
Lecture 1

Remarks: Basic ordering properties are assumed to be true.

Definition 1. *Boundedness:* The subset $A \in R$ is said to be bounded above if $\exists M$ such that $M > x \quad \forall x \in A$. And it is bounded below if $\exists m$ such that $m < x \quad \forall x \in A$. If A has both then it is called bounded.

Definition 2. *Least Upper Bound (lub) Axiom:* If A is nonempty subset of R and it is bounded above, then A has a least upper bound in R .

Theorem 1. *If A is nonempty subset in R and it is bounded below, then it has a greatest lower bound in R .*

Proof. We first create a set T of lower bounds of A

$$T = \{m \mid m < x \quad \forall x \in A\}$$

T is non-empty since A is bounded below. Now, we need to prove that there exists a supremum of T which is also a lower bound of A .

Since, set T is bounded above by all the elements of set A , it should have a least upper bound, say M such that $M > m \quad \forall m \in T$. Also, every element of A is an upper bound of T hence by definition of supremum, we can say $M \leq x \quad \forall x \in A$ hence M is the lower bound of A . This makes it the greatest lower bound. \square

Lemma 1. *Suppose $A \neq \phi$ and $s = \text{lub}(A)$ then for any y such that $y < s$, $\exists a \in A$ such that $y < a \leq s$.*

Proof. Suppose for contradiction, \nexists any element a such that $y < a$. This means that $y \geq a, \forall a \in A \implies y$ is upper bound of set A . But y is already less than least upper bound of set A . Hence contradiction.

Therefore, $\exists a \in A$ such that $y < a \leq s$. \square

Theorem 2. *Archimedean Property:* Given any positive real numbers $x, y \exists n \in \mathbb{N}$ such that $nx > y$.

Proof. Let a set $A = \{nx \mid n \in \mathbb{N}\}$. Suppose for contradiction $nx \leq y$. Then y is the upper bound of the set A .

Let a $x > 0$, then $y - x < y$ hence $y - x$ is not the upper bound of the set A . This means that $\exists m \in \mathbb{N}$ such that $y - x < mx \implies y < mx + x \implies y < (m + 1)x$ which is impossible since $(m + 1)x \in A$ and y is upper bound of the A .

$nx > y$ is true. \square

Theorem 3. *If A and B are the two non empty bounded subsets of R , such that $x \leq y \quad \forall x \in A$ and $\forall y \in B$ then $\sup(A) \leq \inf(B)$*

Proof. Let a be the supremum of A and b be the infimum of B . Therefore, $a \geq x \quad \forall x \in A$ and $b \leq y \quad \forall y \in B$. Also, A is bounded above by B and elements of B are the upper bound for A . Hence, $a \leq y \quad \forall y \in B$. This means that a is the lower bound of B and a is $\sup(A)$. In other words, $\sup(A) \leq \inf(B)$. \square

Theorem 4. *Given any two real number a, b with $a < b$, $\exists \mathbb{Q}$ between a and b .*

Proof. Since $b - a > 0$. Take two positive number $b - a$ and $1 \exists n \in \mathbb{Z}$ such that $n(b - a) > 1$.

TBD □

Theorem 5. *Any monotone increasing sequence of real numbers that is bounded above converges to some real number.*

Proof. Let x_n be a monotone increasing sequence in \mathbb{R} that is bounded above hence there exists a s such that $s = \text{lub}\{x_n \mid n \in \mathbb{N}\}$

Suppose $\epsilon > 0 \implies s - \epsilon < s$ and $s - \epsilon$ is not the upper bound of the x_n .

Using lemma 1, we can say that $\exists x_\epsilon \in x_n$ such that $s - \epsilon < x_\epsilon < s$.

Using monotone condition, for some $n_0 \in \mathbb{N}$, we have,

$$s - \epsilon < x_n < s < s + \epsilon \quad \forall n > n_0 \in \mathbb{N}$$

Hence $|x_n - s| < \epsilon$. x_n converges to $s \quad \forall n > n_0$. □

Remark. *Nested Interval theorem \approx lub \approx theorem 4*

Proof. TBD □

Theorem 6. *Nested Interval theorem: Suppose $\{I_n\}$ is the sequence of closed and bounded non-empty intervals such that $I_1 \supset I_2 \supset I_3 \dots$ then:*

$$1. \bigcap_{n \geq 1} I_n \neq \emptyset.$$

$$2. \text{ If the sequence of the length of the intervals goes to 0 then } \bigcap_{n \geq 1} I_n = \{x\}.$$

Proof. Let I_n be an interval $[a_n, b_n]$ with $a_m < b_n \forall m, n \in \mathbb{N}$. Then $\forall n \in \mathbb{N}$, a_n is the increasing sequence and b_n is the decreasing sequence. b_n is upper bound of a_n hence, $a_n < \inf(b_n)$.

For b_n , a_n is the lower bound of b_n i.e. $\sup(a_n) < b_n$. If we combine all inequalities, we get

$$a_n \leq \sup(a_n) \leq \inf(b_n) \leq b_n \quad \forall n \in \mathbb{N}$$

Using density theorem, we can say that \exists some \mathbb{Q} between $\sup(a_n)$ and $\inf(b_n)$.

Hence, $\bigcap_{n \geq 1} I_n \neq \emptyset$.

Let the length of the interval to be $L = |b_n - a_n|$. Suppose for contradiction, we have two elements in $\bigcap_{n \geq 1} I_n$ instead of one, say x and y .

The distance between x and y is $|y - x|$. Since, $L \rightarrow 0$ hence $\exists n \in \mathbb{N}$ such that for some $n_0 \geq n$, $|L| = |b_{n_0} - a_{n_0}| < \epsilon$ for some $\epsilon > 0$. Since $|L| \rightarrow 0$, we can choose ϵ such that it is smaller than $|y - x|$. Then, if interval contains any one of the point, it can not contain the other. □

Lecture 2

Definition 3. *Decimals representation of Real numbers: Let $z \in \mathbb{R}^+$ be given. Let n_0 be the largest integer such that $n_0 \leq z$. Let n_1 be the largest integer such that $n_0 + \frac{n_1}{10} \leq z$. As such, say n_k is defined for some k . Let n_{k+1} be the largest integer*

such that $n_0 + \frac{n_1}{10^1} + \frac{n_1}{10^2} + \cdots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \leq z$. Consider the set of all such finite sums, i.e. the set of all

$$z_k = n_0 + \frac{n_1}{10^1} + \frac{n_1}{10^2} + \cdots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \leq z$$

This set has a supremum and that is z itself. We symbolically write $z = n_0.n_1n_2\ldots$

Lemma 2. Let p be an integer ≥ 2 . If $0 \leq a_n \leq p-1$, where a_n is an integer then $\sum_{n=0}^{\infty} \frac{a_n}{p^n}$ converges to some x in $[0, 1]$.

Proof. Since $0 \leq a_n \leq p-1$, we can replace all a_n with $p-1$ and then we have

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n} \leq (p-1) \sum_{n=1}^{\infty} \frac{1}{p^n} = 1$$

Therefore, the sequence is bounded, and it is monotonic increasing this means it converges to some $x \leq 1$ and x is already positive. Hence it converges to some x in $[0, 1]$. \square

Lemma 3. Conversely, given any $0 \leq x \leq 1, \exists a_n \in \mathbb{Z}$ and $0 \leq a_n \leq p-1$ such that $x = \sum_{n=0}^{\infty} \frac{a_n}{p^n}$.

Proof. Suppose we have $0 < x \leq 1$ and a_1 is the largest integer such that $\frac{a_1}{p} < x \leq 1$. Since x is bounded above by 1, we have $a_1 < p \implies a_1 \leq p-1$ since a_1 is an integer. Similarly, find a_2 such that $\frac{a_1}{p} + \frac{a_2}{p^2} < x$. This can be achieved by Archimedean property. Also, note that $a_2 \leq p-1$, since we have

$$\begin{aligned} \frac{a_1}{p} + \frac{a_2}{p^2} &< x < 1 \\ \frac{a_1}{p} + \frac{a_2}{p^2} &\leq \frac{p-1}{p} + \frac{a_2}{p^2} < 1 \quad (a_1 \leq p-1) \\ 1 - \frac{1}{p} + \frac{a_2}{p^2} &< 1 \\ \frac{a_2}{p^2} &< \frac{1}{p} \\ a_2 &< p \\ a_2 &\leq p-1 \end{aligned}$$

Inductively, we can define a_n as the largest integer with $a_n \leq p-1$ such that $\sum_{i=1}^n \frac{a_i}{p^i} < x$. Since $a_n < p$ TBD \square

Suppose $\{a_n\}$ is the bounded sequence in \mathbb{R} , we define two sets:

$$\begin{aligned} s_n &:= \inf\{a_n, a_{n+1}, \dots\} \\ S_n &:= \sup\{a_n, a_{n+1}, \dots\} \end{aligned}$$

Notice that $\inf_k(\{a_n\}) \leq s_n \leq S_n \leq \sup_k(\{s_n\})$

Definition 4. *Limit superior and limit inferior: Let a_n be the bounded sequence of real numbers then*

$$\liminf(a_n) = \lim_{n \rightarrow \infty} s_n$$

$$\limsup(a_n) = \lim_{n \rightarrow \infty} S_n$$

Limit superior is the supremum of all subsequential limits of $\{a_n\}$. Similarly, limit inferior is the infimum of all subsequential limits of $\{a_n\}$.

Note that s_n is the increasing sequence and S_n is the decreasing sequence and they are bounded on both sides. Hence, we can also say that using monotone convergence theorem

$$\liminf(a_n) = \lim_{n \rightarrow \infty} s_n = \sup(s_n)$$

$$\limsup(a_n) = \lim_{n \rightarrow \infty} S_n = \inf(S_n)$$

Theorem 7. *A sequence $\{a_n\}$ is bounded above iff $\limsup a_n < \infty$ (is finite).*

Proof. TBD □

Theorem 8. *A sequence $\{a_n\}$ is bounded below iff $\liminf a_n < \infty$ (is finite).*

Proof. TBD □

Theorem 9. *Given any sequence $\{a_n\}$ there exists a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \rightarrow \limsup(a_n)$.*

Proof. TBD □

Theorem 10. *Given any sequence $\{a_n\}$ there exists a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \rightarrow \liminf(a_n)$.*

Proof. TBD □

Lecture 3

Definition 5. *Finite Set: A set is finite if there exists a bijection between the set and the $\{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$.*

Theorem 11. *\mathbb{N} is an infinite set.*

Proof. Negation of above definition would be a set is infinite if there does not exist a bijection between $\{1, 2, \dots, n\}$ and \mathbb{N} . Suppose a function $f(\{1, 2, \dots, n\}) \rightarrow \mathbb{N}$. We can have a natural number $f(1) + f(2) + \dots + f(n) > f(i) \quad \forall i \in \mathbb{N}$ that does not have a preimage in $\{1, 2, \dots, n\}$ hence the map is not bijective. □

Theorem 12. *A set is infinite iff there exists a one-one map from \mathbb{N} to the set.*

Proof. \implies) For $1 \in \mathbb{N}$ there exist a image in X say $f(1)$. Now, take $2 \in \mathbb{N}$ such that there exist a image $f(2) \in X \setminus \{1\}$. This means that $f(1) \neq f(2)$. Since for every $n \in \mathbb{N}$ we can have $f(n)$ in $X \setminus \{1, 2, \dots, n-1\}$ as X is also infinite. Hence, we have constructed a one-one map from $\mathbb{N} \rightarrow X$.

(\Leftarrow Since \mathbb{N} is infinite and we have a one-one mapping from $\mathbb{N} \rightarrow X$ therefore for each $n \in \mathbb{N}$, we have only one $f(n) \in X$ and every $n \in \mathbb{N}$ has a image (it's a map). Hence, X is infinite. □

Definition 6. *Equivalent or Equipotent set: Two sets are equivalent or equipotent if there exists bijection between X and Y .*

For example: Finite sets are equivalent to $\{1, 2, \dots, n\}$ for some fixed $n \in \mathbb{N}$.

Definition 7. *Countably infinite set: A infinite set X is said to be countably infinite if there exists a bijection between X and \mathbb{N} .*

Definition 8. *Uncountably infinite set: A set is said to be uncountably infinite if it is not countably infinite set.*

Example:

1. Countably infinite: bijective map between $\mathbb{Z} \rightarrow \mathbb{N}$. The map will look like

$$n \rightarrow \begin{cases} \frac{n}{2} & n \in \text{even} \\ \frac{-(n+1)}{2} & n \in \text{odd} \end{cases}$$

2. $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ is also equivalent to \mathbb{N} i.e. countably infinite. Map will be $(m \times n) \rightarrow 2^m(2n-1)$ m and n are unique and m such that 2^m is the maximum multiple of 2.

Theorem 13. *If A is an infinite subset of \mathbb{N} then there exists bijection between A and \mathbb{N} .*

Proof. TBD □

Corollary 1. *Monotone Subsequence theorem: Any sequence $\{x_n\}$ of real numbers has a monotone subsequence*

Proof. We define "peak" as any element x_m is called a peak if $x_m \geq x_n$ for all $n > m$. There cases can be two possible cases

1. Infinite peaks: This means that there exists m'_i s say $\{m_1, m_2, \dots\}$ such that $x_{m_i} > x_n$ for all $n > m_i$, and for all $i \in \mathbb{N}$. We can arrange m'_i s in increasing order $m_1 < m_2 < \dots$ and $x_{m_1} > x_{m_2} > x_{m_3} > \dots$ is a decreasing subsequence.
2. Finite peaks: (0 or some $n \in \mathbb{N}$). Assume the peaks are $\{x_{m_1}, x_{m_2}, \dots, x_{m_n}\}$, this means that there exists some $s_1 = m_n + 1, s \in \mathbb{N}$ such that x_{s_1} is not a peak. Therefore, there also exists some $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$. Because of finite peaks x_{s_2} is also not a peak, hence for some $s_3 > s_2$, we have $x_{s_2} < x_{s_3}$. Proceeding with induction, we have $x_{s_1} < x_{s_2} < x_{s_3} < \dots$ is a increasing sequence.

□

Theorem 14. *\mathbb{Q} is a countably infinite set.*

Proof. TBD □

Theorem 15. *Any interval in \mathbb{R} is an uncountable set.*

Proof. TBD □

Definition 9. Suppose $X \neq \phi$. A partial order on X is a relation R on X such that R is

1. Reflexive.
2. Anti-symmetric $\implies aRb, bRa \implies a = b$
3. Transitive.

Examples:

1. $R = " \leq "$ is a partial order.

$\mathcal{P}(A)$ is power set of A and $X, Y \subset A$ then $X \leq Y$ iff $X \subseteq Y$.

Lecture 4

Definition 10. Given $E \subset X$ where X is partially order set, we say E is totally ordered if any two elements of E are comparable i.e. if $e_1, e_2 \in E$, then $e_1 \leq e_2$ or $e_2 \leq e_1$. Totally ordered \equiv linearly order \equiv chain.

Definition 11. Upper bound of E : An element is $x \in X$ is called upper bound of E if for any $x' \in E$, we have $x' \leq x$. x is called the maximal element if $x' \geq x \implies x' = x$.

For maximal element, x should be an upper bound for set E and x should belong to E .

Let $X \neq \phi$. \mathcal{F} is a collection of subsets of X (element of $\mathcal{P}(X)$). An element $F \in \mathcal{F}$ is a upper bound for a subfamily \mathcal{F}' of \mathcal{F} provided every member of \mathcal{F}' is a subset of F .

F will be the maximal element of \mathcal{F} if it is not a proper subset (means not contained in) of any member in \mathcal{F} .

Lemma 4. Zorn's lemma: Let X be a partially order set. If every totally ordered subset of X is bounded above then X has a maximal element.

Definition 12. Cardinality of X : Two sets A and B have same cardinality if there exists a bijection between them. Set X has cardinal number α means that there exists a set Y equivalent to X with number of elements equal to α .

If α and β are cardinal numbers of set X and set Y such that $\alpha \leq \beta$ then there exists a one-one mapping from $X \rightarrow Y$.

Theorem 16. Cantor-Schroeder-Bernstein theorem: If there exists a one-one mapping from $X \rightarrow Y$ and $Y \rightarrow X$ then there exists a bijection between X and Y .

Limits of functions

Definition 13. Limit point: A point $a \in \mathbb{R}$ is called a limit point of a set $X \subseteq \mathbb{R}$ if for every neighbourhood $(a - \epsilon, a + \epsilon), \epsilon > 0$ there exists $x \in X$ such that $a \neq x$.

For a function f defined on $X \subseteq \mathbb{R}$, f converges to some l means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - l| < \epsilon \quad \forall x \in X \ni |x - a| < \delta$$

Limit of functions as limit of sequences-

Proposition 1. *Let $f : X \rightarrow \mathbb{R}$ and let a be a limit of X . Then $\lim_{x \rightarrow a} f(x) = l$ if and only if for every sequence $\{x_n\}_{n \geq 1}$ in X that converges to a and $x_n \neq a$ for all n , the sequence $\{f(x_n)\}_{n \geq 1}$ converges to l .*

A function f is continuous on X if it is continuous in every point in X .

Definition 14. *Continuity of f : A function f is said to be continuous at some point $x \in X$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $y \in X$ and $|y - x| < \delta$, we have $|f(y) - f(x)| < \epsilon$.*

A function f is continuous at a limit point $a \in X$ if and only if $f(a)$ is defined and $\lim_{x \rightarrow a} f(x) = f(a)$.

Proposition 2. *Let f be a real valued function defined on subset X of \mathbb{R} and $a \in X$ is the limit point of X . Then f is continuous at a if and only if for every sequence $\{x_n\}_{n \geq 1}$ that converges to a and $x_n \neq a$ for every n , we have $\lim f(x_n) = f(\lim x_n) = f(a)$. Continuous function preserve convergence (maps convergent sequence into convergent sequences).*

Theorem 17. *Bolzano intermediate value theorem: Let I be an interval and $f : I \rightarrow \mathbb{R}$, if $a, b \in I$ and $\alpha \in \mathbb{R}$ satisfies $f(a) < \alpha < f(b)$ or $f(a) > \alpha > f(b)$ then there exists a point $c \in I$ between a and b such that $f(c) = \alpha$.*

Sequences of functions

Let $X \subseteq \mathbb{R}$. If for every $n = 1, 2, \dots$, we assigned a real valued function f_n defined on X then $\{f_n\}_{n \geq 1}$ is called sequence of functions.

Point-wise convergence

A sequence is called point-wise convergent if for each $x \in X \subseteq \mathbb{R}$, the sequence $f_1(x), f_2(x), \dots$ of real numbers is convergent.

Point-wise limit

A function defined on X is called a point-wise limit if for every $\epsilon > 0$ there exist $n_0 \in \mathbb{Z}$ depending on x and ϵ such that for all $n \geq n_0$, we have $|f(x) - f_n(x)| < \epsilon$.

A series $\sum_{n=1}^{\infty} x_n$ of real numbers converges to $x \in \mathbb{R}$ if the sequence of partial sums

$\{s_n\}_{n \geq 1}$ converges to x where $s_n = \sum_{k=1}^n x_k$ (n th partial sum).

The limit $x = \lim s_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k$ is called the sum of the series. In the case of

functions, if for every $x \in X$, $\sum_{n=1}^{\infty} f(x_n)$ converges and if we define $f(x) = \sum_{n=1}^{\infty} f(x_n)$,

the $f(x)$ is called the sum of the series $\sum_{n=1}^{\infty} f_n$.

Uniform convergence

A sequence of functions $\{f_n\}_{n \geq 1}$ defined on set $X \subseteq \mathbb{R}$ is said to be uniformly convergent on X to f if for every $\epsilon > 0$ there exists a $n_0 \in \mathbb{Z}$ (depending on ϵ only) such that for all $n \geq n_0$, we have $|f(x) - f_n(x)| < \epsilon$ for all $x \in X$.

Thus, every uniformly convergent sequence is pointwise convergent but converse is not true always.

Uniform Convergence for series of functions

The series of functions $\sum_{n=1}^{\infty}$ converges uniformly on X if the sequence $\{s_n\}_{n \geq 1}$ partial sum of functions, where $s_n(x) = \sum_{k=1}^n f_k(x)$, $x \in X$ converges uniformly on X .

Cauchy Criterion of Uniform convergence

The sequence of functions $\{f_n\}_{n \geq 1}$ defined on $X \subseteq \mathbb{R}$ converges uniformly on X if and only if given $\epsilon > 0$ there exists n_0 such that for all $x \in X$, and all $m \geq n_0, n \geq n_0$ we have $|f_m(x) - f_n(x)| < \epsilon$.

Proposition 3. Suppose $\{f_n\}_{n \geq 1}$ is sequence of continuous functions defined on X and it converges uniformly to f then f is continuous.

Basic Topology in \mathbb{R}

Definition 15. Open sets in \mathbb{R} : A subset G of \mathbb{R} is said to be open if for every $x \in G$, there is a neighbourhood $(x - \epsilon, x + \epsilon)$, $\epsilon > 0$ that is contained in G .

Definition 16. Open cover: A open cover of X is a collection of $C = \{G_\alpha \mid \alpha \in I\}$ of open sets in \mathbb{R} whose union contains the set X ,

$$X \subseteq \bigcup_{\alpha} G_\alpha$$

where I is some indexing set.

Definition 17. Sub-cover: If C' is a subcollection of C such that the union of sets in C' also contains the set X then C' is called subcover from C of X . If the number of sets in C' is finite then we call it a finite subcover.

Definition 18. Compact set: A subset X of \mathbb{R} is said to be compact if every open cover of X has a finite sub-cover.

Proposition 4. (Heine-Borel Theorem) Let X be a set of real numbers. Then the following statements are equivalent:

1. X is closed and bounded.
2. X is compact (every open cover has a finite subcover).
3. Every infinite subset of X has a limit point in X .

Proof. 1 \implies 2) Suppose $X = [a, b]$ a infinite interval which is closed and bounded. We define a set S

$$S = \{x \in [a, b] \mid \exists n \in I \ni [a, x] \subseteq \bigcup_{i=1}^{i=n} O_i\}$$

S is non-empty as we can always choose n such that S contains atleast a . Also, b is the upper bound of the set S and then it should possess the least upper bound property. Let $\lambda = \sup(S)$.

Basically, S is a collection of all x such that the interval $[a, x]$ has a finite subcover. Hence, we need to show that λ is largest possible element of S (i.e. $\lambda \in S$) for which $[a, \lambda]$ has a finite subcover and $\lambda = b$.

Since, $\lambda = \sup(S)$ and $\lambda < b$ hence, $\lambda \in X$. This means that there exists atleast one open set O_β that contains λ for some $\beta \in I$. Hence, for some $\epsilon > 0$, the neighbourhood $(\lambda - \epsilon, \lambda + \epsilon) \in O_\beta$. Since, $\lambda - \epsilon$ is not the supremum of set S . Therefore, there exists a element $x \in S$ such that $\lambda > x > \lambda - \epsilon \implies x \in O_\beta$. Also, $x \in S$ hence, by definition of S the finite subcovers of $[a, x]$ are $\{O_1, O_2, \dots, O_n\}$ for some fixed $n \in I$. Hence, adding O_β to this set $\{O_\beta, O_1, O_2, \dots, O_n\}$ is also finite and it is subcover of set $[a, \lambda] \implies \lambda \in S$.

For proving $\lambda = b$, we have $\lambda \leq b$. Suppose for contradiction that $\lambda < b \implies b - \lambda > 0$. Define $y := \lambda + \frac{1}{2} \min(\epsilon, b - \lambda)$. This implies that belongs y to neighbourhood of $\lambda \implies y \in O_\beta$ and $y \in \bigcup \{O_\beta, O_1, O_2, \dots, O_n\} \implies y \leq \lambda$. But by definition $y > \lambda$. Hence, contradiction. Therefore, $y = b$ is the only possibility. This means that $X := [a, b]$ has finite subcover hence X is compact. \square

Proof. $2 \implies 3$) Let a infinite subset G of X where X is compact set. Suppose for contradiction G does not have any limit point in X TBD. \square

Proposition 5. *Let f be real-valued continuous function defined on closed and bounded interval $I = [a, b]$. Then f is bounded on I and it assumes its maximum and minimum values in the interval I i.e. there are points $x_1, x_2 \in I$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in X$.*

Uniformly Continuous function Let f be real-valued continuous function defined on X . Then f is said to be uniformly continuous if given $\epsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in X$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

Proposition 6. *If a real-valued function f is continous on a closed and bounded interval I then f is uniformly continuous on I .*

Sequences in \mathbb{R}

1. A convergent sequence in \mathbb{Q} is a Cauchy sequence in \mathbb{Q} .
2. A Cauchy sequence in \mathbb{Q} is bounded; in particular, every convergent sequence in \mathbb{Q} is bounded.
3. Limit of a sequence say $\{x_n\}_{n \geq 1}$ is unique.

Let F_Q denote the set of all cauchy sequence in \mathbb{Q} .

Definition 19. *A sequence $\{x_n\}_{n \geq 1}$ in F_Q is said to be equivalent to a sequence $\{y_n\}_{n \geq 1}$ in F_Q if and only if $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$. Notation of equivalence is $\{x_n\}_{n \geq 1} \sim \{y_n\}_{n \geq 1}$.*

Proposition 7. *If $\{x_n\}_{n \geq 1} \in F_Q$ then $\lim_{n \rightarrow \infty} x_n = x$ if and only if $\{x_n\} \sim \{x\}$, where $\{x\}$ denotes the constant sequence with each term is equal to x .*

Proof. Follows from definition of equivalence relation.

Since, $\lim_{n \rightarrow \infty} |x_n - x| = 0$ (from definition) $\implies \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x = x$.
Converse, is similar. \square

Proposition 8. *If $\{x_n\}$ and $\{y_n\}$ are in F_Q then so the sequences $\{x_n + y_n\}$ and $\{x_n y_n\}$.*

Proof. Apply cauchy criterion for each sequence and choose ϵ to be $\frac{\epsilon}{2}$. Then try to find ϵ bound for sequences $\{x_n + y_n\}$ and $\{x_n y_n\}$. \square

Proposition 9. *If $\{x_n\}, \{y_n\}, \{x'_n\}$ and $\{y'_n\}$ are in F_Q and $\{x_n\} \sim \{x'_n\}, \{y_n\} \sim \{y'_n\}$ then $\{x_n + x'_n\} \sim \{y_n + y'_n\}$ and $\{x_n y_n\} \sim \{x'_n y'_n\}$.*

Proof. For $\{x_n\} \sim \{x'_n\}, \{y_n\} \sim \{y'_n\}$ follows from writing modulus inequality $|a| - |b| \leq |a - b| \leq |a + b| \leq |a| + |b|$ and then applying sandwich theorem.

For $\{x_n y_n\} \sim \{x'_n y'_n\}$, we know that cauchy sequence in \mathbb{Q} are bounded. Hence, there exists a rational K_1, K_2 such that $|x_n| \leq K_1$ and $|y'_n| \leq K_2$ for all n . We can write

$$|x_n - x'_n| < \frac{\epsilon}{2K_1} \text{ and } |y_n - y'_n| < \frac{\epsilon}{2K_2}$$

$$\begin{aligned} |x_n y_n - x'_n y'_n| &= |x_n y_n - x_n y'_n + x_n y'_n - x'_n y'_n| \\ &= |x_n(y_n - y'_n) + y'_n(x_n - x'_n)| \\ &\leq |x_n|(y_n - y'_n)| + |y'_n|(x_n - x'_n)| \\ &< K_1 \left(\frac{\epsilon}{2K_1} \right) + K_2 \left(\frac{\epsilon}{2K_2} \right) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \end{aligned}$$

\square

Metric Spaces

Definition 20. *Let $X \neq \emptyset$. A metric on X is a function $d : X \times X \rightarrow [0, \infty)$ such that*

1. $d(x, y) = 0 \implies x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$ for any $x, y, z \in X$.

The metric space in which every cauchy sequence converges is called "complete" metric space.

Proposition 10. *The function $f(x) = \frac{x}{1+x}, x \geq 0$, is monotonically increasing.*

Proof. For some $x, y \geq 0$ and $x > y$, we have $\frac{1}{1+x} < \frac{1}{1+y}$ and $1 - \frac{1}{1+x} > 1 - \frac{1}{1+y} \implies \frac{x}{1+x} > \frac{y}{1+y}$. \square

Theorem 18. *For any two real numbers x and y , this inequality holds*

$$\frac{|x + y|}{1 + |x + y|} \leq \frac{|x|}{1 + |x|} + \frac{|y|}{1 + |y|}$$

Proof. Using previous proposition, we can say if $|x+y| \leq |x|+|y|$ and the sequence $\frac{x}{1+x}, x \geq 0$ is monotonically increasing then

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|+|y|}{1+|x|+|y|} = \frac{|x|}{1+|x|+|y|} + \frac{|y|}{1+|x|+|y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$$

□

Proposition 11. *Generalised AM-GM inequality: If $a > 0$ and $b > 0$ and if $0 \leq \lambda \leq 1$ is fixed, then*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$$

Proof. Since, $y = \ln(x)$ is concave, then

$$\begin{aligned} \ln(\lambda a + (1-\lambda)b) &\geq \lambda \ln(a) + (1-\lambda)\ln(b) \\ \ln(\lambda a + (1-\lambda)b) &\geq \ln(a^\lambda b^{1-\lambda}) \end{aligned}$$

Since e^x is increasing function, we have

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$$

□

Remark. When $x \geq 0, y \geq 0$ and $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$$

Proof. Replace $a = x^p, b = y^q$ and take $\lambda = \frac{1}{p} \implies 1-\lambda = 1-\frac{1}{p} = \frac{1}{q}$

□

Theorem 19. (Holder's inequality) Let $x_i \geq 0$ and $y_i \geq 0$ for $i = 1, 2, \dots, n$, and suppose that $p > 1$ and $q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i^p \right)^{1/p} \left(\sum_{i=1}^n y_i^q \right)^{1/q}$$

In the special case when $p = q = 2$, the above inequality reduces to

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i^2 \right)^{1/2} \left(\sum_{i=1}^n y_i^2 \right)^{1/2}$$

This is called **Cauchy-Schwarz inequality**.