ETH zürich

D MAVT

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Technische Mechanik
151-0223-10

Vorlesung 06

Kraft und Moment
Force and moment

Herbstsemester 2025

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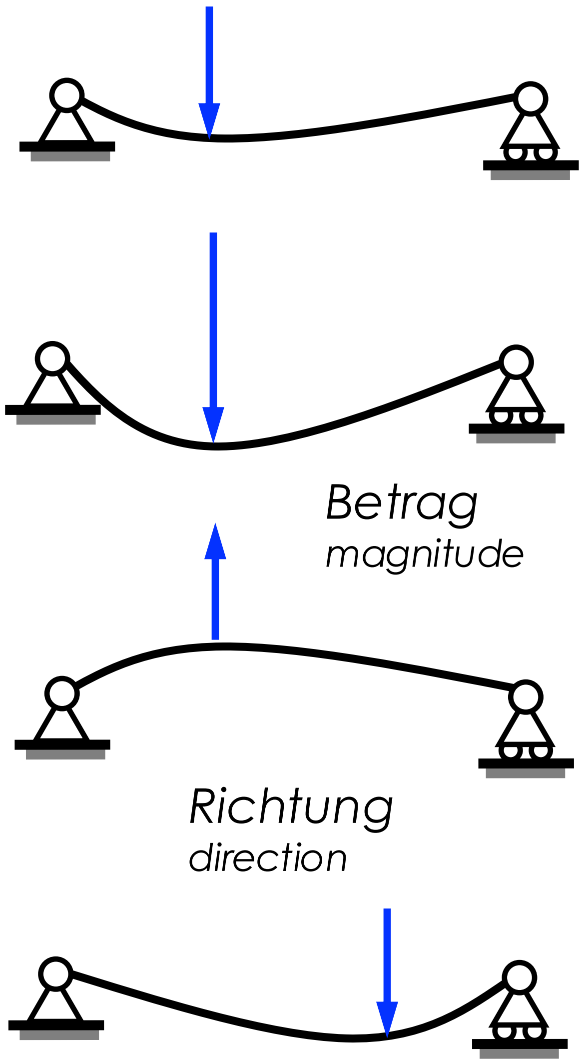
6.1 Definition

Spielen eine Rolle:

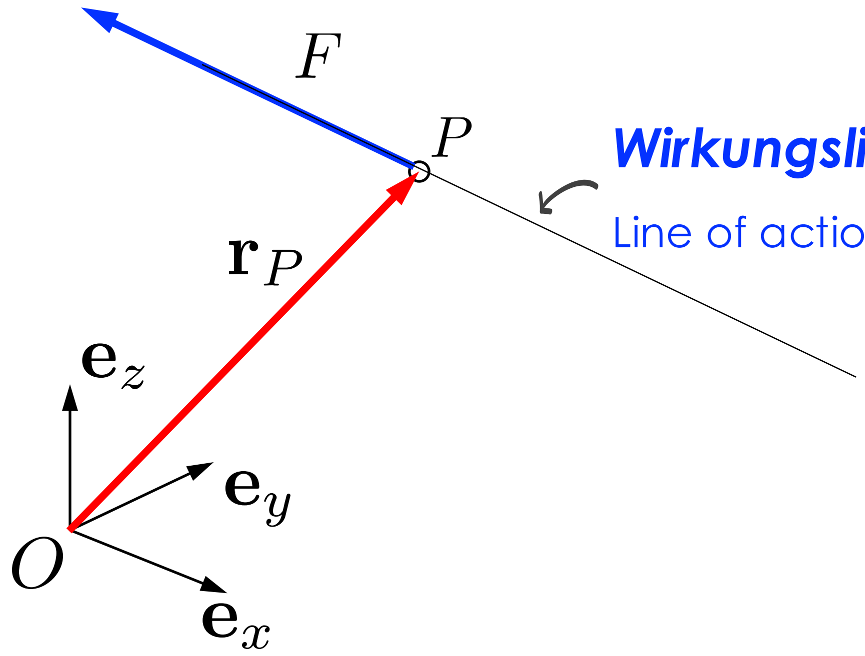
Kräfte manifestieren sich durch ihre Wirkung:
Deformation, Änderung des Bewegungszustand, etc.



Angriffspunkt
Point of application

Die Kraft ist ein Punktgebundener Vektor

The force is a bound vector

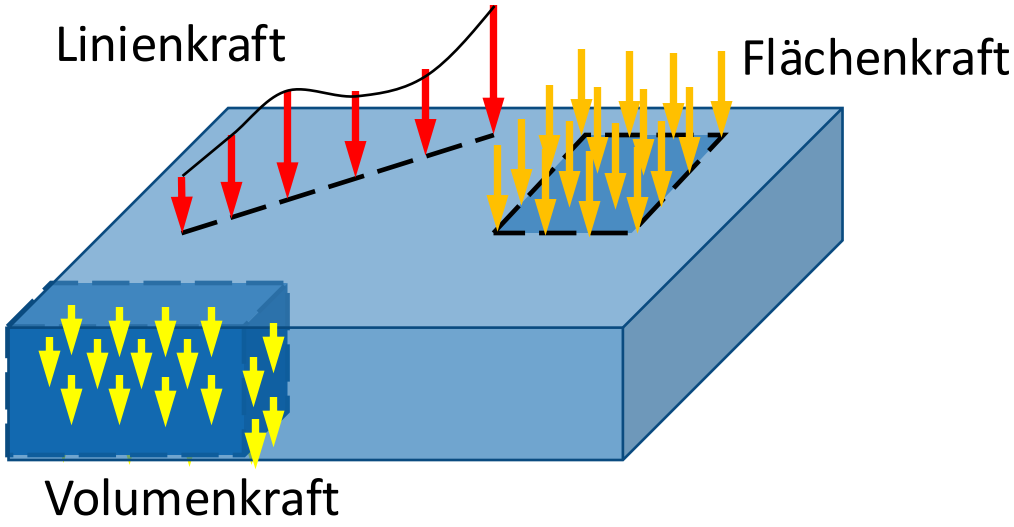


Einheit: Newton

$$1\mathrm{N} = \frac{1\mathrm{Kg}\cdot 1\mathrm{m}}{1\mathrm{s}^2}$$

6.2 Verteilung der Kräfte

6.2. *Force distribution*

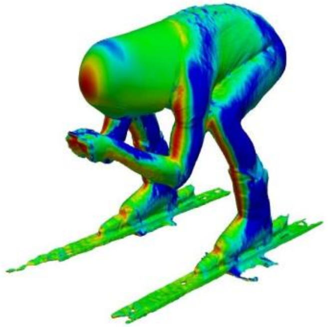


Einzelkraft [N]
Concentrated force

Linienkraft [N/m]
Line force

Schlittschuhkufe Kontaktdruck

Flächenkraft [N/m^2]:
Surface force



Luftwiderstand

Volumenkraft [N/m^3]:
Body force/volume force

Schneebelastung

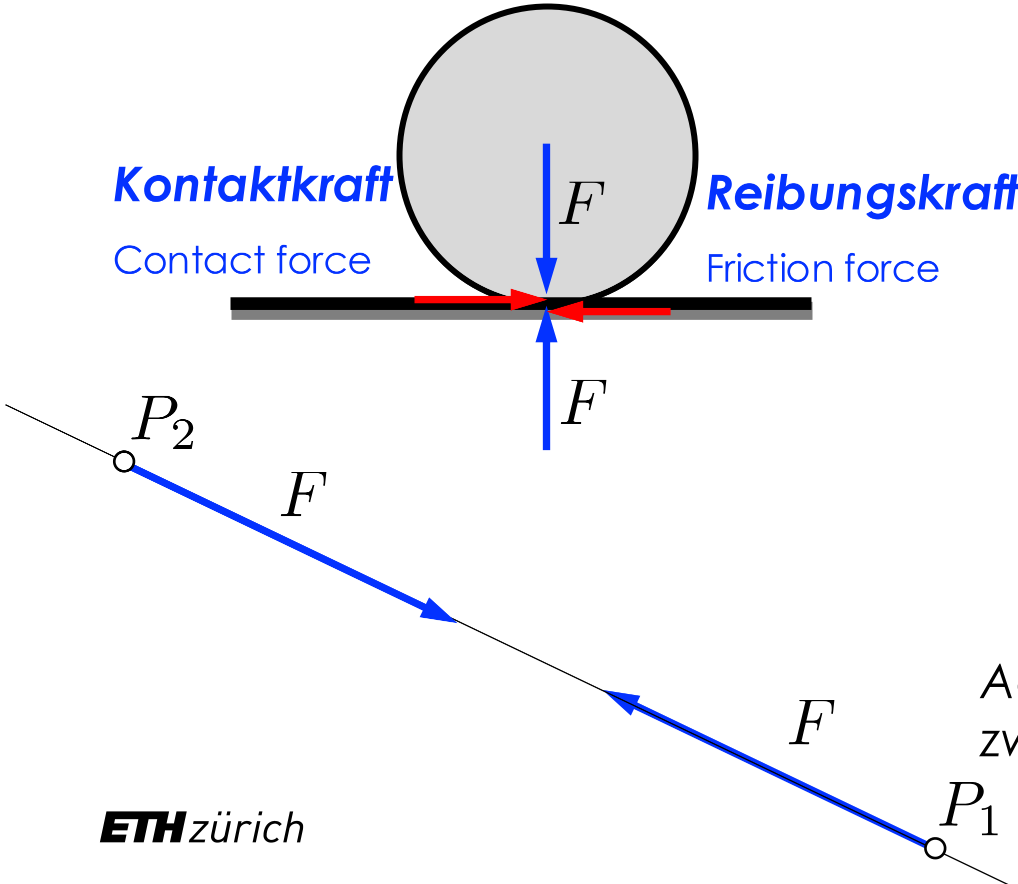
Aufgehängtes Schwimmbad

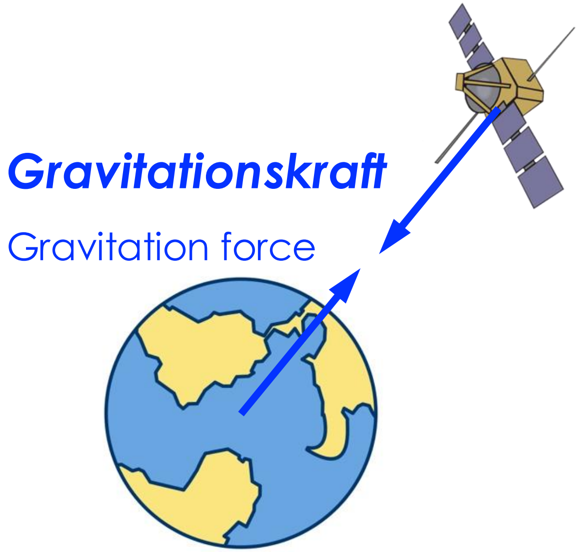
6.3 Reaktionsprinzip (Newton, 1687)

6.3 Principle of action and reaction

*Übt ein materieller Punkt P_1 auf einen anderen materiellen Punkt P_2 die Kraft **F** (in P_2) aus (Actio), so übt P_2 auf P_1 eine Kraft –**F** (in P_1) aus (Reactio). Beide Kräfte haben dieselbe Wirkungslinie.*

When a material point P_1 exerts a force ***F*** on another material point P_2, the point P_2 simultaneously exerts an equal and opposite force –**F** on P_1 (reaction). Both forces act along the same line of action.

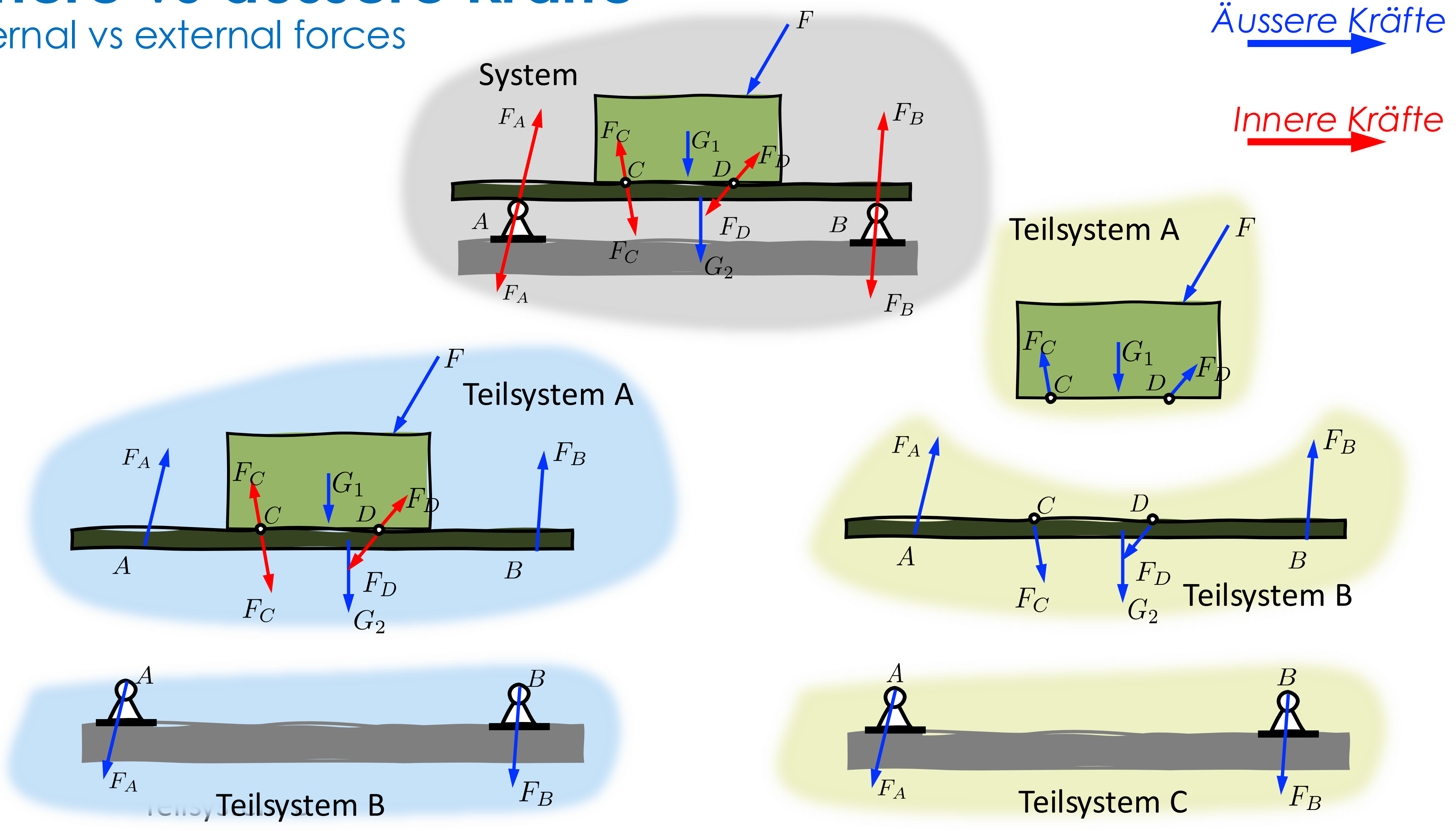




Actio und Reactio liegen auf dem Gerade zwischen den beiden Angriffspunkte

6.4 Innere vs äussere Kräfte

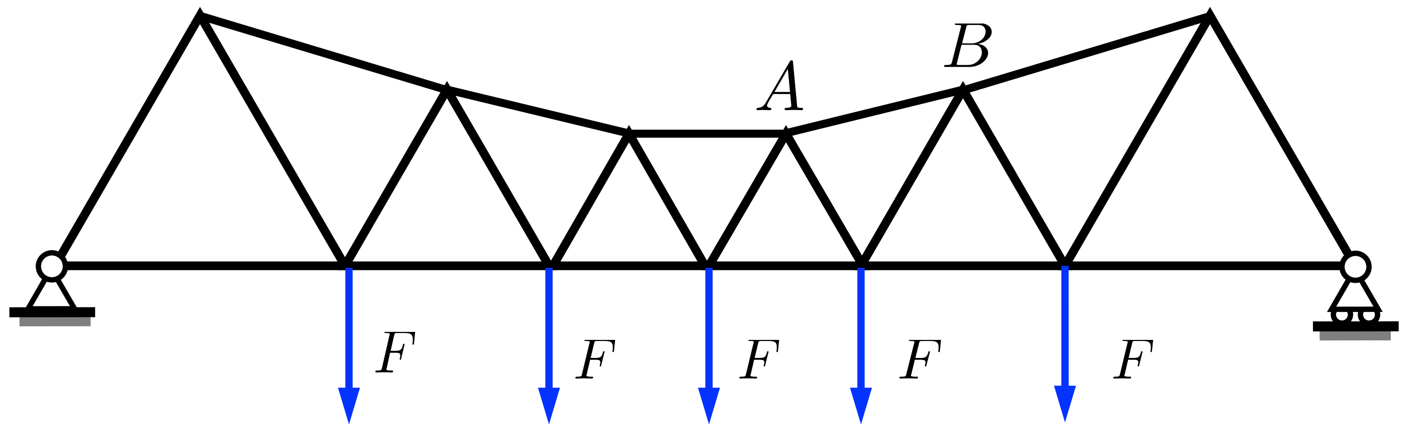
6.4 Internal vs external forces

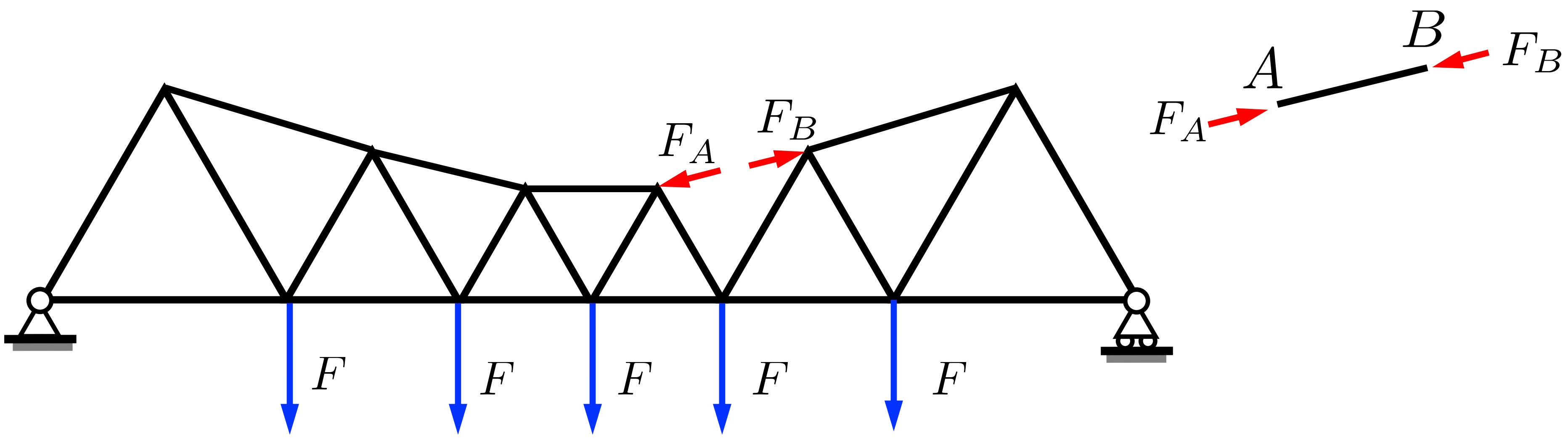


Durch das Reaktionsprinzip ist die **Summe der innere Kräfte null.**

6.5 Systemabgrenzung

6.5 subsystems





Äussere Kräfte: Reaktionskraft nicht am System

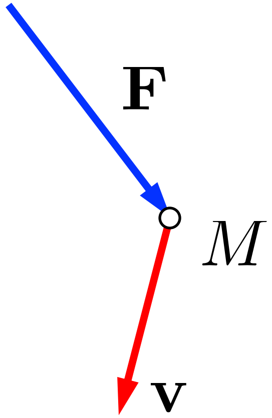
Innere Kräfte: Reaktionskraft auch am System

6.6 Leistung

6.6. *Power*

$$\mathcal{P} = \mathbf{F} \cdot \mathbf{v}$$

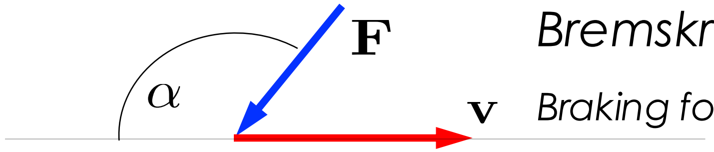
Einheit: $1\mathrm{W} = \dfrac{1\mathrm{N} \cdot 1\mathrm{m}}{1\mathrm{s}}$



Antriebskraft

Drive force

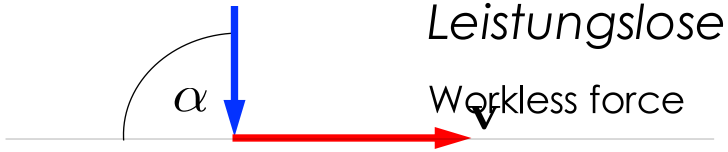
$$0 \leq \alpha < 90^\circ$$



Bremskraft

Braking force

$$90^\circ < \alpha \leq 180^\circ$$

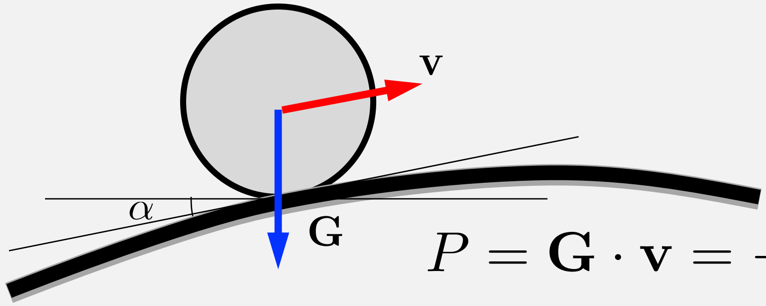


Leistungslose Kraft

Workless force

$$\alpha = 90^\circ$$

Beispiel 6.1: *Rollendes Rad*



$$P = \mathbf{G} \cdot \mathbf{v} = -Gv \sin \alpha$$

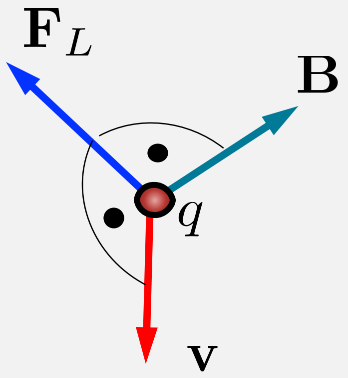
Horizontale Bewegung: $\alpha = 0 \Rightarrow \mathcal{P} = 0$

Beispiel 6.2: *Lorentz Kraft*

Magnetische Feld

$$\mathbf{F}_L = q\mathbf{v} \times \mathbf{B}$$

Elektrische Ladung

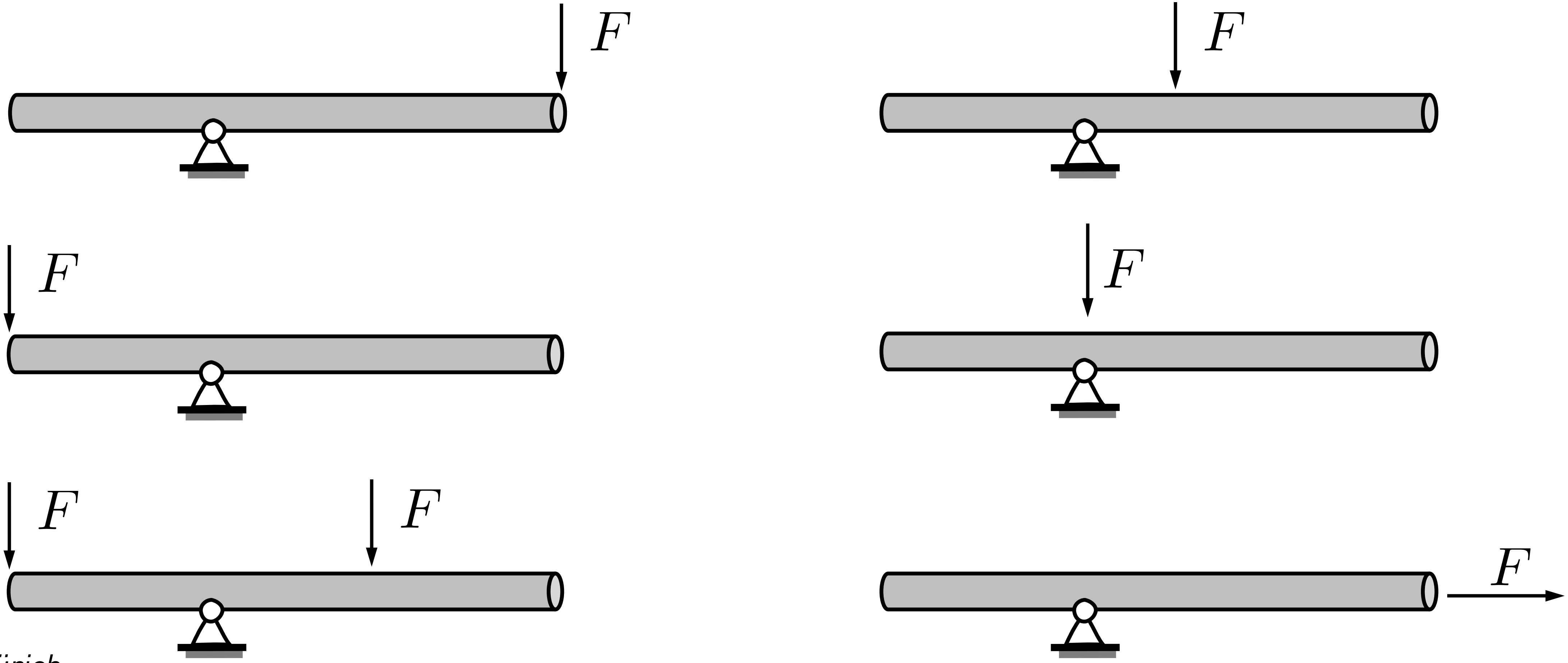


$$\mathbf{F}_L \perp \mathbf{v} \Rightarrow \mathcal{P} = 0$$

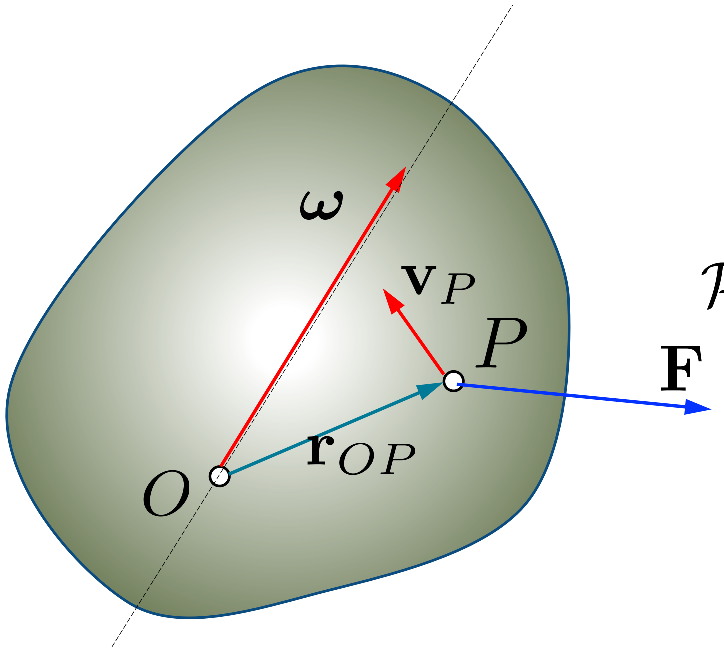
6.7 Moment der Kraft

6.7 moment of the force

Völlig intuitiv: Fähigkeit der Kraft, etwas in Drehung zu setzen



6.7 Moment der Kraft



Betrachten wir eine Rotation:

$$\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r}_{OP}$$

Die Leistung dar Kraft **F,** die an punkt P greift, ist

$$\mathcal{P} = \mathbf{v}_P \cdot \mathbf{F} = (\boldsymbol{\omega} \times \mathbf{r}_{OP}) \cdot \mathbf{F} = (\mathbf{r}_{OP} \times \mathbf{F}) \cdot \boldsymbol{\omega} = \mathbf{M}_O \cdot \boldsymbol{\omega}$$

Moment der Kraft bezüglich O

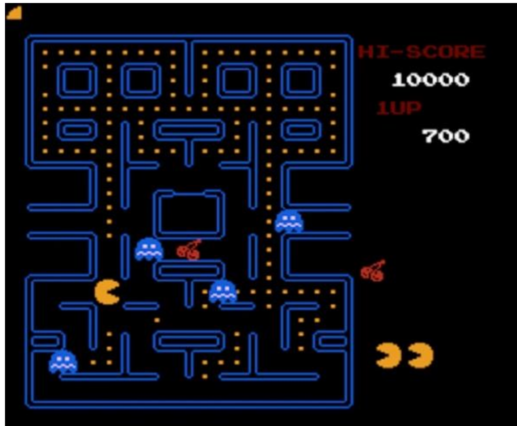
$$\mathbf{M}_O = \mathbf{r}_{OP} \times \mathbf{F}$$

Einheit: Nm

Das Moment ist vom Bezugspunkt abhängig!

The moment depends on the reference point!

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

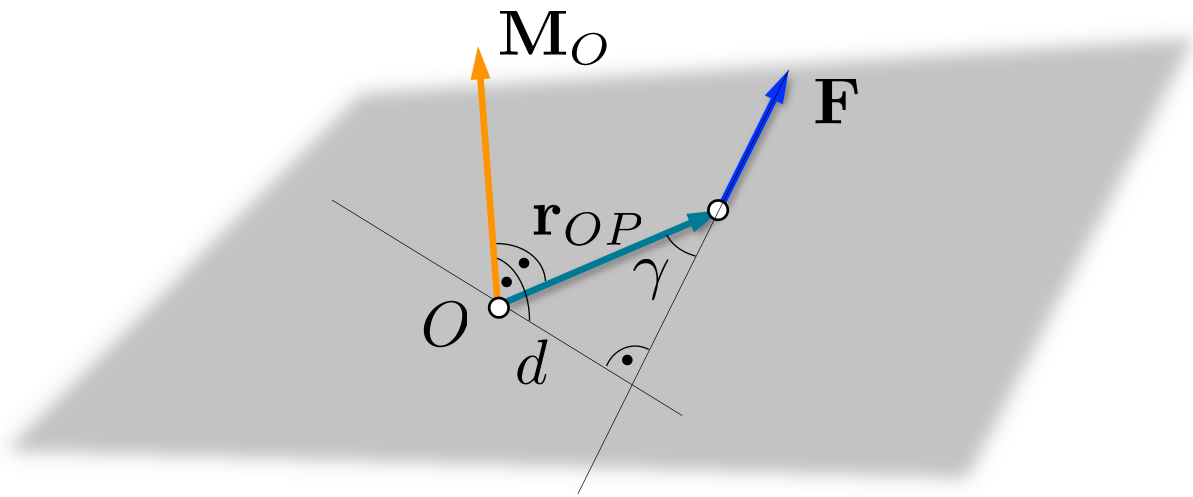


6.7 Moment der Kraft

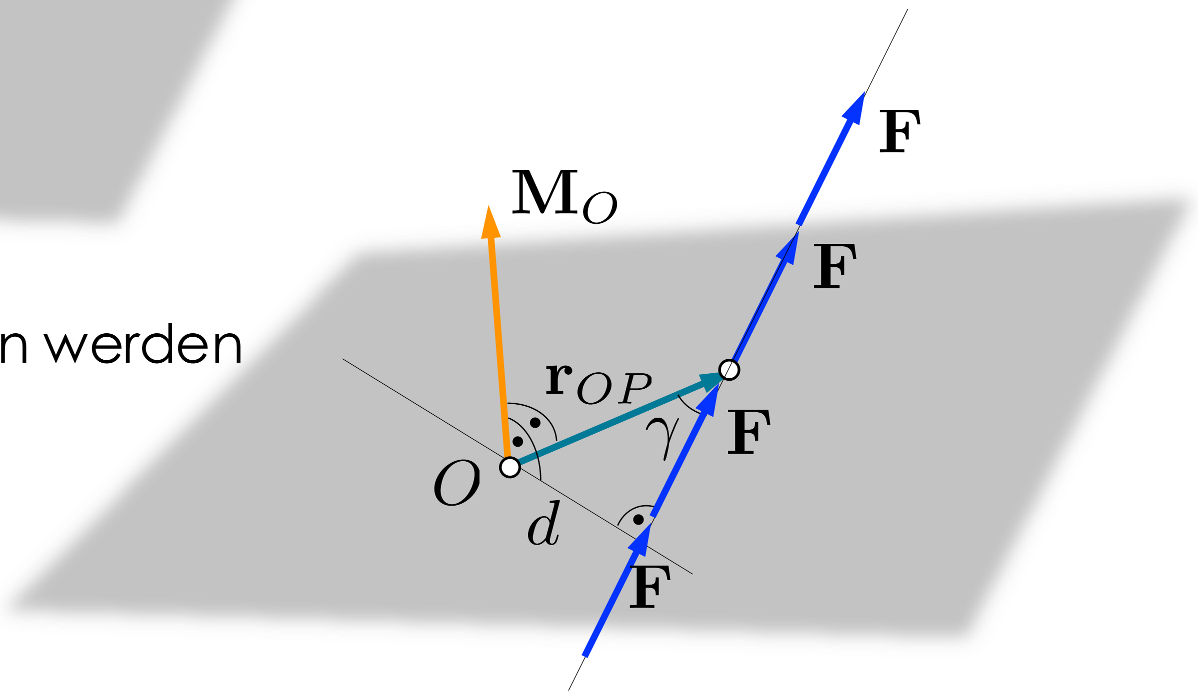
$$\mathbf{M}_O = \mathbf{r}_{OP} \times \mathbf{F}$$

$\mathbf{M}_O$ ist senkrecht zu **F** und $\mathbf{r}_{OP}$

$$|\mathbf{M}_O| = M_O = F r_{OP} \sin\gamma = F d$$



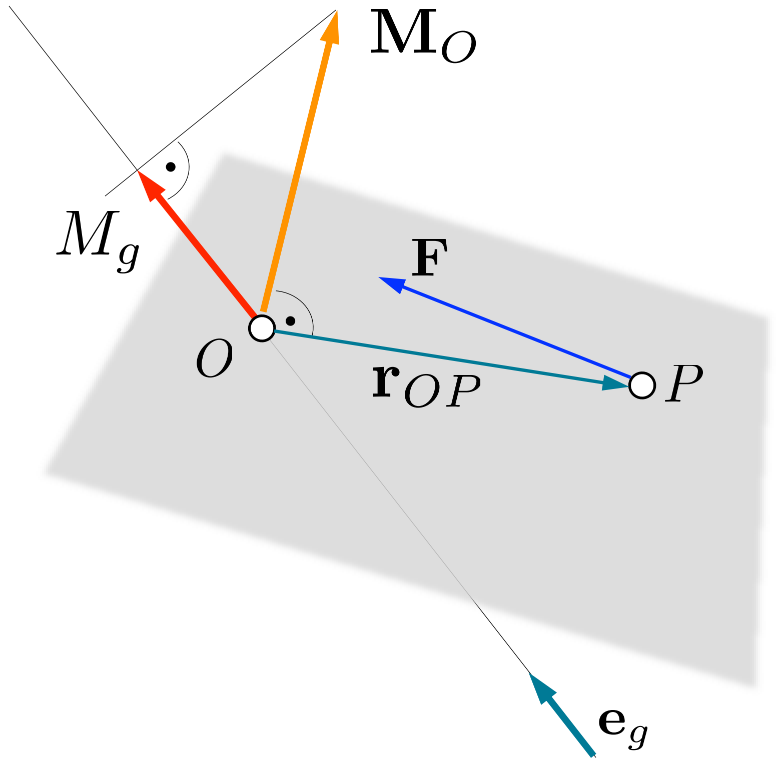
F kann längs ihrer Wirkungslinie verschoben werden

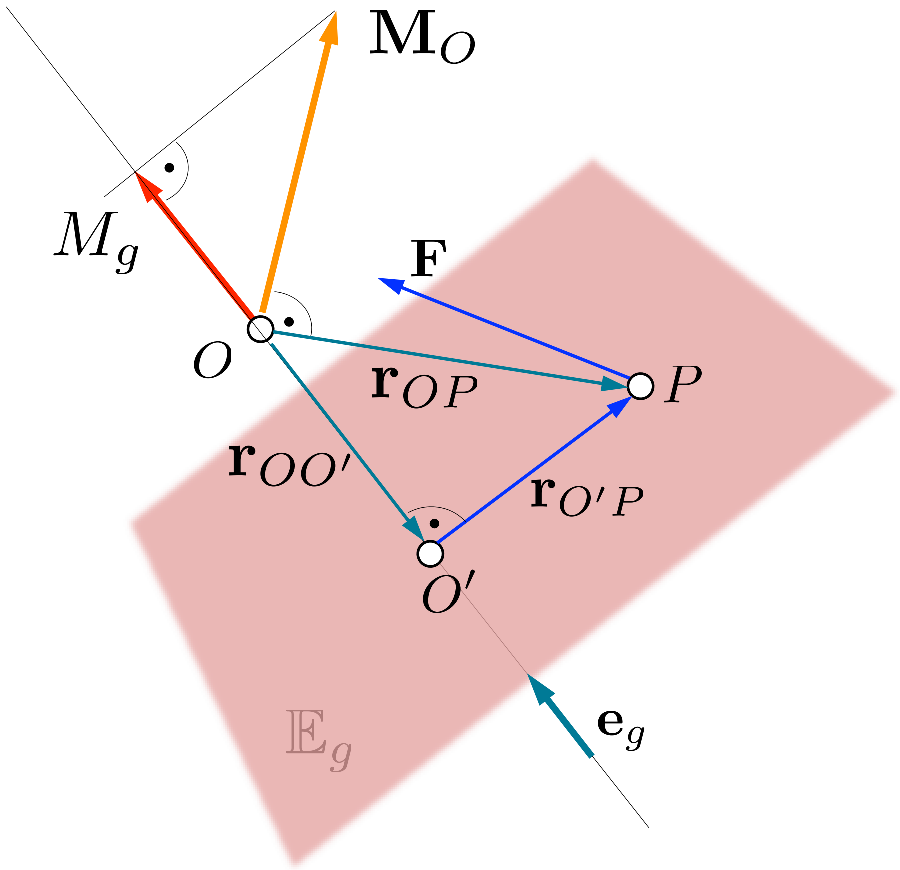


6.8 Moment bezüglich einer Achse

6.8 moment with respect to an axis

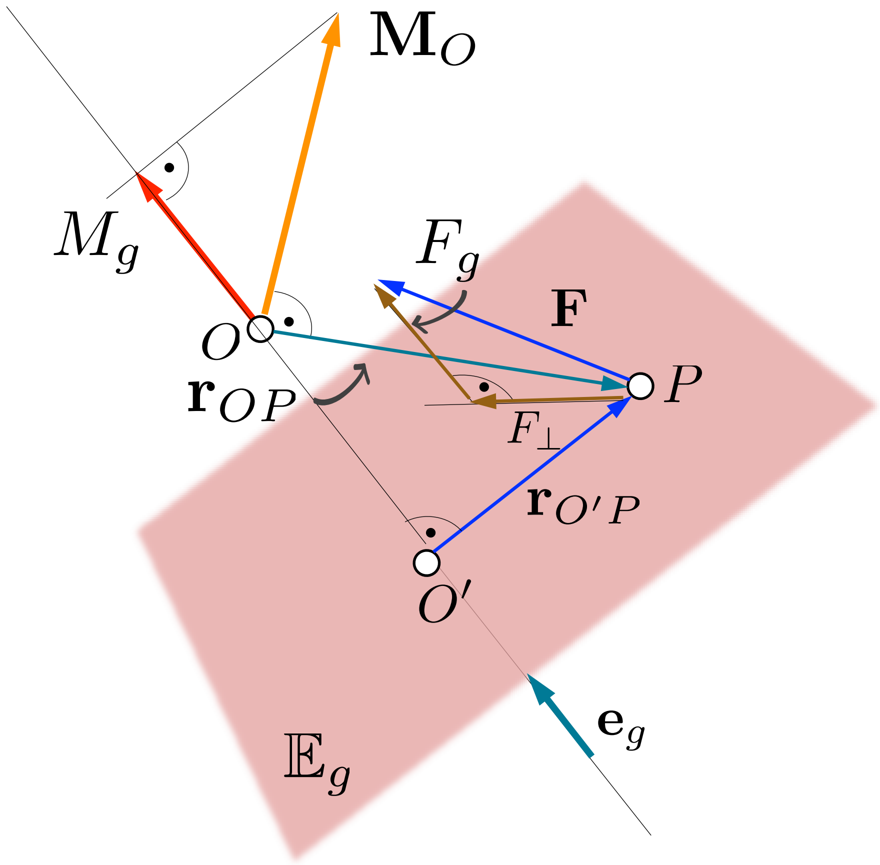
$$M_g = \mathbf{M}_O \cdot \mathbf{e}_g$$

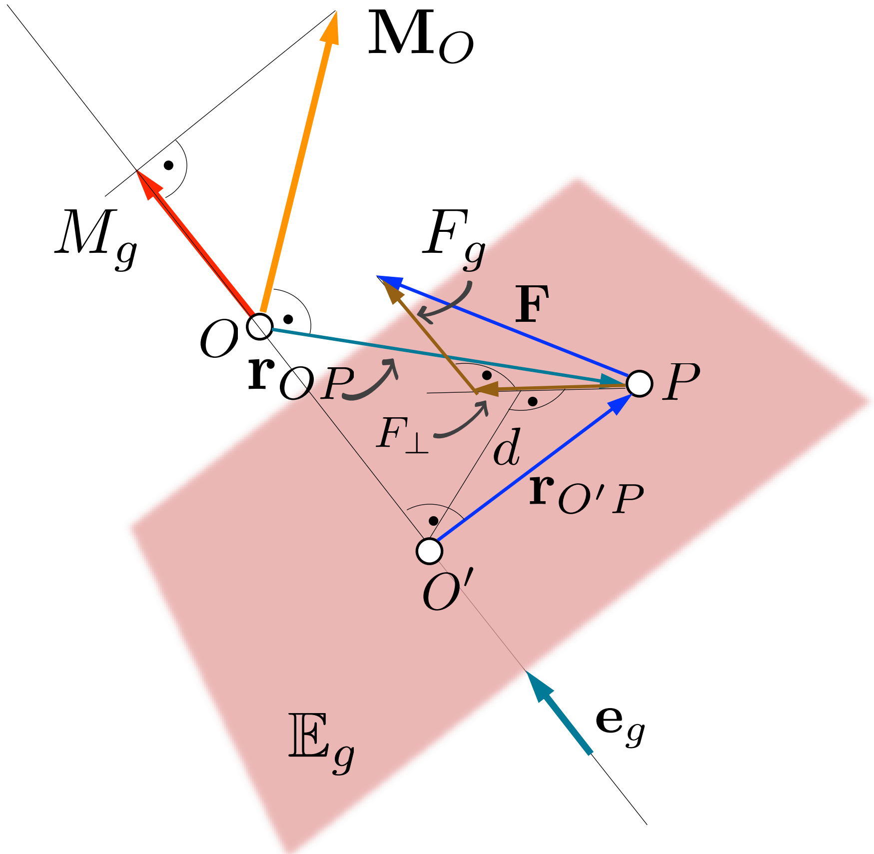




$$(\mathbf{r}_{OP} \times \mathbf{F}) \cdot \mathbf{e}_g = ((\mathbf{r}_{OO'} + \mathbf{r}_{O'P}) \times \mathbf{F}) \cdot \mathbf{e}_g = (\mathbf{r}_{OO'} \times \mathbf{F}) \cdot \mathbf{e}_g + (\mathbf{r}_{O'P} \times \mathbf{F}) \cdot \mathbf{e}_g = (\mathbf{r}_{O'P} \times \mathbf{F}) \cdot \mathbf{e}_g$$

6.8 Moment bezüglich einer Achse





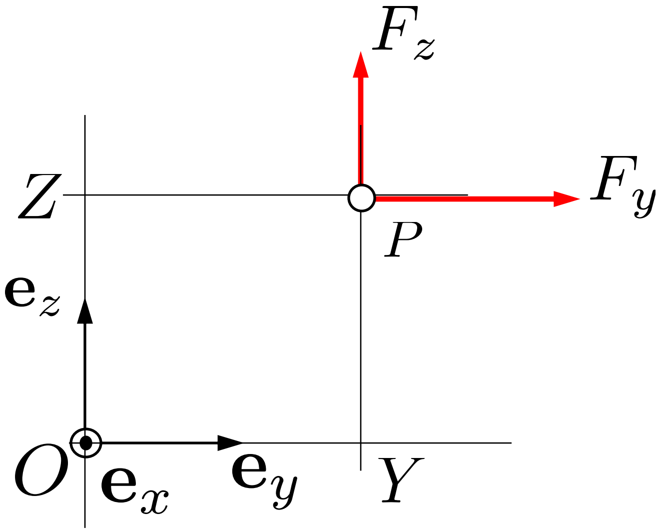
$$M_g = \mathbf{e}_g \cdot \mathbf{M}_O = \mathbf{e}_g \cdot \mathbf{M}_{O'} = \mathbf{e}_g \cdot (\mathbf{r}_{O'P} \times (\mathbf{F}_g + \mathbf{F}_\perp)) =$$
$$\mathbf{e}_g \cdot (\mathbf{r}_{O'P} \times \mathbf{F}_g) + \mathbf{e}_g \cdot (\mathbf{r}_{O'P} \times \mathbf{F}_\perp) = \pm d F_\perp$$

6.8 Moment bezüglich einer Achse

$$\mathbf{M}_O = \mathbf{r}_{OP} \times \mathbf{F} \qquad \mathbf{r}_{OP} = [X\ Y\ Z]^T \qquad \mathbf{F} = [F_x\ F_y\ F_z]^T$$

$$\mathbf{M}_O = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \times \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} YF_z - ZF_y \\ ZF_x - XF_z \\ XF_y - YF_x \end{bmatrix}$$

$$\mathbf{M}_O \cdot \mathbf{e}_x = YF_z - ZF_y$$



1. F_x : Kein Betrag
2. $F_y Z$: Dreht in negative Richtung
3. $F_z Y$: Dreht in positive Richtung

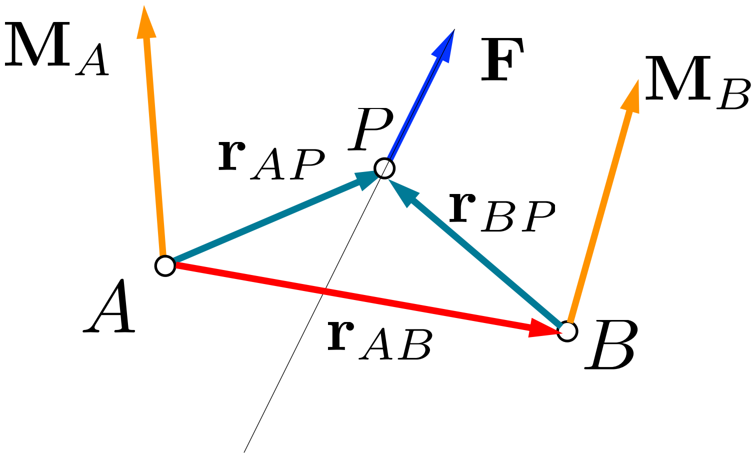
6.9 Transformationformel des Moments

6.9 *Moment transfer formula*

Gegeben sei $\mathbf{M}_B$.
Wie kann man $\mathbf{M}_A$ berechnen?

Nach der Definition:

$$\mathbf{M}_B = \mathbf{r}_{BP} \times \mathbf{F}$$
$$\mathbf{M}_A = \mathbf{r}_{AP} \times \mathbf{F}$$

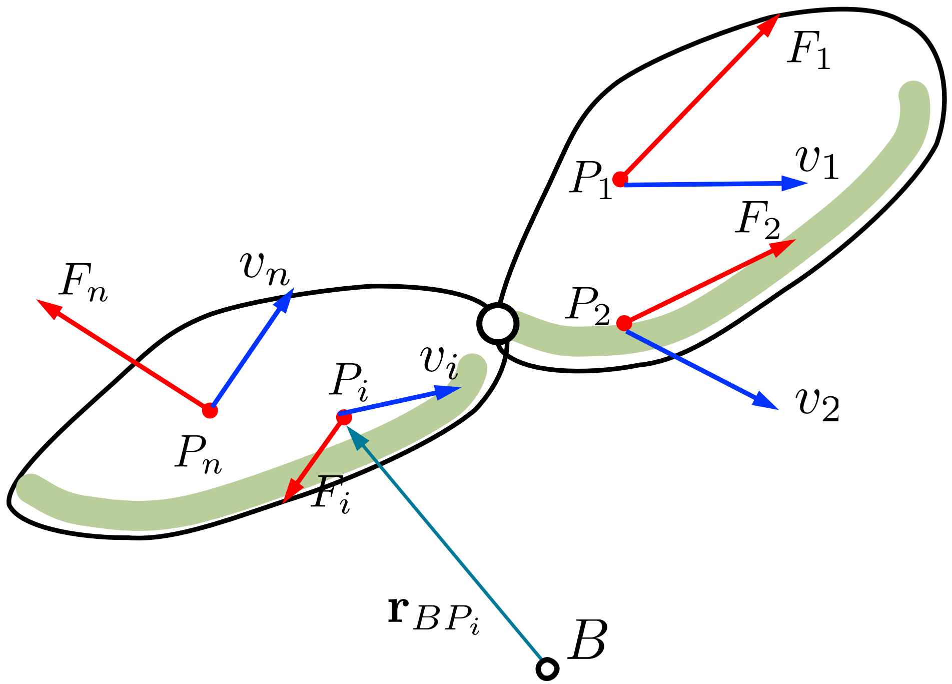


$$\mathbf{M}_A = \mathbf{r}_{AP} \times \mathbf{F} = (\mathbf{r}_{AB} + \mathbf{r}_{BP}) \times \mathbf{F} = \mathbf{r}_{AB} \times \mathbf{F} + \mathbf{r}_{BP} \times \mathbf{F} = \mathbf{r}_{AB} \times \mathbf{F} + \mathbf{M}_B$$

$$\mathbf{M}_A = \mathbf{M}_B + \mathbf{r}_{AB} \times \mathbf{F}$$

6.10 Resultierende Kraft und Moment, Gesamtleistung

6.10 resultant force and moment, total power



$$\mathbf{F}_i, P_i, \;\; i = 1 \ldots n$$

Resultierende Kraft:

Resultant force

$$\mathbf{R} = \sum_{i=1}^{n} \mathbf{F}_i$$

Resultierendes Moment:

Resultant moment

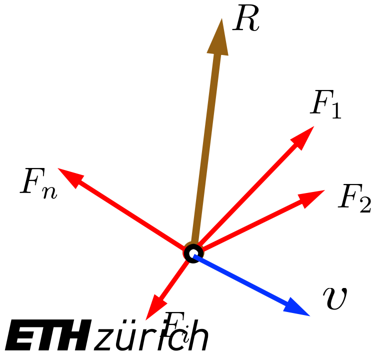
$$\mathbf{M}_B = \sum_{i=1}^{n} \mathbf{r}_{BP_i} \times \mathbf{F}_i$$

Gesamtleistung:

Total power

$$\mathcal{P}_{tot} = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{v}_i$$

Für gleicher Angriffspunkt:

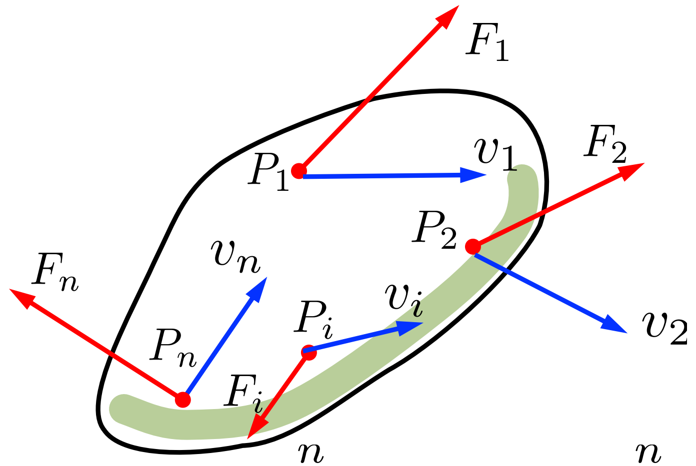


$$\mathbf{v}_i = \mathbf{v} \;\; \forall i$$

$$\mathcal{P}_{tot} = \sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{v}_i = \mathbf{v} \cdot \sum_{i=1}^{n} \mathbf{F}_i = \mathbf{v} \cdot \mathbf{R}$$

6.11 Gesamtleistung des Starrkörpers

6.11 total power for a rigid body



Wir betrachten jetzt <u>einen</u> Starrkörper, wofür eine Kinemate bekannt ist:

$$\{\mathbf{v}_B, \boldsymbol{\omega}\}$$

Die Gesamtleistung wird dann:

$$\mathcal{P}_{tot} = \sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{v}_i = \sum_{i=1}^{n} \mathbf{F}_i \cdot (\mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{BP_i}) = \sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{v}_B + \sum_{i=1}^{n} \mathbf{F}_i \cdot (\boldsymbol{\omega} \times \mathbf{r}_{BP_i}) =$$

$$= \mathbf{R} \cdot \mathbf{v}_B + \sum_{i=1}^{n} \boldsymbol{\omega} \cdot (\mathbf{r}_{BP_i} \times \mathbf{F}_i) = \mathbf{R} \cdot \mathbf{v}_B + \boldsymbol{\omega} \cdot \sum_{i=1}^{n} \mathbf{r}_{BP_i} \times \mathbf{F}_i = \mathbf{R} \cdot \mathbf{v}_B + \boldsymbol{\omega} \cdot \mathbf{M}_B$$

Gesamtleistung eines Starrkörpers:

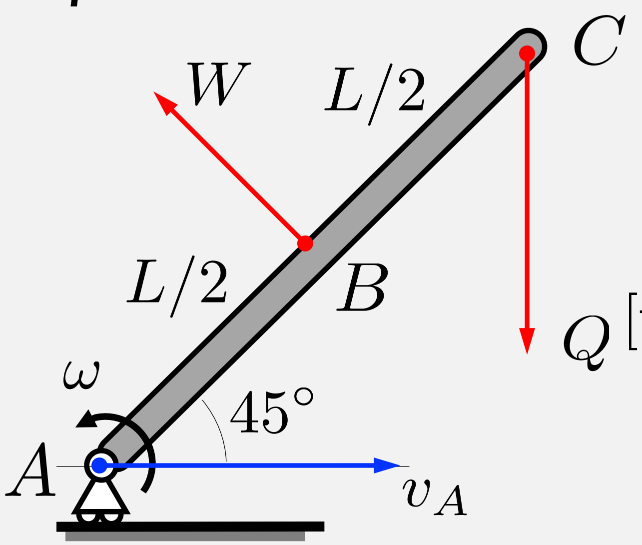
$$\boxed{P_{tot} = \mathbf{R} \cdot \mathbf{v}_B + \mathbf{M}_B \cdot \boldsymbol{\omega}}$$

Analog zu der Kinemate, führen wir die ***Dyname*** ein: $\{\mathbf{R}, \mathbf{M}_B\}$

6.11 Gesamtleistung des Starrkörpers

Beispiel

Gegeben: ω, v_A, W, Q
Gefragt: Gesamtleistung



Strategie 1: gemäss Definition $P_{tot} = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{v}_i$

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{AB}$$

$$[\mathbf{v}_B] = v_A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \omega \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \frac{L}{2} \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = v_A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\sqrt{2}\omega L}{4} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_C = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{AC}$$

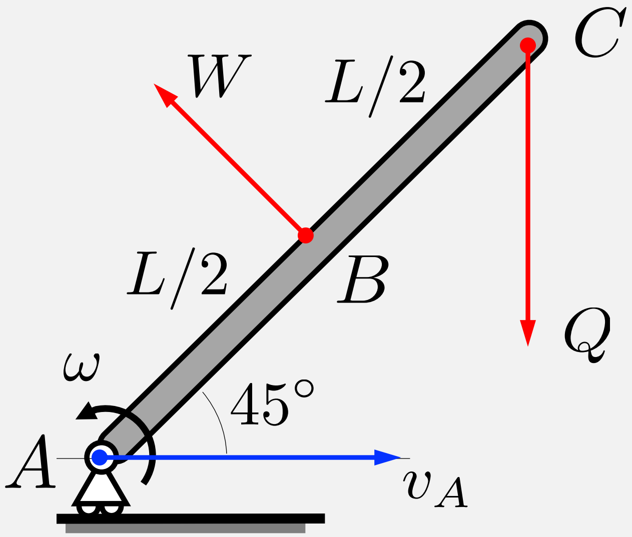
$$[\mathbf{v}_C] = v_A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \omega \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times L \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = v_A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\sqrt{2}\omega L}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$[\mathbf{W}] = \frac{W}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad [\mathbf{Q}] = Q \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathcal{P}_{tot} = \mathbf{W} \cdot \mathbf{v}_B + \mathbf{Q} \cdot \mathbf{v}_C = -\frac{v_A}{\sqrt{2}} W + \frac{\omega L}{2} W - \frac{\omega L \sqrt{2}}{2} Q$$

6.11 Gesamtleistung des Starrkörpers

Beispiel (forts.)



Strategie 2: Formel der Leistung für Starrkörper

$$\mathcal{P}_{tot} = \mathbf{R} \cdot \mathbf{v}_A + \mathbf{M}_A \cdot \boldsymbol{\omega}$$

$$[\mathbf{R}] = [\mathbf{W}] + [\mathbf{Q}] = \frac{W}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + Q \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -W/\sqrt{2} \\ W/\sqrt{2} - Q \\ 0 \end{bmatrix}$$

$$[\mathbf{M}_A] = \begin{bmatrix} 0 \\ 0 \\ WL/2 - \sqrt{2}QL/2 \end{bmatrix}$$

$$\mathcal{P}_{tot} = \mathbf{R} \cdot \mathbf{v}_A + \mathbf{M}_A \cdot \boldsymbol{\omega} = -\frac{v_A}{\sqrt{2}} W + \frac{\omega L}{2} W - \frac{\omega L \sqrt{2}}{2} Q$$