# Black Bodies and the Ultraviolet Catastrophe

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**Abstract**

This paper is an overview of some of the basic mechanisms and theory of black body radiation and how it influenced the development of Quantum Mechanics through the Ultraviolet Catastrophe. The principles of what define a black body will be discussed, along with how energy interacts with a black body in particular how thermal radiation leaves a black body. The derivation of the Rayleigh-Jeans formula will be discussed along with the experimental results which contradicted it, generating the Ultraviolet Catastrophe and in particular how it led to Max Planck's work on Quantum Mechanics. This work will be briefly touched upon. An overview of the history of the development of these concepts and their relation to physics at the time will also be discussed.

# Contents

# 1 Introduction

In the early and mid 1800s the groundwork for our current understanding of thermodynamics was laid including in particular, many of the kinetic theory of gases laws, laws of thermodynamics, concepts of a black body and theories of electromagnetism. This body of work set up the physical understanding necessary to discuss the concept of a black body and to particular discover the Ultraviolet catastrophe leading to a revolution in how physics was understood through Max Planck's concept of energy quantization.

## 1.1 What is a black body?

A black body is material which follows certain restrictions on its emission of light. As the name would suggest, it absorbs effectively all of the electromagnetic radiation which comes into contact with the black body. In appearance, this is why the object is black, since it absorbs all of the light and none is reflected back to the viewer. The light and heat energy a black body comes into contact with will heat up the black body as well since that energy is conserved and since it isn't reflected into the environment, will have to go into the object. As this object is heated it will emit light at different wavelengths. For an intuitive example, consider the heating of steel or the heating of coal.

As heat is applied and energy is transferred to the metal or this coal it begins to glow in visible light. This process will be discussed in more detail and with more rigor later, however a conceptual understanding is useful.

# 2 Classical Theory

## 2.1 What are black bodies? (formally)

As previously stated black bodies are objects which effectively absorb the light which come into contact with the object, and which emits some of the electromagnetic radiation. The Jeans Cube is one such idealized black body. It is important to note that only the hole of the Jeans Cube is the black body. A Jeans Cube is a cube with a small hole through which electromagnetic radiation passes and enters the cube where these waves move around the inside of the cube. The walls of the cube are metallic and conductive. There is a small negligible chance that this radiation will leave the Jeans Cube through that hole. Since this absorbs effectively all of the radiation which goes into it and only emits black body radiation this is a black body. It is worth noting that a black body needs to be in thermal equilibrium with the environment. Experimentally, it was determined that black bodies do not emit just one wavelength of electromagnetic radiation and that in fact it was fairly 'smooth' in the sense that radiation emitted from a black body was more of a continuous function on some intervals and not just confined to some small number of wavelengths. It was through this that it was reasoned that if the total intensity $I$ of black body radiation wanted to be observed simply integrating we can get the total intensity.

$$I = \int_0^\infty I(\lambda) d\lambda$$

Some other experimental laws which are well known connect quite nicely to this. For example the Stefan-Boltzmann law:

$$I = \sigma T^4$$

Where $\sigma$ is some constant with $\sigma = 5.67037 \times 10^{-8}$. This means we can connect this with the previous statement for, though this isn't particularly useful in this setting, since the intensity function is known and derivable.

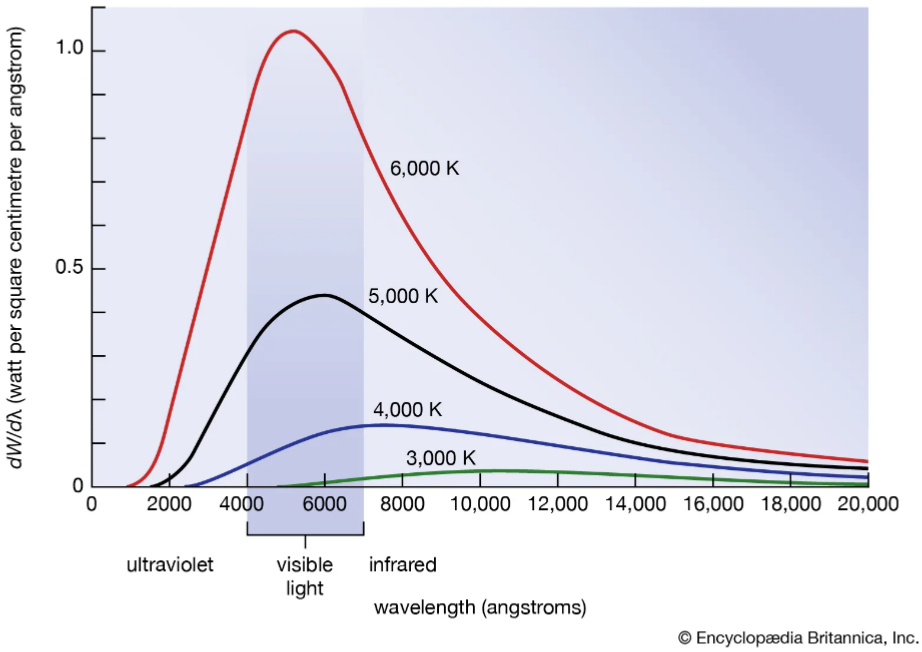
$$T = \sqrt[4]{\frac{\int_0^\infty I(\lambda)\lambda}{\sigma}}$$

### 2.2 Wien's Displacement Law

One of the more important results from the classical theory is Wien's displacement law. This law was discovered by Wilhelm Wien before Max Planck's formulation of quantized energy. However, this law can be derived mathematically from Max Plancks theory. This derivation will not be discussed. The Wien displacement law is written as:

$$\lambda_{peak} = \frac{b}{T}$$

Where $b$ is some constant, and $T$ is a temperature in Kelvins This was determined to be $b = 2.897771955 \times 10^{-3} m \cdot K$. Recall that at each wavelength, a black body radiates a certain intensity and that when we integrate this intensity function with respect to wavelength, we get the total intensity of the object. This law simply refers to what value of $\lambda$ maximizes this intensity function. As we can clearly see, in order to get very large wavelengths we do not need much temperature at all. This corresponds to infrared, microwaves, and radio waves. Likewise, if we wanted to have very small wavelengths such as visible light, x-rays- or even gamma rays, the value of the temperature of the black body is particularly high. It is helpful to see some reference points since this is a non-linear, in particular, a hyperbola. In order to get the visible red light (700 nanometers) a temperature of 4139 Kelvins is required. In order to get ultraviolet light (400 nanometers) a temperature of 7244 kelvins is required, and in order to get an x-ray (10 nm) a temperature of 289,780 Kelvins is required. Finally, according to NASA, the central temperature of the sun is $1.571 \times 10^7$ meaning that the wavelength of light emitted at the center of the sun is about 0.1845 nanometers, though there are likely some conditions on this since the center of the sun is not an ideal environment and has other variables involved. Below is attached a graph of this relationship.




One this which is noticeable here as discussed in the formula and examples, is that as the temperature gets larger the maximizing wavelength decreases. Recall from the introduction section how it was stated that, as something heats up it begins to glow. This helps explains that phenomena. However, that is not to say that just because $\lambda_{peak} > 700$nm that nothing can be seen. After all, $\lambda_{peak}$ is the *peak* intensity for any one wavelength, it does not indicate that no wavelength can be within the visible range. For instance in the graph we can see the object heated to 3000K still has some emission of light in the visible range. It becomes more clear however as it is heated up.

A final point is that, based on the information discussed thus far it is not immediately clear whether the intensity is material dependent or not. Wien's displacement law only relates the location of the peak to the temperature on the wave length spectrum, not the characteristics of the intensity or value of the intensity at some wavelength to temperature.

## 2.3 Rayleigh Jeans Formula Derivation

Energy density is as a concept not much different than the concept of mass density.

$$\rho_m = \frac{m}{V} \quad \rho_E = \frac{E}{V}$$

Mass density being on the left, and energy density with total energy being denoted with $E$ being on the right. Connecting to black body radiation, we can derive that, with $c$ equal to the speed of light:

$$I(\lambda) = \frac{c}{4}\rho_E(\lambda)$$

While a precise derivation will not be shown we can show that the units check out for this principle in terms of dimensional analysis.

$$I = \frac{c}{4}\frac{E}{V} \iff \frac{J/s}{L^2} = \frac{d/s}{4}\frac{J}{L^3} \iff J/s = \frac{J/s}{4}$$

Recall previously that:

$$I = \int_0^\infty I(\lambda)d\lambda \implies I = \int_0^\infty \frac{c}{4}\rho_E(\lambda)d\lambda$$

This means that if we can find a formula for the energy density at $\lambda$ we can find the total intensity on some range of wavelengths (by integrating though those wavelengths as lower and upper bounds only instead of form 0 to infinity) and in total across all wavelengths.

Recall that we are looking at a Jeans Cube for simplicity's sake. Electromagnetic waves reflecting off of these metallic walls will form standing waves. We can determine that the number of standing waves at some point and range $\lambda d\lambda$ will be given by, with V as the volume:

$$N_W(\lambda)d\lambda = \frac{8\pi V}{\lambda^4}d\lambda$$

For clarity, its worth noting that if we wanted the number of standing waves in the volume we would calculate that as:

$$\int_0^\infty \frac{8\pi V}{\lambda^4}d\lambda$$

We can use the Maxwell-Boltzmann energy distribution for a one dimensional oscillator which gives the probability of a particular wave having some energy. This will not be derived. A probability density function is a function which gives a value for the probability of something dependent on an $x$ variable is on the interval $x + dx$. The particular probability density function for this situation however is with $k_b$ as the Boltzmann constant:

$$P(E) = \frac{1}{k_bT}e^{-E/k_bT}$$

We can find the average of any continuously distributed system with a probability density function and a function and its expected value at the quantity at the x coordinate by using the formula

$$f(x)_{avg} = \int_a^b f(x)P(x)dx \quad [1]$$

Applying this to this situation since our probability density function is just $P(E)$ and our expected energy $E$ at some energy (the only dependent variable in P(E) is of course just $E$. This means that our average energy is:

$$E_{avg} = \int_0^\infty E \quad P(E)dE = \int_0^\infty E \quad \frac{e^{-E/k_bT}}{k_bT} = k_bT$$

We recall now that:

$$I = \int_0^\infty \frac{c}{4}\rho_E(\lambda)d\lambda \;\; \& \;\; \rho_E = \frac{E}{V}$$

We can combine these concepts to say that the energy density part of the left integral, $\rho_E(\lambda)d\lambda$ = The number of standing waves at an interval wavelength inside of the box divided by the volume of that box, times the average energy of each standing wave. We have terms for each of these so we can say

$$\rho_E(\lambda)d\lambda = \frac{N_W(\lambda)d\lambda}{V}\cdot k_bT = \frac{8\pi}{\lambda^4}k_bTd\lambda \implies \int_0^\infty \frac{c}{4}\frac{8\pi}{\lambda^4}k_bTd\lambda = \int_0^\infty \frac{2c\pi}{\lambda^4}k_bTd\lambda = I$$

This is a rather remarkable result and extremely useful, since now we can describe the emissions of a black body. We can further use this result to say:

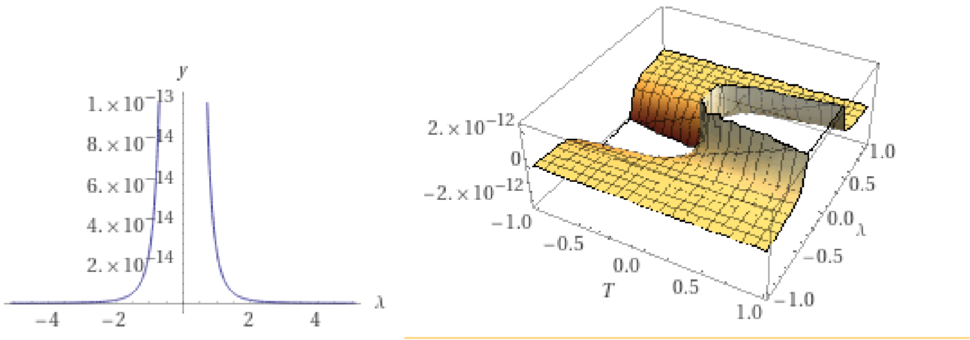
$$I(\lambda) = \frac{2c\pi}{\lambda^4}k_bT$$

This function will tell us what the intensity is at each wavelength, and when we integrate this we should get the total energy.

[1] I very heavily based what I am saying here off a stack exchange post which seemed to be well received and clear. I cited this particular post in my works cited. It also seemed to match up with multiple other sources so it seemed relevant mathematical explanation and reasonable to include it

### 2.4 Catastrophe!

Looking at this formula, there may be an issue which is apparent, but first lets graph it and make sure that it makes sense. Recall from the Wien Displacement Law that we expect a curve reminiscent of the Maxwell-Boltzmann distribution and we also expect this curve to be reasonably smooth, continuous, and have a maximum which corresponds to a maximum intensity at some wavelength. Well unfortunately for Rayleigh and Jeans in the late 1800s this is not what they or we see.




Left image is a 2d case with a constant temperature T=1, right is the general case

Well, as we can see in these graphs the Rayleigh-Jeans formula indicates that not only is there no maximum, but that the intensity goes to $\infty$ as the wavelength gets very small. Remember that the Wien Displacement Law was derived experimentally, which means that Rayleigh-Jeans formula is incorrect and that there is a fundamental flaw in the theory they are using such that there results have an infinite error at increasingly small wavelengths. This was a catastrophe for physics at the time since a new theoretical understanding and approach to energy and physics is necessary to resolve these errors. This was dubbed "The Ultraviolet Catastrophe." This resolution is what Max Plank and his introduction of quantized energy did.

## 3 Introducing Quantum Mechanics

### 3.1 A Quanta of Energy

Max Planck's contribution came in the form of quantizing energy. Instead of the energy being a continuous variable as described by the Maxwell-Boltzmann probability density function it was made discrete. Each wave could only take energy which was an integer multiple of a finite amount of energy. This was necessary to ensure that the average energy at large wavelengths the Rayleigh-Jeans formula (which was a good formula for large wavelengths) approached $k_bT$ and likewise approached 0 for very small wavelengths, in line with the experimental data. Each of these electromagnetic waves have an average energy therefore of

$$E_N = n\epsilon, n \in \mathbb{N}, \epsilon = hf$$

This epsilon is the quantization of energy which has been discussed, and depends on the frequency of the wave $f$, along with a new variable called Planck's constant $h$ which was later determined to be $6.626068 \times 10^{-34}$ Joules per second. Recall that $c = \lambda f$ and therefore we can replace f and get

$$E_N = n\left(\frac{ch}{\lambda}\right)$$

. Note how this is non-linear and how as $\lambda \to 0$ $E_n$ gets larger **if** $n$ is left constant.

### 3.2 Resolution to the Ultraviolet Catastrophe

Using this new concept, Planck derived that the number of electromagnetic waves with Energy $E_n$ is given by:

$$N_n = N(1 - e^{-\epsilon/k_bT})(e^{-n\epsilon/k_bT}) \implies \sum_{n=0}^{\infty} N_n = N$$

Finally we derive that:

$$E_{avg} = \frac{\epsilon}{e^{\epsilon/k_bT} - 1} = \frac{hc/\lambda}{e^{hc/\lambda k_bT} - 1}$$

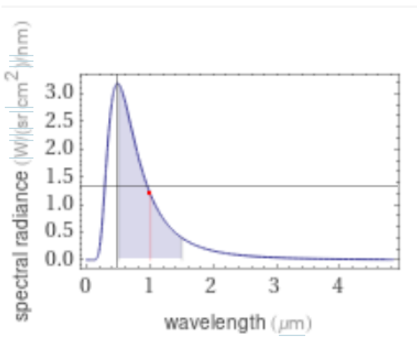
We can then do the same process as before, and check by substituting in our known results, namely that

$$I(\lambda)a\lambda = \frac{c}{4}\rho_E(\lambda)d\lambda, N_W(\lambda) = \frac{8\pi V}{\lambda^4}, E_{avg}$$

There is no problem doing this because by quantizing our energy we have not altered the number of standing waves in the volume, nor have we altered the Volume itself and that is all that is nessecary to obtain this result. Doing this we get that:

$$I(\lambda) = \frac{N_W(\lambda)}{V}E_{avg} = \frac{c}{4}\left(\frac{8\pi}{\lambda^4}\right)\frac{hc/\lambda}{e^{hc/\lambda k_bT} - 1} = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda k_bT} - 1}$$

This right most form is known as Planck's law and resolves the ultraviolet catastrophe. An example graph for this function with certain parameters is:




# 4 Conclusion

As we can see the investigation of black body radiation produced some rather groundbreaking and generally remarkable conclusions. It also opened the door to the development of Quantum Mechanics through the quantization of energy. It is particularly interesting that Wien's Displacement Law was derived before Max Planck's contribution to solve the ultraviolet catastrophe and that the law can be directly derived from this solution to the catastrophe and how this law explains some rather simple behaviors of hot objects.

## 5 Sources

Primary sources (Used for derivations, theory, and exposition):

Modern Physics Third Edition, by Kenneth Krane. ISBN: 978-1-118-06114-5
https://www.youtube.com/watch?v=rCfPQLVzus4
https://www.thermal-engineering.org/what-is-wiens-displacement-law-definition/
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Secondary sources (pictures, data, etc):

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https://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.html
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