5 Riemann Integrals

Riemann Integration: Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $P = \{x_0, x_1, ..., x_n\}$ where $a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$ be a partition of $[a, b]$.
i.e.,$[a, b] = [a, x_1] \cup [x_1, x_2] \cup ... \cup [x_{n-1}, b]$
Consider the i^{th} sub-interval $I_i = [x_{i-1}, x_i]$

$$\begin{aligned}
\text{Define } \triangle x_i &= x_i - x_{i-1} \\
m_i &= \inf_x \{f(x) : x \in I_i\} \\
M_i &= \sup_x \{f(x) : x \in I_i\} \\
m &= \inf_x \{f(x) : x \in [a, b]\} \\
M &= \sup_x \{f(x) : x \in [a, b]\} \\
m \le m_i &\le M_i \le M, i = 1, 2, 3, ..., n
\end{aligned}$$

Riemann Lower Sum: Riemann lower sum of f w.r.t partition P is given by $L(P, f) = \sum_{i=1}^{n} m_i \triangle x_i$.

Riemann Upper Sum: The Riemann upper sum of f w.r.t partition P is given by $U(P, f) = \sum_{i=1}^{n} M_i \triangle x_i$

Consider $m \le m_i \le M_i \le M$

$$\begin{aligned}
m \sum_{i=1}^{n} \triangle x_i \le \sum_{i=1}^{n} m \triangle x_i &\le \sum_{i=1}^{n} M_i \triangle x_i \le \sum_{i=1}^{n} M \triangle x_i \\
m(b-a) \le L(P, f) &\le U(P, f) \le M(b-a)
\end{aligned}$$

Riemann Lower Integral: The Riemann Lower integral of f on $[a, b]$ is given by $\underline{\int_a^b} f dx = \sup_p \{L(P, f)\}$.

Riemann Upper Integral: The Riemann Upper integral of f on $[a, b]$ is given by $\overline{\int_a^b} f dx = \inf_p \{U(P, f)\}$.

Results:

1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function,then both the Riemann lower and upper integrals exists and $\underline{\int_a^b} f dx = \overline{\int_a^b} f dx$

2. For any partition P we have $m(b-a) \le L(P, f) \le \underline{\int_a^b} f dx \le \overline{\int_a^b} f dx \le U(P, f) \le M(b-a)$

Theorems: Let $f : [a, b] \to \mathbb{R}$ be a bounded function then f is said to be Reimann integrable over $[a, b]$ if $\underline{\int_a^b} f dx = \overline{\int_a^b} f dx$. If f is Reimann integrable over $[a, b]$, then we write $\int_a^b f dx = \underline{\int_a^b} f dx = \overline{\int_a^b} f dx$.
The set of all Reimann integrable function over $[a, b]$ is denoted by $\mathcal{R}[a, b]$. f is Reimann integrable over $[a, b]$, we also write $f \in \mathcal{R}[a, b]$.
Refinement of a Partition : Let P be a partition of $[a, b]$. A partition P^* of $[a, b]$ is said to be as Refinement if P is contained in P^*. Suppose we have two distinct partitions P_1 and P_2, then $P_1 \cup P_2$ is a common refinement of P_1 and P_2.
Theorem: Let $f : [a, b] \to \mathbb{R}$ be a bounded function, let P and P^* be two partitions of $[a, b]$ such that $P \subseteq P^*$, then
$L(P, f) \le L(P^*, f)$
$U(P, f) \ge U(P^*, f)$
Result: Let $f : [a, b] \to \mathbb{R}$ be a bounded function,let P_1 and P_2 be two partitions of $[a, b]$, then $L(P_1, f) \le$

$U(P_2, f)$ and $L(P_2, f) \leq U(P_1, f)$.

Norm or Mesh of a Partition: Let P be a partition of $[a, b]$,the length of the largest sub-interval of $[a, b]$, corresponding to P is called the norm or mesh of P, and denoted by $||P||$ or $\mu(P)$.i.e.,$||P|| = \max_i\{\triangle x_i\}$.

Theorem:

1. A necessary and sufficient condition for the integrability of a bounded function f on $[a, b]$ is that for any $\epsilon > 0$ there exists $\delta > 0$ such that $U(P, f) - L(P, f) < \epsilon$ for every partitions P with $||P|| < \delta$.
2. Let $f \in \mathcal{R}[a, b]$, then $|f|$ is also Riemann integrable over $[a, b]$ and $\left|\int\limits_a^b f dx\right| \leq \int\limits_a^b |f| dx$. Converse of the above theorem is not true.
   i.e., If $|f|$ is integrable then f need not be integrable.
   Eg: Dirichlet function.
3. Let $f : [a, b] \to \mathbb{R}$ be integrable, then f^2 is also integrable.
   Converse need not be true.
   Eg: Dirichlet function.
4. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, then f is Riemann integrable iff f^3 is Riemann integrable.

Result:

1. Let $f : [a, b] \to \mathbb{R}$ be a bounded and continuous function such that $f(x) \geq 0, \forall x \in [a, b]$ if $\int\limits_a^b f dx = 0$, then $f \equiv 0$.
2. Let f and g be integrable over $[a, b]$, then $f + g, f - g, fg$ are also Riemann integrable over $[a, b]$.
3. If $f(x) \leq g(x), \forall x \in [a, b]$, then $\int\limits_a^b f dx \leq \int\limits_a^b g(x) dx$.
4. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and let $F(x) = \int\limits_0^x f(t) dt, x \in [a, b]$, then F is continuous on $[a, b]$.
5. Let $f : [a, b] \to \mathbb{R}$ be continuous, then f is $\mathcal{R}(f)$ over $[a, b]$ and if $F(x) = \int\limits_0^x f(t) dt, x \in [a, b]$, then f is differentiable on $[a, b]$.
6. Let $f : [a, b] \to \mathbb{R}$ be monotone, then f is $\mathcal{R}(f)$ over $[a, b]$.
7. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let f has only a countable number of discontinuity in $[a, b]$, then $f \in \mathcal{R}[a, b]$.

Theorem: Let $f : [a, b] \to \mathbb{R}$ be a bounded function, f is $f \in \mathcal{R}[a, b]$ iff f is continuous almost everywhere on $[a, b]$.

Fundamental Theorem of Calculus: Let $f : [a, b] \to \mathbb{R}$, be continuous. Let $F(x)$ be a function such that $F'(x) = f(x), \forall x \in [a, b]$ then $\int\limits_a^b f(x) dx = F(b) - F(a)$.

MVT for Integrals: Let $f : [a, b] \to \mathbb{R}$ be continuous, then there exists a number λ lying between m and M such that $\int\limits_a^b f(x) dx = \lambda(b - a)$ where $\lambda = f(c)$, for some $c \in (a, b)$ and $m = \inf f(x), M = \sup f(x)$.

Corollary: Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable, then there exists a $k \geq 0$ such that $|f(x)| \leq k, \forall x \in [a, b]$ and $\left|\int\limits_a^b f(x) dx\right| \leq k(b - a)$