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IMS22090
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Assignment 1

Ans 1. Given $\epsilon = 10^{-2}$, we can use Taylor's expansion for a_n and b_n. Assume $f(n) = \frac{1}{n+2}$ and $g(n) = \frac{1}{n^2}$. For some $n = N_0$, we can write

$$f(n) = f(N_0) + \frac{f'(N_0)(n - N_0)}{1!} + \frac{f''(N_0)(n - N_0)^2}{2!} + \dots$$
$$a_n = \frac{1}{N_a + 2} + \frac{-(n - N_a)}{(n - N_a)^2 1!}$$

Ans 4. For $f(x) - x = 0$ in the interval $[0, 1]$, if we take $x = 0$, it is simple to observe that $f(0) - 0 = 0$. Hence, $x = 0$ is the solution of the given equation in the interval $[0, 1]$. $f(0) = 0$ also means that 0 is the root of $f(x)$.

Ans 6. Taylor's expansion of $f(x)$ at x_0 is

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!}$$

where $k \in \mathbb{Z}^+$. For $x_0 = 0$ and $n = 3$, we can write

$$sin(x) = sin(0) + cos(0)x - sin(0)\frac{x^2}{2!} - cos(0)\frac{x^3}{3!}$$
$$sin(x) = x - \frac{x^3}{6}$$

Error term will be

$$E = \frac{x^4}{24} sin(\xi(x))$$

where $0 < \xi(x) < x$, for $n = 8$ we have

$$\begin{aligned} sin(x) &= sin(0) + cos(0)x - sin(0)\frac{x^2}{2!} - cos(0)\frac{x^3}{3!} + sin(0)\frac{x^4}{24} + cos(0)\frac{x^5}{120} \\ &\quad - sin(0)\frac{x^6}{720} - cos(0)\frac{x^7}{7!} + sin(0)\frac{x^8}{8!} \\ sin(x) &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} \end{aligned}$$

Error term will be

$$E = \frac{x^9}{9!} cos(\xi(x))$$

where $0 < \xi(x) < x$

Ans 7. For $f(x) = cos(x)$ and $n = 3$, Taylor's expansion at $x_0 = 0$ will look like

$$cos(x) = cos(0) - sin(0)x - cos(0)\frac{x^2}{2} + sin(0)\frac{x^3}{6}$$
$$cos(x) = 1 - \frac{x^2}{2}$$

$$
\begin{aligned}
\int_0^{0.4} f(x)dx &\approx \int_0^{0.4} \left(1 - \frac{x^2}{2}\right) dx \\
&\approx \left[x - \frac{x^3}{6}\right]_0^{0.4} \\
&\approx 0.4 - \frac{(0.4)^3}{6} \\
&\approx 0.389333\ldots
\end{aligned}
$$

(b) Error function will look like for $0 < cos(\xi(x)) \leq 1$

$$E = \left|cos(\xi(x))\frac{x^4}{24}\right| \leq \left|\frac{x^4}{24}\right|$$

Hence, the upper bound of the error is $\frac{x^4}{24}$.

Ans 8. Taylor's expansion of e^x around $x_0 = 0$ with polynomial degree $n = 4$ will be

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

Putting $x = 1$

$$
\begin{aligned}
e &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \\
e &= 2.70833
\end{aligned}
$$

Error term will be

$$error = \left|\frac{e^{\xi(x)}x^{n+1}}{(n+1)!}\right| \leq 10^{-6}$$

Putting $x = 1$

$$error = \left|\frac{e^{\xi(1)}}{(n+1)!}\right|$$

Since $0 < \xi(x) < x = 1$ and e^x is the increasing function, we have $0 < e^{\xi(x)} < e$

$$
\begin{aligned}
\left|\frac{e^{\xi(1)}}{(n+1)!}\right| &\leq \left|\frac{e}{(n+1)!}\right| \leq 10^{-6} \\
(n+1)! &\geq \frac{e}{10^{-6}} \\
(n+1)! &\geq e \times 10^6
\end{aligned}
$$

Ans 9. Second degree Taylor's approximation for $f(x) = \sqrt{x+1}$ at $x_0 = 0$ is

$$
\begin{aligned}
\sqrt{x+1} &= 1 + \frac{1}{2\sqrt{x+1}} + \frac{-1}{4(x+1)^{3/2}} \\
&= 1 + \frac{1}{2\sqrt{2}} - \frac{x^2}{8}
\end{aligned}
$$

Remainder term is

$$R = \frac{x^3}{16(\xi(x)+1)^{5/2}}$$

where $x_0 = 0 < \xi(x) < x$. We can obtain a bound for this term if we choose $x \in [0, \infty]$.