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IMS22090
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Assignment 1

Ans 1. Given $\epsilon = 10^{-2}$

$$
\begin{aligned}
|a_n| &< 10^{-2} \\
|a_n| &< \frac{1}{100} \\
\left|\frac{1}{n+2}\right| &< \frac{1}{100}
\end{aligned}
$$

$n \in \mathbb{N}$, so $n+2$ is positive for all n.

$$
\begin{aligned}
n+2 &> 100 \\
n &> 98 \\
n &= 99 \quad \text{(least positive integer.)}
\end{aligned}
$$

$$
\begin{aligned}
|b_n| &< 10^{-2} \\
\left|\frac{1}{n^2}\right| &< \frac{1}{100}
\end{aligned}
$$

$n \in \mathbb{N}$ and n^2 is positive for all n.

$$
\begin{aligned}
n^2 &> 100 \\
n &> 10 \\
n &= 11 \quad \text{(least positive integer.)}
\end{aligned}
$$

Ans 3. For domain $[0, 1]$, the range of the function $f(x) = sin(x) + x^2 - 1$ is

$$
\begin{aligned}
R &= sin((0,1)) + ((0,1))^2 - 1 \\
R &= (0, sin(1)) + (0,1) - 1 \\
R &= (0, sin(1)) + (-1, 0) \\
R &= (-1, sin(1))
\end{aligned}
$$

Since, in the interval $(0, 1)$ the function is taking the value $(-1, sin(1))$. It is going from negative to positive in y hence it will definately cut the x axis atleast once which will be the root.

Ans 4. For $f(x) - x = 0$ in the interval $[0, 1]$, if we take $x = 0$, it is simple to observe that $f(0) - 0 = 0$. Hence, $x = 0$ is the solution of the given equation in the interval $[0, 1]$. $f(0) = 0$ also means that 0 is the root of $f(x)$.

Ans 6. Taylor's expansion of $f(x)$ at x_0 is

$$
f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!}
$$

where $k \in \mathbb{Z}^+$. For $x_0 = 0$ and $n = 3$, we can write

$$
\begin{aligned}
sin(x) &= sin(0) + cos(0)x - sin(0)\frac{x^2}{2!} - cos(0)\frac{x^3}{3!} \\
sin(x) &= x - \frac{x^3}{6}
\end{aligned}
$$

Error term will be

$$
E = \frac{x^4}{24} sin(\xi(x))
$$

where $0 < \xi(x) < x$, for $n = 8$ we have

$$
\begin{aligned}
sin(x) &= sin(0) + cos(0)x - sin(0)\frac{x^2}{2!} - cos(0)\frac{x^3}{3!} + sin(0)\frac{x^4}{24} + cos(0)\frac{x^5}{120} \\
&\quad - sin(0)\frac{x^6}{720} - cos(0)\frac{x^7}{7!} + sin(0)\frac{x^8}{8!} \\
sin(x) &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!}
\end{aligned}
$$

Error term will be

$$E = \frac{x^9}{9!} cos(\xi(x))$$

where $0 < \xi(x) < x$

Ans 7. For $f(x) = cos(x)$ and $n = 3$, Taylor's expansion at $x_0 = 0$ will look like

$$
\begin{aligned}
cos(x) &= cos(0) - sin(0)x - cos(0)\frac{x^2}{2} + sin(0)\frac{x^3}{6} \\
cos(x) &= 1 - \frac{x^2}{2}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{0.4} f(x)dx &\approx \int_0^{0.4} \left(1 - \frac{x^2}{2}\right) dx \\
&\approx \left[x - \frac{x^3}{6}\right]_0^{0.4} \\
&\approx 0.4 - \frac{(0.4)^3}{6} \\
&\approx 0.389333\ldots
\end{aligned}
$$

(b) Error function will look like for $0 < cos(\xi(x)) \leq 1$

$$E = \left| cos(\xi(x))\frac{x^4}{24} \right| \leq \left| \frac{x^4}{24} \right|$$

Hence, the upper bound of the error is $\frac{x^4}{24}$.

Ans 8. Taylor's expansion of e^x around $x_0 = 0$ with polynomial degree $n = 4$ will be

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

Putting $x = 1$

$$
\begin{aligned}
e &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \\
e &= 2.70833
\end{aligned}
$$

Error term will be

$$error = \left| \frac{e^{\xi(x)} x^{n+1}}{(n+1)!} \right| \leq 10^{-6}$$

Putting $x = 1$

$$error = \left|\frac{e^{\xi(1)}}{(n+1)!}\right|$$

Since $0 < \xi(x) < x = 1$ and e^x is the increasing function, we have $0 < e^{\xi(x)} < e$

$$\begin{aligned}\left|\frac{e^{\xi(1)}}{(n+1)!}\right| \leq \left|\frac{e}{(n+1)!}\right| &\leq 10^{-6} \\ (n+1)! &\geq \frac{e}{10^{-6}} \\ (n+1)! &\geq e \times 10^6\end{aligned}$$

Ans 9. Second degree Taylor's approximation for $f(x) = \sqrt{x+1}$ at $x_0 = 0$ is

$$\begin{aligned}\sqrt{x+1} &= 1 + \frac{1}{2\sqrt{x+1}} + \frac{-1}{4(x+1)^{3/2}} \\ &= 1 + \frac{1}{2\sqrt{2}} - \frac{x^2}{8}\end{aligned}$$

Remainder term is

$$R = \frac{x^3}{16(\xi(x)+1)^{5/2}}$$

where $x_0 = 0 < \xi(x) < x$. We can obtain a bound for this term if we choose $x \in [0, \infty]$.

Ans 10. For $f(x) = \frac{1}{x}$, we have $f'(x)\frac{-1}{x^2}$, $f''(x) = \frac{2}{x^3}$ and $f'''(x) = \frac{-6}{x^4}$
General expression for derivative will be

$$f^{(k)}(x) = \frac{(-1)^k k!}{x^{k+1}}$$

General expression for Taylor's expansion will be

$$\begin{aligned}\frac{1}{x} &= \sum_{k=0}^{n} \frac{(-1)^k k!(x-x_0)^k}{x_0^{k+1} k!} \\ P_n(x) = \frac{1}{x} &= \sum_{k=0}^{n} (-1)^k (x-1)^k \qquad (x_0 = 1)\end{aligned}$$

$f(3)$ using $P_0(x)$

$$f(3) = 1$$

$f(3)$ using $P_1(x)$

$$f(3) = 1 - (3-1) = -1$$

$f(3)$ using $P_2(x)$

$$f(3) = 1 - 2 + (3-1)^2 = 3$$

$f(3)$ using $P_3(x)$

$$f(3) = 1 - 2 + 4 - (3-1)^3 = -5$$

$f(3)$ using $P_4(x)$

$$f(3) = 1 - 2 + 4 - 8 + (3-1)^4 = 11$$

$f(3)$ using $P_5(x)$

$$f(3) = 1 - 2 + 4 - 8 + 16 - (3 - 1)^5 = -21$$

We can observe that the values of $f(3)$ using Taylor's expansion is not converging rather it is oscillating.