Real Sequences:Level 3- Tutorial Problems

1. We say that a sequence (a_n) does NOT converge to l if
   (a) $\forall\varepsilon > 0,\ \forall n_0 \in \mathbb{N},\ \forall n \geq n_0$ we have $|a_n - l| > \varepsilon$
   (b) $\forall\varepsilon > 0,\ \forall n_0 \in \mathbb{N},\ \exists n \geq n_0$ such that $|a_n - l| > \varepsilon$
   (c) $\exists\varepsilon > 0,\ \forall n_0 \in \mathbb{N},\ \exists n \geq n_0$ such that $|a_n - l| > \varepsilon$
   (d) $\exists\varepsilon > 0,\ \forall n_0 \in \mathbb{N},\ \forall n \geq n_0$ we have $|a_n - l| > \varepsilon$
2. Consider a sequence (a_n) of positive numbers satisfying the condition $a_na_{n+2} \leq a_{n+1}^2, \forall n \in \mathbb{N}$ then (a_n) is a
   (a) convergent sequence if $a_1 \neq 2a_2$
   (b) monotonically increasing sequence if $a_1 \neq 2a_2$
   (c) convergent sequence if $a_1 = 2a_2$
   (d) monotonically increasing sequence if $a_1 = 2a_2$
3. Consider a sequence (a_n) of real numbers. Which of the following conditions imply that (a_n) is convergent?
   (a) $|a_{n+1} - a_n| < \frac{1}{n},\ \forall n \in \mathbb{N}$
   (b) $|a_{n+1} - a_n| < \frac{1}{3^n},\ \forall n \in \mathbb{N}$
   (c) $a_n > 0,\ \forall\ n \in \mathbb{N}$ and a_n is monotonically increasing
   (d) $a_n > 0,\ \forall\ n \in \mathbb{N}$ and a_n is monotonically decreasing
4. If $\{a_n\}$ is a sequence converging to l. Let $b_n = \begin{cases} a_{2n}, & \text{if } n \text{ is odd,} \\ a_{3n}, & \text{if } n \text{ is even} \end{cases}$. Then the sequence $\{b_n\}$
   (a) need not converge
   (b) should converge to 0
   (c) should converge to $2l$ or to $3l$
   (d) should converge to l
5. Let $\{x_n\}$ and $\{y_n\}$ be two sequence in $\mathbb{R}$ such that $\lim\limits_{n\to\infty} x_n = 2$ and $\lim\limits_{n\to\infty} y_n = -2$. Then
   (a) $x_n \geq y_n$ for all $n \in \mathbb{N}$
   (b) $x_n^2 \geq y_n$ for all $n \in \mathbb{N}$
   (c) there exists an $m \in \mathbb{N}$ such that $|x_n| \leq y_n^2$ for all $n > m$
   (d) there exists an $m \in \mathbb{N}$ such that $|x_n| = |y_n|$ for all $n > m$
6. Let $\{x_n\}$ be an increasing sequence of irrational numbers in $[0, 2]$. Then
   (a) $\{x_n\}$ converges to 2
   (b) $\{x_n\}$ converges to $\sqrt{2}$
   (c) $\{x_n\}$ converges to some number in $[0, 2]$
   (d) $\{x_n\}$ may not converges
7. Write the logical negation of the following statement about a sequence $\{a_n\}$ of real numbers: "For all $n \in \mathbb{N}$ there exists an $m \in \mathbb{N}$ such that $m > n$ and $a_m \neq a_n$."
   (a) There exists an $n \in \mathbb{N}$ such that $a_m = a_n$ for all $m > n$

(b) For all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $m > n$ and $a_m = a_n$
(c) There exists $n \in \mathbb{N}$ such that $a_m \neq a_n$, for all $m > n$
(d) There exists $n \in \mathbb{N}$ such that $a_m = a_n$, for all $m \leq n$

8. $\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{3n}\right) =$
   (a) 0
   (b) $\log 2$
   (c) $\log 3$
   (d) ∞

9. Consider the following two statements.
   S_1: If (a_n) is any real sequence, then $\left(\frac{a_n}{1+|a_n|}\right)$ has a convergent subsequence.
   S_2: If every subsequence of (a_n) has a convergent subsequence, then (a_n) is bounded.
   Which of the following statements is true?
   (a) Both S_1 and S_2 are true
   (b) Both S_1 and S_2 are false
   (c) S_1 is false but S_2 is true
   (d) S_1 is true but S_2 is false

10. Let $\ell \in \mathbb{R}$, and (a_n) be a real sequence. Then which of the following is equivalent to '$(a_n) \to \ell$ as $n \to \infty$'?
   (a) $\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N}$ such that $|a_n - \ell| < 2\epsilon$ whenever $n \geq n_0$
   (b) $\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N}$ such that $|a_n - \ell| < \epsilon$ whenever $n \geq 2n_0$
   (c) $\forall \epsilon > 0,\ \exists n_0 \in 3\mathbb{N}$ such that $|a_n - a_m| < 2\epsilon$ whenever $m, n \geq n_0$
   (d) $\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N}$ such that $|a_n - a_m| < 2\epsilon$ whenever $m, n \geq n_0$

11. The non-zero values for x_0 and x_1 such that the sequence defined by the recurrence relation $x_{n+2} = 2x_n$, is convergent are
   (a) $x_0 = 1$ and $x_2 = 1$
   (b) $x_0 = \frac{1}{2}$ and $x_1 = \frac{1}{4}$
   (c) $x_0 = \frac{1}{10}$ and $x_1 = \frac{1}{20}$
   (d) none of the above

12. Which of the following sequences converges to e?
   (a) $\left(1 + \frac{1}{2n}\right)^n$
   (b) $\left(2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right)$
   (c) $\left(1 + \frac{1}{n}\right)^n$
   (d) $\left(\frac{2n+1}{2n-2}\right)^n$

13. Let (a_n) be a sequence where all rational numbers are terms (and all terms are rational). Then
   (a) no subsequence of (a_n) converges
   (b) there are uncountably many convergent subsequence of (a_n)
   (c) every limit point of (a_n) is a rational number
   (d) no limit point of (a_n) is a rational number

14. Which of the following statements is false?
   (a) Every bounded sequence is convergent
   (b) Every convergent sequence is bounded
   (c) Every bounded sequence has a limit point
   (d) Every convergent sequence has a unique point

15. $\lim_{n\to\infty} \frac{2n-3}{n+1}$ equals
   (a) 0
   (b) 1
   (c) 2
   (d) e

16. If a sequence is not a Cauchy sequence then it is a
   (a) divergent sequence
   (b) convergent sequence
   (c) bounded sequence
   (d) none of these

17. $\lim_{n\to\infty} \frac{1}{n}\left(1 + 2^{\frac{1}{2}} + 3^{\frac{1}{3}} + \cdots + n^{\frac{1}{n}}\right)$ is
   (a) 1
   (b) 2
   (c) 0
   (d) none of these

18. Let $\{a_n\}$ and $\{b_n\}$ be sequence of real numbers defined as $a_1 = 1$ for $n \geq 1$, $a_{n+1} = a_n + (-1)^n 2^{-n}$, $b_n = \frac{2a_{n+1} - a_n}{a_n}$. Then
   (a) $\{a_n\}$ converges to zero and $\{b_n\}$ is a Cauchy sequence
   (b) $\{a_n\}$ converges to a non-zero number and $\{b_n\}$ is a Cauchy sequence
   (c) $\{a_n\}$ converges to zero and $\{b_n\}$ is not a convergent sequence
   (d) $\{a_n\}$ converges to a non-zero number and $\{b_n\}$ is not a convergent sequence

19. If sequences of real numbers $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are such that, $b_n = a_{2n}$ and $c_n = a_{2n+1}$, then $\{a_n\}_{n=1}^{\infty}$ is convergent implies
   (a) $\{b_n\}_{n=1}^{\infty}$ is convergent but $\{c_n\}_{n=1}^{\infty}$ need not be convergent
   (b) $\{c_n\}_{n=1}^{\infty}$ is convergent but $\{b_n\}_{n=1}^{\infty}$ need not be convergent
   (c) both $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are convergent
   (d) both $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are divergent

20. $\lim_{n\to\infty} (2^n + 3^n)$ is equal to
   (a) 2
   (b) 3
   (c) 5
   (d) 6