2.3 Limit of a function

Limits:Let f be a real function and let $a \in \mathbb{R}$ be any point the left hand limit of f at the point a is given by

$$f(a^-) = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$$

The right hand limit of f at the point a is given by

$$f(a^+) = \lim_{x \to a^+} f(x) = \lim_{x \to a^+} f(a+h)$$

If $f(a^-) = f(a^+)$ then we say that $\lim_{x \to a} f(x)$ exists, we write $\lim_{x \to a} f(x) = f(a^+) = f(a^-)$.
$\epsilon - \delta$ **Definition:** Let f be a real function & let $a \in \mathbb{R}, f$ is said to have the limit l at the point a, if for any $\epsilon > 0 \exists \delta > 0$ such that $0 < |x-a| < \delta \Rightarrow |f(x) - l| < \epsilon$.
Sequential definition: Let f be a real function & let $a \in \mathbb{R}, f$ is said to have the limit l at the point a, for any sequence $(x_n \to a)$, the sequence $(f(x_n)) \to l$.
Algebra of limits: Let f and g be two real functions with limits $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, for some $a \in \mathbb{R}$ then,

1. $\lim_{x \to a}(f+g)(x) = l + m$
2. $\lim_{x \to a}(f-g)(x) = l - m$
3. $\lim_{x \to a}(fg)(x) = lm$
4. $\lim_{x \to a} \frac{f}{g}(x) = \frac{l}{m}, m \neq 0$
5. $\lim_{x \to a} f(x)^{g(x)} = l^m$
6. $\lim_{x \to a} kf(x) = kl$
7. $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(m)$, provided f is continuous.

L'Hospital's Rule: Let f and g be two differential functions, if for some $a \in \mathbb{R}$, $\frac{f(a)}{g(a)}$ is either $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then
$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.
If $\frac{f'(a)}{g'(a)}$ is either $\frac{0}{0}, \frac{\infty}{\infty}$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$ and soon.
If we get any real number or ∞ in finite steps then that number is the limit. If we get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in infinitely many steps, then the limit does not exists.
Logarithmic Method: Let f and g be two real functions and let $a \in \mathbb{R}$, if $(f(a))^{g(a)}$ is in the form $0^0, 1^\infty, \infty^\infty, \infty^0, 0^\infty$ etc then take $k = \lim_{x \to a}(f(x))^{g(x)}$.
In $k = \lim_{x \to a} g(x)$. In $f(x)$ this limit can be found using any known method or results, let it be l.

$$\begin{aligned} \therefore \ln k &= l \\ \Rightarrow k &= e^l \end{aligned}$$

Example: Evaluate $\lim_{x \to \infty}(1 + \frac{1}{x})^x$

$$\begin{aligned} k &= \lim_{x \to \infty} (1 + \frac{1}{x})^x \\ \Rightarrow \ln k &= \lim_{x \to \infty} x . \ln(1 + \frac{1}{x}) \\ &= \lim_{x \to \infty} \frac{\ln(1 + \frac{1}{x})}{\frac{1}{x}} \\ &= \lim_{x \to \infty} \frac{\left(\frac{1}{1+\frac{1}{x}}\right)\left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}} \\ &= 1 \\ \therefore k &= e \end{aligned}$$

Results:

1. $\lim_{x\to 0}\frac{1}{x}$ does not exists.
2. $\lim_{x\to 0}\frac{1}{|x|}=\infty$
3. $$\lim_{n\to\infty} x^n = \begin{cases} 0 & , \text{if} x\in(-1,1) \\ 1 & , x=1 \\ \text{does not exists} & , x=-1 \\ \infty & , x>1 \\ \text{does not exists} & , x<-1 \end{cases}$$
4. $\lim_{x\to 0}\frac{\sin x}{x}=1$
5. $\lim_{x\to \infty}\frac{\sin x}{x}=0$
6. $\lim_{x\to 0}\frac{\tan x}{x}=1$
7. $\lim_{x\to 0^+}\frac{\cos x}{x}=\infty$
8. $\lim_{x\to \infty}\frac{\cos x}{x}=0$
9. $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$
10. $\lim_{x\to 0}\frac{a^x-1}{x}=\log a, a>1$
11. $\lim_{x\to 0}=\frac{e^x-1}{x}=1$
12. $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$
13. $\lim_{x\to 0}\frac{cosx-1}{x}=0$
14. $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$
15. $\lim_{x\to \infty}(1+\frac{1}{x})^x=e$
16. $\lim_{x\to 0}(1+ax)^{\frac{1}{x}}=e^a$
17. $\lim_{x\to \infty}(1+\frac{a}{x})^x=e^a$
18. $\lim_{x\to 0}(1+x)^{\frac{a}{x}}=e^a$
19. $\lim_{x\to \infty}(1+\frac{1}{x})^{ax}=e^a$
20. $\lim_{x\to 0}\sin\frac{1}{x}$ and $\lim_{x\to 0}\cos\frac{1}{x}$ does not exists.
21. $\lim_{x\to 0}x\sin\frac{1}{x}=0$
22. $\lim_{x\to 0}x\cos\frac{1}{x}=0$
23. Let f be a differential function and let $\lim_{x\to\infty}f(x)=l, l<\infty$ then $\lim_{x\to\infty}f^{'}(x)=0$, if it exists.
24. If $\lim_{x\to\infty}f(x)+f^{'}(x)$ exists then $\lim_{x\to\infty}f(x)+f^{'}(x)=\lim_{x\to\infty}f(x)$.
25. $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-...-(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}$
26. $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-...-(-1)^{n+1}\frac{x^{2n-2}}{(2n-2)!}$
27. $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+...$
28. $a^x=1+\frac{x}{1!}\log a+\frac{x^2}{2!}(\log a)^2+...+\frac{x^n}{n!}(\log a)^n+...$
29. $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-...+(-1)^{n+1}\frac{x^n}{n}+..., |x|<1$
30. $\log 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...$
31. $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+...$
32. $\tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+...$

Leibnitz's Rule for Differentiating an integral:

1. Let $G(x,t),\alpha(x),\beta(x)$ be differential w.r.t.x, consider the function $F(x)=\int\limits_{\alpha(x)}^{\beta(x)}G(x,t)dt$, then

$$F'(x)=\int\limits_{\alpha(x)}^{\beta(x)}\frac{\partial G(x,t)}{\partial x}dt+G(x,\beta(x))\beta'(x)-G(x,\alpha(x))\alpha'(x)$$

2. If $F(x) = \int_{\alpha(x)}^{\beta(x)} G(t)dt$

   $F'(x) = G(\beta(x))\beta'(x) - G(\alpha(x))\alpha'(x)$

3. $F(x) = \int_{0}^{x} G(t)dt$

   $F'(x) = G(x)$