Program : **B.Tech**

Subject Name: **Discrete Structure**

Subject Code: **CS-302**

Semester: **3rd**

Subject Notes
CS301 - Discrete Structures

Introduction

Mathematics can be broadly classified into two categories −

1. **Continuous Mathematics** − It is based upon continuous number line or the real numbers. It is characterized by the fact that between any two numbers, there are almost always an infinite set of numbers. For example, a function in continuous mathematics can be plotted in a smooth curve without breaks.
2. **Discrete Mathematics** − It involves distinct values; i.e. between any two points, there are a countable number of points. For example, if we have a finite set of objects, the function can be defined as a list of ordered pairs having these objects, and can be presented as a complete list of those pairs.

UNIT-1

Discrete Mathematics - Sets

German mathematician **G. Cantor** introduced the concept of sets. He had defined a set as a collection of definite and distinguishable objects selected by the means of certain rules or description.

Set theory forms the basis of several other fields of study like counting theory, relations, graph theory and finite state machines. In this chapter, we will cover the different aspects of **Set Theory**.

Set - Definition

A set is an unordered collection of different elements. A set can be written explicitly by listing its elements using set bracket. If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.

Some Example of Sets

1. A set of all positive integers
2. A set of all the planets in the solar system
3. A set of all the states in India
4. A set of all the lowercase letters of the alphabet

Representation of a Set

Sets can be represented in two ways −

1. Roster or Tabular Form
2. Set Builder Notation

1. Roster or Tabular Form

The set is represented by listing all the elements comprising it. The elements are enclosed within braces and separated by commas.

Example 1 − Set of vowels in English alphabet, $A=\{a,e,i,o,u\}$

Example 2 − Set of odd numbers less than 10, $B=\{1,3,5,7,9\}$

2. Set Builder Notation

The set is described by specifying a property that elements of the set have in common. The set is described as $A=\{x:p(x)\}$

Example 1 − The set {a,e,i,o,u}

is written as −

A={x:x is a vowel in English alphabet}

Example 2 − The set {1,3,5,7,9}

is written as −

$B=\{x:1\le x<10 \text{ and } (x\%2)\neq 0\}$

If an element x is a member of any set S, it is denoted by $x\in S$

and if an element y is not a member of set S, it is denoted by $y \notin S$.

Example 3− If $S=\{1,1.2,1.7,2\}, 1\in S$

but $1.5 \notin S$

Some Important Sets

N − the set of all natural numbers = {1,2,3,4,.....}

Z − the set of all integers = {.....,−3,−2,−1,0,1,2,3,.....}

Z⁺ − the set of all positive integers

Q − the set of all rational numbers

R − the set of all real numbers

W − the set of all whole numbers

<u>**Types of Sets**</u>

Sets can be classified into many types. Some of which are finite, infinite, subset, universal, proper, singleton set, etc.

1. **Finite Set-** A set which contains a definite number of elements is called a finite set.

Example − $S=\{x \mid x \in N \text{ and } 70>x>50\}$

2. **Infinite Set-** A set which contains infinite number of elements is called an infinite set.

Example − $S=\{x \mid x \in N \text{ and } x>10\}$

3. **Subset-** A set X is a subset of set Y (Written as $X \subseteq Y$) if every element of X is an element of set Y.

Example 1 − Let, $X=\{1,2,3,4,5,6\}$ and $Y=\{1,2\}$. Here set Y is a subset of set X as all the elements of set Y is in set X. Hence, we can write $Y \subseteq X$

Example 2 − Let, $X=\{1,2,3\}$ and $Y=\{1,2,3\}$. Here set Y is a subset (Not a proper subset) of set X as all the elements of set Y is in set X. Hence, we can write $Y \subseteq X$

4. **Proper Subset-** The term "proper subset" can be defined as "subset of but not equal to". A Set X is a proper subset of set Y (Written as $X \subset Y$) if every element of X is an element of set Y and $|X|<|Y|$

Example − Let, $X=\{1,2,3,4,5,6\}$ and $Y=\{1,2\}$. Here set $Y \subset X$ since all elements in Y are contained in X too and X has at least one element is more than set Y

5. **Universal Set-**It is a collection of all elements in a particular context or application. All the sets in that context or application are essentially subsets of this universal set. Universal sets are represented as U

Example − We may define U as the set of all animals on earth. In this case, set of all mammals is a subset of U, set of all fishes is a subset of U, set of all insects is a subset of U , and so on.

6. **Empty Set or Null Set-** An empty set contains no elements. It is denoted by ∅ . As the number of elements in an empty set is finite, empty set is a finite set. The cardinality of empty set or null set is zero.

Example − $S=\{x \mid x \in N \text{ and } 7<x<8\}=\emptyset$

7. **Singleton Set or Unit Set-** Singleton set or unit set contains only one element. A singleton set is denoted by $\{s\}$

Example − $S=\{x \mid x \in N, 7<x<9\} = \{8\}$

8. **Equal Set-** If two sets contain the same elements they are said to be equal.

Example − If $A=\{1,2,6\}$ and $B=\{6,1,2\}$ they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

9. **Equivalent Set-** If the cardinalities of two sets are same, they are called equivalent sets.

Example − If $A=\{1,2,6\}$ and $B=\{16,17,22\}$, they are equivalent as cardinality of A is equal to the cardinality of B. i.e. $|A|=|B|=3$

10. **Overlapping Set -** Two sets that have at least one common element are called overlapping sets.

In case of overlapping sets −

i. $n(A \cup B)=n(A)+n(B)-n(A \cap B)$
ii. $n(A \cup B)=n(A-B)+n(B-A)+n(A \cap B)$
iii. $n(A)=n(A-B)+n(A \cap B)$
iv. $n(B)=n(B-A)+n(A \cap B)$

Example − Let, $A=\{1,2,6\}$ and $B=\{6,12,42\}$

There is a common element '6', hence these sets are overlapping sets.

11. **Disjoint Set -** Two sets A and B are called disjoint sets if they do not have even one element in common. Therefore, disjoint sets have the following properties −

$n(A \cap B)=\emptyset$

$n(A \cup B)=n(A)+n(B)$

Example − Let, $A=\{1,2,6\}$ and $B=\{7,9,14\}$ there is not a single common element, hence these sets are overlapping sets.

Cardinality of a Set

Cardinality of a set S, denoted by $|S|$, is the number of elements of the set. The number is also referred as the cardinal number. If a set has an infinite number of elements, its cardinality is ∞.

Example − $|\{1,4,3,5\}|=4, |\{1,2,3,4,5,\ldots\}|=\infty$

If there are two sets X and Y,

i. $|X|=|Y|$ denotes two sets X and Y having same cardinality. It occurs when the number of elements in X is exactly equal to the number of elements in Y. In this case, there exists a bijective function 'f' from X to Y.
ii. $X|\le|Y$ denotes that set X's cardinality is less than or equal to set Y's cardinality. It occurs when number of elements in X is less than or equal to that of Y. Here, there exists an injective function 'f' from X to Y.
iii. $|X|<|Y|$ denotes that set X's cardinality is less than set Y's cardinality. It occurs when number of elements in X is less than that of Y. Here, the function 'f' from X to Y is injective function but not bijective.
iv. *If* $|X|\le|Y|$ and $|X|\le|Y|$ then $|X|=|Y|$ The sets X and Y are commonly referred as equivalent sets.

Venn Diagrams

Venn diagram, invented in 1880 by John Venn, is a schematic diagram that shows all possible logical relations between different mathematical sets.

Set Operations

Set Operations include Set Union, Set Intersection, Set Difference, Complement of Set, and Cartesian Product.

1. **Set Union -** The union of sets A and B (denoted by $A \cup B$) is the set of elements which are in A, in B, or in both A and B. Hence, $A \cup B=\{x|x\in A \text{ OR } x\in B\}$

Example − If $A=\{10,11,12,13\}$
and B = {13,14,15}, then $A \cup B=\{10,11,12,13,14,15\}$
(The common element occurs only once)

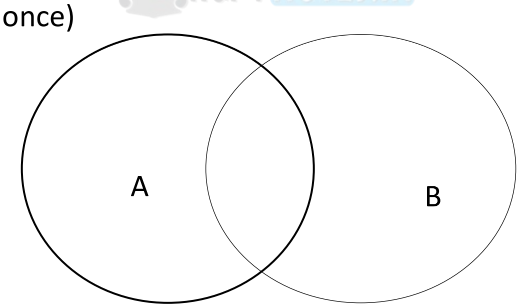

Figure 1.1 Union of two set

2. **Set Intersection -** The intersection of sets A and B (denoted by $A \cap B$) is the set of elements which are in both A and B. Hence, $A \cap B=\{x|x\in A \text{ AND } x\in B\}$

Example − If $A=\{11,12,13\}$ and $B=\{13,14,15\}$, then $A \cap B=\{13\}$

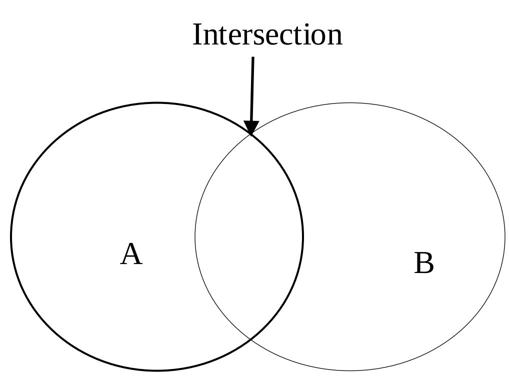

Figure 1.2 Intersection of two set

3. **Set Difference/ Relative Complement -** The set difference of sets A and B (denoted by $A–B$) is the set of elements which are only in A but not in B. Hence, $A−B=\{x|x∈A \text{ AND } x∉B\}$

Example − If $A=\{10,11,12,13\}$
and $B=\{13,14,15\}$, then $(A−B)=\{10,11,12\}$ and $(B−A)=\{14,15\}$. Here, we can see $(A−B)≠(B−A)$

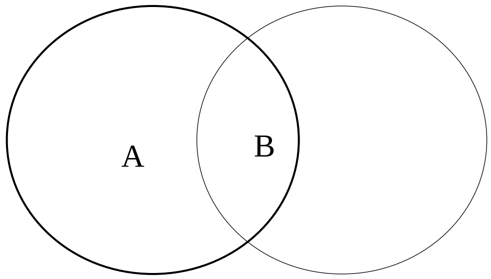

Figure 1.3 Set Difference of two set

4. **Complement of a Set -** The complement of a set A (denoted by $A′$) is the set of elements which are not in set A. Hence, $A′=\{x|x∉A\}$

More specifically, $A′=(U−A)$ where U is a universal set which contains all objects.
Example − If A={x|x belongs to set of odd integers} then A′={y|y does not belong to set of odd integers}

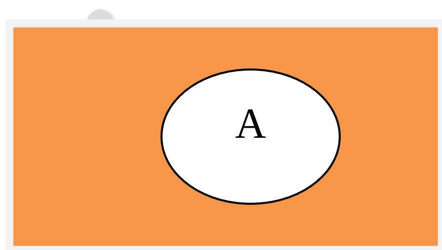

Figure 1.4 Complement of set

5. **Cartesian Product / Cross Product -** The Cartesian product of n number of sets $A1,A2,\ldots An$ denoted as $A1×A2\cdots×An$ can be defined as all possible ordered pairs $(x1,x2,\ldots xn)$ where $x1∈A1,x2∈A2,\ldots xn∈An$

Example − If we take two sets $A=\{a,b\}$ and $B=\{1,2\}$
The Cartesian product of A and B is written as − $A×B=\{(a,1),(a,2),(b,1),(b,2)\}$
The Cartesian product of B and A is written as − $B×A=\{(1,a),(1,b),(2,a),(2,b)\}$

6. **Power Set -** Power set of a set S is the set of all subsets of S including the empty set. The cardinality of a power set of a set S of cardinality n is $2n$. Power set is denoted as $P(S)$

Example –
For a set $S=\{a,b,c,d\}$
let us calculate the subsets −

i. Subsets with 0 elements − $\{∅\}$ (the empty set)
ii. Subsets with 1 element − $\{a\},\{b\},\{c\},\{d\}$
iii. Subsets with 2 elements − $\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}$
iv. Subsets with 3 elements − $\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}$
v. Subsets with 4 elements − $\{a,b,c,d\}$

Hence, $P(S)=\{\{∅\},\{a\},\{b\},\{c\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},\{a,b,c,d\}\}$

$|P(S)|=24=16$

Note – The power set of an empty set is also an empty set. $|P(\{\emptyset\})|=20=1$

Solved problems on union of sets:

1. Let A = {x : x is a natural number and a factor of 18} and B = {x : x is a natural number and less than 6}. Find A ∪ B.

Solution:

A = {1, 2, 3, 6, 9, 18}

B = {1, 2, 3, 4, 5}

Therefore, A ∪ B = {1, 2, 3, 4, 5, 6, 9, 18}

2. Let A = {0, 1, 2, 3, 4, 5}, B = {2, 4, 6, 8} and C = {1, 3, 5, 7} Verify (A ∪ B) ∪ C = A ∪ (B ∪ C)

Solution:

(A ∪ B) ∪ C = A ∪ (B ∪ C)

L.H.S. = (A ∪ B) ∪ C

A ∪ B = {0, 1, 2, 3, 4, 5, 6, 8}

(A ∪ B) ∪ C = {0, 1, 2, 3, 4, 5, 6, 7, 8} .... (1)

R.H.S. = A ∪ (B ∪ C)

B ∪ C = {1, 2, 3, 4, 5, 6, 7, 8}

A ∪ (B ∪ C) = {0, 1, 2, 3, 4, 5, 6, 7, 8} .... (2)

Therefore, from (1) and (2), we conclude that;

(A ∪ B) ∪ C = A ∪ (B ∪ C) [*verified*]

3. Let X = {1, 2, 3, 4}, Y = {2, 3, 5} and Z = {4, 5, 6}.

(i) Verify X ∪ Y = Y ∪ X **(ii) Verify (X ∪ Y) ∪ Z = X ∪ (Y ∪ Z)**

Solution:

(i) X ∪ Y = Y ∪ X

L.H.S = X ∪ Y

= {1, 2, 3, 4} ∪ {2, 3, 4} = {1, 2, 3, 4, 5}

R.H.S. = Y ∪ X

= {2, 3, 5} ∪ {1, 2, 3, 4} = {2, 3, 5, 1, 4}

Therefore, X ∪ Y = Y ∪ X [*verified*]

(ii) (X ∪ Y) ∪ Z = X ∪ (Y ∪ Z)

L.H.S. = (X ∪ Y) ∪ Z

X ∪ Y = {1, 2, 3, 4} U {2, 3, 5}

= {1, 2, 3, 4, 5}

Now (X ∪ Y) ∪ Z

= {1, 2, 3, 4, 5, 6} {4, 5, 6}

= {1, 2, 3, 4, 5, 6}

R.H.S. = X U (Y ∪ Z)

Y ∪ Z = {2, 3, 5} ∪ {4, 5, 6}

= {2, 3, 4, 5, 6}

X ∪ (Y ∪ Z) = {1, 2, 3, 4} ∪ {2, 3, 4, 5, 6}

Therefore, (X ∪ Y) ∪ Z = X ∪ (Y ∪ Z) [*verified*]

Solved problems on intersection of sets:

1. Let A = {x : x is a natural number and a factor of 18} B = {x : x is a natural number and less than 6}
Find A ∪ B and A ∩ B.

Solution:

A = {1, 2, 3, 6, 9, 18} B = {1, 2, 3, 4, 5}

Therefore, A ∩ B = {1, 2, 3}

2. If P = {multiples of 3 between 1 and 20} and Q = {even natural numbers upto 15}. Find the intersection of the two given set P and set Q.

Solution:

P = {multiples of 3 between 1 and 20} So, P = {3, 6, 9, 12, 15, 18}

Q = {even natural numbers upto 15} So, Q = {2, 4, 6, 8, 10, 12, 14}

Therefore, intersection of P and Q is the largest set containing only those elements which are common to both the given sets P and Q

Hence, P ∩ Q = {6, 12}.

3. Let A = {0, 1, 2, 3, 4, 5}, B = {2, 4, 6, 8} and C = {1, 3, 5, 7} Verify (A ∩ B) ∩ C = A ∩ (B ∩ C)

Solution:

(A ∩ B) ∩ C = A ∩ (B ∩ C)

L.H.S. = (A ∩ B) ∩ C

A ∩ B = {2, 4}

(A ∩ B) ∩ C = {∅} ………………. (1)

R.H.S. = A ∩ (B ∩ C)

B ∩ C = {∅}

A ∩ {B ∩ C} = {∅} ………………. (2)

Therefore, from (1) and (2), we conclude that;

(A ∩ B) ∩ C = A ∩ (B ∩ C) [*verified*]

4. Given three sets P, Q and R such that:

P = {x : x is a natural number between 10 and 16},

Q = {y : y is a even number between 8 and 20} and

R = {7, 9, 11, 14, 18, 20}

(i) Find the difference of two sets P and Q

(ii) Find Q - R

(iii) Find R - P

(iv) Find Q – P

Solution:

According to the given statements:

P = {11, 12, 13, 14, 15}

Q = {10, 12, 14, 16, 18}

R = {7, 9, 11, 14, 18, 20}

(i) P – Q = {Those elements of set P which are not in set Q}

= {11, 13, 15}

(ii) Q – R = {Those elements of set Q not belonging to set R}

= {10, 12, 16}

(iii) R – P = {Those elements of set R which are not in set P}

= {7, 9, 18, 20}

(iv) Q – P = {Those elements of set Q not belonging to set P}

= {10, 16, 18}

Proofs of Some General Identities on Sets-

Identities

1. **Commutative Laws:** $A \cap B = A \cap B$ and $A \cup B = B \cup A$
2. **Associative Laws:** $(A \cap B) \cap C = A \cap (B \cap C)$
   $(A \cup B) \cup C = A \cup (B \cup C)$
3. **Distributive Laws:** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. **Intersection and Union with universal set:** $A \cap U = A$ and $A \cup U = U$
5. **Double Complement Law:** $(A^c)^c = A$
6. **Idempotent Laws:** $A \cap A = A$ and $A \cup A = A$
7. **De Morgan's Laws:** $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$
8. **Absorption Laws:** $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$
9. **Alternate Representation for Difference:** $A - B = A \cap B^c$
10. **Intersection and Union with a subset:** if $A \subseteq B$, then $A \cap B = A$ and $A \cup B = B$

Some Proofs of Identities:

1. **$A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$**

Sol. $x\in A\cup(B\cap C)$

$\Leftrightarrow$ $x\in A \vee x\in(B\cap C)$

$\Leftrightarrow$ $x\in A \vee (x\in B \wedge x\in C)$

$\Leftrightarrow$ $(x\in A \vee x\in B) \wedge (x\in A \vee x\in C)$

(distributive law for logical expressions)

$\Leftrightarrow$ $x\in(A\cup B) \wedge x\in(A\cup C)$

$\Leftrightarrow$ $x\in(A\cup B)\cap(A\cup C)$

2. **$A \oplus B = (A - B) \cup (B - A)$**

Sol. $A \oplus B = \{ x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$

$= \{ x : (x \in A - B) \vee (x \in B - A)\}$

$= \{ x : x \in ((A - B) \cup (B - A))\}$

$= (A - B) \cup (B - A)$

3. **Show $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$.**

Sol. Assume $x\in A\cap(B\cup C)$, & show $x\in(A\cap B)\cup(A\cap C)$.

We know that $x\in A$, and either $x\in B$ or $x\in C$.

- Case 1: $x\in B$. Then $x\in A\cap B$, so $x\in(A\cap B)\cup(A\cap C)$.
- Case 2: $x\in C$. Then $x\in A\cap C$, so $x\in(A\cap B)\cup(A\cap C)$.

Therefore, $x\in(A\cap B)\cup(A\cap C)$.

Therefore, $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.

4. **$(A \cup B)^c = A^c \cap B^c$**

Sol. $(x \in \overline{A \cup B})$

$\Rightarrow (x \notin (A \cup B))$

$\Rightarrow (x \notin A$ and $x \notin B)$

$\Rightarrow (x \in \overline{A} \cap \overline{B})$

$(x \in (\overline{A} \cap \overline{B}))$

$\Rightarrow (x \notin A$ and $x \notin B)$

$\Rightarrow (x \notin A \cup B)$

$\Rightarrow (x \in \overline{A \cup B})$

Problems on Operation on Sets

1. If A = {1, 3, 5}, B = {3, 5, 6} and C = {1, 3, 7}

(i) Verify that A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

(ii) Verify A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

Solution:

(i) A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

L.H.S. = A ∪ (B ∩ C)

B ∩ C = {3}

A ∪ (B ∩ C) = {1, 3, 5} ∪ {3} = {1, 3, 5} .... (1)

R.H.S. = (A ∪ B) ∩ (A ∪ C)

A ∪ B = {1, 3, 5, 6}

A ∪ C = {1, 3, 5, 7}

(A ∪ B) ∩ (A ∪ C) = {1, 3, 5, 6} ∩ {1, 3, 5, 7} = {1, 3, 5} .... (2)

From (1) and (2), we conclude that;

A ∪ (B ∩ C) = A ∪ B ∩ (A ∪ C) [*verified*]

(ii) A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

L.H.S. = A ∩ (B ∪ C)

B ∪ C = {1, 3, 5, 6, 7}

A ∩ (B ∪ C) = {1, 3, 5} ∩ {1, 3, 5, 6, 7} = {1, 3, 5} .... (1)

R.H.S. = (A ∩ B) ∪ (A ∩ C)

A ∩ B = {3, 5}

A ∩ C = {1, 3}

(A ∩ B) ∪ (A ∩ C) = {3, 5} ∪ {1, 3} = {1, 3, 5} .... (2)

From (1) and (2), we conclude that;

A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C) [*verified*]

2. Let A = {a, b, d, e}, B = {b, c, e, f} and C = {d, e, f, g}

(i) Verify A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

(ii) Verify A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

Solution:

(i) A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

L.H.S. = A ∩ (B ∪ C)

B ∪ C = {b, c, d, e, f, g}

A ∩ (B ∪ C) = {b, d, e} .... (1)

R.H.S. = (A ∩ B) ∪ (A ∩ C)

A ∩ B = {b, e}

A ∩ C = {d, e}

(A ∩ B) ∪ (A ∩ C) = {b, d, e} .... (2)

From (1) and (2), we conclude that;

A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C) [*verified*]

(ii) A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

L.H.S. = A ∪ (B ∩ C)

B ∩ C = {e, f}

A ∪ (B ∩ C) = {a, b, d, e, f} .... (1)

R.H.S. = (A ∪ B) ∩ (A ∪ C)

$A \cup B = \{a, b, c, d, e, f\}$
$A \cup C = \{a, b, d, e, f, g\}$
$(A \cup B) \cap (A \cup C) = \{a, b, d, e, f\}$ ……………….. (2)
From (1) and (2), we conclude that;
$A \cup (B \cap C) = A \cup B \cap (A \cup C)$ [*verified*]

The Inclusion-Exclusion principle

The **Inclusion-exclusion principle** computes the cardinal number of the union of multiple non-disjoint sets. For two sets A and B, the principle states –

$$|A\cup B|=|A|+|B|-|A\cap B|$$

For three sets A, B and C, the principle states –

$$|A\cup B\cup C|=|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$$

The generalized formula –

$$|\bigcup ni=1Ai|=\sum 1\le i<j<k\le n|Ai\cap Aj|+\sum 1\le i<j<k\le n|Ai\cap Aj\cap Ak|-\cdots+(-1)\pi-1|A1\cap\cdots\cap A2|$$

Numerical on Sets:-

1. **Let A and B be two finite sets such that n(A) = 20, n(B) = 28 and n(A ∪ B) = 36, find n(A ∩ B).**

Solution:

Using the formula n(A ∪ B) = n(A) + n(B) - n(A ∩ B).
then n(A ∩ B) = n(A) + n(B) - n(A ∪ B)
= 20 + 28 - 36
= 48 - 36
= 12

2. If n(A - B) = 18, n(A ∪ B) = 70 and n(A ∩ B) = 25, then find n(B).

Solution:

Using the formula n(A∪B) = n(A - B) + n(A ∩ B) + n(B - A)
70 = 18 + 25 + n(B - A)
70 = 43 + n(B - A)
n(B - A) = 70 - 43
n(B - A) = 27
Now n(B) = n(A ∩ B) + n(B - A)
= 25 + 27
= 52

3. In a group of 60 people, 27 like cold drinks and 42 like hot drinks and each person likes at least one of the two drinks. How many like both coffee and tea?

Solution:

Let A = Set of people who like cold drinks.
B = Set of people who like hot drinks.
Given
(A ∪ B) = 60 n(A) = 27 n(B) = 42 then;
n(A ∩ B) = n(A) + n(B) - n(A ∪ B)
= 27 + 42 - 60
= 69 - 60 = 9
Therefore, 9 people like both tea and coffee.

4. There are 35 students in art class and 57 students in dance class. Find the number of students who are either in art class or in dance class.
(i) When two classes meet at different hours and 12 students are enrolled in both activities.
(ii) When two classes meet at the same hour.

Solution:

n(A) = 35, n(B) = 57, n(A ∩ B) = 12

(Let A be the set of students in art class.
B be the set of students in dance class.)

(i) When 2 classes meet at different hours $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 35 + 57 - 12$

$= 92 - 12$

$= 80$

(ii) When two classes meet at the same hour, $A \cap B = \emptyset$ $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= n(A) + n(B)$

$= 35 + 57$

$= 92$

5. In a group of 100 persons, 72 people can speak English and 43 can speak French. How many can speak English only? How many can speak French only and how many can speak both English and French?

Solution:

Let A be the set of people who speak English.
B be the set of people who speak French.
A - B be the set of people who speak English and not French.
B - A be the set of people who speak French and not English.
$A \cap B$ be the set of people who speak both French and English.

Given,

$n(A) = 72$ $n(B) = 43$ $n(A \cup B) = 100$

Now, $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= 72 + 43 - 100$

$= 115 - 100$

$= 15$

Therefore, Number of persons who speak both French and English = 15

$n(A) = n(A - B) + n(A \cap B)$

$\Rightarrow n(A - B) = n(A) - n(A \cap B)$

$= 72 - 15$

$= 57$

and $n(B - A) = n(B) - n(A \cap B)$

$= 43 - 15$

$= 28$

Therefore, Number of people speaking English only = 57

Number of people speaking French only = 28

6. In a competition, a school awarded medals in different categories. 36 medals in dance, 12 medals in dramatics and 18 medals in music. If these medals went to a total of 45 persons and only 4 persons got medals in all the three categories, how many received medals in exactly two of these categories?

Solution:

Let A = set of persons who got medals in dance.

B = set of persons who got medals in dramatics.
C = set of persons who got medals in music.

Given,

$n(A) = 36$ $n(B) = 12$ $n(C) = 18$

$n(A \cup B \cup C) = 45$ $n(A \cap B \cap C) = 4$

We know that number of elements belonging to exactly two of the three sets A, B, C

$= n(A \cap B) + n(B \cap C) + n(A \cap C) - 3n(A \cap B \cap C)$

$= n(A \cap B) + n(B \cap C) + n(A \cap C) - 3 \times 4$ ....(i)

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
Therefore, $n(A \cap B) + n(B \cap C) + n(A \cap C) = n(A) + n(B) + n(C) + n(A \cap B \cap C) - n(A \cup B \cup C)$
From (i) required number
$= n(A) + n(B) + n(C) + n(A \cap B \cap C) - n(A \cup B \cup C) - 12$
$= 36 + 12 + 18 + 4 - 45 - 12$
$= 70 - 67$
$= 3$

7. Each student in a class of 40 plays at least one indoor game chess, carrom and scrabble. 18 play chess, 20 play scrabble and 27 play carrom. 7 play chess and scrabble, 12 play scrabble and carrom and 4 play chess, carrom and scrabble. Find the number of students who play (i) chess and carrom. (ii) chess, carrom but not scrabble.

Solution:
Let A be the set of students who play chess
B be the set of students who play scrabble
C be the set of students who play carrom
Therefore, We are given $n(A \cup B \cup C) = 40$,
$n(A) = 18$, $n(B) = 20$ $n(C) = 27$,
$n(A \cap B) = 7$, $n(C \cap B) = 12$ $n(A \cap B \cap C) = 4$
We have
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$
Therefore, $40 = 18 + 20 + 27 - 7 - 12 - n(C \cap A) + 4$
$40 = 69 – 19 - n(C \cap A)$
$40 = 50 - n(C \cap A)$ $n(C \cap A) = 50 - 40$
$n(C \cap A) = 10$
Therefore, Number of students who play chess and carrom are 10.
Also, number of students who play chess, carrom and not scrabble
$= n(C \cap A) - n(A \cap B \cap C)$
$= 10 – 4$
$= 6$

Relations

Whenever sets are being discussed, the relationship between the elements of the sets is the next thing that comes up. **Relations** may exist between objects of the same set or between objects of two or more sets.

Definition and Properties

A binary relation R from set x to y (written as xRy or $R(x,y)$) is a subset of the Cartesian product $x \times y$. If the ordered pair of G is reversed, the relation also changes. Generally an n-ary relation R between sets $A1,\dots,$ *and* An is a subset of the n-ary product $A1\times\cdots\times An$. The minimum cardinality of a relation R is Zero and maximum is $n2$ in this case. A binary relation R on a single set A is a subset of $A\times A$
For two distinct sets, A and B, having cardinalities m and n respectively, the maximum cardinality of a relation R from A to B is mn.

Domain and Range

If there are two sets A and B, and relation R have order pair (x, y), then −

i. The **domain** of R, Dom(R), is the set $\{x | (x,y) \in R$ for some y in B$\}$
ii. The **range** of R, Ran(R), is the set $\{y | (x,y) \in R$ for some x in A$\}$

Examples Let, $A=\{1,2,9\}$ and $B=\{1,3,7\}$
Case 1 − If relation R is 'equal to' then $R=\{(1,1),(3,3)\}$
Dom(R) = $\{1,3\}$, $Ran(R)=\{1,3\}$

Case 2 – If relation R is 'less than' then $R=\{(1,3),(1,7),(2,3),(2,7)\}$
Dom(R) = {1,2}, $Ran(R)=\{3,7\}$
Case 3 – If relation R is 'greater than' then $R=\{(2,1),(9,1),(9,3),(9,7)\}$
Dom(R) = {2,9}, $Ran(R)=\{1,3,7\}$

Properties of Relation

1. **Reflexive :** A relation R on a set A is called **reflexive** if (a, a)∈R for every element a∈A. **Example :** Are the following relations on {1, 2, 3, 4} reflexive?
   i. R = {(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)} No
   ii. R = {(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)} Yes
   iii. R = {(1, 1), (2, 2), (3, 3)} No
2. **Irreflexive :** A relation on a set A is called **irreflexive** if (a, a)∉R for every element a∈A.
3. **Symmetric :** A relation R on a set A is called **symmetric** if (b, a)∈R whenever (a, b)∈R for all a, b∈A.
4. **Asymmetric :** A relation R on a set A is called **asymmetric** if (a, b)∈R implies that (b, a)∉R for all a, b∈A.
5. **Antisymmetric :** A relation R on a set A is called **antisymmetric** if a = b whenever (a, b)∈R and (b, a)∈R.

Types of Relations

1. The **Empty Relation** between sets X and Y, or on E, is the empty set ∅
2. The **Full Relation** between sets X and Y is the set $X\times Y$
3. The **Identity Relation** on set X is the set $\{(x,x)|x\in X\}$
4. The Inverse Relation R' of a relation R is defined as – $R'=\{(b,a)|(a,b)\in R\}$

Example – If $R=\{(1,2),(2,3)\}$ then R' will be $\{(2,1),(3,2)\}$

Composition of Relation

Definition: Let R be a relation from the set A to B and S be a relation from B to C, i.e. R ⊆ A×B and S ⊆B×C the composite of *R* and S is the relation consisting of ordered pairs (a,c) where a∈A, c∈C and for which there exists an element b∈B such that (a,b)∈R and (b,c)∈S. We denote the composite of R and S by R ∘ S

$(a,b)\in S\circ R \leftrightarrow \exists x:(a,x)\in R\wedge(x,b)\in S$

Note $(a,b)\in R\wedge(b,c)\in S\rightarrow(a,c)\in S\circ R$

Example $R=\{(1,1),(1,4),(2,3),(3,1),(3,4)\}$

$S=\{(1,0),(2,0),(3,1),(3,2),(4,1)\}$

$S\circ R=\{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$

Representation of Relations using Graph

A relation can be represented using a directed graph. The number of vertices in the graph is equal to the number of elements in the set from which the relation has been defined. For each ordered pair (x, y) in the relation R, there will be a directed edge from the vertex 'x' to vertex 'y'. If there is an ordered pair (x, x), there will be self- loop on vertex 'x'.

Suppose, there is a relation $R=\{(1,1),(1,2),(3,2)\}$ on set $S=\{1,2,3\}$, it can be represented by the following graph –

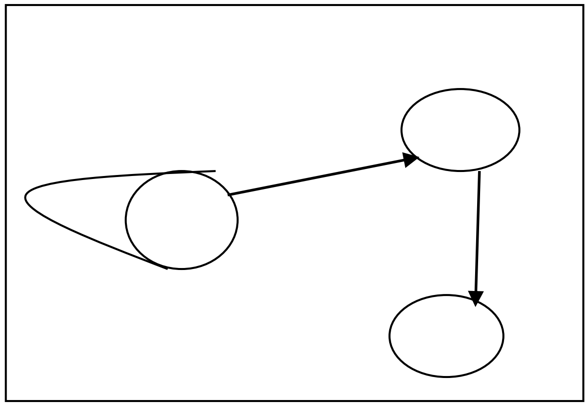

Figure 1.5 Relation as a graph

Equivalance Relation

Equivalence relation on set is a relation which is reflexive, symmetric and transitive.
A relation R, defined in a set A, is said to be an equivalence relation if and only if
(i) R is reflexive, that is, aRa for all a ∈ A.
(ii) R is symmetric, that is, aRb ⇒ bRa for all a, b ∈ A.
(iii) R is transitive, that is aRb and bRc ⇒ aRc for all a, b, c ∈ A.

Partial Order Relation

Partial order relation on set is a relation which is reflexive, antisymmetric and transitive.
A relation R, defined in a set A, is said to be an equivalence relation if and only if
(i) R is reflexive, that is, aRa for all a ∈ A.
(ii) R isantisymmetric if a = b whenever (a, b)∈R and (b, a)∈R

(iii) R is transitive, that is aRb and bRc ⇒ aRc for all a, b, c ∈ A.

A Job Scheduling Problem

We consider the problem of scheduling the execution of a set of a tasks on a multiprocessor computing system which has a set of identical processors.

Job Scheduling Problem means to make a schedule for a finite no. of workers to complete a given set of task.

Let T={T1 , T2 , T3 ----------------Tn } denote a set of tasks to be executed on the computing system. Suppose that the execution of a task occupies one and only one processor. Moreover since the processors are identical, a task can be executed on any one of the processors. Let P(Ti) denoted the execution time of task Ti , the amount of time it takes to execute Ti on a processor. Ti ≠ Tj , Ti ≤ Tj if and only if the execution of task Tj cannot begin until the execution of task Ti has been completed.

Functions

A **Function** assigns to each element of a set, exactly one element of a related set. Functions find their application in various fields like representation of the computational complexity of algorithms, counting objects, study of sequences and strings, to name a few. The third and final chapter of this part highlights the important aspects of functions.

Function - Definition

A function or mapping (Defined as $f:X\rightarrow Y$) is a relationship from elements of one set X to elements of another set Y (X and Y are non-empty sets). X is called Domain and Y is called Codomain of function 'f'. Function 'f' is a relation on X and Y such that for each $x\in X$, there exists a unique $y\in Y$ such that $(x,y)\in R$. 'x' is called pre-image and 'y' is called image of function f. A function can be one to one or many to one but not one to many.

1. **Injective / One-to-one function-** A function $f:A\rightarrow B$ is injective or one-to-one function if for every $b\in B$, there exists at most one $a\in A$ such that $f(s)=t$. This means a function **f** is injective if $a1\neq a2$ implies $f(a1)\neq f(a2)$

Example

a) $f:N\rightarrow N, f(x)=5x$ is injective.
b) $f:N\rightarrow N, f(x)=x2$ is injective.
c) $f:R\rightarrow R, f(x)=x2$ is not injective as $(-x)2=x2$

2. **Surjective / Onto function -** A function $f:A\rightarrow B$ is surjective (onto) if the image of f equals its range. Equivalently, for every $b\in B$, there exists some $a\in A$ such that $f(a)=b$. This means that for any y in B, there exists some x in A such that $y=f(x)$

Example

a) $f:N\rightarrow N, f(x)=x+2$ is surjective.
b) $f:R\rightarrow R, f(x)=x2$ is not surjective since we cannot find a real number whose square is negative.

3. **Bijective / One-to-one Correspondent** - A function $f:A\rightarrow B$ is bijective or one-to-one correspondent if and only if **f** is both injective and surjective.

Problem - Prove that a function $f:R\rightarrow R$ defined by $f(x)=2x-3$ is a bijective function.

Explanation − We have to prove this function is both injective and surjective.

If $f(x1)=f(x2)$, then $2x1-3=2x2-3$ and it implies that $x1=x2$. Hence, f is **injective**.

Here, $2x-3=y$

So, $x=(y+5)/3$ which belongs to R and $f(x)=y$. Hence, f is **surjective**.

Since **f** is both **surjective** and **injective**, we can say **f** is **bijective**.

4. **Inverse of a Function -** The **inverse** of a one-to-one corresponding function $f:A\rightarrow B$, is the function $g:B\rightarrow A$, holding the following property − $f(x)=y\Leftrightarrow g(y)=x$ The function f is called **invertible**, if its inverse function g exists.

Example

A Function $f:Z\rightarrow Z, f(x)=x+5$, is invertible since it has the inverse function $g:Z\rightarrow Z, g(x)=x-5$.

A Function $f:Z\rightarrow Z, f(x)=x2$ is not invertiable since this is not one-to-one as $(-x)2=x2$

<u>**Composition of Functions**</u>

Two functions $f:A\rightarrow B$ and $g:B\rightarrow C$ can be composed to give a composition *gof*. This is a function from A to C defined by $(gof)(x)=g(f(x))$

Example

Let $f(x)=x+2$

and $g(x)=2x$, find $(fog)(x)$ and $(gof)(x)$

.

Solution

$(fog)(x)=f(g(x))=f(2x+1)=2x+1+2=2x+3$

$(gof)(x)=g(f(x))=g(x+2)=2(x+2)+1=2x+5$

Hence, $(fog)(x)\neq(gof)(x)$

Some Facts about Composition

a) If f and g are one-to-one then the function (*gof*) is also one-to-one.
b) If f and g are onto then the function (*gof*) is also onto.
c) Composition always holds associative property but does not hold commutative property.

Example : Let f:R->R be defined by f(x)= 2x+1, x<=0; x^2+1 ,x>0

Let g:R->R be defined by g(x)=3x-7 , x<=0; x^3, x>0

Then find the composition gof. :

Solution:

Let x = -2,-1,0,1,2,3…….

For x =-2,-1,0	for x= 1,2,3….
F(x)=2x+1	f(x)= x^2+1
F(-2)= -3	f(1)=2
F(-1)= 0	f(2)=5
F(0)= 1	f(3)= 10

Let x = -2,-1,0,1,2,3…….

For x =-2,-1,0	for x= 1,2,3….
G(x)=3x-7	g(x)=x^3
G(-1)=-10	g(1)=1
G(-2)=-13	g(2)=8
G(0)=-7	g(3)=27

Gof=g[f(x)]

G[f(-2)]=g(-3)=-16

G[f(-1)] =g(0)=-7

G[f(0)] =g(1)=1

G[f(1)] =g(2)=8

G[f(2)] =g(5)=125

G[f(3)] =g(10)=1000…………

Pigeonhole principle

If the number of pigeon is more than the number of pigeonholes, then some pigeonhole must be occupied by two or more than two pigeons. This statement is called the Pigeon hole principle, it is also called Dirchlet Drawer Principle. This statement is also written as "**If n pigeonholes are occupied by n + 1 or more pigeons, then at least one pigeonhole is occupied by more than one pigeon**".

Example 1 Among 13 people there are two who have their birthdays in the same month.

Example 2 A basket of fruit is being arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruit that should be put in the basket in order to guarantee that either there are at least 8 apples or at least 6 bananas or at least 9 oranges? Answer: 8 + 6 + 9 − 3 + 1 = 21.

Mathematical Induction

Mathematical induction, is a technique for proving results or establishing statements for natural numbers. This part illustrates the method through a variety of examples.

Definition Let P(n) be a mathematical statement about nonnegative integers n and n be a fixed nonnegative integer.

(1) Suppose **P(n_0)** is true i.e.. P(n) is true for n = **n_0**.

2) Whenever **k** is an integer such that **k ≥ n_0 and P(k)** is **true**, then **P(k + 1)** is true.

Then **P(n)** is true for all integers n ≥ n_0.

We note that a proof by mathematical induction consists of three steps.

Step 1. (Basis) Show that P(n_0) is true.

Step 2. (Inductive hypothesis). Write the inductive hypothesis: Let k be an integer such that k ≥ n_0 and P(k) be true.

Step 3. (Inductive step). Show that P(k + 1) is true.

Numericals on Mathematical Induction

1. Using the principle of mathematical induction, prove that

$1^2 + 2^2 + 3^2 + ... + n^2 = (1/6)\{n(n + 1)(2n + 1\}$ for all n ∈ N.

Solution : Let the given statement be P(n). Then,

P(n): $1^2 + 2^2 + 3^2 + ... + n^2 = (1/6)\{n(n + 1)(2n + 1)\}$.

Putting n =1 in the given statement, we get

LHS = $1^2 = 1$ and RHS = $(1/6) \times 1 \times 2 \times (2 \times 1 + 1) = 1$.

Therefore LHS = RHS.

Thus, P(1) is true.
Let P(k) be true. Then,

P(k): $1^2 + 2^2 + 3^2 + ... + k^2 = (1/6)\{k(k + 1)(2k + 1)\}$.

Now, $1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2$

$= (1/6) \{k(k + 1)(2k + 1) + (k + 1)^2$

$= (1/6)\{(k + 1).(k(2k + 1)+6(k + 1))\}$

$= (1/6)\{(k + 1)(2k^2 + 7k + 6\})$

$= (1/6)\{(k + 1)(k + 2)(2k + 3)\}$

$= 1/6\{(k + 1)(k + 1 + 1)[2(k + 1) + 1]\}$

$\Rightarrow$ P(k + 1): $1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2$

$= (1/6)\{(k + 1)(k + 1 + 1)[2(k + 1) + 1]\}$

$\Rightarrow$ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all $n \in N$.

2. Using the principle of mathematical induction, prove that

$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + n(n + 1) = (1/3)\{n(n + 1)(n + 2)\}$.

Solution : Let the given statement be P(n). Then,

P(n): $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + n(n + 1) = (1/3)\{n(n + 1)(n + 2)\}$.

Thus, the given statement is true for n = 1, i.e., P(1) is true.

Let P(k) be true. Then,

P(k): $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + k(k + 1) = (1/3)\{k(k + 1)(k + 2)\}$.

Now, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 +...+ k(k + 1) + (k + 1)(k + 2)$

$= (1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + k(k + 1)) + (k + 1)(k + 2)$

$= (1/3)\, k(k + 1)(k + 2) + (k + 1)(k + 2)$ [using (i)]

$= (1/3)\, [k(k + 1)(k + 2) + 3(k + 1)(k + 2)$

$= (1/3)\{(k + 1)(k + 2)(k + 3)\}$

$\Rightarrow$ P(k + 1): $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 +......+ (k + 1)(k + 2)$

$= (1/3)\{k + 1)(k + 2)(k +3)\}$

$\Rightarrow$ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1)is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all values of $\in$ N.

3. Using the principle of mathematical induction, prove that

$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 +.....+ (2n - 1)(2n + 1) = (1/3)\{n(4n^2 + 6n - 1)$.

Solution : Let the given statement be P(n). Then,

P(n): $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 +...... + (2n - 1)(2n + 1)= (1/3)n(4n^2 + 6n - 1)$.

When n = 1, LHS = $1 \cdot 3 = 3$ and RHS = $(1/3) \times 1 \times (4 \times 1^2 + 6 \times 1 - 1)$

$= \{(1/3) \times 1 \times 9\} = 3$.

LHS = RHS.

Thus, P(1) is true.

Let P(k) be true. Then,

P(k): $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + ... + (2k - 1)(2k + 1) = (1/3)\{k(4k^2 + 6k - 1)$ ......(i)

Now,

$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + ... + (2k - 1)(2k + 1) + \{2k(k + 1) - 1\}\{2(k + 1) + 1\}$

$= \{1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + ... + (2k - 1)(2k + 1)\} + (2k + 1)(2k + 3)$

$= (1/3)\, k(4k^2 + 6k - 1) + (2k + 1)(2k + 3)$ [using (i)]

$= (1/3)\ [(4k^3 + 6k^2 - k) + 3(4k^2 + 8k + 3)]$

$= (1/3)(4k^3 + 18k^2 + 23k + 9)$

$= (1/3)\{(k + 1)(4k^2 + 14k + 9)\}$

$= (1/3)[k + 1)\{4k(k + 1)^2 + 6(k + 1) - 1\}]$

$\Rightarrow$ P(k + 1): $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + ... + (2k + 1)(2k + 3)$

$= (1/3)[(k + 1)\{4(k + 1)^2 + 6(k + 1) - 1)\}]$

$\Rightarrow$ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all $n \in N$.

4. Using the principle of mathematical induction, prove that

$1/(1 \cdot 2) + 1/(2 \cdot 3) + 1/(3 \cdot 4) + ... + 1/\{n(n + 1)\} = n/(n + 1)$

Solution : Let the given statement be P(n). Then,

P(n): $1/(1 \cdot 2) + 1/(2 \cdot 3) + 1/(3 \cdot 4) + ... + 1/\{n(n + 1)\} = n/(n + 1)$.

Putting n = 1 in the given statement, we get

LHS= $1/(1 \cdot 2)$ = and RHS = $1/(1 + 1) = 1/2$.

LHS = RHS.

Thus, P(1) is true.

Let P(k) be true. Then,

P(k): $1/(1 \cdot 2) + 1/(2 \cdot 3) + 1/(3 \cdot 4) + ... + 1/\{k(k + 1)\} = k/(k + 1)$ .....(i)

Now $1/(1 \cdot 2) + 1/(2 \cdot 3) + 1/(3 \cdot 4) + ... + 1/\{k(k + 1)\} + 1/\{(k + 1)(k + 2)\}$

$[1/(1 \cdot 2) + 1/(2 \cdot 3) + 1/(3 \cdot 4) + ... + 1/\{k(k + 1)\}] + 1/\{(k + 1)(k + 2)\}$

$= k/(k + 1)+1/\{ (k + 1)(k + 2)\}$.

$\{k(k + 2) + 1\}/\{(k + 1)^2/[(k + 1)k + 2)]$ using ...(ii)

$= \{k(k + 2) + 1\}/\{(k + 1)(k + 2\}$

$= \{(k + 1)^2\}/\{(k + 1)(k + 2)\}$

$= (k + 1)/(k + 2) = (k + 1)/(k + 1 + 1)$

$\Rightarrow P(k + 1): 1/(1 \cdot 2) + 1/(2 \cdot 3) + 1/(3 \cdot 4) + \dots\dots\dots + 1/\{k(k + 1)\} + 1/\{(k + 1)(k + 2)\}$

$= (k + 1)/(k + 1 + 1)$

⇒ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1)is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all $n \in N$.

5. Using the principle of mathematical induction, prove that

$\{1/(3 \cdot 5)\} + \{1/(5 \cdot 7)\} + \{1/(7 \cdot 9)\} + \dots\dots + 1/\{(2n + 1)(2n + 3)\} = n/\{3(2n + 3)\}$.

Solution : Let the given statement be P(n). Then,

$P(n): \{1/(3 \cdot 5) + 1/(5 \cdot 7) + 1/(7 \cdot 9) + \dots\dots + 1/\{(2n + 1)(2n + 3)\} = n/\{3(2n + 3)$.

Putting n = 1 in the given statement, we get

and LHS = $1/(3 \cdot 5) = 1/15$ and RHS = $1/\{3(2 \times 1 + 3)\} = 1/15$.

LHS = RHS

Thus , P(1) is true.

Let P(k) be true. Then,

$P(k): \{1/(3 \cdot 5) + 1/(5 \cdot 7) + 1/(7 \cdot 9) + \dots\dots\dots + 1/\{(2k + 1)(2k + 3)\} = k/\{3(2k + 3)\}$ ..... (i)

Now, $1/(3 \cdot 5) + 1/(5 \cdot 7) + \dots\dots\dots + 1/[(2k + 1)(2k + 3)] + 1/[\{2(k + 1) + 1\}2(k + 1) + 3$

$= \{1/(3 \cdot 5) + 1/(5 \cdot 7) + \dots\dots + [1/(2k + 1)(2k + 3)]\} + 1/\{(2k + 3)(2k + 5)\}$

$= k/[3(2k + 3)] + 1/[2k + 3)(2k + 5)]$ [using (i)]

$= \{k(2k + 5) + 3\}/\{3(2k + 3)(2k + 5)\}$

$= (2k^2 + 5k + 3)/[3(2k + 3)(2k + 5)]$

$= \{(k + 1)(2k + 3)\}/\{3(2k + 3)(2k + 5)\}$

$= (k + 1)/\{3(2k + 5)\}$

$= (k + 1)/[3\{2(k + 1) + 3\}]$

= P(k + 1): 1/(3 · 5) + 1/(5 · 7) + …….. + 1/[2k + 1)(2k + 3)] + 1/[{2(k + 1) + 1}{2(k + 1) + 3}]

= (k + 1)/{3{2(k + 1) + 3}]

⇒ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for n ∈ N.

6. Using the principle of mathematical induction, prove that

1/(1 · 2 · 3) + 1/(2 · 3 · 4) + …….. + 1/{n(n + 1)(n + 2)} = {n(n + 3)}/{4(n + 1)(n + 2)} for all n ∈ N.

Solution : Let P (n): 1/(1 · 2 · 3) + 1/(2 · 3 · 4) + ……. + 1/{n(n + 1)(n + 2)} = {n(n + 3)}/{4(n + 1)(n + 2)} .
Putting n = 1 in the given statement, we get

LHS = 1/(1 · 2 · 3) = 1/6 and RHS = {1 × (1 + 3)}/[4 × (1 + 1)(1 + 2)] = (1 × 4)/(4 × 2 × 3) = 1/6.

Therefore LHS = RHS.

Thus, the given statement is true for n = 1, i.e., P(1) is true.

Let P(k) be true. Then,

P(k): 1/(1 · 2 · 3) + 1/(2 · 3 · 4) + ……… + 1/{k(k + 1)(k + 2)} = {k(k + 3)}/{4(k + 1)(k + 2)}. …….(i)

Now, 1/(1 · 2 · 3) + 1/(2 · 3 · 4) + ………….. + 1/{k(k + 1)(k + 2)} + 1/{(k + 1)(k + 2)(k + 3)}

= [1/(1 · 2 · 3) + 1/(2 · 3 · 4) + ………..…. + 1/{ k(k + 1)(k + 2}] + 1/{(k + 1)(k + 2)(k + 3)}

= [{k(k + 3)}/{4(k + 1)(k + 2)} + 1/{(k + 1)(k + 2)(k + 3)}]
[using(i)]

= {k(k + 3)^2 + 4}/{4(k + 1)(k + 2)(k + 3)}

= (k^3 + 6k^2 + 9k + 4)/{4(k + 1)(k + 2)(k + 3)}

= {(k + 1)(k + 1)(k + 4)}/{4 (k + 1)(k + 2)(k + 3)}

= {(k + 1)(k + 4)}/{4(k + 2)(k + 3)

⇒ P(k + 1): 1/(1 · 2 · 3) + 1/(2 · 3 · 4) + ………….. + 1/{(k + 1)(k + 2)(k + 3)}

= {(k + 1)(k + 2)}/{4(k + 2)(k + 3)}

⇒ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all n ∈ N.

7. Using the Principle of mathematical induction, prove that

{1 - (1/2)}{1 - (1/3)}{1 - (1/4)} .... {1 - 1/(n + 1)} = 1/(n + 1) for all n ∈ N.

Solution : Let the given statement be P(n). Then,

P(n): {1 - (1/2)}{1 - (1/3)}{1 - (1/4)} .... {1 - 1/(n + 1)} = 1/(n + 1).

When n = 1, LHS = {1 – (1/2)} = ½ and RHS = 1/(1 + 1) = ½.

Therefore LHS = RHS.

Thus, P(1) is true.

Let P(k) be true. Then,

P(k): {1 - (1/2)}{1 - (1/3)}{1 - (1/4)} .... [1 - {1/(k + 1)}] = 1/(k + 1)

Now, [{1 - (1/2)}{1 - (1/3)}{1 - (1/4)} .... [1 - {1/(k + 1)}] · [1 – {1/(k + 2)}]

= [1/(k + 1)] · [{(k + 2) - 1}/(k + 2)}]

= [1/(k + 1)] · [(k + 1)/(k + 2)]

= 1/(k + 2)

Therefore p(k + 1): [{1 - (1/2)}{1 - (1/3)}{1 - (1/4)} .... [1 - {1/(k + 1)}] = 1/(k + 2)

⇒ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all n ∈ N.

8. Using the principle of mathematical induction, prove that
a + ar + ar^2 + ... + ar^{n-1} = (ar^{n-1})/(r - 1) for r > 1 and all n ∈ N.
Solution : Let the given statement be P(n). Then,

P(n): a + ar + ar^2 + ... +ar^{n-1} = {a(r^n -1)}/(r - 1).

When n = 1, LHS = a and RHS = {a(r^1 - 1)}/(r - 1) = a

Therefore LHS = RHS.

Thus, P(1) is true.

Let P(k) be true. Then,

P(k): $a + ar + ar^2 + …… + ar^{k-1} = \{a(r^k - 1)\}/(r - 1)$

Now, $(a + ar + ar^2 + …… + ar^{k-1}) + ar^k = \{a(r^k - 1)\}/(r - 1) + ar^2$ .... [using(i)]

$= a(r^{k+1} - 1)/(r - 1)$.

Therefore,
P(k + 1): $a + ar + ar^2 + …….. +ar^{k-1} + ar^k = \{a(r^{k+1} - 1)\}/(r - 1)$

⇒ P(k + 1)is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1) is true, whenever P(k) is true.
Hence, by the principle of mathematical induction, P(n) is true for all n ∈ N.

9. Let a and b be arbitrary real numbers. Using the principle of mathematical induction, prove that $(ab)^n = a^nb^n$ for all n ∈ N.
Solution : Let the given statement be P(n). Then,

P(n): $(ab)^n = a^nb^n$.
When = 1, LHS = $(ab)^1$ = ab and RHS = a^1b^1 = ab
Therefore LHS = RHS.
Thus, the given statement is true for n = 1, i.e., P(1) is true.

Let P(k) be true. Then,

P(k): $(ab)^k = a^kb^k$.

Now, $(ab)^{k+1} = (ab)^k (ab)$

$= (a^kb^k)(ab)$ [using (i)]

$= (a^k \cdot a)(b^k \cdot b)$ [by commutativity and associativity of multiplication on real numbers]

$= (a^{k+1} \cdot b^{k+1})$.

Therefore P(k+1): $(ab)^{k+1} = ((a^{k+1} \cdot b^{k+1})$

⇒ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all n ∈ N.

10. Using the principle of mathematical induction, prove that $(x^n - y^n)$ is divisible by (x - y)for all n ∈ N.
Solution : Let the given statement be P(n). Then,

P(n): $(x^n - y^n)$ is divisible by $(x - y)$.

When n = 1, the given statement becomes: $(x^1 - y^1)$ is divisible by $(x - y)$, which is clearly true.

Therefore P(1) is true.

Let p(k) be true. Then,

P(k): $x^k - y^k$ is divisible by (x-y).

Now, $x^{k+1} - y^{k+1} = x^{k+1} - x^k y - y^{k+1}$

[on adding and subtracting $x)^k y$]

$= x^k(x - y) + y(x^k - y^k)$, which is divisible by $(x - y)$ [using (i)]

⇒ P(k + 1): $x^{k+1} - y^{k+1}$is divisible by $(x - y)$

⇒ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1) is true, whenever P(k) is true.

Hence, by the Principal of Mathematical Induction, P(n) is true for all n ∈ N.

11. Using the principle of mathematical induction, prove that $(10^{2n-1} + 1)$ is divisible by 11 for all n ∈ N.
Solution : Let P (n): $(10^{2n-1} + 1)$ is divisible by 11.

For n=1, the given expression becomes $\{10^{(2 \times 1 - 1)} + 1\} = 11$, which is divisible by 11.

So, the given statement is true for n = 1, i.e., P (1) is true.

Let P(k) be true. Then,

P(k): $(10^{2k-1} + 1)$ is divisible by 11

⇒ $(10^{2k-1} + 1) = 11$ m for some natural number m.

Now, $\{10^{2(k-1)-1} + 1\} = (10^{2k+1} + 1) = \{10^2 \cdot 10^{(2k-1)} + 1\}$

$= 100 \times \{10^{2k-1} + 1\} - 99$

$= (100 \times 11m) - 99$

$= 11 \times (100m - 9)$, which is divisible by 11

⇒ P (k + 1): $\{10^{2(k+1)} - 1 + 1\}$ is divisible by 11

⇒ P (k + 1) is true, whenever P(k) is true.

Thus, P (1) is true and P(k + 1) is true , whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all n ∈ N.

12. Using the principle if mathematical induction, prove that (7n – 3n) is divisible by 4 for all n ∈ N.
Solution : Let P(n) : ($7^n – 3^n$) is divisible by 4.
For n = 1, the given expression becomes (7^1 - 3^1) = 4, which is divisible by 4.
So, the given statement is true for n = 1, i.e., P(1) is true.
Let P(k) be true. Then,

P(k): (7^k - 3^k) is divisible by 4.

⇒ (7^k - 3^k) = 4m for some natural number m.

Now, {$7^{(k + 1)}$ - 3 (k + 1)} = $7^{(k + 1)} – 7 \cdot 3^k + 7 \cdot 3^k - 3^{(k + 1)}$
(on subtracting and adding 7 · 3k)

= $7(7^k - 3^k) + 3^k (7 - 3)$

= (7 × 4m) + 4 · 3k

= $4(7m + 3^k)$, which is clearly divisible by 4.

∴ P(k + 1): {$7^{(k + 1)}$ - 3 (k + 1)} is divisible by 4.

⇒ P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all n ∈ N.

13. Using the principle if mathematical induction, prove that
($2 \cdot 7^n + 3 \cdot 5^n - 5$) is divisible by 24 for all n ∈ N.
Solution : Let P(n): ($2 \cdot 7^n + 3 \cdot 5^n - 5$) is divisible by 24.
For n = 1, the given expression becomes ($2 \cdot 7^1 + 3 \cdot 5^1 - 5$) = 24, which is clearly divisible by 24.

So, the given statement is true for n = 1, i.e., P(1) is true.

Let P(k) be true. Then,

P(k): ($2 \cdot 7^n + 3 \cdot 5^n - 5$) is divisible by 24.

⇒ ($2 \cdot 7^n + 3 \cdot 5^n - 5$) = 24m, for m = N

Now, ($2 \cdot 7^n + 3 \cdot 5^n - 5$)

= ($2 \cdot 7^k \cdot 7 + 3 \cdot 5^k \cdot 5 - 5$)

= $7(2 \cdot 7^k + 3 \cdot 5^k - 5) = 6 \cdot 5^k + 30$

$= (7 \times 24m) - 6(5^k - 5)$

$= (24 \times 7m) - 6 \times 4p$, where $(5^k - 5) = 5(5^{k-1} - 1) = 4p$
[Since $(5^{k-1} - 1)$ is divisible by $(5 - 1)$]

$= 24 \times (7m - p)$

$= 24r$, where $r = (7m - p) \in N$

⇒ P (k + 1): $(2 \cdot 7^k + 13 \cdot 5^k + 1 - 5)$ is divisible by 24.

⇒ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all $n \in N$.

14. Using the principle of mathematical induction, prove that n(n + 1)(n + 5) is a multiple of 3 for all n ∈ N.
Solution : Let P(n): n(n + 1)(n + 5) is a multiple of 3.

For n = 1, the given expression becomes $(1 \times 2 \times 6) = 12$, which is a multiple of 3.

So, the given statement is true for n = 1, i.e. P(1) is true.

Let P(k) be true. Then,

P(k): k(k + 1)(k + 5) is a multiple of 3

⇒ K(k + 1)(k + 5) = 3m for some natural number m, ... (i)

Now, (k + 1l)(k + 2)(k + 6) = (k + 1)(k + 2)k + 6(k + 1)(k + 2)

= k(k + 1)(k + 2) + 6(k + 1)(k + 2)

= k(k + 1)(k + 5 – 3) + 6(k + 1)(k + 2)

= k(k + 1)(k + 5) – 3k(k + 1) + 6(k + 1)(k + 2)

= k(k + 1)(k + 5) + 3(k + 1)(k +4) [on simplification]

= 3m + 3(k + 1)(k + 4) [using (i)]

= 3[m + (k + 1)(k + 4)], which is a multiple of 3

⇒ P(k + 1): (k + 1)(k + 2)(k + 6) is a multiple of 3

⇒ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all n ∈ N.

15. Using the principle of mathematical induction, prove that $(n^2 + n)$ is even for all n ∈ N.
Solution : Let P(n): $(n^2 + n)$ is even.

For n = 1, the given expression becomes $(1^2 + 1) = 2$, which is even.

So, the given statement is true for n = 1, i.e., P(1)is true.

Let P(k) be true. Then,

P(k): $(k^2 + k)$ is even

⇒ $(k^2 + k) = 2m$ for some natural number m. ... (i)

Now, $(k + 1)^2 + (k + 1) = k^2 + 3k + 2$

$= (k^2 + k) + 2(k + 1)$

$= 2m + 2(k + 1)$ [using (i)]

$= 2[m + (k + 1)]$, which is clearly even.

Therefore, P(k + 1): $(k + 1)^2 + (k + 1)$ is even

⇒ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n)is true for all n ∈ N.