SUBJECT: Linear Algebra - TOPIC: Vector Space

Vector space:
Let V be a non-empty set of vectors and let $\mathbb{F}$ be a scalar field. V is said to be a vector space over the field $\mathbb{F}$ under the operations, vectoe addition $(+ : V \times V \longrightarrow V)$, scalar multiplication $(\cdot : \mathbb{F} \times V \longrightarrow V)$, if the following axioms are satisfied.

1. Addition Axioms
   for $v_1, v_2, v_3 \in V$.
   (a) $v_1 + v_2 \in V$ (closure property)
   (b) $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$
   (c) $v_1 + 0 = v_1 = 0 + v_1$
   (d) $v_1 + (-v_1) = 0 = (-v_1) + v_1$
   (e) $v_1 + v_2 = v_2 + v_1$
2. Multiplicative Axioms
   for $v_1, v_2, v_3 \in V$ and $c_1, c_2, c_3 \in \mathbb{F}$.
   (a) $c_1 v_1 \in V$
   (b) $(c_1 + c_2)v_1 = c_1 v_1 + c_2 v_1$
   (c) $c_1(v_1 + v_2) = c_1 v_1 + c_1 v_2$
   (d) $1v_1 = v_1$, where 1 is the multiplicative identity of $\mathbb{F}$.
   (e) $c_1(c_2 v_1) = (c_1 c_2)v_1 = c_2(c_1 v_1)$.

Vector Space - Examples

(a) $\mathbb{R}$ over $\mathbb{R}$, $\mathbb{C}$ over $\mathbb{C}$

(b) $\mathbb{C}$ over $\mathbb{R}$

(c) $\mathbb{R}^n$ over $\mathbb{R}$, $\mathbb{C}^n$ over $\mathbb{C}$

(d) $M_n(\mathbb{R})$ over $\mathbb{R}$

(e) $P_n(x)$ over $\mathbb{R}$

(f) $P(x)$ over $\mathbb{R}$

(g) $\mathbb{F}(I)$ over $\mathbb{R}$ where $\mathbb{F}(I)$= set of all real valued function on interval I

Subspace of a vector space: Let V be a vector space over $\mathbb{F}$, and let W be a subset of V, W is said to be a subspace of V if W itself is a vector space over $\mathbb{F}$, withrespect to the same operation in V, and having the same additive identity.

Theorem:

1. Let V be a vector space over a field $\mathbb{F}$, let W be a subset of V, W is a subspace of V iff
   (i) $0 \in W$.
   (ii) $c\alpha + \beta \in W$, for all $c \in \mathbb{F}$ and $\alpha, \beta \in W$.
2. Let V be a vector space over a field $\mathbb{F}$, and let V_1 and V_2 be two subspace of V, then $V_1 \cap V_2$, and $V_1 + V_2$ are subspaces of V, where $V_1 + V_2 = \{(v_1 + v_2) : v_1 \in V_1, v_2 \in V_2\}$

To prove $V_1 \cap V_2$ is subspace of V.

$O \in V_1,\ O \in V_2 \quad \Rightarrow \quad O \in V_1 \cap V_2$

Let $\alpha, \beta \in V_1 \cap V_2 \quad \Rightarrow \quad \alpha, \beta \in V_1$ and $\alpha, \beta \in V_2$

$\alpha, \beta \in V_1 \quad \Rightarrow \quad C\alpha + \beta \in V_1,\ C \in \mathbb{F}$

Similarly $C\alpha + \beta \in V_2 \ \ C \in \mathbb{F}$

$\therefore\ C\alpha + \beta \in V_1 \cap V_2$

To prove $V_1 + V_2$ is subspace of V

$O \in V_1,\ O \in V_2 \quad \Rightarrow \quad O = O + O \in V_1 + V_2$

Let $\alpha, \beta \in V_1 + V_2 \quad \Rightarrow \quad \alpha_1 \in V_1,\ \alpha_2 \in V_2$ s.t $\alpha_1 + \alpha_2 = \alpha$

$\beta_1 \in V_1,\ \beta_2 \in V_2$ s.t $\beta_1 + \beta_2 = \beta$

$\alpha, \beta \in V_1 \quad \Rightarrow \quad C\alpha_1 + \beta_1 \in V_1$

Similarly $C\alpha_2 + \beta_2 \in V_2$

$\Rightarrow\ C(\alpha_1 + \alpha_2) + (\beta_1 + \beta_2) \in V_1 + V_2$

$\Rightarrow\ C\alpha + \beta \in V_1 + V_2$

Remark: $V_1 \cup V_2$ need not be a subspace of V.

Eg: $V = \mathbb{R}^2 \qquad \mathbb{F} = \mathbb{R}$.

$V_1 = \{(x, 0) : x \in \mathbb{R}\} \quad V_2 = \{(0, y) : y \in \mathbb{R}\}$.

$(1, 0), (0, 1) \in V_1 \cup V_2$ but $(1, 1) \notin V_1 \cup V_2$.

Note: $V_1 \cup V_2$ is a subspace of V, if either $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$.

Result: Any subspace of $\mathbb{F}^n$ over $\mathbb{F}$ is in the form $\{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : \sum\limits_{i=1}^{n} a_i x_i = 0\}$