SUBJECT: Linear Algebra - TOPIC: Rank of a Matrix

Rank of a Matrix

Let A be a $m \times n$ real matrix. $Rank(A)$ is defined as

1. The order of the largest non-singular square submatrix of A.
2. The number of linearly independent rows or columns of A.
3. The dimension of the row space or column space of A.
4. The number of non zero rows in the row reduced echlon form of A.
5. The order of the identity submatrix in the normal form of A
6. The rank of linear transformation from $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ corresponding to A.

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}_{3\times3}$ $\quad B = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$ $\quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{2\times4}$.

$\rho(A) = 3 \qquad \rho(B) = 2 \qquad \rho(C) = 2.$

Result:

1. Rank of the zero matrix is zero.
2. If A is a non-zero $m \times n$ matrix, then $1 \le rank(A) \le \min\{m, n\}$.
3. If A and B are two $m \times n$ matrices, then $rank(A + B) \le rank(A) + rank(B)$.
4. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then $rank(AB) \le \min\{rank(A), rank(B)\}$.
5. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then
   (i) $rank(A) + rank(B) - n \le rank(AB)$.(Sylvester's inequality)

   (ii) $rank(I - AB) = rank(I - BA)$.
6. If A is a non-zero skew symmetric matrix, then $rank(A) \ge 2$.
   Since A is nonzero skew symmetric matrix, say $g \neq 0$ be element of A, then there exist atleast one 2-rowed minor of matrix A. ie., Minor $\begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix}$ whose determinant is $g^2 \neq 0$.
7. Let $A \in M_{m\times n}(\mathbb{R})$, then $rank(A) = rank(A^T A) = rank(AA^T)$.
8. Let $A \in M_n(\mathbb{R})$, then $rank(A) = n$ iff $det(A) \neq 0$.
   $A \in M_n(\mathbb{R})$
   Necessary Condition: rank $(A) = n$ implies there exist a non-singular square submatrix of order n, which equals A. ie., $|A| \neq 0$
   Sufficient Condition: $|A| \neq 0$ implies A itself is a non-singular square submatrix of order n and the largest matrix. So, rank $(A) = n$
9. If $det(A) = 0$, then $rank(A) < n$.
   (Using Contrapositive of above statement and rank $(A) \le n$)
10. Let $A \in M_n(\mathbb{R})$, then $rank(A) \ge rank(A^2) \ge rank(A^3) \cdots$.
    (Choose $B = A$ in result rank $(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$)
11. Let A be an $m \times n$ matrix with $m \le n$ and $rank(A) = m$. If B is a matrix of order $p \times m$, then $rank(BA) = rank(B)$.
12. Let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A, B, C, D \in M_n(\mathbb{R})$, then

$$rank(T) = \begin{cases} rank(A) + rank(D - (CA^{-1}B)), \\ \qquad\qquad\qquad \text{if } A \text{ is invertible} \\ rank(A) + rank(D), \\ \qquad\qquad\qquad \text{if } A \text{ is not invertible} \end{cases}$$

Nullity of a matrix: Let A be an $m \times n$ matrix, then nullity of A is the dimension of the null space of A, denoted by $dim(N(A))$ or $nullity(A)$. $N(A) = \{x : Ax = 0, x \in \mathbb{R}^n\}$
$nullity(A) = n - rank(A)$
$rank(A) + nullity(A) =$ number of columns.

Range space of A:

1. $Range(A) = \{Ax \in \mathbb{R}^m : x \in \mathbb{R}^n\}$, where A is an $m \times n$ matrix.
2. $Range(A)$ is a subspace of R^m.
3. $rank(A) = dim(Range(A))$.
4. $Columnnullity(A) = n - rank(A) = nullity(A)$.

Row space:

1. Let A be an $m \times n$ matrix, then $rowspace(A) = \{x^T A \in \mathbb{R}^n : for x \in \mathbb{R}^n\}$.
2. Row space is a subspace of $\mathbb{R}^n$.

Result: Let $A \in M_n(\mathbb{R})$, then $nullity(A) \le nullity(A^2) \le nullity(A^3) \cdots$.
(Using rank(A)+ nullity(A)=n and rank (A) $\ge$ rank (A^2) $\ge$ rank (A^3) $\ge \dots$)