Computational Lab (MA318)
Probability Integral Transformation
Assignment 3

Try to solve all the problems

Probability Integral Transformation Method

Let X have continuous cumulative distribution function $F_X(x)$ and define a random variable Y as $Y = F_X(X)$. Then Y is uniformly distributed on $(0, 1)$, that is, $P(Y \leq y) = y,\ 0 < y < 1$.
Proof : We know,

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(F_X(X) \leq y) \\ &= P(X \leq F_X^{-1}(y)) \\ &= P(X \leq F_X^{-1}(F_X(x)) \\ &= P(X \leq x) \\ &= y \end{aligned}$$

* If F_X is monotonic, then F_X^{-1} is well defined by

$$F_X^{-1}(y) = x \quad \Leftrightarrow \quad F_X(x) = y.$$

The Rejection Method

Suppose we have a method for generating a random variable having density function $g(x)$. We can use this as the basis for generating from the continuous distribution having density function $f(x)$ by generating Y from g and then accepting this generated value with a probability proportional to $\frac{f(Y)}{g(Y)}$.

Specifically, let c be a constant such that

$$\frac{f(y)}{g(y)} \leq c \quad \text{for all } y.$$

We then have the following technique (illustrated in Figure 5.1) for generating a random variable having density f.

Algorithm

Step 1: Generate Y having density g.
Step 2: Generate a random number U.
Step 3: If $U \leq \frac{f(Y)}{cg(Y)}$, set $X = Y$. Otherwise, return to Step 1.
Remark: The basic idea is to find an alternative probability distribution G, with density function $g(x)$, from which we already have an efficient algorithm for generating from (e.g., inverse

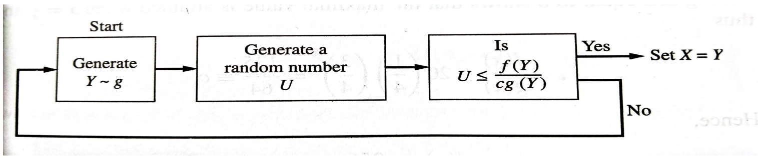

Figure 1: The rejection method for simulating a random variable X having density function f.

transform method or whatever), but also such that the function $g(x)$ is "close" to $f(x)$. In particular, we assume that the ratio $f(x)/g(x)$ is bounded by a constant $c > 0$; $\sup_x f(x)/g(x) \le c$.

Example: Let us use the rejection method to generate a random variable having density function

$$f(x) = 20x(1-x)^3, \quad 0 < x < 1.$$

Since this random variable (which is Beta with parameters 2, 4) is concentrated in the interval $(0,1)$, let us consider the rejection method with

$$g(x) = 1, \quad 0 < x < 1.$$

To determine the smallest constant c such that $\frac{f(x)}{g(x)} \le c$, we use calculus to determine the maximum value of

$$\frac{f(x)}{g(x)} = 20x(1-x)^3.$$

Differentiation of this quantity yields

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = 20\left[(1-x)^3 - 3x(1-x)^2\right].$$

Setting this equal to 0 shows that the maximal value is attained when $x = \frac{1}{4}$, and thus

$$\frac{f(x)}{g(x)} \le 20\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^3 = \frac{135}{64} \equiv c.$$

Hence,

$$\frac{f(x)}{cg(x)} = \frac{256}{27}x(1-x)^3.$$

and thus the rejection procedure is as follows:

Step 1: Generate random numbers U_1 and U_2.

Step 2: If $U_2 \le \frac{256}{27}\, U_1(1-U_1)^3$, stop and set $X = U_1$. Otherwise, return to Step 1.

The average number of times that Step 1 will be performed is $c = \frac{135}{64} \approx 2.11$.

Assignment 3

1. Use the inverse transformation method to generate a random sample of size 1000 from the given distribution function :

$$F_X(x) = \ln(1+x) \quad ; \quad 0 \le x \le e-1.$$

   Also find the mean and median of the generated sample and compare these with the theoretical values.

2. Generate a random sample of size 1000 from the uniform distribution with parameters $(0, 2)$. Also, find the mean and variance of the generated sample and compare these with the theoretical values.

3. Generate a random sample of size 1000 from the one-parameter exponential distribution. Also, find the mean and variance of the generated sample and compare these with the theoretical values.

4. Generate a random sample of size 1000 having the gamma$(\frac{3}{2}, 1)$ density

$$f(x) = kx^{1/2}e^{-x} \quad ; \quad x > 0.$$

where $k = 1/\Gamma(3/2) = 2/\sqrt{\pi}$. Use the following algorithm.
Hint : Use exponential distribution with the same mean as proposal density g(x).

Algorithm 1 Algorithm to Generate Gamma Samples

1: Generate a random number U_1 and set $Y = -\frac{3}{2}\log U_1$.
2: Generate a random number U_2.
3: If $U_2 < \left(\frac{2eY}{3}\right)^{1/2} e^{-Y/3}$, set $X = Y$. Otherwise, return to Step 1.

Also, verify the result.

. . . . end . . . .