IIT-JAM MATHEMATICAL SCIENCE: ONLINE COURSE
SUBJECT: Linear Algebra - TOPIC: Direct sum, Coordinate vector and Transition matrix

Sum and Direct sum of Subspaces

Sum of two subspaces: Let V be a vector space over the field $\mathbb{F}$, and let V_1 and V_2 be its subspaces, sum of V_1 and V_2 is given by $V_1 + V_2\{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\}$

1. $V_1 + V_2$ is a subspace of V.
2. $V_1 + V_2$ is the smallest subspace of V, which contains $V_1 \cup V_2$ and $V_1 + V_2 = span(V_1 \cup V_2)$

Theorem: Let V be finite dimensional vector space over a field $\mathbb{F}$, then
$dim(V_1 + V_2) = dimV_1 + dimV_2 - dim(V_1 \cap V_2)$.

Note: This result cannot be extended in general.

Direct sum: Let V be a vector space over a field $\mathbb{F}$, and let V_1, V_2 be two subspaces, V is said to be the direct sum of V_1 and V_2 if any vector $v_1 \in V_1$, $v_2 \in V_2$. If V is the direct sum of V_1 and V_2, then we write $V = V_1 \bigoplus V_2$.

Theorem: Let V be a vector space over a field $\mathbb{F}$, let V_1, V_2 be two its subspaces, then $V = V_1 \bigoplus V_2$ iff
(i) $V_1 + V_2 = V$. (ii) $V_1 \cap V_2 = \{0\}$.

Result: Let V be a finite dimensional vector space over a field $\mathbb{F}$, and let V_1, V_2 be two its subspaces, then
$dim(V) = dim(V_1) + dim(V_2)$, if $V = V_1 \bigoplus V_2$.

Moreover, If $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for V_1 and V_2 respectively, then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for V.

Coordinate vector: Let V be a n-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{B} = \{v_1, v_2, \cdots, v_n\}$ be an ordered basis for V, let $v \in V$ be any vector, then v can be uniquely expressed as a linear combination of vectors in $\mathcal{B}$, that is $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n, c_i \in \mathbb{F}, \forall i$.
The matrix, $(c_1 \quad c_2 \quad \cdots \quad c_n)^T$ is called the coordinate vector of v with respect to the ordered basis $\mathcal{B}$, it is denoted by $[v]_{\mathcal{B}}$.

Linear transformations or linear maps: Let V and W be two vector space over the same field $\mathbb{F}$, a map $T : V \longrightarrow W$ is said to be linear map if
$T(v_1 + v_2) = T(v_1) + T(v_2), \forall v_1, v_2 \in V$
$T(cv) = cT(v), \forall c \in \mathbb{F}, v \in V$ or
$T(cv_1 + v_2) = cT(v_1) + T(v_2)$

Result: If $T : V \longrightarrow W$ is a linear transformation then $T(0) = 0$.

$$T(O_v + O_v) = T(O_v) + T(O_v)$$
$$T(O_v) = T(O_v) + T(O_v)$$
$$O_w = T(O_v)$$

Kernel and image of a linear tranformation

1. Let V and W be two vector space over the same field $\mathbb{F}$ and map $T : V \longrightarrow W$ is a linear transformation. The kernel of T or the null space of T is given by $ker(T) = \{v \in V : Tv = 0\}$. $ker(T)$ is a subspace of V and $dim(ker(T))$ is called nullity of T.
2. The image space of T or the range space of T is given by $\{Tv : v \in V\}$, that is
$\{w \in W : w = T(v), \text{ for some } v \in V\ \}$.
The range space is a subspace of w, and the $dim(range(T))$ is called the $rank(T)$.

Rank- Nullity Theorem: Let V and W be a finite dimensional vector space over the same field $\mathbb{F}$, and let $T : V \longrightarrow W$ is a linear transformation, then $rank(T) + nullity(T) = dim(V)$

Result: Let V be a vector space over the same field $\mathbb{F}$ and $T : V \longrightarrow V$ is a linear transformation, then

(i) $ker(T) \subseteq kerT^2 \subseteq kerT^3 \subseteq \cdots$ and
$nullity(T) \leq nullityT^2 \leq nullityT^3 \leq \cdots$

(ii) $Range(T) \supseteq Range(T^2) \supseteq Range(T^3) \supseteq \cdots$ and
$rank(T) \geq rank(T^2) \geq rank(T^3) \geq \cdots$.

(iii) Let $\mathbb{F}$ be a field and let $T : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ be a linear transformation then T is in form $T(x_1, x_2, \cdots, x_n) = (\sum_{i=1}^{n}(a_{1i}x_i), \sum_{i=1}^{n}(a_{2i}x_i), \cdots \sum_{i=1}^{n}(a_{mi}x_i))$

Singular and non-singular transformation Let V and W be two vector space over the same field $\mathbb{F}$, and let $T : V \longrightarrow W$ is a linear transformation.

(i) T is $1-1$ or injective if $v_1 \neq v_2 \Rightarrow T(v_1) \neq T(v_2)$.

(ii) T is onto or surjective if $T(V) = W$, that is $img(T) = W$, that is $rank(T) = dim(W)$.

1. T is bijective or invertible if T is both $1-1$ and onto.
2. T is singular if $T(x) = 0$, for some non-zero $v \in V$.
3. T is non-singular if $T(v) = 0 \Leftrightarrow v = 0$.

Result:

1. T is $1-1$ iff $ker(T) = \{0\}$.

2. T is non-singular iff $ker(T) = \{0\}$.
3. If $dimV = dimW < \infty$, then T is $1-1$ iff T is onto.

4. If $dimV > dimW$ and $dimV < \infty$, then T is singular.
$$\begin{aligned} \dim V > \dim W \quad &\Rightarrow \ T \text{ is not one-one} \\ &\Rightarrow \ \ker(T) \neq \{0\} \\ &\Rightarrow \ T \text{ is singular.} \end{aligned}$$

***Definition*:** Let V and W be two vector space over the same field $\mathbb{F}$. V and W are said to be isomorphic if there exist an invertible linear transformation $T : V \longrightarrow W$.

Result:

1. If V and W are finite dimensional vector space over the same field $\mathbb{F}$, then V and W are isomorphic iff $dimV = dimW$.
2. Let V be an n - dimensional vector space over a field $\mathbb{F}$, then $V \cong \mathbb{F}^n$.
3. Let V be an infinite dimensional vector space over a field $\mathbb{F}$ then $V \cong V \times V$. In general $V \cong V^n$
4. $\mathbb{R}$ over $\mathbb{Q}$ is isomorphic to $\mathbb{C}$ over $\mathbb{Q}$.

Theorem:

1. Let V and W are finite dimensional vector space over a field $\mathbb{F}$, and let $T : V \longrightarrow W$ be linear map if $\{v_1, v_2, \cdots, v_n\}$ is a spanning set of V, then $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is a spanning set of $img(T)$.
2. Let V and W be two finite dimensional vector space over the same field $\mathbb{F}$, and let $\{v_1, v_2, \cdots, v_n\}$ be a basis for V, then $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is a basis for $img(T)$ for any non-singular linear map $T : V \longrightarrow W$.

Definition: Let V be a vector space over the same field $\mathbb{F}$, any linear transformation, $T : V \longrightarrow V$ is also called linear operator on V.

Definition: Let V be a vector space over the same field $\mathbb{F}$, any linear transformation, $T : V \longrightarrow \mathbb{F}$ is also called linear functional on V.

Definition:

Matrix representation of a linear transformation: Let V and W be two vector space over the same field $\mathbb{F}$, with $dim(V) = n$, $dim(W) = m$, let $\mathcal{B}_1 = \{v_1, v_2, \cdots, v_n\}$, $\mathcal{B}_2 = \{u_1, u_2, \cdots, u_m\}$ be ordered bases for V and W respectively. Let $T : V \longrightarrow W$ be a linear transformation.

For each v_i, $T(v_i)$ can be uniquely expressed as linear combination of elements of $\mathcal{B}_2$, that is

$$\begin{aligned} T(v_1) &= a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m \\ T(v_2) &= a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m \\ &\vdots \\ T(v_n) &= a_{1n}w_1 + a_{2n}u_2 + \cdots + a_{mm}w_m \end{aligned}$$

The matrix of T with respect to the basis $\mathcal{B}_1$ and $\mathcal{B}_2$ is an $m \times n$ matrix whose columns are the co-ordinate vectors of $T(v_i)$'s with respect to the bases $\mathcal{B}_2$, that is

$$\begin{aligned} M_{\mathcal{B}_2}^{\mathcal{B}_1} &= [[T(v_1)]_{\mathcal{B}_2}, [T(v_2)]_{\mathcal{B}_2}, \cdots, [T(v_n)]_{\mathcal{B}_2}] \\ &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{aligned}$$

Result:

1. For any $v \in V$, we have $M[v]_{\mathcal{B}_1} = [T(v)]_{\mathcal{B}_2}$
   If $[T(v)]_{\mathcal{B}_2} = [c_1 \quad c_2 \quad \cdots \quad c_m]^T$, then
   $T(v) = c_1w_1 + c_2w_2 + \cdots + c_mw_m$.
2. $rank(T) = rank(M)$
   $nullity(T) = nullity(M)$.
3. Let T be a linear operator on a vector space V over a field $\mathbb{F}$, with $dimV = n$, let M be the matrix of T with respect to an ordered basis $\mathcal{B}$ of V, then

(i) Characteristic polynomial of T = Characteristic polynomial of M.
(ii) Minimal polynomial of T = Minimal polynomial of M.
(iii) Eigen values of T = Eigen values of M.
(iv) Eigen vector of T = Eigen vector of M.
(v) $rank(T) = rank(M)$.
(vi) $nullity(T) = nullity(M)$.

4. Let V be a vector space over the same field $\mathbb{F}$ and $dim(V) = n$, let $T : V \longrightarrow W$ be a linear transformation, let A be the matrix of T with respect to the ordered bases $\mathcal{B}_1$ of V and let $\mathcal{B}$ be the matrix of T with respect to the ordered bases $\mathcal{B}_2$ of V, then A and B are similar.

IIT-JAM MATHEMATICAL SCIENCE: ONLINE COURSE
SUBJECT: Linear Algebra - TOPIC:Special Linear Transformations

Some standard linear transformations
Transformation on $\mathbb{R}^2$

1. Projection map (idempotent maps)
   (i) Projection on to x - axis
   $T(x,y) = (x,0)$
   $T^2(x,y) = T(x,0) = (x,0) = T$. Therefore $T^2 = T$.
   (a) Matrix of T with respect to standard basis $M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
   (b) $\chi_T(x) = x(x-1) = M_T(x)$.
   (c) T is diagonalizable.
   (d) Eigen values are $0, 1$ and T is always singular.
   (e) $ker(T) = \{(0,y) : y \in \mathbb{R}\}$.
   (f) $Nullity(T) = 1$.
   (g) $Range(T) = \{(x,0) : x \in \mathbb{R}\}$
   (h) $rank(T) = 1$.
   (i) $\mathbb{R}^2 = kerT \oplus RangeT$.

   (ii) Projection on to y- axis.
   $T(x,y) = (0,y)$
   All the results are analogus to the previous case.
2. Reflection map (Involuntary map)
   (i) Reflection along x - axis
   $T(x,y) = (x,-y)$
   $T^2(x,y) = T(x,-y) = (x,y) = I$. Therefore $T^2 = I$.
   (a) Matrix of T with respect to standard basis $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
   (b) $\chi_T(x) = (x-1)(x+1) = M_T(x)$.
   (c) T is diagonalizable.
   (d) Eigen values are $1, -1$ and T is non-singular.
   (e) $ker(T) = \{(0,0)\}$.
   (f) $Nullity(T) = 1$.
   (g) $Range(T) = \mathbb{R}^2$

   (ii) Reflection along y - axis
   $T(x,y) = (-x,y)$
   All the results are analogus to the previous case.
3. Rotation map (Orthogonal maps)
   (i) Rotation through θ degree anticlockwise
   $T(x,y) = (\cos\theta x - \sin\theta y, \sin\theta x + \cos\theta y)$.

(a) Matrix of T with respect to standard basis $M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(b) $\chi_T(x) = x^2 - 2\cos\theta + 1$.

(c) $m_T(x) = \begin{cases} x-1, & \theta = 2n\pi \\ x+1, & \theta = (2n-1)\pi \\ x^2 - 2\cos\theta x + 1, & \text{otherwise} \end{cases}$

(d) T is diagonalizable over $\mathbb{C}$.

(e) Eigen values are $\cos\theta + i\sin\theta$ and $\cos\theta - i\sin\theta$.

(f) T is non-singular.

(ii) Rotation through θ degree clockwise
$T(x,y) = (\cos\theta x + \sin\theta y, -\sin\theta x + \cos\theta y)$.

1. Matrix of T with respect to standard basis $M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

All the results are analogus to the previous case (replace θ by $-\theta$).

4. Dilation maps (Scalar maps)
$T(x,y) = (\alpha x, \alpha y)$, where α is a fixed real number
$T(x,y) = \alpha(x,y) = \alpha I$.

(a) Matrix of T with respect to standard basis $M = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$.

(b) $\chi_T(x) = (x-\alpha)^2$.

(c) $M_T(x) = (x-\alpha)$.

(d) T is diagonalizable.

(e) Eigen values are $\alpha, -\alpha$.

(f) T is non-singular if $\alpha \neq 0$.

Note : If $|\alpha| < 1$, then T is called a contraction map.

II. Transformations on $\mathbb{R}^3$

1. Projection map.

(i) On x- axis : $T(x,y,z) = (x,0,0)$

(ii) On y- axis : $T(x,y,z) = (0,y,0)$

(iii) On z- axis : $T(x,y,z) = (0,0,z)$

(iv) On xy- axis : $T(x,y,z) = (x,y,0)$

(v) On yz- axis : $T(x,y,z) = (0,y,z)$

(vi) On xz- axis : $T(x,y,z) = (x,0,z)$

$\chi_T(x) = \begin{cases} x^2(x-1) & ,(i),(ii),(iii) \\ x(x-1)^2 & ,(iv),(v),(vi) \end{cases}$
$m_T(x) = x(x-1)$.
T is singular and diagonalizable.

2. Reflection map.

(i) On x- axis : $T(x,y,z) = (x,-y,-z)$

(ii) On y- axis : $T(x,y,z) = (-x,y,-z)$

(iii) On z- axis : $T(x,y,z) = (-x,-y,z)$

(iv) On xy- plane : $T(x,y,z) = (x,y,-z)$

(v) On yz- plane : $T(x,y,z) = (-x,y,z)$

(vi) On xz- plane : $T(x,y,z) = (x,-y,z)$

$\chi_T(x) = \begin{cases} (x-1)(x+1)^2 & ,(i),(ii),(iii) \\ (x-1)^2(x+1) & ,(iv),(v),(vi) \end{cases}$
$m_T(x) = (x-1)(x+1)$
T is non-singular and diagonalizable.

3. Rotation map.

(i) Rotation along xy- plane θ degree anticlockwise : $T(x, y, z) = (\cos\theta x - \sin\theta y, \sin\theta x + \cos\theta y, -z)$.

Matrix of T with respect to standard basis $M = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(ii) Rotation along yz- plane θ degree anticlockwise : $T(x, y, z) = (x, \cos\theta y - \sin\theta z, \sin\theta y + \cos\theta z)$.

Matrix of T with respect to standard basis $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

(iii) Rotation along zx- plane θ degree anticlockwise : $T(x, y, z) = (\cos\theta x - \sin\theta z, y, \sin\theta x + \cos\theta z)$.

Matrix of T with respect to standard basis $M = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$

To get the map when the rotation is through θ degree clockwise, replace θ by $-\theta$ in the above cases.

Eigen values are $1, \cos\theta + i\sin\theta, \cos\theta - i\sin\theta$ $\chi_T(x) = (x-1)(x^2 - 2\cos\theta x + 1)$ $m_T(x) = \begin{cases} (x-1) \\ (x-1)(x+1) \\ (x-1)(x^2 - 2\cos\theta x \end{cases}$

T is non-singular and diagonalizable (over $\mathbb{C}$).

(4) Dilation map.

$T(x, y, z) = (\alpha x, \alpha y, \alpha z)$, for a fixed $\alpha \in \mathbb{R}$. $T = \alpha I$.

III. Derivative map: Let V be a real vector space of different functions over $\mathbb{R}$, the map $D : V \longrightarrow V$ defined by $D(f(x)) = \dfrac{df(x)}{dx}$ is a linear operator on V called the derivative map.

Result: Derivative map is always singular. Since it maps constant functions to zero, $ker(T)$ is non-trivial always.

Particular case: If $V = P_n(x)$, then $D : V \longrightarrow V$ defined by $D(P(x)) = P'(x)$. Here D is nilpotent operator, $D^{n+1} = 0$. Here 0 is the only eigen value of D.

IV. Integral map: Let V be the vector space of all Riemann integer functions over $[a, b]$ over the field $\mathbb{R}$, the map $J : V \longrightarrow V$ defined by $J(f(x)) = \int_a^x f(t)dt$ is a linear map, called the integral map.

Particular case: $J : P_n(x) \longrightarrow P_{n+1}(x)$, $x \in [a, b]$ defined by $J(P(x)) = \int_a^x P(t)dt$ matrix with respect to standard basis.

$$\begin{aligned}
J(1) &= \int_a^x 1dt = x - a \\
J(x) &= \int_a^x xdt = \tfrac{x^2}{2} - \tfrac{a^2}{2} \\
J(x^2) &= \int_a^x x^2dt = \tfrac{x^3}{3} - \tfrac{a^3}{3} \\
&\vdots \\
J(x^n) &= \int_a^x x^n dt = \tfrac{x^{n+1}}{n+1} - \tfrac{a^{n+1}}{n+1}
\end{aligned}$$

$$M = \begin{pmatrix} -a & -\frac{a^2}{2} & \cdots & -\frac{a^{n+1}}{n+1} \\ 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{n+1} \end{pmatrix}_{(n+1)\times(n+1)}$$

Matrix Transformations:

(i) Let $A \in M_n(\mathbb{R})$ be a non-zero fixed matrix, then the linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$T(x) = Ax$ is called a matrix transformation.

properties

(a) $\chi_T(x) = \chi_A(x)$.
(b) $m_T(x) = m_A(x)$.
(c) Eigen values of T = Eigen values of A.
(d) Trace T = Trace A.
(e) Determinant of T = Determinant of A.
(f) Rank of T = Rank of A.
(g) Nullity of T = Nullity of A.
(h) The matrix of T with respect to the standard basis is A itself.
(i) T is non-singular iff A is non-singular.
(j) T is diagonalizable iff A is diagonalizable.

(i) Let $A \in M_n(\mathbb{R})$ be a non-zero fixed matrix, then the linear transformation $T : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ defined by $T(x) = Ax$ is called a matrix transformation.

properties

(a) $\chi_T(x) = (\chi_A(x))^n$.
(b) $m_T(x) = m_A(x)$.
(c) Trace T = n $\cdot$ Trace A.
(d) Determinant of $T = (det(A))^n$.
(e) T is invertible iff A is invertible.
(f) T is diagonalizable iff A is diagonalizable.
(g) If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigen values of A, then $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigen values of T, each of multiplicity n.