2.5 Differentiability

Differentiability: Let f be a real function, let $x_0 \in \mathbb{R}$, let f be continuous at x_0, f is said to be differentiable at x_0 if $f'(x_0^-) = f'(x_0^+)$ where

$$\begin{aligned} f'(x_0^-) &= \lim_{h \to 0} \frac{f(x_0 - h) - f(x_0)}{-h} \\ f'(x_0^+) &= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \end{aligned}$$

If f is differentiable at x_0, we write $f'(x_0) = f'(x_0^-) = f'(x_0^+)$ or $f'(x_0) = \lim_x \to x_0 \dfrac{f(x) - f(x_0)}{x - x_0}$.

If f is differentiable at every point of its domain, then we say that f is a differentiable function, and the derivative is given by $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Example:

1. Polynomial functions are differentiable over $\mathbb{R}$
2. Rationals are differentiable in the domain.
3. Trigonometric functions are differentiable in the domain.
4. Exponential functions are differentiable over $\mathbb{R}$
5. Logarithmic functions are differentiable in $(0, \infty)$

Counter examples:

1. $|x|$ is not differentiable at $x = 0$
2. $|x - a|$ is not differentiable at $x = a$
3. If f is a differential function, then $|f(x)|$ is not differential at simple zeros of f
4. Mini(f, g) and Max(f, g) where both f and g are differentiable, are not differentiable at simple zeros of $(f - g)$

Result:

1. $\dfrac{d}{dx}|x| = \dfrac{|x|}{x}, x \neq 0$
2. $\dfrac{d}{dx}[x] = \begin{cases} 0 & , \mathbb{R} \setminus \mathbb{Z} \\ \text{does not exists} & , \text{otherwise or} x \in \mathbb{Z} \end{cases}$
3. $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \log_e a}$

   $\log_a x = \dfrac{\log_e x}{\log_e a}$

Applications of Derivatives

Increasing and Decreasing functions: Let f be a differential function

1. If $f'(x) > 0, \forall x \in D(f)$, then f is strictly increasing in $D(f)$.
   Eg:$f(x) = e^x$
   $f'(x) = e^x > 0$
2. If $f'(x) \geq 0, \forall x \in D(f)$, then f is increasing in $D(f)$.
   Eg: $f(x) = x^3, x \in \mathbb{R}$
   $f'(x) = 3x^2$
3. If $f'(x) < 0, \forall x \in D(f)$, then f is strictly decreasing in $D(f)$.
4. If $f'(x) \leq 0, \forall x \in D(f)$, then f is decreasing in $D(f)$.

Maxima and Minima: Let f be a continuous function on $\mathbb{R}$, a point $x = x_0$, to be a point of local extrema, it is necessary that either $f'(x_0) = 0$ or $f'(x_0)$ does not exists, such x_0 is called the critical point or stationary point.

First Derivative Test: If x_0 is a critical point of f, then x_0 is a local minima of f if f' changes the sign from negative to positive in a neighbourhood of x_0.
If x_0 is a critical point of f, then x_0 is a local maxi of f if f' changes the sign from positive to negative in the neighbourhood of x_0.
Second Derivative Test: If f is twice differential and x_0 be a critical point of f. x_0 is a point of local minima if $f'(x_0) = 0$ and $f'(x_0) > 0$. x_0 is a point of local maxima if $f'(x_0) = 0$ and $f'(x_0) < 0$. If $f'(x_0) = 0$ and $f''(x_0) = 0$, then the point $x = x_0$ is neither a point of local maxi nor a point of local mini.In this case x_0 is called the point of inflection.
K^{th} **Derivative Test:** Let f be a k times differential function, where $k \in \mathbb{Z}, k \geq 1$. Let c be a point in the $D(f)$, with $f'(c) = 0, f''(c) = 0, ..., f^{k-1}(c) = 0, f^k(c) \neq 0$, then the point $x = c$, has the following possibilities.

k	$f^k(c)$	
even	positive	strict local minima
even	negative	strict local maxima
odd	positive	increasing point of inflection
odd	negative	decreasing point of inflection

Absolute Maxima and Minima: Let $f : [a, b] \to \mathbb{R}$ be a twice differential function, then the absolute maxi or minima are the maxi or mini values of f on $[a, b]$.
The candidates for the points of absolute extreme are the points $x = a, x = b$, and critical points of f in $[a, b]$. Evaluate the values of the function at these points and find extreme among them.
Result: Let $f(x) = a\cos x + b\sin x, a, b, x \in \mathbb{R}$, then $f(x) \in [-\sqrt{a^2 + b^2}, \sqrt{a^2 + b^2}]$.
Result: Let $a, c \in \mathbb{R}$ with $c > 0$ and $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^a \sin\left(\dfrac{1}{|x|^c}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ then

1. f is continuous iff $a > 0$.
2. f is differential (i.e.,$f'(0)$ exists) iff $a > 1$.
3. f' is continuous iff $a - c > 1$.
4. f' is bounded iff $a - c \geq 1$.
5. $f''(0)$ exists iff $a - c > 2$.
6. f'' is continuous iff $a - 2c > 2$.
7. f'' is bounded iff $a - 2c > 2$

Angle between two Functions or Curves: Let f and g be two functions and let (x_0, y_0) be a common point of f and g, i.e., $f(x_0) = y_0, g(x_0) = y_0$ the angle between f and g at point (x_0, y_0) is the angle between the tangents of f and g at the point (x_0, y_0).
Suppose that the slope of the tangent of f at the point (x_0, y_0) is m_1, that of g is m_2. If θ is the angle between these tangents, then $\theta = \tan^{-1}\left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$ where $m_1 = f'(x_0, y_0), m_2 = g'(x_0, y_0)$
Convex functions: Let $f : A \to B$, where $A, B \subseteq \mathbb{R}$ be a function, f is said to be a convex function on A, if $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \forall x, y \in A, \lambda \in [0, 1]$.
Example: $e^x, x^2, x^4, (x - 2)^2$ etc.
If f is a linear function, then $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$
Note: If $f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y) \forall x, y \neq A$, then f is called strictly convex function.
Theorem:

1. Let f be differential on A, then f is convex on A iff f' is increasing on A.
2. Let f be twice differential on A, then f is convex on A, iff $f''(x) \geq 0$ on A.

Result: Let f be convex on A, then $f\left(\sum\limits_{i=1}^{n} \lambda_i x_i\right) = \sum\limits_{i=1}^{n} \lambda_i f(x_i)$ where $\sum\limits_{i=1}^{n} \lambda_i = 1, \lambda_i \in [0, 1], x_i \in A$.
Theorem: Let f be continuous on A, and satisfies the property $f\left(\dfrac{x + y}{2}\right) < \dfrac{f(x) + f(y)}{2}, \forall x, y \in A, x \neq$

y, then f is strictly convex on A.

Result:

1. Let $f : R \to R$ be convex and bounded above then f is constant on $\mathbb{R}$.
2. Let f be a convex function on an open interval I, then f satisfies Lipschitz's conditions locally on I.

Mean Value Theorems

(I) **Rolle's Theorem:** Let $f : [a, b] \to \mathbb{R}$ be a function satisfying the following conditions.

(i) f is continuous on $[a, b]$.
(ii) f is differential on (a, b).
(iii) $f(a) = f(b)$ then there exists atleast one real number $c \in (a, b)$ such that $f^{'}(c) = 0$.

Result:

(i) If x_1 and $x_2(x_1 < x_2)$ are two real zero's of a differential function f then there exists atleast one zero of $f^{'}$ in (x_1, x_2).
(ii) Between any two consecutive zero's of a differential function there exists atleast one zero for its derivative.
(iii) If all roots of a polynomial p, degree $n \geq 2$ are real then all roots of $p^{'}$ are also real.
(iv) If $\dfrac{a_n}{n+1} + \dfrac{a_{n-1}}{n} + ... + a_0$, where $a_i \in \mathbb{R}$, then the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$ has atleast one root in $(0, 1)$.
(v) Let f be continuous differential on $[a, b]$, and twice differential on (a, b) with $f(a) = f(b) = f^{'}(a) = 0$ then there exists $c \in (a, b)$ such that $f^{'}(c) = 0$.
(vi) Let f be continuous differential on $[a, b]$, and twice differential on (a, b) with $f(a) = f(b)$ and $f^{'}(a) = f^{'}(b) = 0$, then there exists $c_1, c_2 \in (a, b)$ with $c_1 \neq c_2$ such that $f^{''}(c_1) = f^{''}(c_2)$.

(II) **Lagrange's Mean Value Theorem:** Let $f : [a, b] \to \mathbb{R}$ be a function satisfying the condition

(i) f is continuous on $[a, b]$.
(ii) f is differential on (a, b).

Then there exists atleast one real number $c \in (a, b)$ such that $\dfrac{f(b) - f(a)}{b - a} = f^{'}(c)$

Another version of LMVT: Let $f : [a, b] \to R$ be a function satisfying the conditions

(i) f is continuous on $[a, b]$.
(ii) f is differential on (a, b). Then $f(a + h) = f(a) + hf^{'}(a + \theta h)$
where $h = b - a, \theta \in (0, 1)$

Result: To find the approximate value of a function f satisfying LMVT, at some point, we use the relation $f(a + h) \approx f(a) + hf^{'}(a)$.

(III) **Cauchy's Mean Value Theorem or Generalised MVT:** Let $f, g : [a, b] \to \mathbb{R}$ be two functions satisfying the conditions,

(i) f, g are continuous on $[a, b]$
(ii) f, g is differential on (a, b)
(iii) $g^{'}(t), \forall t \in (a, b)$
(iv) $g(a) \neq g(b)$, then there exists atleast one $c \in (a, b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f^{'}(c)}{g^{'}(c)}$

Note:

(i) The LMVT is a special case of CMVT in which we take $g(x) = x$.
(ii) Rolle's theorem, is a special case of LMVT, in which we take $f(a) = f(b)$.