Infinite Series: Let (a_n) be a sequence of real numbers then the formal sum, $a_1 + a_2 + ... + a_n + ...$ of the terms of the sequence (a_n) is called the series associated with (a_n), denoted as $\sum\limits_{n=1}^{\infty} a_n$.

i.e., $\sum\limits_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ...$

Define (S_n) as,

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

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$S_n = a_1 + ... + a_n$ this sequence, (S_n) is called the sequence of partial sums of the series (a_n).

Theorem:

1. The infinite series $\sum\limits_{i=1}^{\infty} a_n$ converges iff the sequence of partial sum (S_n) converges.
2. Necessary condition for convergent of an infinite series let $\sum\limits_{n=1}^{\infty} a_n$ be a series associated with (a_n), if series $\sum a_n$ converges to a limit $S, S \in \mathbb{R}$, then $\lim\limits_{n\to\infty} a_n = 0$.

Remarks: The above condition is not sufficient.

Eg: $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

$\sum\limits_{n=1}^{\infty} \frac{1}{n} = \infty$

let $S_2 = 1 + \frac{1}{2} \geq 1 + \frac{1}{2}$

$S_{2^2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \geq 1 + 2.\frac{1}{2}$

$S_{2^3} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

$\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

$\geq 1 + 3.\frac{1}{2}$

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$S_{2^k} \geq 1 + k.\frac{1}{2}$

$\lim\limits_{n\to\infty} S_{2^k} = \infty$

$\therefore (S_n)$ is unbounded, $\sum \frac{1}{n}$ is divergent **Note:** If (a_n) is any sequence such that $\lim\limits_{n\to\infty} a_n \neq 0$,then we can say, $\sum\limits_{n=1}^{\infty} a_n$ does not converges.

Example::

$((-1)^n)$

$S_1 = -1$

$S_2 = 0$

$S_3 = -1$

$S_4 = 0$

The sequence $\lim\limits_{n\to\infty} (-1)^n$ does not exists, $\sum a_n$ does not converges (cannot say divergent), it is bounded.

Theorem(Cauchy' s General Principle for Convergent of a Series): A series $\sum_{n=1}^{\infty} a_n$ converges iff the sequence of partial sums (S_n) is a Cauchy sequence.
i.e., $\sum_{n=1}^{\infty} a_n$ converges, iff for each $\epsilon > 0$ there exists a positive integer N_ϵ such that $|S_n - S_m| < \epsilon, \forall m, n \geq N_\epsilon$.
In other words, $\sum_{n=1}^{\infty} a_n$ converges iff for any $\epsilon > 0$ there exists a positive integer N_ϵ such that $|S_{n+p} - S_n| < \epsilon, \forall n \geq N_\epsilon, p \geq 1$.
i.e., $|a_{n+1} + a_{n+2} + ... + a_{n+p}| < \epsilon, \forall n \geq N_\epsilon, p \geq 1$.
Results: If $\sum_{n=1}^{\infty} a_n = a$, then

1. $\sum_{n=1}^{\infty} ka_n = ka$
2. $\sum_{n=0}^{\infty} a_n = a_0 + a$
3. $\sum_{n=2}^{\infty} a_n = a - a_1$
4. If $\sum_{n=1}^{\infty} a_n = a, \sum_{n=1}^{\infty} b_n = b$ then $\sum(a_n + b_n) = a + b, \sum(a_n - b_n) = a - b$
5. Consider the series, $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$, and $\sum_{n=1}^{\infty}(a_n + b_n)$ or $\sum_{n=1}^{\infty}(a_n - b_n)$, then
   (i) If any two of the above series converges then the third series also converges.
   (ii) If one of the series is convergent and one of them is divergent, then the third one is divergent.
   (iii) If any two of the above series are divergent, then the third one may be convergent or divergent.

Positive Term Series: A series $\sum a_n$ is said to be a positive term series, if $a_n \geq 0, \forall n \in \mathbb{N}$. In positive term series, the sequence of partial sums (S_n) is monotonically increasing and bounded below by a_1.
Theorem:

1. A positive term series $\sum a_n$ converges iff the sequence of the partial sum (S_n) is bounded above.
2. Pringsheim's Theorem
   Let (a_n) be a monotonically decreasing sequence of positive terms, if $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} na_n = 0$.

Test for Convergence of Positive Series:

1. **Comparison Test:** Let (a_n) and (b_n) be two positive term sequences such that $a_n \leq b_n \forall n \in \mathbb{N}$, then
   (i) If $\sum b_n$ converges, then $\sum a_n$ converges.
   (ii) If $\sum a_n$ diverges, then $\sum b_n$ diverges.
2. **Limit Comparison Test:** Let (a_n) and (b_n) be two positive term sequences, if $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = l$, where l is a non-zero finite number, then $\sum a_n$ and $\sum b_n$ behave alike.
   i.e., if $\sum a_n$ and $\sum b_n$ converges or diverges together.
   In addition, if $l = 0$ and $\sum b_n$ converges $\Rightarrow \sum a_n$ converges. If $l = \infty$ and $\sum b_n$ diverges $\Rightarrow \sum a_n$ diverges.
3. **D' Alembert' s Ratio Test:** Let (a_n) be a sequence of positive terms, such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$,then
   (i) $\sum a_n$ converges if $l < 1$.
   (ii) $\sum a_n$ diverges if $l > 1$.
   (iii) Test fails if $l = 1$.
4. **Cauchy 's Root Test:** Let (a_n) be a sequence of positive terms, such that $\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = l$,then
   (i) $\sum a_n$ converges if $l < 1$.

(ii) $\sum a_n$ diverges if $l > 1$.
(iii) Test fails if $l = 1$.

5. **Raabe ' s Test:** Let (a_n) be a sequence of positive terms such that $\lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}} - 1\right) = l$, then
   (i) $\sum a_n$ converges if $l > 1$.
   (ii) $\sum a_n$ diverges if $l < 1$.
   (iii) Test fails if $l = 1$.
6. **Logarithm Test:** Let (a_n) be a sequence of positive terms, such that $\lim_{n\to\infty} n\log\left(\frac{a_n}{a_{n+1}}\right) = l$, then
   (i) $\sum a_n$ converges if $l > 1$.
   (ii) $\sum a_n$ diverges if $l < 1$.
   (iii) Test fails if $l = 1$.
7. **Condensation Test:** Let (a_n) be a sequence of positive terms, then $\sum a_n$ and $\sum 2^n a_{2^n}$ behave alike. i.e., $\sum a_n$ and $\sum 2^n a_{2^n}$ converges or diverges together.
8. **Cauchy' s Integral Test:** Let f be a non-negative monotone decreasing function satisfying , $f(n) = a_n, \forall n \in \mathbb{N}$. Then, the $\sum a_n$ and the improper integral $\int\limits_1^\infty f(x)dx$ behave alike.

   i.e., $\sum a_n$ and $\int\limits_1^\infty f(x)dx$ converges or diverges together.

Alternating Series: A series whose terms are alternating positive and negative is called an alternating series. An alternating series in the form $\sum\limits_{n=1}^{\infty}(-1)^n a_n$, where (a_n) is either a positive sequence or a negative sequence.

Leibnitz' s Test: Let (a_n) be a positive term sequence, and let $\sum(-1)^n a_n$ be an alternating series if $a_n \leq a_{n-1}, \forall n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = 0$, then the series $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ converges.

Absolute Convergence of a Series: A series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

Theorem: Let $\sum a_n$ be any series, if $\sum a_n$ converges absolutely then $\sum a_n$ converges.

i.e.,absolute converges $\Rightarrow$ converges, Converse is not true. Eg: $\sum \frac{(-1)^n}{n}$.

Conditionally Convergent Series: A series $\sum a_n$ is said to converge conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Abel's Test: Let $\sum a_n$ be convergent and (b_n) be a positive monotone decreasing sequence. Then the series $\sum a_n b_n$ convergent.

Result:

1. If $\sum a_n$ and $\sum b_n$ are convergent, then the series $\sum a_n b_n$ converges absolutely if atleast one of $\sum a_n$ and $\sum b_n$ converges absolutely.
2. If $\sum a_n$ and $\sum b_n$ converges absolutely, then the series $\sum a_n b_n$ converges absolutely and $\sum a_n b_n = \sum a_n . \sum b_n$.

Dirichlet Test: Let (b_n) be a positive monotone decreasing sequence, with $\lim_{n\to\infty} b_n = 0$, and let the sequence of partial sums of the series $\sum a_n$ is bounded, then $\sum a_n b_n$ is convergent.

Result:

1. Let $\sum a_n$ be convergent absolutely,let $\sum b_n$ be a bounded sequence, then $\sum a_n b_n$ is convergent.
2. The series $\sum \frac{1}{n^p}$, converges if $p > 1$ (absolutely) diverges if $p \leq 1$

Result: The $\sum \frac{1}{P(n)}$ where $P(n)$ is polynomial of degree k, then the series

1. Converges if $k > 1$
2. Diverges if $k \leq 1$