Lecture 1

Remarks: Basic ordering properties are assumed to be true.

Definition 1. *Boundedness: The subset $A \in R$ is said to be bounded above if $\exists M$ such that $M > x \ \ \forall x \in A$. And it is bounded below if $\exists m$ such that $m < x \ \ \forall x \in A$. If A has both then it is called bounded.*

Definition 2. *Least Upper Bound (lub) Axiom: If A is nonempty subset of R and it is bounded above, then A has a least upper bound in R.*

Theorem 1. *If A is nonempty subset in R and it is bounded below, then it has a greatest lower bound in R.*

Proof. We first create a set T of lower bounds of A

$$T = \{m \mid m < x \ \ \forall x \in A\}$$

T is non-empty since A is bounded below. Now, we need to prove that there exits a supremum of T which is also a lower bound of A.
Since, set T is bounded above by all the elements of set A, it should have a least upper bound, say M such that $M > m \ \ \forall m \in T$. Also, every element of A is an upper bound of T hence by definition of supremum, we can say $M \leq x \ \ \forall x \in A$ hence M is the lower bound of A. This makes it the greatest lower bound. □

Lemma 1. *Suppose $A \neq \phi$ and $s = lub(A)$ then for any y such that $y < s$, $\exists a \in A$ such that $y < a \leq s$.*

Proof. Suppose for contradiction, $\nexists$ any element a such that $y < a$. This means that $y \geq a, \forall a \in A \implies y$ is upper bound of set A. But y is already less than least upper bound of set A. Hence contradiction.
Therefore, $\exists \ a \in A$ such that $y < a \leq s$. □

Theorem 2. *Archimedean Property: Given any positive real numbers $x, y \ \exists \ n \in N$ such that $nx > y$.*

Proof. Let a set $A = \{nx \mid n \in \mathbb{N}\}$. Suppose for contradiction $nx \leq y$. Then y is the upper bound of the set A.
Let a $x > 0$, then $y - x < y$ hence $y - x$ is not the upper bound of the set A. This means that $\exists m \in \mathbb{N}$ such that $y - x < mx \implies y < mx + x \implies y < (m+1)x$ which is impossible since $(m+1)x \in A$ and y is upper bound of the A.
$nx > y$ is true. □

Theorem 3. *If A and B are the two non empty bounded subsets of R, such that $x \leq y \ \ \forall x \in A$ and $\forall y \in B$ then $sup(A) \leq inf(B)$*

Proof. Let a be the supremum of A and b be the infimum of B. Therefore, $a \geq x \ \ \forall x \in A$ and $b \leq y \ \ \forall y \in B$. Also, A is bounded above by B and elements of B are the upper bound for A. Hence, $a \leq y \ \ \forall y \in B$. This means that a is the lower bound of B and a is $sup(A)$. In other words, $sup(A) \leq inf(B)$. □

Theorem 4. *Given any two real number a, b with $a < b$, $\exists \ \mathbb{Q}$ between a and b.*

Proof. Since $b-a>0$. Take two positive number $b-a$ and 1 $\exists$ n $\in \mathbb{Z}$ such that $n(b-a)>1$.
TBD $\square$

Theorem 5. *Any monotone increasing sequence of real numbers that is bounded above converges to some real number.*

Proof. Let x_n be a monotone increasing sequence in $\mathbb{R}$ that is bounded above hence there exits a s such that $s = lub\{x_n \mid n \in \mathbb{N}\}$
Suppose $\epsilon > 0 \implies s-\epsilon < s$ and $s-\epsilon$ is not the upper bound of the x_n.
Using lemma 1, we can say that $\exists x_\epsilon \in x_n$ such that $s-\epsilon < x_\epsilon < s$.
Using monotone condition, for some $n_0 \in \mathbb{N}$, we have,

$$s-\epsilon < x_n < s < s+\epsilon \quad \forall n > n_0 \in \mathbb{N}$$

Hence $|x_n - s| < \epsilon$. x_n converges to s $\forall n > n_0$. $\square$

Remark. *Nested Interval theorem* $\approx$ *lub* $\approx$ *theorem* 4

Proof. TBD $\square$

Theorem 6. *Nested Interval theorem: Suppose* $\{I_n\}$ *is the sequence of closed and bounded non-empty intervals such that* $I_1 \supset I_2 \supset I_3 \ldots$ *then:*

1. $\bigcap\limits_{n\geq 1} I_n \neq \phi$.
2. *If the sequence of the length of the intervals goes to* 0 *then* $\bigcap\limits_{n\geq 1} I_n = \{x\}$.

Proof. Let I_n be an interval $[a_n, b_n]$ with $a_m < b_n \forall m, n \in \mathbb{N}$. Then $\forall n \in \mathbb{N}$, a_n is the increasing sequence and b_n is the decreasing sequence. b_n is upper bound of a_n hence, $a_n < inf(b_n)$.
For b_n, a_n is the lower bound of b_n i.e. $sup(a_n) < b_n$. If we combine all inequalities, we get

$$a_n \leq sup(a_n) \leq inf(b_n) \leq b_n \quad \forall n \in \mathbb{N}$$

Using density theorem, we can say that $\exists$ some $\mathbb{Q}$ between $sup(a_n)$ and $inf(b_n)$.
Hence, $\bigcap\limits_{n\geq 1} I_n \neq \phi$.
Let the length of the interval to be $L = |b_n - a_n|$. Suppose for contradiction, we have two elements in $\bigcap\limits_{n\geq 1} I_n$ instead of one, say x and y.
The distance between x and y is $|y-x|$. Since, $L \to 0$ hence $\exists n \in \mathbb{N}$ such that for some $n_0 \geq n$, $|L| = |b_{n_0} - a_{n_0}| < \epsilon$ for some $\epsilon > 0$. Since $|L| \to 0$, we can choose ϵ such that it is smaller than $|y-x|$. Then, if interval contains any one of the point, it can not contain the other. $\square$

Lecture 2

Definition 3. *Decimals representation of Real numbers: Let* $z \in \mathbb{R}^+$ *be given. Let* n_0 *be the largest integer such that* $n_0 \leq z$. *Let* n_1 *be the largest integer such that* $n_0 + \frac{n_1}{10} \leq z$. *As such, say* n_k *is defined for some* k. *Let* n_{k+1} *be the largest integer* *such that* $n_0 + \frac{n_1}{10^1} + \frac{n_1}{10^2} + \cdots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \leq z$. *Consider the set of all such finite sums, i.e. the set of all*

$$z_k = n_0 + \frac{n_1}{10^1} + \frac{n_1}{10^2} + \cdots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \leq z$$

This set has a supremum and that is z *itself. We symbolically write* $z = n_0.n_1n_2\ldots$

Lemma 2. *Let* p *be an integer* ≥ 2. *If* $0 \leq a_n \leq p-1$, *where* a_n *is an integer then* $\sum\limits_{n=0}^{\infty} \frac{a_n}{p^n}$ *converges to some* x *in* $[0,1]$.

Proof. Since $0 \leq a_n \leq p-1$, we can replace all a_n with $p-1$ and then we have

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n} \leq (p-1)\sum_{n=1}^{\infty} \frac{1}{p^n} = 1$$

Therefore, the sequence is bounded, and it is monotonic increasing this means it converges to some $x \leq 1$ and x is already positive. Hence it converges tp some x in $[0,1]$. □

Lemma 3. *Conversely, given any* $0 \leq x \leq 1, \exists\ a_n \in \mathbb{Z}$ *and* $0 \leq a_n \leq p-1$ *such that* $x = \sum\limits_{n=0}^{\infty} \frac{a_n}{p^n}$.

Proof. Suppose we have $0 < x \leq 1$ and a_1 is the largest integer such that $\frac{a_1}{p} < x \leq 1$. Since x is bounded above by 1, we have $a_1 < p \implies a_1 \leq p-1$ since a_1 is an integer. Similarly, find a_2 such that $\frac{a_1}{p} + \frac{a_2}{p^2} < x$.This can be achieved by Archimedean property. Also, note that $a_2 \leq p-1$, since we have

$$\begin{aligned}
\frac{a_1}{p} + \frac{a_2}{p^2} &< x < 1 \\
\frac{a_1}{p} + \frac{a_2}{p^2} \leq \frac{p-1}{p} + \frac{a_2}{p^2} &< 1 \qquad (a_1 \leq p-1) \\
1 - \frac{1}{p} + \frac{a_2}{p^2} &< 1 \\
\frac{a_2}{p^2} &< \frac{1}{p} \\
a_2 &< p \\
a_2 &\leq p-1
\end{aligned}$$

Inductively, we can define a_n as the largest integer with $a_n \leq p-1$ such that $\sum\limits_{i=1}^{n} \frac{a_i}{p_i} < x$. Since $a_n < p$ TBD □

Suppose $\{a_n\}$ is the bounded sequence in $\mathbb{R}$, we define two sets:

$$\begin{aligned}
s_n &:= inf\{a_n, a_{n+1}, \ldots\} \\
S_n &:= sup\{a_n, a_{n+1}, \ldots\}
\end{aligned}$$

Notice that $\underset{k}{inf}(\{a_n\}) \leq s_n \leq S_n \leq \underset{k}{sup}(\{s_n\})$

Definition 4. *Limit superior and limit inferior: Let a_n be the bounded sequence of real numbers then*

$$\lim inf(a_n) = \lim_{n\to\infty} s_n$$
$$\lim sup(a_n) = \lim_{n\to\infty} S_n$$

Limit superior is the supremum of all subsequential limits of $\{a_n\}$. Similarly, limit inferior is the infimum of all subsequential limits of $\{a_n\}$.

Note that s_n is the increasing sequence and S_n is the decreasing sequence and they are bounded on both sides. Hence, we can also say that using monotone convergence theorem

$$\lim inf(a_n) = \lim_{n\to\infty} s_n = sup(s_n)$$
$$\lim sup(a_n) = \lim_{n\to\infty} S_n = inf(S_n)$$

Theorem 7. *A sequence $\{a_n\}$ is bounded above iff limsup $a_n < \infty$ (is finite).*

Proof. TBD □

Theorem 8. *A sequence $\{a_n\}$ is bounded below iff liminf $a_n < \infty$ (is finite).*

Proof. TBD □

Theorem 9. *Given any sequence $\{a_n\}$ there exists a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to limsup(a_n)$.*

Proof. TBD □

Theorem 10. *Given any sequence $\{a_n\}$ there exists a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to liminf(a_n)$.*

Proof. TBD □

Lecture 3

Definition 5. *Finite Set: A set is finite if there exists a bijection between the set and the $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.*

Theorem 11. $\mathbb{N}$ *is an infinite set.*

Proof. Negation of above definition would be a set is infinite if there does not exists a bijection between $\{1, 2, \ldots, n\}$ and $\mathbb{N}$. Suppose a function $f(\{1, 2, \ldots, n\}) \to \mathbb{N}$. We can have a natural number $f(1) + f(2) + \cdots + f(n) > f(i) \;\; \forall i \in \mathbb{N}$ that does not have a preimage in $\{1, 2, \ldots, n\}$ hence the map is not bijective. □

Theorem 12. *A set is infinite iff there exists a one-one map from $\mathbb{N}$ to the set.*

Proof. $\implies$) For $1 \in \mathbb{N}$ there exist a image in X say $f(1)$. Now, take $2 \in \mathbb{N}$ such that there exist a image $f(2) \in X\backslash\{1\}$. This means that $f(1) \neq f(2)$. Since for every $n \in \mathbb{N}$ we can have $f(n)$ in $X\backslash\{1, 2, \ldots, n-1\}$ as X is also infinite. Hence, we have constructed a one-one map from $\mathbb{N} \to X$.
($\impliedby$ Since $\mathbb{N}$ is infinite and we have a one-one mapping from $\mathbb{N} \to X$ therefore for each $n \in \mathbb{N}$, we have only one $f(n) \in X$ and every $n \in \mathbb{N}$ has a image (it's a map). Hence, X is infinite. □

Definition 6. *Equivalent or Equipotent set: Two sets are equivalent or equipotent if there exists bijection between X and Y.*

For example: Finite sets are equivalent to $\{1, 2, \dots, n\}$ for some fixed $n \in \mathbb{N}$.

Definition 7. *Countably infinite set: A infinite set X is said to be countably infinite if there exists a bijection between X and $\mathbb{N}$.*

Definition 8. *Uncountably infinite set: A set is said to be uncountably infinite if it is not countably infinite set.*

Example:

1. Countably infinite: bijective map between $\mathbb{Z} \to \mathbb{N}$. The map will look line

$$n \to \begin{cases} \frac{n}{2} & n \in even \\ \frac{-(n+1)}{2} & n \in odd \end{cases}$$

2. $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is also equivalent to $\mathbb{N}$ i.e. countably infinite. Map will be $(m \times n) \to 2^m(2n-1)$ m and n are unique and m such that 2^m is the maximum multiple of 2.

Theorem 13. *If A is an infinite subset of $\mathbb{N}$ then there exists bijection between A and $\mathbb{N}$.*

Proof. TBD □

Corollary 1. *Monotone Subsequence theorem: Any sequence $\{x_n\}$ of real numbers has a monotone subsequence*

Proof. We define "peak" as any element x_m is called a peak if $x_m \geq x_n$ for all $n > m$. There cases can be two possible cases

1. Infinite peaks: This means that there exists $m_i's$ say $\{m_1, m_2, \dots\}$ such that $x_{m_i} > x_n$ for all $n > m_i$, and for all $i \in \mathbb{N}$. We can arrange $m_i's$ in increasing order $m_1 < m_2 < \dots$ and $x_{m_1} > x_{m_2} > x_{m_3} > \dots$ is a decreasing subsequence.

2. Finite peaks: (0 or some $n \in \mathbb{N}$). Assume the peaks are $\{x_{m_1}, x_{m_2}, \dots, x_{m_n}\}$, this means that there exists some $s_1 = m_n + 1, s \in \mathbb{N}$ such that x_{s_1} is not a peak. Therefore, there also exists some $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$. Because of finite peaks x_{s_2} is also not a peak, hence for some $s_3 > s_2$, we have $x_{s_2} < x_{s_3}$. Proceeding with induction, we have $x_{s_1} < x_{s_2} < x_{s_3} < \dots$ is a increasing sequence.

□

Theorem 14. $\mathbb{Q}$ *is a countably infinite set.*

Proof. TBD □

Theorem 15. *Any interval in $\mathbb{R}$ is an uncountable set.*

Proof. TBD □

Definition 9. *Suppose $X \neq \phi$. A partial order on X is a relation R on X such that R is*

1. *Reflexive.*
2. *Anti-symmetric $\implies aRb, bRa \implies a = b$*
3. *Transitive.*

Examples:

1. $R =$ " $\leq$ " is a partial order.

$\mathcal{P}(\mathcal{A})$ is power set of A and $X, Y \subset A$ then $X \leq Y$ iff $X \subseteq Y$.

Lecture 4

Definition 10. *Given $E \subset X$ where X is partially order set, we say E is totally ordered if any two elements of E are comparable i.e. if $e_1, e_2 \in E$, then $e_1 \leq e_2$ or $e_2 \leq e_1$. Totally ordered $\equiv$ linearly order $\equiv$ chain.*

Definition 11. *Upper bound of E: An element is $x \in X$ is called upper bound of E if for any $x' \in E$, we have $x' \leq x$. x is called the maximal element if $x' \geq x \implies x' = x$.*
For maximal element, x should be an upper bound for set E and x should belong to E.

Let $X \neq \phi$. $\mathcal{F}$ is a collection of subsets of X (element of $\mathcal{P}$(X)). An element $F \in \mathcal{F}$ is a upper bound for a subfamily $\mathcal{F}'$ of $\mathcal{F}$ provided every member of $\mathcal{F}'$ is a subset of F.
F will be the maximal element of $\mathcal{F}$ if it is not a proper subset (means not contained in) of any member in $\mathcal{F}$.

Lemma 4. *Zorn's lemma: Let X be a partially order set. If every totally ordered subset of X is bounded above then X has a maximal element.*

Definition 12. *Cardinality of X : Two sets A and B have same cardinality if there exists a bijection between them. Set X has cardinal number α means that there exists a set Y equivalent to X with number of elements equal to α.*

If α and β are cardinal numbers of set X and set Y such that $\alpha \leq \beta$ then there exists a one-one mapping from $X \to Y$.

Theorem 16. *Cantor-Schroeder-Bernstein theorem: If there exists a one-one mapping from $X \to Y$ and $Y \to X$ then there exists a bijection between X and Y.*

Limits of functions

Definition 13. *Limit point: A point $a \in \mathbb{R}$ is called a limit point of a set $X \subseteq \mathbb{R}$ if for every neighbourhood $(a - \epsilon, a + \epsilon), \epsilon > 0$ there exists $x \in X$ such that $a \neq x$.*

For a function f defined on $X \subseteq \mathbb{R}$, f converges to some l means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - l| < \epsilon \; \forall x \in X \ni |x - a| < \delta$$

Limit of functions as limit of sequences-

Proposition 1. *Let $f : X \to \mathbb{R}$ and let a be a limit of X. Then $lim_{x \to a} f(x) = l$ if and only if for every sequence $\{x_n\}_{n \geq 1}$ in X that converges to a and $x_n \neq a$ for all n, the sequence $\{f(x_n)\}_{n \geq 1}$ converges to l.*

A function f is continuous on X if it is continuous in every point in X.

Definition 14. *Continuity of f: A function f is said to be continuous at some point $x \in X$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $y \in X$ and $|y - x| < \delta$, we have $|f(y) - f(x)| < \epsilon$.*

A function f is continuous at a limit point $a \in X$ if and only if $f(a)$ is defined and $lim_{x \to a} f(x) = f(a)$.

Proposition 2. *Let f be a real valued funtion defined on subset X of $\mathbb{R}$ and $a \in X$ is the limit point of X. Then f is continuous at a if and only if for every sequence $\{x_n\}_{n \geq 1}$ that converges to a and $x_n \neq a$ for every n, we have $lim f(x_n) = f(lim x_n) = f(a)$. Continuous function preserve convergence (maps convergent sequence into convergent sequences).*

Theorem 17. *Bolzano intermediate value theorem: Let I be an interval and $f : I \to \mathbb{R}$, if $a, b \in I$ and $\alpha \in \mathbb{R}$ satisfies $f(a) < \alpha < f(b)$ or $f(a) > \alpha > f(b)$ then there exists a point $c \in I$ between a and b such that $f(c) = \alpha$.*

Sequences of functions

Let $X \subseteq \mathbb{R}$. If for every $n = 1, 2, \ldots$, we assigned a real valued function f_n defined on X then $\{f_n\}_{n \geq 1}$ is called sequence of functions.

Point-wise convergence

A sequence is called point-wise convergent if for each $x \in X \subseteq \mathbb{R}$, the sequence $f_1(x), f_2(x), \ldots$ of real numbers is convergent.

Point-wise limit

A function defined on X is called a point-wise limit if for every $\epsilon > 0$ there exist $n_0 \in \mathbb{Z}$ depending on x and ϵ such that for all $n \geq n_0$, we have $|f(x) - f_n(x)| < \epsilon$.

A series $\sum\limits_{n=1}^{\infty} x_n$ of real numbers converges to $x \in \mathbb{R}$ if the sequence of partial sums $\{s_n\}_{n \geq 1}$ converges to x where $s_n = \sum\limits_{k=1}^{n} x_k$ (n_{th} partial sum).

The limit $x = lim s_n = lim_{n \to \infty} \sum\limits_{k=1}^{n} x_k$ is called the sum of the series. In the case of functions, if for every $x \in X$, $\sum\limits_{n=1}^{\infty} f(x_n)$ converges and if we define $f(x) = \sum\limits_{n=1}^{\infty} f(x_n)$, the $f(x)$ is called the sum of the series $\sum\limits_{n=1}^{\infty} f_n$.

Uniform convergence

A sequence of functions $\{f_n\}_{n\geq 1}$ defined on set $X \subseteq \mathbb{R}$ is said to be uniformly convergent on X to f if for every $\epsilon > 0$ there exists a $n_0 \in \mathbb{Z}$ (depending on ϵ only) such that for all $n \geq n_0$, we have $|f(x) - f_n(x)| < \epsilon$ for all $x \in X$.
Thus, every uniformly convergent sequence is pointwise convergent but converse is not true always.

Uniform Convergence for series of functions

The series of functions $\sum\limits_{n=1}^{\infty}$ converges uniformly on X if the sequence $\{s_n\}_{n\geq 1}$ partial sum of functions, where $s_n(x) = \sum\limits_{k=1}^{n} f_k(x), x \in X$ converges uniformly on X.

Cauchy Criterion of Uniform convergence

The sequence of functions $\{f_n\}_{n\geq 1}$ defined on $X \in \mathbb{R}$ converges uniformly on X if and only if given $\epsilon > 0$ there exists n_0 such that for all $x \in X$, and all $m \geq n_0, n \geq n_0$ we have $|f_m(x) - f_n(x)| < \epsilon$.

Proposition 3. *Suppose $\{f_n\}_{n\geq 1}$ is sequence of continuous functions defined on X and it converges uniformly to f then f is continous.*

Basic Topology in $\mathbb{R}$

Definition 15. *Open sets in $\mathbb{R}$: A subset G of $\mathbb{R}$ is said to be open if for every $x \in G$, there is a neighbourhood $(x - \epsilon, x + \epsilon), \epsilon > 0$ that is contained in G.*

Definition 16. *Open cover: A open cover of X is a collection of $C = \{G_\alpha \mid \alpha \in I\}$ of open sets in $\mathbb{R}$ whose union contains the set X,*

$$X \subseteq \bigcup_{\alpha} G_\alpha$$

where I is some indexing set.

Definition 17. *Sub-cover: If C' is a subcollection of C such that the union of sets in C' also contains the set X then C' is called subcover from C of X. If the number of sets in C' is finite then we call it a finite subcover.*

Definition 18. *Compact set: A subset X of $\mathbb{R}$ is said to be compact if every open cover of X has a finite sub-cover.*

Proposition 4. *(Heine-Borel Theorem) Let X be a set of real numbers. Then the following statements are equivalent:*

1. *X is closed and bounded.*
2. *X is compact (every open cover has a finite subcover).*
3. *Every infinite subset of X has a limit point in X.*

Proof. 1 $\implies$ 2) Suppose $X = [a, b]$ a infinite interval which is closed and bounded. We define a set S

$$S = \{x \in [a, b] \mid \exists n \in I \ni [a, x] \subseteq \bigcup_{i=1}^{i=n} O_i\}$$

S is non-empty as we can always choose n such that S contains atleast a. Also, b is the upper bound of the set S and then it should possess the least upper bound property. Let $\lambda = sup(S)$.

Basically, S is a collection of all x such that the interval $[a, x]$ has a finite subcover. Hence, we need to show that λ is largest possible element of S (i.e. $\lambda \in S$) for which $[a, \lambda]$ has a finite subcover and $\lambda = b$.

Since, $\lambda = sup(S)$ and $\lambda < b$ hence, $\lambda \in X$. This means that there exists atleast one open set O_β that contains λ for some $\beta \in I$. Hence, for some $\epsilon > 0$, the neighbourhood $(\lambda - \epsilon, \lambda + \epsilon) \in O_\beta$. Since, $\lambda - \epsilon$ is not the supremum of set S. Therefore, there exists a element $x \in S$ such that $\lambda > x > \lambda - \epsilon \implies x \in O_\beta$. Also, $x \in S$ hence, by definition of S the finite subcovers of $[a, x]$ are $\{O_1, O_2, \ldots, O_n\}$ for some fixed $n \in I$. Hence, adding O_β to this set $\{O_{beta}, O_1, O_2, \ldots, O_n\}$ is also finite and it is subcover of set $[a, \lambda] \implies \lambda \in S$.

For proving $\lambda = b$, we have $\lambda \leq b$. Suppose for contradiction that $\lambda < b \implies b - \lambda > 0$. Define $y := \lambda + \frac{1}{2}\min(\epsilon, b - \lambda)$. This implies that belongs y to neighbourhood of $\lambda \implies y \in O_\beta$ and $y \in \bigcup\{O_\beta, O_1, O_2, \ldots, O_n\} \implies y \leq \lambda$. But by definition $y > \lambda$. Hence, contradiction. Therefore, $y = b$ is the only possibility. This means that $X := [a, b]$ has finite subcover hence X is compact. □

Proof. $2 \implies 3$) Let a infinite subset G of X where X is compact set. Suppose for contradiction G does not have any limit point in X TBD. □

Proposition 5. *Let f be real-valued continuous function defined on closed and bounded interval $I = [a, b]$. Then f is bounded on I and it assumes its maximum and minimum values in the interval I i.e. there are points $x_1, x_2 \in I$ such that $f(X_1) \leq f(x) \leq f(x_2)$ for all $x \in X$.*

Uniformly Continuous function Let f be real-valued continuous function defined on X. Then f is said to be uniformly continuous if given $\epsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in X$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

Proposition 6. *If a real-valued function f is continous on a closed and bounded interval I then f is uniformly continuous on I.*

Sequences in $\mathbb{R}$

1. A convergent sequence in $\mathbb{Q}$ is a Cauchy sequence in $\mathbb{Q}$.

2. A Cauchy sequence in $\mathbb{Q}$ is bounded; in particular, every convergent sequence in $\mathbb{Q}$ is bounded.

3. Limit of a sequence say $\{x_n\}_{n\geq 1}$ is unique.

Let F_Q denote the set of all cauchy sequence in $\mathbb{Q}$.

Definition 19. *A sequence $\{x_n\}_{n\geq 1}$ in F_Q is said to be equivalent to a sequence $\{y_n\}_{n\geq 1}$ in F_Q if and only if $\lim_{n\to\infty} |x_n - y_n| = 0$. Notation of equivalence is $\{x_n\}_{n\geq 1} \sim \{y_n\}_{n\geq 1}$.*

Proposition 7. *If $\{x_n\}_{n\geq 1} \in F_Q$ then $\lim_{n\to\infty} x_n = x$ if and only if $\{x_n\} \sim \{x\}$, where $\{x\}$ denotes the constant sequence with each term is equal to x.*

Proof. Follows from definition of equivalence relation.
Since, $\lim_{n\to\infty}|x_n - x| = 0$ (from definition) $\implies$ $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x = x$. Converse, is similar. □

Proposition 8. *If $\{x_n\}$ and $\{y_n\}$ are in F_Q then so the sequencs $\{x_n + y_n\}$ and $\{x_n y_n\}$.*

Proof. Apply cauchy criterion for each sequence and choose ϵ to be $\frac{\epsilon}{2}$. Then try to find ϵ bound for sequences $\{x_n + y_n\}$ and $\{x_n y_n\}$. □

Proposition 9. *If $\{x_n\}$, $\{y_n\}$, $\{x'_n\}$ and $\{y'_n\}$ are in F_Q and $\{x_n\} \sim \{x'_n\}$, $\{y_n\} \sim \{y'_n\}$ then $\{x_n + x'_n\} \sim \{y_n + y'_n\}$ and $\{x_n y_n\} \sim \{x'_n y'_n\}$.*

Proof. For $\{x_n\} \sim \{x'_n\}$, $\{y_n\} \sim \{y'_n\}$ follows from writing modulus inequality $|a| - |b| \leq |a - b| \leq |a + b| \leq |a| + |b|$ and then applying sandwich theorem.

For $\{x_n y_n\} \sim \{x'_n y'_n\}$, we know that cauchy sequence in $\mathbb{Q}$ are bounded. Hence, there exists a rational K_1, K_2 such that $|x_n| \leq K_1$ and $|y'_n| \leq K_2$ for all n. We can write

$$|x_n - x'_n| < \frac{\epsilon}{2K_1} \text{ and } |y_n - y'_n| < \frac{\epsilon}{2K_2}$$

$$\begin{aligned}
|x_n y_n - x'_n y'_n| &= |x_n y_n - x_n y'_n + x_n y'_n - x'_n y'_n| \\
&= |x_n(y_n - y'_n) + y'_n(x_n - x'_n)| \\
&\leq |x_n||(y_n - y'_n)| + |y'_n||(x_n - x'_n)| \\
&< K_1\left(\frac{\epsilon}{2K_1}\right) + K_2\left(\frac{\epsilon}{2K_2}\right) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

□

Inequalities

Proposition 10. *The function $f(x) = \frac{x}{1+x}, x \geq 0$, is monotonically increasing.*

Proof. For some $x, y \geq 0$ and $x > y$, we have $\frac{1}{1+x} < \frac{1}{1+y}$ and $1 - \frac{1}{1+x} > 1 - \frac{1}{1+y} \implies \frac{x}{1+x} > \frac{y}{1+y}$. □

Theorem 18. *For any two real numbers x and y, this inequality holds*

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$$

Proof. Using previous proposition, we can say if $|x+y| \leq |x| + |y|$ and the sequence $\frac{x}{1+x}, x \geq 0$ is monotonically increasing then

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|+|y|}{1+|x|+|y|} = \frac{|x|}{1+|x|+|y|} + \frac{|y|}{1+|x|+|y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$$

□

Proposition 11. *Generalised AM-GM inequality: If $a > 0$ and $b > 0$ and if $0 \leq \lambda \leq 1$ is fixed, then*

$$a^{\lambda}b^{1-\lambda} \leq \lambda a + (1-\lambda)b$$

Proof. Since, $y = ln(x)$ is concave, then

$$\begin{aligned} ln(\lambda a + (1-\lambda)b) &\geq \lambda ln(a) + (1-\lambda)ln(b) \\ ln(\lambda a + (1-\lambda)b) &\geq ln(a^{\lambda}b^{1-\lambda}) \end{aligned}$$

Since e^x is increasing function, we have

$$a^{\lambda}b^{1-\lambda} \leq \lambda a + (1-\lambda)b$$

□

Remark. *When $x \geq 0, y \geq 0$ and $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$$

Proof. Replace $a = x^p$, $b = y^q$ and take $\lambda = \frac{1}{p} \implies 1 - \lambda = 1 - \frac{1}{p} = \frac{1}{q}$ □

Theorem 19. *(Holder's inequality) Let $x_i \geq 0$ and $y_i \geq 0$ for $i = 1, 2, \ldots, n$, and suppose that $p > 1$ and $q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \left(\sum_{i=1}^{n} y_i^q\right)^{1/q}$$

In the special case when $p = q = 2$, the above inequality reduces to

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} y_i^2\right)^{1/2}$$

This is called ***Cauchy-Schwarz inequality.***

Proof. For the case of $x_i = 0$ and $y_i = 0$, it is trivially true.
For the case of $x_i > 0$ and $y_i > 0$, we can write the given inequality as

$$\sum_{i=1}^{n} \left(\frac{x_i}{\left(\sum_{i=1}^{n} x_i^p\right)^{1/p}} \frac{y_i}{\left(\sum_{i=1}^{n} y_i^q\right)^{1/q}} \right) \leq 1$$

Replace $x_i' = \frac{x_i}{\left(\sum_{i=1}^{n} x_i^p\right)^2}$ and $y_i' = \frac{y_i}{\left(\sum_{i=1}^{n} y_i^q\right)^2}$, also $x_i' \geq 0$ and $y_i' \geq 0$, we get $\sum_{n=1}^{n} x_i' y_i' \leq 1$.
Now, apply the Youngs inequality for $i = 1, 2, \ldots, n$ and sum them up to get

$$\sum_{i=1}^{n} x_i' y_i' \leq \frac{\sum x^p}{p} + \frac{\sum y^q}{q}$$

Since, $x_i > 0$ and $y_i > 0$ hence $\sum_{n=1}^{n} x'^p \neq 0 \neq \sum_{n=1}^{n} y'^q$ and it is equivalent to prove it for (some constant)

$$\sum_{n=1}^{n} x'^p = 1 = \sum_{n=1}^{n} y'^q$$

Hence, we have

$$\sum_{i=1}^{n} x'_i y'_i \leq \frac{1}{p} + \frac{1}{q} = 1$$

This proves the required inequality. □

Theorem 20. *(Minkowski's inequality) Let $x_i \geq 0$ and $y_i \geq 0$ for $i = 1, 2, \ldots, n$ and suppose that $p \geq 1$. Then*

$$\left(\sum_{i=1}^{n}(x_i + y_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^q\right)^{1/q}$$

Proof. If $p = 1$, it is trivially true. So, assume $p > 1$, we have

$$\sum_{i=1}^{n}(x_i + y_i)^p = \sum_{i=1}^{n} x_i(x_i + y_i)^{p-1} + \sum_{i=1}^{n} y_i(x_i + y_i)^{p-1}$$

Apply Holder's inequality for both terms on RHS

$$\begin{aligned}
\sum_{i=1}^{n}(x_i + y_i)^p &\leq \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \left(\sum_{i=1}^{n}(x_i + y_i)^{(p-1)q}\right)^{1/q} \\
&\quad + \left(\sum_{i=1}^{n} y_i^p\right)^{1/p} \left(\sum_{i=1}^{n}(x_i + y_i)^{(p-1)q}\right)^{1/q} \\
&\leq \left[\left(\sum_{i=1}^{n} x_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p\right)^{1/p}\right] \left(\sum_{i=1}^{n}(x_i + y_i)^p\right)^{1/q}
\end{aligned}$$

Divide both sides by $\left(\sum_{i=1}^{n}(x_i + y_i)^p\right)^{1/q}$ as it is $\neq 0$ and we will get the required inequality. For the case of $\left(\sum_{i=1}^{n}(x_i + y_i)^p\right)^{1/q} = 0$, proof is self-evident. □

Theorem 21. *(Minkowski Inequality for infinite sums) Suppose that $p \geq 1$ and let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be sequences of non-negative terms such that $\sum_{n=1}^{\infty} x_n^p$ and $\sum_{n=1}^{\infty} y_n^p$ are convergent. Then $\sum_{n=1}^{\infty}(x_n + y_n)^p$ is convergent. Moreover,*

$$\left(\sum_{n=1}^{\infty}(x_i + y_i)^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} x_i^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} y_i^q\right)^{1/q}$$

Proof. For any positive integer m, we can use Minkowski inequality for finite sums

$$\begin{aligned}\left(\sum_{n=1}^{m}(x_i+y_i)^p\right)^{1/p} &\leq \left(\sum_{n=1}^{m}x_i^p\right)^{1/p}+\left(\sum_{n=1}^{m}y_i^q\right)^{1/q}\\ &\leq \left(\sum_{n=1}^{\infty}x_i^p\right)^{1/p}+\left(\sum_{n=1}^{\infty}y_i^q\right)^{1/q}\end{aligned}$$

Since, the sequence $\{(\sum_{n=1}^{m}(x_i+y_i)^p)^{1/p}\}$ is increasing sequence(w.r.t m) of nonnegative real numbers and is bounded above by the sum in above equation. Hence, it is convergent and as $m \to \infty$ the limit is also bounded above by the sum. This proves the inequality. □

Theorem 22. *Let $p > 1$. For $a \geq 0$ and $b \geq 0$, we have*

$$(a+b)^p \leq 2^{p-1}(a^p+b^p)$$

Proof. TBD □

Metric Spaces

Definition 20. *Let $X \neq \phi$. A metric on X is a function $d : X \times X \to [0, \infty)$ such that*

1. *d(x,y) = 0 $\implies$ $x = y$*
2. *d(x,y) = d(y,x)*
3. *d(x,y) $\leq$ d(x,z) + d(z,y) for any $x, y, z \in X$.*

The metric space in which every cauchy sequence converges is called "complete" metric space.

Examples of metric spaces

1. **The space of all bounded sequences.** Let X be the set of all infinite sequences of numbers that are bounded $\{x_i\}_{i\geq 1}$ such that $sup|\{x_i\}| < \infty$. The metric on X is defined as $d(x,y) = sup_i|x_i - y_i|$ $(\leq sup_i|x_i - z_i| + sup_i|y_i - z_i|)$.

2. **The space l_p.** Let X be the set of all sequences $x = \{x\}_{i\geq 1}$ such that

$$\left(\sum_{i=1}^{\infty}|x_p|^p\right)^{1/p} < \infty, \quad p \geq 1$$

If $\{x\}_{n\geq 1}$ and $\{y\}_{n\geq 1}$ are two sequences belong to X then we define the metric as

$$d(x,y) = \left(\sum_{i=1}^{\infty}|x_i - y_i|^p\right)^{1/p}$$

Minkowski inequality for infinite sequences makes it a suitable metric.

3. **The space of bounded functions.** Let S be any nonempty set and $\mathcal{B}(\mathcal{S})$ denote the set of all real or complex-valued functions on S, each of which is bounded. i.e.
$$\sup_{x\in S}|f(x)| < \infty$$
It can also be shown that $\sup_{x\in S}|f(x) - g(x)| < \infty$. We defined metric as $d(f,g) = \sup_{x\in S}|f(x) - g(x)|, \quad f,g \in \mathcal{B}(\mathcal{S})$. The metric d is called **uniform metric** or **supremum metric**.

4. **The space of continuous functions.** Let X be the set of all continuous functions defined on $[a,b]$, an interval in $\mathbb{R}$. For $f,g \in X$, define
$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$
We can also defined another metric on the set X as
$$d(f,g) = \int_a^b |f(x) - g(x)|dx$$
Assume $h(x) \in X = C[a,b]$, we have
$$\begin{aligned} d(f,g) &= \int_a^b |f(x) - h(x) + h(x) - g(x)|dx \\ &\le \int_a^b \left(|f(x) - h(x)| + |h(x) - g(x)|\right) dx \\ &\le d(f,h) + d(h,g) \end{aligned}$$

5. **Metric on extended real line.** Let $X = \mathbb{R}\cup\{\infty\}\cup\{-\infty\}$. Define $f : X \to \mathbb{R}$ by the rule
$$f(x) = \begin{cases} \frac{x}{1+|x|} & \text{if } -\infty < x < \infty \\ 1 & \text{if } x = \infty \\ -1 & \text{if } x = -\infty \end{cases}$$
We define metric using f as
$$d(x,y) = |f(x) - f(y)|, \quad x,y \in X$$
Proof. TBD $\square$