Spacecraft Assignment 2

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1 Problem Statement - 1

Numerically solve the governing equations derived in the class for the pendulum attached to a cart problem. Provide different forcing (a(t), acceleration) to the cart, and see if you can stabilize the pendulum in an inverted manner for some initial conditions. Justify why you were/not able to stabilize the pendulum. Plot phase-space to show the dynamics of the system.

1.1 Objectives

- Numerically solve the governing equations of a pendulum attached to a accelerating cart.
- Integrate the equations to get the points and other state variables.
- Try to stabilize the pendulum in an inverted manner using different initial conditions and acceleration.
- Justify how we were/not able to stabilize the system.
- Plot phase space diagram.

1.2 Methodology

1.2.1 Solving for the Equations

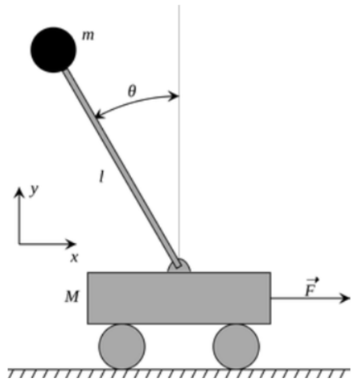
We use the Lagrangian equations (L = T - U) to form the general equations.

$$\ddot{x} = \lambda(\theta)\big[mgl^2\dot{\theta}^2 \sin\theta - mgl\sin\theta\cos\theta - l\mu\dot{x} + ul\big] \tag{1}$$

$$\ddot{\theta} = \lambda(\theta)\big[(m+M)g\sin\theta - mgl\dot{\theta}^2\sin\theta\cos\theta + \mu\dot{x}\cos\theta - u\cos\theta\big] \tag{2}$$

where,

$$\lambda(\theta) = \frac{1}{l(M + m\sin^2\theta)}. \tag{3}$$



We take some generalized coordinates,

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix}. \tag{4}$$

We then form the matrix,

$$\dot{\mathbf{x}} = \frac{d}{dt}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \lambda(x_1)\big[ml^2 {x_3}^2 \sin x_1 - mgl \sin x_1 \cos x_1 - l\mu x_2 + ul\big] \\ \lambda(x_1)\big[(m+M)g \sin x_1 - ml{x_3}^2 \sin x_1 \cos x_1 + \mu x_2 \cos x_1 - u \cos x_1\big] \end{bmatrix}. \tag{5}$$

The friction coefficient originating from the friction force $\bar{F}$ is taken to be zero.

1.2.2 Simulating the Problem

To simulate the inverted pendulum on a cart we solve the equation iteratively. We use the numerical method, fourth order Runge-Kutta, to integrate the system of non-linear differential equations.
The equations involved are:

$$k_1 = hf(x_n, y_n), \tag{6}$$

$$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}), \tag{7}$$

$$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}), \tag{8}$$

$$k_4 = hf(x_n + h, y_n + k_3), \tag{9}$$

$$y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(h^5). \tag{10}$$

1.2.3 Linearizing the Equations

The non-linear equations of motion can be linearised to the form,

$$\dot{\mathbf{x}} = \mathcal{A}\mathbf{x} + \mathcal{B}u \tag{11}$$

The inverted pendulum in our example has two equilibrium points. We take the point of unstable equilibrium, corresponding to the pendulum being carefully balanced upright. It's unstable in the sense that any perturbation from its point, will cause it to fall down.

$$\begin{cases} x_0 &= \text{any} \\ x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{cases} . \tag{12}$$

The matrices A and B is given by the Jacobian Matrices.

$$\mathcal{A} = \begin{bmatrix} \frac{\partial f_0}{\partial x_0}\Big|_0 & \frac{\partial f_0}{\partial x_1}\Big|_0 & \frac{\partial f_0}{\partial x_2}\Big|_0 & \frac{\partial f_0}{\partial x_3}\Big|_0 \\ \frac{\partial f_1}{\partial x_0}\Big|_0 & \frac{\partial f_1}{\partial x_1}\Big|_0 & \frac{\partial f_1}{\partial x_2}\Big|_0 & \frac{\partial f_1}{\partial x_3}\Big|_0 \\ \frac{\partial f_2}{\partial x_0}\Big|_0 & \frac{\partial f_2}{\partial x_1}\Big|_0 & \frac{\partial f_2}{\partial x_2}\Big|_0 & \frac{\partial f_3}{\partial x_3}\Big|_0 \\ \frac{\partial f_3}{\partial x_0}\Big|_0 & \frac{\partial f_3}{\partial x_1}\Big|_0 & \frac{\partial f_3}{\partial x_2}\Big|_0 & \frac{\partial f_3}{\partial x_3}\Big|_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mg}{M} & \frac{-\mu}{M} & 0 \\ 0 & \frac{g(m+M)}{lM} & \frac{\mu}{ML} & 0 \end{bmatrix} \tag{13}$$

$$\mathcal{B} = \begin{bmatrix} \frac{\partial f_0}{\partial u}\Big|_0 \\ \frac{\partial f_1}{\partial u}\Big|_0 \\ \frac{\partial f_2}{\partial u}\Big|_0 \\ \frac{\partial f_3}{\partial u}\Big|_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ \frac{-1}{Ml} \end{bmatrix} . \tag{14}$$

1.2.4 Stabilizing the Inverted Pendulum

We define a controller, $u = -\mathcal{K}\mathbf{x}$, and get,

$$\dot{\mathbf{x}} = (\mathcal{A} - \mathcal{B}\mathcal{K})\mathbf{x}, \tag{15}$$

Here K is the gain matrix. It is a proportional gain as the control variable (external force u) is proportional to a state vector.
We add the control force in each integration step, creating a feedback loop. To retrieve a K matrix such that the eigenvalues of (A - BK) becomes the wanted eigenvalues poles, scipy's $'signal.place\_poles'$ package is utilised. This package returns the gain matrix given the poles, A and B. This controlled force helps in stabilizing the inverted pendulum.

1.3 Results and Summary

The image below shows how the state variables(x, θ, $\dot{x}$, $\dot{\theta}$) changes with 'time'. It can be seen that after a time all of them becomes constant showing that the system does becomes stable, i.e., we get a inverted pendulum.
The phase-space diagrams forming a open loop shows a non-conservative system We have also made a animation to better understand the problem and the controlling force.

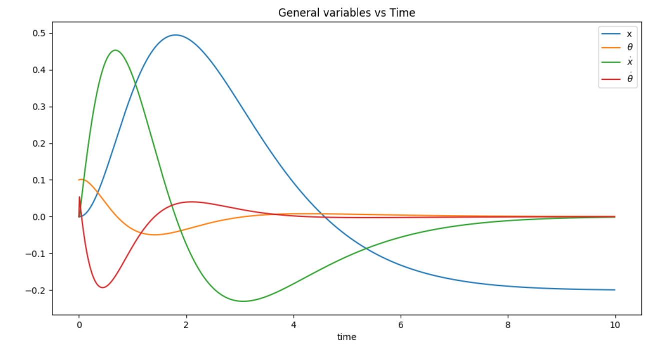


Figure 1: State variables vs Time

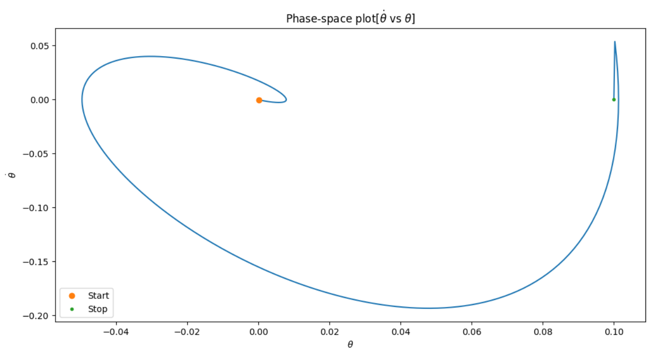


Figure 2: $\dot{\theta}$ vs θ

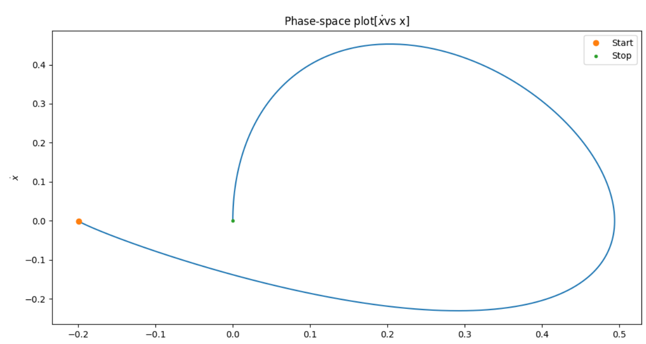


Figure 3: $\dot{x}$vs x

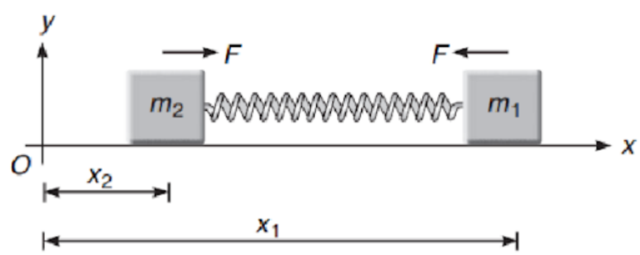
2 Problem Statement - 2

Numerically solve the governing equations for the motion of two particles connected with a spring for different initial conditions. Explore some interesting dynamics which can be identified. Think about all the different parameter ranges.
Plot the phase portraits to represent different dynamics.

2.1 Objectives

- Solve the governing equation for 2 body spring problem.
- Integrate the equation to get values for the state variables.
- Plot the Phase space diagrams.

2.2 Methodology



2.2.1 Solving the Equation

We do the same as we did in the above problem, using Lagrangian equations. Here,

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x1 \\ x2 \\ \dot{x1} \\ \dot{x2} \end{bmatrix}. \tag{16}$$

Here x1, x2 corresponds to the two point particles of mass m and M respectively, joined by a spring of natural length l and spring constant k.
For the 2 body spring problem we have the general equations,

$$\ddot{x1} = \frac{k}{m} * (x2 - x1 - l) \tag{17}$$

$$\ddot{x2} = \frac{k}{M} * (x1 - x2 + l) \tag{18}$$

From this we get the matrix,

$$\dot{\mathbf{x}} = \frac{d}{dt}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \frac{k}{m} * (x2 - x1 - l) \\ \frac{k}{M} * (x1 - x2 + l) \end{bmatrix}. \tag{19}$$

2.2.2 Simulating the Problem

We use the fourth order Runge-Kutta method to integrate the above equations to get the values of the state variables(x1, x2, $\dot{x1}$, $\dot{x2}$) with time.
This gives the plot given below,

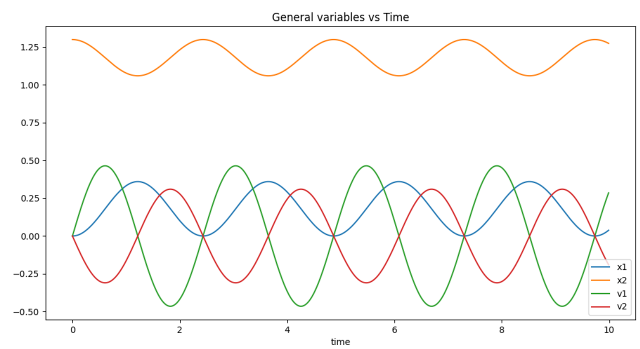


Figure 4: State variables vs Time

We can see that all the state variables vary as a sine wave with time with different parameters.

2.3 Results and Summary

We can observe from the phase-space diagram given below that they are circles and that they form a loop. This shows a conservative system. A harmonic oscillator has the same phase-space diagram.

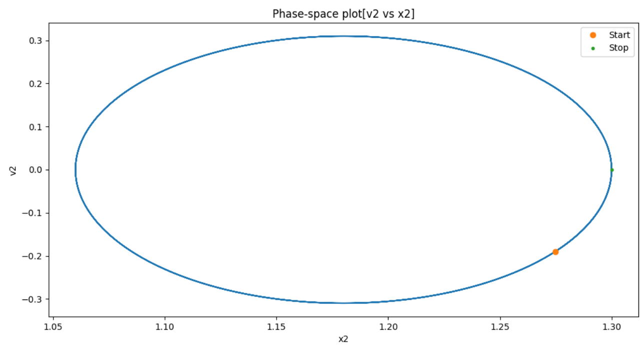


Figure 5: $\dot{\theta}$ vs θ

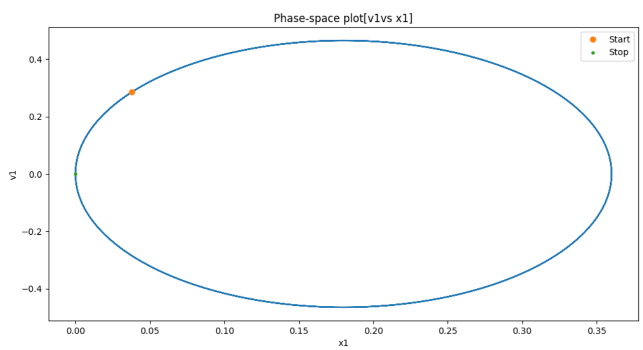


Figure 6: $\dot{x}$ vs x