SUBJECT: Linear Algebra - TOPIC: Matrices And Determinants

Matrices and Determinants: The set of all $n \times n$ matrices with real entries is usually denoted as $M_n(\mathbb{R})$.
$(M_n(\mathbb{R}), +)$ is an abelian group.
$(M_n(\mathbb{R}), +, \cdot)$ is a non commutative ring with unity, which is not an integral domain.

Result:

1. Let $A, B \in M_n(\mathbb{R})$, $A \cdot B = 0$ need not imply that either $A = 0$ or $B = 0$.
   Eg: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
2. Let A, B be an $m \times n$ matrix, and B be an $n \times p$ matrix then AB is of $m \times p$. Also $AB = 0$ does not imply that either $A = 0$ or $B = 0$
   Eg: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{3\times 1} \quad AB = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times 1}$

Determinant of a square matrix: Determinant is a map from $M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ given by $det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ satisfying the following properties.

1. $det(A^T) = det(A)$
   $\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}$
2. $det(A') = -det(A)$, wher A' is the matrix obtained from A by interchanging any two rows or columns.
   $\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} = -\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix} \quad R_1 \Leftrightarrow R_2$
3. $det(A) = 0$ if any two rows or columns of A are proportional.
   $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 7 & 9 \end{vmatrix} = 0, \quad \text{Since } R_2 = 2R_1$
4. $det(kA) = k^n det(A)$, $k \in \mathbb{R}$
   $\begin{vmatrix} 2 & 6 & 4 \\ 10 & 20 & 14 \\ 22 & 26 & 28 \end{vmatrix} = \det\left(2\begin{bmatrix} 1 & 3 & 2 \\ 5 & 10 & 7 \\ 11 & 13 & 14 \end{bmatrix}\right) = 2^3 \begin{vmatrix} 1 & 3 & 2 \\ 5 & 10 & 7 \\ 11 & 13 & 14 \end{vmatrix}$
5. If every element of a row or column of A can be expressed as the sum of two or more terms, thn determinant of A can be expressed as the sum of two or more determinants.
   $\begin{vmatrix} 10+5 & 13+5 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 10 & 13 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 5 & 5 \\ 2 & 3 \end{vmatrix}$
6. $det(A'') = det(A)$, where A'' is the matrix obtained from A by operating $R_i \to R_i \pm kR_j$, $i \neq j$ or $C_i \to C_i \pm kC_j$, $i \neq j$
   $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 8 & 11 \end{vmatrix} \quad R_2 \to R_2 + 2R_1$

Properties:

1. Determinant is a continuous map from $M_n(\mathbb{R}) \longrightarrow \mathbb{R}$.
2. $det(AB) = det(A) \cdot det(B)$, where $A, B \in M_n(\mathbb{R})$.
3. $det(A^n) = (det(A))^n, A \in M_n(\mathbb{R})$.
   (Check for $n = 2$, use induction).

4. $det(A^{-1}) = \frac{1}{det(A)}$, $det(adj(A)) = (det(A))^{n-1}$.
   $AA^{-1} = I \quad \therefore \ |A||A^{-1}| = 1 \ \Rightarrow \ |A^{-1}| = \dfrac{1}{|A|}$
   We know
   $$A^{-1}|A| = adj(A)$$
   $$\therefore \ |A^{-1}|A|| = |adj(A)| \ \Rightarrow \ |A|^n|A^{-1}| = |adj(A)|$$
   $$|A|^n\frac{1}{|A|} = |adj(A)| \ \Rightarrow \ |adj(A)| = |A|^{n-1}$$
5. $det(I) = 1$, I is the identity matrix.
6. Determinant of a triangular matix is just the product of diagonal entries.
7. Let $A \in M_n(\mathbb{R})$. If any row or column of A is combination of sum of other rows or columns, then $det(A) = 0$.
8. Let A be an $m \times n$ matrix and B be an $n \times m$ matrix, then $det(I_m + (AB)_m) = det((BA)_n + I_n)$.
9. Let $A, B \in M_n(\mathbb{R})$, $AB = 0 \Rightarrow$ either $det(A) = 0$ or $det(B) = 0$.
   $|AB| = 0 \ \Rightarrow \ |A||B| = 0$.
   $\mathbb{R}$ is an integral domain. So, $|A|$ or $|B| = 0$

Inverse of a square matrix: Let $A \in M_n(\mathbb{R})$, then A is said to be invertible if there exist a matrix B in $M_n(\mathbb{R})$ such that $AB = BA = I$

The set of all invertible matrices in $M_n(\mathbb{R})$ is $GL_n(\mathbb{R})$, which is a group under multiplication.

Theorem 1: $A \in M_n(\mathbb{R})$ is invertible iff $det(A) \neq 0$

Proof:

Necessary Condition: Let B be inverse of A.

$\therefore \ AB = I \ \Rightarrow \ |A||B| = 1$

$\therefore \ |A| \neq 0$.

Sufficient Condition: We have $|A| \neq 0$.

Let $B = \dfrac{1}{|A|}adj(A)$ be inverse

$\therefore \ AB = A\left(\dfrac{1}{|A|}adj(A)\right) = \dfrac{1}{|A|}A \ adj(A) = \dfrac{|A|^I}{|A|} = I$

Similarly $BA = I$

Definition: Let $A \in M_n(\mathbb{R})$ with $det(A) \neq 0$, then the inverse of A is given by $A^{-1} = \frac{adj(A)}{det(A)}$ where $adj(A)$ is adjugate of A or classical adjoint of A, $adj(A) = (cof(A))^T$ and $(cof(A)) = [A_{i,j}]$ where $A_{i,j}$ is the cofactor of the element $a_{i,j}$.

Result:

1. $A \in M_n(\mathbb{R})$ is invertible iff A^n is invertible.
   A is invertible $\Rightarrow \ |A| \neq 0, \ \ A \in M_n(\mathbb{R})$.
   $\Rightarrow \ |A|^n \neq 0 \ \Rightarrow \ |A^n| \neq 0 \ \therefore \ A^n$ is invertible
   Similarly, A^n is invertible $\Rightarrow \ |A^n| \neq 0 \ \Rightarrow \ |A|^n \neq 0 \ \Rightarrow \ |A| \neq 0$.
2. Let A be a matrix with integer entries, then $adj(A)$ is also a matrix with integer entries.
3. If $A \in M_n(\mathbb{Z})$ then A^{-1} is a matrix with integer entries iff $|A| \in \{-1, 1\}$.
   (if $A \in M_n(\mathbb{R})$, the result doesnot hold.
   Eg: $A = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ -1 & \frac{1}{2} \end{bmatrix}$. $|A| = 1$ but $A^{-1} \notin M_n(\mathbb{Z})$.)

Determinant of block matrix:

1. Consider the block matrix $T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ where A is an $n \times n$ and B is an $m \times m$ matrix, then T is an $(m+n) \times (m+n)$ matrix, and $det(T) = det(A).det(B)$.
2. Let $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are $n \times n$ matrices and $det(D) \neq 0$ then $det(P) = det(D).det(A - BD^{-1}C)$.

3. Let $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are $n \times n$ matrices which commute pairwise, then $det(P) = det(AD - BC)$.