Program : **B.Tech**

Subject Name: **Discrete Structure**

Subject Code: **CS-302**

Semester: **3rd**

UNIT-5

Partially Ordered Set (POSET) - A partially ordered set consists of a set with a binary relation which is reflexive, antisymmetric and transitive. "Partially ordered set" is abbreviated as POSET.

Examples

1. **The set of real numbers under binary operation less than or equal to (≤) is a poset.**

Solution - Let the set $S=\{1,2,3\}$ and the operation is ≤

The relations will be $\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,3)\}$

This relation R is reflexive as $\{(1,1),(2,2),(3,3)\}\in R$

This relation R is anti-symmetric, as

$\{(1,2),(1,3),(2,3)\}\in R$ *and* $\{(1,2),(1,3),(2,3)\}\notin R$

This relation R is also transitive as $\{(1,2),(2,3),(1,3)\}\in R$

Hence, it is a **poset**.

The vertex set of a directed acyclic graph under the operation 'reachability' is a poset.

Hasse Diagram - The Hasse diagram of a poset is the directed graph whose vertices are the element of that poset and the arcs covers the pairs (x, y) in the poset. If in the poset $x<y$, then the point x appears lower than the point y in the Hasse diagram. If $x<y<z$ in the poset, then the arrow is not shown between x and z as it is implicit.

Example

The poset of subsets of $\{1,2,3\}=\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$ is shown by the following Hasse diagram –

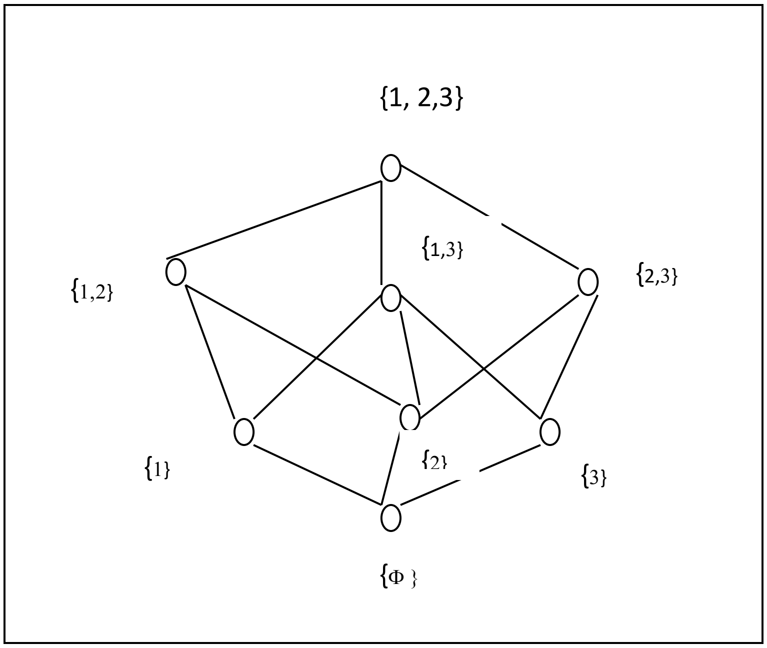

Figure 5.1 Hasse Diagram

Linearly Ordered Set - A Linearly ordered set or Total ordered set is a partial order set in which every pair of element is comparable. The elements $a,b \in S$ are said to be comparable if either $a \le b$ or $b \le a$ holds. Trichotomy law defines this total ordered set. A totally ordered set can be defined as a distributive lattice having the property $\{a \vee b, a \wedge b\} = \{a,b\}$ for all values of a and b in set S.

Example

The powerset of $\{a,b\}$ ordered by \subseteq is a totally ordered set as all the elements of the power set $P=\{\emptyset,\{a\},\{b\},\{a,b\}\}$ are comparable.

Example of non-total order set

A set *S*={1,2,3,4,5,6} under operation x divides y is not a total ordered set.

Here, for all $(x,y) \in S, x|y$ have to hold but it is not true that 2 | 3, as 2 does not divide 3 or 3 does not divide 2. Hence, it is not a total ordered set.

Isomorphic Ordered Set - Two partially ordered sets are said to be isomorphic if their "structures" are entirely analogous. Formally, partially ordered sets $P=(X, \le)$ and $Q=(X', \le')$ are isomorphic if there is a bijection f from X to X' such that $x_1 \le x_2$ precisely when $f(x_1) \le' f(x_2)$.

Well Ordered Set - A well-ordered set is a totally ordered set in which every nonempty subset has a least member.

Lattice - A lattice is a poset $(L,\leq)$ for which every pair $\{a,b\}\in L$ has a least upper bound (denoted by $a\vee b$) and a greatest lower bound (denoted by $a\wedge b$). LUB ($\{a,b\}$) is called the join of a and b. GLB ($\{a,b\}$) is called the meet of a and b.

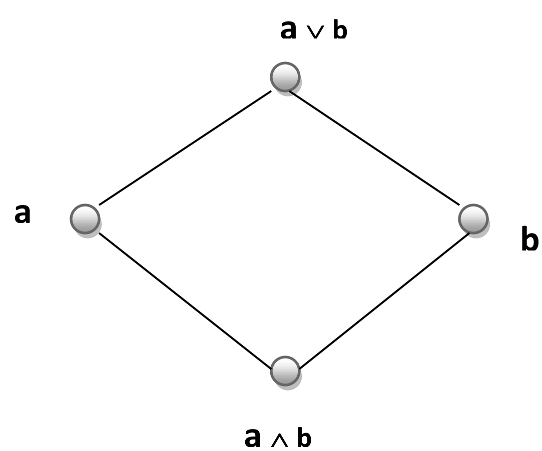

Figure 5.2 Lattice

Example

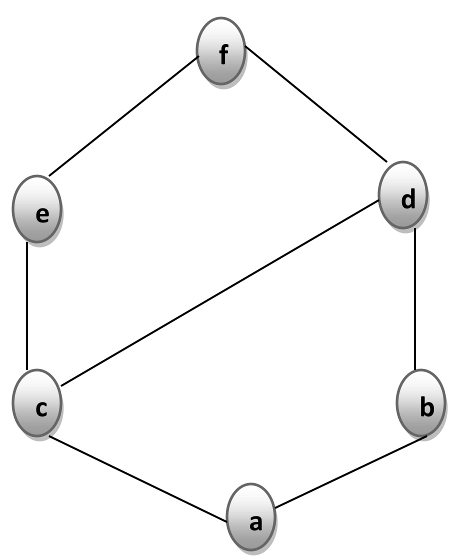

Figure 5.2.1 Example of Lattice

This above figure is a lattice because for every pair $\{a,b\}\in L$, a GLB and a LUB exists.

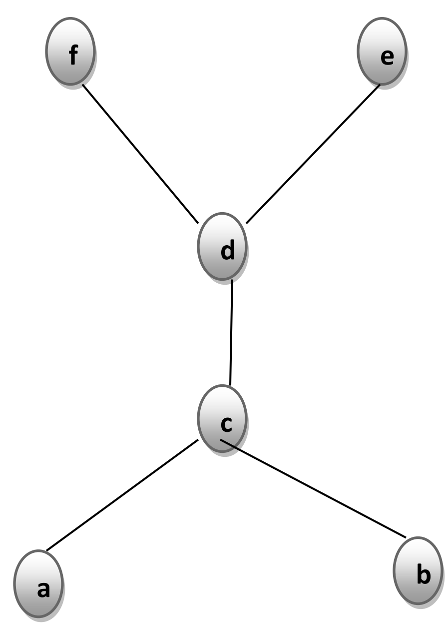

Figure 5.2.2 Lattice

This above figure is a not a lattice because $GLB(a,b)$ and $LUB(e,f)$ does not exist.

Some other lattices are discussed below −

1. **Bounded Lattice -** A lattice L becomes a bounded lattice if it has a greatest element 1 and a least element 0.
2. **Complemented Lattice -** A lattice L becomes a complemented lattice if it is a bounded lattice and if every element in the lattice has a complement. An element x has a complement x' if $\exists x(x \land x' = 0 and x \lor x' = 1)$
3. **Distributive Lattice -** If a lattice satisfies the following two distribute properties, it is called a distributive lattice.

a) $a \lor (b \land c) = (a \lor b) \land (a \lor c)$

b) $a \land (b \lor c) = (a \land b) \lor (a \land c)$

4. **Modular Lattice -** If a lattice satisfies the following property, it is called modular lattice.

$a \land (b \lor (a \land d)) = (a \land b) \lor (a \land d)$

<u>Properties of Lattices</u>

1. **Idempotent Properties**
   a) $a \lor a = a$
   b) $a \land a = a$
2. **Absorption Properties**
   a) $a \lor (a \land b) = a$
   b) $a \land (a \lor b) = a$
3. **Commutative Properties**
   a) $a \lor b = b \lor a$
   b) $a \land b = b \land a$
4. **Associative Properties**
   a) $a \lor (b \lor c) = (a \lor b) \lor c$
   b) $a \land (b \land c) = (a \land b) \land c$

Dual of a Lattice - The dual of a lattice is obtained by interchanging the '∨ ' and '∧ ' operations.

Example

The dual of $[a \vee (b \wedge c)]$ *is* $[a \wedge (b \vee c)]$

Counting Theory - In daily lives, many a times one needs to find out the number of all possible outcomes for a series of events. For instance, in how many ways can a panel of judges comprising of 6 men and 4 women be chosen from among 50 men and 38 women? How many different 10 lettered PAN numbers can be generated such that the first five letters are capital alphabets, the next four are digits and the last is again a capital letter. For solving these problems, mathematical theory of counting are used. **Counting** mainly encompasses fundamental counting rule, the permutation rule, and the combination rule.

The Rules of Sum and Product

The **Rule of Sum** and **Rule of Product** are used to decompose difficult counting problems into simple problems.

1. **The Rule of Sum** − If a sequence of tasks $T1, T2, \ldots, Tm$ can be done in $w1, w2, \ldots wm$ ways respectively (the condition is that no tasks can be performed simultaneously), then the number of ways to do one of these tasks is $w1+w2+\cdots+wm$. If we consider two tasks A and B which are disjoint (i.e. $A \cap B = \emptyset$), then mathematically $|A \cup B| = |A| + |B|$
2. **The Rule of Product** − If a sequence of tasks $T1, T2, \ldots, Tm$ can be done in $w1, w2, \ldots wm$ ways respectively and every task arrives after the occurrence of the previous task, then there are $w1 \times w2 \times \cdots \times wm$ ways to perform the tasks. Mathematically, if a task B arrives after a task A, then $|A \times B| = |A| \times |B|$

Example

Question − **A boy lives at X and wants to go to School at Z. From his home X he has to first reach Y and then Y to Z. He may go X to Y by either 3 bus routes or 2 train routes. From there, he can either choose 4 bus routes or 5 train routes to reach Z. How many ways are there to go from X to Z?**

Solution − From X to Y, he can go in 3+2=5 ways (Rule of Sum). Thereafter, he can go Y to Z in 4+5=9 ways (Rule of Sum). Hence from X to Z he can go in 5×9=45ways (Rule of Product).

Permutations - A **permutation** is an arrangement of some elements in which order matters. In other words a Permutation is an ordered Combination of elements.

Examples

1. From a set S ={x, y, z} by taking two at a time, all permutations are − *xy,yx,xz,zx,yz,zy*
2. We have to form a permutation of three digit numbers from a set of numbers $S=\{1,2,3\}$

Different three digit numbers will be formed when we arrange the digits. The permutation will be = 123, 132, 213, 231, 312, 321

Number of Permutations

The number of permutations of 'n' different things taken 'r' at a time is denoted by *nPr*

$nPr=n!(n-r)!$

where $n!=1.2.3....(n-1).n$

Proof − Let there be 'n' different elements.

There are n number of ways to fill up the first place. After filling the first place (n-1) number of elements is left. Hence, there are (n-1) ways to fill up the second place. After filling the first and second place, (n-2) number of elements is left. Hence, there are (n-2) ways to fill up the third place. We can now generalize the number of ways to fill up r-th place as [n – (r–1)] = n–r+1

So, the total no. of ways to fill up from first place up to r-th-place −

$nPr=n(n-1)(n-2).....(n-r+1)$

$=[n(n-1)(n-2)...(n-r+1)][(n-r)(n-r-1)...3.2.1]/[(n-r)(n-r-1)...3.2.1]$

Hence,

$nPr=n!/(n-r)!$

Some important formulas of permutation

1. If there are n elements of which $a1$ are alike of some kind, $a2$ are alike of another kind; $a3$ are alike of third kind and so on and ar are of rth kind, where $(a1+a2+...ar)=n$ Then, number of permutations of these n objects is = $n!/[(a1!(a2!)...(ar!)]$
2. Number of permutations of n distinct elements taking n elements at a time = $nPn=n!$
3. The number of permutations of n dissimilar elements taking r elements at a time, when x particular things always occupy definite places = $n-xpr-x$
4. The number of permutations of n dissimilar elements when r specified things always come together is − $r!(n-r+1)!$
5. The number of permutations of n dissimilar elements when r specified things never come together is − $n!-[r!(n-r+1)!]$
6. The number of circular permutations of n different elements taken x elements at time = npx/x
7. The number of circular permutations of n different things = npn/n

Some Problems

Problem 1 − From a bunch of 6 different cards, how many ways we can permute it?

Solution − As we are taking 6 cards at a time from a deck of 6 cards, the permutation will be $6P6=6!=720$

Problem 2 − In how many ways can the letters of the word 'READER' be arranged?

Solution − There are 6 letters word (2 E, 1 A, 1D and 2R.) in the word 'READER'.

The permutation will be $=6!/[(2!)(1!)(1!)(2!)]=180$.

Problem 3 − In how ways can the letters of the word 'ORANGE' be arranged so that the consonants occupy only the even positions?

Solution − There are 3 vowels and 3 consonants in the word 'ORANGE'. Number of ways of arranging the consonants among themselves $=3P3=3!=6$.

The remaining 3 vacant places will be filled up by 3 vowels in $3P3=3!=6$ ways. Hence, the total number of permutation is $6\times6=36$

Combinations - A **combination** is selection of some given elements in which order does not matter. The number of all combinations of n things, taken r at a time is −

$nCr=n!r!(n-r)!$

Problem 1 - Find the number of subsets of the set {1,2,3,4,5,6} having 3 elements.

Solution - The cardinality of the set is 6 and we have to choose 3 elements from the set. Here, the ordering does not matter. Hence, the number of subsets will be $6C3=20$

Problem 2 - There are 6 men and 5 women in a room. In how many ways we can choose 3 men and 2 women from the room?

Solution - The number of ways to choose 3 men from 6 men is $6C3$ and the number of ways to choose 2 women from 5 women is $5C2$

Hence, the total number of ways is − $6C3\times5C2=20\times10=200$

Problem 3 - How many ways can you choose 3 distinct groups of 3 students from total 9 students?

Solution - Let us number the groups as 1, 2 and 3

For choosing 3 students for 1^{st} group, the number of ways − $9C3$

The number of ways for choosing 3 students for 2^{nd} group after choosing 1st group − $6C3$

The number of ways for choosing 3 students for 3rd group after choosing 1st and 2nd group − $3C3$

Hence, the total number of ways $=9C3\times6C3\times3C3=84\times20\times1=1680$

Pascal's Identity

Pascal's identity, first derived by Blaise Pascal in 19th century, states that the number of ways to choose k elements from n elements is equal to the summation of number of ways to choose (k-1) elements from (n-1) elements and the number of ways to choose elements from n-1 elements.

Mathematically, for any positive integers k and n: $nCk=n-1Ck-1+n-1Ck$

Proof −

$n-1Ck-1+n-1Ck$

$=(n-1)!(k-1)!(n-k)!+(n-1)!k!(n-k-1)!$

$=(n-1)!(kk!(n-k)!+n-kk!(n-k)!)$

$=(n-1)!nk!(n-k)!$

$=n!k!(n-k)!$

$=nCk$

<u>Probability</u> - Closely related to the concepts of counting is Probability. We often try to guess the results of games of chance, like card games, slot machines, and lotteries; i.e. we try to find the likelihood or probability that a particular result with be obtained.

Probability can be conceptualized as finding the chance of occurrence of an event. Mathematically, it is the study of random processes and their outcomes. The laws of probability have a wide applicability in a variety of fields like genetics, weather forecasting, opinion polls, stock markets etc.

Basic Concepts

Probability theory was invented in the 17th century by two French mathematicians, Blaise Pascal and Pierre de Fermat, who were dealing with mathematical problems regarding of chance.

Before proceeding to details of probability, let us get the concept of some definitions.

Random Experiment – An experiment in which all possible outcomes are known and the exact output cannot be predicted in advance is called a random experiment. Tossing a fair coin is an example of random experiment.

Sample Space − When we perform an experiment, then the set S of all possible outcomes is called the sample space. If we toss a coin, the sample space $S=\{H,T\}$

Event − Any subset of a sample space is called an event. After tossing a coin, getting Head on the top is an event.

The word "probability" means the chance of occurrence of a particular event. The best we can say is how likely they are to happen, using the idea of probability.

Probabilityofoccurenceofanevent=TotalnumberoffavourableoutcomeTotalnumberofOutcomes

As the occurrence of any event varies between 0% and 100%, the probability varies between 0 and 1.

Steps to find the probability

Step 1 − Calculate all possible outcomes of the experiment.

Step 2 − Calculate the number of favorable outcomes of the experiment.

Step 3 − Apply the corresponding probability formula.

Tossing a Coin

If a coin is tossed, there are two possible outcomes − Heads (H) or Tails (T)

So, Total number of outcomes = 2

Hence, the probability of getting a Head (H)

on top is 1/2 and the probability of getting a Tails (T)

on top is 1/2

Throwing a Dice

When a dice is thrown, six possible outcomes can be on the top − 1,2,3,4,5,6.

The probability of any one of the numbers is 1/6

The probability of getting even numbers is 3/6 = 1/3

The probability of getting odd numbers is 3/6 = 1/3

Taking Cards From a Deck

From a deck of 52 cards, if one card is picked find the probability of an ace being drawn and also find the probability of a diamond being drawn.

Total number of possible outcomes − 52

Outcomes of being an ace – 4

Probability of being an ace = 4/52 = 1/13

Probability of being a diamond = 4/52 = 1/13

Probability Axioms

1. The probability of an event always varies from 0 to 1. $[0 \le P(x) \le 1]$
2. For an impossible event the probability is 0 and for a certain event the probability is 1.
3. If the occurrence of one event is not influenced by another event, they are called mutually exclusive or disjoint.
4. If $A1, A2....An$ are mutually exclusive/disjoint events, then $P(Ai \cap Aj) = \emptyset$ for $i \ne j$ and $P(A1 \cup A2 \cup ....An) = P(A1) + P(A2) + ...P(An)$

Properties of Probability

1. If there are two events x and $\bar{x}$ which are complementary, then the probability of the complementary event is –

$$p(\bar{x}) = 1 - p(x)$$

2. For two non-disjoint events A and B, the probability of the union of two events –

$$P(A \cup B) = P(A) + P(B)$$

3. If an event A is a subset of another event B (i.e. $A \subset B$), then the probability of A is less than or equal to the probability of B. Hence, $A \subset B$ implies $P(A) \le p(B)$

<u>Conditional Probability</u>

The conditional probability of an event B is the probability that the event will occur given an event A has already occurred. This is written as $P(B|A)$.

Mathematically – $P(B|A) = P(A \cap B)/P(A)$

If event A and B are mutually exclusive, then the conditional probability of event B after the event A will be the probability of event B that is $P(B)$

Problem 1 - In a country 50% of all teenagers own a cycle and 30% of all teenagers own a bike and cycle. What is the probability that a teenager owns bike given that the teenager owns a cycle?

Solution - Let us assume A is the event of teenagers owning only a cycle and B is the event of teenagers owning only a bike.

So, $P(A) = 50/100 = 0.5$

and $P(A \cap B) = 30/100 = 0.3$

from the given problem.

$P(B|A)=P(A \cap B)/P(A)=0.3/0.5=0.6$

Hence, the probability that a teenager owns bike given that the teenager owns a cycle is 60%.

Problem 2 - In a class, 50% of all students play cricket and 25% of all students play cricket and volleyball. What is the probability that a student plays volleyball given that the student plays cricket?

Solution - Let us assume A is the event of students playing only cricket and B is the event of students playing only volleyball.

So, $P(A)=50/100=0.5$

and $P(A \cap B)=25/100=0.25$

from the given problem.

0.25/0.5=0.5

Hence, the probability that a student plays volleyball given that the student plays cricket is 50%.

Problem 3 - Six good laptops and three defective laptops are mixed up. To find the defective laptops all of them are tested one-by-one at random. What is the probability to find both of the defective laptops in the first two pick?

Solution- Let A be the event that we find a defective laptop in the first test and B be the event that we find a defective laptop in the second test.

Hence, $P(A \cap B)=P(A)P(B|A)=3/9\times2/8=1/21$

Bayes' Theorem

Theorem − If A and B are two mutually exclusive events, where $P(A)$ is the probability of A and $P(B)$ is the probability of B, $P(A|B)$ is the probability of A given that B is true. $P(B|A)$ is the probability of B given that A is true, then Bayes' Theorem states −

P(A|B)=P(B|A)P(A)∑ni=1P(B|Ai)P(Ai)

Application of Bayes' Theorem

1. In situations where all the events of sample space are mutually exclusive events.

2. In situations where either $P(A_i \cap B)$ for each A_i or $P(A_i)$ and $P(B|A_i)$ for each A_i is known.

Problem - Consider three pen-stands. The first pen-stand contains 2 red pens and 3 blue pens; the second one has 3 red pens and 2 blue pens; and the third one has 4 red pens and 1 blue pen. There is equal probability of each pen-stand to be selected. If one pen is drawn at random, what is the probability that it is a red pen?

Solution - Let *Ai* be the event that i^{th} pen-stand is selected. Here, i = 1,2,3.

Since probability for choosing a pen-stand is equal, $P(Ai)=1/3$

Let B be the event that a red pen is drawn.

The probability that a red pen is chosen among the five pens of the first pen-stand,

$P(B|A1)=2/5$

The probability that a red pen is chosen among the five pens of the second pen-stand,

$P(B|A2)=3/5$

The probability that a red pen is chosen among the five pens of the third pen-stand,

$P(B|A3)=4/5$

According to Bayes' Theorem,

$P(B)=P(A1).P(B|A1)+P(A2).P(B|A2)+P(A3).P(B|A3)$

$=1/3.2/5+1/3.3/5+1/3.4/5$

$=3/5$

Binomial Theorem - The binomial theorem states a formula for expressing the powers of sums.

The formal expression of the Binomial Theorem is as follows:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Recurrence Relation - It Show how recursive techniques can derive sequences and be used for solving counting problems. The procedure for finding the terms of a sequence in a recursive manner is called **recurrence relation**. We study the theory of linear recurrence relations and their solutions. Finally, we introduce generating functions for solving recurrence relations.

Definition

A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing Fn as some combination of Fi with $i<n$).

Example – Fibonacci series – $Fn=Fn-1+Fn-2$, Tower of Hanoi – $Fn=2Fn-1+1$

Linear Recurrence Relations

A linear recurrence equation of degree k or order k is a recurrence equation which is in the format$xn=A1xn-1+A2xn-1+A3xn-1+\ldots Akxn-k$ (An is a constant and $Ak\neq 0$) on a sequence of numbers as a first-degree polynomial.

These are some examples of linear recurrence equations –

Recurrence relations	Initial values	Solutions
$F_n = F_{n-1} + F_{n-2}$	$a_1 = a_2 = 1$	Fibonacci number
$F_n = F_{n-1} + F_{n-2}$	$a_1 = 1, a_2 = 3$	Lucas Number
$F_n = F_{n-2} + F_{n-3}$	$a_1 = a_2 = a_3 = 1$	Padovan sequence
$F_n = 2F_{n-1} + F_{n-2}$	$a_1 = 0, a_2 = 1$	Pell number

How to solve linear recurrence relation

Suppose, a two ordered linear recurrence relation is – $Fn=AFn-1+BFn-2$

where A and B are real numbers.

The characteristic equation for the above recurrence relation is –

$x2-Ax-B=0$

Three cases may occur while finding the roots –

Case 1 – If this equation factors as $(x-x1)(x-x1)=0$

and it produces two distinct real roots $x1$ and $x2$, then $Fn=axn1+bxn2$

is the solution. [Here, a and b are constants]

Case 2 – If this equation factors as $(x-x1)2=0$

and it produces single real root $x1$, then $Fn=axn1+bnxn1$

is the solution.

Case 3 – If the equation produces two distinct complex roots, $x1$

and x_2 in polar form $x_1=r\angle\vartheta$ and $x_2=r\angle(-\vartheta)$, then $F_n=r^n(a\cos(n\vartheta)+b\sin(n\vartheta))$

is the solution.

Problem 1 - Solve the recurrence relation $F_n=5F_{n-1}-6F_{n-2}$ where $F_0=1$ and $F_1=4$

Solution - The characteristic equation of the recurrence relation is –

$x^2-5x+6=0$,

So, $(x-3)(x-2)=0$

Hence, the roots are –

$x_1=3$ and $x_2=2$

The roots are real and distinct. So, this is in the form of case 1

Hence, the solution is –

$F_n=ax_1^n+bx_2^n$

Here, $F_n=a3^n+b2^n$ (As $x_1=3$ and $x_2=2$)

Therefore,

$1=F_0=a3^0+b2^0=a+b$

$4=F_1=a3^1+b2^1=3a+2b$

Solving these two equations, we get $a=2$

and $b=-1$

Hence, the final solution is –

$\boldsymbol{F_n=2.3^n+(-1).2^n=2.3^n-2^n}$

Problem 2 Solve the recurrence relation – $F_n=10F_{n-1}-25F_{n-2}$ where $F_0=3$ and $F_1=17$

Solution The characteristic equation of the recurrence relation is –

$x^2-10x-25=0$

So $(x-5)^2=0$

Hence, there is single real root $x_1=5$

As there is single real valued root, this is in the form of case 2

Hence, the solution is –

$Fn=axn1+bnxn1$

$3=F0=a.50+b.0.50=a$

$17=F1=a.51+b.1.51=5a+5b$

Solving these two equations, we get $a=3$

and $b=2/5$

Hence, the final solution is – $Fn=3.5n+(2/5).n.2n$

Problem 3 Solve the recurrence relation $Fn=2Fn-1-2Fn-2$ where $F0=1$ and $F1=3$

Solution The characteristic equation of the recurrence relation is –

$$x2-2x-2=0$$

Hence, the roots are –

$x1=1+i$

and $x2=1-i$

In polar form,

$x1=r\angle\vartheta$

and $x2=r\angle(-\vartheta)$, where $r=2-\surd$ and $\vartheta=\pi 4$

The roots are imaginary. So, this is in the form of case 3.

Hence, the solution is –

$Fn=(2-\surd)n(acos(n.\sqcap/4)+bsin(n.\sqcap/4))$

$1=F0=(2-\surd)0(acos(0.\sqcap/4)+bsin(0.\sqcap/4))=a$

$3=F1=(2-\surd)1(acos(1.\sqcap/4)+bsin(1.\sqcap/4))=2-\surd(a/2-\surd+b/2-\surd)$

Solving these two equations we get $a=1$

and $b=2$

Hence, the final solution is –

$F_n = (\sqrt{2})^n (\cos(n.\pi/4) + 2\sin(n.\pi/4))$

Non-Homogeneous Recurrence Relation and Particular Solutions

A recurrence relation is called non-homogeneous if it is in the form

$F_n = AF_{n-1} + BF_{n-2} + f(n)$

where $f(n) \neq 0$

Its associated homogeneous recurrence relation is $F_n = AF_{n-1} + BF_{n-2}$

The solution (a_n) of a non-homogeneous recurrence relation has two parts.

First part is the solution (a_h) of the associated homogeneous recurrence relation and the second part is the particular solution (a_t)

.

$a_n = a_h + a_t$

Solution to the first part is done using the procedures discussed in the previous section.

To find the particular solution, we find an appropriate trial solution.

Let $f(n) = cx^n$; let $x^2 = Ax + B$ be the characteristic equation of the associated homogeneous recurrence relation and let x_1 and x_2 be its roots.

a) If $x \neq x_1$ and $x \neq x_2$, then $a_t = Ax^n$
b) If $x = x_1$, $x \neq x_2$, then $a_t = Anx^n$
c) If $x = x_1 = x_2$, then $a_t = An^2x^n$

Example

Let a non-homogeneous recurrence relation be $F_n = AF_{n-1} + BF_{n-2} + f(n)$

with characteristic roots $x_1 = 2$ and $x_2 = 5$. Trial solutions for different possible values of $f(n)$

are as follows –

f(n)	Trial solutions
4	A

5.2^n	$An2^n$
8.5^n	$An5^n$
4^n	$A4^n$
$2n^2+3n+1$	An^2+Bn+C

Problem

Solve the recurrence relation $F_n=3F_{n-1}+10F_{n-2}+7.5^n$ where $F_0=4$ and $F_1=3$

Solution

This is a linear non-homogeneous relation, where the associated homogeneous equation is $F_n=3F_{n-1}+10F_{n-2}$

and $f(n)=7.5^n$

The characteristic equation of its associated homogeneous relation is –

$x^2-3x-10=0$

Or, $(x-5)(x+2)=0$

Or, $x_1=5$

and $x_2=-2$

Hence $a_h=a.5^n+b.(-2)^n$, where a and b are constants.

Since $f(n)=7.5^n$, i.e. of the form $c.x^n$, a reasonable trial solution of at will be Anx^n

$a_t=Anx^n=An5^n$

After putting the solution in the recurrence relation, we get –

$An5^n=3A(n-1)5^{n-1}+10A(n-2)5^{n-2}+7.5^n$

Dividing both sides by 5^{n-2} , we get

$An5^2=3A(n-1)5+10A(n-2)5^0+7.5^2$

Or, $25An=15An-15A+10An-20A+175$

Or, $35A=175$

Or, $A=5$

So, $F_n = An5^n = 5n5^n = n5^{n+1}$

The solution of the recurrence relation can be written as –

$F_n = a_h + a_t = a.5^n + b.(-2)^n + n5^{n+1}$

Putting values of $F_0 = 4$

and $F_1 = 3$, in the above equation, we get $a = -2$ and $b = 6$

Hence, the solution is –

$F_n = n5^{n+1} + 6.(-2)^n - 2.5^n$

Generating Functions - Generating Functions represents sequences where each term of a sequence is expressed as a coefficient of a variable x in a formal power series.

Mathematically, for an infinite sequence, say $a_0, a_1, a_2, \ldots, a_k, \ldots,$ the generating function will be –

$G_x = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k + \cdots = \sum_{k=0}^{\infty} a_kx^k$

Some Areas of Application

Generating functions can be used for the following purposes –

a) For solving a variety of counting problems. For example, the number of ways to make change for a Rs. 100 note with the notes of denominations Rs.1, Rs.2, Rs.5, Rs.10, Rs.20 and Rs.50
b) For solving recurrence relations
c) For proving some of the combinatorial identities
d) For finding asymptotic formulae for terms of sequences

Problem 1 - What are the generating functions for the sequences $\{a_k\}$ with $a_k = 2$ and $a_k = 3k$?

Solution When $a_k = 2$, generating function, $G(x) = \sum_{k=0}^{\infty} 2x^k = 2 + 2x + 2x^2 + 2x^3 + \ldots$

When $a_k = 3k, G(x) = \sum_{k=0}^{\infty} 3kx^k = 0 + 3x + 6x^2 + 9x^3 + \ldots\ldots$

Problem 2 - What is the generating function of the infinite series; 1,1,1,1,…?

Solution Here, $a_k = 1$, for $0 \le k \le \infty$

Hence, $G(x) = 1 + x + x^2 + x^3 + \ldots \cdots = \frac{1}{(1-x)}$

Some Useful Generating Functions

a) For $a_k = a^k, G(x) = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \ldots\ldots\cdots = 1/(1-ax)$
b) For $a_k = (k+1), G(x) = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 \ldots\ldots\cdots = \frac{1}{(1-x)^2}$
c) For $a_k = c^n_k, G(x) = \sum_{k=0}^{\infty} c^n_k x^k = 1 + c^n_1 x + c^n_2 x^2 + \ldots\ldots\cdots + x^2 = (1+x)^n$
d) For $a_k = \frac{1}{k!}, G(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots\ldots\cdots = e^x$