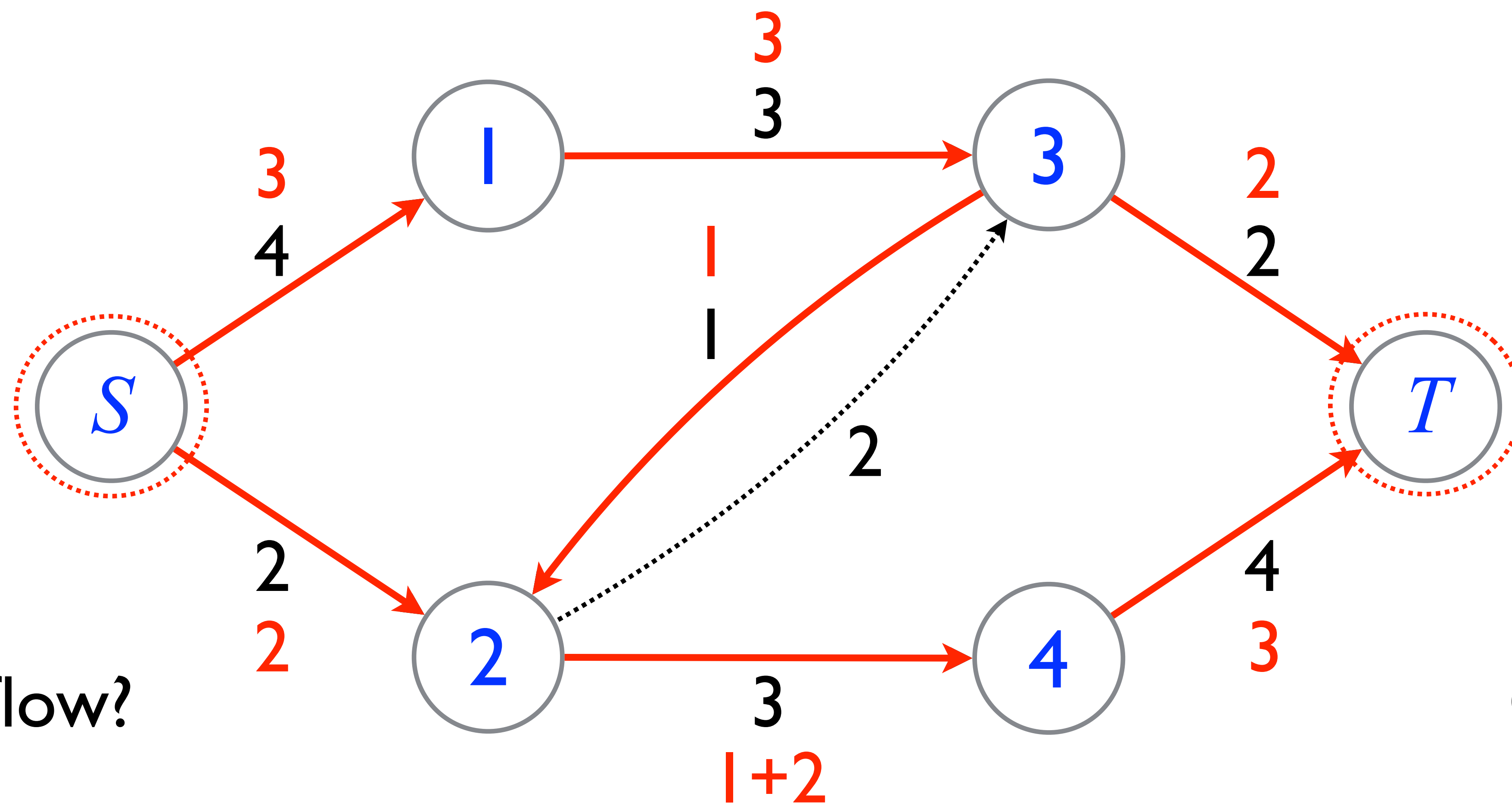
Network Flow



Maximum flow?

Optimum?

Each edge e is associated with a capacity c.

Push as much flow f as possible from S to T.

$$f(e) \leq c(e)$$

$$\Sigma_u f(u, v) = \Sigma_w f(v, w), \text{ where } v \notin \{S, T\}$$

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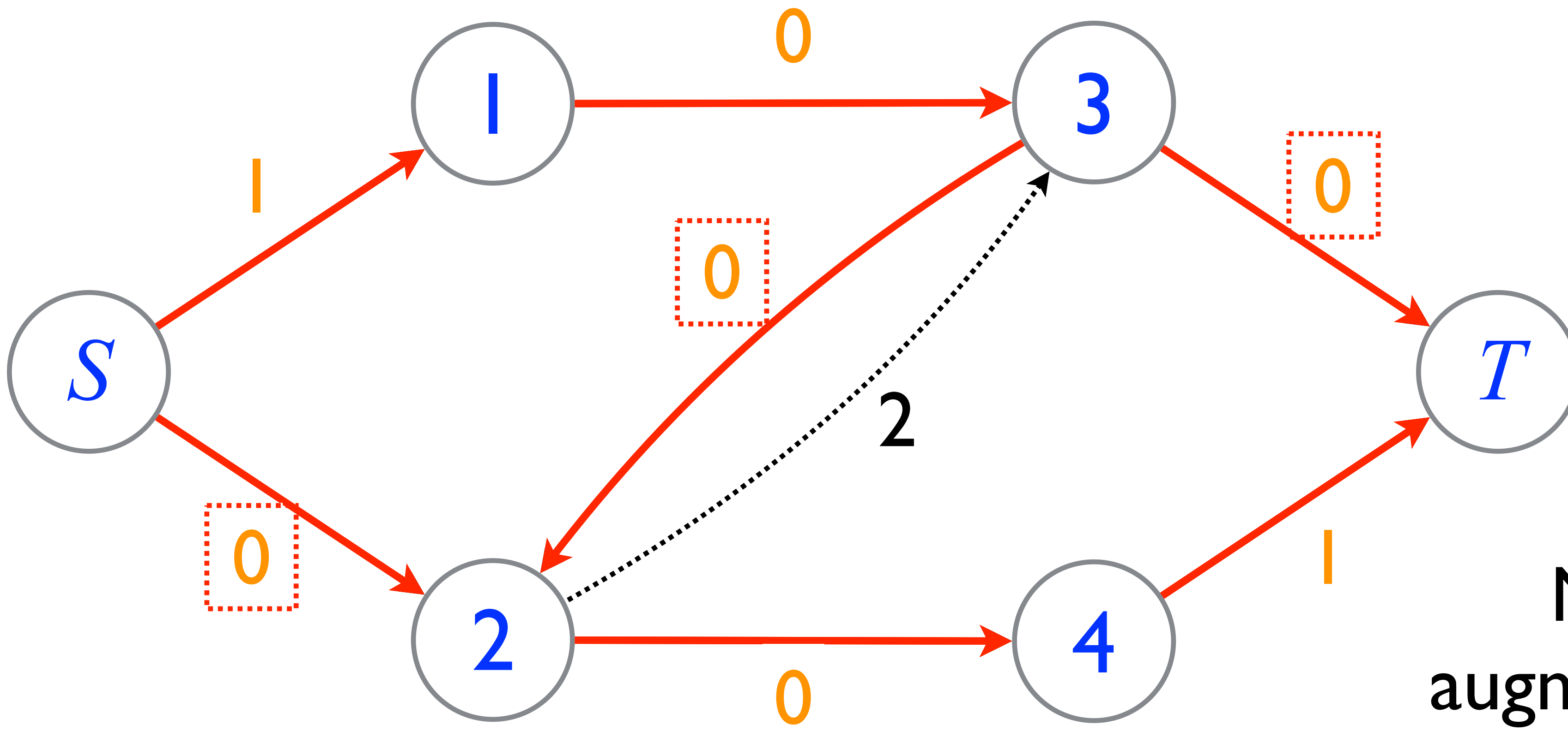
Ford-Fulkerson Algorithm



2

1

2

No more augmenting path!

Find a path p from S to T, where $\forall e \in p.\ r(e) = c(e) - f(e) > 0.$

$$\forall e \in p.\ c(e) = c(e) - min(f(p)).$$

$$\text{MaxFlow} = \text{MaxFlow} + min(f(p)).$$

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MaxFlow Class

```
public class MaxFlow
{
  private Map<Edge,Double> m_flows;
  private double d_maxFlow;

  public MaxFlow(Graph graph)
  {
    init(graph);
  }

  public void init(Graph graph)
  {
    m_flows = new HashMap<>();
    d_maxFlow = 0;

    for (Edge edge : graph.getAllEdges())
      m_flows.put(edge, 0d);
  }
}
```

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MaxFlow Class

```
public void updateResidual(List<Edge> path, double flow)
{
  for (Edge edge : path) updateResidual(edge, flow);
  d_maxFlow += flow;
}

public void updateResidual(Edge edge, double flow)
{
  Double prev = m_flows.get(edge);
  if (prev == null) prev = 0d;
  m_flows.put(edge, prev + flow);
}

public double getResidual(Edge edge)
{
  return edge.getWeight() - m_flows.get(edge);
}

public double getMaxFlow()
{
  return d_maxFlow;
}
```

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FordFulkerson Class

```
private Subgraph getAugmentingPath(Graph graph, MaxFlow mf,
                                   Subgraph sub, int source, int target)
{
  if (source == target) return sub;
  Subgraph tmp;

  for (Edge edge : graph.getIncomingEdges(target))
  {
    if (sub.contains(edge.getSource())) continue;  // cycle
    if (mf.getResidual(edge) <= 0)      continue;  // no residual
    tmp = new Subgraph(sub);
    tmp.addEdge(edge);

    tmp = getAugmentingPath(graph, mf, tmp, source, edge.getSource());
    if (tmp != null) return tmp;
  }

  return null;
}
```

Complexity?

FordFulkerson Class

```
public MaxFlow getMaximumFlow(Graph graph, int source, int target)
{
  MaxFlow mf = new MaxFlow(graph);
  Subgraph sub = new Subgraph();
  double min;

  while ((sub = getAugmentingPath(graph, mf, sub, source, target)) != null)
  {
    min = getMin(mf, sub.getEdges());
    mf.updateResidual(sub.getEdges(), min);
  }

  return mf;
}
```

```
private double getMin(MaxFlow mf, List<Edge> path)
{
  double min = mf.getResidual(path.get(0));

  for (int i=1; i<path.size(); i++)
    min = Math.min(min, mf.getResidual(path.get(i)));

  return min;
}
```

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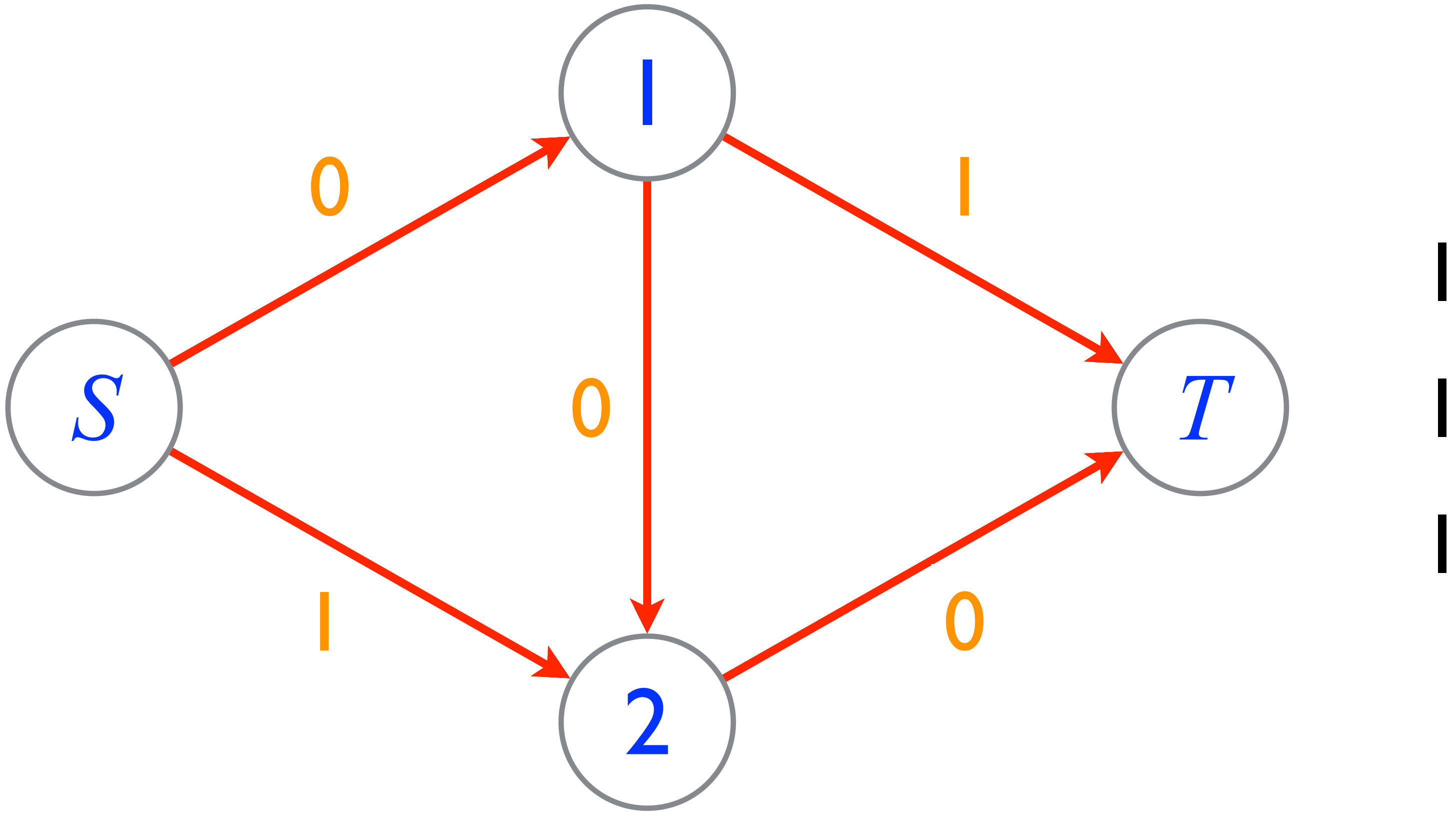
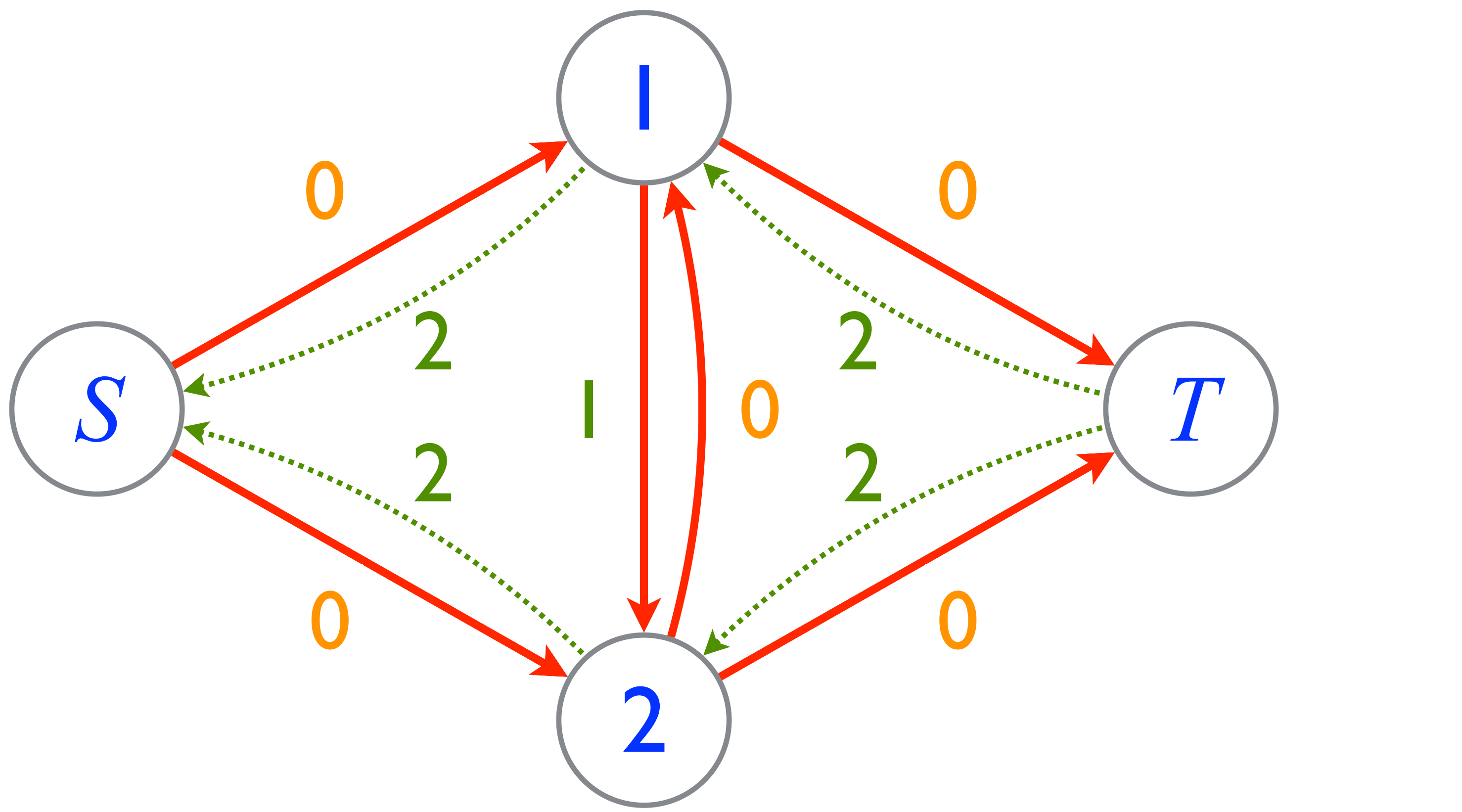
Ford-Fulkerson: Backward Pushing



Ford-Fulkerson: Backward Pushing



```java
protected void updateBackward(Graph graph, Subgraph sub, MaxFlow mf, double min)
{
  boolean found;

  for (Edge edge : sub.getEdges())
  {
    found = false;

    for (Edge rEdge : graph.getIncomingEdges(edge.getSource()))
    {
      if (rEdge.getSource() == edge.getTarget())
      {
        mf.updateResidual(rEdge, -min);
        found = true; break;
      }
    }

    if (!found)
    {
      Edge rEdge = graph.setDirectedEdge
                   (edge.getTarget(), edge.getSource(), edge.getWeight());
      mf.updateResidual(rEdge, -min);
    }
  }
}
```

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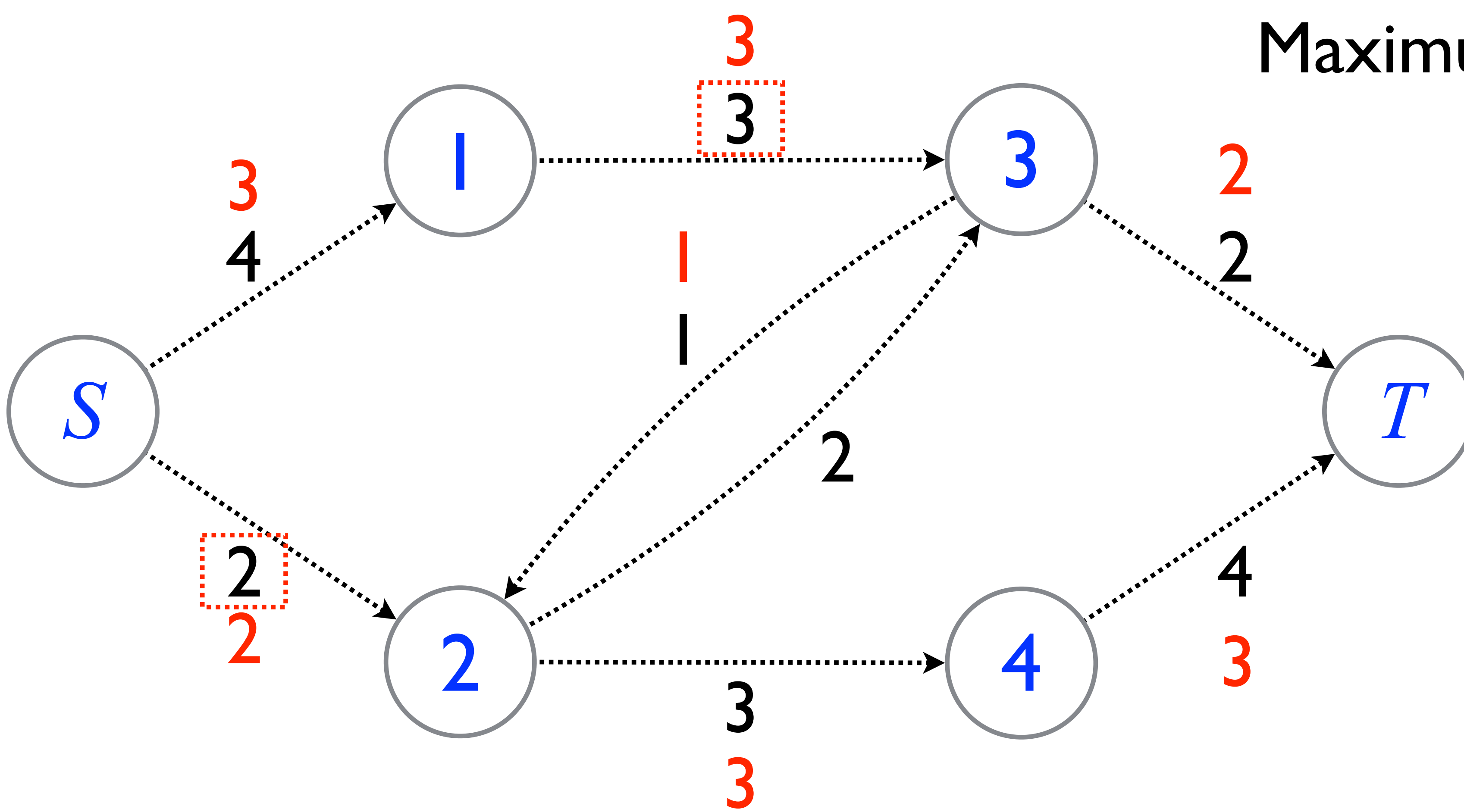
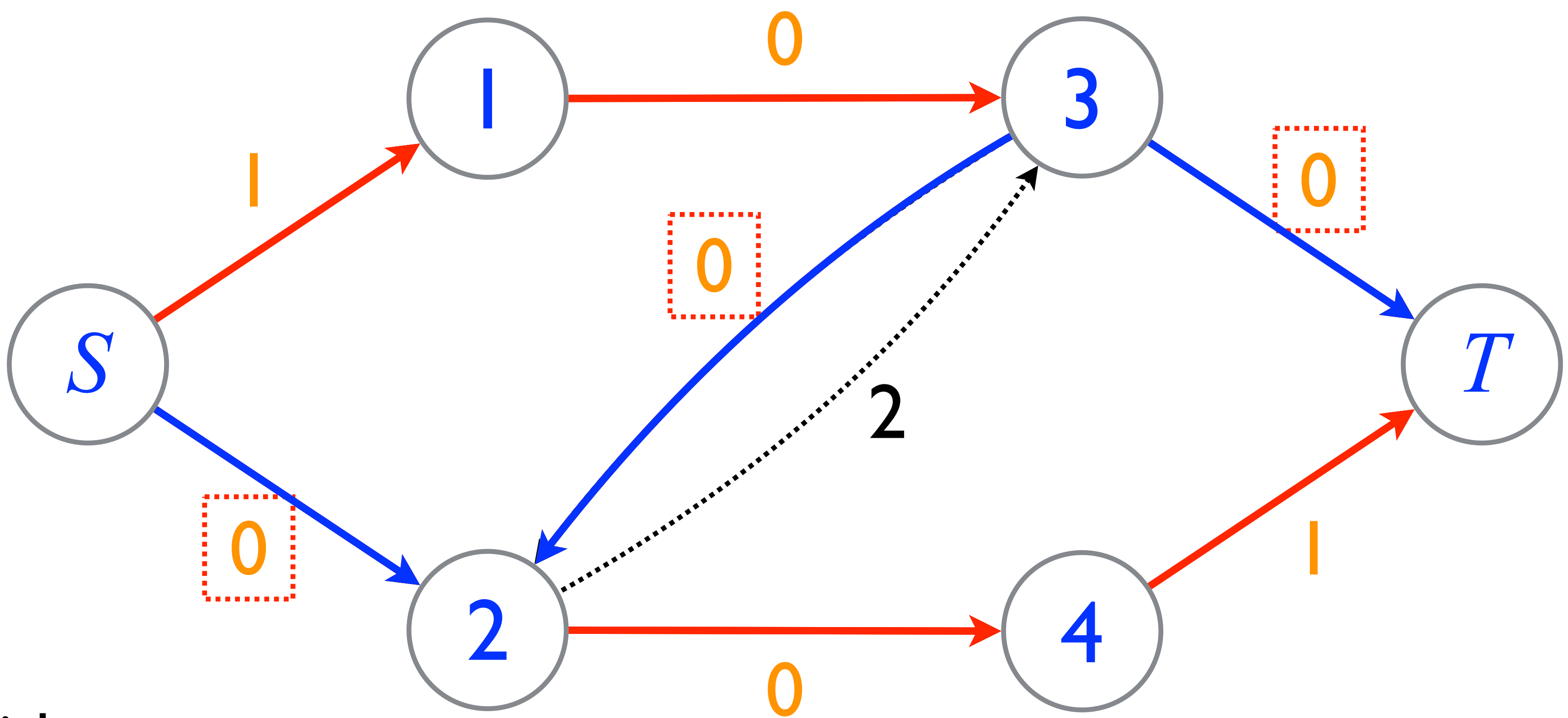
Max-Flow Min-Cut Theorem

- S-T cut
  - A set of edges whose removal disconnects S to T.
  - The capacity of a cut = $\Sigma\, c(e)$, $\forall e$ in the cut.

Minimum cut
vs.
Maximum flow



Finding Min-Cut



- Algorithm
  1. Find a path p from S to T, remove any edge e in p, and put e to a set C.
  2. Repeat #1 until no path is found, and return C.
- How can we find a min-cut?
  - Instead of removing any edge in #1, remove an edge with the minimum original (not residual) weight.

Min-Cut vs. Max-Flow

- Lemma 1
  - There is no extra edge in the min-cut.
  - Prove?
- Lemma 2
  - All edges in the min-cut must be parts of augmenting paths.
  - Prove?
- Lemma 3
  - Each edge in the min-cut is the minimum weighted edge in each augmenting path.
  - Prove?

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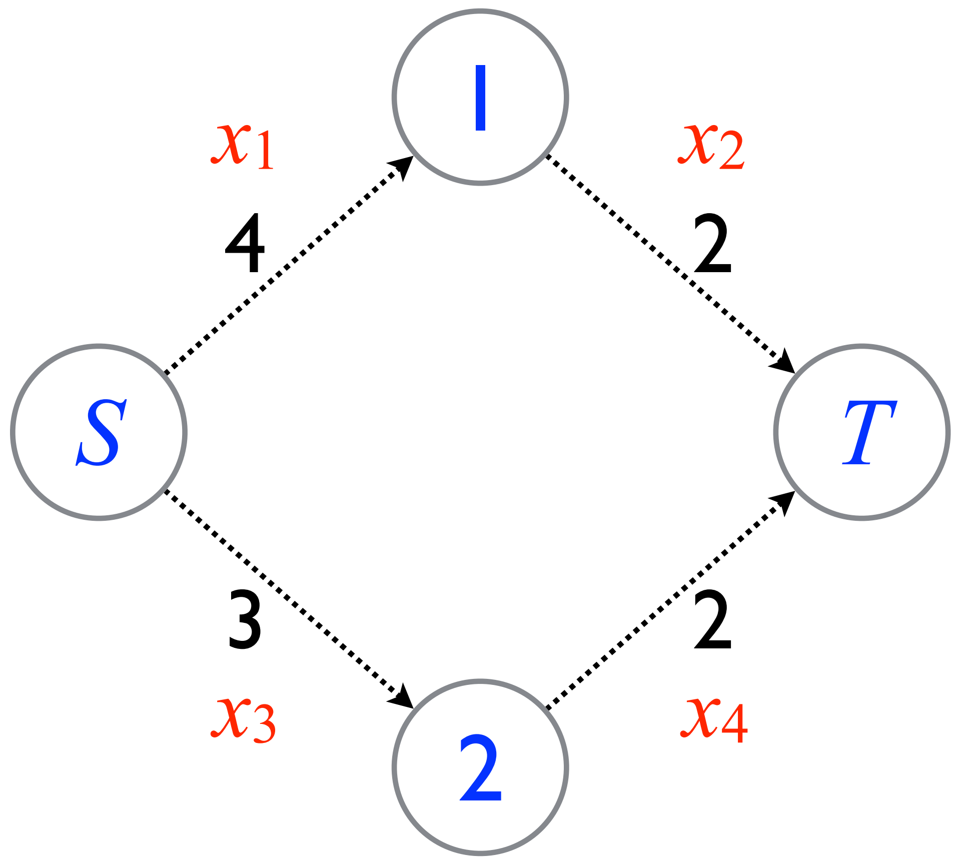
Network Flow Using Simplex



Max-Flow

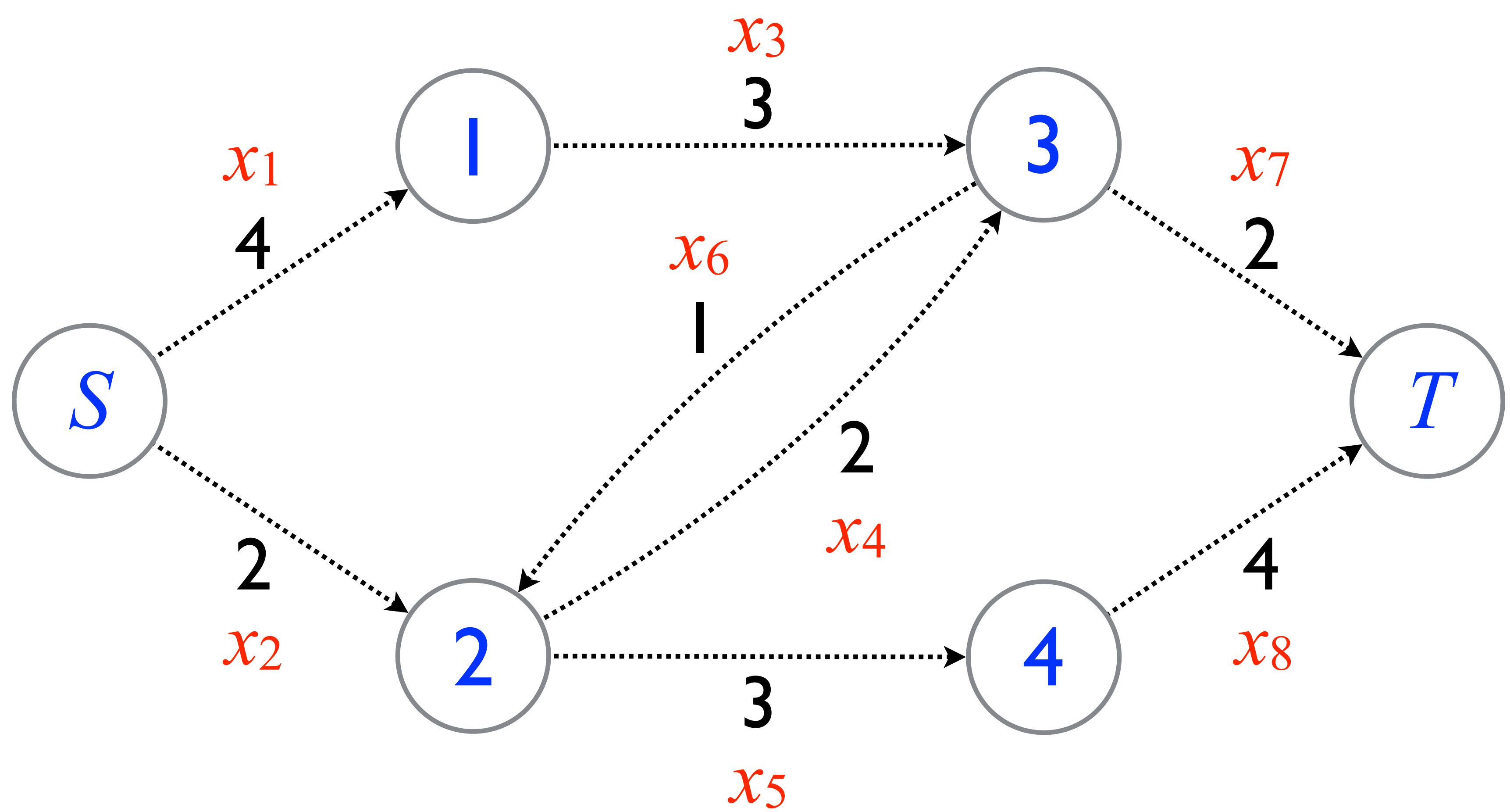
Maximize:

$$x_2 + x_4$$

Subject to:

$$x_1 \leq 4$$
$$x_2 \leq 2$$
$$x_3 \leq 3$$
$$x_4 \leq 2$$
$$x_2 - x_1 \leq 0$$
$$x_4 - x_3 \leq 0$$

Network Flow Using Simplex



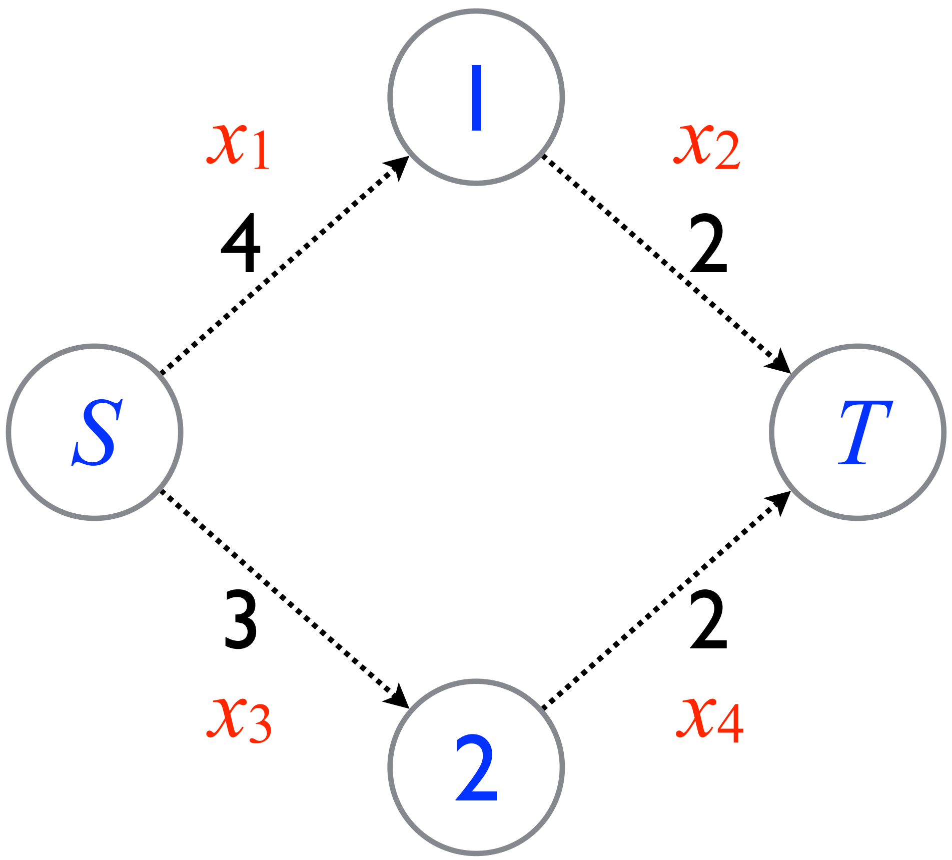
Maximize:

$$x_7 + x_8$$

Subject to:

$$x_1 \le 4 \qquad x_5 \le 3 \qquad x_3 - x_1 \le 0$$
$$x_2 \le 2 \qquad x_6 \le 1 \qquad x_4 + x_5 - x_2 - x_6 \le 0$$
$$x_3 \le 3 \qquad x_7 \le 2 \qquad x_6 + x_7 - x_3 - x_4 \le 0$$
$$x_4 \le 2 \qquad x_8 \le 4 \qquad x_8 - x_5 \le 0$$

Network Flow Using Simplex



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$$x_4 \leq 2$$
$$x_2 - x_1 \leq 0$$
$$x_4 - x_3 \leq 0$$

Min-Cut

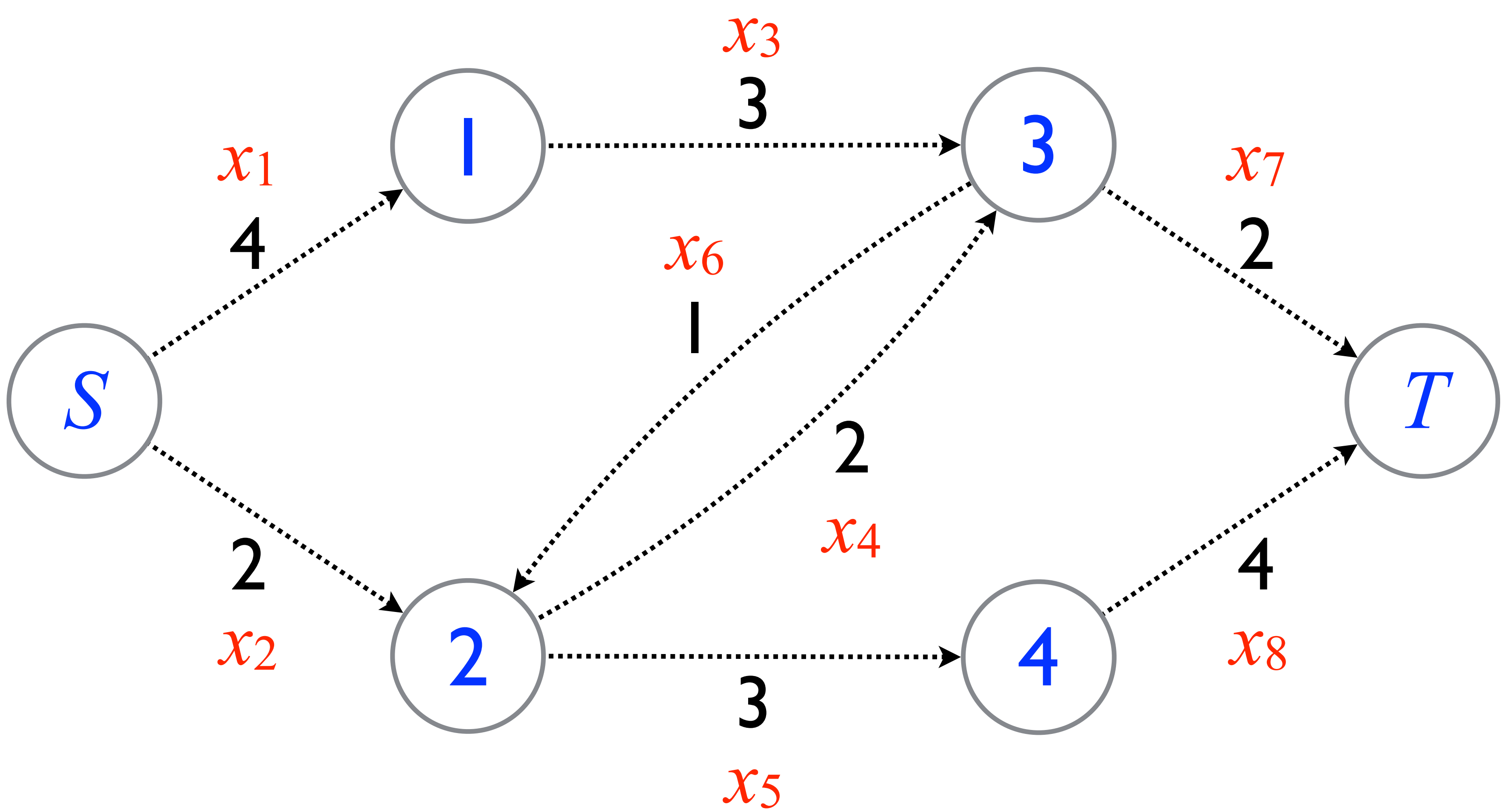
Minimize:

$$4x_1 + 2x_2 + 3x_3 + 2x_4$$

Subject to:

$$x_1 + y_1 \geq 1$$
$$x_3 + y_3 \geq 1$$
$$x_2 - y_1 \geq 0$$
$$x_4 - y_3 \geq 0$$

Network Flow Using Simplex



Minimize:

$$4x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 + x_6 + 2x_7 + 4x_8$$

Subject to:

$$x_1 + y_1 \geq 1$$
$$x_2 + y_2 \geq 1$$
$$x_3 - y_1 + y_3 \geq 0$$
$$x_4 - y_2 + y_3 \geq 0$$
$$x_5 - y_2 + y_4 \geq 0$$
$$x_6 - y_3 + y_2 \geq 0$$
$$x_7 - y_3 \geq 0$$
$$x_8 - y_4 \geq 0$$

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