Mathematics Extended Essay

Bridging the gap between quadratic forms connected to Markov triples and continued fractions

Research question: What are the special characteristics of quadratic forms associated to Markov triples and their relationship to their continued fraction roots?

Word count: 3998

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1 Introduction

This essay is about the connection of a special type of numbers, called Markov numbers, and continued fractions. I was always intrigued by special sequences of numbers like the Fibonacci sequence and the beauty of continued fractions. I decided to find links between my interests and came across a university document (see source 3), which proposed a student project idea (Project 9.4 in the source) that aims at exploring the application of continued fractions to Markov triples, which will be the focus of this essay. The source suggests that the relationship exists between the quadratic forms associated to Markov triples (called Markov forms) and simple periodic continued fractions. I will define these terms, by first introducing Markov triples and numbers, generating new triples, and connecting the quadratic forms to those triples. Then, the continued fraction roots to those quadratic forms will be explored as the central part of this essay. My own theorems and proofs will bridge the gap between quadratic forms and continued fractions. Theorems, proofs, definitions, and equations will be named and labelled based on which section they are in. For example, a proof in section 3 will start with 'Proof 3' or an equation will be referred to as 'equation (3.1)'.

2 Definition of Markov numbers, Markov triples and continued fractions

A Markov number is any number in the triple (x, y, z) of positive integer solutions to the Diophantine equation $x^2 + y^2 + z^2 = 3xyz$, known as the Markov equation. A Diophantine equation is a polynomial equation, usually involving two or more unknowns, such that only integer solutions are of interest. The Markov equation is considered rather than the more general Diophantine equation $x^2 + y^2 + z^2 = kxyz$, where $k \in \mathbb{Z}^+$, because for $k \neq 1,3$, the only solution[1] to this general Diophantine equation is $(0,0,0)$. Hence our main interest for this essay is the Markov equation.

[1] M. Rabideau, R. Schiffler, Continued Fractions and Orderings on the Markov Numbers 30 Aug 2019

The first triples to satisfy the Markov equation are the ones with repeated values: $(1,1,1)$ and $(2,1,1)$. By definition, these two solutions are called singular triples. Singular triples are triples that contain repeating values. All the other triples are non-singular, which means that they have no repeated values in them. The first non-singular triples are $(5,2,1), (13,5,1), (34,13,1), (29,5,2), (433,29,5), (169,29,2)$. The set of Markov numbers is defined as the union of the solutions to Markov triples or in other words, the union of the elements of Markov triples. Therefore, the first Markov numbers in increasing order are $1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985$ and so on. Markov triples can be arranged as a binary tree, called the Markov number tree.

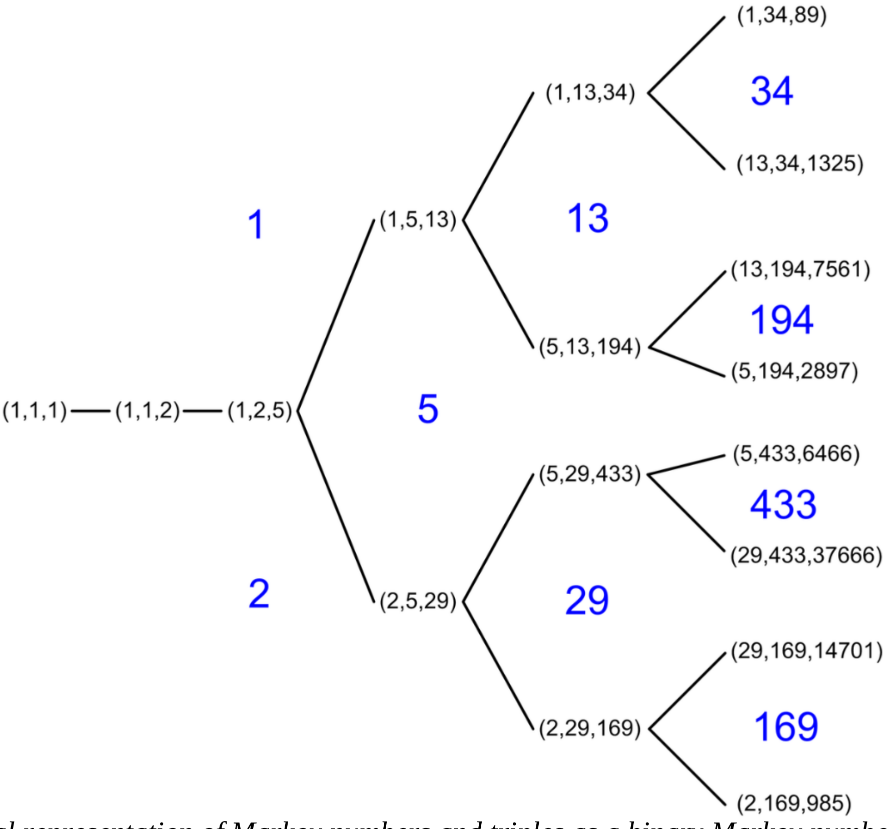

Figure 1 – visual representation of Markov numbers and triples as a binary Markov number tree[2]

[2] KurtSchwitters (2010, April 8) *Tree of Markoff numbers* [Illustration] Wikipedia https://commons.wikimedia.org/wiki/File:MarkoffNumberTree.png

Two types of continued fractions will be defined below.

<u>Definition 2.1</u>

A *Simple Continued Fraction*[3] is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}}$$

where a_i are non-negative integers, for $i > 0$ and a_0 can be any integer. The above expression will be denoted using the list notation in this essay:

$$[a_0;\ a_1, a_2, \dots, a_n].$$

<u>Definition 2.2</u>

A *Periodic Continued Fraction*[4] is an infinite simple continued fraction in which

$$a_l = a_{l+k}$$

for a fixed positive k and all $l \geq L$. The set of partial quotients

$$a_L, a_{L+1}, \dots, a_{L+k-1}$$

is called the ***period*** and the continued fraction may be written as

$$[a_0;\ a_1, a_2, \dots, a_{L-1}, \overline{a_L, a_{L+1}, \dots, a_{L+k-1}}].$$

<u>Note:</u> Periodic Continued Fractions are irrational numbers[5,6]. Source 3 (see Bibliography) proves this on source page 25 Theorem 5.2. It refers to periodic continued fractions as roots to

[3] https://pi.math.cornell.edu/~gautam/ContinuedFractions.pdf, Ch 1, 1.3
[4] https://pi.math.cornell.edu/~gautam/ContinuedFractions.pdf, Ch 5, 5.2
[5] https://pi.math.cornell.edu/~gautam/ContinuedFractions.pdf, p. 25 Theorem 5.2
[6] Weisstein, Eric W. "Periodic Continued Fraction." From MathWorld--A Wolfram Web Resource. https://mathworld.wolfram.com/PeriodicContinuedFraction.html

quadratic equations with integral coefficients which is good for us because Markov forms (see later) will also be quadratic equations with irrational roots. I will prove this in section 6.

3 Iteratively constructing new Markov triples

By using the technique of Vieta jumping, we can construct new Markov triples from existing ones. Vieta jumping is a method that assumes the existence of a solution to a certain number theory problem involving quadratic equations, and by using Vieta's formula, the second solution to the quadratic equation can be obtained (see sources 11 and 12). Therefore, if we have one Markov triple as the solution to the Markov equation $x^2 + y^2 + z^2 = 3xyz$, we can find all the other solutions. A trivial solution to this equation is $(1,1,1)$.

If (x, y, z) is a Markov triple, then let the Diophantine equation $x^2 + y^2 + z^2 = 3xyz$ be a quadratic equation in x. Then, $x^2 - 3xyz + y^2 + z^2 = 0$.

According to the technique of Vieta jumping, assume that the two roots of this quadratic equation are (x, y) and z is just a constant. Fix y and substitute a for x ($a \in \mathbb{R}$), where a is a new variable.

$$a^2 - (3yz)a + (y^2 + z^2) = 0 \tag{3.1}$$

Let the two roots of this equation be a_1 and a_2. We know that $a_1 = x$ is a root. Using Vieta's, the following two equations are obtained:

$$a_1 + a_2 = 3yz \tag{3.2}$$

$$a_1 a_2 = y^2 + z^2. \tag{3.3}$$

From equation (3.1),

$$a_2 = 3yz - a_1 = 3yz - x$$

From equation (3.2),

$$a_2 = \frac{y^2 + z^2}{a_1} = \frac{y^2 + z^2}{x}.$$

If x is the root of the original quadratic equation (3.1), then $3yz - x = \frac{y^2+z^2}{x}$ is also the root of that equation. Since the two values found for a_2 also satisfy the original Diophantine equation $x^2 + y^2 + z^2 = 3xyz$, x can be substituted by the two values found for a_2. Therefore, if (x, y, z) is a Markov triple, then $(3yz - x, y, z)$ and $(\frac{y^2+z^2}{x}, y, z)$ are also Markov triples. For simplicity, $3yz - x$ will be used for generating new Markov triples.

When the assumption was made that (x, y) is the root of the equation, it was arbitrary. It could have been (x, z) or (y, z) as well. Therefore, three other Markov triples can be created from an existing one. If one Markov triple is (x, y, z) and the new triple is $(3yz - x, y, z)$, then by permuting the three numbers in the original triple in 3! ways, we can generate $\frac{3!}{2} = \binom{3}{2} = 3$ new triples. (Division by 2 is necessary, because exchanging the values of y and z does not yield a new solution.) But why is the tree binary when 3 other triples are generated? This is answered in the next proof.

<u>Theorem 3.1:</u> *Every time we apply the transformation $(x, y, z) \rightarrow (3yz - x, y, z)$ for non-singular Markov triples, two new Markov triples are generated.*

<u>Proof 3.1:</u>

If we start from a non-singular Markov triple (x, y, z), where $z \leq y \leq x$, after applying the transformation $(x, y, z) \rightarrow (3yz - x, y, z)$ for the first time, we get:

$$(x, y, z) \rightarrow (3yz - x, y, z) = (x', y', z').$$

Then we apply the transformation again:

$$(x', y', z') \to (3y'z' - x', y', z'),$$

where $x' = 3yz - x,\ y' = y,\ z' = z.$

After substitution,

$$(3y'z' - x', y', z') = (3yz - (3yz - x), y, z) = (x, y, z). Q.E.D.$$

This result means that after applying the transformation to any non-singular Markov triple, only two new Markov triples are generated, and the third triple is identical to the one that generated the triple itself (this is evidence for the binary tree structure of the Markov tree).

After defining the terms needed to understand Markov triples, we met our first goal, which was to generate new Markov triples from existing ones.

4 Markov equations as quadratic forms

Now we can construct infinitely many Markov triples, so now we move on to establish special quadratic forms to all triples to later find connections to continued fractions. To every Markov triple (x, y, z), where $x \geq y \geq z,$ there is a way of associating a quadratic form

$$au^2 + buv + cv^2, \tag{4.1}$$

called a Markov form[7], where a, b, c are integers that depend on (x, y, z). This is also called Markov's Theorem on Quadratic Forms[8].

Let k and l be unique integers such that $ky \equiv \pm\, z \bmod x$ and $lx = k^2 + 1$, k being the least positive integer that satisfies the first equivalence. ($a \equiv b \bmod c$ means "a and b have the same remainder when divided by c", and it can be read as "a is congruent to b modulo c", where '≡' is the sign of congruency)

[7] A.A. Markov. Sur les formes binaires indefinies, I. Math. Ann. 15 (1879), 281-309
[8] M. Nadine. Markov's Theorem on Quadratic forms, Notes 2020 April 9

From this, the Markov form associated with the Markov triple (x, y, z) is

$$f_{x,y,z}(u, v) = xu^2 + (3x - 2k)uv + (l - 3k)v^2. \quad (4.2)$$

This relationship was proved by A. Markoff (see source 5). Regarding the value of k, source 5 says $0 < k < \frac{x}{2}$ and source 6 claims that $0 < k < x$. In both cases, more values of k would satisfy a Markov form, therefore I chose to find the minimum positive value of k to obtain a single Markov form for every triple, focusing on only one scenario that can be better generalized within the word limit.

<u>Example</u>

Let us convert a Markov triple to a Markov form, using the triple $(5,2,1)$.

$$x = 5, y = 2, z = 1$$

$$2k \equiv \pm 1 \; mod \; 5$$

$$k = 2$$

$$l = \frac{k^2 + 1}{x} = 1$$

$$f_{5,2,1}(u, v) = 5u^2 + 11uv - 5v^2$$

What about greater triples?

It is easy to see the value of k in the example above. However, triples with greater terms, finding the value of k is no longer straightforward. It is possible to set up a new Diophantine equation and solve that for k, by choosing the smallest possible positive integer value out of the infinitely many solutions. Since the terms of all Markov triples are co-prime[9] integers, a linear Diophantine equation always exists.

Example

Let the Markov triple be $(29, 5, 2) = (x, y, z)$. Then,

$$5k \equiv \pm 2 \; mod \; 29$$

can be rewritten as

$$5k = 29n \pm 2 \, , n \in \mathbb{Z}.$$

In general,

$$yk = xn \pm z.$$

This is a linear Diophantine equation that can be solved by the Euclidean algorithm[10].

We apply the algorithm first for the equation

$$5k = 29n + 2. \tag{4.3}$$

$$29 = 5 \times 5 + 4$$

$$5 = 1 \times 4 + 1.$$

$$1 = 5 - 4 = 5 - (29 - 5 \times 5) = 6 \times 5 - 29$$

$$2 = 2 \times (6 \times 5) - 2 \times (29) = \mathbf{12} \times 5 - \mathbf{2} \times 29$$

$$k = 12, n = 2.$$

The general solution for this linear Diophantine equation (4.3) is

[9] J. Enouen. What are the Markov and Lagrange Spectra? July 2018
[10] D. Yesilyurt. Solving Linear Diophantine Equations and Linear Congruental Equations, 2012 June 1

$$k(p) = 12 - 29m$$

$$n(p) = 2 - 5m$$

$$m \in \mathbb{Z}.$$

And then we apply the same algorithm again for $5k = 29n - 2$. We get

$$k = 17, n = 3.$$

The general solution is

$$k(p) = 17 - 29m$$

$$n(p) = 3 - 5m$$

$$m \in \mathbb{Z}.$$

We want to find the minimum value for $k, (k > 0)$ satisfying the congruency.

$$\min(k) = 12\,, k \in \mathbb{Z}^{+}$$

The following steps of obtaining the Markov form are straightforward. The Markov form (4.2) associated to all triples is of the form

$$f_{x,y,z}(u, v) = xu^2 + (3x - 2k)uv + (l - 3k)v^2.$$

For our specific triple $(29, 5, 2)$,

$$x = 29, y = 5, z = 2, k = 12, l = \frac{12^2 - 1}{29} = 5$$

After substituting all the values, we obtain the Markov form

$$f_{29,5,2}(u, v) = 29u^2 + 63uv - 31v^2$$

for the triple (29, 5, 2). (see Table 1 line 5)

The following table contains the first fifteen Markov numbers and Markov forms.

Markov triple	Markov form
$(1, 1, 1)$	$u^2 + uv - v^2$
$(2,1,1)$	$2u^2 + 4uv - 2v^2$
$(5,2,1)$	$5u^2 + 11uv - 5v^2$
$(13,5,1)$	$13u^2 + 29uv - 13v^2$
$(29,5,2)$	$29u^2 + 63uv - 31v^2$
$(34,13,1)$	$34u^2 + 76uv - 34v^2$
$(89,34,1)$	$89u^2 + 199uv - 89v^2$
$(169,29,2)$	$169u^2 + 367uv - 181v^2$
$(194,13,5)$	$194u^2 + 432uv - 196v^2$
$(233,89,1)$	$233u^2 + 521uv - 233v^2$
$(433,29,5)$	$433u^2 + 941uv - 463v^2$
$(610,233,1)$	$610u^2 + 1364uv - 610v^2$
$(985, 169,2)$	$985u^2 + 2139uv - 1055v^2$
$(1325,34,13)$	$1325u^2 + 2961uv - 1327v^2$
$(2897,194,5)$	$2897u^2 + 6451uv - 2927v^2$

Table 1 – Markov triples and their corresponding Markov forms

Markov forms are quadratic equations, which can be solved by simple periodic continued fractions. The first Markov form in Table 1 corresponding to the triple (1,1,1) has a leading coefficient 1, therefore its continued fraction solution is easy to obtain. Let $v = 1$.

$$u^2 + uv - v^2 = 0$$

$$u^2 + u - 1 = 0$$

$$u(u + 1) = 1$$

$$u = \frac{1}{u+1}$$

Then, simply plug in the value of u to the denominator on RHS, recursively obtaining the periodic continued fraction.

$$u = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ldots u}}}}$$

$$u = [0; \bar{1}]$$

The solution obtained is only one of the two solutions to the quadratic equation. As it will be seen in section 6, we will look at the case when $v = 1$. Before we move on to solving Markov forms with continued fractions, we will have to prove that real roots always exist to Markov forms and explore some general properties of Markov forms.

In this section we showed how to connect a Markov form to any Markov triple.

5 General properties of Markov forms

To find continued fraction solutions to Markov forms, some properties need to be discussed and proved. We must prove that continued fraction roots always exist to Markov forms. It is clearly visible from Table 1 that the coefficient of v^2 is always negative, for all triples. It means that all Markov forms will have rational roots because their discriminant is positive. For Markov forms,

$$\Delta = b^2 - 4ac > 0,$$

because $b^2 \geq 0, a = x > 0, c < 0$.

All simple and periodic continued fractions are real numbers. Therefore, we must show that Markov forms have real roots. This is easy to achieve if we can see that the discriminant of each form is positive. The discriminant of a Markov form (4.1) of the form $au^2 + buv + cv^2$ is equal to

$$\frac{b^2 - 4ac}{2a}$$

where $a > 0$, so it is enough to show that $c < 0$, therefore the discriminant is positive thus we have real roots.

Theorem 5.1

In a Markov form (4.1) $au^2 + buv + cv^2$, c, the coefficient c of v^2 is always negative for all Markov triples.

Proof 5.1

By the defining equation (4.2) of the Markov form

$$f_{x,y,z}(u,v) = xu^2 + (3x - 2k)uv + (l - 3k)v^2,$$

$$c = l - 3k. \tag{5.1}$$

The general solution to a Diophantine equation $yk = xn \pm z$ is

$$k(p) = k_1 - xm$$

$$n(p) = n_1 - ym, (m \in \mathbb{Z})$$

where k_1 and n_1 are the first solutions found by the Euclidean algorithm.

Our first goal is to show that there exists a value for k, such that $k > 0$ and $k < x$.

If $k_1 > x$, then subtract mx from k_1 such that $2x \geq k_1 - mx \geq x$.

Taking away x from the inequality, $x \geq k_1 - (m+1)x \geq 0$, so the value of the least positive k is

$$k = k_1 - (m+1)x \leq x.$$

Therefore, there exists a positive integer k, where $k \leq x$.

The main goal is to show that $c = l - 3k < 0$.

Let k be the least positive integer satisfying the congruency.

Then, $k \leq x$, so there exists a p such that $p \in \mathbb{Z}, p \geq 0$, and $x = k + p$.

We know by the definition of (4.2) that

$$lx = k^2 + 1,$$

expressing l gives us

$$l = \frac{k^2 + 1}{x}$$

Substituting into (5.1),

$$l - 3k = \frac{k^2 + 1}{x} - 3k = \frac{k^2 - 3kx + 1}{x}.$$

This fraction is negative only if the numerator is negative, since $x > 0$.

By substituting $x = k + p$ to the numerator we obtain

$$k^2 - 3k(k + p) + 1 = -2k^2 - 3kp + 1 < 0,$$

since the maximum value of this expression (-1) is obtained when

$k = 1$ ($k \in \mathbb{Z}^+$) and $p = 0$.

$$c = l - 3k < 0. Q.E.D.$$

It is also important to see that in every Markov form (4.1) $au^2 + buv + cv^2$, b is always positive. This will be useful in the next section when we want to see that each Markov form has one positive and one negative real root. To prove this statement later in theorem 6.1 and conclude that $b = 3x - 2k$ is always positive, a quick proof will follow.

Theorem 5.2

In every Markov form (4.1) $au^2 + buv + cv^2$*, b is positive, thus in (4.2)* $3x - 2k > 0$.

The easiest way is to conclude that x is greater than k. Since k is the smallest positive integer to satisfy $yk \equiv \pm z \; mod \; x$, it is enough to prove that a solution exists to this congruency such that $k < x$.

Proof 5.2

We know that in Markov-triples the terms are co-prime. It means that

$$yk \equiv 0 \; mod \; x$$

happens first when $k = x$.

Let $k = 1$. Then,

$$yk_1 \equiv y \; mod \; x$$

then increase k such that

$$yk_2 \equiv y \; mod \; x$$

holds again. It means that both yk_1 and yk_2 give a remainder of y when divided by x. So we can think of them as $yk_1 = xn + y$ and $yk_2 = xm + y$ given that $(n, m) \in \mathbb{Z}, m = n + 1$. Then set up the following inequality:

$$xn + y < xm = xn + x < xm + y = xn + x + y.$$

Note: $x \geq y$ holds for all Markov triples (x, y, z), $x \geq y \geq z$.

The result obtained from the inequality tells us that there must be a number xm, being the multiple of x, between two consecutive whole numbers giving the same remainder when divided by x.

As mentioned above,

$$yk \equiv 0 \, mod \, x$$

happens first when $k = x$. So it means that for all ks, $k = \{1,2,3, \dots, x-1\}$ there is a unique remainder when yk is divided by x (the inequality tells us that no two consecutive identical remainders occur without having a zero remainder between them). For $k = \{1,2,3, \dots, x-1\}$ altogether $x-1$ different remainders occur, which must contain z, since $z < x$. The remainders will also be in the range of $\{1,2,3, \dots, x-1\}$, each occurring once. The first time when

$$yk \equiv 0 \, mod \, x$$

holds is when $k = x$. For each of the values of $k = \{1,2,3, \dots, x-1\}$, one remainder of $\{1,2,3, \dots, x-1\}$ will be assigned. This set of numbers includes z, since $z < x$. Therefore, the least value of k satisfying

$$yk \equiv \pm z \, mod \, x$$

is smaller than x, since $k \in \{1,2,3, \dots, x-1\}$ such that $yk \equiv \pm z \, mod \, x$ holds.

$$x > k.$$

Therefore,

$$b = 3x - 2k > 0. Q.E.D.$$

In this section we explored some general properties of Markov forms that ensure that their roots can be written in continued fraction forms. This was vital before we start constructing their roots. Theorem 5.1 stated that the roots to Markov forms are always real, which allows us to generalize continued fraction roots to Markov forms, since continued fraction roots are always real. Theorem 5.2 will be needed to prove Theorem 6.1 later, but it belongs here, because the fact that b is always positive for all Markov forms is a general property of the forms.

Note: I noticed that for all the triples that contain a one, the coefficients a, c of the Markov form (5.1)

$$au^2 + buv + cv^2$$

are equal but have opposite signs. I proved it in the Appendices section under Theorem 5.3 out of curiosity.

6 Solving Markov forms with continued fractions (case $v = 1$)

The case $v = 1$ for all Markov forms will be investigated that belong to any Markov triple (x, y, z), where x is the greatest term of the triple.

$$f_{x,y,z}(u, v) = xu^2 + (3x - 2k)uv + (l - 3k)v^2 =$$

$$= f_{x,y,z}(u, 1) = xu^2 + (3x - 2k)u + (l - 3k).$$

In general, Markov forms are in the form

$$au^2 + buv - cv^2, (a, b, c) \in \mathbb{Z}^+. \tag{6.1}$$

From now on, we will refer to (4.1) as the form above (6.1) because we proved that the coefficient of v^2 is negative.

The generalized method for solving any quadratic equation of the form (6.1)

$$au^2 + bu - c, (v = 1)$$

where

$$b^2 - 4ac > 0,$$

is to let

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and let the periodic continued fraction solution be

$$[a_0; \overline{a_1, a_2, \dots, a_n}]$$

I devised my own algorithm which will generate the terms of the continued fraction solution:

<u>Step 1.</u>

Take the integer part of x and set it equal to the first term of the continued fraction.

$$a_0 = [x].$$

<u>Step 2.</u>

Subtract the integer part of x from itself,

$$x - [x]$$

and take its reciprocal. Now we have

$$\frac{1}{x - [x]}$$

Take the integer part of this fraction to obtain the next term in the continued fraction sequence

$$a_1 = \left[\frac{1}{x - [x]}\right].$$

<u>Step 3.</u>

Let

$$x_2 = \frac{1}{x - [x]}$$

and go back to Step 2, repeating the process with $x_2, x_3, \dots, x_n$ until we get

$$x_p - [x_p]$$

equal to a previously occurred

$$x_q - [x_q]$$

such that $q < p$.

Using the algorithm above we can compute the positive continued fraction roots to each Markov form. Only positive roots can be found with this algorithm, because $x - [x]$ is always positive$(x > 0)$. The table below contains only the positive root to Markov forms. Due to the word limitation of the essay, only the positive continued fraction roots will be discussed.

Continued fraction root u	Decimal root u	Markov form ($v = 1$)
$[0;\overline{1}]$	0.6180339887	$u^2 + uv - v^2$
$[0;\overline{2}]$	0.4142135624	$2u^2 + 4uv - 2v^2$
$[0;\overline{2,1,1,2}]$	0.3866068747	$5u^2 + 11uv - 5v^2$
$[0;\overline{2,1,1,1,1,2}]$	0.3826416996	$13u^2 + 29uv - 13v^2$
$[0;\overline{2,2,2,1,1,2}]$	0.4133966975	$29u^2 + 63uv - 31v^2$
$[0;\overline{2,1,1,1,1,1,1,2}]$	0.3820645628	$34u^2 + 76uv - 34v^2$
$[0;\overline{2,1,1,1,1,1,1,1,1,2}]$	0.3819803891	$89u^2 + 199uv - 89v^2$
$[0;\overline{2,2,2,2,2,1,1,2}]$	0.4141895125	$169u^2 + 367uv - 181v^2$
$[0;\overline{2,1,1,2,2,1,1,1,1,2}]$	0.3865890813	$194u^2 + 432uv - 196v^2$
$[0;\overline{2,1,1,1,1,1,1,1,1,1,1,2}]$	0.3819681089	$233u^2 + 521uv - 233v^2$
$[0;\overline{2,2,2,1,1,2,2,1,1,2}]$	0.4133931413	$433u^2 + 941uv - 463v^2$
$[0;\overline{2,1,1,1,1,1,1,1,1,1,1,1,1,2}]$	0.3819663173	$610u^2 + 1364uv - 610v^2$
$[0;\overline{2,2,2,2,2,2,2,1,1,2}]$	0.4142128544	$985u^2 + 2139uv - 1055v^2$
$[0;\overline{2,1,1,1,1,2,2,1,1,1,1,1,1,2}]$	0.3826413196	$1325u^2 + 2961uv - 1327v^2$
$[0;\overline{2,1,1,2,2,1,1,2,2,1,1,1,1,2}]$	0.3866067949	$2897u^2 + 6451uv - 2927v^2$

Table 2 - Simple periodic continued fraction roots of Markov forms

Simple periodic fractions of the form $[0; a_0, a_1, \ldots, a_n]$ are positive. It seems that there always exists a periodic continued fraction root to Markov forms. We should prove that this is true for all Markov forms in order to generalize later the continued fraction pattern found for the chain of roots to Markov forms. When both roots are calculated to quadratic Markov forms, we can notice that there is always one positive and one negative root. The positive one is always less than one(see Table 2). It is clear from the Table 2 that continued fraction roots of Markov forms are irrational, starting with zero. We should prove that this holds for all Markov forms to generalize our findings.

Lemma 6.1

Roots of Markov forms are irrational.

A Markov form (4.1) $f_{x,y,z}(u,1) = xu^2 + (3x - 2k)u + (l - 3k)$ has solutions

$$\frac{2k - 3x \pm \sqrt{9x^2 + 4k^2 - 4xl}}{2x} = \frac{2k - 3x \pm \sqrt{9x^2 + 4k^2 - 4x\left(\frac{k^2+1}{x}\right)}}{2x} =$$

$$= \frac{2k - 3x \pm \sqrt{9x^2 - 4}}{2x} = \frac{2k - 3x}{2x} \pm \frac{\sqrt{9x^2 - 4}}{2x}$$

If $\sqrt{9x^2 - 4}$ is irrational, the whole fraction is irrational, because $\frac{2k-3x}{2x}$ is rational and $2x$ is also rational. Simply $rational + \frac{irrational}{rational} = rational + irrational = irrational.$

The only way $\sqrt{9x^2 - 4}$ can be rational is when $9x^2 - 4$ is a perfect square[11]. Observe that $9x^2 - 4$ is never a perfect square. Perfect squares are integers and $x \in \mathbb{Z}^+$. The greatest perfect square smaller than $9x^2 - 4$ is

$$(3x - 1)^2 = 9x^2 - 6x + 1$$

and the smallest perfect square greater than $9x^2 - 4$ is

$$(3x)^2 = 9x^2.$$

$$9x^2 - 6x + 1 < 9x^2 - 4 < 9x^2.$$

[11] see Appendices Lemma 6.1.1 which shows that only perfect squares have integer and rational roots

There are no perfect squares between $9x^2 - 6x + 1$ and $9x^2$ since they are consecutive ones(($3x - 1$) and $3x$ are consecutive integers). $9x^2 - 4$ is not a perfect square and therefore solutions to Markov forms are always irrational.

Theorem 6.1

For every Markov form with $v = 1$ there is one positive and one negative root. Both roots are irrational, and the positive root is less than one, so it can always be written in the form

$$[0; \overline{a_1, a_2, \dots, a_n}].$$

Proof 6.1

The two roots α, β to the Markov form (4.2, $v = 1$)

$$f(u, 1) = xu^2 + (3x - 2k)u + l - 3k$$

are:

$$\alpha, \beta = \frac{2k - 3x \pm \sqrt{9x^2 + 4k^2 - 4xl}}{2x}$$

where $x, l, k \in \mathbb{Z}^+$ and

$$4k^2 - 4xl = -4$$

since

$$l = \frac{k^2 + 1}{x}$$

and

$$4k^2 - 4xl = 4k^2 - 4x\frac{k^2 + 1}{x} = -4.$$

Thus, one root is

$$\alpha = \frac{2k - 3x + \sqrt{9x^2 - 4}}{2x}$$

and the other root is

$$\beta = \frac{2k - 3x - \sqrt{9x^2 - 4}}{2x}$$

To see that α is positive, consider its numerator and group together positive and negative terms and see how their values relate.

The positive term in the numerator of α is

$$2k + \sqrt{9x^2 - 4}$$

and the negative term is $-3x$.

It is easy to see that their sum is positive by squaring both values.

$$positive\ term\ squared = (2k + \sqrt{9x^2 - 4})^2 = 4k^2 + 4k\sqrt{9x^2 - 4} + 9x^2 - 4$$

$$negative\ term\ squared = (3x)^2 = 9x^2.$$

Subtract $9x^2$ from both terms and since

$$4k^2 \geq 4$$

and

$$4k\sqrt{9x^2 - 4} > 0,$$

so

$$positive\ term\ squared > negative\ term\ squared$$

and α is therefore positive since both its numerator and denominator are positive.

The same strategy is to be applied when proving that β is negative.

The negative term in the numerator of β is

$$-3x - \sqrt{9x^2 - 4}$$

and the positive term is just $2k$. Referring back to Theorem 3.3, $3x > 2k$. Therefore the numerator of β is negative and its denominator $2x$ is positive. Thus β is negative.

$$\alpha > 0, \beta < 0.$$

Each Markov form has one positive and one negative rational root.

To see that $\alpha < 1$, consider

$$\alpha = \frac{2k - 3x + \sqrt{9x^2 - 4}}{2x}$$

where

$$3x > \sqrt{9x^2 - 4} \; and \; 2x > 2k > 2k - 3x + \sqrt{9x^2 - 4},$$

thus the denominator is greater than the numerator, so the fraction is smaller than one.

$$0 < \alpha < 1$$

According to Lemma 6.1, both

$$\alpha \; and \; \beta$$

are irrational.

Therefore, there is always a solution to Markov forms of the form $[0; \overline{a_1, a_2, \dots, a_n}]$. $Q.E.D.$

The relationship between periodic continued fraction roots and the Markov forms exists through the convergents of the continued fractions.

Definition 6.1

We call $[a_0;\ \dots, a_m]$ (for $0 \le m \le n$) the $\boldsymbol{m^{th}}$ $\boldsymbol{convergent}$ to $[a_0;\ \dots, a_n]$.

Convergents can be written[12] as a rational fraction $\frac{p_n}{q_n}$ where p_n and q_n are defined by

$$p_0 = a_0, p_1 = a_1 a_0 + 1, p_n = a_n p_{n-1} + p_{n-2}\ for\ n \le 2 \tag{6.2}$$

$$q_0 = 1, q_1 = a_1, q_n = a_n q_{n-1} + q_{n-2}\ for\ n\ \le 2 \tag{6.3}$$

$$[a_0; a_1 \dots, a_n] = \frac{p_n}{q_n}$$

Example:

Calculating all convergents of the following simple continued fraction:

$$a = [a_0; a_2, a_3, a_4] = [0; 2,1,1,2], n = 4$$

$$p_0 = 0, p_1 = 1, p_2 = 1, p_3 = 2, p_4 = 5$$

$$q_0 = 1, q_1 = 2, q_2 = 3, q_3 = 5, q_4 = 13$$

$$\frac{p_0}{q_0} = \frac{0}{1} = 0$$

$$\frac{p_1}{q_1} = \frac{1}{2} = 0 + \frac{1}{2}$$

[12] https://pi.math.cornell.edu/~gautam/ContinuedFractions.pdf p.9 Theorem 2.3

$$\frac{p_2}{q_2} = \frac{1}{3} = 0 + \cfrac{1}{2 + \cfrac{1}{1}}$$

$$\frac{p_3}{q_3} = \frac{2}{5} = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1}}}$$

$$\frac{p_4}{q_4} = \frac{5}{13} = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}$$

The red-colored terms appear in the list notation of a.

The numbers p_n and q_n satisfy[13]

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \tag{6.4}$$

This identity from source 3 will be required when we set up two equations (6.7) and (6.8) for two unknows later.

Note: the n^{th} convergent of $[a_0;\ \ldots, a_n] = [a_0;\ \ldots, a_n]$.

Periodic continued fractions have the following property[14]:

If $a = [\overline{a_1; a_2, \ldots, a_n}]$, then a is a solution of

$$q_n u^2 + (q_{n-1} - p_n)u - p_{n-1} = 0. \tag{6.5}$$

We will use this identity alone to develop the connection of Markov forms to their roots. The property above is mentioned in sources 9 and 10 which is useful to us if we rewrite our continued fraction roots. The property holds for continued fractions of the form $[\overline{a_1; a_2, \ldots, a_n}] = [a_1; a_2, \ldots, a_n, a_1, a_2, \ldots, a_n \ldots]$. We proved in Theorem 6.1 that all continued

[13] https://pi.math.cornell.edu/~gautam/ContinuedFractions.pdf p.9 Theorem 2.5

[14] https://crypto.stanford.edu/pbc/notes/contfrac/periodic.html

fraction solutions are of the form $[0;\overline{a_1, a_2, \dots, a_n}]$. If we want the property to hold for our solutions as well, we have to connect our solutions to fractions of the form $[\overline{a_1; a_2, \dots, a_n}]$, because the property in the sources only holds for fractions of this kind. The way I did that is by recognizing that

$$[0;\overline{a_1, a_2, \dots, a_n}] = \frac{1}{[\overline{a_1; a_2, \dots, a_n}]}$$

which comes from the definition of continued fractions. Now we can rearrange it as

$$[\overline{a_1; a_2, \dots, a_n}] = \frac{1}{[0;\overline{a_1, a_2, \dots, a_n}]}$$

and do the following. Let $u = [\overline{a_1; a_2, \dots, a_n}]$, and let u be a solution to (6.5)

$$q_n u^2 + (q_{n-1} - p_n)u - p_{n-1} = 0.$$

Now, $\frac{1}{u} = \frac{1}{[\overline{a_1;a_2,\dots,a_n}]} = [0;\overline{a_1, a_2, \dots, a_n}]$, so we should find an equation with one of its roots being $\frac{1}{u}$.

Let u_1 *and* u_2 be the solutions to the original quadratic (6.5)

$$q_n u^2 + (q_{n-1} - p_n)u - p_{n-1} = 0.$$

Using Vieta's,

$$u_1 + u_2 = \frac{-(q_{n-1} - p_n)}{q_n}$$

and

$$u_1 u_2 = \frac{-p_{n-1}}{q_n}.$$

Let the new equation be a quadratic of the form $\alpha x^2 + \beta x + c$ with roots $\frac{1}{u_1}$ and $\frac{1}{u_2}$.

By using Vieta's, the new coefficients are:

$$\frac{-\beta}{\alpha} = \frac{1}{u_1} + \frac{1}{u_2} = \frac{u_1 + u_2}{u_1 u_2} = \frac{\frac{-(q_{n-1} - p_n)}{q_n}}{\frac{-p_{n-1}}{q_n}} = \frac{-(q_{n-1} - p_n)}{-p_{n-1}} = \frac{q_{n-1} - p_n}{p_{n-1}}$$

$$\frac{c}{\alpha} = (\frac{1}{u_1})(\frac{1}{u_2}) = \frac{1}{u_1 u_2} = \frac{-q_n}{p_{n-1}}$$

By comparing coefficients,

$$-\beta = q_{n-1} - p_n \rightarrow \boldsymbol{\beta = p_n - q_{n-1}}$$

$$\boldsymbol{c = -q_n}$$

$$\boldsymbol{\alpha = p_{n-1}}$$

The new equation with reciprocal roots is the following:

$$p_{n-1}u^2 + (p_n - q_{n-1})u - q_n \,. \tag{6.6}$$

What we have shown here can be formulated as a theorem.

<u>**Theorem 6.2**</u>

If $[\overline{a_1; a_2, \dots, a_n}]$ *is a solution to (6.5)*

$$q_n u^2 + (q_{n-1} - p_n)u - p_{n-1} = 0,$$

then $[0; \overline{a_1, a_2, \dots, a_n}]$ *is a solution to (6.6)*

$$p_{n-1}u^2 + (p_n - q_{n-1})u - q_n \,.$$

We can observe that $p_{n-1}u^2 + (p_n - q_{n-1})u - q_n$ is identical to the Markov form of the form

$$f(u,1) = xu^2 + (3x - 2k)u + l - 3k$$

with roots $[0; \overline{a_1, a_2, \dots, a_n}]$.

Theorem 6.2 enables us to find the solution to any Markov form just by using its coefficients.

If we have the Markov form (6.1, $v = 1, a = x$)

$$f(u, 1) = xu^2 + bu - c$$

equivalent to (6.6)

$$p_{n-1}u^2 + (p_n - q_{n-1})u - q_n$$

with roots of the form $[0; \overline{a_1, a_2, \dots, a_n}]$, converted back to an equation whose roots are $[\overline{a_1, a_2, \dots, a_n}]$ which is (6.5)

$$q_n u^2 + (q_{n-1} - p_n)u - p_{n-1}$$

is equal to $cu^2 - bu - x$.

By Theorem 6.2. We know from the simple continued fraction convergents' identity that

$$a = [a_1; a_2, a_3, \dots, a_n] = \frac{p_n}{q_n}$$

and $[\overline{a_1; a_2, a_3, \dots, a_n}]$ is a solution to the quadratic (6.5)

$$q_n u^2 + (q_{n-1} - p_n)u - p_{n-1}.$$

By equating coefficients of this quadratic and $cu^2 - bu - x$,

$$x = p_{n-1}, b = \ (p_n - q_{n-1}), c = q_n \ .$$

By using identities of convergents (6.4) and substituting xc for $p_{n-1}q_n$,

$$p_n q_{n-1} - p_{n-1}q_n = (-1)^{n-1}$$

$$p_n q_{n-1} - xc = (-1)^{n-1}$$

We have two equations for the unknowns $p_n \ and \ q_{n-1}$.

$$p_n q_{n-1} = (-1)^{n-1} + xc \tag{6.7}$$

$$p_n - q_{n-1} = b \tag{6.8}$$

Using (6.8)

$$q_{n-1} = p_n - b$$

and substituting back to (6.7)

$$p_n(p_n - b) = (-1)^{n-1} + xc$$

$$p_n{}^2 - bp_n - xc - (-1)^{n-1} = 0.$$

After solving this equation for both $(-1)^{n-1} = 1$ and $(-1)^{n-1} = -1$, we choose the solution for which p_n is an integer.

$$p_n = \frac{\mathrm{b} \pm \sqrt{b^2 - 4(-xc - (-1)^{n-1})}}{2} = \frac{\mathrm{b} \pm \sqrt{b^2 + 4xc + 4(-1)^{n-1}}}{2} \tag{6.9}$$

$$q_n = c. \tag{6.10}$$

$[\overline{a_1; a_2, \dots, a_n}]$ is a solution to (6.5)

$$q_n u^2 + (q_{n-1} - p_n)u - p_{n-1}$$

but the Markov form (6.1, $v = 1, a = x$)

$$f(u, 1) = xu^2 + bu - c$$

is of the form (6.6)

$$p_{n-1}u^2 + (p_n - q_{n-1})u - q_n$$

so its solution by Theorem 6.2 is

$$[0; \overline{a_1, a_2, \dots, a_n}] = \frac{1}{[\overline{a_1; a_2, \dots, a_n}]}$$

whose simple continued fraction period $[0; a_1, a_2, a_3, \dots, a_n]$ equals

$$\frac{1}{\frac{p_n}{q_n}} = \frac{q_n}{p_n}$$

Expressing

$$\frac{q_n}{p_n}$$

in terms of the coefficients of the Markov form using (6.9) and (6.10) equals

$$\frac{2c}{\mathrm{b} \pm \sqrt{b^2 + 4xc + 4(-1)^{n-1}}} \, .$$

Since

$$\frac{q_n}{p_n} = \frac{2c}{\mathrm{b} \pm \sqrt{b^2 + 4xc + 4(-1)^{n-1}}} \qquad (6.11)$$

to get the periodic continued fraction solution, rewrite the rational fraction $\frac{q_n}{p_n}$ as a continued fraction. Every rational fraction has a continued fraction expansion which is finite[15].

We are only looking for positive continued fraction roots in this essay. Therefore,

$$\frac{q_n}{p_n} = \frac{2c}{\mathrm{b} \pm \sqrt{b^2 + 4xc + 4(-1)^{n-1}}} > 0$$

which means that only

[15] https://pi.math.cornell.edu/~gautam/ContinuedFractions.pdf Theorem 2.1 p. 7

$$\frac{q_n}{p_n} = \frac{2c}{\mathrm{b} + \sqrt{b^2 + 4xc + 4(-1)^{n-1})}}$$

is a valid solution for our purposes (note that $2c$ is positive). It is easy to see that

$$\mathrm{b} \le \sqrt{b^2 + 4xc + 4(-1)^{n-1})}$$

since $(x, c) \epsilon \mathbb{Z}^+$ and equality can only occur if $x = c = 1$ and n is even, therefore $4xc = 4(-1)^{n-1}$, they cancel each other and only $b = \sqrt{b^2}$ remains. Otherwise, $4xc > 4(-1)^{n-1}$ and

$$\mathrm{b} \le \sqrt{b^2 + 4xc + 4(-1)^{n-1}}.$$

If we allowed the negative sign in the denominator of equation (6.11), the denominator could be zero (which must be ruled out anyways) or negative, which is not what we are looking for. We will only use the positive solution

$$\frac{q_n}{p_n} = \frac{2c}{\mathrm{b} + \sqrt{b^2 + 4xc + 4(-1)^{n-1}}} \tag{6.12}$$

If we rewrite $\frac{q_n}{p_n}$ as a simple finite continued fraction, we get $[0; a_1, a_2, a_3, \dots, a_n]$.

Since we know that the simple continued fraction period of the periodic continued fraction $[\overline{a_1; a_2, a_3, \dots, a_n}]$ equals

$$[a_1; a_2, a_3, \dots, a_n] \; = \frac{p_n}{q_n}$$

and $[\overline{a_1; a_2, a_3, \dots, a_n}]$ is a root of (6.5)

$$q_n u^2 + (q_{n-1} - p_n)u - p_{n-1}$$

therefore

$$[0;\overline{a_1,a_2,\dots,a_n}]=\frac{1}{\overline{[a_1;a_2,\dots,a_n]}}\text{ and }\frac{q_n}{p_n}=[0;a_1,a_2,a_3,\dots,a_n]$$

where $[0;\overline{a_1,a_2,\dots,a_n}]$ is a root of the Markov form (6.6)

$$p_{n-1}u^2+(p_n-q_{n-1})u-q_n=f(u,1)=xu^2+bu-c$$

Now compute (6.12) $\frac{q_n}{p_n}=\frac{2c}{\mathrm{b}+\sqrt{b^2+4xc+4(-1)^{n-1})}}$, for one odd and one even value of n, and then using the result for which the denominator is an integer ($p_n,q_n\in\mathbb{Z}^+$), the whole expression will be equal to $[0;a_1,a_2,a_3,\dots,a_n]$. Then, mark the period $[0;\overline{a_1,a_2,a_3,\dots,a_n}]$.

In conclusion, for a Markov form (6.1, $v=1,a=x$)

$$f(u,1)=xu^2+bu-c$$

the *period* of the periodic continued fraction solution equals (6.12)

$$\frac{q_n}{p_n}=\frac{2c}{\mathrm{b}+\sqrt{b^2+4xc+4(-1)^{n-1})}}=[0;a_1,a_2,a_3,\dots,a_n].$$

To obtain the final periodic continued fraction solution, mark the period:

$$[0;\overline{a_1,a_2,a_3,\dots,a_n}].$$

<u>Example</u>

Markov triple	Markov form$(v=1)$
(5,2,1)	$5u^2+11u-5$

Table 3 – finding the continued fraction root of a Markov form

$$f(u,1)=xu^2+bu-c=5u^2+11u-5.$$

The period of the periodic continued fraction solution to the Markov form is

$$\frac{q_n}{p_n} = \frac{2c}{\mathrm{b} + \sqrt{b^2 + 4xc + 4(-1)^{n-1})}} = \frac{10}{11 + \sqrt{121 + 100 + 4}} = \frac{5}{13} = [0; 2,1,1,2].$$

or

$$\frac{q_n}{p_n} = \frac{2c}{\mathrm{b} + \sqrt{b^2 + 4xc + 4(-1)^{n-1})}} = \frac{10}{11 + \sqrt{121 + 100 - 4)}}$$

but $\sqrt{121 + 100 - 4}$ is not an integer therefore we use the first case when n is odd.

Mark the period.

$$[0; \overline{2,1,1,2}].$$

Indeed, the periodic continued fraction solution to the Markov form in Table 3 equals

$$[0; \overline{2,1,1,2}]$$

and can be checked in Table 2.

We reached our end goal, because we found a way to calculate the continued fraction solution to Markov forms, simply by using the forms' coefficients and the formula we developed. Therefore, we solved Markov forms with continued fractions.

7 Conclusion

In this essay, we first looked at Markov triples $(x, y, z) \epsilon \mathbb{Z}^+$ satisfying the equation

$$x^2 + y^2 + z^2 = 3xyz$$

and showed how to generate infinitely many triples (see Theorem 3.1). Then, we defined what a Markov form is:

$$f_{x,y,z}(u, v) = xu^2 + (3x - 2k)uv + (l - 3k)v^2$$

investigating the single case when $v = 1$ and calculated the corresponding Markov forms to the generated Markov triples (see Table 1). After associating Markov forms to Markov triples, we looked at the properties of Markov forms and proved them to ensure that positive periodic continued fraction roots always exist to them (see Theorems 5.1, 5.2). Then, as the focus of the essay, using the idea of convergents of continued fractions (see equations 6.2, 6.3), Theorems 6.1, 6.2 and the following identities:

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \tag{6.4}$$

If $a = [\overline{a_1; a_2, \dots, a_n}]$, then a is a solution of

$$q_n u^2 + (q_{n-1} - p_n)u - p_{n-1} = 0. \tag{6.5}$$

a relationship was developed between Markov forms and their periodic continued fraction roots which says:

for a Markov form (6.1, $v = 1, a = x$)

$$f(u, 1) = xu^2 + bu - c$$

the *period* of the periodic continued fraction solution equals (6.12)

$$\frac{q_n}{p_n} = \frac{2c}{\mathrm{b} + \sqrt{b^2 + 4xc + 4(-1)^{n-1}}} = [0; a_1, a_2, a_3, \dots, a_n].$$

To obtain the final periodic continued fraction solution, mark the period:

$$[0; \overline{a_1, a_2, a_3, \dots, a_n}].$$

This connection enables us to solve Markov forms using continued fractions, bridging the gap between the two areas of mathematics. Since our research was very focused on a certain case of Markov forms $v = 1$, and only their relationship to continued fraction solutions, in the Appendices section part ‘Conjectures and further research suggestions’ I propose exciting ideas for further exploration. Sources 3, 9, 10 were the most useful in reaching my aim by providing identities and suggesting new concepts, whereas Sources 5, 6 established the connection between Markov forms and triples and the rest was descriptive to help the reader understand concepts.

Notations

$\boldsymbol{mod}$: $a \equiv b \ mod \ c$ means "a and b have the same remainder when divided by c".

$\boldsymbol{x \in} \mathbb{Z}$: x is an integer.

$\boldsymbol{x \in} \mathbb{Z}^+$: x is a positive integer.

RHS: right hand side of the equation

LHS: left hand side of the equation

$[\boldsymbol{a_0}; \boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \dots, \boldsymbol{a_n}]$: simple continued fraction with terms $a_0, a_1, a_2, a_3, \dots, a_n$ (see Definition 2.1)

$[\boldsymbol{a_0}; \overline{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \dots, \boldsymbol{a_n}}]$: periodic continued fraction with terms $a_0, a_1, a_2, a_3, \dots, a_n$ (see Definition 2.2)

$\boldsymbol{\frac{p_n}{q_n}}$: n^{th} convergent of the simple continued fraction $[a_0; a_1, a_2, a_3, \dots, a_n]$ (see Definition 6.2)

Δ: discriminant of a quadratic equation

Appendices

<u>Lemma 6.1.1</u>

The square root of any integer that is not a perfect square is irrational.

Assume that

$$\sqrt{n} = \frac{a}{b}$$

where $n, a, b \in \mathbb{Z}^{+} and\ \frac{a}{b}$ is a reduced fraction, so $a\ and\ b$ are relative primes. Since n is not a perfect square, assume that $\frac{a}{b}$ is rational. Squaring both sides,

$$n = \frac{a^2}{b^2}$$

Since $a\ and\ b$ are relative primes, their squares are also relative primes. Therefore n cannot be an integer, which is contradiction, since we assumed that $n \in \mathbb{Z}^{+}$.

Theorem 5.3

Markov forms associated to Markov triples that contain a '1' have the same coefficient for terms u^2 and v^2, but with an opposite sign.

Example.

$$(1,1,1) \to u^2 + uv - v^2$$

or

$$(5,2,1) \to 5u^2 + 11uv - 5v^2.$$

We see that from the general Markov quadratic form

$$au^{2} + buv - cv^2$$

when the Markov triple is of the form $(x, y, z = 1), x \geq y \geq z = 1$, then $x = a = -c$.

We know from the defining equation of the coefficients and their relationship to constants k and l that

$$c = l - 3k = \frac{k^2 + 1}{x} - 3k = \frac{k^2 - 3kx + 1}{x}.$$

If we could show that $k^2 - 3kx + 1 = -x^2$, then $c = -x \rightarrow x = a = c$, concluding the theorem.

Before starting the proof, we need a preparatory lemma.

Lemma 5.3

$$y^2 \equiv -1 \; mod \; x$$

As we observe the triples containing a 1,

$$(1,1,1), (2,1,1), (5,2,1), (13,5,1), (34,13,1), \ldots$$

we see that $z = 1$ and each new triple is created in the way

$$(x, y, z) \rightarrow (3xz - y, x, z) = (3x - y, x, z) \; (since \; (z = 1))$$

Let x_{n-1} be the biggest term of a Markov triple of the form $(x_{n-1}, y, 1)$.

The next triple will be then $(x_n, \; x_{n-1}, 1)$.

The next one is $(x_{n+1}, \; x_n, 1)$.

This shows a recursive structure, because of the transformations

$$(x, y, z) \rightarrow (3xz - y, x, z) = (x', y', z')$$

we see that the 2nd triple's $y' = x$, which means that if we start from the first triples containing 1s, we obtain a recursive relationship.

$$(1,1,1) = (x_1, x_0, 1)$$

$$(2,1,1) = (3x_1 - x_0, x_1, 1) = (x_2, x_1, 1)$$

$$(5,2,1) = (3x_2 - x_1, x_2, 1) = (x_3, x_2, 1) \ldots$$

Thus, the middle term of the $(n+1)^{th}$ Markov triple is the previous triple's biggest term.

This recursive relationship can be defined as

$$x_n = 3x_{n-1} - x_{n-2}.$$

And by definition, in a Markov triple (x, y, z), if $x = x_n = 3x_{n-1} - x_{n-2}$, then $y = x_{n-1}$ and $z = 1$. This relationship is thus simply derived from the construction algorithm of new Markov triples and can be easily verified by checking the Markov tree.

Take the original equation of Markov triples

$$x^2 + y^2 + z^2 = 3xyz$$

Plug in $z = 1, x = x_n, y = x_{n-1}$.

Then the following equation is obtained:

$${x_n}^2 + {x_{n-1}}^2 + 1 = 3x_n x_{n-1}.$$

Check both LHS and RHS $mod\ x_n$.

RHS:

$$3x_n x_{n-1} \equiv 0\ mod\ x_n$$

LHS:

$${x_n}^2 \equiv 0\ mod\ x_n, 1 \equiv 1\ mod\ x_n \rightarrow {x_{n-1}}^2 \equiv -1\ mod\ x_n$$

$${x_{n-1}}^2 = y^2 \equiv -1\ mod\ x_n$$

Therefore, in a Markov triple $(x, y, z), z = 1, x \geq y \geq 1$,

$$y^2 \equiv -1\ mod\ x. Q.E.D.$$

<u>Proof 5.3</u>

The defining equation of k states that

$$yk \equiv \pm 1 \, mod \, x$$

According to lemma 5.3,

$$y^2 \equiv -1 \, mod \, x$$

therefore $k = y$.

The original Markov triple equation with $z = 1$

$$x^2 + y^2 + 1 = 3xy$$

can be expressed in terms of k as

$$x^2 + k^2 + 1 = 3xk$$

$$k^2 - 3xk + 1 = -x^2.$$

$$c = l - 3k = \frac{k^2 + 1}{x} - 3k = \frac{k^2 - 3kx + 1}{x} = \frac{-x^2}{x} = -x = -a.$$

$$a = -c.$$

$$Q.E.D.$$

This is true for all Markov triples $(x, y, z), z = 1, x \geq y \geq 1$. These values can be found in the upper branch of the Markov tree.

Conjectures and further research suggestions

This essay could have been developed in many different directions and I decided to investigate the relationship of continued fractions to Markov forms. However, there are other patterns in the continued fraction roots that might be interesting from the perspective of number theory. First of all, an interesting question would be to see why the continued fraction roots only contain ones and twos. Also, we could explore in a further research paper if there is any direct relationship between Markov triples and the continued fractions roots of their Markov forms.

Consider the following triples with their continued fraction solutions:

Triple	continued fraction solution to the triple's Markov form
(194,13,5)	$[0;\overline{2,1,1,2,2,1,1,1,1,2}]$
(13,5,1)	$[0;\overline{2,1,1,1,1,2}]$
(5,2,1)	$[0;\overline{2,1,1,2}]$

Table 4 – continued fraction concatenation conjecture example

<u>Observation</u>:

The greatest terms of (13,5,1) and (5,2,1) become the smaller terms of (194,13,5). The continued fraction solution to (194,13,5) equals

$$[0;\overline{2,1,1,2,2,1,1,1,1,2}] = [0;\overline{2,1,1,2}] + [0;\overline{2,1,1,1,1,2}]$$

where the $+$ operation concatenates the periods of the two periodic continued fractions. This relationship seems to hold generally for all Markov triples (visible in Table 2) which I formulate here as a conjecture. This relationship might be bridged by using the *continuants* of continued fractions, which is a new concept related to them. For more information on continuants of continued fractions, see source 3. We could also find relationships between Markov forms and their negative roots written in the continued fraction form

$$-[0;\overline{a_1, a_2, \ldots, a_n}].$$

As I mentioned in the essay, our research was limited to positive roots because of the word limitation. The last thing I would add is to investigate if there are any patterns in the continued fraction roots of Markov forms when the value of v of the form

$f_{x,y,z}(u,v) = xu^2 + (3x - 2k)uv + (l - 3k)v^2$,

is not one, but any integer. I personally found no nice patterns in the roots when I tried values other than one.

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