4 Power Series

Taylor series : Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable in all orders at $x = a$. Then, the Taylor series expansion of f about $x = a$ is given by : $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + ...$ in some neighbourhood of 'a'.

Meclaurin series : $f(x) = f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}(x)^2 + ... + \frac{f^{(n)}(0)}{n!}(x)^n + ...$

Power series : A series of the form $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is a power series about the centre $x = x_0$, a_n is a real sequence.

Theorem : There exist $R \in \mathbb{R} \cup \{\infty\}$ such that every power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely for $|x-x_0| < R$ and diverges for $|x-x_0| > R$.

This R is known as the radius of convergence of the power series $\sum a_n(x-x_0)^n$ and $|x-x_0| = R$ is the circle of convergence.

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}$$

Result : The power series $\sum_{n=0}^{\infty} p(n)(x-x_0)^n$, where $p(n)$ is a polynomial of degree 'n' has radius of convergence R=1. Also, radius of convergence of $\sum_{n=0}^{\infty} \frac{1}{p(n)}(x-x_0)^n = 1$

- Suppose R is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. Then
i) If the series converges for $x = x_1$, then $|x_1 - x_0| \leq R$.
ii) If the series diverges for $x = x_2$, then $|x_2 - x_0| \geq R$.
iii) Suppose x_1, x_2 in (i) & (ii) satisfies $|x_1 - x_0| = |x_2 - x_0|$, then , $R = |x_1 - x_0|$
- Suppose R is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. Then,

(i) Radius of curvature of $\sum_{n=0}^{\infty} a_n(x-x_0)^{n+k}, k \in \mathbb{N}$ is R.

(ii) Radius of curvature of $\sum_{n=0}^{\infty} p(n)a_n(x-x_0)^n$ is R. [$p(n)$: polynomial of degree n]

(iii) Radius of curvature of $\sum_{n=0}^{\infty} \frac{1}{p(n)}a_n(x-x_0)^n$ is R .

(iv) Radius of curvature of $\sum_{n=0}^{\infty} n!a_n(x-x_0)^n$ is 0.

(v) Radius of curvature of $\sum_{n=0}^{\infty} \frac{1}{n!}a_n(x-x_0)^n$ is ∞.

(vi) Radius of curvature of $\sum_{n=0}^{\infty} a_n^k(x-x_0)^n, k \in \mathbb{N}$ is R^k.

(vii) Radius of curvature of $\sum_{n=0}^{\infty} a_n(x-x_0)^{nk}$ is $R^{1/k}$

(viii) Let $\langle a_n \rangle$ be a bounded sequence. Then radius of convergence of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is ≥ 1

Taylor's theorem : Let $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is a power series which converges for $|x-x_0| < R$, then $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ is infinitely differentiable at $x = x_0$ also each derivative $f^{(n)}$ at x_0 is given by $f^{(n)}(x) = n!a_n$. Then, $f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x-x_0)^n$ is the Taylor's expansion of f about the centre x_0.

The n^{th} partial sum of the Taylor series is the n^{th} degree polynomial

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Now, f(x) can be written as $f(x) = T_n(x) + R_n(x)$ where $R_n(x)$ is the reminder term.

Theorem: If $f^{n+1}(x)$ is continuous on an open interval I that contains a, and x is in I, then there exists a c between a and x such that

$$R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$$