GENERAL TOPOLOGY

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Abstract. We shall learn some general topology.

Definition 1. *Induced metric: A metric which is derived from a norm. A normed space is a special metric space whose metric is derived from a norm.*

Example 1. *$\mathscr{C}[a,b]$: Set of all bounded continuous real function on a closed interval form the normed space with norm defined as*

$$\|f\| = \int_a^b |f(x)|\ dx \quad or, \quad \|f\| = \sup|f(x)|$$

and the induced metric is

$$\|f-g\| = \int_a^b |f(x)-g(x)|\ dx \quad or, \quad \|f-g\| = \sup|f(x)-g(x)|$$

Definition 2. *Distance of a point x from a set A:*

$$d(x,A) = \inf\{d(x,a) \mid \forall a \in A\}$$

Diameter of the set:

$$d(A) = \sup\{d(a_1,a_2) \mid \forall a_1, a_2 \in A\}$$

Definition 3. *Bounded mapping: A mapping f of a non-empty set into a metric space is said to be bounded if its range is bounded i.e. $\exists M \in \mathbb{R}$ such that $|f(x)| \le M$*

Example 2. *A pseudo metric which is not a metric*

$$f, g \in \mathbb{R}^2 \textit{ and } d(f,g) := \textit{difference between their } x \textit{ coordinates}$$

Definition 4. *Interval: A set $A \subset \mathbb{R}$ is an interval if*

$$\forall x, y \in A \textit{ and } \forall t \in \mathbb{R}\colon x \le t \le y \implies t \in A$$

Theorem 1. *Union of intervals with non empty intersection is an interval.*

Proof. Let $\{I_i\}$ be the set of interval and $a \in \cap_i I_i$.
Proof Idea: Take any two points in the union and show that they contains every point in between them (take general point and show

that it will belong to the union).
Let $x, y \in \cup_i I_i$ and let $t \in \mathbb{R}$: $x \leq t \leq y$ then there are following possiblities:
$t < a$,
$t = a$ or,
$t > a$.
All are trivial to show that they lie in union. □

1. Topological Spaces

Definition 5. *Topology: A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:*

- *ϕ and X are in $\mathcal{T}$.*
- *The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.*
- *The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.*

A set X with topology $\mathcal{T}$ is called an topological space $(X, \mathcal{T})$.

Definition 6. *Open set of X: For the topological space $(X, \mathcal{T})$, a subset U of X is an open set of X if U belongs to the collection $\mathcal{T}$.*

Example 3. *Discrete Topology: If X is any set then collection of all subsets of X is a topology on X, called* **discrete topology.**

Example 4. *Indiscrete or trivial topology: The topology consisting of only ϕ and the whole set X is called* **trivial topology.**

Example 5. *Finite complement topology: Let X be a set and $\mathcal{T}$ be the collection of all subset U of X such that $X - U$ is either finite or X. Then $\mathcal{T}$ is called* **finite complement topology***. (This topology is consists of subset of X whose complement is either finite or X.)*

Proof. Let $\{U_i\}$ be the indexed family of subsets of X belongs to $\mathcal{T}$. ϕ and X are obviously there. Assume each $\bigcup_i U_i$ is non-empty (trivial for empty case):

$$X - \bigcup_i U_i = \bigcap_i (X - U_i)$$

Since each U_i is in $\mathcal{T}$, $X - U_i$ is finite. and $\bigcap \liminf_i X - U_i$ is contained in every $X - U_i$ hence it is finite.

To show $\bigcap_i^n X - U_i$ is in $\mathcal{T}$,

$$X - \bigcap_i^n U_i = \bigcup_i^n (X - U_i)$$

Rhs is finite union of finite sets hence it is finite. □

Example 6. *Let X be set and $\mathcal{T}_c$ be the collection of all subsets U of X such that U^c is either countable or all of X. Then $\mathcal{T}_c$ is a topology of X.*

Proof. ϕ and X are trivial inside $\mathcal{T}_c$. Let U_i be the indexed family of subsets of X. Assume $\bigcup\limits_i U_i$ is non-empty (trivial for empty case). To show that $\bigcup\limits_i U_i$ is in $\mathcal{T}_c$

$$X - \bigcup_i U_i = \bigcap_i (X - U_i)$$

Since, $X - U_i$ is countable for each i and $\bigcap\limits_i (X - U_i)$ is in U_i for each i. Hence, $\bigcap\limits_i (X - U_i)$ is countable.
To show that $\bigcap\limits_i U_i$ is in $\mathcal{T}_c$, use the same argument as last example and the fact that finite union of countable sets is countable. □

Definition 7. *Finer or strictly finer topology: For a set X, if $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X such that $\mathcal{T} \subset \mathcal{T}'$ then we say $\mathcal{T}'$ is* **finer** *than $\mathcal{T}$ and if $\mathcal{T}'$ properly contains $\mathcal{T}$ then we say it's* **strictly finer**.

Then $\mathcal{T}$ is called **coarser** *than $\mathcal{T}'$ or,* **strictly coarser** *if it is contained in $\mathcal{T}'$ properly.*

Definition 8. *Comparable: We say $\mathcal{T}$ is comparable with $\mathcal{T}'$ if either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$.*

2. Basis for a Topology

Definition 9. *If X is a set, a* **basis** *for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that*

- *For each $x \in X$, there is atleast one basis element $B \in \mathcal{B}$ such that $x \in B$.*
- *If x belongs to the intersection of two basis elements B_1 and B_2, then there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.*

We define a topology $\mathcal{T}$ generated by $\mathcal{B}$ as: A subset U of X is said to be open in X (e.g. an element of topology on X) if for all $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$.

Remark 1. *Each element of the basis is an element of the topology.*

Example 7. *If X is any set then the collection of all one element subsets of X is a basis for the discrete topology on X.(Power set of X).*

Proof. Trivial to see. □

Lemma 1. *The collection $\mathcal{T}$ generated by the basis $\mathcal{B}$ is a topology.*

Proof. Let the collection $\mathcal{T} = \{U_i\}_{i \in I}$. Condition for the set U_i to belong to the collection is that for each $x \in U_i$ there exists an element $B \in \mathcal{B}$ and $x \in B \subset U_i$.
Membership of ϕ and X: For ϕ, it is vacuously true (true due to non-availability of elements in the set). For X, for each $x \in X$, there exists $B \in \mathcal{B}$ (by definition of basis) such that $x \in B$ and $B \subset X$. □