The Tesseract
Linear Algebra Art Project

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1 Introduction

The **Tesseract**, or hypercube, is the four-dimensional analogue of the cube. Just as a cube is bounded by 6 squares, a tesseract is bounded by 8 cubes. Because human perception is constrained to three spatial dimensions, a "true" tesseract cannot be seen directly. Instead, we must rely on mathematical projections of $\mathbb{R}^4$ onto $\mathbb{R}^3$ or $\mathbb{R}^2$—to visualize its structure.

We use Python script to simulate the rotation and projection of a hypercube. This mathematical process mirrors the techniques used by contemporary artists to bridge the dimensional gap. From the infinity mirror sculptures of **Nicky Alice** to the kinetic installations of **Julius von Bismarck** at CERN, and the monumental architecture of **Johann Otto von Spreckelsen**, the fourth dimension finds expression in modern art.

2 The Tesseract

The tesseract, or hypercube Q_4, is the four-dimensional analogue of the cube. It is defined geometrically as the Cartesian product of four closed intervals centered at the origin. Given a scaling factor s (`SCALE`), the set of points forming the solid tesseract is:

$$Q_4 = [-s, s] \times [-s, s] \times [-s, s] \times [-s, s] \subset \mathbb{R}^4 \tag{1}$$

While the solid tesseract contains infinite points, the visualization focuses on its **1-skeleton** (the wireframe structure). This structure consists of:

- **16 Vertices**
- **32 Edges**
- **24 Square Faces**
- **8 Cubic Cells**

The Initial Vertex matrix V_0 contains all 2^4 permutations of the coordinate signs $\{-s, s\}$:

$$V_0 = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_{16} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} -s & -s & -s & -s & \cdots & s \\ -s & -s & -s & -s & \cdots & s \\ -s & -s & s & s & \cdots & s \\ -s & s & -s & s & \cdots & s \end{bmatrix} \tag{2}$$

The columns of V_0 correspond to the vertices of the hypercube. For example:

- $\mathbf{v}_1 = [-s, -s, -s, -s]^T$:

- $\mathbf{v}_2 = [-s, -s, -s, +s]^T$: A neighbor varying only in the w dimension.
- $\mathbf{v}_{16} = [+s, +s, +s, +s]^T$: The diagonally opposite to the first corner.

This matrix V_0 serves as the input for the first frame of the animation loop, providing the base coordinates on which the rotation matrices operate.

3 Linear Algebra Operations

3.1 Rotations in 4D Space

Unlike 3D rotations which occur around an *axis*, 4D rotations occur around a *plane*. The code utilizes a rotation matrix $M \in SO(4)$. Where SO(4) stands for the **Special Orthogonal group of dimension 4**. It is the mathematical group representing all rotations in 4-dimensional Euclidean space ($\mathbb{R}^4$) that fix the origin. For example a rotation in the xw-plane is represented by:

$$R_{xw}(\theta) = \begin{bmatrix} \cos\theta & 0 & 0 & -\sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\theta & 0 & 0 & \cos\theta \end{bmatrix} \tag{3}$$

3.2 Dimensionality Reduction using Projection Matrix

- **Orthogonal Projection**: A linear transformation that maps the 4D tesseract onto a 3D/2D subspace by truncating the 1/2-coordinate, effectively casting a "flat" shadow.
- **Perspective Projection (Projective Geometry):** Unlike the orthogonal method, this method tries to simulate how a 4D observer would perceive the object in 3D/2D space.

  1. **4D → 3D**
     The vector is first projected onto a 3D hyperplane. Dividing by the 4D depth (w) creates the characteristic "tesseract" effect where the "far" cell shrinks to fit inside the "near" cell.
     $$\mathbf{v}_{3D} = \frac{1}{d_4 - w} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4}$$
     where d_4 is the 4D camera distance.
  2. **3D → 2D**
     The resulting 3D structure is then projected onto the 2D plane by divides by the new depth (z_{3D}).
     $$\mathbf{v}_{2D} = \frac{1}{d_3 - z_{3D}} \begin{bmatrix} x_{3D} \\ y_{3D} \end{bmatrix} \tag{5}$$
     where d_3 is the 3D camera distance.

4 Python Simulation

- **Orthogonal Projection** This view lacks depth scaling. Visually, the "inner" and "outer" cubes appear identical in size. As the object rotates, it resembles two rigid, interlinked cubes sliding past one another.
- **Perspective Projection:** This view creates the "cube-within-a-cube" effect, by dividing coordinates by the 4D depth, parts of the tesseract "further away" in the fourth dimension shrink toward the center. The code utilizes color as a visual proxy for the perceiving depth.

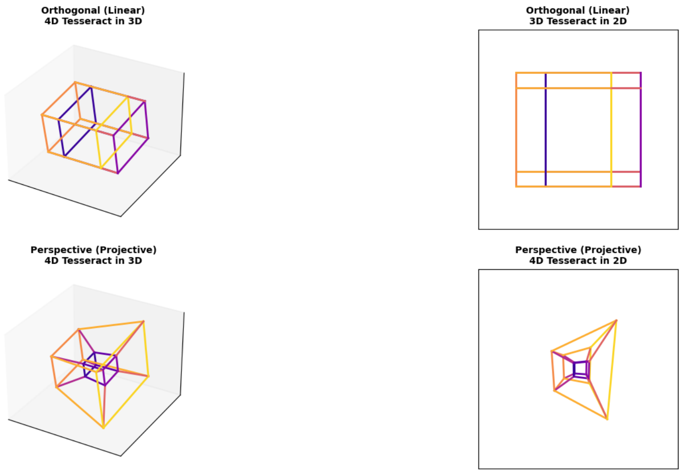

Figure 1: **The Tesseract (Hypercube)**

A vertex shifting from blue to yellow indicates it is traveling "forward" through the fourth dimension, this allows the viewer to witness the "inside-out" rotation characteristic of 4D objects.

5 The Tesseract in Art and Architecture

5.1 Julius von Bismarck & Benjamin Maus: Round About Four Dimensions

Commissioned for the CERN Science Gateway, the kinetic sculpture *Round About Four Dimensions* is a physical manifestation of the rotating 3d-projected hypercube.

- **The Installation:** The sculpture consists of a flexible, motorized aluminum lattice that continuously turns itself inside-out.
- **Mathematical Connection:** This movement physically replicates the **Stereographic projection** of a rotating hypersphere. Just as our Python script updates vertex positions frame-by-frame to show a tumbling tesseract, this sculpture adjusts its actuators to simulate the continuous deformation of a 4D object passing through 3D space.

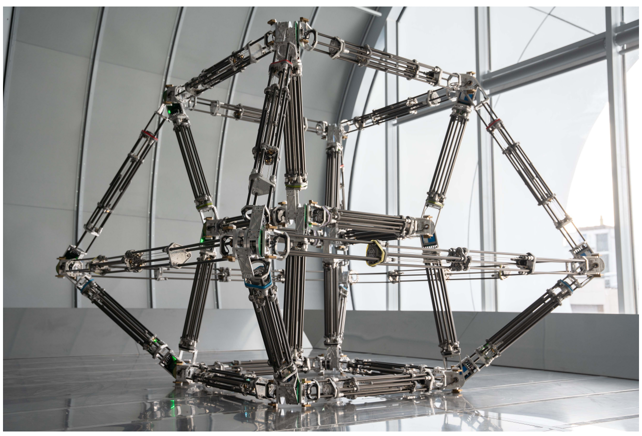

Figure 2: **Round About Four Dimensions**

5.2 Nicky Alice: The Infinity Mirror Tesseract

Contemporary artist Nicky Alice utilizes light and reflection to simulate the **Perspective projection** used in our code.

- **The Installation:** Cube-shaped sculptures lined with mirrors and LEDs is used.
- **Mathematical Connection:** The mirrors create an infinite recursion of the cube's image. This visual effect matches perspective projection math: as the "distance" (d) increases in the reflection, the "scale" of the cube shrinks (scale $= 1/(d-w)$), creating a vanishing point that implies a fourth spatial dimension.

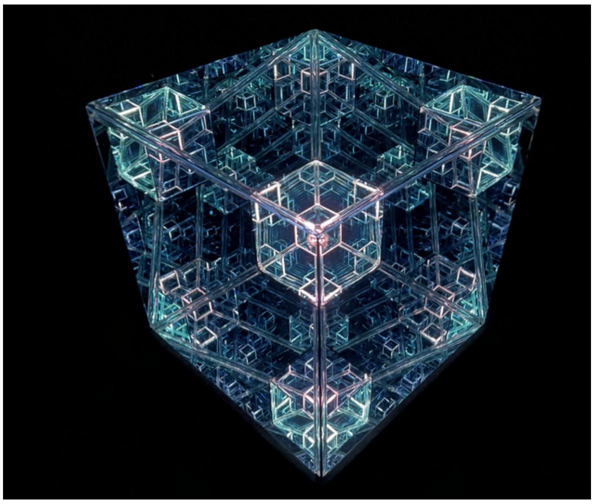

Figure 3: **The Infinity Mirror Tesseract**

5.3 Johann Otto von Spreckelsen: La Grande Arche de la Défense

Located in Paris, this massive monument is a modern architectural interpretation of the tesseract. It is 110-meter hypercube, corresponding to our Initial Vertex Matrix V_0 with a scaling factor of $s = 55$ meters, projected into 3D space.

Figure 4: **La Grande Arche de la Défense**

References

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[3] Nicky Alice. (2024). *Tesseract 5D Hypercube Art Sculpture*. Infinity Mirror Installation. https://www.nickyalice.com/product/tesseract-5d-hypercube-art-sculpture-27-made-to-order/

[4] Johan Otto von Spreckelsen. *La Grande Arche de la Défense*. 1989. Puteaux, France. https://fr.wikipedia.org/wiki/Arche_de_la_D%C3%A9fense