SUBJECT: Linear Algebra - TOPIC: Linear independence and dependence, Basis, Dimension

Linear dependance and Linear independence:
Linear combination: Let V be a vector space over a field $\mathbb{F}$,a vector $v \in V$is said to be a linear combination of the vectors $v_1, v_2 \cdots, v_n$,if there exist scalars $c_1, c_2, \cdots c_n \in \mathbb{F}$such that $c_1v_1 + c_2v_2 + \cdots + c_nv_n = v$
Linear dependence or Linear independence: The vectors $v_1, v_2 \cdots, v_n \in V$ are said to be linearly independent if $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ implies $c_1 = c_2 = \cdots = c_n = 0$. Otherwise, the vectors $v_1, v_2 \cdots, v_n \in V$ are said to be linearly dependent.
Observations:

1. The set $\{0\}$ is linearly dependent.
2. The empty set $\emptyset$ is linearly independent.
3. If any one of the $\{v_i$'s in $v_1, v_2 \cdots, v_n\}$ is zero, then the vectors $v_1, v_2 \cdots, v_n$ are linearly dependent. $(0V_1 + 0V_2 + \cdots + 1V_i + 0V_{i+1} + \cdots + 0V_n,$ where $V_i = \overrightarrow{0}$ is a nontrivial linear combination resulting in zero vector).
4. If $v_i = kv_j$, for some i and j and $k \neq 0$, $i \neq j$, then the vectors $\{v_1, v_2 \cdots, v_n\}$ are linearly dependent. $(0V_1 + 0V_2 + \cdots + V_i + \cdots + (-k)V_j + \cdots + 0V_n,$ where $V_i = kV_j)$ is a nontrivial linear combination resulting in zero vector.
5. If any one of the $\{v_i$'s is linearly combination of some other $\{v_j$'s in the $\{v_1, v_2 \cdots, v_n\}$, then the set is linearly dependent.
   $V_i = \alpha_1 V_1 + \cdots + \alpha_j V_j$
   $|V_i - \alpha_1 V_1 - \cdots - \alpha_j V_j = 0$
   Linearly dependent.
6. Any subset of linearly independent set is linearly independent.
7. Any superset of linearly dependent set is linearly dependent.

Spanning set: Let V be a vector space over a field $\mathbb{F}$, and let $S = \{v_1, v_2 \cdots, v_n\}$ be a subset of V, S is said to be a spanning set of V if any vector $v \in V$ can be expressed as linear combination of elements of S. $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n,$ for some $c_1 \in \mathbb{F}$
Linear span of a set: Let Vbe a vector space over a field $\mathbb{F}$and let $S = \{v_1, v_2 \cdots, v_n\}$ be a subset of V. The linear span of S is the set of all finite linear combinations of the vectors in S. ie., $L(S) = \{c_1v_1 + c_2v_2 + \cdots + c_nv_n : c_i \in \mathbb{F}, v_i \in S, \forall i = 1, 2, \cdots, n\}$ If $L(S) = V$ then S is a spanning set of V.
Result: For any non-empty subset S of V, $L(S)$ is a subspace of V.

Basis of a vector space
Let V be a vector space over a field $\mathbb{F}$, and let $S = \{v_1, v_2 \cdots, v_n\}$, S is said to be a basis of V, if

(i) S is linearly independent.
(ii) S spans V, that is $L(S) = V$.

Result: Every non-trivial vector space has a basis.
Dimension: The cardinality of a basis of a vector space is called the dimension of the vector space. If it is finite, then the vector space is said to be finite dimensional otherwise the vector space the vector space is of infinite dimension.

Vector space	**Standard basis**	**Dimension**
$\{0\}$		0
$\mathbb{F}^n$ over $\mathbb{F}$	$\{e_i : e_i = (0, 0, \cdots, 0, 1, 0, \cdots, 0)\}$	n
$P(x)$	$\{1, x, x^2 \cdots\}$	∞
$\mathbb{F}$ over $\mathbb{F}$	$\{1\}$	1
$P_n(x)$	$\{1, x, x^2 \cdots x^n\}$	$n+1$
$\mathbb{F}(x)$	$\{1, x, x^2 \cdots, \sin x, \cos x, \cdots, \sin 2x, \cdots\}$	∞
$M_{m\times n}(\mathbb{F})$	$\left\{M_{ij} : M_{ij} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}\right\}$	mn
$\mathbb{Q}(\sqrt{2})(\mathbb{Q})$	$\{1, \sqrt{2}\}$	2
$\mathbb{C}$ over $\mathbb{R}$	$\{1, i\}$	2
$\mathbb{R}$ over $\mathbb{Q}$	$\{1, \sqrt{2}, \sqrt{3}, \cdots, \sqrt[3]{2}, \sqrt[3]{3}, \cdots, 2^{\frac{1}{n}}, 3^{\frac{1}{n}} \cdots, e, \pi, \cdots\}$	∞

Result:

1. Let V be a vector space over the field $\mathbb{F}$, and $dimV = n$, then
   (i) Any subset of V having more than n elements is linearly dependent.
   Let $B = \{V_1, \ldots, V_n\}$ is basis for V. Let $A = \{U_1, \ldots, U_p\}$ such that $p > n$.
   If A is L.I, and since B generates V.
   $P \le n$ contradiction.
   (ii) Let S be a subset of V having n elements then S is linearly independent iff $L(S) = V$.
2. Let S be a subset of V, and let $L(S) = V$, where V is a vector space over $\mathbb{F}$, then any maximum number of linearly independent vectors in S form a basis for V. Suppose we delet every vector that is linear combination of other vectors in S, then remaining set of vectors form a basis for V. That is any spanning set contains a basis.
3. Let V be a vector space over the field $\mathbb{F}$, and let $dimV = n$, let $S = \{v_1, v_2 \cdots, v_m\}$, $m \le n$ be a linearly independent set of vectors then S is a part of some basis for V. That is S can be extended as a basis for V.

Dimension of subspace: Let V be a vector space over the field $\mathbb{F}$ with $dimV = n$, let W be a subspace of V, then W is also finite dimensional and $dimW \le dimV$. $dimW = dimV$ iff $V = W$.

Hyperspace: If $dimW = n - 1$, where W is a subspace of V, with $dimv = n$, then W is called a hyperspace of V.

Result:

1. V be a vector space over the field $\mathbb{F}$ with dimension n, let $\{v_1, v_2 \cdots, v_n\}$ be a basis for V. Let A be any non-singular matrix of order n with entries in $\mathbb{F}$, then the set $\{Av_1, Av_2, \cdots, Av_n\}$ is also a basis for V.
2. Let $\{R_1, R_2, \cdots, R_n\}$ be the set of rows of A, and $\{c_1, c_2, \cdots, c_n\}$ be the set of column of A, then both the sets form basis for R^n if A is invertible.
3. Consider the matrix space $M_n(\mathbb{F})$, let $A \in M_n(\mathbb{F})$. Let $W = span\{I, A, A^2, \cdots\}$, then $dimW \le n$.
4. Let V be a vector space over the field $\mathbb{F}$, with $dimV = n$. Let $\mathbb{F}$ be finite field with cardinality $\mathbb{F}$ is equal to P^k, then $|V| = (P^k)^n = P^{nk}$.
   (i) The number of ordered basis for V is given by $(P^{nk} - 1)(P^{nk} - P^k)(P^{nk} - P^{2k}) \cdots (P^{nk} - P^{(n-1)k})$.
   (ii) The number of unordered basis for V is given by $\frac{(P^{nk}-1)(P^{nk}-P^k)(P^{nk}-P^{2k})\cdots(P^{nk}-P^{(n-1)k})}{n!}$.
   (iii) The number of one dimensional subspace of V is given by $\dfrac{(P^{nk} - 1)}{P^k - 1}$.
   (iv) Number of m dimensional subspaces of V is given by

$\frac{(P^{nk}-1)(P^{nk}-P^k)(P^{nk}-P^{2k})\cdots(P^{nk}-P^{(m-1)k})}{(P^{mk}-1)(P^{mk}-P^k)(P^{mk}-P^{2k})\cdots(P^{mk}-P^{(m-1)k})}.$