Black Bodies and the Ultraviolet Catastrophe

Devon R

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Abstract

This paper is an overview of some of the basic mechanisms and theory of black body radiation and how it influenced the development of quantum mechanics through the ultraviolet catastrophe. The principles of what define a black body will be discussed, along with how energy interacts with a black body in particular how thermal radiation leaves a black body. The derivation of the Rayleigh-Jeans formula will be discussed along with the experimental results which contradicted it, generating the ultraviolet catastrophe and how it led to Max Planck's work on quantum mechanics. This work will be briefly touched upon. An overview of the history of the development of these concepts and their relation to physics at the time will also be discussed.

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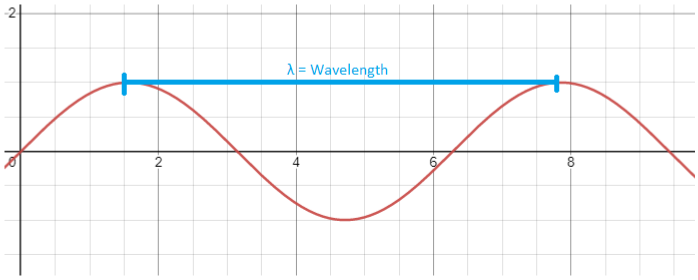
1 Introduction

In the early and mid 1800s the groundwork for our current understanding of thermodynamics was laid including, many of the kinetic theory of gases laws, laws of thermodynamics, concepts of a black body and theories of electromagnetism. This body of work set up the physical understanding necessary to discuss the concept of a black body and to discover the Ultraviolet catastrophe leading to a revolution in how physics was understood through Max Planck's concept of energy quantization. It is critical to recall that visible light is just one type of electromagnetic radiation. Radio, Ultraviolet, Infrared, X-Rays and other electromagnetic waves all are electromagnetic radiation.



λ (Wavelength) is just the peak to peak distance on a wave.

These wavelengths will determine whether something is visible light or not or whether it falls into any of the other categories of electromagnetic radiation that exist.

1.1 What is a black body?

A black body is material which follows certain restrictions on its emission of light. As the name would suggest, it absorbs effectively all of the electromagnetic radiation which comes into contact with the black body. In appearance, this is why the object is black, since it absorbs all of the light and none is reflected back to the viewer. The light and heat energy a black body comes into contact with will heat up the black body as well since that energy must be conserved and since it isn't reflected into the environment, will have to go into the object. As this object is heated it will emit light at different wavelengths. For an intuitive example consider black paint or a dark piece of coal. Light which touches black paint or goes into a dark piece of coal generally does not return, so it appears dark.

Consider the heating of steel or the heating of coal. As heat is applied and energy is transferred these begins to glow in visible light. This is electromagnetic radiation that is emitted because of temperature. This process will be discussed in more detail and with more rigor later, however a conceptual understanding is useful.

2 Classical Theory

2.1 What are black bodies? (formally)

As previously stated black bodies are objects which effectively absorb the light which come into contact with the object, and which emits some of the electromagnetic radiation. This electromagnetic radiation is emitted at various intensities at various wavelengths of light. It is very useful for our purpose to consider an idealized black body, and the Jeans Cube is one such idealized black body. A Jeans Cube is a cube with a small hole through which electromagnetic radiation passes and enters the cube where these waves move around the inside of the cube. It is important to note that only the hole of the Jeans Cube is the black body, not the entire cube. The walls of the cube are metallic and conductive. There is a small negligible chance that this radiation will leave the Jeans Cube through that hole. Since this absorbs effectively all of the radiation which goes into it and only emits black body radiation this is a black body.

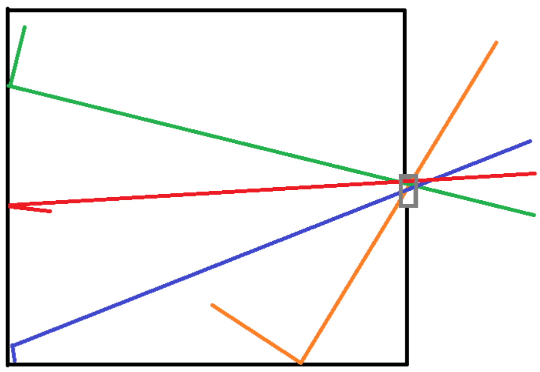


Image of a Jean's Cube

Above is a graphical representation of this Jean's Cube. As we can see light of various wavelengths (hence the different colors) enter though this small hole. Once they enter this hole though they bound around the inside of the box reflecting back and forth. There is a very low chance that this reflection process will have this light exit the hole on any single step. Since this absorbs all light and reflects none of that light back, this hole in the cube is a black body. Though the reader should not become particularly attached to this representation as this is also a single slit so the electromagnetic waves entering the box will also propagate as a single slit would. However, the key concept is that the light is extremely unlikely to leave the box. It is also worth noting that this need not be a box. However it is most convenient to consider it a box, so we will throughout the text.

It is worth noting that a black body needs to be in thermal equilibrium in the context that we are considering. This means that the temperature of the black body is uniform. Experimentally, it was determined that black bodies do not emit just one wavelength of electromagnetic radiation and that in fact it was fairly 'smooth' in the sense that radiation emitted from a black body was more of a continuous function on some intervals of wavelength and not just confined to some small number of wavelengths. So, if some 'variable filter' were set up to only admit one wavelength of light through it and were placed next to a black body, as the wavelength of the filter changes as the operator desires we would always see something. It was through this that it was reasoned that if the total intensity I of black body radiation wanted to be observed simply integrating we can get the total intensity.

$$I = \int_0^\infty I(\lambda) d\lambda$$

Some other experimental laws which are well known connect quite nicely to this. For example the Stefan-Boltzmann law which relates temperature to the intensity:

$$I = \sigma T^4$$

Where σ is some constant with $\sigma = 5.67037 \times 10^{-8}$. This means we can connect this with the previous statement for, though this isn't useful in this setting, since the intensity function is known and derivable.

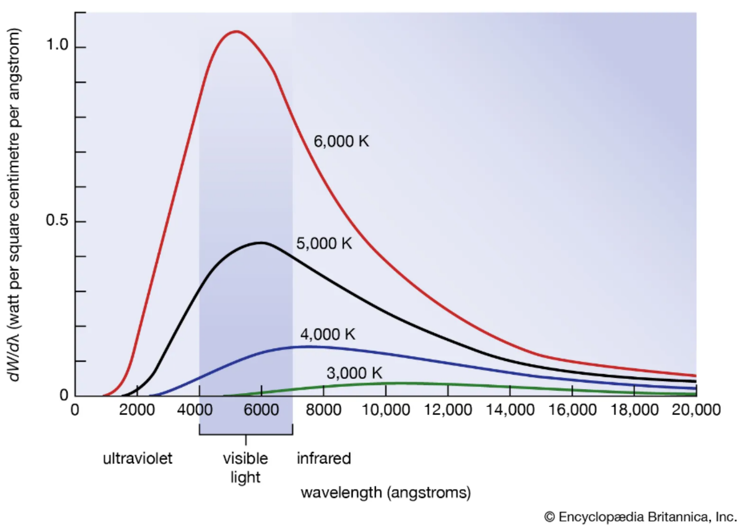
$$T = \sqrt[4]{\frac{\int_0^\infty I(\lambda)\lambda}{\sigma}}$$

2.2 Wien's Displacement Law

One of the more important results from the classical theory is Wien's displacement law. This law was discovered by Wilhelm Wien before Max Planck's formulation of quantized energy. However, this law can be derived mathematically from Max Plancks theory. This derivation will not be discussed. The Wien displacement law is written as:

$$\lambda_{peak} = \frac{b}{T}$$

Where b is some constant, and T is a temperature in Kelvins. This was determined to be $b = 2.897771955 \times 10^{-3} m \cdot K$. Recall that at each wavelength, a black body radiates a certain intensity and that when we integrate this intensity function with respect to wavelength, we get the total intensity of the object. This law simply refers to what value of λ maximizes this intensity function. As we can clearly see, in order to get very large wavelengths we do not need much temperature at all. This corresponds to infrared, microwaves, and radio waves. Likewise, if we wanted to have very small wavelengths such as visible light, x-rays- or even gamma rays, the value of the temperature of the black body is high. It is helpful to see some reference points since this is a non-linear hyperbola. In order to get the visible red light (700 nanometers) a temperature of 4139 Kelvins is required. In order to get ultraviolet light (400 nanometers) a temperature of 7244 kelvins is required, and in order to get an x-ray (10 nm) a temperature of 289,780 Kelvins is required. Finally, according to NASA, the central temperature of the sun is 1.571×10^7 meaning that the wavelength of light emitted at the center of the sun is about 0.1845 nanometers, though there are likely some conditions on this since the center of the sun is not an ideal environment and has other variables involved. Below is attached a graph of this relationship.



One this which is noticeable here as discussed in the formula and examples, is that as the temperature gets larger the maximizing wavelength decreases. Recall from the introduction section how it was stated that, as something heats up it begins to glow. This helps explains that phenomena. However, that is not to say that just because $\lambda_{peak} > 700$nm that nothing can be seen. After all, λ_{peak} is the *peak* intensity for any one wavelength, it does not indicate that no wavelength can be within the visible range. For instance in the graph we can see the object heated to 3000K still has some emission of light in the visible range. It becomes more clear however as it is heated up.

Similarly as an object cools down, we expect to see the wavelength increase. Consider if we were to over many millions or billions of years were to track the peak intensity of a star. As the star loses energy, the star may have started emitting radiation at the x-ray or gamma ray wavelength, but over time could move into the radio or infrared as it, or its remnants begin to die and lose energy. While specifics are more in the astronomy field and unsuitable for in depth analysis in this text, it is an interesting connection none the less.

A final point is that, based on the information discussed thus far it is not immediately clear whether the intensity is material dependent or not. Wien's displacement law only relates the location of the peak to the temperature on the wave length spectrum, not the characteristics of the intensity or value of the intensity at some wavelength to temperature and will require requires further discussion.

2.3 Rayleigh Jeans Formula Derivation

The Rayleigh-Jeans formula is very simply an attempt to write out a function $I(\lambda)$. When doing this we expect this result to have a peak which corresponds to the Wien displacement law and experimental results which show that each wavelength emits a certain amount of intensity and that this is related to temperature. So this formula should have a temperature variable and a wavelength variable at minimum. In order to do this a few new concepts like Energy Density and probability density functions need to be introduced.

Energy density is as a concept not much different than the concept of mass density.

$$\rho_m = \frac{m}{V} \quad \rho_E = \frac{E}{V}$$

Mass density being on the left with mass denoted by m, and energy density with total energy being denoted with E being on the right. Connecting to black body radiation, we can derive that, with c equal to the speed of light:

$$I(\lambda) = \frac{c}{4}\rho_E(\lambda)$$

While a precise derivation will not be shown we can show that the units check out for this principle in terms of dimensional analysis. Since some volume is some distance cubed for the Jean's Cube, and Energy's units is joules we can make those substitutions. Likewise, c is just some other distance over seconds.

$$I = \frac{c}{4}\frac{E}{V} \iff \frac{\frac{d}{s}}{4}\frac{J}{d^3} \iff \frac{Jd}{4sd^3} \iff \frac{J}{4sd^2}$$

Area in this context is d^2 and J/s is power, intensity is J/s over area, so this checks out. Over course this is just the dimensional analysis for the units though however.
Recalling the following, we conclude :

$$I = \int_0^\infty I(\lambda)d\lambda \iff I = \int_0^\infty \frac{c}{4}\rho_E(\lambda)d\lambda$$

This means that if we can find a formula for the energy density at λ we can find the total intensity on some range of wavelengths (by integrating though those wavelengths as lower and upper bounds only instead of from 0 to infinity) and in total across all wavelengths.

Recall that we are looking at a Jeans Cube for simplicity's sake. Electromagnetic waves reflecting off of these metallic walls will form standing waves. We can determine that the number of standing waves at some point and range $\lambda d\lambda$ will be given by, with V as the volume:

$$N_W(\lambda)d\lambda = \frac{8\pi V}{\lambda^4}d\lambda$$

For clarity, its worth noting that if we wanted the number of standing waves in the volume we would calculate that as:

$$\int_0^\infty \frac{8\pi V}{\lambda^4}d\lambda$$

We can use the Maxwell-Boltzmann energy distribution for a one dimensional oscillator which gives the probability of a particular wave having some energy. This will not be derived. A probability density function is a function which gives a value for the probability of something dependent on an x variable is on the interval $x + dx$. The probability density function for this situation however is with k_b as the Boltzmann constant:

$$P(E) = \frac{1}{k_b T}e^{-E/k_b T}$$

We can find the average of any continuously distributed system with a probability density function and a function and its expected value at the quantity at the x coordinate by using the formula

$$f(x)_{avg} = \int_a^b f(x)P(x)dx \quad [1]$$

Applying this to this situation since our probability density function is just $P(E)$ and our expected energy E at some energy (the only dependent variable in P(E) is of course just E. This means that our average energy is:

$$E_{avg} = \int_0^\infty E \quad P(E)dE = \int_0^\infty E \quad \frac{e^{-E/k_b T}}{k_b T} = k_b T$$

We recall now that:

$$I = \int_0^\infty \frac{c}{4}\rho_E(\lambda)d\lambda \;\; \& \;\; \rho_E = \frac{E}{V}$$

The total energy in the box is just the average energy of each wave across all wavelengths, times the number of waves in the box. We can then get the energy density by dividing by the volume. We can go further and say that the energy density at some wavelength which is $\rho_E(\lambda)d\lambda$ is the number of waves at some wavelength, times the average energy of each wave, divided by the volume of the cube, then by integrating $\rho_E(\lambda)$ across all wavelengths we would get the energy density across every wavelength. This however, is not exactly what we want since we want to relate this concept to the intensity function.

$$\rho_E(\lambda)d\lambda = \frac{N_W(\lambda)d\lambda}{V}\cdot k_b T = \frac{8\pi}{\lambda^4}k_b T d\lambda \implies \int_0^\infty \frac{c}{4}\frac{8\pi}{\lambda^4}k_b T d\lambda = \int_0^\infty \frac{2c\pi}{\lambda^4}k_b T d\lambda = I$$

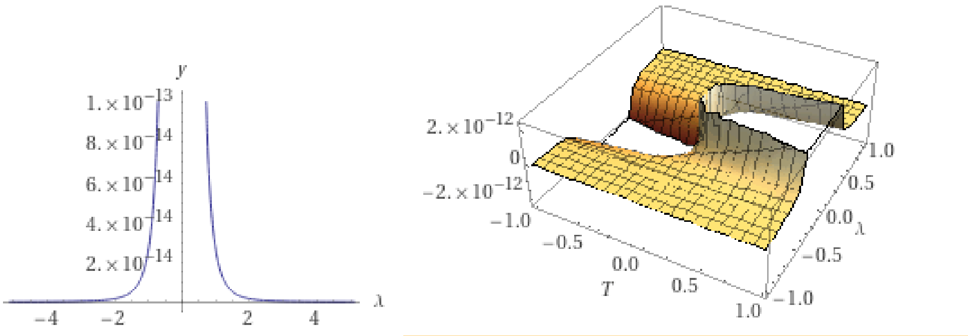
This is a rather remarkable result and extremely useful, since now we can describe the emissions of a black body. We can further use this result to say using our previous relation of intensity at some wavelength to the energy density at some wavelength and again by integrating the function we get the total intensity:

$$I(\lambda) = \frac{c}{4}\left(\frac{8\pi}{\lambda^4}k_b T\right) = \frac{2c\pi}{\lambda^4}k_b T$$

[1] I very heavily based what I am saying here off a stack exchange post which seemed to be well received and clear. I cited this specific post in my works cited. It also seemed to match up with multiple other sources so it seemed relevant mathematical explanation and reasonable to include it

2.4 Catastrophe!

Looking at this formula, there may be an issue which is apparent, but first lets graph it and make sure that it makes sense. Recall from the Wien Displacement Law that we expect a curve reminiscent of the Maxwell-Boltzmann distribution and we also expect this curve to be reasonably smooth, continuous, and have a maximum which corresponds to a maximum intensity at some wavelength. Well unfortunately for Rayleigh and Jeans in the late 1800s this is not what they or we see.



Left image is a 2d case with a constant temperature T=1, right is the general case

As we can see in these graphs the Rayleigh-Jeans formula indicates that not only is there no maximum, but that the intensity goes to ∞ as the wavelength gets very small. Remember that the Wien Displacement Law was derived experimentally, which means that Rayleigh-Jeans formula is incorrect and that there is a fundamental flaw in the theory they are using such that there results have an infinite error at increasingly small wavelengths. This was a catastrophe for physics at the time since a new theoretical understanding and approach to energy and physics is necessary to resolve these errors. This was dubbed "The Ultraviolet Catastrophe." This resolution is what Max Planck and his introduction of quantized energy did.

3 Introducing Quantum Mechanics

3.1 A Quanta of Energy

Max Planck's contribution came in the form of quantizing energy. Instead of the energy being a continuous variable as described by the Maxwell-Boltzmann probability density function it was made discrete. Each wave could only take energy which was an integer multiple of a finite amount of energy. This was necessary to ensure that the average energy at large wavelengths the Rayleigh-Jeans formula (which was a good formula for large wavelengths) approached $k_b T$ and likewise approached 0 for very small wavelengths, in line with the experimental data. Each of these electromagnetic waves have an average energy therefore of

$$E_N = n\epsilon, n \in \mathbb{N}, \epsilon = hf$$

This epsilon is the quantization of energy which has been discussed, and depends on the frequency of the wave f, along with a new variable called Planck's constant h which was later determined to be 6.626068×10^{-34} Joules per second. Recall that $c = \lambda f$ and therefore we can replace f and get

$$E_N = n\left(\frac{ch}{\lambda}\right)$$

Note how this is non-linear and how as $\lambda \to 0$ E_n gets larger **if** n is left constant.

3.2 Resolution to the Ultraviolet Catastrophe

Only a brief overview without much justification will be given to explain Planck's solution. However, parts will be highlighted to communicate the overall key points of the derivation. Using this new concept of energy quantization along with relating the energy quanta to the frequency of the electromagnetic wave and his constant, Planck derived that the number of electromagnetic waves with Energy E_n is given by, with N being the total number of electromagnetic waves in the black body:

$$N_n = N(1 - e^{-\epsilon/k_bT})(e^{-n\epsilon/k_bT})$$

Then through some more steps we can derive that:

$$E_{avg} = \frac{\epsilon}{e^{\epsilon/k_bT} - 1} = \frac{hc/\lambda}{e^{hc/\lambda k_bT} - 1}$$

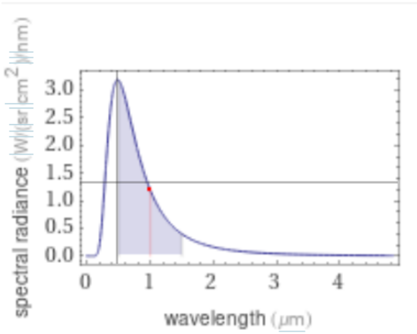
Since Planck's theory only altered the average energy calculation, and not the number of waves at some wavelength, or the means of relating the intensity to the energy density in the Rayleigh-Jeans derivation, We can then do the same process as before, and check by substituting in our new known results, namely that

$$I(\lambda)d\lambda = \frac{c}{4}\rho_E(\lambda)d\lambda, N_W(\lambda) = \frac{8\pi V}{\lambda^4}, E_{avg} = \frac{hc/\lambda}{e^{hc/\lambda k_bT} - 1}$$

There is no problem doing this because by quantizing our energy we have not altered the number of standing waves in the volume, nor have we altered the volume itself and that is all that is necessary to obtain this result. Doing this process with these definitions as done previously gets that:

$$I(\lambda) = \frac{N_W(\lambda)}{V}E_{avg} = \frac{c}{4}\left(\frac{8\pi}{\lambda^4}\right)\frac{hc/\lambda}{e^{hc/\lambda k_bT} - 1} = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda k_bT} - 1}$$

This right most form is known as Planck's law and resolves the ultraviolet catastrophe. An example graph for this function with certain parameters is:



As we can see, this matches much better with the experimental results. In fact, it matches almost perfectly. We can also see that there exists a peak to this graph, as predicted by Wien's Displacement Law, and Wien's Displacement Law can be derived from Planck's Law.

4 Conclusion

As we can see the investigation of black body radiation produced some rather groundbreaking and generally remarkable conclusions. It also opened the door to the development of quantum mechanics through the quantization of energy. It is particularly interesting that Wien's Displacement Law was derived before Max Planck's contribution to solve the ultraviolet catastrophe and that the law can be directly derived from this solution to the catastrophe and how this law explains some rather simple behaviors of hot objects.

5 Sources

Primary sources (Used for derivations, theory, and exposition):

Source 1: Modern Physics Third Edition, by Kenneth Krane. ISBN: 978-1-118-06114-5
Source 2: https://www.youtube.com/watch?v=rCfPQLVzus4
Source 3: https://www.thermal-engineering.org/what-is-wiens-displacement-law-definition/
Source 4: https://chemistry.stackexchange.com/questions/42253/derivation-of-mean-speed-from-maxwell-boltzmann-distribution

Secondary sources (pictures, data, etc):

Source 5: https://energosteel.com/en/the-steel-tempering/
Source 6:https://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.html
Source 7 :https://www.britannica.com/science/Wiens-law