COCHIN
Near Lulu Mall
Edappally
Ph: 0484 4060755
9400 31 57 55

IIT-JAM and OTHER MSc ENTRANCE EXAMS IN MATHEMATICAL SCIENCE
SUBJECT: Real Analysis

Real Sequences: A real sequence is a function $f : \mathbb{N} \to \mathbb{R}$. i.e., any function $f : \mathbb{N} \to \mathbb{R}$ is a sequence. Any sequence is a continuous function.
Range of a Sequence: The range or range set of a sequence is the set of all distinct terms of a sequence.
Bounds of a Sequence: Let (a_n) be a sequence (a_n) is said to be bounded above if there exists a real number, k, such that $a_n \leq k, \forall n$. Similarly, (a_n) is said to be bounded below, if there exists k, such that $a_n \geq k, \forall n$. (a_n) is said to be a bounded sequence if it is both bounded above and below.
Subsequence: Let (a_n) be a sequence. Any sequence obtained from (a_n) by removing some terms of (a_n), is a subsequence of (a_n), usually denoted as (a_{n_k}).
Limit Point of a Sequence: Let (a_n) be a sequence, let $l \in \mathbb{R}, l$ is said to be a limit point of (a_n) , if every neighbourhood of l contains infinite number of terms of (a_n).
i.e., l is a limit point of (a_n) if $a_n \in (l - \epsilon, l + \epsilon)$ for infinitely many m, for any $\epsilon > 0$.
Bolzano Weierstrass Theorem for Sequence: Every bounded sequence has atleast one limit point.
Result: The set of all limit points of a bounded sequence is bounded.
Limit inf and Limit sup: Let (a_n) be a sequence and let
$S_1 = \{a_1, a_2, ...\}$
$S_2 = \{a_2, a_3, ...\}$
$S_3 = \{a_3, a_4, ...\}$
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$S_k = \{a_k, a_{k+1}, ...\}$
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Limit inferior of (a_n) is defined as
$\underline{\lim} a_n = \sup\{\inf S_1, \inf S_2, ...\}$
$\overline{\lim} a_n = \inf\{\sup S_1, \sup S_2, ...\}$
Results:

1. If (a_n) is a bounded sequence, then, $\inf a_n \leq \underline{\lim} a_n \leq \overline{\lim} a_n \leq \sup a_n$
2. If (a_n) is a bounded sequence,then, $\underline{\lim} a_n$ and $\overline{\lim} a_n$ are finite.
3. If (a_n) is any sequence, then $\underline{\lim}(-a_n) = -\overline{\lim} a_n$.
4. $\underline{\lim} a_n = -\overline{\lim}(-a_n)$
5. $\underline{\lim} a_n \leq \overline{\lim} a_n$ for any sequence.
6. Let $(a_n), (b_n)$ be two bounded sequences, then , $\underline{\lim} a_n + \underline{\lim} b_n \leq \underline{\lim}(a_n + b_n) \leq \underline{\lim} a_n + \overline{\lim} b_n \leq \overline{\lim}(a_n + b_n) \leq \overline{\lim} a_n + \overline{\lim} b_n$.
7. Let $(a_n), (b_n)$ are positive bounded sequences, then , $\underline{\lim} a_n . \underline{\lim} b_n \leq \underline{\lim}(a_n . b_n) \leq \underline{\lim} a_n . \overline{\lim} b_n \leq \overline{\lim}(a_n . b_n) \leq \overline{\lim} a_n . \overline{\lim} b_n$.
8. (a_n) be an arbitrary sequence, let S be the set of all limit points of (a_n), then

$$\underline{\lim}\, a_n = \begin{cases} \inf S & , S \neq \phi, (a_n), \text{is bounded below} \\ \infty & , S = \phi, (a_n), \text{is bounded below} \\ -\infty & , S \neq \phi, (a_n) \text{not bounded below} \end{cases}$$

$$\overline{\lim}\, a_n = \begin{cases} \sup S & , S \neq \phi, (a_n), \text{is bounded above} \\ -\infty & , S = \phi, (a_n), \text{is bounded above} \\ \infty & , S \neq \phi, (a_n) \text{not bounded above} \end{cases}$$

Convergence of Sequences:

(I) **Convergent Sequences:** Let (a_n) be a sequence and let $l \in R, (a_n)$ is said to converge to l, if for any $\epsilon > 0$,there exists a natural number N_ϵ such that $|a_n - l| < \epsilon, \forall n \geq N_\epsilon$.
If (a_n) converges to l, then we write $\lim\limits_{n \to \infty} a_n = l$ or $(a_n) \to l$

Theorem:

- Every bounded sequence with a unique limit point is convergent.
- A bounded sequence (a_n) converges to a real number, l iff $\underline{lim} a_n = \overline{lim} a_n = l$.
- A necessary and sufficient condition for a sequence to converge is that it is bounded and has a unique limit point.

Corollary: Every convergent sequence is bounded. The converse need not be true.
Eg:$((-1)^n)$

(II) **Divergent Sequences:** Let (a_n) be a sequence ,(a_n) is said to diverge to $+\infty$, if $\underline{lim} a_n = \overline{lim} a_n = \infty$.
Similarly, (a_n) is said to diverge to $-\infty$, if $\underline{lim} a_n = \overline{lim} a_n = -\infty$.

(III) **Oscillating Sequences:**

(i) **Finitely Oscillating Sequences:** Let (a_n) be a sequence , (a_n) is said to oscillate finitely, if it is bounded but not convergent.
In this case, $\underline{lim} a_n \neq \overline{lim} a_n$.
Eg: $((-1)^n), \sin\left(\frac{n\pi}{3}\right)$

(ii) **Infinitely Oscillating Sequence:** Let (a_n) be a sequence, (a_n) is said to oscillate infinitely if (a_n) is not divergent and either $\underline{lim} a_n$ or $\overline{lim} a_n$ is infinity.
Eg: $((-2)^n), (n(-1)^n)$

Cauchy Sequences: Let (a_n) be a sequence, (a_n) is said to be a Cauchy sequence if for any $\epsilon > 0$ there exists a natural number N_ϵ such that $|a_n - a_m| < \epsilon, \forall n, m > N_\epsilon$.

Theorem:

1. Every Cauchy sequence is bounded.
2. (a_n) is convergent iff it is a Cauchy sequence. **Remark:** Let (a_n) be a sequence such that for any $\epsilon > 0$ there exists a natural number N_ϵ, such that $|a_{n+1} - a_n| < \epsilon, \forall n \geq N_\epsilon$, then (a_n) need be a Cauchy sequence.

   Eg: $(\sqrt{n}), a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \dfrac{1}{\sqrt{n} - \sqrt{n+1}}$
3. Let (a_n) be a sequence,let $\alpha \in (0,1)$ be given such that $|a_{n+1} - a_n| \leq \alpha^n, \forall n \in \mathbb{N}$ or $|a_{n+1} - a_n| \leq \alpha |a_n - a_{n-1}|, \forall n \in \mathbb{N}$, then (a_n) is a Cauchy sequence.

Monotone Sequences: Let (a_n) be a sequence, (a_n) is said to be a monotone sequence, if it is either monotone increasing $(a_n \leq a_{n+1}, \forall n)$ or monotone decreasing $(a_n \geq a_{n+1}, \forall n)$.

Theorem (Monotone convergent theorem): A monotone sequence (a_n) is convergent iff it is bounded.

Remark:

1. A monotone increasing sequence which is bounded above convergent to its supremum and a monotone decreasing sequence which is bounded below convergent to its infimum.
2. A monotone sequence which is not convergent is divergent.

Theorems on Convergence: Let (a_n) and (b_n) be two sequences such that $(a_n) \to a, (b_n) \to b$ then

1. $(a_n \pm b_n) \to a \pm b$

2. $(a_n b_n) \to ab$
3. $\left(\dfrac{a_n}{b_n}\right) \to \left(\dfrac{a}{b}\right), b \neq 0$

   If $b = 0, a_n \neq kb_n$ then $\overline{lim}\left(\dfrac{a_n}{b_n}\right) = a$
4. Let (a_n) be a sequence of positive terms and $(a_n) \to a$, then $a \geq 0$
5. Let a_n and b_n be two convergence sequences with $\lim\limits_{n\to\infty} a_n, \lim\limits_{n\to\infty} b_n = b$ such that $a_n \leq b_n, \forall n$ then $a \leq b$.
6. Let (a_n) be a sequence such that $(a_n) \to 0$ and let (b_n) be a bounded sequence,then $(a_n b_n) \to 0$

   Remark:

   If $(a_n) \to a, a \neq 0, (b_n)$ is a bounded sequence, then $(a_n b_n)$ need not be convergent.
7. **Sandwich Theorem or Squeeze Theorem:**

   Let $(a_n), (b_n)$ and (c_n) be three sequences such that $a_n \leq b_n \leq c_n, \forall n \in \mathbb{N}$. If $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = l$, then, $\lim\limits_{n\to\infty} b_n = l$.
8. **Cauchy' s Theorems:**

   (i) Let (a_n) be a sequence, such that $\lim\limits_{n\to\infty} a_n = l, (l \in \mathbb{R} \cup \{\pm\infty\})$ then

   $$\lim_{n\to\infty}\left(\frac{a_1 + a_2 + ... + a_n}{n}\right) = l$$

   (ii) Let (a_n) be a sequence of positive terms with $\lim\limits_{n\to\infty} a_n = l, (l \in \mathbb{R} \cup \{\pm\infty\})$ then

   (a) $\lim\limits_{n\to\infty}\left(\dfrac{a_1 + a_2 + ... + a_n}{n}\right) = l$

   (b) $\lim\limits_{n\to\infty}(a_1 a_2 ... a_n)^{\frac{1}{n}} = l$

   (iii) Let (a_n) be a sequence of positive terms then $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}$ exists iff $a_n^{\frac{1}{n}}$ exists. If they exists,then both the limits are equal.
9. **Cesero's Theorem:** Let (a_n) and (b_n) be two sequences such that $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$, then $\lim\limits_{n\to\infty}\dfrac{a_1 b_n + a_2 b_{n-1} + ... + a_n b_1}{n} = ab$.
10. Let (a_n) be a sequence, let $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = l$. If $l > 1$ then $\lim\limits_{n\to\infty} a_n = \infty$
11. Let (a_n) be a sequence of positive terms , let $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = l$. If $l < 1$ then $\lim\limits_{n\to\infty} a_n = 0$
12. Cantor's Nested Interval Theorem

    Let $I_n = [a_n, b_n]$ be convergent sequence with respective limits a and b, $|a_n - b_n| \to 0$, let $I_{n+1} \subseteq I_n$,let $\lim\limits_{n\to\infty} I_n = \bigcap\limits_{n=1}^{\infty} I_n = a(= b)$.

Results:

1. Let (a_n) be a sequence, satisfying the conditions $|a_n - a_{n-1}| \leq b_n, \forall n$,where $\sum b_n$ is convergent absolutely,then (a_n) is a Cauchy sequence.
2. $\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{k=1}^{n} f\left(\dfrac{k}{n}\right) = \int\limits_0^1 f(x)dx$
3. $\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{k=1}^{a_n} f\left(\dfrac{k}{n}\right) = \int\limits_0^a f(x)dx$
4. $\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{k=1}^{n^2} f\left(\dfrac{k}{n}\right) = \int\limits_0^\infty f(x)dx$