Lecture 1

Remarks: Basic ordering properties are assumed to be true.

Definition 1. *Boundedness: The subset $A \in R$ is said to be bounded above if $\exists M$ such that $M > x \ \ \forall x \in A$. And it is bounded below if $\exists m$ such that $m < x \ \ \forall x \in A$. If A has both then it is called bounded.*

Definition 2. *Least Upper Bound (lub) Axiom: If A is nonempty subset of R and it is bounded above, then A has a least upper bound in R.*

Theorem 1. *If A is nonempty subset in R and it is bounded below, then it has a greatest lower bound in R.*

Proof. We first create a set T of lower bounds of A

$$T = \{m \mid m < x \ \ \forall x \in A\}$$

T is non-empty since A is bounded below. Now, we need to prove that there exits a supremum of T which is also a lower bound of A.
Since, set T is bounded above by all the elements of set A, it should have a least upper bound, say M such that $M > m \ \ \forall m \in T$. Also, every element of A is an upper bound of T hence by definition of supremum, we can say $M \leq x \ \ \forall x \in A$ hence M is the lower bound of A. This makes it the greatest lower bound. $\square$

Lemma 1. *Suppose $A \neq \phi$ and $s = lub(A)$ then for any y such that $y < s$, $\exists a \in A$ such that $y < a \leq s$.*

Proof. Suppose for contradiction, $\nexists$ any element a such that $y < a$. This means that $y \geq a, \forall a \in A \implies y$ is upper bound of set A. But y is already less than least upper bound of set A. Hence contradiction.
Therefore, $\exists \ a \in A$ such that $y < a \leq s$. $\square$

Theorem 2. *Archimedean Property: Given any positive real numbers $x, y \ \exists \ n \in N$ such that $nx > y$.*

Proof. Let a set $A = \{nx \mid n \in \mathbb{N}\}$. Suppose for contradiction $nx \leq y$. Then y is the upper bound of the set A.
Let a $x > 0$, then $y - x < y$ hence $y - x$ is not the upper bound of the set A. This means that $\exists m \in \mathbb{N}$ such that $y - x < mx \implies y < mx + x \implies y < (m+1)x$ which is impossible since $(m+1)x \in A$ and y is upper bound of the A.
$nx > y$ is true. $\square$

Theorem 3. *If A and B are the two non empty bounded subsets of R, such that $x \leq y \ \ \forall x \in A$ and $\forall y \in B$ then $sup(A) \leq inf(B)$*

Proof. Let a be the supremum of A and b be the infimum of B. Therefore, $a \geq x \ \ \forall x \in A$ and $b \leq y \ \ \forall y \in B$. Also, A is bounded above by B and elements of B are the upper bound for A. Hence, $a \leq y \ \ \forall y \in B$. This means that a is the lower bound of B and a is $sup(A)$. In other words, $sup(A) \leq inf(B)$. $\square$

Theorem 4. *Given any two real number a, b with $a < b$, $\exists \ \mathbb{Q}$ between a and b.*

Proof. Since $b - a > 0$. Take two positive number $b - a$ and 1 $\exists$ n $\in \mathbb{Z}$ such that $n(b - a) > 1$.
TBD $\square$

Theorem 5. *Any monotone increasing sequence of real numbers that is bounded above converges to some real number.*

Proof. Let x_n be a monotone increasing sequence in $\mathbb{R}$ that is bounded above hence there exits a s such that $s = lub\{x_n \mid n \in \mathbb{N}\}$
Suppose $\epsilon > 0 \implies s - \epsilon < s$ and $s - \epsilon$ is not the upper bound of the x_n.
Using lemma 1, we can say that $\exists x_\epsilon \in x_n$ such that $s - \epsilon < x_\epsilon < s$.
Using monotone condition, for some $n_0 \in \mathbb{N}$, we have,

$$s - \epsilon < x_n < s < s + \epsilon \quad \forall n > n_0 \in \mathbb{N}$$

Hence $|x_n - s| < \epsilon$. x_n converges to s $\;\forall n > n_0$. $\square$

Remark. *Nested Interval theorem* $\approx$ *lub* $\approx$ *theorem* 4

Proof. TBD $\square$

Theorem 6. *Nested Interval theorem: Suppose $\{I_n\}$ is the sequence of closed and bounded non-empty intervals such that $I_1 \supset I_2 \supset I_3 \dots$ then:*

1. $\bigcap\limits_{n\geq 1} I_n \neq \phi$.

2. *If the sequence of the length of the intervals goes to* 0 *then* $\bigcap\limits_{n\geq 1} I_n = \{x\}$.

Proof. Let I_n be an interval $[a_n, b_n]$ with $a_m < b_n \forall m, n \in \mathbb{N}$. Then $\forall n \in \mathbb{N}$, a_n is the increasing sequence and b_n is the decreasing sequence. b_n is upper bound of a_n hence, $a_n < inf(b_n)$.
For b_n, a_n is the lower bound of b_n i.e. $sup(a_n) < b_n$. If we combine all inequalities, we get

$$a_n \leq sup(a_n) \leq inf(b_n) \leq b_n \quad \forall n \in \mathbb{N}$$

Using density theorem, we can say that $\exists$ some $\mathbb{Q}$ between $sup(a_n)$ and $inf(b_n)$.
Hence, $\bigcap\limits_{n\geq 1} I_n \neq \phi$.
Let the length of the interval to be $L = |b_n - a_n|$. Suppose for contradiction, we have two elements in $\bigcap\limits_{n\geq 1} I_n$ instead of one, say x and y.
The distance between x and y is $|y - x|$. Since, $L \to 0$ hence $\exists n \in \mathbb{N}$ such that for some $n_0 \geq n$, $|L| = |b_{n_0} - a_{n_0}| < \epsilon$ for some $\epsilon > 0$. Since $|L| \to 0$, we can choose ϵ such that it is smaller than $|y - x|$. Then, if interval contains any one of the point, it can not contain the other. $\square$

Lecture 2

Definition 3. *Decimals representation of Real numbers: Let $z \in \mathbb{R}^+$ be given. Let n_0 be the largest integer such that $n_0 \leq z$. Let n_1 be the largest integer such that $n_0 + \frac{n_1}{10} \leq z$. As such, say n_k is defined for some k. Let n_{k+1} be the largest integer* *such that* $n_0 + \frac{n_1}{10^1} + \frac{n_1}{10^2} + \cdots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \leq z$. *Consider the set of all such finite sums, i.e. the set of all*

$$z_k = n_0 + \frac{n_1}{10^1} + \frac{n_1}{10^2} + \cdots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \leq z$$

This set has a supremum and that is z *itself. We symbolically write* $z = n_0.n_1n_2\ldots$

Lemma 2. *Let* p *be an integer* ≥ 2. *If* $0 \leq a_n \leq p-1$, *where* a_n *is an integer then* $\sum\limits_{n=0}^{\infty} \frac{a_n}{p^n}$ *converges to some* x *in* $[0,1]$.

Proof. Since $0 \leq a_n \leq p-1$, we can replace all a_n with $p-1$ and then we have

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n} \leq (p-1) \sum_{n=1}^{\infty} \frac{1}{p^n} = 1$$

Therefore, the sequence is bounded, and it is monotonic increasing this means it converges to some $x \leq 1$ and x is already positive. Hence it converges tp some x in $[0,1]$. $\square$

Lemma 3. *Conversely, given any* $0 \leq x \leq 1, \exists\ a_n \in \mathbb{Z}$ *and* $0 \leq a_n \leq p-1$ *such that* $x = \sum\limits_{n=0}^{\infty} \frac{a_n}{p^n}$.

Proof. Suppose we have $0 < x \leq 1$ and a_1 is the largest integer such that $\frac{a_1}{p} < x \leq 1$. Since x is bounded above by 1, we have $a_1 < p \implies a_1 \leq p-1$ since a_1 is an integer. Similarly, find a_2 such that $\frac{a_1}{p} + \frac{a_2}{p^2} < x$.This can be achieved by Archimedean property. Also, note that $a_2 \leq p-1$, since we have

$$\begin{aligned}
\frac{a_1}{p} + \frac{a_2}{p^2} &< x < 1 \\
\frac{a_1}{p} + \frac{a_2}{p^2} \leq \frac{p-1}{p} + \frac{a_2}{p^2} &< 1 \qquad (a_1 \leq p-1) \\
1 - \frac{1}{p} + \frac{a_2}{p^2} &< 1 \\
\frac{a_2}{p^2} &< \frac{1}{p} \\
a_2 &< p \\
a_2 &\leq p-1
\end{aligned}$$

Inductively, we can define a_n as the largest integer with $a_n \leq p-1$ such that $\sum\limits_{i=1}^{n} \frac{a_i}{p_i} < x$. Since $a_n < p$ TBD $\square$

Suppose $\{a_n\}$ is the bounded sequence in $\mathbb{R}$, we define two sets:

$$\begin{aligned}
s_n &:= inf\{a_n, a_{n+1}, \ldots\} \\
S_n &:= sup\{a_n, a_{n+1}, \ldots\}
\end{aligned}$$

Notice that $\underset{k}{inf}(\{a_n\}) \leq s_n \leq S_n \leq \underset{k}{sup}(\{s_n\})$

Definition 4. *Limit superior and limit inferior: Let a_n be the bounded sequence of real numbers then*

$$\lim inf(a_n) = \lim_{n\to\infty} s_n$$
$$\lim sup(a_n) = \lim_{n\to\infty} S_n$$

Limit superior is the supremum of all subsequential limits of $\{a_n\}$. Similarly, limit inferior is the infimum of all subsequential limits of $\{a_n\}$.

Note that s_n is the increasing sequence and S_n is the decreasing sequence and they are bounded on both sides. Hence, we can also say that using monotone convergence theorem

$$\lim inf(a_n) = \lim_{n\to\infty} s_n = sup(s_n)$$
$$\lim sup(a_n) = \lim_{n\to\infty} S_n = inf(S_n)$$

Theorem 7. *A sequence $\{a_n\}$ is bounded above iff limsup $a_n < \infty$ (is finite).*

Proof. TBD □

Theorem 8. *A sequence $\{a_n\}$ is bounded below iff liminf $a_n < \infty$ (is finite).*

Proof. TBD □

Theorem 9. *Given any sequence $\{a_n\}$ there exists a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to limsup(a_n)$.*

Proof. TBD □

Theorem 10. *Given any sequence $\{a_n\}$ there exists a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to liminf(a_n)$.*

Proof. TBD □

Lecture 3

Definition 5. *Finite Set: A set is finite if there exists a bijection between the set and the $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.*

Theorem 11. $\mathbb{N}$ *is an infinite set.*

Proof. Negation of above definition would be a set is infinite if there does not exists a bijection between $\{1, 2, \ldots, n\}$ and $\mathbb{N}$. Suppose a function $f(\{1, 2, \ldots, n\}) \to \mathbb{N}$. We can have a natural number $f(1) + f(2) + \cdots + f(n) > f(i) \quad \forall i \in \mathbb{N}$ that does not have a preimage in $\{1, 2, \ldots, n\}$ hence the map is not bijective. □

Theorem 12. *A set is infinite iff there exists a one-one map from $\mathbb{N}$ to the set.*

Proof. $\implies$) For $1 \in \mathbb{N}$ there exist a image in X say $f(1)$. Now, take $2 \in \mathbb{N}$ such that there exist a image $f(2) \in X \backslash \{1\}$. This means that $f(1) \neq f(2)$. Since for every $n \in \mathbb{N}$ we can have $f(n)$ in $X \backslash \{1, 2, \ldots, n-1\}$ as X is also infinite. Hence, we have constructed a one-one map from $\mathbb{N} \to X$.
($\impliedby$ Since $\mathbb{N}$ is infinite and we have a one-one mapping from $\mathbb{N} \to X$ therefore for each $n \in \mathbb{N}$, we have only one $f(n) \in X$ and every $n \in \mathbb{N}$ has a image (it's a map). Hence, X is infinite. □

Definition 6. *Equivalent or Equipotent set: Two sets are equivalent or equipotent if there exists bijection between X and Y.*

For example: Finite sets are equivalent to $\{1, 2, \ldots, n\}$ for some fixed $n \in \mathbb{N}$.

Definition 7. *Countably infinite set: A infinite set X is said to be countably infinite if there exists a bijection between X and $\mathbb{N}$.*

Definition 8. *Uncountably infinite set: A set is said to be uncountably infinite if it is not countably infinite set.*

Example:

1. Countably infinite: bijective map between $\mathbb{Z} \to \mathbb{N}$. The map will look line

$$n \to \begin{cases} \frac{n}{2} & n \in even \\ \frac{-(n+1)}{2} & n \in odd \end{cases}$$

2. $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is also equivalent to $\mathbb{N}$ i.e. countably infinite. Map will be $(m \times n) \to 2^m(2n-1)$ m and n are unique and m such that 2^m is the maximum multiple of 2.

Theorem 13. *If A is an infinite subset of $\mathbb{N}$ then there exists bijection between A and $\mathbb{N}$.*

Proof. TBD □

Corollary 1. *Monotone Subsequence theorem: Any sequence $\{x_n\}$ of real numbers has a monotone subsequence*

Proof. We define "peak" as any element x_m is called a peak if $x_m \geq x_n$ for all $n > m$. There cases can be two possible cases

1. Infinite peaks: This means that there exists $m_i's$ say $\{m_1, m_2, \ldots\}$ such that $x_{m_i} > x_n$ for all $n > m_i$, and for all $i \in \mathbb{N}$. We can arrange $m_i's$ in increasing order $m_1 < m_2 < \ldots$ and $x_{m_1} > x_{m_2} > x_{m_3} > \ldots$ is a decreasing subsequence.

2. Finite peaks: (0 or some $n \in \mathbb{N}$). Assume the peaks are $\{x_{m_1}, x_{m_2}, \ldots, x_{m_n}\}$, this means that there exists some $s_1 = m_n + 1, s \in \mathbb{N}$ such that x_{s_1} is not a peak. Therefore, there also exists some $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$. Because of finite peaks x_{s_2} is also not a peak, hence for some $s_3 > s_2$, we have $x_{s_2} < x_{s_3}$. Proceeding with induction, we have $x_{s_1} < x_{s_2} < x_{s_3} < \ldots$ is a increasing sequence.

□

Theorem 14. *$\mathbb{Q}$ is a countably infinite set.*

Proof. TBD □

Theorem 15. *Any interval in $\mathbb{R}$ is an uncountable set.*

Proof. TBD □

Definition 9. *Suppose $X \neq \phi$. A partial order on X is a relation R on X such that R is*

1. *Reflexive.*
2. *Anti-symmetric $\implies aRb, bRa \implies a = b$*
3. *Transitive.*

Examples:

1. $R =$ " $\leq$ " is a partial order.

$\mathcal{P}(\mathcal{A})$ is power set of A and $X, Y \subset A$ then $X \leq Y$ iff $X \subseteq Y$.

Lecture 4

Definition 10. *Given $E \subset X$ where X is partially order set, we say E is totally ordered if any two elements of E are comparable i.e. if $e_1, e_2 \in E$, then $e_1 \leq e_2$ or $e_2 \leq e_1$. Totally ordered $\equiv$ linearly order $\equiv$ chain.*

Definition 11. *Upper bound of E: An element is $x \in X$ is called upper bound of E if for any $x' \in E$, we have $x' \leq x$. x is called the maximal element if $x' \geq x \implies x' = x$.*
For maximal element, x should be an upper bound for set E and x should belong to E.

Let $X \neq \phi$. $\mathcal{F}$ is a collection of subsets of X (element of $\mathcal{P}$(X)). An element $F \in \mathcal{F}$ is a upper bound for a subfamily $\mathcal{F}'$ of $\mathcal{F}$ provided every member of $\mathcal{F}'$ is a subset of F.
F will be the maximal element of $\mathcal{F}$ if it is not a proper subset (means not contained in) of any member in $\mathcal{F}$.

Lemma 4. *Zorn's lemma: Let X be a partially order set. If every totally ordered subset of X is bounded above then X has a maximal element.*

Definition 12. *Cardinality of X : Two sets A and B have same cardinality if there exists a bijection between them. Set X has cardinal number α means that there exists a set Y equivalent to X with number of elements equal to α.*

If α and β are cardinal numbers of set X and set Y such that $\alpha \leq \beta$ then there exists a one-one mapping from $X \to Y$.

Theorem 16. *Cantor-Schroeder-Bernstein theorem: If there exists a one-one mapping from $X \to Y$ and $Y \to X$ then there exists a bijection between X and Y.*

Limits of functions

Definition 13. *Limit point: A point $a \in \mathbb{R}$ is called a limit point of a set $X \subseteq \mathbb{R}$ if for every neighbourhood $(a - \epsilon, a + \epsilon), \epsilon > 0$ there exists $x \in X$ such that $a \neq x$.*

For a function f defined on $X \subseteq \mathbb{R}$, f converges to some l means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - l| < \epsilon \ \forall x \in X \ni |x - a| < \delta$$

Limit of functions as limit of sequences-

Proposition 1. *Let $f : X \to \mathbb{R}$ and let a be a limit of X. Then $lim_{x\to a}f(x) = l$ if and only if for every sequence $\{x_n\}_{n\geq 1}$ in X that converges to a and $x_n \neq a$ for all n, the sequence $\{f(x_n)\}_{n\geq 1}$ converges to l.*

A function f is continuous on X if it is continuous in every point in X.

Definition 14. *Continuity of f: A function f is said to be continuous at some point $x \in X$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $y \in X$ and $|y - x| < \delta$, we have $|f(y) - f(x)| < \epsilon$.*

A function f is continuous at a limit point $a \in X$ if and only if $f(a)$ is defined and $lim_{x\to a}f(x) = f(a)$.

Proposition 2. *Let f be a real valued funtion defined on subset X of $\mathbb{R}$ and $a \in X$ is the limit point of X. Then f is continuous at a if and only if for every sequence $\{x_n\}_{n\geq 1}$ that converges to a and $x_n \neq a$ for every n, we have $lim f(x_n) = f(lim x_n) = f(a)$. Continuous function preserve convergence (maps convergent sequence into convergent sequences).*

Theorem 17. *Bolzano intermediate value theorem: Let I be an interval and $f : I \to \mathbb{R}$, if $a, b \in I$ and $\alpha \in \mathbb{R}$ satisfies $f(a) < \alpha < f(b)$ or $f(a) > \alpha > f(b)$ then there exists a point $c \in I$ between a and b such that $f(c) = \alpha$.*

Sequences of functions

Let $X \subseteq \mathbb{R}$. If for every $n = 1, 2, \dots,$ we assigned a real valued function f_n defined on X then $\{f_n\}_{n\geq 1}$ is called sequence of functions.

Metric Spaces

Definition 15. *Let $X \neq \phi$. A metric on X is a function $d : X \times X \to [0, \infty)$ such that*

1. *d(x,y) = 0 $\implies$ $x = y$*
2. *d(x,y) = d(y,x)*
3. *d(x,y) $\leq$ d(x,z) + d(z,y) for any $x, y, z \in X$.*