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ICS 2018 Problem Sheet #6

Problem 6.1: completeness of → and ¬
Proof that the two elementary Boolean functions → (implication) and ¬ (negation) are universal, i.e., they are sufficient to express all possible Boolean functions.

Here
For A ,B we have

A	B	$A \rightarrow B$
0	0	1
0	1	1
1	0	0
1	1	1

Negation

A	$\neg A$
0	1
1	0

Now

So,

A	B	$\neg A$	$\neg B$	$(\neg A \rightarrow B)$	$(A \rightarrow \neg B)$	$\neg(A \rightarrow \neg B)$
0	0	1	1	0	1	0
0	1	1	0	1	1	0
1	0	0	1	1	1	0
1	1	0	0	1	0	1

Therefore, from the table we can observe that:
$(\neg A \rightarrow B)$ is same as $(A \vee B)$
$\neg(A \rightarrow \neg B)$ is same as $(A \wedge B)$

Since the functions AND and OR can be expresses by the elementary Boolean functions $(\rightarrow)$implication and negation $(\neg)$. We can find the other Boolean functions with these functions.

For Equivalence:
$(\neg A \rightarrow B) \rightarrow (\neg(A \rightarrow \neg B))$

For Exclusive OR
$\neg((\neg A \rightarrow B) \rightarrow (\neg(A \rightarrow \neg B)))$

For NOT AND
$(A \rightarrow \neg B)$

For NOT OR
$\neg(\neg A \rightarrow B)$

Problem 6.2: conjunctive and disjunctive normal form
Consider the following Boolean formula:

$$\phi(P, Q, R, S) = (\neg P \lor Q) \land (\neg Q \lor R) \land (\neg R \lor S) \land (\neg S \lor P)$$

a) How many interpretations of the variables P, Q, R and S satisfy φ? Provide a proof for your answer.

Here

P	Q	R	S	ϕ
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

Total no interpretations is $2^4 = 16$.
So ϕ has Boolean value 1 only in 2 interpretations, on the rest 14 interpretation ϕ is 0.
So ϕ is satisfied for boolean value 1 in only 2 interpretations.

b) Given the interpretations that satisfy φ, write the formula for φ in disjunctive normal form (DNF).

So here we have two conditions where $\phi = 1$
That are when
So
When we have,

P	Q	R	S	ϕ
1	1	1	1	1

$(P \land Q \land R \land S)$

When we have,

P	Q	R	S	ϕ
0	0	0	0	1

$(\neg P \wedge \neg Q \wedge \neg R \wedge \neg S)$

So by DNF
We have
$\phi = (P \wedge Q \wedge R \wedge S) \vee (\neg P \wedge \neg Q \wedge \neg R \wedge \neg S)$

c) Using the equivalence laws for Boolean expressions, derive the DNF representation of φ algebraically from the CNF representation. Write the derivation down stepwise.

Here we have CNF:

$\phi(P, Q, R, S) = (\neg P \vee Q) \wedge (\neg Q \vee R) \wedge (\neg R \vee S) \wedge (\neg S \vee P)$

Using Distributive Property, we derive,

$\phi = (\neg P \vee Q) \wedge (\neg Q \vee R) \wedge (\neg R \vee S) \wedge (\neg S \vee P)$

Converting from CNF to DNF in steps

We know for for any $(A \wedge \neg A) = 0$
Therefore, we will be using this throughout to get the solution;

We have
For $(\neg P \vee Q) \wedge (\neg Q \vee R)$
$= (\neg P \wedge (\neg Q \vee R)) \vee (Q \wedge (\neg Q \vee R))$
$= (\neg P \wedge \neg Q) \vee (\neg P \wedge R) \vee (Q \wedge \neg Q) \vee (Q \wedge R)$
$= (\neg P \wedge \neg Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

For $(\neg P \vee Q) \wedge (\neg Q \vee R) \wedge (\neg R \vee S)$
$= \big((\neg P \wedge \neg Q) \vee (\neg P \wedge R) \vee (Q \wedge R)\big) \wedge (\neg R \vee S)$
$= ((S \wedge (\neg P \wedge \neg Q)) \vee (S \wedge (\neg P \wedge R)) \vee (S \wedge (Q \wedge R)) \vee ((\neg R \wedge (\neg P \wedge \neg Q)) \vee (\neg R \wedge (\neg P \wedge R)) \vee (\neg R \wedge (Q \wedge R))$
$= (\neg R \wedge \neg P \wedge \neg Q) \vee ((S \wedge \neg P \wedge \neg Q) \vee (S \wedge \neg P \wedge R) \vee (S \wedge Q \wedge R))$

For $(\neg P \vee Q) \wedge (\neg Q \vee R) \wedge (\neg R \vee S) \wedge (\neg S \vee P)$
$=((\neg R \wedge \neg P \wedge \neg Q) \vee (S \wedge \neg P \wedge \neg Q) \vee (S \wedge \neg P \wedge R) \vee (S \wedge Q \wedge R)) \wedge (\neg S \vee P)$
$=\big((\neg R \wedge \neg P \wedge \neg Q) \vee (S \wedge \neg P \wedge \neg Q) \vee (S \wedge \neg P \wedge R) \vee (S \wedge Q \wedge R)\big) \wedge (\neg S \vee P)$
$=\big(\neg S \wedge (\neg R \wedge \neg P \wedge \neg Q)) \vee (\neg S \wedge (S \wedge \neg P \wedge \neg Q)) \vee (\neg S \wedge (S \wedge \neg P \wedge R)) \vee (\neg S \wedge (S \wedge Q \wedge R)\big) \wedge \big(P \wedge (\neg R \wedge \neg P \wedge \neg Q)) \vee (P \wedge (S \wedge \neg P \wedge \neg Q)) \vee (P \wedge (S \wedge \neg P \wedge R)) \vee (P \wedge (S \wedge Q \wedge R)\big)$
$=\big(\neg S \wedge (\neg R \wedge \neg P \wedge \neg Q)\big) \wedge \big(P \wedge (S \wedge Q \wedge R)\big)$

Using Associative Property, we have simplified
$=(P \wedge Q \wedge R \wedge S) \vee (\neg P \wedge \neg Q \wedge \neg R \wedge \neg S)$

Here all the terms except $(P \wedge \mathrm{Q} \wedge \mathrm{R} \wedge \mathrm{S})$ and $(\neg P \wedge \neg \mathrm{Q} \wedge \neg \mathrm{R} \wedge \neg S)$ have the value 0 because of this property [$(\mathrm{A} \wedge \neg \mathrm{A}) = 0$].

Therefore the DNF form is:
$\phi(\mathrm{P}, \mathrm{Q}, \mathrm{R}, \mathrm{S}) = (P \wedge \mathrm{Q} \wedge \mathrm{R} \wedge \mathrm{S}) \vee (\neg P \wedge \neg \mathrm{Q} \wedge \neg \mathrm{R} \wedge \neg \mathrm{S})$