Lecture 1

Remarks: Basic ordering properties are assumed to be true.

Definition 1. *Boundedness: The subset $A \in R$ is said to be bounded above if $\exists M$ such that $M > x \;\; \forall x \in A$. And it is bounded below if $\exists m$ such that $m < x \;\; \forall x \in A$. If A has both then it is called bounded.*

Definition 2. *Least Upper Bound (lub) Axiom: If A is nonempty subset of R and it is bounded above, then A has a least upper bound in R.*

Theorem 1. *If A is nonempty subset in R and it is bounded below, then it has a greatest lower bound in R.*

Proof. We first create a set T of lower bounds of A

$$T = \{m \mid m < x \;\; \forall x \in A\}$$

T is non-empty since A is bounded below. Now, we need to prove that there exits a supremum of T which is also a lower bound of A.
Since, set T is bounded above by all the elements of set A, it should have a least upper bound, say M such that $M > m \;\; \forall m \in T$. Also, every element of A is an upper bound of T hence by definition of supremum, we can say $M \leq x \;\; \forall x \in A$ hence M is the lower bound of A. This makes it the greatest lower bound. $\square$

Lemma 1. *Suppose $A \neq \phi$ and $s = lub(A)$ then for any y such that $y < s$, $\exists a \in A$ such that $y < a \leq s$.*

Proof. Suppose for contradiction, $\nexists$ any element a such that $y < a$. This means that $y \geq a, \forall a \in A \implies y$ is upper bound of set A. But y is already less than least upper bound of set A. Hence contradiction.
Therefore, $\exists \; a \in A$ such that $y < a \leq s$. $\square$

Theorem 2. *Archimedean Property: Given any positive real numbers $x, y \; \exists \; n \in N$ such that $nx > y$.*

Proof. Let a set $A = \{nx \mid n \in \mathbb{N}\}$. Suppose for contradiction $nx \leq y$. Then y is the upper bound of the set A.
Let a $x > 0$, then $y - x < y$ hence $y - x$ is not the upper bound of the set A. This means that $\exists m \in \mathbb{N}$ such that $y - x < mx \implies y < mx + x \implies y < (m+1)x$ which is impossible since $(m+1)x \in A$ and y is upper bound of the A.
$nx > y$ is true. $\square$

Theorem 3. *If A and B are the two non empty bounded subsets of R, such that $x \leq y \;\; \forall x \in A$ and $\forall y \in B$ then $sup(A) \leq inf(B)$*

Proof. Let a be the supremum of A and b be the infimum of B. Therefore, $a \geq x \;\; \forall x \in A$ and $b \leq y \;\; \forall y \in B$. Also, A is bounded above by B and elements of B are the upper bound for A. Hence, $a \leq y \;\; \forall y \in B$. This means that a is the lower bound of B and a is $sup(A)$. In other words, $sup(A) \leq inf(B)$. $\square$

Theorem 4. *Given any two real number a, b with $a < b$, $\exists \; \mathbb{Q}$ between a and b.*

Proof. Since $b - a > 0$. Take two positive number $b - a$ and 1 $\exists$ n $\in \mathbb{Z}$ such that $n(b - a) > 1$.
TBD $\square$

Theorem 5. *Any monotone increasing sequence of real numbers that is bounded above converges to some real number.*

Proof. Let x_n be a monotone increasing sequence in $\mathbb{R}$ that is bounded above hence there exits a s such that $s = lub\{x_n \mid n \in \mathbb{N}\}$
Suppose $\epsilon > 0 \implies s - \epsilon < s$ and $s - \epsilon$ is not the upper bound of the x_n.
Using lemma 1, we can say that $\exists x_\epsilon \in x_n$ such that $s - \epsilon < x_\epsilon < s$.
Using monotone condition, for some $n_0 \in \mathbb{N}$, we have,

$$s - \epsilon < x_n < s < s + \epsilon \quad \forall n > n_0 \in \mathbb{N}$$

Hence $|x_n - s| < \epsilon$. x_n converges to s $\forall n > n_0$. $\square$

Remark. *Nested Interval theorem* $\approx$ *lub* $\approx$ *theorem* 4

Proof. TBD $\square$

Theorem 6. *Nested Interval theorem: Suppose* $\{I_n\}$ *is the sequence of closed and bounded non-empty intervals such that* $I_1 \supset I_2 \supset I_3 \ldots$ *then:*

1. $\bigcap\limits_{n \geq 1} I_n \neq \phi$.

2. *If the sequence of the length of the intervals goes to* 0 *then* $\bigcap\limits_{n \geq 1} I_n = \{x\}$.

Proof. Let I_n be an interval $[a_n, b_n]$ with $a_m < b_n \forall m, n \in \mathbb{N}$. Then $\forall n \in \mathbb{N}$, a_n is the increasing sequence and b_n is the decreasing sequence. b_n is upper bound of a_n hence, $a_n < inf(b_n)$.
For b_n, a_n is the lower bound of b_n i.e. $sup(a_n) < b_n$. If we combine all inequalities, we get

$$a_n \leq sup(a_n) \leq inf(b_n) \leq b_n \quad \forall n \in \mathbb{N}$$

Using density theorem, we can say that $\exists$ some $\mathbb{Q}$ between $sup(a_n)$ and $inf(b_n)$.
Hence, $\bigcap\limits_{n \geq 1} I_n \neq \phi$.
Let the length of the interval to be $L = |b_n - a_n|$. Suppose for contradiction, we have two elements in $\bigcap\limits_{n \geq 1} I_n$ instead of one, say x and y.
The distance between x and y is $|y - x|$. Since, $L \to 0$ hence $\exists n \in \mathbb{N}$ such that for some $n_0 \geq n$, $|L| = |b_{n_0} - a_{n_0}| < \epsilon$ for some $\epsilon > 0$. Since $|L| \to 0$, we can choose ϵ such that it is smaller than $|y - x|$. Then, if interval contains any one of the point, it can not contain the other. $\square$

Lecture 2

Definition 3. *Decimals representation of Real numbers: Let* $z \in \mathbb{R}^+$ *be given. Let* n_0 *be the largest integer such that* $n_0 \leq z$*. Let* n_1 *be the largest integer such that* $n_0 + \frac{n_1}{10} \leq z$*. As such, say* n_k *is defined for some* k*. Let* n_{k+1} *be the largest integer* *such that* $n_0 + \frac{n_1}{10^1} + \frac{n_1}{10^2} + \cdots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \leq z$. *Consider the set of all such finite sums, i.e. the set of all*

$$z_k = n_0 + \frac{n_1}{10^1} + \frac{n_1}{10^2} + \cdots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \leq z$$

This set has a supremum and that is z *itself. We symbolically write* $z = n_0.n_1n_2\ldots$

Lemma 2. *Let* p *be an integer* ≥ 2. *If* $0 \leq a_n \leq p-1$, *where* a_n *is an integer then* $\sum_{n=0}^{\infty} \frac{a_n}{p^n}$ *converges to some* x *in* $[0,1]$.

Proof. Since $0 \leq a_n \leq p-1$, we can replace all a_n with $p-1$ and then we have

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n} \leq (p-1)\sum_{n=1}^{\infty} \frac{1}{p^n} = 1$$

Therefore, the sequence is bounded, and it is monotonic increasing this means it converges to some $x \leq 1$ and x is already positive. Hence it converges tp some x in $[0,1]$. ☐

Lemma 3. *Conversely, given any* $0 \leq x \leq 1, \exists\ a_n \in \mathbb{Z}$ *and* $0 \leq a_n \leq p-1$ *such that* $x = \sum_{n=0}^{\infty} \frac{a_n}{p^n}$.

Proof. Suppose we have $0 < x \leq 1$ and a_1 is the largest integer such that $\frac{a_1}{p} < x \leq 1$. Since x is bounded above by 1, we have $a_1 < p \implies a_1 \leq p-1$ since a_1 is an integer. Similarly, find a_2 such that $\frac{a_1}{p} + \frac{a_2}{p^2} < x$.This can be achieved by Archimedean property. Also, note that $a_2 \leq p-1$, since we have

$$\begin{aligned}
\frac{a_1}{p} + \frac{a_2}{p^2} &< x < 1 \\
\frac{a_1}{p} + \frac{a_2}{p^2} \leq \frac{p-1}{p} + \frac{a_2}{p^2} &< 1 \qquad (a_1 \leq p-1) \\
1 - \frac{1}{p} + \frac{a_2}{p^2} &< 1 \\
\frac{a_2}{p^2} &< \frac{1}{p} \\
a_2 &< p \\
a_2 &\leq p-1
\end{aligned}$$

Inductively, we can define a_n as the largest integer with $a_n \leq p-1$ such that $\sum_{i=1}^{n} \frac{a_i}{p_i} < x$. Since $a_n < p$ TBD ☐

Suppose $\{a_n\}$ is the bounded sequence in $\mathbb{R}$, we define two sets:

$$\begin{aligned}
s_n &:= inf\{a_n, a_{n+1}, \ldots\} \\
S_n &:= sup\{a_n, a_{n+1}, \ldots\}
\end{aligned}$$

Notice that $\underset{k}{inf}(\{a_n\}) \leq s_n \leq S_n \leq \underset{k}{sup}(\{s_n\})$

Definition 4. *Limit superior and limit inferior: Let a_n be the bounded sequence of real numbers then*

$$\lim inf(a_n) = \lim_{n\to\infty} s_n$$
$$\lim sup(a_n) = \lim_{n\to\infty} S_n$$

Limit superior is the supremum of all subsequential limits of $\{a_n\}$. Similarly, limit inferior is the infimum of all subsequential limits of $\{a_n\}$.

Note that s_n is the increasing sequence and S_n is the decreasing sequence and they are bounded on both sides. Hence, we can also say that using monotone convergence theorem

$$\lim inf(a_n) = \lim_{n\to\infty} s_n = sup(s_n)$$
$$\lim sup(a_n) = \lim_{n\to\infty} S_n = inf(S_n)$$

Theorem 7. *A sequence $\{a_n\}$ is bounded above iff limsup $a_n < \infty$ (is finite).*

Proof. TBD □

Theorem 8. *A sequence $\{a_n\}$ is bounded below iff liminf $a_n < \infty$ (is finite).*

Proof. TBD □

Theorem 9. *Given any sequence $\{a_n\}$ there exists a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to limsup(a_n)$.*

Proof. TBD □

Theorem 10. *Given any sequence $\{a_n\}$ there exists a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to liminf(a_n)$.*

Proof. TBD □

Lecture 3

Definition 5. *Finite Set: A set is finite if there exists a bijection between the set and the $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.*

Theorem 11. $\mathbb{N}$ *is an infinite set.*

Proof. Negation of above definition would be a set is infinite if there does not exists a bijection between $\{1, 2, \ldots, n\}$ and $\mathbb{N}$. Suppose a function $f(\{1, 2, \ldots, n\}) \to \mathbb{N}$. We can have a natural number $f(1) + f(2) + \cdots + f(n) > f(i) \ \ \forall i \in \mathbb{N}$ that does not have a preimage in $\{1, 2, \ldots, n\}$ hence the map is not bijective. □

Theorem 12. *A set is infinite iff there exists a one-one map from $\mathbb{N}$ to the set.*

Proof. $\implies$) For $1 \in \mathbb{N}$ there exist a image in X say $f(1)$. Now, take $2 \in \mathbb{N}$ such that there exist a image $f(2) \in X\backslash\{1\}$. This means that $f(1) \neq f(2)$. Since for every $n \in \mathbb{N}$ we can have $f(n)$ in $X\backslash\{1, 2, \ldots, n-1\}$ as X is also infinite. Hence, we have constructed a one-one map from $\mathbb{N} \to X$.
($\impliedby$ Since $\mathbb{N}$ is infinite and we have a one-one mapping from $\mathbb{N} \to X$ therefore for each $n \in \mathbb{N}$, we have only one $f(n) \in X$ and every $n \in \mathbb{N}$ has a image (it's a map). Hence, X is infinite. □

Definition 6. *Equivalent or Equipotent set: Two sets are equivalent or equipotent if there exists bijection between X and Y.*

For example: Finite sets are equivalent to $\{1, 2, \dots, n\}$ for some fixed $n \in \mathbb{N}$.

Definition 7. *Countably infinite set: A infinite set X is said to be countably infinite if there exists a bijection between X and $\mathbb{N}$.*

Definition 8. *Uncountably infinite set: A set is said to be uncountably infinite if it is not countably infinite set.*

Example:

1. Countably infinite: bijective map between $\mathbb{Z} \to \mathbb{N}$. The map will look line

$$n \to \begin{cases} \frac{n}{2} & n \in even \\ \frac{-(n+1)}{2} & n \in odd \end{cases}$$

2. $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is also equivalent to $\mathbb{N}$ i.e. countably infinite. Map will be $(m \times n) \to 2^m(2n-1)$ m and n are unique and m such that 2^m is the maximum multiple of 2.

Theorem 13. *If A is an infinite subset of $\mathbb{N}$ then there exists bijection between A and $\mathbb{N}$.*

Proof. TBD □

Corollary 1. *Monotone Subsequence theorem: Any sequence $\{x_n\}$ of real numbers has a monotone subsequence*

Proof. We define "peak" as any element x_m is called a peak if $x_m \geq x_n$ for all $n > m$. There cases can be two possible cases

1. Infinite peaks: This means that there exists $m_i's$ say $\{m_1, m_2, \dots\}$ such that $x_{m_i} > x_n$ for all $n > m_i$, and for all $i \in \mathbb{N}$. We can arrange $m_i's$ in increasing order $m_1 < m_2 < \dots$ and $x_{m_1} > x_{m_2} > x_{m_3} > \dots$ is a decreasing subsequence.

2. Finite peaks: (0 or some $n \in \mathbb{N}$). Assume the peaks are $\{x_{m_1}, x_{m_2}, \dots, x_{m_n}\}$, this means that there exists some $s_1 = m_n + 1, s \in \mathbb{N}$ such that x_{s_1} is not a peak. Therefore, there also exists some $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$. Because of finite peaks x_{s_2} is also not a peak, hence for some $s_3 > s_2$, we have $x_{s_2} < x_{s_3}$. Proceeding with induction, we have $x_{s_1} < x_{s_2} < x_{s_3} < \dots$ is a increasing sequence.

□

Theorem 14. *$\mathbb{Q}$ is a countably infinite set.*

Proof. TBD □

Theorem 15. *Any interval in $\mathbb{R}$ is an uncountable set.*

Proof. TBD □

Definition 9. *Suppose $X \neq \phi$. A partial order on X is a relation R on X such that R is*

1. *Reflexive.*

2. *Anti-symmetric $\implies aRb, bRa \implies a = b$*

3. *Transitive.*

Examples:

1. $R =$ " $\leq$ " is a partial order.

$\mathcal{P}(\mathcal{A})$ is power set of A and $X, Y \subset A$ then $X \leq Y$ iff $X \subseteq Y$.

Lecture 4

Definition 10. *Given $E \subset X$ where X is partially order set, we say E is totally ordered if any two elements of E are comparable i.e. if $e_1, e_2 \in E$, then $e_1 \leq e_2$ or $e_2 \leq e_1$. Totally ordered $\equiv$ linearly order $\equiv$ chain.*

Definition 11. *Upper bound of E: An element is $x \in X$ is called upper bound of E if for any $x' \in E$, we have $x' \leq x$. x is called the maximal element if $x' \geq x \implies x' = x$.*
For maximal element, x should be an upper bound for set E and x should belong to E.

Let $X \neq \phi$. $\mathcal{F}$ is a collection of subsets of X (element of $\mathcal{P}$(X)). An element $F \in \mathcal{F}$ is a upper bound for a subfamily $\mathcal{F}'$ of $\mathcal{F}$ provided every member of $\mathcal{F}'$ is a subset of F.
F will be the maximal element of $\mathcal{F}$ if it is not a proper subset (means not contained in) of any member in $\mathcal{F}$.

Lemma 4. *Zorn's lemma: Let X be a partially order set. If every totally ordered subset of X is bounded above then X has a maximal element.*

Definition 12. *Cardinality of X : Two sets A and B have same cardinality if there exists a bijection between them. Set X has cardinal number α means that there exists a set Y equivalent to X with number of elements equal to α.*

If α and β are cardinal numbers of set X and set Y such that $\alpha \leq \beta$ then there exists a one-one mapping from $X \to Y$.

Theorem 16. *Cantor-Schroeder-Bernstein theorem: If there exists a one-one mapping from $X \to Y$ and $Y \to X$ then there exists a bijection between X and Y.*

Limits of functions

Definition 13. *Limit point: A point $a \in \mathbb{R}$ is called a limit point of a set $X \subseteq \mathbb{R}$ if for every neighbourhood $(a - \epsilon, a + \epsilon), \epsilon > 0$ there exists $x \in X$ such that $a \neq x$.*

For a function f defined on $X \subseteq \mathbb{R}$, f converges to some l means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - l| < \epsilon \ \forall x \in X \ni |x - a| < \delta$$

Limit of functions as limit of sequences-

Proposition 1. *Let $f : X \to \mathbb{R}$ and let a be a limit of X. Then $lim_{x\to a} f(x) = l$ if and only if for every sequence $\{x_n\}_{n\geq 1}$ in X that converges to a and $x_n \neq a$ for all n, the sequence $\{f(x_n)\}_{n\geq 1}$ converges to l.*

A function f is continuous on X if it is continuous in every point in X.

Definition 14. *Continuity of f: A function f is said to be continuous at some point $x \in X$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $y \in X$ and $|y - x| < \delta$, we have $|f(y) - f(x)| < \epsilon$.*

A function f is continuous at a limit point $a \in X$ if and only if $f(a)$ is defined and $lim_{x\to a} f(x) = f(a)$.

Proposition 2. *Let f be a real valued funtion defined on subset X of $\mathbb{R}$ and $a \in X$ is the limit point of X. Then f is continuous at a if and only if for every sequence $\{x_n\}_{n\geq 1}$ that converges to a and $x_n \neq a$ for every n, we have $lim f(x_n) = f(lim x_n) = f(a)$. Continuous function preserve convergence (maps convergent sequence into convergent sequences).*

Theorem 17. *Bolzano intermediate value theorem: Let I be an interval and $f : I \to \mathbb{R}$, if $a, b \in I$ and $\alpha \in \mathbb{R}$ satisfies $f(a) < \alpha < f(b)$ or $f(a) > \alpha > f(b)$ then there exists a point $c \in I$ between a and b such that $f(c) = \alpha$.*

Sequences of functions

Let $X \subseteq \mathbb{R}$. If for every $n = 1, 2, \ldots$, we assigned a real valued function f_n defined on X then $\{f_n\}_{n\geq 1}$ is called sequence of functions.

Point-wise convergence

A sequence is called point-wise convergent if for each $x \in X \subseteq \mathbb{R}$, the sequence $f_1(x), f_2(x), \ldots$ of real numbers is convergent.

Point-wise limit

A function defined on X is called a point-wise limit if for every $\epsilon > 0$ there exist $n_0 \in \mathbb{Z}$ depending on x and ϵ such that for all $n \geq n_0$, we have $|f(x) - f_n(x)| < \epsilon$.

A series $\sum_{n=1}^{\infty} x_n$ of real numbers converges to $x \in \mathbb{R}$ if the sequence of partial sums $\{s_n\}_{n\geq 1}$ converges to x where $s_n = \sum_{k=1}^{n} x_k$ (n_{th} partial sum).

The limit $x = lim s_n = lim_{n\to\infty} \sum_{k=1}^{n} x_k$ is called the sum of the series. In the case of functions, if for every $x \in X$, $\sum_{n=1}^{\infty} f(x_n)$ converges and if we define $f(x) = \sum_{n=1}^{\infty} f(x_n)$, the $f(x)$ is called the sum of the series $\sum_{n=1}^{\infty} f_n$.

Uniform convergence

A sequence of functions $\{f_n\}_{n\geq 1}$ defined on set $X \subseteq \mathbb{R}$ is said to be uniformly convergent on X to f if for every $\epsilon > 0$ there exists a $n_0 \in \mathbb{Z}$ (depending on ϵ only) such that for all $n \geq n_0$, we have $|f(x) - f_n(x)| < \epsilon$ for all $x \in X$.
Thus, every uniformly convergent sequence is pointwise convergent but converse is not true always.
Uniform Convergence for series of functions
The series of functions $\sum\limits_{n=1}^{\infty}$ converges uniformly on X if the sequence $\{s_n\}_{n\geq 1}$ partial sum of functions, where $s_n(x) = \sum\limits_{k=1}^{n} f_k(x), x \in X$ converges uniformly on X.
Cauchy Criterion of Uniform convergence
The sequence of functions $\{f_n\}_{n\geq 1}$ defined on $X \in \mathbb{R}$ converges uniformly on X if and only if given $\epsilon > 0$ there exists n_0 such that for all $x \in X$, and all $m \geq n_0, n \geq n_0$ we have $|f_m(x) - f_n(x)| < \epsilon$.

Proposition 3. *Suppose $\{f_n\}_{n\geq 1}$ is sequence of continuous functions defined on X and it converges uniformly to f then f is continous.*

Basic Topology in $\mathbb{R}$

Definition 15. *Open sets in $\mathbb{R}$: A subset G of $\mathbb{R}$ is said to be open if for every $x \in G$, there is a neighbourhood $(x - \epsilon, x + \epsilon), \epsilon > 0$ that is contained in G.*

Definition 16. *Open cover: A open cover of X is a collection of $C = \{G_\alpha \mid \alpha \in I\}$ of open sets in $\mathbb{R}$ whose union contains the set X,*

$$X \subseteq \bigcup_{\alpha} G_\alpha$$

where I is some indexing set.

Definition 17. *Sub-cover: If C' is a subcollection of C such that the union of sets in C' also contains the set X then C' is called subcover from C of X. If the number of sets in C' is finite then we call it a finite subcover.*

Definition 18. *Compact set: A subset X of $\mathbb{R}$ is said to be compact if every open cover of X has a finite sub-cover.*

Proposition 4. *(Heine-Borel Theorem) Let X be a set of real numbers. Then the following statements are equivalent:*

1. *X is closed and bounded.*

2. *X is compact (every open cover has a finite subcover).*

3. *Every infinite subset of X has a limit point in X.*

Proof. 1 $\implies$ 2) Suppose $X = [a, b]$ a infinite interval which is closed and bounded. We define a set S

$$S = \{x \in [a, b] \mid \exists n \in I \ni [a, x] \subseteq \bigcup_{i=1}^{i=n} O_i\}$$

S is non-empty as we can always choose n such that S contains atleast a. Also, b is the upper bound of the set S and then it should possess the least upper bound property. Let $\lambda = sup(S)$.

Basically, S is a collection of all x such that the interval $[a, x]$ has a finite subcover. Hence, we need to show that λ is largest possible element of S (i.e. $\lambda \in S$) for which $[a, \lambda]$ has a finite subcover and $\lambda = b$.

Since, $\lambda = sup(S)$ and $\lambda < b$ hence, $\lambda \in X$. This means that there exists atleast one open set O_β that contains λ for some $\beta \in I$. Hence, for some $\epsilon > 0$, the neighbourhood $(\lambda - \epsilon, \lambda + \epsilon) \in O_\beta$. Since, $\lambda - \epsilon$ is not the supremum of set S. Therefore, there exists a element $x \in S$ such that $\lambda > x > \lambda - \epsilon \implies x \in O_\beta$. Also, $x \in S$ hence, by definition of S the finite subcovers of $[a, x]$ are $\{O_1, O_2, \ldots, O_n\}$ for some fixed $n \in I$. Hence, adding O_β to this set $\{O_{beta}, O_1, O_2, \ldots, O_n\}$ is also finite and it is subcover of set $[a, \lambda] \implies \lambda \in S$.

For proving $\lambda = b$, we have $\lambda \leq b$. Suppose for contradiction that $\lambda < b \implies b - \lambda > 0$. Define $y := \lambda + \frac{1}{2}\min(\epsilon, b - \lambda)$. This implies that belongs y to neighbourhood of $\lambda \implies y \in O_\beta$ and $y \in \bigcup\{O_\beta, O_1, O_2, \ldots, O_n\} \implies y \leq \lambda$. But by definition $y > \lambda$. Hence, contradiction. Therefore, $y = b$ is the only possibility. This means that $X := [a, b]$ has finite subcover hence X is compact. □

Proof. 2 $\implies$ 3) Let a infinite subset G of X where X is compact set. Suppose for contradiction G does not have any limit point in X TBD. □

Proposition 5. *Let f be real-valued continuous function defined on closed and bounded interval $I = [a, b]$. Then f is bounded on I and it assumes its maximum and minimum values in the interval I i.e. there are points $x_1, x_2 \in I$ such that $f(X_1) \leq f(x) \leq f(x_2)$ for all $x \in X$.*

Uniformly Continuous function Let f be real-valued continuous function defined on X. Then f is said to be uniformly continuous if given $\epsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in X$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

Proposition 6. *If a real-valued function f is continous on a closed and bounded interval I then f is uniformly continuous on I.*

Sequences in $\mathbb{R}$

1. A convergent sequence in $\mathbb{Q}$ is a Cauchy sequence in $\mathbb{Q}$.

2. A Cauchy sequence in $\mathbb{Q}$ is bounded; in particular, every convergent sequence in $\mathbb{Q}$ is bounded.

3. Limit of a sequence say $\{x_n\}_{n\geq 1}$ is unique.

Let F_Q denote the set of all cauchy sequence in $\mathbb{Q}$.

Definition 19. *A sequence $\{x_n\}_{n\geq 1}$ in F_Q is said to be equivalent to a sequence $\{y_n\}_{n\geq 1}$ in F_Q if and only if $\lim_{n\to\infty} |x_n - y_n| = 0$. Notation of equivalence is $\{x_n\}_{n\geq 1} \sim \{y_n\}_{n\geq 1}$.*

Proposition 7. *If $\{x_n\}_{n\geq 1} \in F_Q$ then $\lim_{n\to\infty} x_n = x$ if and only if $\{x_n\} \sim \{x\}$, where $\{x\}$ denotes the constant sequence with each term is equal to x.*

Proof. Follows from definition of equivalence relation.
Since, $\lim_{n\to\infty} |x_n - x| = 0$ (from definition) $\implies$ $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x = x$. Converse, is similar. $\square$

Proposition 8. *If $\{x_n\}$ and $\{y_n\}$ are in F_Q then so the sequencs $\{x_n + y_n\}$ and $\{x_n y_n\}$.*

Proof. Apply cauchy criterion for each sequence and choose ϵ to be $\frac{\epsilon}{2}$. Then try to find ϵ bound for sequences $\{x_n + y_n\}$ and $\{x_n y_n\}$. $\square$

Proposition 9. *If $\{x_n\}$, $\{y_n\}$, $\{x'_n\}$ and $\{y'_n\}$ are in F_Q and $\{x_n\} \sim \{x'_n\}$, $\{y_n\} \sim \{y'_n\}$ then $\{x_n + x'_n\} \sim \{y_n + y'_n\}$ and $\{x_n y_n\} \sim \{x'_n y'_n\}$.*

Proof. For $\{x_n\} \sim \{x'_n\}$, $\{y_n\} \sim \{y'_n\}$ follows from writing modulus inequality $|a| - |b| \leq |a - b| \leq |a + b| \leq |a| + |b|$ and then applying sandwich theorem.

For $\{x_n y_n\} \sim \{x'_n y'_n\}$, we know that cauchy sequence in $\mathbb{Q}$ are bounded. Hence, there exists a rational K_1, K_2 such that $|x_n| \leq K_1$ and $|y'_n| \leq K_2$ for all n. We can write

$$|x_n - x'_n| < \frac{\epsilon}{2K_1} \text{ and } |y_n - y'_n| < \frac{\epsilon}{2K_2}$$

$$\begin{aligned}
|x_n y_n - x'_n y'_n| &= |x_n y_n - x_n y'_n + x_n y'_n - x'_n y'_n| \\
&= |x_n(y_n - y'_n) + y'_n(x_n - x'_n)| \\
&\leq |x_n||(y_n - y'_n)| + |y'_n||(x_n - x'_n)| \\
&< K_1\left(\frac{\epsilon}{2K_1}\right) + K_2\left(\frac{\epsilon}{2K_2}\right) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

$\square$

Inequalities

Proposition 10. *The function $f(x) = \frac{x}{1+x}, x \geq 0$, is monotonically increasing.*

Proof. For some $x, y \geq 0$ and $x > y$, we have $\frac{1}{1+x} < \frac{1}{1+y}$ and $1 - \frac{1}{1+x} > 1 - \frac{1}{1+y} \implies \frac{x}{1+x} > \frac{y}{1+y}$. $\square$

Theorem 18. *For any two real numbers x and y, this inequality holds*

$$\frac{|x + y|}{1 + |x + y|} \leq \frac{|x|}{1 + |x|} + \frac{|y|}{1 + |y|}$$

Proof. Using previous proposition, we can say if $|x + y| \leq |x| + |y|$ and the sequence $\frac{x}{1+x}, x \geq 0$ is monotonically increasing then

$$\frac{|x + y|}{1 + |x + y|} \leq \frac{|x| + |y|}{1 + |x| + |y|} = \frac{|x|}{1 + |x| + |y|} + \frac{|y|}{1 + |x| + |y|} \leq \frac{|x|}{1 + |x|} + \frac{|y|}{1 + |y|}$$

$\square$

Proposition 11. *Generalised AM-GM inequality: If $a > 0$ and $b > 0$ and if $0 \leq \lambda \leq 1$ is fixed, then*

$$a^{\lambda} b^{1-\lambda} \leq \lambda a + (1-\lambda) b$$

Proof. Since, $y = ln(x)$ is concave, then

$$ln(\lambda a + (1-\lambda)b) \geq \lambda ln(a) + (1-\lambda) ln(b)$$
$$ln(\lambda a + (1-\lambda)b) \geq ln(a^{\lambda} b^{1-\lambda})$$

Since e^x is increasing function, we have

$$a^{\lambda} b^{1-\lambda} \leq \lambda a + (1-\lambda) b$$

□

Remark. *When $x \geq 0, y \geq 0$ and $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$xy \leq \frac{1}{p} x^p + \frac{1}{q} y^q$$

Proof. Replace $a = x^p$, $b = y^q$ and take $\lambda = \frac{1}{p} \implies 1 - \lambda = 1 - \frac{1}{p} = \frac{1}{q}$ □

Theorem 19. *(Holder's inequality) Let $x_i \geq 0$ and $y_i \geq 0$ for $i = 1, 2, \ldots, n$, and suppose that $p > 1$ and $q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p \right)^{1/p} \left(\sum_{i=1}^{n} y_i^q \right)^{1/q}$$

In the special case when $p = q = 2$, the above inequality reduces to

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} \left(\sum_{i=1}^{n} y_i^2 \right)^{1/2}$$

This is called ***Cauchy-Schwarz inequality.***

Proof. For the case of $x_i = 0$ and $y_i = 0$, it is trivially true.
For the case of $x_i > 0$ and $y_i > 0$, we can write the given inequality as

$$\sum_{i=1}^{n} \left(\frac{x_i}{\left(\sum_{i=1}^{n} x_i^p \right)^{1/p}} \frac{y_i}{\left(\sum_{i=1}^{n} y_i^q \right)^{1/q}} \right) \leq 1$$

Replace $x_i' = \frac{x_i}{\left(\sum_{i=1}^{n} x_i^p \right)^2}$ and $y_i' = \frac{y_i}{\left(\sum_{i=1}^{n} y_i^q \right)^2}$, also $x_i' \geq 0$ and $y_i' \geq 0$, we get $\sum_{n=1}^{n} x_i' y_i' \leq 1$.
Now, apply the Youngs inequality for $i = 1, 2, \ldots, n$ and sum them up to get

$$\sum_{i=1}^{n} x_i' y_i' \leq \frac{\sum x^p}{p} + \frac{\sum y^q}{q}$$

Since, $x_i > 0$ and $y_i > 0$ hence $\sum\limits_{n=1}^{n} x'^p \neq 0 \neq \sum\limits_{n=1}^{n} y'^q$ and it is equivalent to prove it for (some constant)

$$\sum_{n=1}^{n} x'^p = 1 = \sum_{n=1}^{n} y'^q$$

Hence, we have

$$\sum_{i=1}^{n} x_i' y_i' \leq \frac{1}{p} + \frac{1}{q} = 1$$

This proves the required inequality. □

Theorem 20. *(Minkowski's inequality) Let $x_i \geq 0$ and $y_i \geq 0$ for $i = 1, 2, \ldots, n$ and suppose that $p \geq 1$. Then*

$$\left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^q\right)^{1/q}$$

Proof. If $p = 1$, it is trivially true. So, assume $p > 1$, we have

$$\sum_{i=1}^{n} (x_i + y_i)^p = \sum_{i=1}^{n} x_i (x_i + y_i)^{p-1} + \sum_{i=1}^{n} y_i (x_i + y_i)^{p-1}$$

Apply Holder's inequality for both terms on RHS

$$\begin{aligned}
\sum_{i=1}^{n} (x_i + y_i)^p &\leq \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \left(\sum_{i=1}^{n} (x_i + y_i)^{(p-1)q}\right)^{1/q} \\
&\quad + \left(\sum_{i=1}^{n} y_i^p\right)^{1/p} \left(\sum_{i=1}^{n} (x_i + y_i)^{(p-1)q}\right)^{1/q} \\
&\leq \left[\left(\sum_{i=1}^{n} x_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p\right)^{1/p}\right] \left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{1/q}
\end{aligned}$$

Divide both sides by $\left(\sum\limits_{i=1}^{n} (x_i + y_i)^p\right)^{1/q}$ as it is $\neq 0$ and we will get the required inequality. For the case of $\left(\sum\limits_{i=1}^{n} (x_i + y_i)^p\right)^{1/q} = 0$, proof is self-evident. □

Theorem 21. *(Minkowski Inequality for infinite sums) Suppose that $p \geq 1$ and let $\{x_n\}_{n \geq 1}$ and $\{y_n\}_{n \geq 1}$ be sequences of non-negative terms such that $\sum\limits_{n=1}^{\infty} x_n^p$ and $\sum\limits_{n=1}^{\infty} y_n^p$ are convergent. Then $\sum\limits_{n=1}^{\infty} (x_n + y_n)^p$ is convergent. Moreover,*

$$\left(\sum_{n=1}^{\infty} (x_i + y_i)^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} x_i^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} y_i^q\right)^{1/q}$$

Proof. For any positive integer m, we can use Minkowski inequality for finite sums

$$\begin{aligned}\left(\sum_{n=1}^{m}(x_i+y_i)^p\right)^{1/p} &\leq \left(\sum_{n=1}^{m}x_i^p\right)^{1/p} + \left(\sum_{n=1}^{m}y_i^q\right)^{1/q}\\ &\leq \left(\sum_{n=1}^{\infty}x_i^p\right)^{1/p} + \left(\sum_{n=1}^{\infty}y_i^q\right)^{1/q}\end{aligned}$$

Since, the sequence $\{(\sum_{n=1}^{m}(x_i+y_i)^p)^{1/p}\}$ is increasing sequence(w.r.t m) of nonnegative real numbers and is bounded above by the sum in above equation. Hence, it is convergent and as $m \to \infty$ the limit is also bounded above by the sum. This proves the inequality. □

Theorem 22. *Let $p > 1$. For $a \geq 0$ and $b \geq 0$, we have*

$$(a+b)^p \leq 2^{p-1}(a^p + b^p)$$

Proof. TBD □

Metric Spaces

Definition 20. *Let $X \neq \phi$. A metric on X is a function $d : X \times X \to [0, \infty)$ such that*

1. *d(x,y) = 0 $\implies$ $x = y$*
2. *d(x,y) = d(y,x)*
3. *d(x,y) $\leq$ d(x,z) + d(z,y) for any $x, y, z \in X$.*

The metric space in which every cauchy sequence converges is called "complete" metric space.

Examples of metric spaces

1. **The space of all bounded sequences.** Let X be the set of all infinite sequences of numbers that are bounded $\{x_i\}_{i\geq 1}$ such that $sup|\{x_i\}| < \infty$. The metric on X is defined as $d(x,y) = sup_i|x_i - y_i|$ ($\leq sup_i|x_i - z_i| + sup_i|y_i - z_i|$).

2. **The space l_p.** Let X be the set of all sequences $x = \{x\}_{i\geq 1}$ such that

$$\left(\sum_{i=1}^{\infty}|x_p|^p\right)^{1/p} < \infty, \quad p \geq 1$$

   If $\{x\}_{n\geq 1}$ and $\{y\}_{n\geq 1}$ are two sequences belong to X then we define the metric as

$$d(x,y) = \left(\sum_{i=1}^{\infty}|x_i - y_i|^p\right)^{1/p}$$

   Minkowski inequality for infinite sequences makes it a suitable metric.

3. **The space of bounded functions.** Let S be any nonempty set and $\mathcal{B}(\mathcal{S})$ denote the set of all real or complex-valued functions on S, each of which is bounded. i.e.

$$\sup_{x\in S}|f(x)|<\infty$$

It can also be shown that $\sup_{x\in S}|f(x)-g(x)|<\infty$. We defined metric as $d(f,g)=\sup_{x\in S}|f(x)-g(x)|, \quad f,g\in\mathcal{B}(\mathcal{S})$. The metric d is called **uniform metric** or **supremum metric**.

4. **The space of continuous functions.** Let X be the set of all continuous functions defined on $[a,b]$, an interval in $\mathbb{R}$. For $f,g\in X$, define

$$d(f,g)=\sup_{x\in[a,b]}|f(x)-g(x)|$$

We can also defined another metric on the set X as

$$d(f,g)=\int_a^b|f(x)-g(x)|dx$$

Assume $h(x)\in X=C[a,b]$, we have

$$\begin{aligned}d(f,g)&=\int_a^b|f(x)-h(x)+h(x)-g(x)|dx\\&\le\int_a^b\left(|f(x)-h(x)|+|h(x)-g(x)|\right)dx\\&\le d(f,h)+d(h,g)\end{aligned}$$

5. **Metric on extended real line.** Let $X=\mathbb{R}\cup\{\infty\}\cup\{-\infty\}$. Define $f:X\to\mathbb{R}$ by the rule

$$f(x)=\begin{cases}\frac{x}{1+|x|} & \text{if } -\infty<x<\infty\\ 1 & \text{if } x=\infty\\ -1 & \text{if } x=-\infty\end{cases}$$

We define metric using f as

$$d(x,y)=|f(x)-f(y)|,\quad x,y\in X$$

Proof. For injectivity, if $f(x)=f(y)$, we need to prove that $x=y$. Notice that $xy>0$ is the only possible case. Since, $xy<0$ (x and y are of opposite sign) is not possible because then $f(x)\neq f(y)$.

$$\begin{aligned}\frac{x}{1+|x|}&=\frac{y}{1+|y|}\\x+x|y|&=y+y|x|\\x&=y\quad\because \text{xy}>0\end{aligned}$$

Hence, f is one - one function. The metric d is positive, $d(x,y)=d(y,x)$ $d=0\implies f(x)=f(y)\implies x=y$ ($\because f$ is one-one). On the other hand, if

$x = y$ then it is obvious that $f(x) = f(y) \implies d = 0$.
For triangle inequality, we can write

$$\begin{aligned}
\left|\frac{x}{1+|x|} - \frac{1}{1+|y|}\right| &= \left|\frac{x}{1+|x|} - \frac{z}{1+|z|} + \frac{z}{1+|z|} - \frac{y}{1+|y|}\right| \\
&= \left|\left(\frac{x}{1+|x|} - \frac{z}{1+|z|}\right) + \left(\frac{z}{1+|z|} - \frac{y}{1+|y|}\right)\right| \\
&\leq \left|\frac{x}{1+|x|} - \frac{z}{1+|z|}\right| + \left|\frac{z}{1+|z|} - \frac{y}{1+|y|}\right| \\
&\leq d(x,z) + d(z,y)
\end{aligned}$$

Hence, it is a metric. □

6. In the above example, we can choose $f(x) = \tan^{-1} x$. Since, it is one - one function, and $\tan^{-1}(\infty) = \pi/2, \tan^{-1}(-\infty) = -\pi/2$. Then the metric will be

$$d(x,y) = |\tan^{-1} x - \tan^{-1} y|, \quad x, y \in X$$

7. Let d be a metric on nonempty set X. The function defined as

$$e(x,y) = \min\{1, d(x,y)\}, \qquad \forall x, y \in X$$

is a metric.

Proof. The above expression says two things $e(x,y) \leq 1$ and $e(x,y) \leq d(x,y)$. We need to prove that $e(x,y) \leq e(x,z) + e(z,y)$. Since, if either of the terms on RHS or both are less than 1, the inequality is satisfied because $e(x,y) \leq 1$. If both the terms on RHS are less than 1 then, we have

$$e(x,y) \leq \min\{1, d(x,y)\} \leq d(x,y) \leq d(x,z) + d(z,y) = e(x,z) + e(z,y)$$

□

Proposition 12. *Let (X,d) be a metric space. Define $d' : X \times X \to \mathbb{R}$ by*

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

Then d' is a metric on X. Besides, $d'(x,y) < 1$ for all $x, y \in X$.

Proof. All axioms of metric spaces are satisfied, we just need to show the triangle inequality. We need to prove that $d'(x,y) \leq d'(x,z) + d'(z,y)$. Since, we know that $\frac{x}{1+x}, x \geq 0$ is monotonically increasing sequence 10. And we also know that $d(x,y) \leq d(x,z) + d(z,y)$. Hence, combining these two arguments, we have

$$\begin{aligned}
d'(x,y) = \frac{d(x,y)}{1+d(x,y)} &\leq \frac{d(x,z) + d(z,y)}{1 + d(x,z) + d(z,y)} \\
&= \frac{d(x,z)}{1 + d(x,z) + d(z,y)} + \frac{d(z,y)}{1 + d(x,z) + d(z,y)} \\
&\leq \frac{d(x,z)}{1 + d(x,z)} + \frac{d(z,y)}{1 + d(z,y)} (\because d(x,z), d(z,y) \geq 0) \\
&= d'(x,z) + d'(z,y)
\end{aligned}$$

□

Examples The space of all sequences of numbers. Let X be the space of all sequence of numbers. Let $x = \{x_i\}_{i\geq 1}$ and $y = \{y_i\}_{i\geq 1}$ be elements of X. Define

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}$$

is a metric.

Proof. All the metric space axioms are satisfied, we only need to show for triangle inequality. Since, we know that $\frac{1}{2}, \frac{1}{2^2}, \ldots$ are positive numbers hence we can conclude that

$$\frac{1}{2}\frac{|x|}{(1+|x|)} < \frac{1}{2}\frac{|y|}{(1+|y|)}, \qquad x < y$$

We can sum both the sides to a similar inequality as above and the fact that $|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$ makes it clear to get the required result. $\square$

Remark. *The above expression is not true in $\mathbb{R}^2$. In $\mathbb{R}^2$, if we define metric $d(x,y) = |x_2 - y_2|$ and take $x = (0,0)$ and $y = (1,0)$ then we have $d(x,y) = 0$ but $x \neq y$. Hence, d is not a metric in $\mathbb{R}^2$.*

Definition 21. *Pseudometric: Let X be a nonempty set. A pseudometric on X is a mapping of $X \times X \to \mathbb{R}$ that satisfies the axioms*

1. $d(x,y) \geq 0$.
2. $d(x,y) = 0 \;\; if \;\; x = y$
3. $d(x,y) = d(y,x), \quad \forall x, y \in X$
4. $d(x,y) \leq d(x,z) + d(z,y), \quad \forall x, y, z \in X$.

Example Let X be the set of all Riemann integrable functions on $[a,b]$. For $f, g \in X$, define

$$d(f,g) = \int_a^b |f(x) - g(x)| dx$$

This is a pseudometric on X but it is not a metric on X.
Counter example
Let

$$f(x) = \begin{cases} 0 & \text{if } \; x = a \\ 1 & \text{if } \; a < x \leq b \end{cases}$$

and $g(x) = 1 \;\; \forall x \in [a,b]$. We can see that $d(f,g) = 0$ but $f \neq g$.

A equivalence relation on a pseudometric space generates a metric space. Let (X,d) be a pseudometric space. Define a relation R on X by

$$xRy \;\; \text{if and only if} \;\; d(x,y) = 0$$

Then the relation R is an equivalence relation on X (for transitivity, xRy,yRz then use triangle inequality for d to get $d(z,x) \leq 0 \implies d(z,x) = 0$).

The set of all equivalence classes denoted by X/R forms a metric space with the metric defined as

$$\tilde{d}([x],[y]) = d(x,y)$$

where $x \in [x], y \in [y]$ and the metric is independent of the choice of the representatives.

Proof. (For the statement that the metric is well defined and it does not depend on the choice of the representative.)
Let $x, x' \in [x]$ and $y, y' \in [y]$. We need to show that $d(x,y) = d(x',y')$. Using triangle inequality, we can say

$$\begin{aligned} d(x,y) &\leq d(x,x') + d(x',y') + d(y',y) \\ d(x,y) &\leq d(x',y') \quad (\because x \sim x', y \sim y') \end{aligned}$$

By similar argument, we can show $d(x',y') \leq d(x,y)$. Hence, $d(x',y') = d(x,y)$ and the metric does not depend upon the choice of representative. It is easy to show that this is a metric. $\square$

Sequences in metric spaces

Definition 22. *Let d be a metric on a set X and $\{x_n\}$ be a sequence in the set X. An element $x \in X$ is said to be a limit of $\{x_n\}$ if, for every $\epsilon > 0$, there exist a natural number n_0 such that for all $n \geq n_0$,*

$$d(x_n, x) < \epsilon$$

Hence, $\{x_n\}$ converges to x.

Remark. *The limit x of the sequence is always unique.*

Proof. Suppose there exists two limit points for the sequence $\{x_n\}$, x_1 and x_2. This means for given $\epsilon > 0$, there exists n_1, n_2 such that $d(x_n, x_1) < \frac{\epsilon}{2}\ \ \forall n_1 \geq n_0$ and $d(x_n, x_2) < \frac{\epsilon}{2}\ \forall n_2 \geq n_0$. Using triangle inequality, $d(x_1,x_2) \leq d(x_2,x_n) + d(x_n,x_1) = \epsilon$. Hence $x_1 = x_2$. $\square$

Examples

1. Let $X \in \mathbb{R}^n$ with metric

$$d(x,y) = d_p(x,y) = \left(\sum_{j=1}^{n} |x_j - y_j|^p\right)^{1/p},$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are in $\mathbb{R}^n$ and $p \geq 1$. Let $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$ be a sequence in $\mathbb{R}^n$ converges to $x = (x_1, x_2, \ldots, x_n)$.

$$\lim_{k\to\infty} \left(\sum_{j=1}^{n} |x_j^{(k)} - x_j|\right)^{1/p} = \epsilon \ \ \forall k \geq k_0$$

For every $\epsilon > 0$, there exists an integer $k_0(\epsilon)$ such that $|x_j^{(k)} - x_j| < \epsilon\ \ \forall k \geq k_0$. Hence, the sequence converges coordinatewise.

On the other hand, if for each $j = 1, 2, \ldots, n$ there exists the integers k_j such that $|x_j^{(k)} - x_j| < \frac{\epsilon}{n^{1/p}}$ for all $k \geq k_j$ then $k' = \max(k_1, k_2, \ldots, k_n)$ we have

$$\left(\sum_{j=1}^{n} |x_j^{(k)} - x_j| \right)^{1/p} < \epsilon \quad \forall k \geq k'$$

Thus, convergence of sequences in $(\mathbb{R}^n, d_p)$ is equivalent to coordinatewise convergence.

2. If metric is $d_\infty(x_n, x) = \max_j |x_j^{(k)} - x_j|$ ($\{x_n\}_{n\geq 1}$ same as above example) then for every $\epsilon > 0$ there exists an integer k_0 such that we have $\max_j |x_j^{(k)} - x_j| < \epsilon$ for all $k \geq k_0$. Hence, $|x_j^{(k)} - x_j| < \epsilon$ for each $j = 1, 2, \ldots, n$. On the other hand, if for each $j = 1, 2, \ldots, n$, there exists an integer k_j such that $|x_j^{(k)} - x_j| < \epsilon$ for $k \geq k_j$ then we can choose $k' = \max(k_1, k_2, \ldots, k_n)$ so that

$$\max_j |x_j^{(k)} - x_j| < \epsilon \quad \forall k \geq k'$$

Hence, in $(\mathbb{R}^n, d_\infty)$ convergence **implies** coordinatewise convergence.

3. Similarly, convegence in the metric space with discrete metric also **implies** coordinatewise convergence.

4. For space of all sequences with metric

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j^{(k)} - x_j|}{1 + |x_j^{(k)} - x_j|}$$

TBD

5. In this example **coordinatewise convergence does not imply convergence.**
Let $X = l_p (p \geq 1)$ and let $d(x, y) = (\sum_{k=1}^{\infty} |x_k - y_k|^p)^{1/p}$, where $\{x_k\}_{k\geq 1}$ and $\{y_k\}_{k\geq 1}$ are in l_p. Let $\{\{x_k\}_{k\geq 1}\}_{n\geq 1}$ be a sequence in l_p that converges to $x \in l_p$

$$\lim_{n\to\infty} d(x^{(n)}, x) = \lim_{n\to\infty} \left(\sum_{k=1}^{\infty} |x_k^{(n)} - x_k|^p \right)^{1/p}$$

Then for $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$\left(\sum_{k=1}^{\infty} |x_k^{(n)} - x_k|^p \right)^{1/p}$$

for all $n \geq n_0(\varepsilon)$. Since $\varepsilon > 0$, it follows that $\lim_{n\to\infty} x_k^{(n)} = x_k$ for each k.
The converse is not true. Counter example: Let $x_k^{(n)} = x_k + \delta_{kn}$ where

$$\delta_{kn} = \begin{cases} 0 & \text{if } \; k \neq n \\ 1 & \text{if } \; k = n \end{cases}$$

Here $|x_k^{(n)} - x_k| = \delta_{kn} = 0$ if $n > k$. Hence, $\lim_{n\to\infty} x_k^{(n)} = x_k$ for each k. But

$$d(x^{(n)}, x) = \left(\sum_{k=1}^{\infty} |x_k^{(n)} - x_k|^p \right)^{1/p} = 1$$

for each n.. Thus $x^n \nrightarrow x$ in the l_p metric.
But the below theorem holds.

Theorem 23. *Let $\{x^{(n)}\}n \geq 1$ be a sequence in l_p such that $\lim_{n\to\infty} x_k^{(n)} = x_k$ for each k, where $x = \{x_k\}_{k\geq 1}$ is an element of l_p. Suppose also that for every $\varepsilon > 0$ there exists an integer $m_0(\varepsilon)$ such that*

$$\left(\sum_{k=m+1}^{\infty} |x_k^{(n)}|^p \right)^{1/p} < \varepsilon$$

for $m \geq m_0(\varepsilon)$ and for all n. Then $\lim_{n\to\infty} d(x^{(n)}, x) = 0$.

Proof. TBD(cumbersome to write) □

Cauchy Sequences

Definition 23. *Let d be a metric on set X. A sequence $\{x_n\}_{n\geq 1}$ in the set X is said to be cauchy if, for every $\varepsilon > 0$ there exists a positive integer n_0 such that $d(x_n, x_m) < \varepsilon$ for all $n \geq n_0$ and $m \geq n_0$.*

Remark. *In $\mathbb{R}$ and $\mathbb{C}$, a sequence is convergent if and only if it is cauchy.*

Remark. *In general metric spaces, every convergent sequence is cauchy, converse is not necessarily true.*

Proposition 13. *A convergent sequence in a metric space is a Cauchy sequence.*

Proof. Assume X is a metric space and d is the metric defined on X. Let $\{x_n\}_{n\geq 1}$ is a convergent sequence and x is the limit. For given $\varepsilon > 0$ there exists a positive integer n_0 such that $d(x_n, x) < \varepsilon$ for all $n \geq n_0$. Suppose $n, m \geq n_0$ such that $d(x_n, x) < \varepsilon$ and $d(x_m, x) < \varepsilon$. Then using triangle inequality, we can say $d(x_n, x_m) < \varepsilon$ hence the sequence is cauchy. □

Converse, is however, not true. Counter example: for the set X with metric $|x - y|$, sequence of rationals such as $1.4, 1.41, 1.414, \ldots$ limit is $\sqrt{2}$ (means its converging and therefore its cauchy) which is irrational hence sequence is converging to a limit which does not belong to the set of rationals.
Hence, Cauchy sequence does not need to converge to a point in the space.

Definition 24. *A metric space (X, d) is said to be complete if every Cauchy sequences in X is convergent.*

Example: $\mathbb{R}, \mathbb{C}$ and $\mathbb{R}^n$ are complete with their usual metrics. A set of rationals with metric $|x - y|$ for all $x, y \in \mathbb{Q}$ is incomplete as there exits a sequence in $\mathbb{Q}$ $(1.4, 1.41, 1.414, \ldots)$ converges to $\sqrt{2}$ which does not belong to $\mathbb{Q}$.

Definition 25. *Let $\{x_n\}_{n\geq 1}$ be a given sequence in a metric space (X, d) and let $\{n_k\}_{k\geq 1}$ is sequence of positive integer $n_1 < n_2 < \dots$ Then the sequence $\{x_{n_k}\}_{k\geq 1}$ is called a subsequence of $\{x_n\}_{n\geq 1}$. If the subsequence converges then its limit is called subsequential limit.*

Remark. *A sequence $\{x_n\}_{n\geq 1}$ in X converges to x if and only if all of its subsequences converges x.*

Proposition 14. *If a cauchy sequence of points in a metric space (X, d), contains a convergent subsequence, then the sequence converges to the same limit as the subsequence.*

Proof. Let $\{x_n\}_{n\geq 1}$ be the cauchy sequence in the metric space (X, d) then for given $\varepsilon > 0$ there exists a positive integer n_0 such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$. Let $\{x_{n_k}\}_{k\geq 1}$ be the convergent subsequence and its limit to be x_n. Therefore,

$$d(x_{n_m}, x_n) < \varepsilon \quad \forall n, m \geq n_0$$

Using triangle inequality, we can say (let x is limit of main sequence)

$$d(x, x_n) \leq d(x, x_{n_m}) + d(x_{n_m}, x_n) < d(x, x_{n_m}) + \varepsilon$$

whenever $m, n \geq n_0$. Taking $m \to \infty$,

$$d(x, x_n) < \varepsilon$$

whenever $n \geq n_0$. So, $\{x_n\}_{n\geq 1}$ converges to x. ($d(x, x_{n_m}) < \varepsilon$ since, $\{x_{n_m}\}$ is a convergent subsequence $\implies$ cauchy and distance between two is less than ε for cauchy sequences.) $\square$