Devansh Tripathi
IMS22090
Lecturer: Dr. Asha K. Dond

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Assignment 1

Ans 1. Given $\epsilon = 10^{-2}$

$$|a_n| < 10^{-2}$$
$$|a_n| < \frac{1}{100}$$
$$\left|\frac{1}{n+2}\right| < \frac{1}{100}$$

$n \in \mathbb{N}$, so $n+2$ is positive for all n.

$$n + 2 > 100$$
$$n > 98$$
$$n = 99 \quad \text{(least positive integer.)}$$

$$|b_n| < 10^{-2}$$
$$\left|\frac{1}{n^2}\right| < \frac{1}{100}$$

$n \in \mathbb{N}$ and n^2 is positive for all n.

$$n^2 > 100$$
$$n > 10$$
$$n = 11 \quad \text{(least positive integer.)}$$

Ans 2. Given is

$$|a_n - L| \leq \mu |a_{n-1} - L|$$

If we replace $n \to n-1$, we get

$$|a_{n-1} - L| \leq \mu |a_{n-2} - L|$$

Similarly, performing the replace operation k times, we get

$$|a_n - L| \leq \mu|a_{n-1} - L| \leq \mu^2|a_{n-2} - L| \leq \mu^3|a_{n-3} - L| \leq \ldots \mu^k|a_{n-k} - L|$$

We choose k such that $n - k \geq N \implies n - N \geq k$. Therefore, we get

$$|a_n - L| \leq \mu^{n-k}|a_{n-k} - L|$$

Taking limits both sides, $0 \leq |a_n - L| \leq 0 \implies |a_n - L| = 0 \implies a_n \to L$.

Ans 3. For domain $[0, 1]$, the range of the function $f(x) = sin(x) + x^2 - 1$ is

$$\begin{aligned}
R &= sin((0,1)) + ((0,1))^2 - 1 \\
R &= (0, sin(1)) + (0,1) - 1 \\
R &= (0, sin(1)) + (-1, 0) \\
R &= (-1, sin(1))
\end{aligned}$$

Since, in the interval $(0, 1)$ the function is taking the value $(-1, sin(1))$. It is going from negative to positive in y hence it will definately cut the x axis atleast once which will be the root.

Ans 4. For $f(x) - x = 0$ in the interval $[0, 1]$, if we take $x = 0$, it is simple to observe that $f(0) - 0 = 0$. Hence, $x = 0$ is the solution of the given equation in the interval $[0, 1]$. $f(0) = 0$ also means that 0 is the root of $f(x)$.

Ans 5. Since $g(x) = 0$, has atleast n roots assume $x_1, x_2, \dots x_n$. This means that $g(x_1) = 0, g(x_2) = 0 \dots g(x_n) = 0$
Using LMVT for x_2 and x_1, we have $g'(c_1) = \frac{g(x_2)-g(x_1)}{x_2-x_1} = 0$ for some $c \in [x_1, x_2]$. Hence, after generalizing this argument from $[x_1, x_2]$ to $[x_{n-1}, x_n]$ we will get $n-1$ $c_i's$ such that $g'(c_i) = 0$. Hence, $g'(x)$ will have atleast $n-1$ roots.

Ans 6. Taylor's expansion of $f(x)$ at x_0 is

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$$

where $k \in \mathbb{Z}^+$. For $x_0 = 0$ and $n = 3$, we can write

$$\begin{aligned} sin(x) &= sin(0) + cos(0)x - sin(0)\frac{x^2}{2!} - cos(0)\frac{x^3}{3!} \\ sin(x) &= x - \frac{x^3}{6} \end{aligned}$$

Error term will be

$$E = \frac{x^4}{24} sin(\xi(x))$$

where $0 < \xi(x) < x$, for $n = 8$ we have

$$\begin{aligned} sin(x) &= sin(0) + cos(0)x - sin(0)\frac{x^2}{2!} - cos(0)\frac{x^3}{3!} + sin(0)\frac{x^4}{24} + cos(0)\frac{x^5}{120} \\ &\quad - sin(0)\frac{x^6}{720} - cos(0)\frac{x^7}{7!} + sin(0)\frac{x^8}{8!} \\ sin(x) &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} \end{aligned}$$

Error term will be

$$E = \frac{x^9}{9!} cos(\xi(x))$$

where $0 < \xi(x) < x$

Ans 7. For $f(x) = cos(x)$ and $n = 3$, Taylor's expansion at $x_0 = 0$ will look like

$$\begin{aligned} cos(x) &= cos(0) - sin(0)x - cos(0)\frac{x^2}{2} + sin(0)\frac{x^3}{6} \\ cos(x) &= 1 - \frac{x^2}{2} \end{aligned}$$

$$\begin{aligned} \int_0^{0.4} f(x)dx &\approx \int_0^{0.4} \left(1 - \frac{x^2}{2}\right) dx \\ &\approx \left[x - \frac{x^3}{6}\right]_0^{0.4} \\ &\approx 0.4 - \frac{(0.4)^3}{6} \\ &\approx 0.389333\dots \end{aligned}$$

(b) Error function will look like for $0 < cos(\xi(x)) \leq 1$

$$E = \left| cos(\xi(x))\frac{x^4}{24} \right| \leq \left| \frac{x^4}{24} \right|$$

Hence, the upper bound of the error is $\frac{x^4}{24}$.

Ans 8. Taylor's expansion of e^x around $x_0 = 0$ with polynomial degree $n = 4$ will be

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

Putting $x = 1$

$$\begin{aligned} e &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \\ e &= 2.70833 \end{aligned}$$

Error term will be

$$error = \left| \frac{e^{\xi(x)} x^{n+1}}{(n+1)!} \right| \leq 10^{-6}$$

Putting $x = 1$

$$error = \left| \frac{e^{\xi(1)}}{(n+1)!} \right|$$

Since $0 < \xi(x) < x = 1$ and e^x is the increasing function, we have $0 < e^{\xi(x)} < e$

$$\begin{aligned} \left| \frac{e^{\xi(1)}}{(n+1)!} \right| &\leq \left| \frac{e}{(n+1)!} \right| \leq 10^{-6} \\ (n+1)! &\geq \frac{e}{10^{-6}} \\ (n+1)! &\geq e \times 10^6 \end{aligned}$$

Ans 9. Second degree Taylor's approximation for $f(x) = \sqrt{x+1}$ at $x_0 = 0$ is

$$\begin{aligned} \sqrt{x+1} &= 1 + \frac{1}{2\sqrt{x+1}} + \frac{-1}{4(x+1)^{3/2}} \\ &= 1 + \frac{1}{2\sqrt{2}} - \frac{x^2}{8} \end{aligned}$$

Remainder term is

$$R = \left| \frac{x^3}{16(\xi(x)+1)^{5/2}} \right| \leq \left| \frac{x^3}{16} \right|$$

where $x_0 = 0 < \xi(x) < x$. We can obtain a bound for this term if we fix x.

Ans 10. For $f(x) = \frac{1}{x}$, we have $f'(x)\frac{-1}{x^2}$, $f''(x) = \frac{2}{x^3}$ and $f'''(x) = \frac{-6}{x^4}$
General expression for derivative will be

$$f^{(k)}(x) = \frac{(-1)^k k!}{x^{k+1}}$$

General expression for Taylor's expansion will be

$$\frac{1}{x} = \sum_{k=0}^{n} \frac{(-1)^k k!(x-x_0)^k}{x_0^{k+1} k!}$$

$$P_n(x) = \frac{1}{x} = \sum_{k=0}^{n} (-1)^k (x-1)^k \qquad (x_0 = 1)$$

$f(3)$ using $P_0(x)$

$$f(3) = 1$$

$f(3)$ using $P_1(x)$

$$f(3) = 1 - (3-1) = -1$$

$f(3)$ using $P_2(x)$

$$f(3) = 1 - 2 + (3-1)^2 = 3$$

$f(3)$ using $P_3(x)$

$$f(3) = 1 - 2 + 4 - (3-1)^3 = -5$$

$f(3)$ using $P_4(x)$

$$f(3) = 1 - 2 + 4 - 8 + (3-1)^4 = 11$$

$f(3)$ using $P_5(x)$

$$f(3) = 1 - 2 + 4 - 8 + 16 - (3-1)^5 = -21$$

We can observe that the values of $f(3)$ using Taylor's expansion is not converging rather it is oscillating.