Lecture 1: Theory of Groups and Rings

Definition 1. *Fibre of f over b: For a function $f : A \to B$, the pre-image of $b \in B$ is called the fibre of f over b.*

Definition 2. *Equivalence class of $a \in A$ is defined to be $\{x \mid xRa\}$. The elements of equivalence class of $a \in A$ are said to be equivalent to a and any element of this class is called the representative of this class. They are denoted by $[a]$.*

Lemma 1. *Any two equivalence classes are either disjoint or equal.*

Proof. Suppose, we have two equivalence classes $[a]$ and $[b]$ such that $[a] \neq [b]$. We need to prove that $[a] \cap [b] = \phi$ Suppose for contradiction that $\exists x$ such that $x = [a] \cap [b]$. This means that xRa and xRb. Using symmetry of equivalence relation, we have aRx and xRb and by transitivity we can say aRb. Hence, $[a] = [b]$ which is a contradiction.
We have prove that if they are not equal then they are disjoint. Other way can be proved with similar argument. $\square$

Definition 3. *Partition of A: A partition of A is any collection $\{A_i \mid i \in I\}$ of non-empty subsets of A such that it follows:*

1. $\bigcup\limits_{i \in I} A_i = A$, *and*

2. $A_i \bigcap A_j = \phi \quad \forall i, j \in I$ *and* $i \neq j$.

Remark. *The notion of an equivalence relation on A and a partition of A are the same.*

For a set A, every equivalence relation on A *induces* the partition on set A using equivalence classes. In other words, every equivalence class associated with a equivalence relation forms partition of A.
If R is the equivalence relation on A then the induced partition P will be

$$P = \{\{b \mid bRa, \forall b \in A\} \mid a \in A\}$$

Also, with the given partition P, we can define relation R as

$$R = \{bRa \mid \exists p_i \in P \text{ such that } a, b \in p_i \ \forall i \in I\}$$

We can say a partition P is made up of equivalence classes p_i.

Proposition 1. *Let A be a nonempty set.*

1. *If R defines a equivalence relation on A then the set of equivalence classes of R forms a partition of A.*

2. *If $\{A_i \mid i \in I\}$ is a partition of A then there is an equivalence relation on A whose equivalence classes are sets $A_i, i \in I$.*

Proof. 1. Suppose P is a set of equivalence classes of R, defined as

$$P = \{\{b \mid bRa, b \in A\} \mid a \in A\}$$

We need to prove that P defines partition of A. For some $p_i \in P$, it will be nonempty since it will have atleast a which is related to itself. Now, using lemma 1, we can say that $p_i \cap p_j = \phi$ for some $p_i, p_j \in P$ and $i, j \in \mathbb{N}$ given that $i \neq j$.
Also, we know that every point of set A will be in some equivalence class (reflexivity so atleast in the equivalence of itself). If we take union of all those classes we will get A as each point has a atleast a equivalence class.

2. Given the collection of sets, $Q = \{A_i \mid i \in I\}$ as a partition of A, we need to show that there exists an equivalence relation on A with equivalence classes as sets $A_i, i \in I$.
We can define relation R as

$$R = \{(a, b) \mid bRa \text{ and } \exists A_i \in Q \text{ such that } a, b \in A_i\}$$

It is reflexive (obvious), symmetric (obvious) and transitive (aRb such that $a, b \in A_i$ and bRc such that $b, c \in A_i \implies aRc$ such that $a, c \in A_i$). Hence, it is a equivalence relation and the corresponding equivalence class for some i will be

$$q_i = \{b \mid bRa_i, \forall b \in A \; \& \; a_i \in A\}$$

Since with the same arguments given in proof of part (1) that $q_i, i \in I$ is nonempty, disjoint and exhausts the set A, we can say it forms the partition of A and since every equivalence relation induces a unique partition, we can also say that $q_i, i \in I$ are precisely the sets $A_i, i \in I$.

□

Properties of Integers

Definition 4. *Well Ordering of $\mathbb{Z}$: If A is any nonempty subset of $\mathbb{Z}^+$, there is some element $m \in A$ such that $m \leq a$, $\forall a \in A$. m is called the minimum element of A.*

Definition 5. *$a|b = a$ divides b. $b = ac$ for some $c \in \mathbb{Z}$.*

Definition 6. *For some $a, b \in \mathbb{Z}$, denote $d = gdc(a, b)$ and $l = lcm(a, b)$ then $dl = ab$.*

Definition 7. *The Division Algorithm: If $a, b \in \mathbb{Z} \backslash \{0\}$, then there exists a unique $q, r \in \mathbb{Z}$ such that $a = bq + r$ where $0 \leq r \leq |b|$. And q is called quotient and r is called remainder.*

Definition 8. *The Euclidean Algorithm: Suppose $a, b \in \mathbb{Z} \backslash \{0\}$, we can use this* *algorithm to find the gcd of these two number in the following way:*

$$\begin{aligned}
a &= q_0 b + r_0 \\
b &= q_1 r_0 + r_1 \\
r_0 &= q_2 r_1 + r_2 \\
&\vdots \\
r_{n-2} &= q_{n+1} r_{n-1} + r_n \\
r_{n-1} &= q_{n+2} r_n
\end{aligned}$$

where r_n is the gcd of (a, b). Such an r_n exists because $|b| > |r_0| > |r_1| \ldots$ is a decreasing sequence of strictly postive integers hence it cannot go on to infinite elements.

$\mathbb{Z}$-linear combination of a and b: For $a, b \in \mathbb{Z}\backslash\{0\}$, we have $x, y \in \mathbb{Z}$ such that we can write $gcd(a, b)$ as linear combination of x, y

$$gcd(a, b) = ax + by$$

.

Theorem 1. *Fundamental Theorem of Arithmetic: If $n \in \mathbb{Z}$, $n > 1$, then n can be factored uniquely into the product of primes, i.e. there are distinct primes $p_1, p_2, \ \ldots, p_s$ and positive integers $\alpha_1, \alpha_2, \ \ldots, \alpha_s$, such that*

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$$

This factorization is unique in the sense that the set of $p_i's$ is unique and no other set of primes and the exponent can generate the same number.

Proof. We will use induction to prove the first part that every $n > 1$ can be written as the product of primes.
For $n = 2$ it is true as $2 = 2^1$. Suppose for all the numbers less than n can be written as the product of primes. Now, for n we can have two cases:

Case: 1 If n is prime then it is obvious that it's true.

Case: 2 If n is composite then n can be written as $n = ab$ where $0 < a, b < n$ by definition of composite numbers. And by our assumption for induction it is true thatn a, b can be written as product of primes as they are less than n. Hence, $n = ab$ can also be a product of primes.
Now, about the uniqueness of the primes factors. Let's assume that there exists some primes $q_i's$ and the exponents $\beta_i's$ such that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s} = q_1^{\beta_1} q_2^{\beta_2} \ldots q_s^{\beta_s}$$

Since p_1 divides the left side, it should also divides the right side. Hence, $p_1 | q_i$ for some i. But p_1 and q_i are primes $\implies p_1 = q_i$. WLOG, we can choose $i = 1 \implies p_1 = q_1$.

$$p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s} = p_1^{\beta_1} q_2^{\beta_2} \ldots q_s^{\beta_s}$$

Now, we can have $\alpha_1 > \beta_1$ and we can cancel $p_1^{\beta_1}$ from both sides.

$$p_1^{\alpha_1 - \beta_1} p_2^{\alpha_2} \dots p_s^{\alpha_s} = q_2^{\beta_2} \dots q_s^{\beta_s}$$

But observe that now p_1 divides left side but not the right side. Hence $\alpha_1 \not> \beta_1$. Similar argument for $\alpha_1 < \beta_1$. Therefore, $\alpha_1 = \beta_1$.
Using induction we can show that both sides are equivalent. Hence, primes and their coefficients are unique. □

We can also define lcm and gcd using fundamental theorem of arithmetic as:

$$gcd(a,b) = p_1^{\min(\alpha_1,\beta_1)} p_2^{\min(\alpha_2,\beta_2)} \dots$$
$$lcm(a,b) = p_1^{\max(\alpha_1,\beta_1)} p_2^{\max(\alpha_2,\beta_2)} \dots$$

Definition 9. *Euler ϕ- function: For $n \in \mathbb{Z}^+$ let $\phi(n)$ be the number of positive integers $a \leq n$ with $(a,n) = 1$. For primes p, $\phi(p) = p - 1$, and more generally, $\forall a \geq 1$ we have*

$$\phi(p^a) = p^a - p^{a-1} = p^{a-1}(p-1)$$

The function ϕ is multiplicative *in the sense that $\phi(ab) = \phi(a)\phi(b)$ if $(a,b) = 1$. So for some $n = p_1\alpha_1 p_2\alpha_2 \ \dots \ p_s\alpha_s$ we can write*

$$\phi(n) = \phi(p_1\alpha_1 p_2\alpha_2 \ \dots \ p_s\alpha_s) = \phi(p_1\alpha_1)\phi(p_2\alpha_2) \ \dots \ \phi(p_s\alpha_s)$$

Theorem 2. *If n is composite then there are integers a and b such that $n \mid ab$ but $n \nmid a$ or $n \nmid b$.*

Proof. Since n is composite then $n = x_1^{n_1} x_2^{n_2} \ \dots y_1^{n_1'} y_2^{n_2'} \ \dots$ where x, y are primes $\in (0,n)$ and $n_i's \geq 1 \ \forall i \in \mathbb{N}$. We have to prove the existence of the integers a, b such that $n \mid ab$ but $n \nmid a$ or $n \nmid b$.
We can constuct such integers given the prime factorization of n. If we define $a = x_1^{n_1} x_2^{n_2} \ \dots$ and $b = y_1^{n_1'} y_2^{n_2'} \ \dots$ then we have satisfied the needed conditions. □