CS461 -
Written Assignment - 2
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* Problem Statement

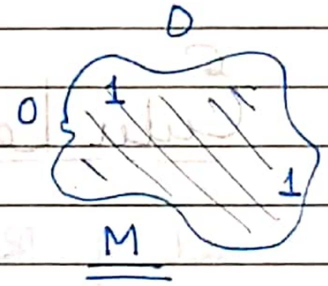
Can we define an implicit function for a ~~fu~~ surface, given a set of points.
(Poisson Surface Reconstruction)

* Solution

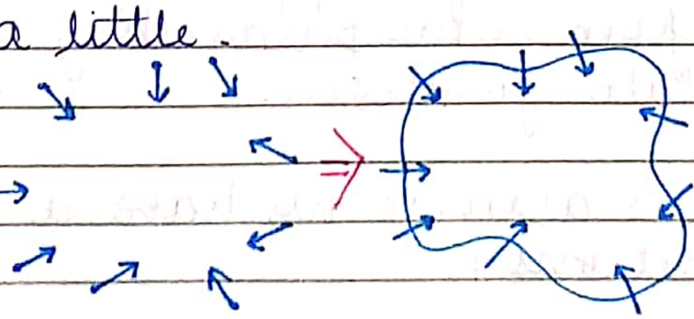
We need to define a function that represents the figure. Hence, the function should be positive inside the model and negative outside.

The key idea is to reconstruct the surface of the model by solving for the indicator function of the form,

$$\chi_M(p) = \begin{cases} 1 & \text{if } p \in M \\ 0 & \text{otherwise} \end{cases}$$



In practice, we define the indicator function to be -1/2 outside the model and 1/2 inside, so that the surface is a zero level set. We also smooth the function a little.



We need to achieve the following.

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In order to achieve this, we can make use of the relationship between normal field and the shape boundary, i.e. the gradient of the smooth indicator function.

>> Let us represent the indicator gradient by $\nabla \chi_M$, and the point normals by a vector field V.

Thus we need to find a function χ whose gradient best approximates V.

$$\min_{\chi} \| \nabla \chi - V \|^2$$

Now we convert this problem into a poisson problem by applying the divergence operator.

$$\nabla \cdot (\nabla \chi) = \nabla \cdot V \iff \Delta \chi = \nabla V$$

Δ ↳ Laplacian

Euler Lagrange Formation

Let $f: \mathbb{R} \to \mathbb{R}$, imagine we didnt know f, but we did know g i.e. df/dx.

But what if g is not analytically integrable?
We then look for approximate solutions, drawn from some parametrized family of candidate functions.

>> Let us assume we have a family of functions F.

Lets minimize the mean squared approximation error over some interval Ω and functions $f \in F$.

$$\text{minimize} \int_{\Omega} \left| \frac{df}{dx} - g \right|^2 dx$$

We know that,

$\int_{\Omega} L(x, f(x), f'(x))\, dx$ are obtained as the solution to the PDE;

$$\gg \frac{\partial L}{\partial f} - \frac{d}{dx} \cdot \frac{\partial L}{\partial f'} = 0$$

In our case,

$$L(x, f(x), f'(x)) = (f'(x) - g(x))^2$$

So, $\frac{\partial L}{\partial f} = 0$

$$\frac{\partial L}{\partial f'} = 2(f'(x) - g(x))$$

$$\frac{d}{dx} \cdot \frac{\partial L}{\partial f'} = 2(f''(x) - g'(x))$$

Thus, $\frac{\partial L}{\partial f} - \frac{d}{dx}\left(\frac{\partial L}{\partial f'}\right) = 0$

$$\Rightarrow \boxed{f'' = g'}$$

Now, we will need to discretize this.

» Sample n consecutive points $\{x_i\}$ from Ω. For simplicity's sake, let us assume that they are evenly spaced, thus

$$|x_{i+1} - x_i| = h$$

We want to maximize $\sum_i \left(f'(x_i) - g(x_i)\right)^2$

Thus we get that

$$f'(x_i) \approx \frac{f_{i+1} - f_i}{h} = \frac{1}{h}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix} f_i \\ f_{i+1}\end{bmatrix}$$

All the derivatives can be listed as a matrix ~~fun~~ multiplication $A.f = g$ where

$$A = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix}, \quad f = [f_1, f_2 \cdots f_n]^T$$

$$g = [g_1, g_2 \cdots g_n]^T$$

where, f and g are discrete approximation of the corresponding continuous function.

<u>Back to the problem</u>

We need to solve a set of equations, $Af = g$ in a least squares sense,

minimize $\|r\|^2 = \|g - Af\|^2$

The minimum is acheived when all the directional derivatives are zero giving the normal equations,

$$\boxed{A^T A f = A^T g}$$

The discrete operator A we constructed is full rank (invertible) and thus, gives an unique solution $A^{-1}g$ for f.

In higher dimensions

We have a function $f : \mathbb{R}^p \to \mathbb{R}^q$. The same discretization process can be applied on the domain as before.

Thus we can obtain discrete analogues of the gradient $\nabla(A)$, divergence $\nabla\cdot(-A^T)$ and laplacian $\Delta = (\nabla)\cdot\nabla(-A^TA)$

Note: We can convert a continuous variation problem into discrete one using mappings like conts function → discrete vector of values, conts operator → discrete matrix, function composition → matrix multiplication etc.

* **Conclusion:** To solve $\Delta x = \nabla . v$, we can discretize the system by converting the function to a vector of values at sample point and then solve for a least square fit.