

# THEORY OF ODE

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ABSTRACT. We shall learn some theory related to ordinary differential equations.

## Part 1. Preliminaries

### 1. PRELIMINARIES FROM REAL ANALYSIS

**Definition 1** (Pointwise convergence). Let  $I$  be any interval in  $\mathbb{R}$ . Let  $f_n : I \rightarrow \mathbb{R}, n = 1, 2, \dots$ , be a sequence of functions. We say that the  $\{f_n\}$  converges pointwise to a function  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  if the sequence  $\{f_n(x)\}$  converges to  $f(x)$  for every  $x \in I$ .

**Remark 1.** Uniform convergence preserves continuity, interchange of limit and integral.

**Theorem 1.** *Uniform limit of the sequence of continuous function is continuous.*

**Remark 2.** Converse of the above theorem is not always true.

**Theorem 2** (Cauchy Criterion). *Let  $\{f_n\}_{n \geq 1}$  a sequence of function defined on a metric space  $(X, d_X)$  with values in a complete metric space  $(Y, d_Y)$ . Then there exists a function  $f : X \rightarrow Y$  such that*

$$f_n \rightarrow f \text{ uniformly on } X$$

*if and only if the following condition is satisfied: For every  $\varepsilon > 0$ , there exists an integer  $n_0$  such that*

$$m, n \geq n_0 \text{ implies } d_Y(f_m(x), f_n(x)) < \varepsilon$$

*for every  $x \in X$ .*

*Proof.* ( $\implies$ ) Assume that the sequence  $\{f_n\}$  converges uniformly on  $X$ . For given  $\varepsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that  $\forall m > n_0$  and for all  $x \in X$

$$d_Y(f_m, f) \leq \frac{\varepsilon}{2}$$

and there exists  $n_1 \in \mathbb{N}$  such that  $\forall n \geq n_1$

$$d_Y(f_n, f) \leq \frac{\varepsilon}{2}$$

Take  $n_3 = \min\{n_0, n_1\}$  then for all  $m, n \geq n_3$

$$\begin{aligned} d_Y(f_m, f_n) &\leq d_Y(f_m, f) + d_Y(f, f_n) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &\leq \varepsilon \text{ for all } x \in X \end{aligned}$$

( $\Leftarrow$ ) Conversely, suppose that  $m, n \geq n_0$  implies that  $d_Y(f_m, f_n) < \varepsilon$  for all  $x \in X$ . Then for each  $x \in X$ , the sequence  $\{f_n(x)\}_{n \geq 1}$  is cauchy in a complete space  $Y$  and therefore converges in  $Y$ . Let  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  for each  $x \in X$ . For  $k > 0$  then

$$d_Y(f_n(x), f_{n+k}(x)) < \frac{\varepsilon}{2}$$

for every  $k = 0, 1, \dots$  and every  $x \in X$ .

$$d_Y(f_n(x), f(x)) = \lim_{n \rightarrow \infty} d_Y(f_n(x), f_{n+k}(x)) \leq \frac{\varepsilon}{2} < \varepsilon$$

$$d_Y(f_n(x), f(x)) < \varepsilon$$

for every  $x \in X$  and  $f_n \rightarrow f$  uniformly over  $X$ .  $\square$

**Theorem 3** (Weierstrass M-test). *Let  $f_1, f_2, \dots$  be a sequence of real valued functions defined on a set  $X$  and suppose that*

$$|f_n(x)| \leq M_n$$

*for all  $x \in X$  and all  $n = 1, 2, \dots$ . If  $\sum_{n=1}^{\infty} M_n$  converges, then  $\sum_{n=1}^{\infty} f_n$  converges uniformly.*

*Proof.* If  $\sum_{n=1}^{\infty} M_n$  converges, then for given  $\varepsilon > 0$ ,

$$\left| \sum_{k=n}^m f_k(x) \right| \leq \sum_{k=n}^m |f_k(x)| \leq \sum_{k=n}^m M_k < \varepsilon$$

for every  $x \in X$  and provided that  $m, n$  are sufficiently large. Now, by cauchy criterion, uniform convergence follows.  $\square$

**Remark 3.** For Uniform boundedness and equicontinuity, the underline space is assumed to be compact hence equicontinuity here is same as uniform equicontinuity.

**Remark 4.** Some authors define pointwise and uniform equicontinuity separately and then on compact space they prove the equivalence.

**Definition 2** (Uniform Boundedness). A sequence of functions  $\{f_n\}$  defined on  $I(\text{compact})$  is said to be uniformly bounded if there exists a constant  $M > 0$  such that  $|f_k(x)| \leq M$ , for all  $x \in I$ , for all  $k \in \mathbb{N}$ .

**Definition 3** (Equicontinuity). A sequence of functions  $\{f_k\}$  defined on  $I(\text{compact})$  is said to be equicontinuity on  $I$ , if for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|f_k(x) - f_k(y)| < \varepsilon$  whenever  $x, y \in I$  and  $|x - y| < \delta$ , and for all  $k$ .

**Remark 5.** If the family of functions is equicontinuous, then each member in the family is uniformly continuous. Converse may not be true always. For e.g.  $f_k(x) = x^k, 0 \leq x \leq 1, k = 1, 2, \dots$  is not an equicontinuous family; however each member is uniformly continuous function (continuous on closed and bounded interval).

**Example 1.** Each finite set of functions defined on compact set is equicontinuous. If the set is singleton then its trivial. For two element set, take minimum of the  $\delta$ 's and then for finite set take minimum of all  $\delta$ 's.

**Theorem 4.** For the sequence of functions  $f_n$  and  $f$  defined on compact set  $E$  and  $f_n \rightarrow f$  uniformly then the family of functions  $A = \{f_n : n \in \mathbb{N}\}$  is equicontinuous.

*Proof.* Since  $f_n \rightarrow f$  uniformly, by cauchy criterion of uniform continuity  $\exists n_0 \in \mathbb{N}$  such that  $\forall m, n \geq n_0$

$$|f_m(x) - f_n(x)| < \frac{\varepsilon}{3}$$

Look at the family of functions  $B = \{f_1, f_2, \dots, f_{n_0}\}$ , since it is finite, it is equicontinuous. For  $k \in \mathbb{N}$

$$|f_k(x) - f_k(y)| < \frac{\varepsilon}{3}$$

whenever  $|x - y| < \delta$  for all  $f_k \in B$  and for all  $x, y \in E$ . Now, for each  $f_n \in A$

$$\begin{aligned} |f_n(x) - f_n(y)| &\leq |f_n(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(y)| + |f_{n_0}(y) - f_n(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &< \varepsilon \text{ whenever } |x - y| < \delta \end{aligned}$$

□

**Theorem 5** (Arzela-Ascoli). Let  $\{f_k\}$  be a sequence of functions in  $C[a, b]$  which is uniformly bounded and equicontinuous. Then, there exists a subsequence  $\{f_{k_n}\}$  of  $\{f_k\}$  such that  $\{f_{k_n}\}$  converges uniformly to a function  $f \in C[a, b]$ .

*Proof.* TODO □

**Definition 4** (Lipschitz continuity). A function  $f: D \subset \mathbb{R} \rightarrow \mathbb{R}$ , is said to be *locally* Lipschitz in  $D$  if for any  $x_0 \in D$ , there exists a neighbourhood  $N_{x_0}$  of  $x_0$  and an  $\alpha = \alpha(x_0) > 0$  such that

$$|f(x) - f(y)| \leq \alpha(x_0)|x - y|, \text{ for all } x, y \in N_{x_0}$$

The function  $f: D \subset \mathbb{R} \rightarrow \mathbb{R}$  is said to be *globally* Lipschitz in  $D$  if there exists  $\alpha > 0$  such that

$$|f(x) - f(y)| \leq \alpha|x - y|, \text{ for all } x, y \in D$$

**Remark 6.** The smallest  $\alpha$  satisfying the above equation is called *Lipschitz constant* of  $f$ . It should be finite.

**Remark 7.** If  $f$  is globally Lipschitz, then it is uniformly continuous. (Take  $\delta = \frac{\varepsilon}{\alpha}$ ).

**Theorem 6** (Sufficient condition for Lipschitz continuity). *Suppose  $D$  is an open interval in  $\mathbb{R}$  and  $f: D \rightarrow \mathbb{R}$  is differentiable on  $D$  and  $\alpha = \sup_{x \in D} |f'(x)| < \infty$ . Then,  $f$  is Lipschitz with a Lipschitz constant less than or equal to  $\alpha$ .*

*Proof.* Use mean value theorem:

$$\frac{|f(b) - f(a)|}{|b - a|} = f'(c) \leq \alpha$$

And Lipschitz constant is by definition the smallest  $\alpha$  satisfying above inequality. Therefore, Lipschitz constant  $\leq \alpha$ . □

**Example 2.** Example of Lipschitz continuous functions: polynomials, polynomials of sine and cosine, exponential functions etc.

**Definition 5** (Lipschitz continuity for vector valued map). A function  $\mathbf{f}(t, \mathbf{y}): (a, b) \times D \rightarrow \mathbb{R}^n$  is said to be Lipschitz continuous (globally) with respect to  $\mathbf{y}$  if there exists  $\alpha > 0$  such that

$$|\mathbf{f}(t, \mathbf{y}_1) - \mathbf{f}(t, \mathbf{y}_2)| \leq \alpha|\mathbf{y}_1 - \mathbf{y}_2|$$

for all  $(t, \mathbf{y}_1)$  and  $(t, \mathbf{y}_2)$  in  $(a, b) \times D$ .  $\alpha$  should be finite.

**Theorem 7** (Sufficient condition for Lipschitz continuity of  $\mathbf{f}(t, \mathbf{y})$ ). *Let  $\mathbf{f}: (a, b) \times D \rightarrow \mathbb{R}^n$  be a  $C^1$  vector valued function, where  $D$  is a convex domain in  $\mathbb{R}^n$  such that*

$$\sup_{(t, \mathbf{y}) \in (a, b) \times D} \left| \frac{\partial f_i}{\partial y_j}(t, \mathbf{y}) \right| = \alpha < \infty,$$

for  $i, j = 1, 2, \dots, n$ . Then,  $\mathbf{f}(t, \mathbf{y})$  is Lipschitz continuous on  $(a, b) \times D$  with respect to  $\mathbf{y}$  having a Lipschitz constant less than or equal to a multiple of  $\alpha$ .

**Remark 8** (Convex Domain). A set is called a convex domain if it contains all the line segments between any two points of the set.

**Remark 9.** Lipschitz continuity is a smoothness property stronger than continuity, but weaker than differentiability, locally. There can be functions which are not differentiable but still they can be Lipschitz continuous.

**Theorem 8** (Calculus Lemma). Let  $(a, b)$  be a finite or infinite interval and  $h: (a, b) \rightarrow \mathbb{R}$  satisfy either

- (i)  $h$  is bounded above and non-decreasing or
- (ii)  $h$  is bounded below and non-increasing,

then,  $\lim_{t \rightarrow b} h(t)$  exists.

*Proof.* Let  $\alpha = \sup_{t \in (a, b)} h(t)$ . For some  $\varepsilon > 0$ ,  $\alpha - \varepsilon$  is not a supremum.

Hence there exists  $t_0$  such that  $\alpha - \varepsilon < h(t_0) < \alpha$ . For some  $t \in (a, b)$  such that  $t \geq t_0$  then

$$\alpha - \varepsilon < h(t_0) \leq h(t) < \alpha$$

Then  $\alpha - h(t) < \varepsilon$ . Also above can be written as  $\alpha < h(t_0) + \varepsilon \leq h(t) + \varepsilon < \alpha + \varepsilon$ . Then  $h(t) + \varepsilon < \alpha + \varepsilon \implies h(t) < \alpha + \varepsilon \implies h(t) - \alpha < \varepsilon$ . Therefore

$$|h(t) - \alpha| < \varepsilon \quad \forall t \geq t_0 \implies \lim_{t \rightarrow b} h(t) = \alpha$$

□

**Theorem 9** (Change of Variable Formula). Let  $g: [c, d] \rightarrow \mathbb{R}$  be a  $C^1$  function and let  $[a, b]$  be any interval containing the image of  $g$ , that is  $g[c, d] \subset g[a, b]$ . If  $f: [a, b] \rightarrow \mathbb{R}$  is a continuous function, then

$$\int_c^d f(g(t))g'(t)dt = \int_{g(c)}^{g(d)} f(x)dx.$$

**Theorem 10** (Generalized Leibnitz Formula). Let  $\alpha, \beta: [a, b] \rightarrow \mathbb{R}$  be differentiable functions and  $c, d$  be real numbers satisfying

$$c \leq \alpha(t), \beta(t) \leq d, \text{ for all } t \in [a, b].$$

Let  $f: [a, b] \times [c, d] \rightarrow \mathbb{R}$  be a continuous function such that  $\frac{\partial f}{\partial t}(t, s)$  is also continuous. Define

$$F(t) = \int_{\alpha(t)}^{\beta(t)} f(t, s)ds.$$

Then,  $F$  is differentiable and

$$\frac{dF}{dt} = \int_{\alpha(t)}^{\beta(t)} \frac{\partial f}{\partial t}(t, s) ds + f(t, \beta(t)) \frac{d\beta}{dt} - f(t, \alpha(t)) \frac{d\alpha}{dt}.$$

**Definition 6** (Banach space). A normed linear space which is a complete metric space (metric space by the norm) is called a Banach space.

**Example 3** (Examples of Banach space).  $\mathbb{R}^n$  under supremum norm and  $p$ -norm. Function space  $C[a, b]$  with supremum norm. However, it is *not* a complete space with respect to  $\|\cdot\|_1$  norm.

**Remark 10.** A sequence  $\{f_n\} \subset C[a, b]$ ,  $f_n \rightarrow f$  in supremum norm is equivalent to saying that  $f_n$  converges uniformly to  $f$  ( $\exists \delta > 0$  which works in supremum norm for  $f$ , will work for all  $f$  in the sequence.)

**Theorem 11** (Banach Fixed Point theorem). Suppose  $(X, d)$  is a complete metric space and  $T: X \rightarrow X$  is a contraction, that is, there exists an  $\alpha \in (0, 1)$  such that

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all  $x, y \in X$ . Then,  $T$  has a unique fixed point  $x^* \in X$ . Further, the sequence  $\{x_k\}$  defined by  $x_k = Tx_{k-1}$ ,  $x_0 \in X$  is arbitrary and  $k = 1, 2, \dots$ , converges to  $x^*$ .

**Remark 11.** If we omit the contraction condition then  $f(x) = x + 1$  show there does not exist any fixed point. If completion condition is dropped then  $f: (0, 1) \rightarrow (0, 1)$  defined by  $f(x) = mx$ ,  $0 < m < 1$  is a contraction with  $\alpha = m$  but no fixed point.

*Proof.* Choose any  $x_0 \in X$  and define the sequence  $x_1 = Tx_0$ ,  $x_2 = T^2x_0, \dots, x_k = T^kx_0, \dots$ . We need to show that sequence  $\{x_k\}$  is a Cauchy sequence. To see this

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \alpha d(x_n, x_{n-1})$$

By induction,

$$d(x_{n+1}, x_n) \leq \alpha^n d(x_1, x_0)$$

Consider, for  $m < n$  and using triangle inequality

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + \dots + d(x_{m+1}, x_m) \leq \alpha^m \frac{1 - \alpha^{n-m-1}}{1 - \alpha} d(x_1, x_0)$$

and the rhs  $\rightarrow 0$  as  $n, m \rightarrow \infty$ . Hence the sequence is Cauchy and by completeness, there exists  $x^* \in X$  such that  $x_k \rightarrow x^*$ . By continuity of  $T$ , we get  $Tx_k \rightarrow Tx^*$ . By continuity of  $T$ , we get  $Tx_k \rightarrow Tx^*$ . But  $Tx_k = x_{k+1} \rightarrow x^*$ . Thus,  $Tx^* = x^*$ . If there exists  $y^*$  with same properties as  $x^*$  then by uniqueness of limits  $x^* = y^*$ .  $\square$

**Remark 12.** Here, fixed point can be constructed with any desired accuracy as limiting value is fixed point. And secondly, any point can be taken as an initial guess.

**Corollary 1.** *Let  $T: X \rightarrow X$  where  $X$  is a complete metric space, be such that  $T^k$  is a contraction for some  $k \geq 1$ . Then,  $T$  has a unique fixed point.*

*Proof.* Since,  $T^k$  is contraction map hence it has a unique fixed point (by Banach fixed point theorem). Let  $x^*$  be that point i.e.  $T^k x^* = x^*$ . Applying  $T$ , we get  $T^k(Tx^*) = Tx^*$  implies  $Tx^*$  is the fixed point for  $T^k \implies Tx^* = x^*$ . Therefore,  $x^*$  is the fixed point of  $T$ . For uniqueness, assume  $x_1$  is another fixed point for  $T \implies Tx_1 = x_1$ . Applying  $T$  repeatedly, we get  $T^k x_1 = x_1$  but fixed point for  $T^k$  is  $x^*(\text{unique}) \implies x^* = x_1$ .  $\square$

## 2. PRELIMINARIES FROM LINEAR ALGEBRA

**Definition 7** (Normed Linear Space). A norm, denoted by  $\|\cdot\|$  on a vector space or a linear space  $X$  is a mapping from  $X \rightarrow \mathbb{R}$  that satisfies

- $\|x\| \geq 0$ ;  $\|x\| = 0$  if and only if  $x = 0$ ,
- $\|ax\| = |a|\|x\|$ ,
- (Triangle inequality)  $\|x + y\| \leq \|x\| + \|y\|$ ,

for all  $x, y \in X$  and scalar  $a$ .

**Example 4.**  $\mathbb{R}^n, \mathbb{C}^n$

**Remark 13.** Every normed linear space is a metric space with metric induced by the norm. A metric space can be equipped with different norms which are fundamentally different.  $C[0, 1]$  with sup norm is complete while with the integral norm  $\left(\int_0^1 |f(x)| dx\right)$  is not complete.

**Definition 8** (Vector Field). A vector field on a space (most commonly Euclidean space) is a function  $\vec{F}$  that assigns a vector to each point of the space.

**Definition 9** (Matrix norm).  $|A|$  should satisfy the following criterion

- $|A| \geq 0$ ;  $|A| = 0$  if and only if  $A = 0$ ,
- $|aA| = |a||A|$ ,
- (Triangle inequality)  $|A + B| \leq |A| + |B|$ ,
- $|AB| \leq |A||B|$

for all  $A, B \in M_n(\mathbb{R})$  and scalars  $a$ .

**Remark 14.**  $M_n(\mathbb{R})$  is a complete metric space.

**2.1. Matrix Exponential  $e^A$ .** Let  $A \in \mathbb{M}_n\mathbb{R}$ , define the sequence of matrices

$$S_k = I + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!}.$$

For  $k > l$ ,

$$|S_k - S_l| \leq \sum_{j=l+1}^k \frac{|A|^j}{j!} \rightarrow 0 \text{ as } l, k \rightarrow \infty.$$

Thus  $\{S_k\}$  is a Cauchy sequence and converges to some  $S \in M_n(\mathbb{R})$ .

**Definition 10.** Given  $A \in M_n(\mathbb{R})$ ,  $e^A$  is defined as

$$e^A = S$$

where  $S = \lim_{k \rightarrow \infty} \sum_{j=0}^k \frac{A^j}{j!}$ .

**Remark 15.**  $e^A \in M_n(\mathbb{R})$ . Also,  $|e^A| \leq e^{|A|}$  (substitute the values to see this.)

**Remark 16.** For diagonal matrix  $A = \text{diag}(\lambda_1, \dots, \lambda_n) \implies A^j = \text{diag}(\lambda_1^j, \dots, \lambda_n^j)$  and  $e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!}$  then

$$e^A = \sum_{j=0}^{\infty} \frac{1}{j!} \text{diag}(\lambda_1^j, \dots, \lambda_n^j) \implies e^A = \text{diag}\left(\sum_{j=0}^{\infty} \frac{\lambda_1^j}{j!}, \dots, \sum_{j=0}^{\infty} \frac{\lambda_n^j}{j!}\right),$$

therefore,

$$e^A = \text{diag}(e^{\lambda_1}, \dots, e^{\lambda_n}).$$

Here are important observations:

- (1) If  $A \sim B \implies \exists$  non-singular matrix  $P$  such that  $B = PAP^{-1}$ . Then  $e^A \sim e^B$  ( $\because B^j = PA^jP^{-1} \implies e^B = P \left( \sum_{j=0}^{\infty} \frac{A^j}{j!} \right) P^{-1} \implies e^B = Pe^AP^{-1}$ ).
- (2) If  $A$  represents a block diagonal matrix  $A = \text{diag}(A_1, \dots, A_k)$  where all  $A_i$ 's are square matrix on the diagonal (their size maybe different) then

$$e^A = \text{diag}(e^{A_1}, e^{A_2}, \dots, e^{A_k}).$$

**Remark 17.** (a)  $|A| \leq \max\{|A_1|, \dots, |A_k|\} \implies$

(b)  $|e^A| \leq \max\{e^{|A_1|}, \dots, e^{|A_k|}\}$ . ((a)suspicious identity to me!!)

(c) Equality in above cases holds for Euclidean norm.



**Computing  $e^A$ .** If  $A$  is diagonalizable then we look for matrix  $P$  so that  $PAP^{-1} = B$  is diagonal matrix hence computing  $e^A$  is easy.

If  $A$  is not diagonalizable then we look for  $P$  so that  $PAP^{-1}$  is a block diagonal, with easily computable  $e^{A_i}$ .

If  $T$  is a linear transformation and it is invariant on all coordinate axes then  $A$  is diagonalizable with respect to standard basis.

If usual coordinate axes are invariant under  $T$  then we look for  $n$  distinct directions, if possible, which are invariant under  $T$  and then take these directions as new bases. The matrix with respect to this basis will be diagonal matrix.

**Remark 18.** The set of all eigenvalues of  $A$  is known as *spectrum* of  $A$ . It is denoted by  $\sigma(A)$ .

The eigenvalues of  $A$  are the roots of the *characteristics polynomials*  $\det(\lambda I - A)$  which is a real polynomial in  $\lambda$  of degree  $n$ . The roots maybe real or complex. If the eigenvalue is real then there exists a corresponding real eigenvector and if it complex then there exists a corresponding complex eigenvector.

### 3. DOUBTS

**Ques 1.** How to prove theorem 7? Is there a mean value theorem for vector valued functions that I can apply here?

**Ques 2.** Proof for change of variable formula in Theorem 9?

**Ques 2.** Proof for generalized Leibnitz formula in Theorem 10?

## Part 2.

### First and Second order linear equations

#### 4. FIRST ORDER EQUATIONS

**4.1. General form of IVP and BVP.** General form of IVP of first order ODE is

$$\left. \begin{array}{l} \dot{y} = f(t, y) \\ y(t_0) = y_0 \end{array} \right\}$$

General form of IVP of second order ODE is

$$\left. \begin{array}{l} \ddot{y} = f(t, y, \dot{y}) \\ y(t_0) = y_0 \\ \dot{y}(t_0) = y_1 \end{array} \right\}$$

General form of BVP

$$\left. \begin{array}{l} \ddot{y} = f(t, y, \dot{y}) \text{ for } t \in (a, b) \\ \alpha_1 y(a) + \beta_1 \dot{y}(a) = \gamma_1 \\ \alpha_2 y(b) + \beta_2 \dot{y}(b) = \gamma_2 \end{array} \right\}$$

**Definition 11** (Solution of ODE). Let  $f$  be defined on the rectangle in  $R := (a, b) \times (c, d)$  containing the initial data  $(t_0, y_0)$ . A solution to the IVP is a function  $y : (\bar{a}, \bar{b}) \rightarrow \mathbb{R}$  which is differentiable and satisfies the IVP together with initial condition.

**Remark 19.** Interval  $(\bar{a}, \bar{b})$  is referred as interval of existence of solution. For all  $t \in (\bar{a}, \bar{b}) \subset (a, b)$ ,  $y(t) \in (c, d)$ . If  $(\bar{a}, \bar{b}) = (a, b)$  then  $y$  is called the global solution otherwise it is a local solution.

For vector valued functions, definition of solution can be extended. Let  $\mathbf{f} : (a, b) \times \Omega \rightarrow \mathbb{R}^n$  be a vector valued continuous function so that  $\mathbf{f} = (f_1, \dots, f_n)$  and each  $f_i$  is a real valued continuous (we assume this throughout the notes) function, where  $\Omega \subset \mathbb{R}^n$  is open domain. For a given initial value  $y_0 \in \Omega$ , the IVP is given by

$$\left. \begin{array}{l} \dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \\ \mathbf{y}(t_0) = \mathbf{y}_0 \end{array} \right\}$$

**4.2. First Order linear equations.** A general first order ODE can be written as

$$f(t, y, \dot{y}) = h(t)$$

where  $h(t)$  is the function of  $t$  only. From linear algebra it can be shown that  $f$  takes the form (TODO)

$$f(t, y, \dot{y}) = p_0(t)\dot{y} + p_1(t)y$$

Thus the linear differential operator is given by  $L = p_0(t)\frac{d}{dt} + p_1(t)$ . General linear and homogeneous ODE is given by

$$Ly = 0$$

and non-homogeneous ODE is given by

$$Ly = q(t)$$

where  $p_0, p_1$  and  $q$  are the given functions of  $t$ .

**Remark 20.** Due to the linear structure of the operator, it follows the superposition principle i.e. if  $y_1$  and  $y_2$  are the two solutions of the homogeneous equation then  $\alpha y_1 + \beta y_2$  is also a solution of the homogeneous equation.

**Definition 12** (Singular equations). The equations in which coefficient of the highest order term vanished at one or more points are called singular equations. E.g. Bessel's eqn, Lagrange eqn, Legendre eqn.

**Definition 13** (Regular equations). The coefficient of highest order term is never 0. Their general form is given by  $Ly := \dot{y} + p(t)y = q(t)$ .

**Remark 21.** A continuous function defined on a interval in  $\mathbb{R}$  whose modulus is a constant, then  $f$  itself is constant.

4.2.1. **Integration factor.** For non-homogeneous equation

$$\dot{y} + p(t)y = q(t)$$

we can find a function  $h(t)$  such that  $\dot{h}(t) = \dot{y} + p(t)y$  then we  $\dot{h}(t) = q(t)$  which can be integrated easily to find the solution. Let a differentiable function  $\mu(t)$  then  $d(\mu y)/dt = \mu \dot{y} + \dot{\mu}y$ . Multiplying both sides by  $\mu(t)$  in main non-homogeneous equation and then comparing with the derivative of  $\mu(t)$ , we get  $\dot{\mu}(t) = \mu(t)p(t)$ . Here  $\mu(t)$  is called *integrating*

$$\mu(t)\dot{y}(t) + \mu(t)p(t)y(t) = \mu(t)q(t). \quad (3.1.16)$$

If  $\mu$  is positive, then any solution of (3.1.15) is a solution of (3.1.16) and vice versa. The term on the left hand side of (3.1.16) can be written as  $\frac{d}{dt}(\mu y)$ , provided  $\mu$  satisfies  $\dot{\mu}(t) - p(t)\mu(t) = 0$ . Thus, (3.1.16) is exact. Note that the equation satisfied by  $\mu$  is a homogeneous linear DE in  $\mu$  and hence,  $\mu(t) = \exp\left(\int^t p(\tau)d\tau\right)$  is a solution and it is positive. Thus, (3.1.16) becomes

FIGURE 1. why  $\mu$  has to be positive? Check this.

*factor* associated with homogeneous part.

### 4.3. Exact Differential equations.

**Definition 14** (Exact Differential equations). If the differential equation  $\dot{y} = f(t, y)$  can be written as  $\frac{d}{dt}\phi(t, y(t)) = 0$  for a two variable function  $\phi$  in a domain in the  $t-y$  plane, then the differential equation is said to be an exact differential equation.

**Necessary condition for DE to be exact.** Consider the differential form  $Mdt + Ndy$ , this can be written as general first order equation as

$$M(t, y) + N(t, y)\dot{y} = 0$$

Above DE is exact *if and only if* there exists  $\phi$  such that  $\frac{d}{dt}\phi(t, y) = 0$ . Therefore,

$$\frac{\partial}{\partial t}\phi(t, y) + \frac{\partial}{\partial y}\phi(t, y)\dot{y} = M(t, y) + N(t, y)\dot{y}$$

and this implies

$$M = \frac{\partial \phi}{\partial t} \text{ and } N = \frac{\partial \phi}{\partial y}.$$

Assuming  $\phi$  is double differentiable, we get

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

**Theorem 12.** Assume  $M, N$  are defined on a rectangle  $D = (a, b) \times (c, d)$  and  $M, N \in C^1(D)$ . Then, there exists a function  $\phi$  defined in  $D$ , such that  $M = \frac{\partial \phi}{\partial t}$  and  $N = \frac{\partial \phi}{\partial y}$  if and only if  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$ .

*Proof.* ( $\implies$ ) Assume there exists a function  $\phi$  following the needed conditions given in theorem. The first relation  $M = \frac{\partial \phi}{\partial t}$  suggests that  $\phi(t, y) = \int Mdt + h(y)$  for some  $h$ . Hence,

$$N = \frac{\partial \phi}{\partial y} = \int \frac{\partial M}{\partial y} dt + \frac{\partial h(y)}{\partial y} \implies \frac{\partial h}{\partial y} = N - \int \frac{\partial M}{\partial y} dt$$

Since, LHS is a function of  $y$  alone hence RHS should be a function of  $y$  alone. Therefore,

$$\frac{\partial}{\partial t} \left( N - \int \frac{\partial M}{\partial y} dt \right) = 0 \implies \frac{\partial N}{\partial t} - \frac{\partial M}{\partial y} = 0$$

( $\Leftarrow$ ) Assume  $M$  and  $N$  satisfies  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$ . Let a point  $(t_0, y_0) \in D$ . Define  $\phi(t, y) = \int_{t_0}^t M(s, y)ds + h(y)$  where  $h$  has to be determined.

$$\begin{aligned}\frac{\partial \phi(t, y)}{\partial y} &= \int_{t_0}^t \frac{\partial M(s, y)}{\partial y} ds + \frac{\partial h(y)}{\partial y} \\ &= \int_{t_0}^t \frac{\partial N(s, y)}{\partial t} ds + \frac{\partial h(y)}{\partial y} \\ &= N(t, y) - N(t_0, y) + \frac{\partial h(y)}{\partial y}\end{aligned}$$

If we want  $\frac{\partial \phi}{\partial y}$  to be equal to  $N(t, y)$  then  $\frac{\partial h(y)}{\partial y} = N(t_0, y) \implies h(y) = \int_{y_0}^y N(t_0, \xi)d\xi$ . Therefore,

$$\phi(t, y) = \int_{t_0}^t M(s, y)ds + \int_{y_0}^y N(t_0, \xi)d\xi$$

$\phi$  is determined upto a constant. Hence on changing the point  $(t_0, y_0)$  the constant term in  $\phi$  will change. We may discard all the constant in expression of  $\phi$ .  $\square$

**Definition 15** (Alternate definition of Exact ODE). The DE,  $M(t, y) + N(t, y)\dot{y} = 0$  is said to be exact if  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$ .

**Solution of exact DEs.** If a DE is exact then it can be written as  $\frac{\partial \phi}{\partial t}(t, y) = 0$  and the solution of this DE can be written as  $\phi = \text{constant}$ .

## 5. SECOND ORDER LINEAR EQUATIONS

A second order linear differential equation is given by

$$Ly \equiv \ddot{y} + p(t)\dot{y} + q(t)y = r(t).$$

The operator  $Ly = L(t, y, \dot{y}, \ddot{y})$  is a multi-linear operator (linear in each of its variable). The IVP for second order linear ODE is given by

$$\left. \begin{aligned} \ddot{y} + p(t)\dot{y} + q(t)y &= r(t), t \in I \\ y(t_0) &= y_0, \dot{y}(t_0) = y_1 \end{aligned} \right\}$$

**Theorem 13.** Let  $p, q, r$  be continuous functions defined in a compact interval  $I(t_0)$  and  $y_0, y_1$  be any real numbers. Then, the IVP above has a unique solution  $y$  defined in  $I(t_0)$  satisfying  $y(t_0) = y_0$  and  $\dot{y}(t_0) = y_1$

**Lemma 1.** Wronskian  $W(z, w) \equiv 0$  if and only if  $z$  and  $w$  are dependent. Alternatively,  $W \neq 0$  if and only if  $z$  and  $w$  are independent.

*Proof.* Let  $t_0 \in I$  be a point in interval  $I$  on which  $z$  and  $w$  are defined. Then the Wronskian is given by

$$W = \begin{vmatrix} z(t_0) & w(t_0) \\ \dot{z}(t_0) & \dot{w}(t_0) \end{vmatrix}$$

This comes from the linear system (for  $y_0$  and  $y_1$  check next theorem)

$$\begin{cases} z(t_0)\alpha + w(t_0)\beta = y_0 \\ \dot{z}(t_0)\alpha + \dot{w}(t_0)\beta = y_1 \end{cases}$$

( $\Leftarrow$ ) If  $z = kw$  or  $w = kz$  then the  $W \equiv 0$ .

( $\Rightarrow$ ) Assume that  $W \equiv 0 \implies z\dot{w} - \dot{z}w = 0$

$$\begin{aligned} z \frac{dw}{dt} &= \frac{dz}{dt} w \\ \int \frac{dw}{w} \times dt &= \int \frac{dz}{z} \times dt \\ \ln w &= \ln z + c \\ w &= e^{\ln z} e^c \\ w &= ze^c \end{aligned}$$

Therefore,  $w$  and  $z$  are dependent.  $\square$

### 5.1. Homogeneous equations.

**Theorem 14.** Let  $z, w$  be two solutions of the linear homogeneous second order DE. Then, for any  $\alpha, \beta \in \mathbb{R}$ , the function  $y = \alpha z + \beta w$  is also a solution.

Further, if  $z$  and  $w$  are two linearly independent solution of the homogeneous DE in an interval  $I(t_0)$ , then every solution  $y$  can be written as  $\alpha z + \beta w$ .

*Proof.* First part of the theorem is trivial to verify. Let  $y$  be a solution of the homogeneous DE. For some  $t_0 \in I$ ,  $y(t_0) = y_0$  and  $\dot{y}(t_0) = y_1$ . We want to show that there exists  $\alpha$  and  $\beta$  such that  $y(t) = \alpha z(t) + \beta w(t)$ . Since  $z$  and  $w$  are independent,  $W \neq 0$  then there exists a unique solution to the linear system

$$\begin{cases} z(t_0)\alpha + w(t_0)\beta = y_0 \\ \dot{z}(t_0)\alpha + \dot{w}(t_0)\beta = y_1 \end{cases}$$

Hence, there exists  $\alpha$  and  $\beta$ . Here,  $z$  and  $w$  are basis elements for the solution space of second order linear homogeneous DE. Therefore, all the solution for this equation can be written in this form.  $\square$

**Corollary 2.**  $\dim(S) = 2$  where  $S$  is the solution space of second order linear homogeneous DE.

*Proof.* By previous theorem, it is clear that  $\{z, w\}$  is a spanning and independent set. Hence, it is a basis of the solution space  $S$  implies  $\dim(S) = 2$ . TODO.  $\square$

**Remark 22.** For  $n$ -th order linear homogeneous DE, the solution space is  $n$ -dimensional.

**5.1.1. Method to find independent solutions for homogeneous linear ODE.** The main idea is to remove the term involving the first order derivative  $\dot{y}$  via an integrating factor.

Solution form we look for  $y = u(t)v(t)$  implies  $\dot{y} = \dot{u}v + u\dot{v}$  and  $\ddot{y} = u\ddot{v} + 2\dot{u}\dot{v} + \ddot{u}v$ . Substituting in  $Ly = 0$ , we get

$$(u\ddot{v} + 2\dot{u}\dot{v} + \ddot{u}v) + p(t)(\dot{u}v + u\dot{v}) + q(t)uv = 0$$

$$u\ddot{v} + (2\dot{u} + p(t)u)\dot{v} + (\ddot{u} + p(t)\dot{u})v + q(t)uv = 0$$

Choose  $u$  such that coefficient of  $\dot{v}$  vanishes

$$2\dot{u} + p(t)u = 0$$

**Note.** Here,  $u$  never vanishes if it is not 0 initially. From above equation  $u$  can be found. The final equation after putting the above condition can be solved for  $v$  hopefully in some situations.

**5.1.2. Method of Reduction of order.** This methods can be used to find the another linearly independent solution of the linear homogeneous equation given a non trivial solution to the equation.

Give  $y_1(t)$  to be a non-trivial solution, to produce another independent solution we assume the form  $y_2(t) = c(t)y_1(t)$ . We can find Wronskian to show that  $y_1(t)$  and  $y_2(t)$  are independent with their general form that we have assumed.

## 5.2. Non-homogeneous equations.

$$Ly := \ddot{y} + p(t)\dot{y} + q(t)y = r(t) \quad (1)$$

**Theorem 15.** Assume that the functions  $p$ ,  $q$  and  $r$  in (1) are continuous. Let  $S$  and  $\tilde{S}$  be respectively, the solution set of homogeneous equation and non homogeneous equation. Then,  $S$  is a linear space of dimension 2 and  $\tilde{S}$  is an affine space given by  $\tilde{S} = S + y_p$  for any particular solution  $y_p$  of (1).

*Proof.* The assumption of continuity is needed for existence of solution. We have already proved in theorem 15 that the dimension of  $S$  is 2. Since, any linear combination of solution is a solution and scalar times the solution is a solution,  $S$  is linear.

For some arbitrary  $y \in \tilde{S}$  and let some  $y_p$  be a particular solution for (1). Since  $Ly = r(t)$  and  $Ly_p = r(t)$  implies that  $L(y - y_p) = 0$  (due

to linearity of operator  $L$ ) then  $y - y_p$  is a solution of homogeneous equation. Hence, by theorem 14,  $y - y_p = \alpha z + \beta w$  where  $z$  and  $w$  are two independent solutions of homogeneous equation. Also,  $y$  is arbitrary

$$y = \alpha z + \beta w + y_p$$

$$\tilde{S} = S + y_p$$

□

**5.2.1. Method of variation of parameters.** This method can be used to find the particular solution, given the two independent solutions of corresponding homogeneous equation.

If  $y_1(t)$  and  $y_2(t)$  are the two independent solutions of homogeneous equation. We seek the solution for the non homogeneous equation, to be of the form

$$y(t) = c_1(t)y_1(t) + c_2(t)y_2(t)$$

where  $c_1(t)$  and  $c_2(t)$  are the functions to be determined. Compute  $Ly(t) = r(t)$ , and then assume  $\dot{c}_1 y_1 + \dot{c}_2 y_2 = 0$  to be one equation. Another equation can be found by substituting above assumption in  $Ly(t) = r(t)$ . Then by solving this system, we can find  $c_1(t)$  and  $c_2(t)$  and  $y_p(t) = y(t)$ .

**Why  $\dot{c}_1 y_1 + \dot{c}_2 y_2 = 0$ ?** Since, we just want a particular solution to the non-homogeneous equation under some condition hence we will make an assumption such that our calculation become easy. By this assumptions, we will get rid of all terms with derivative of  $c_1$  and  $c_2$ .

**5.2.2. Method of Undetermined coefficient.** (For special Non homogeneous terms) Consider the equation

$$Ly := \ddot{y} + p\dot{y} + qy = e^{at}$$

- (1) For RHS term to be  $e^{at}$ , we can have form of the particular solution as  $y_p(t) = Ae^{at}/Ate^{at}/At^2e^{at} \dots$
- (2) For RHS term to be  $\sin(bt)$ , we can have particular solution for to be  $y_p(t) = A\sin(bt) + B\cos(bt)$  or  $ty_p(t)$  or  $t^2y_p(t)$  and so on.
- (3) For RHS term to be  $a_0 + a_1t + a_2t^2 + \dots a_nt^n$ , we can have the particular solution form to be  $y_p(t) = A_0 + A_1t + \dots + A_nt^n$  or  $ty_p(t)$  or  $t^2y_p(t)$  and so on.

## 6. DOUBTS

**Ques 1.** In linear equations with constant coefficient, how is  $\dot{y} \propto y$ ?



**Ques 2.** If we have the equation  $\dot{y} + p(t)y = q(t)$  and the equation  $\mu(t)\dot{y} + \mu(t)p(t)y = \mu(t)q(t)$  then how to show that they will have the same solution set?

**Ques 3.** Author mentioned that if  $\mu(t) > 0$  then both above equations will have the same solution. But how to show this?