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Lecturer:

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Lecture 1: Theory of Groups and Rings

Definition 1. *Fibre of f over b :* For a function $f : A \rightarrow B$, the pre-image of $b \in B$ is called the fibre of f over b .

Definition 2. *Equivalence class of $a \in A$ is defined to be $\{x \mid xRa\}$. The elements of equivalence class of $a \in A$ are said to be equivalent to a and any element of this class is called the representative of this class. They are denoted by $[a]$.*

Lemma 1. *Any two equivalence classes are either disjoint or equal.*

Proof. Suppose, we have two equivalence classes $[a]$ and $[b]$ such that $[a] \neq [b]$. We need to prove that $[a] \cap [b] = \phi$. Suppose for contradiction that $\exists x$ such that $x \in [a] \cap [b]$. This means that xRa and xRb . Using symmetry of equivalence relation, we have aRx and xRb and by transitivity we can say aRb . Hence, $[a] = [b]$ which is a contradiction.

We have proved that if they are not equal then they are disjoint. Other way can be proved with similar argument. \square

Definition 3. *Partition of A :* A partition of A is any collection $\{A_i \mid i \in I\}$ of non-empty subsets of A such that it follows:

1. $\bigcup_{i \in I} A_i = A$, and
2. $A_i \cap A_j = \phi \quad \forall i, j \in I \text{ and } i \neq j$.

Remark. *The notion of an equivalence relation on A and a partition of A are the same.*

For a set A , every equivalence relation on A induces the partition on set A using equivalence classes. In other words, every equivalence class associated with an equivalence relation forms a partition of A .

If R is the equivalence relation on A then the induced partition P will be

$$P = \{\{b \mid bRa, \forall b \in A\} \mid a \in A\}$$

Also, with the given partition P , we can define relation R as

$$R = \{bRa \mid \exists p_i \in P \text{ such that } a, b \in p_i \quad \forall i \in I\}$$

We can say a partition P is made up of equivalence classes p_i .

Proposition 1. *Let A be a nonempty set.*

1. *If R defines an equivalence relation on A then the set of equivalence classes of R forms a partition of A .*
2. *If $\{A_i \mid i \in I\}$ is a partition of A then there is an equivalence relation on A whose equivalence classes are sets $A_i, i \in I$.*

Proof. 1. Suppose P is a set of equivalence classes of R , defined as

$$P = \{\{b \mid bRa, b \in A\} \mid a \in A\}$$

We need to prove that P defines partition of A . For some $p_i \in P$, it will be nonempty since it will have atleast a which is related to itself. Now, using lemma 1, we can say that $p_i \cap p_j = \phi$ for some $p_i, p_j \in P$ and $i, j \in \mathbb{N}$ given that $i \neq j$.

Also, we know that every point of set A will be in some equivalence class (reflexivity so atleast in the equivalence of itself). If we take union of all those classes we will get A as each point has a atleast a equivalence class.

2. Given the collection of sets, $Q = \{A_i \mid i \in I\}$ as a partition of A , we need to show that there exists an equivalence relation on A with equivalence classes as sets $A_i, i \in I$.

We can define relation R as

$$R = \{(a, b) \mid bRa \text{ and } \exists A_i \in Q \text{ such that } a, b \in A_i\}$$

It is reflexive (obvious), symmetric (obvious) and transitive (aRb such that $a, b \in A_i$ and bRc such that $b, c \in A_i \implies aRc$ such that $a, c \in A_i$). Hence, it is a equivalence relation and the corresponding equivalence class for some i will be

$$q_i = \{b \mid bRa_i, \forall b \in A \text{ \& } a_i \in A\}$$

Since with the same arguments given in proof of part (1) that $q_i, i \in I$ is nonempty, disjoint and exhausts the set A , we can say it forms the partition of A and since every equivalence relation induces a unique partition, we can also say that $q_i, i \in I$ are precisely the sets $A_i, i \in I$. □

Properties of Integers

Property 1. *Well Ordering of \mathbb{Z} : If A is any nonempty subset of \mathbb{Z}^+ , there is some element $m \in A$ such that $m \leq a, \forall a \in A$. m is called the minimum element of A .*

Proof. Proof for well ordering property of \mathbb{Z} i.e. existence and uniqueness of minimal element.

For a nonempty subset A of \mathbb{Z}^+ , we can prove this by induction. Suppose $A = \{a_1\}$ then $a_1 \leq a_1$ hence a_1 is the minimal element. Suppose there exists a minimal element m in the set A have n element $a_1, a_2, \dots a_n$.

For the case when $A = \{a_1, a_2, \dots a_{n+1}\}$, we already have m as the minimal element for $\{a_1, a_2, \dots a_n\}$ hence there are three cases: if $a_{n+1} = m$, $a_{n+1} > m$ and $a_{n+1} < m$ and in all three cases there exists a minimal element in set A .

For uniqueness of minimal element, suppose for contradiction there exists two minimal elements in set A , say m_1, m_2 such that $m_1 \neq m_2$. Then there are two possibilities $m_1 > m_2$ and $m_1 < m_2$. If $m_1 > m_2$ then assume m_1 to be the minimal element hence this case is not possible. Similarly, $m_1 < m_2$ is also not possible assuming m_2 be the minimal element. Hence, contradiction. Therefore, $m_1 = m_2$. □

Definition 4. $a|b = a$ divides b . $b = ac$ for some $c \in \mathbb{Z}$.

Definition 5. For some $a, b \in \mathbb{Z}$, denote $d = \gcd(a, b)$ and $l = \text{lcm}(a, b)$ then $dl = ab$.

Definition 6. The Division Algorithm: If $a, b \in \mathbb{Z} \setminus \{0\}$, then there exists a unique $q, r \in \mathbb{Z}$ such that $a = bq + r$ where $0 \leq r < |b|$. And q is called quotient and r is called remainder.

Definition 7. The Euclidean Algorithm: Suppose $a, b \in \mathbb{Z} \setminus \{0\}$, we can use this algorithm to find the gcd of these two number in the following way:

$$\begin{aligned} a &= q_0b + r_0 \\ b &= q_1r_0 + r_1 \\ r_0 &= q_2r_1 + r_2 \\ &\vdots \\ r_{n-2} &= q_{n+1}r_{n-1} + r_n \\ r_{n-1} &= q_{n+2}r_n \end{aligned}$$

where r_n is the gcd of (a, b) . Such an r_n exists because $|b| > |r_0| > |r_1| \dots$ is a decreasing sequence of strictly positive integers hence it cannot go on to infinite elements.

\mathbb{Z} -linear combination of a and b : For $a, b \in \mathbb{Z} \setminus \{0\}$, we have $x, y \in \mathbb{Z}$ such that we can write $\gcd(a, b)$ as linear combination of x, y

$$\gcd(a, b) = ax + by$$

.

Theorem 1. Fundamental Theorem of Arithmetic: If $n \in \mathbb{Z}$, $n > 1$, then n can be factored uniquely into the product of primes, i.e. there are distinct primes p_1, p_2, \dots, p_s and positive integers $\alpha_1, \alpha_2, \dots, \alpha_s$, such that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$$

This factorization is unique in the sense that the set of p_i 's is unique and no other set of primes and the exponent can generate the same number.

Proof. We will use induction to prove the first part that every $n > 1$ can be written as the product of primes.

For $n = 2$ it is true as $2 = 2^1$. Suppose for all the numbers less than n can be written as the product of primes. Now, for n we can have two cases:

Case: 1 If n is prime then it is obvious that it's true.

Case: 2 If n is composite then n can be written as $n = ab$ where $0 < a, b < n$ by definition of composite numbers. And by our assumption for induction it is true that a, b can be written as product of primes as they are less than n . Hence, $n = ab$ can also be a product of primes.

Now, about the uniqueness of the primes factors. Let's assume that there exists some primes q'_i s and the exponents β'_i s such that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s} = q_1^{\beta_1} q_2^{\beta_2} \dots q_s^{\beta_s}$$

Since p_1 divides the left side, it should also divide the right side. Hence, $p_1 | q_i$ for some i . But p_1 and q_i are primes $\implies p_1 = q_i$. WLOG, we can choose $i = 1 \implies p_1 = q_1$.

$$p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s} = p_1^{\beta_1} q_2^{\beta_2} \dots q_s^{\beta_s}$$

Now, we can have $\alpha_1 > \beta_1$ and we can cancel $p_1^{\beta_1}$ from both sides.

$$p_1^{\alpha_1 - \beta_1} p_2^{\alpha_2} \dots p_s^{\alpha_s} = q_2^{\beta_2} \dots q_s^{\beta_s}$$

But observe that now p_1 divides left side but not the right side. Hence $\alpha_1 \not> \beta_1$. Similar argument for $\alpha_1 < \beta_1$. Therefore, $\alpha_1 = \beta_1$.

Using induction we can show that both sides are equivalent. Hence, primes and their coefficients are unique. \square

We can also define lcm and gcd using fundamental theorem of arithmetic as:

$$\gcd(a, b) = p_1^{\min(\alpha_1, \beta_1)} p_2^{\min(\alpha_2, \beta_2)} \dots$$

$$\text{lcm}(a, b) = p_1^{\max(\alpha_1, \beta_1)} p_2^{\max(\alpha_2, \beta_2)} \dots$$

Definition 8. Euler ϕ - function: For $n \in \mathbb{Z}^+$ let $\phi(n)$ be the number of positive integers $a \leq n$ with $(a, n) = 1$. For primes p , $\phi(p) = p - 1$, and more generally, $\forall a \geq 1$ we have

$$\phi(p^a) = p^a - p^{a-1} = p^{a-1}(p - 1)$$

The function ϕ is multiplicative in the sense that $\phi(ab) = \phi(a)\phi(b)$ if $(a, b) = 1$. So for some $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$ we can write

$$\begin{aligned} \phi(n) &= \phi(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}) = \phi(p_1^{\alpha_1}) \phi(p_2^{\alpha_2}) \dots \phi(p_s^{\alpha_s}) \\ &= p_1^{\alpha_1 - 1} (p_1 - 1) p_2^{\alpha_2 - 1} (p_2 - 1) \dots p_s^{\alpha_s - 1} (p_s - 1) \end{aligned}$$

Theorem 2. If n is composite then there are integers a and b such that $n | ab$ but $n \nmid a$ or $n \nmid b$.

Proof. Since n is composite then $n = x_1^{n_1} x_2^{n_2} \dots y_1^{n'_1} y_2^{n'_2} \dots$ where x, y are primes $\in (0, n)$ and $n'_i \geq 1 \forall i \in \mathbb{N}$. We have to prove the existence of the integers a, b such that $n | ab$ but $n \nmid a$ or $n \nmid b$.

We can construct such integers given the prime factorization of n . If we define $a = x_1^{n_1} x_2^{n_2} \dots$ and $b = y_1^{n'_1} y_2^{n'_2} \dots$ then we have satisfied the needed conditions. \square

Theorem 3. If p is a prime then \sqrt{p} is not a rational number.

Proof. Suppose for contradiction, \sqrt{p} is a rational number. Then there exist x, y such that $\sqrt{p} = \frac{x}{y}$ and $(x, y) = 1$. Then $p = (\frac{x}{y})^2$ but p is a prime hence the only factorization it has is $p = p \times 1$ and factorization is unique by fundamental theorem of arithmetic. Hence contradiction, \sqrt{p} is an irrational number. \square

Ques. If p is a prime then prove that there do not exist nonzero integers a and b such that $a^2 = pb^2$.

Ans. If $a^2 = pb^2$ then $a = \pm\sqrt{p}b$ and using theorem 3, we can say \sqrt{p} is an irrational number and the product of an irrational and an integer can never be an integer. Hence, there does not exist nonzero $a, b \in \mathbb{Z}$ such that $a^2 = pb^2$.

Lecture 2

$\mathbb{Z}/n\mathbb{Z}$: Integers modulo n Let n be a fixed positive integer. Define a relation R on \mathbb{Z} as

$$aRb \text{ iff } n \mid (b - a)$$

R is the equivalence relation as can be verified. We call $a \equiv b \pmod{n}$, (read as: a is congruent to $b \pmod{n}$) if aRb .

The equivalence class of a is denoted by \bar{a} and called *congruent class or residue class of $a \pmod{n}$* .

$$\begin{aligned} n \mid (b - a) &\implies b - a = nk \text{ for some } k \in \mathbb{Z} \\ &\implies b = a + kn \text{ and } b \in \bar{a} \end{aligned}$$

For example: $\bar{0}$ = perfectly divisible by n . These residue classes partitions the \mathbb{Z} . The set of all these equivalence classes under this equivalence relation will be denoted by $\mathbb{Z}/n\mathbb{Z}$, called *integers modulo n* or *integer mod n* .

The process of finding the equivalence class \pmod{n} of some integer a is referred to as *reducing $a \pmod{n}$* .

Addition and multiplication for elements of $\mathbb{Z}/n\mathbb{Z}$:

$$\bar{a} + \bar{b} = \overline{a + b} \quad \text{and} \quad \bar{a} \cdot \bar{b} = \overline{ab}$$

This means that we can take any *representative* element from class \bar{a} and any *representative* element from class \bar{b} and then do usual addition (or multiplication) then find the class in which the result lies.

For example: If we take $\mathbb{Z}/2\mathbb{Z}$, then we have two classes $\bar{0}, \bar{1}$ (it's 0 to $n - 1$, $n = 2$ here) then we can take 4 and 7 from $\bar{0}$ and $\bar{1}$ respectively. $4 + 7 = 11$ and 11 lies in $\bar{1}$ class hence $\bar{0} + \bar{1} = \overline{4 + 7} = \bar{1}$.

The result is well defined and does not depend upon the choice of representatives as shown by the theorem below.

Theorem 4. *The operation of addition and multiplication on $\mathbb{Z}/n\mathbb{Z}$ defined above are well defined i.e. they do not depend on the choice of representative for the classes involved. More precisely, if $a_1, a_2 \in \mathbb{Z}$ and $b_1, b_2 \in \mathbb{Z}$ with $\bar{a}_1 = \bar{b}_1$ and $\bar{a}_2 = \bar{b}_2$, then $\overline{a_1 + a_2} = \overline{b_1 + b_2}$ and $\overline{a_1 a_2} = \overline{b_1 b_2}$, i.e. if*

$$a_1 \equiv b_1 \pmod{n} \quad \text{and} \quad a_2 \equiv b_2 \pmod{n}$$

then

$$a_1 + a_2 \equiv b_1 + b_2 \pmod{n} \quad \text{and} \quad a_1 a_2 \equiv b_1 b_2 \pmod{n}$$

Proof. Since $a_1 \equiv b_1 \pmod{n}$ that means $n \mid b_1 - a_1$ and $b_1 = a_1 + nt$. Similarly, for a_2 , we have $b_2 = a_2 + ns$. On adding the equations, we get $b_1 + b_2 = a_1 + a_2 + n(t + s)$ and $b_1 b_2 = n(nst + a_1 t + a_2 s) + a_1 a_2$. Hence, $a_1 + a_2 \equiv b_1 + b_2 \pmod{n}$ and $a_1 a_2 \equiv b_1 b_2 \pmod{n}$. \square

Definition 9. A subset residue classes of $\mathbb{Z}/n\mathbb{Z}$ with multiplicative inverse lies in $\mathbb{Z}/n\mathbb{Z}$ itself:

$$(\mathbb{Z}/n\mathbb{Z})^\times = \{\bar{a} \in \mathbb{Z}/n\mathbb{Z} \mid \exists \bar{c} \in \mathbb{Z}/n\mathbb{Z} \text{ with } \bar{a} \cdot \bar{c} = \bar{1}\}$$

Proposition 2. Any representative of \bar{a} is coprime to n .

$$(\mathbb{Z}/n\mathbb{Z})^\times = \{\bar{a} \in \mathbb{Z}/n\mathbb{Z} \mid (a, n) = 1\}$$

If a is integer which is coprime to n then we can write $ax + ny = 1$ using Euclidean algorithm for some $x, y \in \mathbb{Z} \implies 1 - ax = ny$ that means $ax \equiv 1 \pmod{n} \implies \bar{a}\bar{x} = \bar{1}$ hence \bar{x} is the multiplicative inverse of \bar{a} . Efficient way of calculating multiplicative inverse.

Ques. Prove that the distinct equivalence classes in $\mathbb{Z}/n\mathbb{Z}$ are precisely $\bar{0}, \bar{1}, \dots, \overline{n-1}$.

Ans. Division algorithm says that for $a, b \in \mathbb{Z} \setminus \{0\}$, we have unique $q, r \in \mathbb{Z} \setminus \{0\}$ such that $b = aq + r$ where $0 \leq r < |a|$. Hence, r can only be $0, 1, 2, \dots, n-1$ which corresponds to equivalence classes.

Theorem 5. If $\bar{a}, \bar{b} \in (\mathbb{Z}/n\mathbb{Z})^\times$, then $\bar{a} \cdot \bar{b} \in (\mathbb{Z}/n\mathbb{Z})^\times$.

Proof. Since, $\bar{a}, \bar{b} \in (\mathbb{Z}/n\mathbb{Z})^\times$, there exists \bar{a}' and \bar{b}' such that $\bar{a} \cdot \bar{a}' = \bar{1}$ and $\bar{b} \cdot \bar{b}' = \bar{1}$. If we multiply both the equations then $\bar{a} \cdot \bar{a}' \cdot \bar{b} \cdot \bar{b}' = \bar{1}$, Assume $\bar{a} \cdot \bar{b} = \bar{c} \in \mathbb{Z}/n\mathbb{Z}$ then we get $\bar{c} \cdot \bar{a}' \cdot \bar{b}' = \bar{1}$. Hence, there exist $\bar{c}' = \bar{a}' \cdot \bar{b}'$ such that $\bar{c} \cdot \bar{c}' = \bar{1}$. Therefore $\bar{a} \cdot \bar{b} \in \mathbb{Z}/n\mathbb{Z}$. \square

Lecture 3

Definition 10. Binary operation: A binary operation $*$ on a set G is a function $* : G \times G \rightarrow G$. For any $a, b \in G$, we can write $a * b$ for $*(a, b)$.

If $*$ is an binary operation on G and H is a subset of G . If restriction of $*$ on H is a binary operation on H i.e. $a, b \in H \implies a * b \in H$ also then H is closed under $*$.

If $*$ is associative (or commutative) on G then it will be associative (or commutative) on H also.

Definition 11. Group: A group is an ordered pair $(G, *)$ where G is a set and $*$ is a binary operation on G satisfying following axioms.

1. $(a * b) * c = a * (b * c), \forall a, b, c \in G$ i.e. $*$ is associative.
2. There exists an element e in G , called identity of G , such that for all $a \in G$ we have $a * e = e * a = a$.

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3. for each $a \in G$ there is an element a^{-1} of G , called an inverse of a , such that $a * a^{-1} = a^{-1} * a = e$.

The group G is called an abelian (or commutative) if $a * b = b * a$ for all $a, b \in G$. G is called finite group if it is a finite set.

Example:

1. $(V, +)$ where V is a vector space and $+$ is vector addition, is an additive group since operation defined is $+$. It is abelian group since $+$ is commutative.
2. For $n \in \mathbb{Z}^+$, $\mathbb{Z}/n\mathbb{Z}$ is a group under operation $+$ with $\bar{0}$ as identity and for \bar{a} inverse is $\overline{-a}$, such that $\bar{a} + \overline{-a} = \bar{1}$. And we can prove that $+$ is an associative operation.
3. For $n \in \mathbb{Z}^+$, the set $(\mathbb{Z}/n\mathbb{Z})^\times$ of equivalence classes \bar{a} which have multiplicative inverses $(\text{mod } n)$ is an abelian group under multiplication of residue classes. We assume here that multiplication is well defined and associative. (We can prove that). Identity will be $\bar{1}$ and by the definition of $(\mathbb{Z}/n\mathbb{Z})^\times$ inverse exists in the set itself.

Definition 12. *Direct Product: If $(A, *)$ and $(B, @)$ are two groups, then $A \times B$ is called direct product, whose elements are those in the Cartesian product*

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

and whose operations are defined component-wise

$$(a_1, b_1)(a_2, b_2) = (a_1 * a_2, b_1 @ b_2)$$

The new set $A \times B$ will also be a group.

It can be prove easily as A and B both contains the inverse and identity element.

Proposition 3. *If G is a group under the operation $*$, then*

1. the identity element of G is unique.
2. for each $a \in G$, a^{-1} is uniquely determined.
3. $(a^{-1})^{-1} = a$ for all $a \in G$.
4. $(a * b)^{-1} = (b^{-1}) * (a^{-1})$
5. for any $a_1, a_2, \dots, a_n \in G$ the value of $a_1 * a_2 \cdots * a_n \in G$ is independent of how the expression is bracketed (generalised associativity).

Proof. 1. Suppose for contradiction, there are two identities e_1, e_2 such that $e_1 \neq e_2$. Then $e_1.e_2 = e_2$ (if e_1 is identity) and $e_1.e_2 = e_1$ (if e_2 is identity). But the result of $e_1.e_2$ should be same as left hand side is same for both equations. Hence $e_1 = e_2$.

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2. Assume there exists two inverse of a , say b, c . If e is the identity element then we have $a * b = e$ and $a * c = e$. Also,

$$\begin{aligned} c &= c * e \\ c &= c * (a * b) \\ c &= (c * a) * b \\ c &= e * b \\ c &= b \end{aligned}$$

3. For some $a \in G$ inverse will be $(a)^{-1} \in G$ such that $aa^{-1} = e$ (e is identity). Now, interchanging the position of the elements $a^{-1}a = e$, we have inverse of a^{-1} is $a \implies (a^{-1})^{-1} = a$
4. Assume $c = (a * b)^{-1}$. Since $c \in G$, using property of inverse we have

$$\begin{aligned} c * (a * b) &= (a * b)^{-1}(a * b) = e \\ (c * a) * b &= e \end{aligned}$$

Right multiply b^{-1} on both sides

$$\begin{aligned} (c * a) * (b * b^{-1}) &= e * b^{-1} \\ c * a &= b^{-1} \end{aligned}$$

Right multiply a^{-1}

$$\begin{aligned} c * (a * a^{-1}) &= b^{-1} * a^{-1} \\ c &= b^{-1} * a^{-1} \end{aligned}$$

□