Lecture 1: Theory of Groups and Rings

Definition 1. *Fibre of f over b: For a function $f : A \to B$, the pre-image of $b \in B$ is called the fibre of f over b.*

Definition 2. *Equivalence class of $a \in A$ is defined to be $\{x \mid xRa\}$. The elements of equivalence class of $a \in A$ are said to be equivalent to a and any element of this class is called the representative of this class. They are denoted by $[a]$.*

Lemma 1. *Any two equivalence classes are either disjoint or equal.*

Proof. Suppose, we have two equivalence classes $[a]$ and $[b]$ such that $[a] \neq [b]$. We need to prove that $[a] \cap [b] = \phi$ Suppose for contradiction that $\exists x$ such that $x = [a] \cap [b]$. This means that xRa and xRb. Using symmetry of equivalence relation, we have aRx and xRb and by transitivity we can say aRb. Hence, $[a] = [b]$ which is a contradiction.
We have prove that if they are not equal then they are disjoint. Other way can be proved with similar argument. □

Definition 3. *Partition of A: A partition of A is any collection $\{A_i \mid i \in I\}$ of non-empty subsets of A such that it follows:*

1. $\bigcup_{i \in I} A_i = A$, *and*
2. $A_i \bigcap A_j = \phi \quad \forall i, j \in I$ *and* $i \neq j$.

Remark. *The notion of an equivalence relation on A and a partition of A are the same.*

For a set A, every equivalence relation on A *induces* the partition on set A using equivalence classes. In other words, every equivalence class associated with a equivalence relation forms partition of A.
If R is the equivalence relation on A then the induced partition P will be

$$P = \{\{b \mid bRa, \forall b \in A\} \mid a \in A\}$$

Also, with the given partition P, we can define relation R as

$$R = \{bRa \mid \exists p_i \in P \text{ such that } a, b \in p_i \ \forall i \in I\}$$

We can say a partition P is made up of equivalence classes p_i.

Proposition 1. *Let A be a nonempty set.*

1. *If R defines a equivalence relation on A then the set of equivalence classes of R forms a partition of A.*
2. *If $\{A_i \mid i \in I\}$ is a partition of A then there is an equivalence relation on A whose equivalence classes are sets $A_i, i \in I$.*

Proof. 1. Suppose P is a set of equivalence classes of R, defined as

$$P = \{\{b \mid bRa, b \in A\} \mid a \in A\}$$

We need to prove that P defines partition of A. For some $p_i \in P$, it will be nonempty since it will have atleast a which is related to itself. Now, using lemma 1, we can say that $p_i \cap p_j = \phi$ for some $p_i, p_j \in P$ and $i, j \in \mathbb{N}$ given that $i \neq j$. Union property TBD.

2. Given the collection of sets, $Q = \{A_i \mid i \in I\}$ as a partition of A, we need to show that there exists an equivalence relation on A with equivalence classes as sets $A_i, i \in I$.
We can define relation R as

$$R = \{(a, b) \mid bRa \text{ and } \exists A_i \in Q \text{ such that } a, b \in A_i\}$$

It is reflexive (obvious), symmetric (obvious) and transitive (aRb such that $a, b \in A_i$ and bRc such that $b, c \in A_i \implies aRc$ such that $a, c \in A_i$). Hence, it is a equivalence relation and the corresponding equivalence class for some i will be

$$q_i = \{b \mid bRa_i, \forall b \in A \ \&\ a_i \in A\}$$

Since with the same arguments given in proof of part (1) that $q_i, i \in I$ is nonempty, disjoint and exhausts the set A, we can say it forms the partition of A and since every equivalence relation induces a unique partition, we can also say that $q_i, i \in I$ are precisely the sets $A_i, i \in I$.

$\square$