SUBJECT: Linear Algebra - TOPIC: Eigen Values/Vectors

Characteristic polynomial of a square matrix: Let $A \in M_n(\mathbb{F})$, where $\mathbb{F}$ is an arbitrary field. The characteristic polynomial of A is denoted by $\Delta_A(x)$ or $\chi_A(x)$, which is given by $\chi_A(x) = det(xI - A) = x^n - S_1x^{n-1} + S_2x^{n-2} - \cdots + (-1)^n S_n$, where S_k is the sum of the principal minors of A of order k and x is a parameter.
Principal minor of a square matrix: Let $A \in M_n(\mathbb{F})$, a principal minor of order k is the minor obtained by removing $(n-k)^{\text{th}}$ rows and the corresponding $(n-k)^{\text{th}}$ columns.
Notes: The characteristic polynomial of $A \in M_n(\mathbb{F})$ is an n^{th} degree monic polynomial in x.
Def:[Characteristic equation]: Let $A \in M_n(\mathbb{F})$, characteristic equation of A is given by $\chi_A(x) = 0$. That is $det(xI - A) = x^n - S_1x^{n-1} + S_2x^{n-2} - \cdots + (-1)^n S_n = 0$.
Cayley Hamilton theorem: Every square matrix satisfies its characteristic equation. Let $A \in M_n(\mathbb{F})$ and $\chi_A(x)$ be the characteristic polynomial of A. Then $\chi_A(A) = 0$, that is $A^n - S_1A^{n-1} + S_2A^{n-2} - \cdots + (-1)^n S_n = 0$.
Result:

1. Let $A \in M_2(\mathbb{F})$, then $\chi_A(x) = x^2 - tr(A)x + det(A)$.
2. Let $A \in M_3(\mathbb{F})$, then $\chi_A(x) = x^3 - tr(A)x^2 + (M_{1,1} + M_{2,2} + M_{3,3})x - det(A)$, where $M_{i,i}$ is the minor of $a_{i,i}$.
3. Let $A \in M_n(\mathbb{F})$ be a triangular matrix with diagonal entrie $a_{i,i}, i = 1, 2, \cdots n$. then
$$\chi_A(x) = (x - a_{1,1})(x - a_{2,2}) \cdots (x - a_{n,n}) = \sum_{i=1}^{n} (x - a_{i,i})$$

Minimal polynomial of a square matrix: Let $A \in M_n(\mathbb{F})$ and let $J_A(x)$ be the collection of all polynomials in x for which A is a root.
$J_A(x) = \{P(x) \in \mathbb{F}[x] : P(A) = 0\}$
The least degree monic polynomial in $J_A(x)$ which divides every elements of $J_A(x)$ is called the minimal polynomial of A, denotd by $m_A(x)$.
Notes:

1. $\chi_A(x) \in J_A(x)$. Therefore $m_A(x)$ divides $\chi_A(x)$.
2. $m_A(x)$ and $\chi_A(x)$ have the same irreducible factors over $\mathbb{F}$.

Result: Let $D = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where A and B are square matrices (need not be of same order) then $\chi_D(x) = \chi_A(x) \cdot \chi_B(x)$ and $m_D(x) = LCM\{m_A(x), m_B(x)\}$.
Eigen values and Eigen vectors of a square matrix:

1. Let $A \in M_n(\mathbb{F})$ and let $\lambda \in \mathbb{F}$, λ is said to be an eigen value of A if there exist a non zero vector $x \in \mathbb{F}^n$ such that $Ax = \lambda x$, such an x is called eigen vector corresponding to the eigen value λ.
2. Let $A \in M_n(\mathbb{F})$ and let $\chi_A(x)$ be the characteristic polynomial of A. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ (need not be distinct) be roots of the characteristic polynomial. λ_i's are called the eigen value of A if $\lambda_i \in \mathbb{F}$.

Result:

1. If $\mathbb{F} = \mathbb{C}$, the complex field, then A has n eigen values counting multiplicities
Eg: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & i & 3 \\ 0 & 0 & 1 \end{bmatrix}$ Eigen values: $1, 1, i$
2. If $\mathbb{F} = \mathbb{R}$, then A may or maynot have eigen values.

Eg: $\begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}$ No real eigen values

Eigen space: Let $A \in M_n(\mathbb{F})$ and let $\lambda \in \mathbb{F}$ be an eigen value of A. The eigen space of λ is the subspace of $\mathbb{F}^n$ given by $E_\lambda = \{x : (A - \lambda I) = 0\} = N(A - \lambda I)$.

Algebraic multiplicity and Geometric multiplicity: Let $A \in M_n(\mathbb{F})$ and let $\chi_A(x)$ be the characteristic polynomial of A. Let $\lambda \in \mathbb{F}$ be an eigen value of A, the multipicity of λ as a root of the characteristic polynomial is called the algebraic multiplicity of λ, it is denoted by AM. The dimension of eigen space E_λ is called the geometric multiplicity of λ, denoted by GM. GM is the number of linearly independent eigen vectors corresponding to λ.

Result:

1. For any eigen value λ of A, $1 \leq GM \leq AM$.
2. If x_1 and x_2 are two eigen vectors of A corresponding to the eigen value λ, then $c_1x_1+c_2x_2$ where $c_1, c_2 \in \mathbb{F}$ is also an eigen vector of A corresponding to λ provided c_1 and c_2 cannot be zero simultaniously or $c_1x_1 + c_2x_2 \neq 0$.

Theorem: Let $A \in M_n(\mathbb{F})$ and let $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{F}$ be distinct eigen values of A if $x_1, x_2, \cdots, x_k$ are the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, \cdots, \lambda_k$, then $x_1, x_2, \cdots, x_k$ are linearly independent. That is eigen vector corresponding to distinct eigen values are linearly independent.

Result:

1. Let $A \in M_n(\mathbb{F})$ and let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the n solutions of the characteristic equation $\chi_A(x) = 0$ (need not be eigen values) then,

   $S_1 = \sum\limits_{i=1}^{n} \lambda_i = trace(A)$

   $S_2 = \sum\limits_{\substack{i,j=1 \\ i<j}}^{n} \lambda_i\lambda_j$

   $S_3 = \sum\limits_{\substack{i,j,k=1 \\ i<j}}^{n} \lambda_i\lambda_j\lambda_k$

   $\vdots$

   $S_n = \prod_{i=1}^{n} \lambda_i = det(A)$
2. Let $A \in M_n(\mathbb{F})$ be a triangular matrix, then it has eigen values and they are the diogonal entries. (Consider a $n \times n$ triangular matrix and find $|A - \lambda I|$)

Result:

1. If $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, $a, b \in \mathbb{R}$, then eigen values of A are $(a-b)$ and $(a+b)$.
2. If $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $a, b \in \mathbb{R}$, then the roots of characteristic polynomial of A are $(a-ib)$ and $(a+ib)$.
3. Let $A \in M_n(\mathbb{F})$ and let λ be an eigen value of A then
   (i) $k\lambda$ is an eigen value of $kA, k \in \mathbb{F}$.
   $Ax = \lambda x \quad \Rightarrow \quad (kA)x = (k\lambda)x$
   (ii) λ^m is an eigen value of $A^m, m \in \mathbb{N}$.
   $Ax = \lambda x \quad \Rightarrow \quad A^2x = \lambda Ax = \lambda^2 x$. (use induction)
   (iii) $P(\lambda)$ is an eigen value of $P(A)$, where P is a polynomial in $\mathbb{F}[x]$.

Result:

1. Let $A \in M_n(\mathbb{F})$, if $\sum\limits_{j=1}^{n}(a_{i,j}) = k$, $\forall i$, then k is an eigen value of A and then corresponding eigen vector is given by $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

2. Let $A \in M_n(\mathbb{F})$, and each element of A is non negetive and $A \neq 0$, then A has a maximal positive eigen value which is called perron root. If λ is the perron root of A, then

$$\min_i \sum_{j=1}^{n} a_{i,j} \leq \lambda \leq \max_i \sum_{j=1}^{n} a_{i,j}$$

If $\sum_{j=1}^{n} a_{i,j} = k$, $\forall i$, then $\lambda = k$.

Theorem: Let $A \in M_n(\mathbb{F})$, then the following are equivalent

1. λ is an eigen value of A.
2. $(\lambda I - A)$ is singular.
3. λ is a root of the characteristic polynomial of A.

Identity matrix, $A = I_n$

1. $\chi_A(x) = (x-1)^n$.
2. $m_A(x) = (x-1)$.
3. Eigen values are $\underbrace{1, 1, \cdots, 1}_{\text{n times}}$.
4. Eigen vectors are any non zero $x \in \mathbb{F}^n$.
5. $AM = GM = n$, number of linearly independent eigen vectors.

Scalar matrix, $A = kI_n, k \in \mathbb{F}$

1. $\chi_A(x) = (x-k)^n$.
2. $m_A(x) = (x-k)$.
3. Eigen values are $\underbrace{k, k, \cdots, k}_{\text{n times}}$.
4. Eigen vectors are any non zero $x \in \mathbb{F}^n$.
5. $AM = GM = n$.
6. $E_k = \mathbb{F}^n$.

Result: Let $A \in M_n(\mathbb{F})$ and let any non zero $x \in \mathbb{F}^n$ is an eigen value of A corresponding to its eigen value, then A is in the form $A = kI_n$, $k \in \mathbb{F}$.

Zero matrix:

1. $\chi_A(x) = (x)^n$.
2. $m_A(x) = (x)$.
3. Eigen values are $\underbrace{0, 0, \cdots, 0}_{\text{n times}}$.
4. Eigen vectors are any non zero $x \in \mathbb{F}^n$.
5. $AM = GM = n$.
6. $E_0 = \mathbb{F}^n$

Nilpotent matrices: Let $A \in M_n(\mathbb{F})$ be a nilpotent matrix with index k, that is $A^k = 0, k \leq n$

1. $\chi_A(x) = (x)^n$.
2. $m_A(x) = (x)^k$.
3. Eigen values are $\underbrace{0, 0, \cdots, 0}_{\text{n times}}$.
4. $E_0 = \{x \in \mathbb{F}^n : Ax = 0\}$, is a proper subspace of $\mathbb{F}^n$. Hence its dimension is $< n$.
5. $GM < n$ if $A \neq 0$.

Theorem: Let $A \in M_n(\mathbb{F})$, A is nilpotent iff all its eigen values are zeroes.

Necessary Condition: A is nilpotent.

ie., $A^k = 0 \quad$ for some $k \in \mathbb{N}$.

Let λ be an eigenvalue of A with eigenvector x.

$Ax = \lambda x$

$A^k x = \lambda^k x \quad \Rightarrow \quad \lambda^k = 0 \quad \Rightarrow \quad \lambda = 0$

Sufficient Condition: All eigenvalues are zero.

$\chi(x) = (x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_n) = x^n \quad (\because \ \lambda_i = 0 \ \ i = 1, 2, \ldots, n)$

By Cayley-Hamilton Theorem,

$\chi(A) = A^n = 0 \quad \Rightarrow \quad A$ is nilpotent.

Idempotent matrices: $A^2 = A, A \neq I, A \neq 0.$

1. $\chi_A(x) = (x)^{k_1}(x-1)^{k_2}, k_1 + k_2 = n$ and $k_1, k_2 \in \mathbb{N}$.
2. $m_A(x) = (x)(x-1)$.
3. Eigen values are $\underbrace{0, 0, \cdots, 0}_{k_1 \text{ times}}$ and Eigen values are $\underbrace{1, 1, \cdots, 1}_{k_2 \text{ times}}$.
4. $GM(0) = AM(0) = k_1$ and $GM(1) = AM(1) = k_2$.

Involuntary matrices $A^2 = I, A \neq \pm I$.

1. $\chi_A(x) = (x-1)^{k_1}(x+1)^{k_2}, k_1 + k_2 = n$ and $k_1, k_2 \in \mathbb{N}$.
2. $m_A(x) = (x+1)(x-1)$.
3. Eigen values are $\underbrace{1, 1, \cdots, 1}_{k_1 \text{ times}}$ and $\underbrace{-1, -1, \cdots, -1}_{k_2 \text{ times}}$.
4. $GM(1) = AM(1) = k_1$ and $GM(-1) = AM(-1) = k_2$.

Real symmetric matrices: Let $A \in M_n(\mathbb{R}), A^T = A$

1. All eigen values of A are real.
   λ be eigenvalue of real symmetric matrix A.
   $$Ax = \lambda x \quad \Rightarrow \quad x^T A x = x^T \lambda x$$
   $$= \lambda x^T x.$$
   $$\lambda = \frac{x^T A x}{x^T x} \in \mathbb{R} \quad [\because \; x^T A x \in \mathbb{R}]$$
2. Eigen vectors corresponding to distinct eigen values are orthogonal.

Hermitian matrices: Let $A \in M_n(\mathbb{C}), A^* = A$

1. All eigen values of A are real.
   (Similar to above)
   $$\lambda = \frac{x^* A x}{x^* x}$$
   Now $\lambda^* = \dfrac{x^* A^* x}{x^* x} = \dfrac{x^* A x}{x^* x} = \lambda.$
   $\therefore \; \lambda$ is real.
2. Eigen vectors corresponding to distinct eigen values are orthogonal.

Real skew symmetric matrices: Let $A \in M_n(\mathbb{R}), A^T = -A$

1. Roots of the characteristic polynomial are either zero or purely imaginary
   Let λ be eigen value with eigen vector x.
   $$Ax = \lambda x.$$
   $$\Rightarrow \; iAx = i\lambda x$$
   Now iA is hermitian matrix, so its eigenvalue
   $i\lambda$ is real.
   $\Rightarrow \; \lambda = 0$ or λ is purely imaginary.
2. If $det(A) \neq 0$, then A has no real eigen value.
   $\det(A) \neq 0 \quad \Rightarrow \quad \lambda \neq 0.$
   $\therefore \; \lambda$ is purely imaginary.

Skew hermitian matrices: Let $A \in M_n(\mathbb{C}), A^* = -A$

1. All eigen values of A are either zero or purely imaginary.
   (Same as above).

Real orthogonal matrices: Let $A \in M_n(\mathbb{C}), A^T A = AA^T = I$

1. If A has real eigen values of then they are either 1 or -1.
   Let λ be an eigenvalue with eigenvector x.
   $$Ax = \lambda x.$$
   Taking transpose on each side
   $$x^T A^T = \lambda x^T$$

$\Rightarrow\ x^TA^TAx = \lambda x^T\lambda x.$

$x^TIx = \lambda^2 x^Tx \quad [\because\ A^TA = I]$

$(1-\lambda^2)(x^Tx) = 0,$

$\lambda^2 = 1 \qquad \lambda = \pm 1.$

2. All the roots of characteristic equation of A are of unit modulus.

Unitary matrices: Let $A \in M_n(\mathbb{R}), A^*A = AA^* = I$

1. Eigen values of A are of unit modulus.
   (Same as above proof and $\lambda\bar{\lambda} = |\lambda|^2$)