SUBJECT: Linear Algebra - TOPIC: Some Special Matrices

Symmetric Matrices :

1. Let $A \in M_n(\mathbb{R})$, A is said to be symmetric if $A^T = A$.
2. Let $A, B \in M_n(\mathbb{R})$ such that A and B are symmetric, then $A+B$, $A-B$, kA, $AB+BA$ A^n are symmetric.
   (AB is not always symmetric. Eg: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.
   Here $AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ which is not symmetric).
3. AB is symmetric iff $AB = BA$.
   Necessary Condition: AB is symmetric
   $AB = (AB)^T = B^TA^T = BA$ (Since A, B are symmetric)
   Sufficient Condition: $AB = BA$
   $AB = B^TA^T$ (Since A, B are symmetric)
   $AB = (AB)^T \quad \Rightarrow AB$ is symmetric
4. If A is symmetric and invertible, then A^{-1} is also symmetric.
   $AA^{-1} = I \quad \Rightarrow \; (A^{-1})^TA^T = I$ (reversal law)
   $\Rightarrow \; (A^{-1})^TA = I$ (Since A is symmetric)
   $\Rightarrow \; (A^{-1})^T = A^{-1}$ (ie., A^{-1} is symmetric)

5. For any square matrix A, $A + A^T$, AA^T, A^TA are symmetric.
6. $det \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$.
7. $det \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} = 3abc - a^3 - b^3 - c^3$.

Skew- Symmetric Matrices:

1. Let $A \in M_n(\mathbb{R})$, A is said to be skew-symmetric if $A^T = -A$.
2. Let Let $A, B \in M_n(\mathbb{R})$ such that A and B are skew-symmetric, then $A+B$, $A-B$, kA, $AB-BA$ are skew-symmetric.
   AB need not skew symmetric. Eg: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
   $AB = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ which not skew symmetric
3. A^n is symmetric if n is even and skew-symmetric if n is odd.
4. For any square matrix A, $A - A^T$ is always sew-symmetric.
5. The diogonal elements of a skew-symmetric matrices are all zeroes.
   $A^T = A \quad \Rightarrow \; a_{ij} = -a_{ji}$
   for diagonal element $a_{ii} = -a_{ii} \quad \Rightarrow \; 2a_{ii} = 0 \quad \Rightarrow \; a_{ii} = 0$.
6. If A is skew symmetric, then $I - A$ is invertible.
7. Determinent of odd order skew-symmetric matrix are 0.

We know $|A^T| = |A|$
Here $A^T = -A$. So, $|A^T| = (-1)^n|A|$
if n is odd, $|A| = -|A| \Rightarrow |A| = 0$.

8. Determinent of even order skew-symmetric matrix is non negative.
9. Determinent of even order skew-symmetric matrix with integer entries is a perfect square.
10. Any square matrix A of order n can be uniquely expressed as the sum of symmetric and skew-symmetric matrices. That is, $A = \left(\frac{A+A^T}{2}\right) + \left(\frac{A-A^T}{2}\right)$

The vector space $M_n(\mathbb{R})$ can be expressed as the direct sum of the set of all symmetric matrices and set of all skew-symmetric matrices.

Hermitian Matrices(Self adjoint Matrices):

1. Let $A \in M_n(\mathbb{C})$, A is said to be hermitian if $A^* = A$, where $A^* = (\overline{A})^T = \overline{(A^T)}$.
   Eg: $\begin{bmatrix} 1 & 1+2i & 5i \\ 1-2i & 2 & 6 \\ -5i & 6 & 3 \end{bmatrix}$.
2. Real symmetric matrices are hermitian matrices.
   For real symmetric matrix A, $\bar{A} = A$, so, $A^* = A^T$.
3. The diagonal entries of a hermitian matrix are real.
   Let $a_{ii} + ib_{ii}$ be diagonal element.
   $A^* = A \Rightarrow (a_{ii} + ib_{ii})^* = a_{ii} + ib_{ii}$
   $a_{ii} - ib_{ii} = a_{ii} + ib_{ii} \Rightarrow b_{ii} = -b_i \Rightarrow b_{ii} = 0$
4. Trace and determinant of hermitian matrix are real.
   All eigen values are real

The result those are true for real symmetric matrices are also true for hermitian matrices.

Skew-Hermitian Matrices:

1. Let $A \in M_n(\mathbb{C})$, A is said to be skew-hermitian if $A^* = -A$, where $A^* = (\overline{A})^T = \overline{(A^T)}$.
   Eg: $\begin{bmatrix} 0 & 1+2i \\ -1+2i & 5i \end{bmatrix}$
2. Real skew-symmetric matrices are skew-hermitian matrices.
3. Diagonal entries of a skew-hermitian matrix are either zero or purely imaginary.
   Let $a_{ii} + ib_{ii}$ be diagonal element.
   $A^* = -A \Rightarrow a_{ii} - ib_{ii} = -a_{ii} - ib_{ii}$
   $\therefore a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$.
4. $A \in M_n(\mathbb{C})$ is skew-hermitian iff iA is hermitian
   Necessary Condition: $A^* = -A$, A is skew hermitian
   Multiply $-i$, $-iA^* = iA$. $\Rightarrow (iA)^* = iA$.
   Sufficient Condition: iA is hermitian ie., $(iA)^* = iA$.
   $-iA^* = iA$.
   Multiply i, $A^* = -A$, A is skew hermitian

The result those are true for real skew-symmetric matrices are also true for skew-hermitian matrices.

Cartisean decomposition of a square matrix: Let $A \in M_n(\mathbb{C})$, then A can be expressed as $A = B + iC$, where B and C are hermitian matrices given by $B = \left(\frac{A+A^*}{2}\right)$ and $C = \left(\frac{A-A^*}{2i}\right)$.

Orthogonal matrices:

1. Let $A \in M_n(\mathbb{R})$, A is said to be orthogonal if $AA^T = A^TA = I$.
2. If A is orthogonal, then $det(A) \in \{\pm 1\}$.
   $|AA^T| \Rightarrow |A||A^T| = 1 \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$.
   Converse not true. Eg: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
3. If A is orthogonal, then $A^{-1} = A^T$.
   $AA^T = I \Rightarrow A^T = A^{-1}$
4. If A is orthogonal, then rows of A are orthonormal.

5. If A is orthogonal, then columns of A are orthonormal.
6. The rows of A or columns of A forms an orthonormal basis for R^n.
7. If A and B are two orthogonal matrices of same order, then A^{-1}, A^T, A^n, AB are also orthogonal.
8. kA and $A+B$ need not be orthogonal.
   Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
   Here $5A$ is not orthogonal $|5A| \neq 1$
   $A+B$ is not orthogonal $|A+B| \neq 1$
9. The set of all orthogonal matrices of order n is a group under matrix multiplication, denoted by $OL_n(\mathbb{R})$.
10. The general form of orthogonal matrix is given by $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, $\theta \in [0, 2\pi)$.
    Here $A^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$.
    check for $n = 2$ and use induction
11. An orthogonal matrix of order n can be expressed as the direct sum of 2×2 orthogonal matrix and [1].

Unitary matrices:

1. Let $A \in M_n(\mathbb{C})$, A is said to be orthogonal if $AA^* = A^*A = I$.
   Eg: $\dfrac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$
2. Real orthogonal matrices are unitary.

The result those are true for real orthogonal matrices are also true for unitary matrices.

Normal matrices:

1. Let $A \in M_n(\mathbb{C})$, A is said to be orthogonal if $AA^* = A^*A$.
   Eg: $\begin{bmatrix} i & i \\ i & -i \end{bmatrix}$
2. Real symmetric, real skew-symmetric and real orthogonal matrics are normal.
3. Hermitian, skew-hermitian and unitary matrices are normal.
4. Let A be a hermitian and let $P(x)$ be a complex polynomial, then $P(A)$ is normal.

Idempotent matrices (Projection)

1. Let $A \in M_n(\mathbb{C})$, A is said to be a idempotent matrix if $A^2 = A$.
2. A non-identity idempotent matrix is singular.
   $A^2 = A \quad \Rightarrow \quad |A||A| = |A| \quad \Rightarrow \quad |A|(|A| - 1) = 0$
   $\qquad |A| = 0$ or 1.
   Determinant of idempotent metrix is zero or one
   Let $A \neq I$, $|A| = 1$ be idempotent.
   $\qquad$ ie., $A^2 = A \quad \Rightarrow \quad A^2 - A = 0$.
   Using distributive law $A(A - I) = 0$.
   Since $|A| = 1$, $|A - I| = 0$.
   $\qquad \Rightarrow \quad A = I$ contradiction.
   $\therefore$ I is only non-singular idempotent matrix.
3. If A is idempotent then $A^n = A$ and $Rank(A) = Trace(A)$.
4. If A is idempotent, then A^T, $A^n, n \in \mathbb{N}$, $I - A$, are also idempotent.
5. If A and B are two idempotent matrix of same order then $A+B$ is idempotent iff $AB + BA = 0$.
6. AB is idempotent if $AB = BA$.
   We have $AB = BA$.
   $(AB)^2 = ABAB$
   $\qquad = ABBA$ (using $AB = BA$)
   $\qquad = AB^2A = ABA = AAB = A^2B = AB$
   AB is idempotent

Periodic matrices: Let $A \in M_n(\mathbb{C})$, A is said to be a periodic, if there exist a $k \in \mathbb{N}$ and $k > 1$ such that $A^k = A$. The least such k is called the period of A.

Eg: $A = \begin{bmatrix} 2 & 5 & 14 \\ 1 & 3 & 8 \\ -1 & -2 & -6 \end{bmatrix}$ Here $A^4 = A$

Nilpotent matrices:

1. Let $A \in M_n(\mathbb{C})$, A is said to be a nilpotent if $A^k = 0$ for some $k \leq n$ and $k \in \mathbb{N}$. The least positive K is called the index of A.

   Eg: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $A^3 = 0$ $\therefore$ Index of $A = 3$
2. If A is nilpotent, then $det(A) = 0$.
   (All eigenvalues are zero)
3. If A is nilpotent, then $(I - A)$ is invertible.

Involuntary matrices:

1. Let $A \in M_n(\mathbb{C})$, A is said to be a involuntary matrix if $A^2 = I$ or $A^{-1} = A$.

   Eg: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
2. If A is involuntary and $A \in M_n(\mathbb{R}$, then $det(A) = \pm 1$.
   $|A|^2 = 1 \quad \Rightarrow \quad |A| = \pm 1$
3. If A is involuntary then $\frac{A+I}{2}$ is idempotent.

   $$\left(\frac{A+I}{2}\right)^2 = \frac{(A+I)^2}{4} = \frac{A^2 + AI + IA + I^2}{4} = \frac{I + 2A + I}{4} = \frac{2A + 2I}{4} = \frac{A+I}{2}$$

Vandor Mondae matrix:

$A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$, $a, b, c \in \mathbb{R}$ and $det(A) = (a-b)(b-c)(c-a)$

In general, $B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{pmatrix}$ and $det(B) = (a_1 - a_2)(a_2 - a_3) \cdots (a_n - a_1)$.