1 fields and relativity

There are apparently electric fields and magnetic fields. An electric field will produce a force on a charge. Magnetic fields are said to be produced by moving charges, and exert force on other moving charges. This document shows how a magnetic field effect can be equivalent to an electric field effect viewed from another fram of reference using special relativity. To start, the electric force between two charges q_1 and q_2 is,

$$F_e = \frac{1}{4\pi\varepsilon_o r^2} q_1 q_2$$

we can normalize the force on a "per positive coulomb" bases, and define something called the electric field. If we make $q_1 = 1C$ for the equation above:

$$F_e = \frac{1}{4\pi\varepsilon_o r^2}(1) q_2$$

The stuff to the left of q_2 is the electric field.

$$F_e = \vec{E} q_1$$

force on q_1 can now be thought of as the electric field at a point times its magnitude.

$$\vec{E}(r) = \frac{q_2}{4\pi\varepsilon_o r^2}$$

with that said, the purpose of this document is to look at magnetic fields, and it will be shown that the magnetic field can be considered an electric field effect.

2 An example

Consider the situation below,

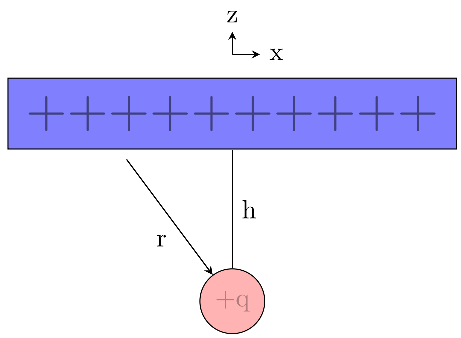


Figure 1: Line density example

where lets say a line density of positive charges is acting single charge q. The electric force on q can be found by integrating over the entire line density, considering the amount of charge that exists per dx as dq. The x component of force cancels out for this case, we only have to consider the z component. lets take θ as the angle between r and h.

$$F_z = \int_{-\infty +\infty} F_e cos\theta = \int_{-\infty +\infty} \frac{dq}{4\pi\varepsilon_o(x^2+h^2)} cos\theta = \int_{-\infty +\infty} \frac{dq}{4\pi\varepsilon_o(x^2+h^2)} \frac{h}{\sqrt{(x^2+h^2)}}$$

to make the integral all with respect to one variable we express the amount of charge per meter as $\lambda = \frac{dq}{dx}$:

$$F_z = \int \frac{\lambda dx}{4\pi\varepsilon_o(x^2+h^2)} \frac{h}{\sqrt{(x^2+h^2)}}$$

this integral can be solved with a formula derived with trigonometry, I will just show the result:

$$\int \frac{dx}{(x^2+h^2)^{3/2}} = \frac{x}{h^2\sqrt{x^2+h^2}}$$

the final force becomes:

$$F_z = \left[\frac{\lambda x}{4\pi\epsilon_o h^2\sqrt{x^2+h^2}}\right]_{-\infty}^{+\infty}$$

or, due to symmetry:

$$F_z = 2\left[\frac{\lambda x}{4\pi\epsilon_o h^2\sqrt{x^2+h^2}}\right]_{0}^{+\infty}$$

which gives,

$$F_z = \frac{\lambda}{2\pi\varepsilon_o h} \tag{1}$$

We can use equation 2 to look at an equivalent example that is usually solved using only the magnetic field.

Figure 2: A picture of the Soyuz rocket contracting in length when its speed increases relative to the observer.

3 Relativity and Magnetism

relativity provides a completely different way to look at magnetism. To start, look at figure 2 below.

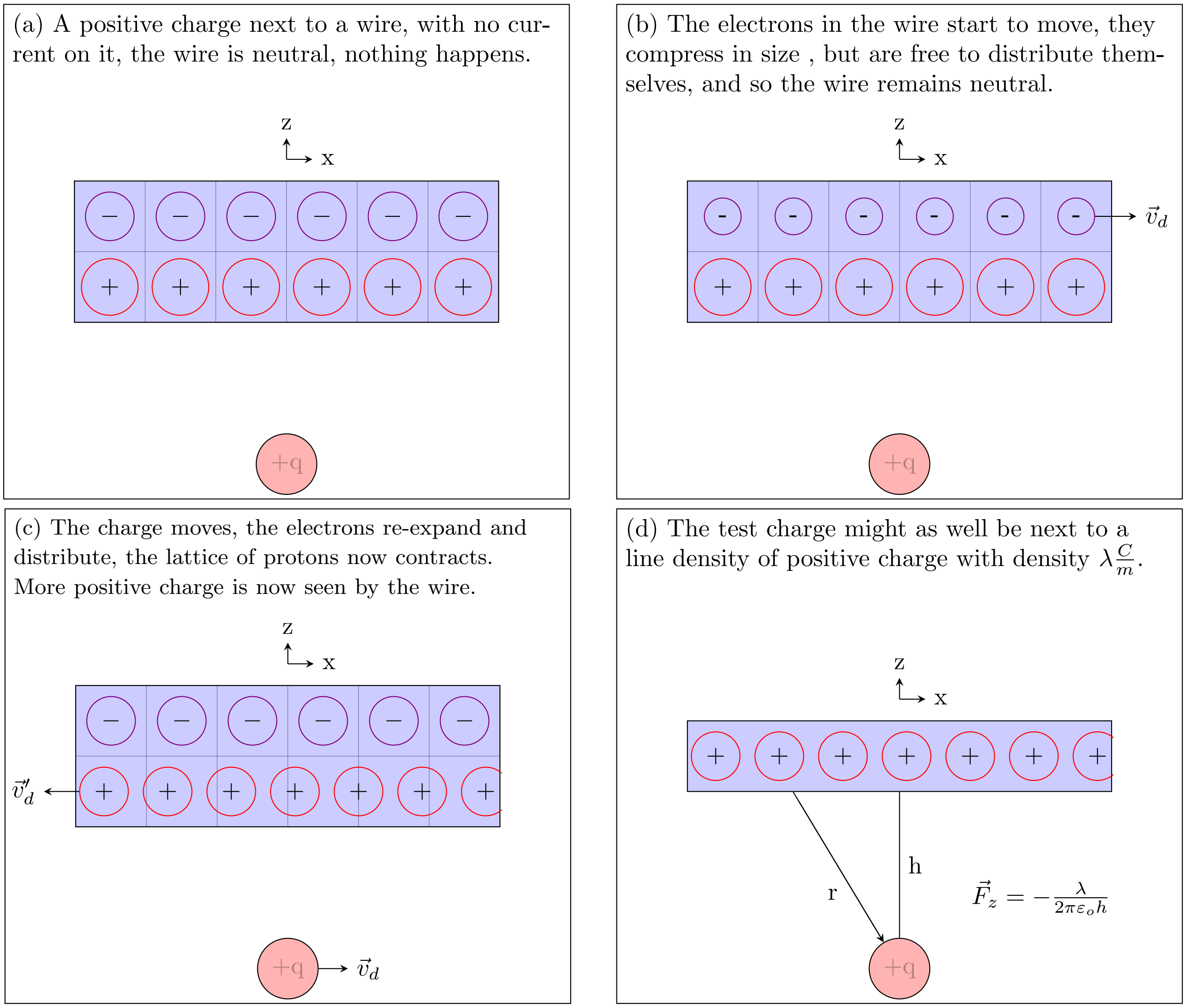
where a rocket that has a length L while stationary, becomes a rocket of length $\frac{2}{3}L$ when moving at $\frac{3}{4}c$, where c is the speed of light (299792458 $\frac{m}{s}$). At first inspection this may seem odd, but then the perhaps you could say, well of course it *looks* smaller, light takes a certain amount of time to each our eyes and the rocket is moving at a speed comparable to this time, but surely it is not *actually* smaller? we can put this question on hold and consider a similar one. In most high school programs, we are shown the astronaut paradox, where an astronaut with a twin brother can leave earth, travel at high speeds relative to light, and arrive back on earth younger then his twin. Here is a quick example to show how it is explained, consider the fact that it takes light eight minutes to leave the sun and arrive on earth. Lets say the astronaut left earth and travelled to the sun at the speed of light. He arrives eight minutes later. If he looks back at earth to measure his age from a stopwatch that started when he left, the stop watch would read $0:00$, because he had travelled at the same speed as the image of $0:00$. So by the stopwatch on earth, he got there instantly. Again, surely he is just using the wrong clock to measure how much he aged? The paradox serves to illustrate the difference between *special* relativity and *general* relativity.For both questions just raised, whether the space ship is actually smaller, or that the amount the astronaut has aged when he arrives back to earth will be less then 16 minutes,science will tell you that yes the spaceship can be considered as actually smaller, and the astronaut will have aged less then 16 minutes. Understanding of this can be found in *general* relativity but its calculation can be done using *special* relativity. Here, like most texts, I will not touch general relativity as it is too difficult, but I am told it is true else modern technology like satellite communication would not be possible.

We can consider the length of objects to contract as they move, along with the properties associated with its length. The amount of this contraction, relative to the stationary length L_o is given by,

$$L = \frac{L_o}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2}$$

4 Magnetism and length contraction

now consider the previous electric field example in figure 3, except instead of a line density of charge, we replace it with an infinite wire.



(a) A positive charge next to a wire, with no current on it, the wire is neutral, nothing happens.

(b) The electrons in the wire start to move, they compress in size , but are free to distribute themselves, and so the wire remains neutral.

(c) The charge moves, the electrons re-expand and distribute, the lattice of protons now contracts. More positive charge is now seen by the wire.

(d) The test charge might as well be next to a line density of positive charge with density $\lambda \frac{C}{m}$.

Figure 3: Demonstration of how due to length contraction a neutral wire (a), that carries a current (b), will exert a force on a charge nearby if it moves (c), a situation that is equivalent to the example of figure 1

in looking at figure 3 (d), for this case the notion of a magnetic field is not needed to conclude that the charge will experience a force. Lets calculate this equivalent line density for a current, and compare it with the magnetic force.

Solved Example

For a small copper wire carrying $10,000,000$ $[A]$, with a radius of 5cm , what would be the equivalent charge line density and force on the test charge in figure 3 (d)? Consider the charge as 100 meters away from the wire, $h = 100$m

density of copper:	ρ_c	$= 8.94cm^3$
atomic weight of copper:	A_{w_c}	$= 63.546$
Avogadro's number:	A_v	$= 6.0221413E23$

the moles per m^3 would be:

$$N = \rho_c / A_{w_c}$$

we can approximately say, that for every atom, there is one free electron, and that the electron density would be

$$n = NA_v$$

the coulomb density is this number multiplied the charge of one electron,

$$\rho_{rest} = n(1.6 \times 10^{-19})$$

which would have to be equal to the postive charge density for a neutral wire, this is all in an effort to relate back to our familiar equation for drift velocity based on a current

$$\vec{v}_d = \frac{I}{\pi r^2 \rho}$$

the new positive charge density will be the uncontracted charge density squeezed into a space where the length in direction of motion is contracted per meter to L_γ,

$$L_\gamma = \frac{1[m]}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \rho_\gamma = \frac{\rho_{rest}}{LWH}$$

where only one of the spatial dimensions of length (L), width (W), and height (H) is contracted.

$$\rho_+ = \rho_{rest}/1 \cdot 1 \cdot L_{new}$$

the charge density in the frame moving at $\vec{v}_d$ is now equivalent to the difference between the new positive charge density per m^3 and the rest frame charge density for the electrons.

$$\rho_{total} = \rho_\gamma - \rho_{rest}$$

which can be made equivalent to a line density of

$$\lambda = \rho_{total}/\pi r^2$$

the force is then

$$\vec{F}_z = -\frac{\lambda}{2\pi\varepsilon_o z} = 37N$$

if someone said use the magnetic field concept, you would find $\vec{B}$ of a wire, and a velocity of $v = 2v_d$ (because the model of drift velocity was derived based on the assumption that the protons are moving at the drift velocity in the opposite direction relative to the electron, confusing yes i know)

$$\vec{B}_{wire} = \frac{\mu_o I}{2\pi z} \quad \text{and plug it into:} \quad F = qvB = 37N$$

Code for the previous example:

```
clear all
close all
clc
% units
meters=1;
cm=(1/100)*meters;
c=299792457.8;          % speed of light

mu=4*pi*1e-7;
eps_o=(1/(mu*c*c));
format long e
%% Moving Cat frame electric force

I=10E3;             % current in amps
r=.05*cm;           % radius of wire
A_wire=pi*r^2;      % area of the wire
z=100;              % distance of cat from wire
q_cat=1;            % charge of the cat

rho_c=(8.94)/(cm^3);    % grams/cm^3
rho_c=10000;
Aw_c=63.546;            % g/mol of copper;
Av=6.0221413e+23;       % Atoms/mol;

mol_per_m_cubed=rho_c/Aw_c;
Atomic_density=mol_per_m_cubed*Av;
num_e=Atomic_density;
columb_density=Atomic_density*1.6E-19;


% drift velocity for a given current
v_d=I/(A_wire*columb_density);
v_cat=.004*c;
lam_o=columb_density*A_wire;
gamma_vd=sqrt(1-(v_d/c)^2);
gamma_cat_p=sqrt(1-((v_d+(v_cat-v_d))/c)^2);
gamma_cat_e=sqrt(1-((v_d)/c)^2);
gamma_diff=sqrt(1-((v_cat-v_d)/c)^2);
gamma_sum=sqrt(1-((v_cat+v_d)/c)^2);
ratio_v_to_c=v_d/c;


lam_neg_e_frame=lam_o*gamma_vd;
% linear charge density of positive ions seen by the electrons
lam_pos_e_frame=lam_o/gamma_vd;

% net charge in the electron frame
lam_tot_e_frame=lam_pos_e_frame-lam_neg_e_frame;
%% CAT FRAME
% linear charge density in the electron frame of reference
lam_neg_cat_frame=lam_o*(gamma_cat_e)/gamma_diff;
```

Figure 4: matlab code from ENGR 459 assignment

5 More general calculations

the calculations involving number of moles in the previous example are not really needed, but are useful because now any metal can be modelled. To make the model more general, we assume a model where no net charge is seen by a stationary observer on a wire whether current is flowing or not. This is important because you may think that, for a current carrying wire, the electrons move, and so contract, but the positive lattice remains stationary, so you would see a net negative charge. experiments tell us this does not happen, and its related to the fact that the electrons are free to move around, each individual electron contracts according to relativity, but not necessarily the distance between them. for the positive lattice however, the story is different, it must contract. Now lets assume we have the following line density at rest.

$$\lambda_o$$

and with a wire carrying no current, the density of positive and negative charges should be equal.

$$\lambda_o^+ = \lambda_o^-$$

when current is flowing in the wire, the amount of positive and negative charges will depend on the velocity of the observer. For any given velocity v the observer will see

$$\lambda_v^+ = \frac{\lambda_o}{\sqrt{1 - \frac{(v_d + (v - v_d))^2}{c^2}}}$$

of positive charge and,

for the reasons previously discussed we are going to assume a model, where drift velocity is in the direction of electron flow, and that when current is flowing in the wire, it is equivalent to looking at a wire where protons move in the opposite direction of electrons and have equal charge density. assuming this model, for any given velocity the observer will see a line density,

$$\lambda_v^- = \lambda_o \frac{\sqrt{1 - \frac{v_d^2}{c^2}}}{\sqrt{1 - \frac{(v - v_d)^2}{c^2}}}$$

of negative charge. The total line density can then be calculated as $\lambda' = \lambda_v^+ - \lambda_v^-$. Now we can compare a solution based on the magnetic field of a current carrying wire, $\vec{B} = \mu_o \frac{I}{2\pi z}$ with the relativistic one.

$$f_m = qv\vec{B} \quad \text{or} \quad f_r = \frac{\lambda'}{2\pi\varepsilon_o z}$$

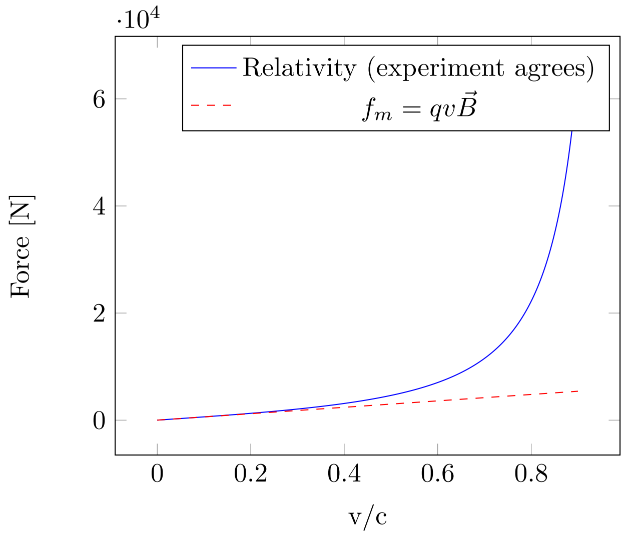


Figure 5: figure showing agreement between relativistic and classical calculations for magnetic forces for $\approx v < .2c$

6 EMF and Two loops

Unfortunately, the magnetic field cannot actually be replaced by the electric field in more complex situations, like figure 6 below. In the picture, the top loop is moving towards the bottom loop. Shown in the top right is the relativistic force plotted on top of the lorentz force. For speeds $<< c$, they agree exactly for the same reason as the previous example. There is also an emf induced in each coil from this motion, this emf cannot be calculated using only Coulomb's law, so the magnetic force needs to exist in this situation. In summary, electric and magnetic effects can be altered in different reference frames with relativity, but they both have to exist in order to replicate experiment. The previous line density example was a special case where a pure magnetic effect in one reference frame became a pure electric effect in another, in most situations, like for the two loops below, both fields still need to be solved for. This is apparently where we are at so far in combining relativity with electromagnetism.

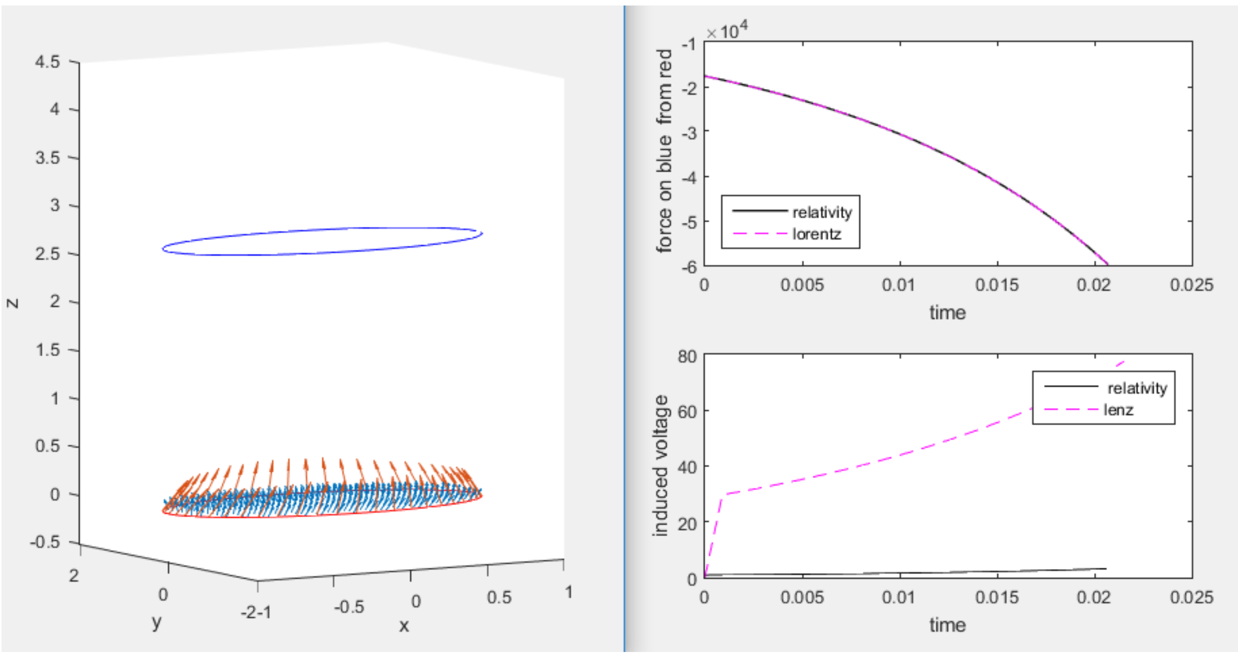


Figure 6: Relativity force calculations for two loops