2.4 Continuity

Continuity: Let $f : A \to B$ be a function, $A, B \subseteq \mathbb{R}$, let $a \in A$, f is said to be left continuous at $x = a$ if $f(a^-) = f(a^+)$ and f is said to be right continuous at $x = a$ if $f(a^+) = f(a^-)$. f is said to be continuous at $x = a$, if $f(a^-) = f(a^+) = f(a)$. i.e., f is said to be continuous at $x = a$, $\lim_{x\to a} f(x) = f(a)$.
ϵ - δ Definition: Let $f : A \to B$ be a real function, and let $a \in A, f$ is said to be continuous at $x = a$,if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$.
Sequential Definition: Let $f : A \to B$ be a real function and let $a \in A, f$ is said to be continuous at $x = a$, if for any $(x_n) \to a$ the image sequence $(f(x_n)) \to f(a)$.
Example:

1. Every polynomial are continuous on $\mathbb{R}$.
2. Rational functions are continuous on their domains.
3. Trigonometric functions are continuous on their domains.
4. Modulus function is continuous.
5. Exponential function is continuous on $\mathbb{R}$.
6. Log is continuous on $(0, \infty)$

Counter examples:

1. Greatest or least integer function is discontinuous at every integer points.
2. Fraction function is discontinuous at integer points.
3. Signum function is discontinuous at $x = 0$
4. $sin\frac{1}{x}$ and $cos\frac{1}{x}$ are not continuous at $x = 0$.

Discontinuity: Let $f : A \to B$ be a real function, let $a \in A, f$ is said to be discontinuous at $x = a$, if it is not continuous at $x = a$.

1. **Removable discontinuity:** Let $f : A \to B$ be a real function, let $a \in A, f$ is said to have a removable discontinuity at $x = a$, if $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} f(x) \neq f(a)$. To remove this discontinuity we have to redefine $f(a) = \lim_{x\to a} f(x)$.
2. **Non-Removable discontinuity:**
   (i) **First kind/simple discontinuity:** Let $f : A \to B$ be a real function, let $a \in A, f$ is said to have a first kind discontinuity at $x = a$, if both $f(a^-)$ and $f(a^+)$ exists, but they are not equal .
   (ii) **Second kind discontinuity:** Let $f : A \to B$ be a real function, let $a \in A, f$ is said to have a second kind discontinuity at $x = a$, if either $f(a^-)$ or $f(a^+)$ does not exists.

Result:

1. Monotone functions cannot have a discontinuity of the second kind.
2. The set of discontinuity of monotone function is almost countable.
3. Let f be monotonically increasing on (a, b), then $f(x^-)$ and $f(x^+)$ exists at every point $x \in (a, b)$ and $f(x^-) \leq f(x) \leq f(x^+)$ if $a < x < y < b$ then $f(x^+) \leq f(y^-)$.
4. Let f be monotonically decreasing on (a, b), then $f(x^-)$ and $f(x^+)$ exists at any point $x \in (a, b)$ and $f(x^-) \geq f(x) \geq f(x^+)$.

Algebra of continuous function: Let f and g be two constant functions, then $f + g, f - g, fg, \frac{f}{g}$, fog, gof, fof, gog etc are also continuous provided they exists.
Theorems on Continuity:

(I) **Extreme Value Theorem**

   (i) Let $f : [a, b] \to \mathbb{R}$ be a continuous function, then f is bounded on $[a, b]$.
   (ii) Let $f : [a, b] \to \mathbb{R}$ be a continuous function, then f attains its bounds atleast once in $[a, b]$.
   (iii) Let $f : [a, b] \to \mathbb{R}$ be continuous, then f has the least and the largest values, say $l = \inf f(x), L = \sup f(x)$. Also, the range of f is the interval $[l, L]$

Intermediate Value Theorem:

   (i) Let $f : [a, b] \to R$ be a continuous function, if $f(a) \neq f(b)$ then f assumes every value between $f(a)$ and $f(b)$.

Note: If $l = \inf(f), L = \sup(f)$, then f assumes every value between $[l, L]$.

(ii) Location Roots Theorem:
Let $f : [a, b] \to R$ be a continuous function, and let $f(a)$ and $f(b)$ be of opposite signs, then there exists atleast one point $c \in (a, b)$ such that $f(c) = 0$.
Note: If there exists two distinct points x_0 and y_0 with $x_0 < y_0$ in $[a, b]$ such that $f(x_0)$ and $f(y_0)$ are of opposite signs. Then there exists atleast one point $z_0 \in (x_0, y_0)$ such that $f(z_0) = 0$.

(iii) Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $c \in (a, b)$ such that $f(c) \neq 0$, then there exists $\delta > 0$ such that $f(x)$ has the same sign as that of $f(c), \forall x \in (c - \delta, c + \delta)$

Result: Let $f : [a, b] \to \mathbb{R}$ be continuous,

(i) If f is monotonically increasing, then $R(f) = [f(a), f(b)]$
(ii) If f is monotonically decreasing, then $R(f) = [f(b), f(a)]$

Observations:

(a) If a function does not satisfy IVT(IVP) on an interval $[a, b]$, then the function is discontinuous on $[a, b]$.
(b) If a function satisfies IVP, on an interval $[a, b]$, then it need not be continuous on $[a, b]$.

Eg:$\sin \frac{1}{x}, x \in [0, 1]$ not continuous at 0, it attains all values in $[0, 1]$.

Darboux function: A function which satisfies IVP, in a certain interval is called a Darboux function.
Result:

(i) Let $f : [a, b] \to \mathbb{R}$ be a differential function, then its derivative f' is a Darboux function.
(ii) Let $f : [a, b] \to \mathbb{R}$ be a continuous function, if $f(a) = f(b)$, then there exists atleast one pair a' and b' in (a, b) such that $f(a') = f(b')$

(II) **Fixed Point Theorem**
Let $f : [a, b] \to [a, b]$ be a continuous function, then there exists atleast one point c in $[a, b]$ such that $f(c) = c$.
i.e., f has atleast one fixed point.
Result:

(i) $f : [a, b] \to [a, b]$ be a continuous function, then it has a unique fixed point.
(ii) If $f(x) = x$, every point is a fixed point.
(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be a differential function such that $f'(x) \neq 1, \forall x \in \mathbb{R}$ then f has atmost one fixed point.
(iv) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, if f is bounded, then it has atleast one fixed point.
(v) Let $f : \mathbb{R} \to \mathbb{R}$ be a differential function such that $|f'(x)| \leq r < 1, \forall x \in \mathbb{R}$ then there exists a unique fixed point x_0 of the function f such that $x_0 = \lim_{n\to\infty} x_n$, where x_n is a sequence in which x_1 is arbitrary and $x_{n+1} = f(x_n), \forall n = 1, 2, ..., n, ...$
(vi) Let f be a real function which is continuous and periodic then it attains its supremum and infimum. Moreover it is bounded.
(vii) Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant, periodic continuous function, then it has a smallest positive period, called the fundamental period.
(viii) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic functions with T_1 and T_2 such that $\dfrac{T_1}{T_2} \notin Q$, then f is a constant function.
(ix) Let $f : [a, b] \to \mathbb{R}$ be a continuous function, defined
$m(x) = \inf_t\{f(t); t \in [a, x]\}$
$M(x) = \sup_t\{f(t); t \in [a, x]\}$
$m(x)$ and $M(x)$ are also continuous in $[a, b]$
(x) Let $f : A \to B$, where $A, B \subseteq \mathbb{R}$ be a function, and let $x_0 \in A$ be an isolated point of A, then

f is continuous at x_0.

(xi) Let $f : \mathbb{N} \to R$ be any function, then f is continuous on $\mathbb{N}$.
Any such function is a sequence and any sequence is a continuous function.

Maximum and Minimum functions: Let f and g be two real functions, then we can define two functions given by

$\min(f,g)(x) = \dfrac{1}{2}\left[(f(x)+g(x)) - |f(x)-g(x)|\right]$

$\max(f,g)(x) = \dfrac{1}{2}\left[(f(x)+g(x)) + |f(x)-g(x)|\right]$ Eg:Draw the graphs of $\min\{x, x^2\}$, $\max\{x, x^2\}$