TUTIORIAL - 1

1.) Big $O(n)$

$f(n) \Rightarrow O(g(n))$

if

$f(n) \leq g(n) \times c \;\forall\, n \geq n_0$

for some constant, $c \geq 0$

$g(n)$ is 'tight' upper bound of $f(n)$

eg.) $f(n) \Rightarrow n^2+n$

$g(n) \Rightarrow n^3$

$n^2+n \leq c * n^3$

$n^2+n = O(n^3)$

ii) Big Omega (Ω)

when $f(n) = \Omega(g(n))$ means $g(n)$ is "tight" lowerbound of $f(n)$ i.e $f(n)$ can go beyound $g(n)$

i.e $f(n) = \Omega g(n)$

if and only if

$f(n) \geq c.g(n)$

$\forall n_2 > n_0$ and c = constant > 0

ex.) $f(n) \Rightarrow n^3 + 4n^2$

$g(n) \Rightarrow n^2$

i.e $f(n) \geq c * g(n)$

$n^3 + 4n^2 = \Omega(n^2)$

iii.) Big Theta (Θ)

When $f(n) = \Theta(g(n))$ gives the tight upperbound and lowerbound both.

i.e $f(n) = \Theta(g(n))$

if and only if

$$C_1 * g(n_1) \leq f(n) \leq C_2 * g(n_2)$$

for all $n \geq max(n_1, n_2)$, some constant $C_1 > 0$ & $C_2 > 0$

i.e $f(n)$ can never go beyound $C_2 g(n)$ and will never come down of $C_1 g(n)$

Ex.) $3n+2 = \Theta(n)$ as $3n+2 \geq 3n$

$3n+2 \leq 4n$, $C_1 = 3$, $C_2 = 4$ & $n_0 = 2$

iv) Small o(Θ)

when $f(n) = o g(n)$ gives the upper bound

i.e $f(n) = o g(n)$

if and only if

$$f(n) < c * g(n)$$

$\forall n \geq n_0 \quad n > 0$

Ex) $f(n) = n^2 : g(n) = n^3$

$f(n) < c * g(n)$

$n^2 = o(n^3)$

(V Small Omega (ω)

It gives the 'lower bound'

ie $f(n) = \omega(g(n))$

Where $g(n)$ is lower bound of $f(n)$ if and only if $f(n) > c * g(n)$ $\forall\, n > n_0$ & some constant, $c > 0$

Q2)

for $i \Rightarrow 1, 2, 4, 6, 8 \,.\,.\,.\,.\,.$ n times

i.e Series is a GP

So $a = 1$, $r = 2/1$

k^{th} value of GP:

$$t_k = ar^{k-1}$$

$$t_k = 1(2)^{k-1}$$

$$2n = 2^k$$

$$\log_2(2n) = k \log 2$$

$$\log_2(2) + \log_2 n = k$$

$$\log_2 n + 2 = k$$

so time complexity $T(n) \Rightarrow O(\log n)$ Ans

Q3) $T(n) = 3T(n-1)$ — ①

$T(n) = 1$

Put $n \Rightarrow n-1$ in 1

$T(n-1) \Rightarrow 3T(n-2)$ — ②

Put (2) in ①

$T(n) \Rightarrow 3 \times 3T(n-2)$

$T(n) = 9T(n-2) \rightarrow 3$

Put $n \Rightarrow n-2$ in (1)

$T(n-2) = 3T(n-3)$

Put in (3)

$T(n) = 27\,T(n-3) \rightarrow 4$

Generalising Series

$T(k) = 3^k\,T(n-k)$ — (5)

for k^{th} terms, Let $n-k=1$

$k = n-1$

put in (5)

$T(n) = 3^{n-1}\,T(1)$

$T(n) = 3^{n-1}$

$T(n) = O(3^n)$

Q4) $T(n) = 2T(n-1) - 1$ — ①

Put $n = n-1$

$T(n-1) = 2T(n-2) - 1$ — ②

put in ①

$T(n) = 2\times(2T(n-2)-1)-1$

$= 4T(n-2) - 2 - 1$ — ③

put $n = n-2$ in ①

$T(n-2) = 2T(n-3) - 1$

Put in ①

$T(n) = 8T(n-3) - 4 - 2 - 1$ — ④

Generalising series

$T(n) = 2^k T(n-k) - 2^{k-1} - 2^{k-2} - \cdots 2^0$

k^{th} term

Let $n-k = 1$

$k = n-1$

$T(n) = 2^{k-1} - 2^{k-1}\left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-1}}\right)$

i.e Series in G.P

$a = \frac{1}{2}$, $r = 1/2$

So, $T(n) = 2^{n-1}\left(1 - \left(1/2 \frac{(1-(1/2)^{n-1})}{1-1/2}\right)\right)$

$= 2^{n-1}\left(1 - 1 + \left(\frac{1}{2}\right)^{n-1}\right)$

$= \frac{2^{n-1}}{2^{n-1}}$

$T(n) = O(1)$ Ans

Q5) $i = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \ldots$

$s = 1 + 3 + 6 + 10 + 15 + \ldots$

Sum of s, $1 + 3 + 6 + 10 + \ldots + n$ — ①

Also $s = 1 + 3 + 6 + 10 + \ldots T_{n-1} + T_n$ — ②

$0 = 1 + 2 + 3 + 4 + \ldots n - T_n$

$T_k = 1 + 2 + 3 + 4 + \ldots + k$

$T_k = \frac{1}{2} k(k+1)$

for k iterations

$1 + 2 + 3 + \ldots + k <= n$

$\frac{k(k+1)}{2} <= n$

$\frac{k^2 + k}{2} <= n$

$O(k^2) <= n$

$k = O(\sqrt{n})$

$T(n) = O(\sqrt{n})$ Ans

Q6) $i^2 = n$

$i = \sqrt{n}$

$i = 1, 2, 3, 4 \ldots \sqrt{n}$

$T(n) = \frac{\sqrt{n} * (\sqrt{n} + 1)}{2}$

$T(n) = \frac{n * \sqrt{n}}{2}$

$T(n) = O(n)$ Ans

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Q6) Since, for $k = k^2$

$k = 1, 2, 4, 8, \ldots k$

$\therefore$ Series is in GP

so $a = 1$, $r = 2$

$$\frac{a(r^n - 1)}{r - 1}$$

$$= \frac{1(2^k - 1)}{1}$$

$$n = 2^k - 1$$

$$n + 1 = 2^k$$

$$\log_2(n) = k$$

i	j	k
1	log n	log(n) * log(n)
2	log n	log(n) * log(n)
⋮	⋮	⋮
n	log(n)	log(n) * log(n)

T.C $\Rightarrow O(n * \log n * \log n)$

$\Rightarrow O(n \log^2(n))$ Ans

Q8) for (i=1 to n)

we get j=n times every turn.

$\therefore i * j = n^2$

Now,

$T(n) = n^2 + T(n-3)$

$T(n-3) = (n^2 3)^2 + T(n-6);$

$T(n-6) = (n^3 6)^2 + T(n-9);$

and $T(1) = 1;$

Now substitute each value in T(n)

$T(n) = n^2 + (n-3)^2 + (n-6)^2 + \ldots + 1$

Let

$k^n - 3k = 1$

$k = (n-1)/3$ total terms $= k+1$

$T(n) = n^2 + (n-3)^2 + (n-6)^2 + \ldots + 1$

$T(n) \simeq kn^2$

$T(n) \simeq (k-1)/3 - n^2$

so,

$T(n) = O(n^3)$ Ans

Q9) for $i=1$ $j = 1+2+ \ldots (n \geq j+i)$

$i=2$ $j = 1+3+5 \ldots (n \geq j+i)$

$i=3$ $j = 1+4+7 \ldots (n \geq j+i)$

n^{th} terms of AP is

$T(n) = a + d * m$

$T(w) = a1 + d * w$

$(n-1)/d = n$

for $i = 1$ $(n-1)/1$ times

$i = 2$ $(n-1)/2$ times

$i = n-1$

We get

$$T(n) = i_1 j_1 + i_2 j_2 + \cdots + i_{n-1} j_{n-1}$$

$$= \frac{(n-1)}{2} + \left(\frac{n-2}{2}\right) + \frac{(n-3)}{3} + \cdots + 1$$

$$= n + n/2 + n/3 + \cdots + \frac{n}{n-1} \cdots n \times 1$$

$$= n\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n-1}\right] - n * 1$$

$$= n \times \log n - n + 1$$

$\because \int 1/x = \log x$

$T(n) = O(n \log n)$ - Ans

Q10 As given n^k & c^n

Relationship b/w n^k & c^n is

$$n^k = O(c^n)$$

$$n^k \leq a(c^n)$$

$\forall\ n \geq n_0$ & constant, $a > 0$

for $n_0 = 1$; $c = 2$

$\Rightarrow 1^k \leq a^2$

$\Rightarrow n_0 = 1$ & $c = 2$ Ans