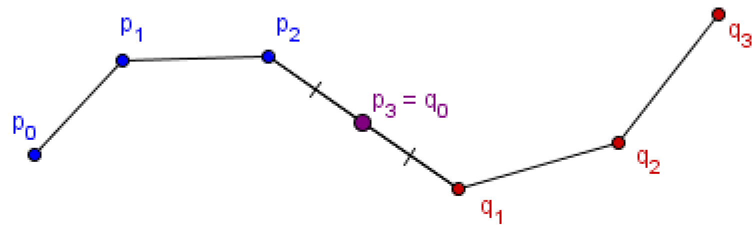
CS461 : WRITTEN ASSIGNMENT - 1

Devaishi Tiwari (170101021)

Problem Statement

Given 7 points, defining two cubic Bezier curves P{p0, p1, p2, p3} and Q{q0, q1, q2, q3}, how to ensure that the resulting merged curve is smooth (in terms of continuity).



Bezier Curves

As discussed in the lectures, general equation of a n-degree bezier curve can be written as,

$$P(a) = \sum_{i=0}^{n} {}^{n}C_i . a^i . (1 - a)^{n-i} . P_i ,$$

where Pi's are the control points, $\mathbf{P(0) = P_0}$ and $\boldsymbol{P(1) = P_3}$.

Ensuring Continuity

We will discuss three types of continuity namely, **positional continuity (C_0), tangential continuity (C_1)** and **curvature continuity (C_2)**. Let us discuss these one by one.

In our case, the equation of two bezier curves P{p0, p1, p2, p3} and Q{q0, q1, q2, q3},

$$P(a) = (1 - a)^3 p_0 + 3a(1 - a)^2 p_1 + 3a^2(1 - a)\, p_2 + a^3 p_3$$
$$Q(a) = (1 - a)^3 q_0 + 3a(1 - a)^2 q_1 + 3a^2(1 - a)\, q_2 + a^3 q_3$$

1. POSITIONAL CONTINUITY

When the last and first control points, of the curve segments P and Q respectively, are at the same coordinates, then these two curve segments are connected. This is known as positional continuity or **C_0 continuity**.

Mathematically speaking, to ensure positional continuity, P(1) should be equal to Q(0). We know that P(1) = p_3 and Q(0) = q_0, and $\boldsymbol{p_3 = q_0}$ is a sufficient and necessary condition.

Since we are given only seven points in the question $p_3 = q_0$, and thus C_0 continuity is present.

2. TANGENTIAL CONTINUITY (Continuity of Velocity)

Even when the two curves are connected, the transition from one curve into another might not always be smooth. For this transition to be smooth, the slope of tangents for both the curves should be equal at the connecting point($p_3 = q_0$). This is known as tangential continuity or **C_1 continuity**.

Thus, slope of tangent to P at point p_3 should be equal to the slope of tangent to Q at q_0. Mathematically speaking, **P'(a)** at **p_3** should be equal to **Q'(a)** at **q_0.**

$$P'(a) = \frac{d}{dt}(P(a))$$
$$= \frac{d}{dt}((1-a)^3 p_0 + 3a(1-a)^2 p_1 + 3a^2(1-a) p_2 + a^3 p_3)$$
$$= 3(1-a)^2 p_0 + [3(1-a)^2 - 6a(1-a)] p_1 + [6a(1-a) - 3a^2] p_2 + 3a^2 p_3$$
$$= 3(1-a)^2 p_0 + 3(1-a)(1-3a) p_1 + (6a - 9a^2) p_2 + 3a^2 p_3$$

Thus, $P'(0) = 3p_1 - 3p_0 = \mathbf{3(p_1 - p_0)}$ and $P'(1) = 3p_3 - 3p_2 = \mathbf{3(p_3 - p_2)}$

Similarly, $Q'(a) = 3(1-a)^2 q_0 + 3(1-a)(1-3a) q_1 + (6a - 9a^2) q_2 + 3a^2 q_3$, $Q'(0) = 3(q_1 - q_0)$ and $Q'(1) = 3(q_3 - q_2)$.

Hence, for C_1 continuity, we should ensure that, **P'(1) = Q'(0),**

=> $3(p_3 - p_2) = 3(q_1 - q_0)$ *[due to C_0 continuity, $p_3 = q_0$]*

=> $q_1 = 2p_3 - p_2$

3. CURVATURE CONTINUITY (Continuity of Acceleration)

Curvature continuity or C_0 continuity ensures that the curvature of the curve P and Q is equal at the connecting point($p_3 = q_0$). Hence, the double differentiation of the equation of curves should be equal at p_3, **P"(a)** at **p_3** should be equal to **Q"(a)** at **q_0.**

$$P''(a) = \frac{d}{dt}(P'(a))$$
$$= \frac{d}{dt}(3(1-a)^2 p_0 + 3(1-a)(1-3a) p_1 + (6a - 9a^2) p_2 + 3a^2 p_3)$$
$$= 6(1-a) p_0 + [-3(1-3a) - 9(1-a)] p_1 + (6 - 18a) p_2 + 6a p_3$$
$$= 6(1-a) p_0 + (18a - 12) p_1 + (6 - 18a) p_2 + 6a p_3$$

Thus, $P''(0) = 6p_0 - 12p_1 + 6p_2$ and $P''(1) = 6p_1 - 12p_2 + 6p_3$

Similarly, $Q'(a) = 6(1-a) q_0 + (18a - 12) q_1 + (6 - 18a) q_2 + 6a q_3$, $Q''(0) = 6q_0 - 12q_1 + 6q_2$ and $Q''(1) = 6q_1 - 12q_2 + 6q_3$.

Hence, for C_2 continuity, we should ensure that, **P"(1) = Q"(0),**

=> $6p_1 - 12p_2 + 6p_3 = 6q_0 - 12q_1 + 6q_2$

=> $p_1 - 2p_2 + p_3 = q_0 - 2q_1 + q_2$

=> $p_1 - 2p_2 = -2q_1 + q_2$ *[due to C_0 continuity, $p_3 = q_0$]*

=> $q_2 = p_1 - 2p_2 + 2(2p_3 - p_2)$ *[due to C_1 continuity, $q_1 = 2p_3 - p_2$]*

=> $q_2 = 4(p_3 - p_2) + p_1$