Lecture 1

Remarks: Basic ordering properties are assumed to be true.

Definition 1. *Boundedness: The subset $A \in R$ is said to be bounded above if $\exists M$ such that $M > x \ \ \forall x \in A$. And it is bounded below if $\exists m$ such that $m < x \ \ \forall x \in A$. If A has both then it is called bounded.*

Definition 2. *Least Upper Bound (lub) Axiom: If A is nonempty subset of R and it is bounded above, then A has a least upper bound in R.*

Theorem 1. *If A is nonempty subset in R and it is bounded below, then it has a greatest lower bound in R.*

Proof. We first create a set T of lower bounds of A

$$T = \{m \mid m < x \ \ \forall x \in A\}$$

T is non-empty since A is bounded below. Now, we need to prove that there exits a supremum of T which is also a lower bound of A.
Since, set T is bounded above by all the elements of set A, it should have a least upper bound, say M such that $M > m \ \ \forall m \in T$. Also, every element of A is an upper bound of T hence by definition of supremum, we can say $M \leq x \ \ \forall x \in A$ hence M is the lower bound of A. This makes it the greatest lower bound. $\square$

Lemma 2. *Suppose $A \neq \phi$ and $s = lub(A)$ then for any $y \in A$ such that $y < s$, $\exists a \in A$ such that $y < a \leq s$.*

Proof. Suppose for contradiction, $\nexists$ any element a such that $y < a$. This means that $y \geq a, \forall a \in A \implies y$ is upper bound of set A. But y is already less than least upper bound of set A. Hence contradiction.
Therefore, $\exists \ a \in A$ such that $y < a \leq s$. $\square$

Theorem 3. *Archimedean Property: Given any positive real numbers $x, y \ \exists \ n \in N$ such that $nx > y$.*

Proof. Let a set $A = \{nx \mid n \in \mathbb{N}\}$. Suppose for contradiction $nx \leq y$. Then y is the upper bound of the set A.
Let a $x > 0$, then $y - x < y$ hence $y - x$ is not the upper bound of the set A. This means that $\exists m \in \mathbb{N}$ such that $y - x < mx \implies y < mx + x \implies y < (m+1)x$ which is impossible since $(m+1)x \in A$ and y is upper bound of the A.
$nx > y$ is true. $\square$

Theorem 4. *If A and B are the two non empty bounded subsets of R, such that $x \leq y \ \ \forall x \in A$ and $\forall y \in B$ then $sup(A) \leq inf(B)$*

Proof. Let a be the supremum of A and b be the infimum of B. Therefore, $a \geq x \ \ \forall x \in A$ and $b \leq y \ \ \forall y \in B$. Also, A is bounded above by B and elements of B are the upper bound for A. Hence, $a \leq y \ \ \forall y \in B$. This means that a is the lower bound of B and a is $sup(A)$. In other words, $sup(A) \leq inf(B)$. $\square$

Theorem 5. *Given any two real number a, b with $a < b$, $\exists \ \mathbb{Q}$ between a and b.*

Proof. Since $b - a > 0$. Take two positive number $b - a$ and 1 $\exists$ n $\in \mathbb{Z}$ such that $n(b - a) > 1$.
TBD $\square$

Theorem 6. *Any monotone increasing sequence of real numbers that is bounded above converges to some real number.*

Proof. Let x_n be a monotone increasing sequence in $\mathbb{R}$ that is bounded above by s i.e. $s = lub\{x_n \mid n \in \mathbb{N}\}$
Suppose $\epsilon > 0 \implies s - \epsilon < s$ and $s - \epsilon$ is not the upper bound of the x_n.
Using lemma 1, we can say that $\exists x_\epsilon \in x_n$ such that $s - \epsilon < x_\epsilon < s$.
Using monotone condition, for some $n_0 \in \mathbb{N}$, we have,

$$s - \epsilon < x_n < s < s + \epsilon \quad \forall n > n_0 \in \mathbb{N}$$

Hence $|x_n - s| < \epsilon$. x_n converges to $s \;\; \forall n > n_0$. $\square$

Remark. *Nested Interval theorem* $\approx$ *lub* $\approx$ *theorem* 4

Proof. TBD $\square$

Theorem 7. *Nested Interval theorem: Suppose* $\{I_n\}$ *is the sequence of closed and bounded non-empty intervals such that* $I_1 \supset I_2 \supset I_3 \ldots$ *then:*

1. $\bigcap_{n \geq 1} I_n \neq \phi$.
2. *If the sequence of the length of the intervals goes to* 0 *then* $\bigcap_{n \geq 1} I_n = \{x\}$.

Proof. Let I_n be an interval $[a_n, b_n]$ with $a_m < b_n \forall m, n \in \mathbb{N}$. Then $\forall n \in \mathbb{N}$, a_n is the increasing sequence and b_n is the decreasing sequence. b_n is upper bound of a_n hence, $a_n < inf(b_n)$.
For b_n, a_n is the lower bound of b_n i.e. $sup(a_n) < b_n$. If we combine all inequalities, we get

$$a_n \leq sup(a_n) \leq inf(b_n) \leq b_n \;\; \forall n \in \mathbb{N}$$

Using density theorem, we can say that $\exists$ some $\mathbb{Q}$ between $sup(a_n)$ and $inf(b_n)$.
Hence, $\bigcap_{n \geq 1} I_n \neq \phi$.
Let the lenght of the interval to be $L = |b_n - a_n|$. Suppose for contradiction, we have two elements in $\bigcap_{n \geq 1} I_n$ instead of one, say x and y.
The distance between x and y is $|y - x|$. Since, $L \to 0$ hence $\exists n \in \mathbb{N}$ such that for some $n_0 \geq n$, $|L| = |b_{n_0} - a_{n_0}| < \epsilon$ for some $\epsilon > 0$. Since $|L| \to 0$, we can choose ϵ such that it is smaller than $|y - x|$. Then, if interval contains any one of the point, it can not contain the other. $\square$

Lecture 2

Definition 3. *Decimals representation of Real numbers: Let* $z \in \mathbb{R}^+$ *be given. Let* n_0 *be the largest integer such that* $n_0 \leq z$. *Let* n_1 *be the largest integer such that* $n_0 + \frac{n_1}{10} \leq z$. *As such, say* n_k *is defined for some* k. *Let* n_{k+1} *be the largest integer* *such that* $n_0 + \frac{n_1}{10^1} + \frac{n_1}{10^2} + \cdots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \leq z$. *Consider the set of all such finite sums, i.e. the set of all*

$$z_k = n_0 + \frac{n_1}{10^1} + \frac{n_1}{10^2} + \cdots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \leq z$$

This set has a supremum and that is z *itself. We symbolically write* $z = n_0.n_1n_2\ldots$

Lemma 8. *Let* p *be an integer* ≥ 2. *If* $0 \leq a_n \leq p-1$, *where* a_n *is an integer then* $\sum_{n=0}^{\infty} \frac{a_n}{p_n}$ *converges to some* x *in* $[0,1]$.

Proof. Since $0 \leq a_n \leq p-1$, we can replace all a_n with $p-1$ and then we have

$$\sum_{n=1}^{\infty} \frac{a_n}{p_n} \leq (p-1)\sum_{n=1}^{\infty} \frac{1}{p_n} = 1$$

Therefore, the sequence is bounded, and it is monotonic increasing this means it converges to some $x \leq 1$ and x is already positive. Hence it converges tp some x in $[0,1]$. □

Lemma 9. *Conversely, given any* $0 \leq x \leq 1, \exists\ a_n \in \mathbb{Z}$ *and* $0 \leq a_n \leq p-1$ *such that* $x = \sum_{n=0}^{\infty} \frac{a_n}{p_n}$.

Proof. Suppose we have $0 < x \leq 1$ and a_1 is the largest integer such that $\frac{a_1}{p} < x \leq 1$. Since x is bounded above by 1, we have $a_1 < p \implies a_1 \leq p-1$ since a_1 is an integer. Similarly, find a_2 such that $\frac{a_1}{p} + \frac{a_2}{p^2} < x$.This can be achieved by Archimedean property. Also, note that $a_2 \leq p-1$, since we have

$$\begin{aligned}
\frac{a_1}{p} + \frac{a_2}{p^2} &< x < 1 \\
\frac{a_1}{p} + \frac{a_2}{p^2} \leq \frac{p-1}{p} + \frac{a_2}{p^2} &< 1 \qquad (a_1 \leq p-1) \\
1 - \frac{1}{p} + \frac{a_2}{p^2} &< 1 \\
\frac{a_2}{p^2} &< \frac{1}{p} \\
a_2 &< p \\
a_2 &\leq p-1
\end{aligned}$$

Inductively, we can define a_n as the largest integer with $a_n \leq p-1$ such that $\sum_{i=1}^{n} \frac{a_i}{p_i} < x$. Since $a_n < p$ TBD □

Suppose $\{a_n\}$ is the bounded sequence in $\mathbb{R}$, we define two sets:

$$\begin{aligned}
s_n &:= inf\{a_n, a_{n+1}, \ldots\} \\
S_n &:= sup\{a_n, a_{n+1}, \ldots\}
\end{aligned}$$

Notice that $\underset{k}{inf}(\{a_n\}) \leq s_n \leq S_n \leq \underset{k}{sup}(\{s_n\})$

Definition 4. *Lim superior and limit inferior: Let a_n be the bounded sequence of real numbers then*

$$\lim inf(a_n) = \lim_{n\to\infty} s_n$$
$$\lim sup(a_n) = \lim_{n\to\infty} S_n$$

Limit superior is the supremum of all subsequential limits of $\{a_n\}$. Similarly, limit inferior is the infimum of all subsequential limits of $\{a_n\}$.

Note that s_n is the increasing sequence and S_n is the decreasing sequence and they are bounded on both sides. Hence, we can also say that using monotone convergence theorem

$$\lim inf(a_n) = \lim_{n\to\infty} s_n = sup(s_n)$$
$$\lim sup(a_n) = \lim_{n\to\infty} S_n = inf(S_n)$$

Theorem 10. *A sequence $\{a_n\}$ is bounded above iff limsup $a_n < \infty$ (is finite).*

Proof. □