2 Functions of One Variable

2.1 Relations and Functions

Relations: Let $A\&B$ be two non-empty sets, a relation from A to B is a subset of the cartesian product $A \times B$, i.e., any subset of $A \times B$ is a relation from A to B.
A relation on A is a subset of $A \times A$. Let R be a relation $A \to B$.

1. If $R = \phi$,then it is called the empty relation.
2. If $R = A \times B$,it is the universal relation.
3. If $A = B$,and if $R = \{(a,a); \forall a \in A\}$, is called the identity relation.

Classification of Relations: Let A be a non-empty set, and R be a relation on A,then

1. R is said to be reflexive if $(a,a) \in R, \forall a \in A$.
2. R is said to be symmetric if $(a,b) \in R \Rightarrow (b,a) \in R, \forall a,b \in A$.
3. R said to be transitive if (a,b) and $(b,c) \in R \Rightarrow (a,c) \in R$.
4. R is said to be an equivalence relation on A, if R is said to be reflexive, symmetric & transitive.
5. R is said to be antisymmetric, if both (a,b) and $(b,a) \in R$, iff $a = b$.
6. R is said to be a partially ordered relation,if R is reflexive,antisymmetric & transitive.

Equivalence class: Let R be an equivalence relation on a non-empty set A, then there exist a partition of A into disjoint classes, called equivalence classes.
For $a \in A$,the equivalence class of a is given by $[a] = \{b \in A; aRb\}$.

Properties:

1. $a \in [a]$
2. If $b \in [a]$,then $[a] = [b]$
3. If $b \notin [a]$,then $[a] \cap [b] = \phi$

Result: Let A & B be two non-empty finite sets,with $|A|=n$,$|B|=m$,then

1. No.of relations from A to B is 2^{mn}
2. No.of relations on A is 2^{n^2}
3. No.of reflexive relations on A is 2^{n^2-n}
4. No.of symmetric relations on A is $2^{n(n+1)/2}$
5. No.of relations on A which are both reflexive & symmetric $2^{(n^2-n)/2}$

Functions: Let A & B be two non-empty sets,and let f is a relation from A to B, f is said to be a function from A to B if every element of A has a unique image in B under f.
If B is contained in $\mathbb{R}$,then f is said to be a real valued function and if $A, B \subseteq \mathbb{R}$, then f is said to be a real function.

Classification of functions:

1. **One-one or injective functions:** Let $f : A \to B$ be a function. For x,$y \in A$,if $x \neq y \Rightarrow f(x) \neq f(y)$,then f is said to be a one-one function. If f is not one-one, then f is called a many-one function.
2. **Onto or surjective functions:** Let $f : A \to B$ be a function. If every element of B is the image of some element of A,then f is said to be an onto function. In other words, then f is said to be onto if $f(A) = B$.If f is not an onto function,then f is called an into function.
3. **Bijective function:** Let $f : A \to B$ be a function, f is said to be bijective if f is both injective & surjective.

Result: Let $f : A \to B$ be a function where A and B be two non-empty finite sets with $n(A)$=n and $n(B)$=m then

1. No.of functions from A to $B = m^n$
2. No.of one-one function from A to B
$$= \begin{cases} {}^mP_n & , m \geqslant n \\ 0 & , m < n \end{cases}$$
3. No.of onto function from A to B

$$= \begin{cases} \sum_{i=1}^{m}(-1)^{m-i}({}^mC_i i^n) & ,m \le n \\ 0 & ,m > n \end{cases}$$

If $m = 2$,then no.of onto function $f : A \to B$ is $2^n - 2$.

4. No.of bijective function from A to B is $\begin{cases} n! & ,m = n \\ 0 & ,m \neq n \end{cases}$
5. If $m = n$, then $f : A \to B$ is one-one $\implies$ f is onto.

Result: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

1. If f satisfies the relation, $f(x + y) = f(x) + f(y), \forall x, y \in R$, then the function is in the form, $f(x) = ax, a \in \mathbb{R}$.
2. If f satisfies the relation,$f(x + y) = f(x) + f(y)$, then f is in the form a^x.
3. If f satisfies the relation, $f(xy) = f(x) + f(y)$, then f is in the form $f(x) = \log_a x$.
4. If f satisfies the relation $f(x).f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ then $f(x) = 1 \pm x^a$.
5. If f satisfies the relation $f(xy) = f(x)f(y)$ then $f(x) = x^a$

Compositions of functions: Let A, B, C be three non-empty sets, $f : A \to B$ and $g : B \to C$ be two functions, then we can define a function A to C, $gof = A \to C$ by $(gof)(x) = g(f(x)) \forall x \in A$.
If $A = C$, then both the compositions gof & fog are defined.
If $A = B = C$, then the compositions gof, fog, fof, gog etc are also defined.

Results:

1. $fog \neq gof$, in general.
2. If $f\&g$ are one-one,then gof and fog are also one-one, provided they exists.
3. If $f\&g$ are onto, then gof and fog are also onto, provided they exists.
4. If gof is one-one, then f is one-one.
5. If gof is onto, then g is onto.

Inverse of Functions: Let A & B be two non-empty sets, and let $f : A \to B$ be a function,f is said to have a left inverse if there exist a function $g : B \to A$ such that $gof = I_A$. f is said to have a right inverse if there exist a function $g : B \to A$ such that $fog = I_B$.f is said to be invertible if there exist a function $g : B \to A$ such that $gof : I_A$ and $fog = I_B$. If f is invertible, then we write $g = f^{-1}$.

Result:

1. A function $f : A \to B$ is invertible,if f is both one-one and onto.
2. If $f : A \to B$ is one-one then it has a left inverse.
3. If $f : A \to B$ is onto then it has a right inverse.
4. Let A and B be two non-empty sets, if there exist a bijection between A and B then A and B are said to be numerically equivalent.
5. Two finite sets A and B are numerically equivalent iff $n(A) = n(B)$.

Even and Odd Functions: Let f be a real function,f is said to be an even function if $f(-x) = f(x), \forall x \in D(f)$ and f is said to be an odd function if $f(-x) = -f(x), \forall x \in D(f)$.

1. $f(x) = 0$ is the only function which is both even and odd.
2. Non-zero constant functions are even functions.
3. Sum of two even functions is even.
4. Difference of two even functions is even.
5. Sum or difference of two odd functions is odd.
6. Sum of an even function with an odd function need not be even or odd.
7. Difference of an even function with an odd function need not be even or odd.
8. Product of two even functions is even.
9. Product of two odd functions is even.
10. Product of an even function with an odd function is odd.
11. Composition of two even functions is even.
12. Composition of two odd functions is odd.

13. gof is even if f is even.
14. Composition of two even or odd function is even if atleast one of them is even.
15. Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function,then, $f(x) + f(-x)$ is an even function and $f(x) - f(-x)$ is an odd function
    Also, $f(x) = \left(\frac{f(x)+f(-x)}{2}\right) + \left(\frac{f(x)-f(-x)}{2}\right)$
    i.e., any function can be expressed as the sum of an even and an odd function.

Periodic Functions: Let f be a real function, f is said to be periodic, if $f(x + T) = f(x)$, for some $T \in \mathbb{R}$, then, T is called the period of f.

Example:

1. The trigonometric function $\sin x, \cos x, \mathrm{cosec}x, \sec x$ are periodic with period $2k\pi, k \in \mathbb{Z}$ and fundamental period is 2π.
2. The trigonometric function $\tan x$ and $\cot x$ are periodic with period $k\pi, k \in \mathbb{Z}$, their fundamental period is π.
3. Fundamental period of the function $\sin^n x, \cos^n x, \sec^n x, \mathrm{cosec}^n x$ is π if n is even and 2π if n is odd.
4. Fundamental period of $\tan^n x, \cot^n x$ is $\pi, \forall n$.

   Fundamental period of $\sin(ax), a \neq 0$ is $\left(\frac{2\pi}{a}\right)$.In general let f be a periodic function with fundamental period T, then the fundamental period of the function $f(ax)$ is $\frac{T}{a}$.
5. The function $\{x\} = x - [x]$ is periodic with period $k, k \in \mathbb{Z}$ and fundamental period is 1.
6. Constant functions are periodic with period $a, a \in \mathbb{R}$, and does not have fundamental periods.

Results: Let f & g be two periodic functions with periods T_1 & T_2 respectively, then $f + g, f - g, fg$ are also periodic if there exist positive integers a & b such that $aT_1 = bT_2$. If $f + g, f - g$ and fg are periodic, then period of these function $\leqslant LCM(T_1, T_2)$.

Note: Sum of two periodic functions need not be periodic.

Example: $\sin x, \sin 2x$ are periodic with fundamental periods 2π and 1 respectively. But their sum is not period, as there does not exist positive integers such that $a.2\pi = b.1$.

Monotone Functions: f is said to be monotone if it is either monotonically increasing or monotonically decreasing.

1. If $x < y \Rightarrow f(x) \leq f(y), \forall x, y \in D(f)$, then f is said to be monotonically increasing.
2. If $x < y \Rightarrow f(x) \geq f(y), \forall x, y \in D(f)$, then f is said to be monotonically decreasing.
3. If $x < y \Rightarrow f(x) < f(y), \forall x, y \in D(f)$, then f is said to be strictly monotonically increasing.
4. If $x < y \Rightarrow f(x) > f(y), \forall x, y \in D(f)$, then f is said to be strictly monotonically decreasing.

Results:

1. Constant functions are both monotonically increasing and monotonically decreasing. They are usually considered as non-decreasing functions.
2. $f(x) = x, e^x, \log_a x, x \in (0, \infty)$ are monotonically increasing functions.
3. $f(x) = \frac{1}{x}$ is monotonically decreasing in the intervals $(-\infty, 0), (0, \infty)$.
4. Lines with positive slope are monotonically increasing and the lines with negative slope are monotonically decreasing.
5. Let $f : A \to B, g : B \to C$ be two real functions, then the function $f + g, f - g, fg$ are defined only if $A \cap C \neq \phi$. If $A \cap C \neq \phi$, then the domains of $f + g, f - g, fg$ are $A \cap C$ itself.
   The $D(\frac{f}{g})$ is given by $A \cap C \setminus \{x \in C | g(x) = 0\}$

Bounded functions: Let f be a real function, f is said to be bounded, if there exist a positive number, such that $|f(x)| \leq M, \forall x \in D(f)$.

Note(Partial fractions):

If $a < b$, $\frac{1}{(x-a)(x-b)} = \frac{1}{(b-a)}\left[\frac{1}{(x-a)} - \frac{1}{(x-b)}\right]$

2.2 Special Functions

Polynomial Functions: A polynomial function with real coefficient is in the from $f(x) = a_0 + a_1x + ... + a_nx^n, a_i \in \mathbb{R}, a_n \neq 0$.

End behavior of polynomials:

Let $f(x) = a_0 + a_1x + ... + a_nx^n, a_i \in \mathbb{R}, a_n \neq 0$ be a polynomial. Depending on the values of both a_n and n, the end behavior of a polynomial is described as follows.

1. Case 1: If $a_n > 0$ and n is even, then $lim_{x\to\pm\infty} f(x) = \infty$

2. Case 2: If $a_n < 0$ and n is even, then $lim_{x\to\pm\infty} f(x) = -\infty$

3. Case 3: If $a_n > 0$ and n is odd, then $lim_{x\to\infty} f(x) = \infty$ and $lim_{x\to-\infty} f(x) = -\infty$

4. Case 4: If $a_n < 0$ and n is odd, then $lim_{x\to\infty} f(x) = -\infty$ and $lim_{x\to-\infty} f(x) = \infty$

Properties:

1. $D(f) = \mathbb{R}$
2. $R(f) =$ depends on f
3. Degree of $f = n$
4. If n is odd,then $R(f) = \mathbb{R}$
5. Degree of a non-zero constant polynomial is zero and degree of the zero polynomial is usually considered as $(-\infty)$.
6. If n is odd,then the equation $f(x) = 0$ has atleast one real solution.Also, the equation $f(x) = y \in \mathbb{R}$, has atleast one solution.In short,if n is odd, then f is an onto function.
7. If n is even then the equation $f(x) = 0$ has atmost n real roots.
8. If the equation $f(x) = 0$ has a pure complex solution,$'\alpha'$, then $\overline{\alpha}$ is also a solution of the equation. i.e.,non-real roots occurs in conjugate pairs.
9. The irrational roots occur in conjugate pairs.

Descat's rule of signs: Let $f(x) = a_0 + a_1x + ... + a_nx^n, a_i \in \mathbb{R}$, be a polynomial function in one real variable x. Then the maximum no.of positive real roots of the equation $f(x) = 0$ is equal to the no.of sign changes in $f(x)$.

The maximum no.of negative real roots of the equation $f(x) = 0$ is equal to the no.of sign changes in $f(-x)$.

Rational Functions: A function f is in the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with real coefficient, is called a rational function.

$D(f) = R - \{x \in \mathbb{R} | q(x) = 0\}$ and $R(f)$ depends on f.

Example: $f(x) = \frac{1}{x}, D(f) = R(f) = \mathbb{R} - \{0\}$

Modulus function/Absolute value function:

A function of the form $f(x) = |x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$ $D(f) = \mathbb{R}$ and $R(f) = [0, \infty)$

$$|x - a| = \begin{cases} (x - a) & , x \geq a \\ -(x - a) & , x < a \end{cases}$$

Signum function:

$$f(x) = \frac{|x|}{x} or \frac{x}{|x|} = \begin{cases} 1 & , x > 0 \\ 0 & , x = 0 \\ -1 & , x < 0 \end{cases}$$

$D(f) = \mathbb{R}, R(f) = \{-1, 0, 1\}$
Signum function is a simple function as its range is a finite set.
Greatest Integer function/Floor function: $f(x) = \lfloor x \rfloor$, which is the greatest integer $\leq x$.
$D(f) = \mathbb{R}, R(f) = \mathbb{Z}$.
Least integer function/Ceil function: $f(x) = \lceil x \rceil$, which is least integer $\geq x$.
$D(f) = \mathbb{R}, R(f) = \mathbb{Z}$.
Fraction function: $f(x) = \{x\} = x - [x]$
$D(f) = \mathbb{R}, R(f) = [0, 1)$, it is a possible function with fundamental period 1.
Exponential function: A function of the form $f(x) = a^x, a > 1$ is called an exponential function
$D(f) = \mathbb{R}, R(f) = (0, \infty)$
Example: $f(x) = e^x$,where e is the Euler's number given by $e = 1 + \frac{1}{1!} + \frac{1}{2!} + ...\infty$
Logarithmic function: A function of the form $f(x) = \log_a x, a > 1$, is called a logarithmic function.
$D(f) = (0, \infty), R(f) = \mathbb{R}$
If $a = e, f(x) = \log_e x = \ln x$,which is called the natural logarithmic function.
If $a = 10, f(x) = \log_{10} x = \log x$,called the common logarithmic function.
Properties:

1. $\log_b a = \frac{1}{\log_a b}$
2. $\log_a bc = \log_a b + \log_a c$
3. $\log_a b = \frac{\log_c b}{\log_c a}$
4. $\log_a b.\log_b c = \log_a c$
5. $\log_a \left(\frac{b}{c}\right) = \log_a b - \log_a c$
6. $\log_a b^c = c \log_a b$
7. $a^{log_a b} = b, a > 1$
8. $\log_a a^b = b$

Trigonometric functions:

	Domain	Range	Principle region in which inverse exists
$\sin x$	$\mathbb{R}$	$[-1, 1]$	$[\frac{-\pi}{2}, \frac{\pi}{2}]$
$\cos x$	$\mathbb{R}$	$[-1, 1]$	$[0, \pi]$
$\tan x$	$\mathbb{R} \setminus \{(2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\}$	$\mathbb{R}$	$[\frac{-\pi}{2}, \frac{\pi}{2}]$
cosec$x$	$\mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\}$	$\mathbb{R} \setminus (-1, 1)$	$[\frac{-\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$
$\sec x$	$\mathbb{R} \setminus \{(2k+1)\frac{\pi}{2}, k \in \mathbb{Z}$	$\mathbb{R} \setminus (-1, 1)$	$[0, \pi] \setminus \{\frac{\pi}{2}\}$
$\cot x$	$\mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\}$	R	$(0, \pi)$

Transformation of functions
Let $f(x)$ be a real function.

1. $f(x - a)$ moves the function a units to the right
2. $f(x + a)$ moves the function a units to the left
3. $f(x) + a$ moves the function a units up
4. $f(x) - a$ moves the function a units down
5. $f(ax)$- horizontal stretch $(if a > 1)$ and horizontal shrink if$(0 < a < 1)$
6. $af(x)$- vertical stretch $(if a > 1)$ and vertical shrink if $(0 < a < 1)$
7. $f(-x)$-reflection over the y axis
8. $-f(x)$-reflection over the x axis