Probability and Statistics

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Lecture-1

Text Book and References

Text Book Devore, J. L., : Probability and Statistics for Engineering and the Sciences, 9th Edition, Cengage Learning, 2022.

Reference Books

- Milton, J. S. and Arnold J. C., Introduction to Probability and Statistics: Principles and Applications for Engineering and the Computing Sciences, 4th Edition, Tata McGraw-Hill, 2007.
- Walpole, R. E., Myers, R. H., Myers, S. L., Ye, K. E., Probability and Statistics for Engineers and Scientists, 9th Edition, Pearson Education, 2016.
- Johnson, R. A., Miller Freund Probability and Statistics for Engineers, 8th Edition, PHI, 2010.
- Meyer, P. L., Introductory Probability and Statistical appl . 2nd Edition, Addison-Wesley, 1970.
- Ross, S. M., Introduction to Probability Models, 11th Edition, Academic Press, 2014.

Scope and Objectives

- Probability and statistics form an exciting sub-area of mathematical science.
- Relevance in almost all disciplines concerned with data and uncertainty.
- Probability theory deals with real life problems, which either inherently involve the chance phenomena or describe the behavior of a system.
- Statistics deals with data through an experiment.
- Statistical analysis is built up on the concepts of probability theory.
- Interpretation of a process in many engineering aspects often depends on the ideas of probability and statistics coupled with computational aspects.
- In this fundamental course, the aim is to: **build up skills in understanding the concepts of random variable, probability distribution, statistical inference, regression and correlation among several other related topics.**

Applications

- Mathematics (Groundwater modeling, Aerodynamics etc.)
- Economics (Share market)
- Biology
- Medical Science
- Business and many more.

Probabilities

Three methods are widely accepted to assign the probability.

Personal Probability

Based on personal judgment or feeling.
Advantage: It is always applicable.
Disadvantage: Accuracy depends upon the accuracy of the information available and the ability of the scientist to assess the information correctly.

Relative Frequency Interpretation

To decide the probability of an event, we perform the experiment large number of times independently. Probability of occurrence of event A:

$$P[A] = f/n,$$

$n=$ no. of trials, $f =$ no. of times the event occurred in n trials, for large n.

Probabilities

Classical Probability

Assumption that all outcomes of the experiment are equally likely for an event A:

$$P(A) = \frac{\text{number of ways event A can occur}}{\text{no. of ways experiment can run}}$$

Definitions

Experiment: any process (physical action) that yields a result or an observation.

Deterministic experiment: if the result can be predicted with certainty prior to the performance of the experiment.

Random experiment: if the outcome can not be predicted with certainty but all possible outcomes can be determined prior to the performance of the experiment.
Ex.- Tossing a fair coin, throwing a dice, etc.

Outcome: a particular result of an experiment.

Sample spaces

Sample space: sample space of an experiment, denoted by $\mathcal{S}$, is the set of all possible outcomes of that experiment.

Ex: simplest experiment to which probability applies is one with two possible outcomes.

One such experiment consists of examining a single fuse to see whether it is defective.
Tossing a fair coin:

$$\mathcal{S} = \{D, ND\}; \{H, T\}.$$

Ex.- Four univ.- 1, 2, 3, 4 are participating in a holiday tournament. In the first round, 1 will play 2 and 3 will play 4. Then the two winners will play for the championship, and the two losers will also play. One possible outcome can be 1324 (1 beats 2 and 3 beats 4 in first-round games, and then 1 beats 3 and 2 beats 4).
List all possible outcomes in sample space $\mathcal{S}$.

Sample spaces

Ex.- Tossing a coin, and then tossing a coin for head and die for tail, then

$$\mathcal{S} = \{HH, HT, T1, \ldots, T6\}$$

Example of infinite sample space:

Ex.- Suppose a coin is tossed till the head appears. Then

$$\mathcal{S} = \{H, TH, TTH, \ldots\}.$$

Events

In our study of probability, we will be interested not only in the individual outcomes of $\mathcal{S}$ but also in various collections of outcomes from $\mathcal{S}$.

Events

An event is any collection (subset) of outcomes contained in the sample space $\mathcal{S}$. If $\mathcal{S} = \{H, T\}$, then the sets $\phi, \{H\}, \{T\}, \{H, T\}$ are events.

Simple/Elementary and Compound Events:
Simple if it consists of exactly one outcome, ex.- $\{H\}, \{T\}$

Compound if it consists of more than one outcome, $\{H, T\}$.

Definitions

Equally likely events: Are the events with the same chance of occurrence.
Ex.- If $\mathcal{S} = \{H, T\}$, then $\{H\}$ and $\{T\}$ equally likely.

Mutually exclusive events: Two events A, B in $\mathcal{S}$ are called mutually exclusive if $A \cap B$ is an empty set. OR, happening of one event excludes the happening of other. Same example as above.

More than 2 events: events $A_1, A_2, \ldots$ are mutually exclusive if for any $i \neq j$,

$$A_i \cap A_j = \phi.$$

Example

Consider an experiment in which each of three vehicles taking a particular freeway exit turns left (L) or right (R) at the end of the exit ramp.

The eight possible outcomes that comprise the sample space are

$$\mathcal{S} = \{LLL, RLL, LRL, LLR, LRR, RLR, RRL, RRR\}.$$

Some compound events:
$A = \{RLL, LRL, LLR\}$ = the event that exactly one of the three vehicles turns right

$B = \{LLL, RLL, LRL, LLR\}$ = the event that at most one of the vehicles turns right

$C = \{LLL, RRR\}$ = the event that all three vehicles turn in the same direction

Suppose that when the experiment is performed, the outcome is $\{LLL\}$. Then the simple event E_1 has occurred and so have the events B and C (but not A).

Some Relations from Set Theory

An event is just a set, so relationships and results from elementary set theory can be used to study events. **Recap:**

Complement of an event A, denoted by A': set of all outcomes that are not contained in A.

Union of two events $A \cup B$: (read as A or B) is the event consisting of all outcomes that are either in A or in B or in both events (i.e., all outcomes in at least one of the events.)

Some Relations from Set Theory

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Union of two events $A \cup B$: (read as A or B) is the event consisting of all outcomes that are either in A or in B or in both events (i.e., all outcomes in at least one of the events.)

Intersection of two events $A \cap B$: (read as A and B) is the event consisting of all outcomes that are in both A and B.

When $A \cap B = \phi$, A and B are said to be mutually exclusive or disjoint events.

Extension to more than two events: For any three events A, B, and C, the event $A \cup B \cup C$ is the set of outcomes contained in at least one of the three events, whereas $A \cap B \cap C$ is the set of outcomes contained in all three events.

Given events $A_1, A_2, A_3, \ldots$, these events are said to be mutually exclusive (or pairwise disjoint) if no two events have any outcomes in common.

Product Rule for Ordered Pairs

If the first element or object of an ordered pair can be selected in q_1 ways, and for each of these q_1 ways the second element of the pair can be selected in q_2 ways, then the number of pairs is $q_1 \times q_2$.

General Product Rule of counting:

Suppose a set consists of ordered collections of r elements (r-tuples) and that there are q_1 possible choices for the first element; for each choice of the first element, there are q_2 possible choices of the second element; . . . ; for each possible choice of the first $r-1$ elements, there are q_r choices of the rth element. Then there are $q_1 \times q_2 \times \ldots q_r$ (r tuples).

If an experiment is taking place in r stages and q_i denote the number of ways ith stage can occur then the experiment can be performed in $q_1 \times q_2 \times \ldots q_r$ ways.

Application of Product Rule

Permutations: A permutation of n distinct objects taken r at a time (r permutation of n objects) is an ordered selection or arrangement of r of the objects.
Number of permutations of any r objects from given n distinct objects is:

$$P_{r,n} = (n!)/(n-r)!.$$

Number of permutations of n different things taken r at a time with unlimited Repetitions is n^r.

Application of Product Rule

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Number of permutations of n different things taken r at a time with unlimited Repetitions is n^r.
Permutations of indistinguishable objects : If we are permuting n objects of m distinguishable types such that there are q_i (indistinguishable) objects of ith type for $i = 1, 2, \ldots, m$, then number of distinct arrangements of these n objects is

$$\frac{(n)!}{(q_1!)(q_2!)\ldots(q_m!)}$$

Ex: How many different words can be made from the letters in the word Statistics?
Soln: $10!/(3!3!2!) = 50400$.

Combinations

A combinations of n distinct objects taken r at a time (r combination of n objects) is

$$C_{r,n} = \frac{P_{r,n}}{r!} = \frac{n!}{(n-r)!r!}.$$

Relation between permutation and combinations is:

$$C_{r,n} r! = P_{r,n}.$$

Ex: In how many ways, 2 players can be selected from a group of 10 players.
Soln: $10!/(8!2!)$.

Thanks!! Questions??

Axioms, Interpretations, and Properties of Probability

Axioms, Interpretations, and Properties of Probability

Given an experiment and a S.S $\mathcal{S}$, objective of probability is to assign to each event A, a real number P(A), called the probability of the event A. This gives a precise measure of the chance that A will occur. So, P(.) is a function defined on subsets of $\mathcal{S}$

All assignments should satisfy the following axioms: (basic properties of probability)

Axiom 1

For any event A, $P(A) \geq 0$. (Non-negativity)

Axioms, Interpretations, and Properties of Probability

Axiom 2

$P(\mathcal{S}) = 1$ (Axioms of certainty)

S.S., by defn., is the event that must occur when the experiment is performed (contains all possible outcomes). Axiom 2 says: maximum possible probability of S.S. =1.

Axiom 3

If $A_1, A_2, A_3, \ldots$ are finite or an infinite collection of mutually exclusive or disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)$$

(Axiom of additivity)

Axioms, Interpretations, and Properties of Probability

Axiom A3 formalizes the idea that if we wish the probability that at least one of a number of events will occur and no two of the events can occur simultaneously, then prob. of at least one occurring is the sum of the prob. of individual events.

<u>Properties of Probability</u>

Proposition (i):

$P(\varnothing) = 0$ where $\varnothing$ is the null event (the event containing no outcomes.

Proof:

Axioms, Interpretations, and Properties of Probability

Proof:

Proposition (ii): $\boldsymbol{P[A'] = 1 - P[A]}$

Proof:

General addition rule

The probability that at least one of the two events will occur when the events are not necessarily mutually exclusive.

For any two events A and B of a S.S.;

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

Proof:

More Probability Properties

The probability of a union of more than two events can be computed analogously.

For any three events A, B, and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) \\ - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Theorem: If $A \subseteq B$, then $P[A] \leq P[B]$

Proof: Decompose B into two mutually exclusive events as

Example-1

Consider tossing a thumbtack in the air. When it comes to rest on the ground, either its point will be up (the outcome U) or down (the outcome D). The sample space for this event is therefore $\mathcal{S} = \{U, D\}$.

The axioms specify $P(\mathcal{S}) = 1$, so the probability assignment will be completed by determining $P(U)$ and $P(D)$.

Since U and D are disjoint and their union is $\mathcal{S}$, the foregoing proposition implies that

$$1 = P(\mathcal{S}) = P(U) + P(D)$$

Example-1

cont'd

It follows that $P(D) = 1 – P(U)$.

One possible assignment of probabilities is

$$P(U) = .5,\ P(D) = .5,$$

whereas another possible assignment is

$$P(U) = .75,\ P(D) = .25.$$

In fact, letting p represent any fixed number between 0 and 1, $P(U) = p$, $P(D) = 1 – p$ is an assignment consistent with the axioms.

Example -2

Consider testing batteries coming off an assembly line one by one until one having a voltage within prescribed limits is found.

The simple events are

$E_1 = \{S\}$, $E_2 = \{FS\}$,

$E_3 = \{FFS\}$, $E_4 = \{FFFS\}$, ...

Suppose the probability of any particular battery being satisfactory is .99.

Example-2

cont'd

Then,

$P(E_1) = .99$, $P(E_2) = (.01)(.99)$,

$P(E_3) = (.01)^2(.99)$, . . . is an assignment of probabilities to the simple events that satisfies the axioms.

As, E_i are disjoint and $\mathcal{S} = E_1 \cup E_2 \cup E_3 \cup \ldots$,

$$1 = P(\mathcal{S}) = P(E_1) + P(E_2) + P(E_3) + \cdots$$

$$= .99[1 + .01 + (.01)^2 + (.01)^3 + \cdots]$$

Interpreting Probability

Examples show: axioms do not completely determine an assignment of probabilities to events.

The axioms serve only to rule out assignments inconsistent with our intuitive notions of probability.

In Example 1, two particular assignments were suggested.

The appropriate or correct assignment depends on the nature of the thumbtack and also on one's interpretation of probability.

Most frequently used and most easily understood interpretation is based on the notion of **relative frequencies.**

Interpreting Probability

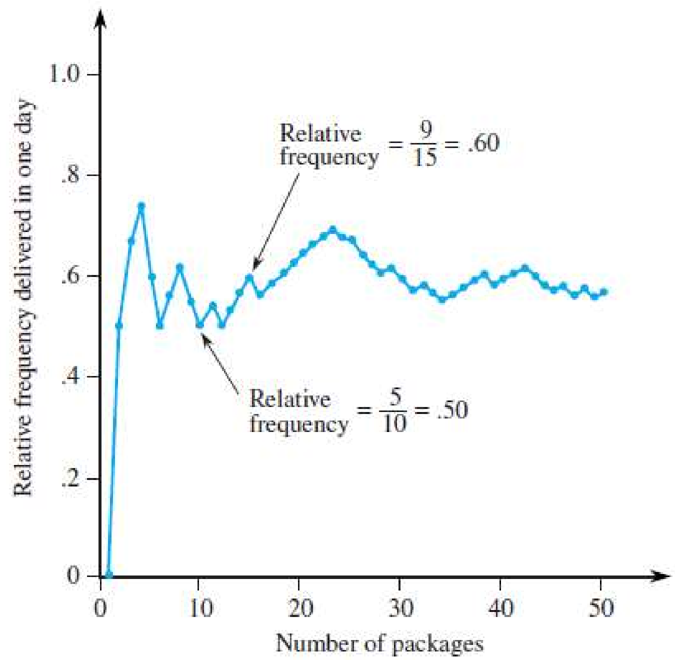
For example, let *A* be the event that a package sent within the state of California for 2nd day delivery actually arrives within one day.

The results from sending 10 such packages (the first 10 replications) are as follows:

Package #	1	2	3	4	5	6	7	8	9	10
Did *A* occur?	N	Y	Y	Y	N	N	Y	Y	N	N
Relative frequency of *A*	0	.5	.667	.75	.6	.5	.571	.625	.556	.5

Interpreting Probability

Figure 2.2(a) shows how the relative frequency $n(A)/n$ fluctuates rather substantially over the course of the first 50 replications.

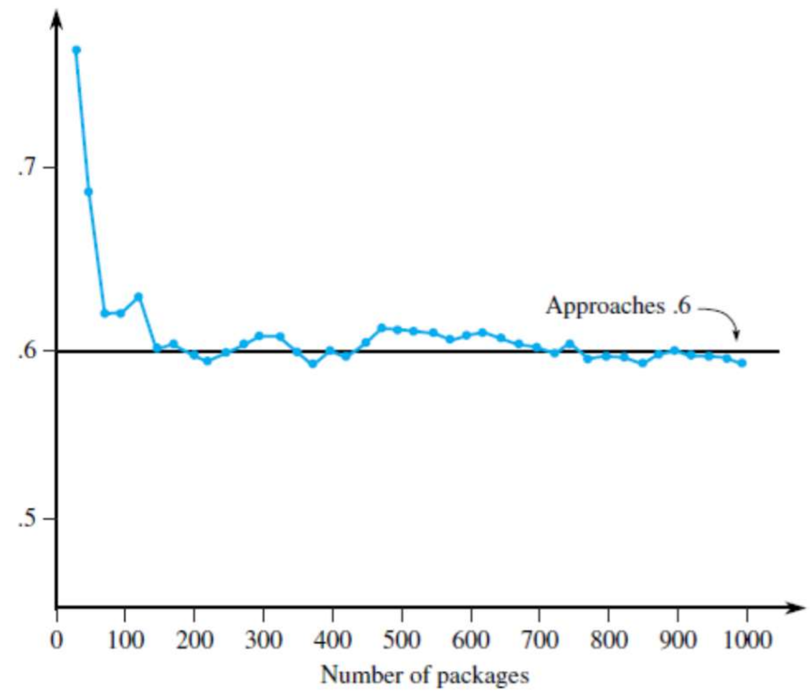


Behavior of relative frequency (a) Initial fluctuation

Figure 2.2

Interpreting Probability

But as the number of replications continues to increase, Figure 2.2(b) illustrates how the relative frequency stabilizes.



Behavior of relative frequency (b) Long-run stabilization

Figure 2.2

Interpreting Probability

This indicates that any relative frequency of this sort will stabilize as the number of replications n increases.

That is, as n gets arbitrarily large,

$$n(A)/n$$

approaches a limiting value referred to as the *limiting relative frequency* of the event A, $P(A)$.

Conditional Probability
Lecture-3

Conditional Probability

> Probabilities assigned to various events depend on what is known about the *experimental situation* when the assignment is made.

> Subsequent to initial assignment, partial information relevant to the outcome of the experiment may become available. Such information may cause us to revise some of our probability assignments.

For a particular event A, $P(A)$ represents the probability, assigned to A

Now, think of $P(A)$ as the original or *unconditional* prob.

Conditional Probability

Today, we study how the information “an event B has occurred” *affects the probability assigned to A.*

Example: A might refer to an individual having a particular disease in the presence of certain symptoms.

If a blood test is performed & result is negative (say B = negative blood test), then prob. of having the disease will change.

Use the notation $P(A \mid B)$ to represent the **conditional prob. of *A* given that the event *B* has occurred.** B is the “conditioning event.”

Example

Complex components are assembled in a plant that uses two different assembly lines, A and A'.

Line A uses older equipment than A', so it is somewhat slower and less reliable.

Suppose on a given day line A has assembled 8 components: 2 have been identified as defective (event B) & 6 as non-defective (B').

Whereas A' has produced 1 defective and 9 non-defective components.

Example

cont'd

Summarized in the table as:

		Condition	
		B	B'
Line	A	2	6
	A'	1	9

Unaware of this information, the sales manager randomly selects 1 of these 18 components for a demonstration.

Prior to the demonstration

$$P(\text{line } A \text{ component selected}) = P(A) = \frac{N(A)}{N} = .44$$

Example

cont'd

However, if chosen component turns out to be defective, then the event *B* has occurred.

So the component must have been 1 of the 3 in the *B* column of the table.

Since these 3 components are equally likely among themselves after *B* has occurred,

$$P(A \mid B) = \frac{2}{3} = \frac{2/18}{3/18} = \frac{P(A \cap B)}{P(B)}$$

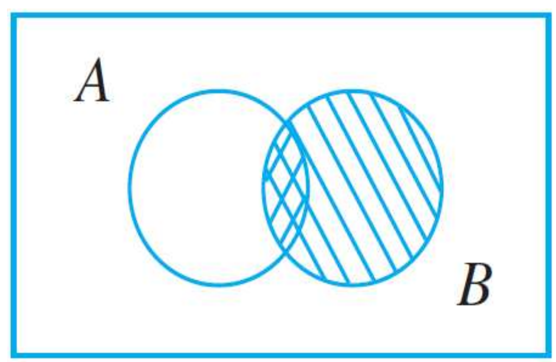
Conditional Probability

So, conditional prob. is written as a ratio of unconditional prob. where:

Numerator is prob. of the intersection of two events & denominator is prob. of conditioning event *B*.

See, the Venn diagram:

Given that *B* has occurred, relevant sample space is no longer *S* but consists of outcomes in *B*



Definition of Conditional Prob.

For any two events A and B with $P(B) > 0$, c**onditional prob. of *A* given that *B* has occurred** is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- Satisfy all the axioms:

Defn. of Cond. Prob.

Ex:

Assuming all individuals buying a certain digital camera, 60% include an optional memory card, 40% include an extra battery, & 30% include both a card and battery.

Consider randomly selecting a buyer and let

A = {memory card purchased} ; B = {battery purchased}.

Then $P(A) = .60$, $P(B) = .40$,

$P(\text{both purchased}) = P(A \cap B) = .30$

Ex

Given that: selected individual purchased an extra battery, prob. that an optional card was also purchased is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{.30}{.40} = .75$$

That is, of all those purchasing an extra battery, 75% purchased an optional memory card. Similarly,

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{.30}{.60} = .50$$

Notice that $P(A \mid B) \neq P(A)$ and $P(B \mid A) \neq P(B)$.

Multiplication Rule for $P(A \cap B)$

Multiplication Rule for $P(A \cap B)$

Definition of conditional prob. yields the following result,

The Multiplication Rule

$$P(A \cap B) = P(A \mid B) \cdot P(B)$$

Important: $P(A \cap B)$ is desired, whereas both $P(B)$ and $P(A \mid B)$ can be specified from the problem description.

Consideration of $P(B \mid A)$ gives $P(A \cap B) = P(B \mid A) \cdot P(A)$

Ex

Four individuals have responded to a request by a blood bank for blood donations. None of them has donated before, so their blood types are unknown.

Suppose only type O+ is desired and only one of the four actually has this type. If the potential donors are selected in random order for typing, what is the prob. that at least three individuals must be typed to obtain the desired type?
Solve:

B = {first type not O+} and

A = {second type not O+}, $P(B) = \frac{3}{4}$.

Given that the first type is not O+, two of the three individuals left are not O+, so $P(A \mid B) = \frac{2}{3}$.

The multiplication rule now gives

P(at least three individuals are typed) = *P*(*A* ∩ *B*)

$$= P(A \mid B) \cdot P(B)$$

$$= \frac{2}{3} \cdot \frac{3}{4} = \frac{6}{12}$$

$$= .5$$

Extension to more than 2 events

The rule is easily extended to experiments involving more than two stages.

For example,

$$P(A_1 \cap A_2 \cap A_3) = P(A_3 \,|\, A_1 \cap A_2) \cdot P(A_1 \cap A_2)$$

$$= P(A_3 \,|\, A_1 \cap A_2) \cdot P(A_2 \,|\, A_1) \cdot P(A_1)$$

where A_1 occurs first, followed by A_2, and finally A_3.

Lecture-4

Total Probability, Bayes' Theorem

Ex 1:

Urn L contains **x** red balls and y black balls

Urn F contains z red balls and w black balls. A ball is randomly chosen from the Urn L and placed in the Urn F. Then a ball is randomly selected from the Urn F and placed in the Urn L.

(i) Find the prob. that a red ball is selected from the Urn L and a red ball is selected from the Urn F?

Solution: P(event in question)

= P(Red ball from Urn L $\cap$ Red ball from Urn F)

$$=\left(\frac{z+1}{z+w+1}\right)\left(\frac{x}{x+y}\right)$$

(ii) At the conclusion of the selection process, what is the probability that the number of red and black balls in the Urn L are identical to the numbers at the beginning?

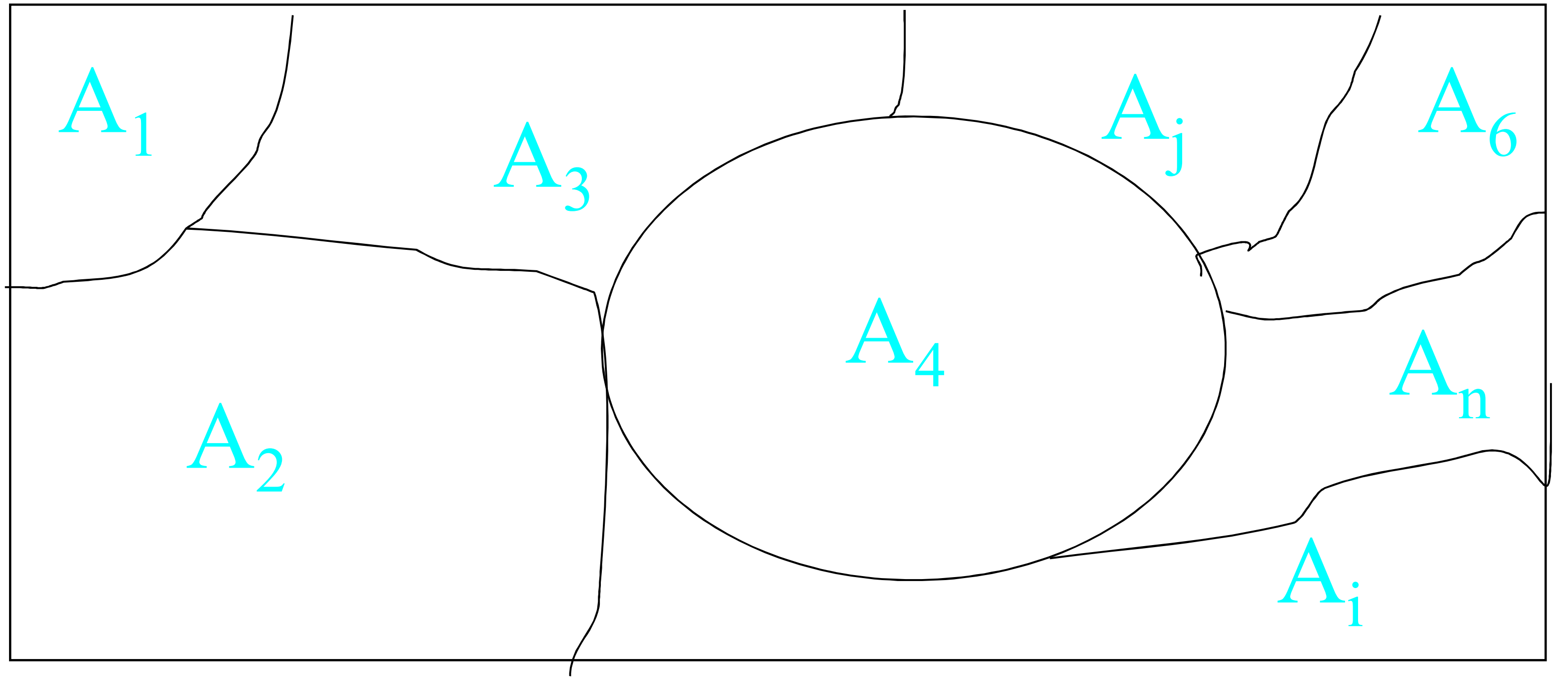
P(event in question)=
= P(Red ball from Urn L $\cap$ Red ball from Urn F)
+ P(black ball from Urn L $\cap$ black ball from Urn F)

$$\left(\frac{z+1}{z+w+1}\right)\left(\frac{x}{x+y}\right)+\left(\frac{w+1}{z+w+1}\right)\left(\frac{y}{x+y}\right)$$

Partition of sample space:

Events $A_1, A_2, \ldots, A_n$ represent a partition of the sample space S if:

1. $A_i \cap A_j = \Phi$ for all $i \neq j$.
2. $A_1 \cup A_2 \ldots\ldots \cup A_n = S$
3. $P[A_i] > 0$ for all i



Total Probability Law

The Law of Total Probability

Let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events. Then for any other event B,

$$P(B) = P(B|A_1)P(A_1) + \cdots + P(B|A_k)P(A_k)$$
$$= \sum_{i=1}^{k} P(B|A_i)P(A_i) \qquad (2.5)$$

Ex 2:

Urn L contains x red & y black balls

Urn F contains z red & w black balls.

A ball is chosen at random from urn L and put in the Urn F then a ball is chosen from urn F.

Find the prob. that ball chosen from urn F is red.

A = ball chosen from the Urn L is Red
A′= ball chosen from the Urn L is black

B = ball chosen from the Urn F is Red

P(B)= P(B|A)P(A)+P(B|A′)P(A′)

$$=\left(\frac{z+1}{z+w+1}\right)\left(\frac{x}{x+y}\right)+\left(\frac{z}{z+w+1}\right)\left(\frac{y}{x+y}\right)$$

Ex3:

An individual has 3 different email accounts. Most of her messages, in fact 70%, come into account #1, whereas 20% come into account #2 and the remaining 10% into account #3.

Of the messages into account #1, only 1% are spam, whereas the corresponding percentages for accounts #2 and #3 are 2% and 5%, respectively.

What is the probability that a randomly selected message is spam?

Ex3:

cont'd

Lets define some notations:
A_i = {message is from account # i} for i = 1, 2, 3.
B = {message is spam}

Then the given percentages imply that

$$P(A_1) = .70,\ P(A_2) = .20,\ P(A_3) = .10$$

$$P(B \mid A_1) = .01,\ P(B \mid A_2) = .02,\ P(B \mid A_3) = .05$$

Now using the law of total probability:

$$P(B) = (.01)(.70) + (.02)(.20) + (.05)(.10) = .016$$

In the long run, 1.6% of this individual's messages will be spam.

Baye's Theorem:

The computation of a posterior probability $P(A_j | B)$ from given prior probabilities $P(A_j)$ and conditional probabilities $P(B | A_i)$ occupies a central position in elementary probability.

Let $A_1, A_2, \ldots, A_k$ be a collection of k mutually exclusive and exhaustive events, union is the sample space, with *prior* probabilities $P(A_i)$ ($i = 1,\ldots, k$).

Then for any other event B for which $P(B) > 0$, the *posterior* probability of A_j given that B has occurred is

$$P(A_j | B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B | A_j)P(A_j)}{\sum_{i=1}^{k} P(B | A_i) \cdot P(A_i)} \quad j = 1, \ldots, k$$

Baye’s Theorem:

The transition from the second to the third expression in above equation relies on using the:

multiplication rule in the numerator

law of total probability in the denominator.

As long as there are relatively few events in the partition, posterior probabilities can be computed without referring explicitly to Bayes’ theorem.

Ex4:

Four units of a bulb making factory, respectively, produce 3%, 2%, 1% and 0.5% defective bulbs. A bulb selected at random from the entire output is found defective.

Find the probability that it is produced by the fourth unit of the factory.

Solution: Define the events as

A : the bulb is defective,

B1 : the bulb is made by 1st unit,

B2 : the bulb is made by second unit,

B3 : the bulb is made by third unit,

B4 : the bulb is made by fourth unit.

Ex4:

From the given data:

P(B1) = 0.25= P(B2) = P(B3) = P(B4)

P(A|B1) = 0.03, P(A|B2) = 0.02,

P(A|B3) = 0.01, P(A|B4) = 0.005.

Then the required probability is:

P(B4|A) = P(B4)P(A|B4)/P(A) where

P(A)=P(B1)P(A|B1)+P(B2)P(A|B2)+P(B3)P(A|B3)+P(B4)P(A|B4)

=(0.25*0.005)/(0.25)[0.03+0.02+0.01+0.005]

= 1/13.

Ex 5:

In answering a question on a multiple choice test, a student either knows the answer or guesses.

Let 3/4 be the prob. that he knows the answer &

1/4 be the prob. that he guesses.

Assuming that a student who guesses at the answer will be correct with prob. 1/4.

What is the prob. that the student knows the answer given that he answered it correctly?

Ans. B1=He knows the answer; B2= He guesses the answer; A= He answers correctly

$$P(B1|A) = P(B1)P(A|B1)/[P(B1)P(A|B1)+P(B2)P(A|B2)]$$

$$= (3/4)(1)/[(3/4)(1)+(1/4)(1/4)] = 12/13$$

Some Results:

Ex: Show that $P(B \cup C|A) = P(B|A) + P(C|A) - P(B \cap C|A)$

Some Results:

Ex: Show that

$$P(C|A \cup B) = [P(A)P(C|A) + P(B)P(C|B) - P(A \cap B)P(C|A \cap B)]/P(A \cup B)$$

Ex:

Two cards are drawn from a deck of cards without replacement. Find the prob. that the 2^{nd} card is an ace given that the 1^{st} card is a red card.

Soln. The 1^{st} red card may or may not be an ace.

Let A: event that the 1^{st} red card is an ace

B: event that the 1^{st} red card is not an ace, and

C: event that the second card is an ace.

Then prob.=

Lecture-5
Independence

Independence

Two events A and B are said to be independent if occurrence or non-occurrence of A does not affect the occurrence or non-occurrence of the other.

Definition

Two events *A* and *B* are **independent** if $P(A \mid B) = P(A)$ and are **dependent** otherwise.

So,

$P(A \cap B) = P(A)P(B)$ is the mathematical condition for the independence of the events A and B.

Note that, three events A, B, C are independent provided these are pairwise independent and

$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

Independence

However, using the definition of conditional probability and the multiplication rule,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$$

The right-hand side of equation is $P(B)$ if and only if

$$P(A|B) = P(A) \text{ (independence)}$$

Note that: it is also straightforward to show that if A and B are independent, then so are the following pairs of events:

Results: (1) A' and B (2) A and B' and (3) A' and B'

Proof (results 1&3):

Ex: Let A and B are mutually exclusive events such that $P(A)P(B)>0$. Then show that A and B are not independent events.

Solution:

Ex: Show that impossible event is independent of every other events.

Solution:

Ex1:

It is known that 30% of a certain company's washing machines require service while under warranty, whereas only 10% of its dryers need such service.

If someone purchases both a washer and a dryer made by this company, what is the probability that both machines will need warranty service?

Ex1:

Let

A: event that washer needs service while under warranty,

B: defined analogously for the dryer.

Then $P(A) = .30$ and $P(B) = .10$.

Assuming that the two machines will function independently of one another, the desired prob. is

$$P(A \cap B) = P(A) \cdot P(B) = (.30)(.10) = .03$$

Independence of More Than Two Events

The notion of independence of two events can be extended to collections of more than two events.

Definition

Events $A_1, \ldots, A_n$ are **mutually independent** if for every k ($k = 2, 3, \ldots, n$) and every subset of indices $i_1, i_2, \ldots, i_k$,

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \cdots \cdot P(A_{i_k})$$

*Note that: A_i are independent, implies that complements of A_i are independent. Therefore, one can replace one or more of the A_i by their complements. There are $2^n - n - 1$ conditions which are to be checked for independence of n events.

Ex: A transmitter is sending a message by using a code, namely, a sequence of 0's and 1's. Each transmitted bit (0 or1) must pass through three relays to reach the receiver. At each relay, the probability is 0.20 that the bit sent will be different from the bit received (a reversal). Assume that the relays operate independently of one another.

Transmitter $\rightarrow$Relay 1 $\rightarrow$ Relay 2 $\rightarrow$ Relay 3 $\rightarrow$ Receiver

(i) If a 1 is sent from the transmitter, what is the probability that a 1 is sent by all three relays?

(ii) If a 1 is sent from the transmitter, what is the probability that a 1 is received by the receiver?

Solution: (i) If a 1 is sent from the transmitter, what is the prob. that a 1 is sent by all three relays?

A= '1' is received and '1' is sent by Relay 1

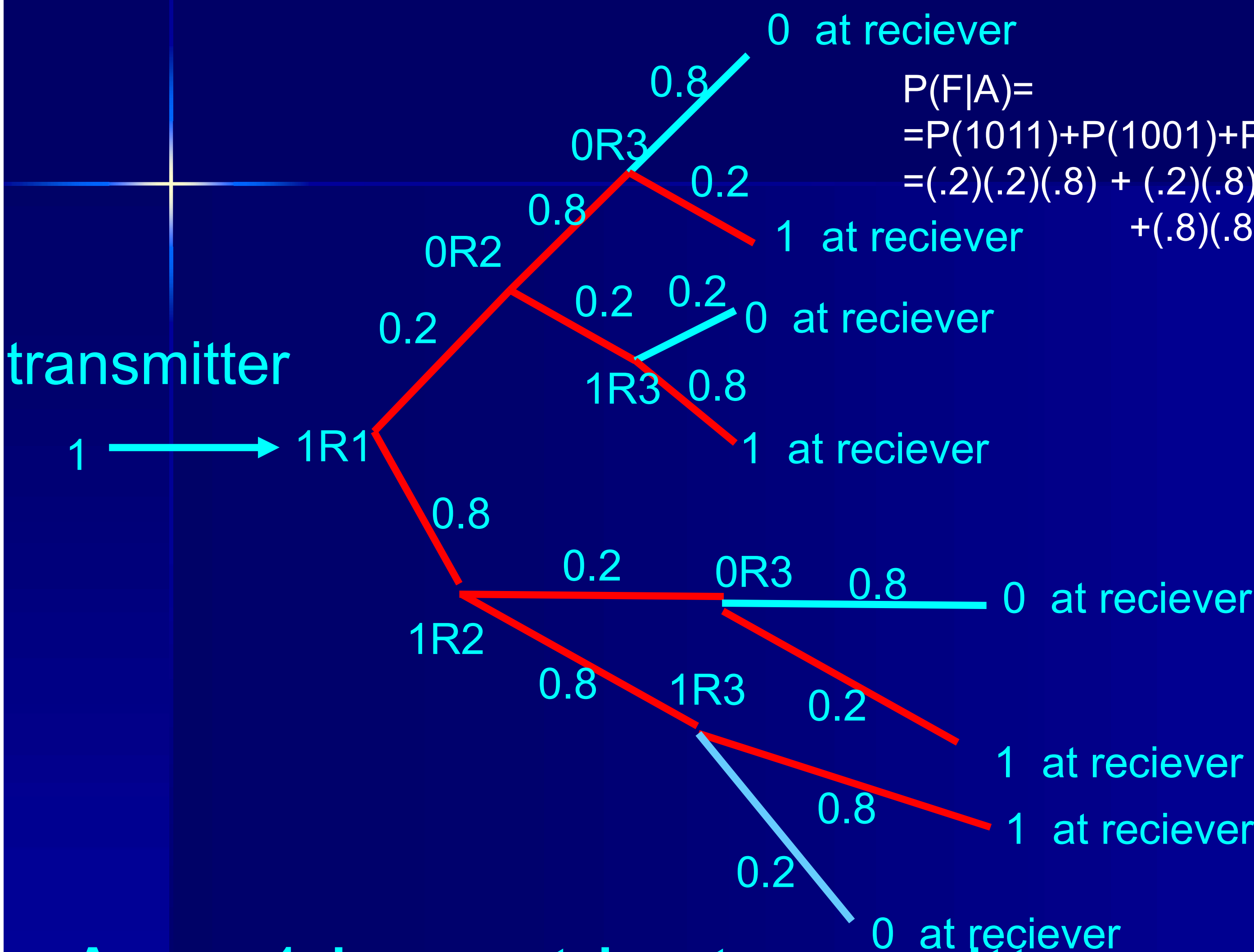
B= '1 ' is received and '1' is sent by Relay 2

C= '1 ' is received and '1' is sent by Relay 3

$P(A)=P(B)=P(C)=0.8$

Hence, $P(A\cap B\cap C)=P(A)P(B)P(C)$ as they are independent.

Solution: (ii) If a “1”is sent from the transmitter, what is the prob. that a “1” is received by the receiver?



P(F|A)=
$=P(1011)+P(1001)+P(1101)+P(1111)$
$=(.2)(.2)(.8) + (.2)(.8)(.2) + (.2)(.2)(.8)$
$+(.8)(.8)(.8)$

A = 1 is sent by transmitter

F = 1 is received by receiver

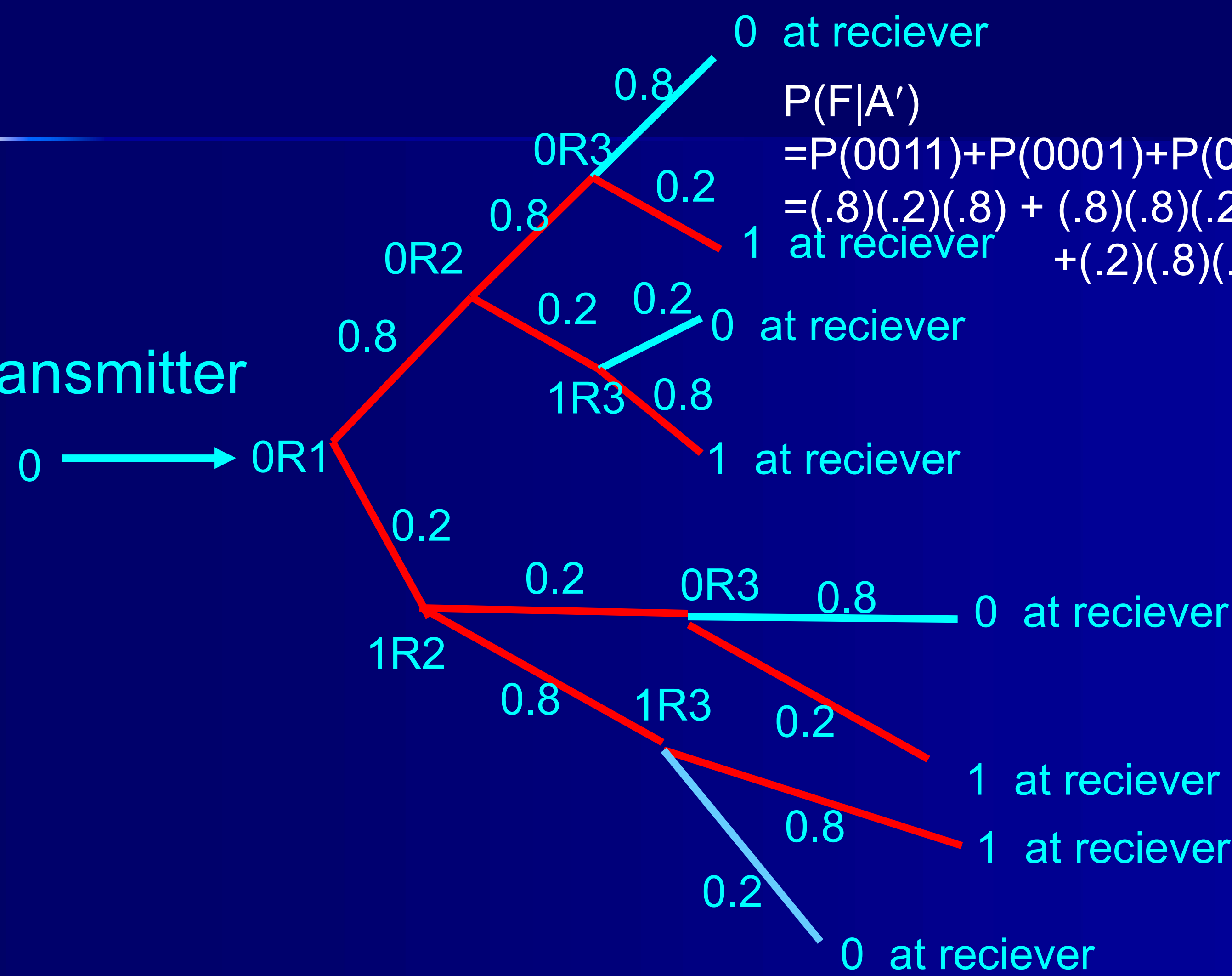
(iii) Suppose 70% of all bits sent from the transmitter are 1's. If a 1 is received by the receiver, what is the prob. that a 1 was sent?

A = 1 is sent by transmitter

F = 1 is received by reciever

$$P(A \mid F) = \frac{P(F \mid A)P(A)}{P(F \mid A)P(A) + P(F \mid A')P(A')}$$

*$P(F|A')$ is in general not equal to $1-P(F|A)$



$$P(F|A') = P(0011)+P(0001)+P(0101)+P(0111) = (.8)(.2)(.8) + (.8)(.8)(.2) + (.2)(.2)(.2) + (.2)(.8)(.8)$$

$$P(A \mid F) = \frac{P(F \mid A)P(A)}{P(F \mid A)P(A) + P(F \mid A')P(A')}$$

$$= \frac{(0.608)(.7)}{(0.608)(.7) + (0.392)(.3)}$$