GENERAL TOPOLOGY

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Abstract. We shall learn some general topology.

Definition 1. Induced metric: A metric which is derived from a norm. A normed space is a special metric space whose metric is derived from a norm.

Example 1. $\mathscr{C}[a,b]$: Set of all bounded continuous real function on a closed interval form the normed space with norm defined as

$$\|f\| = \int_a^b |f(x)|\, dx \quad \text{or,} \quad \|f\| = \sup |f(x)|$$

and the induced metric is

$$\|f-g\| = \int_a^b |f(x)-g(x)|\, dx \quad \text{or,} \quad \|f-g\| = \sup |f(x)-g(x)|$$

Definition 2. Distance of a point x from a set A:

$$d(x,A) = \inf\{d(x,a) \mid \forall a \in A\}$$

Diameter of the set:

$$d(A) = \sup\{d(a_1,a_2) \mid \forall a_1, a_2 \in A\}$$

Definition 3. Bounded mapping: A mapping f of a non-empty set into a metric space is said to be bounded if its range is bounded i.e. $\exists M \in \mathbb{R}$ such that $|f(x)| \leq M$

Example 2. A pseudo metric which is not a metric

$$f, g \in \mathbb{R}^2 \text{ and } d(f,g) := \text{difference between their } x \text{ coordinates}$$

Definition 4. Interval: A set $A \subset \mathbb{R}$ is an interval if

$$\forall x, y \in A \text{ and } \forall t \in \mathbb{R}\colon x \leq t \leq y \implies t \in A$$

Theorem 1. *Union of intervals with non empty intersection is an interval.*

Proof. Let $\{I_i\}$ be the set of interval and $a \in \cap_i I_i$.
Proof Idea: Take any two points in the union and show that they contains every point in between them (take general point and show

that it will belong to the union).
Let $x, y \in \cup_i I_i$ and let $t \in \mathbb{R}$: $x \leq t \leq y$ then there are following possiblities:
$t < a$,
$t = a$ or,
$t > a$.
All are trivial to show that they lie in union. □

1. Topological Spaces

Definition 5. Topology: A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

- ϕ and X are in $\mathcal{T}$.
- The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
- The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X with topology $\mathcal{T}$ is called an topological space $(X, \mathcal{T})$.

Definition 6. Open set of X: For the topological space $(X, \mathcal{T})$, a subset U of X is an open set of X if U belongs to the collection $\mathcal{T}$.

Example 3. Discrete Topology: If X is any set then collection of all subsets of X is a topology on X, called **discrete topology.**

Example 4. Indiscrete or trivial topology: The topology consisting of only ϕ and the whole set X is called **trivial topology.**

Example 5. Finite complement topology: Let X be a set and $\mathcal{T}$ be the collection of all subset U of X such that $X - U$ is either finite or X. Then $\mathcal{T}$ is called **finite complement topology**. (This topology is consists of subset of X whose complement is either finite or X.)

Proof. Let $\{U_i\}$ be the indexed family of subsets of X belongs to $\mathcal{T}$. ϕ and X are obviously there. Assume each $\bigcup\limits_i U_i$ is non-empty (trivial for empty case):

$$X - \bigcup_i U_i = \bigcap_i (X - U_i)$$

Since each U_i is in $\mathcal{T}$, $X - U_i$ is finite. and $\bigcap \liminf_i X - U_i$ is contained in every $X - U_i$ hence it is finite.

To show $\bigcap\limits_i^n X - U_i$ is in $\mathcal{T}$,

$$X - \bigcap_i^n U_i = \bigcup_i^n (X - U_i)$$

Rhs is finite union of finite sets hence it is finite. ☐

Example 6. Let X be set and $\mathcal{T}_c$ be the collection of all subsets U of X such that U^c is either countable or all of X. Then $\mathcal{T}_c$ is a topology of X.

Proof. ϕ and X are trivial inside $\mathcal{T}_c$. Let U_i be the indexed family of subsets of X. Assume $\bigcup\limits_i U_i$ is non-empty (trivial for empty case). To show that $\bigcup\limits_i U_i$ is in $\mathcal{T}_c$

$$X - \bigcup_i U_i = \bigcap_i (X - U_i)$$

Since, $X - U_i$ is countable for each i and $\bigcap\limits_i (X - U_i)$ is in U_i for each i. Hence, $\bigcap\limits_i (X - U_i)$ is countable.

To show that $\bigcap\limits_i U_i$ is in $\mathcal{T}_c$, use the same argument as last example and the fact that finite union of countable sets is countable. ☐

Definition 7. Finer or strictly finer topology: For a set X, if $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X such that $\mathcal{T} \subset \mathcal{T}'$ then we say $\mathcal{T}'$ is **finer** than $\mathcal{T}$ and if $\mathcal{T}'$ properly contains $\mathcal{T}$ then we say it's **strictly finer**. Then $\mathcal{T}$ is called **coarser** than $\mathcal{T}'$ or, **strictly coarser** if it is contained in $\mathcal{T}'$ properly.

Definition 8. Comparable: We say $\mathcal{T}$ is comparable with $\mathcal{T}'$ if either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$.

2. Basis for a Topology

Definition 9. If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

- For each $x \in X$, there is atleast one basis element $B \in \mathcal{B}$ such that $x \in B$.
- If x belongs to the intersection of two basis elements B_1 and B_2, then there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

We define a topology $\mathcal{T}$ generated by $\mathcal{B}$ as: A subset U of X is said to be open in X (e.g. an element of topology on X) if for all $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$.

Remark 1. Each element of the basis is an element of the topology.

Example 7. If X is any set then the collection of all one element subsets of X is a basis for the discrete topology on X.(Power set of X).

Proof. Trivial to see. (Caution: Do not take element of the topology on X. For basis, condition is on the elements of the set X hence take element of X and then check basis conditions.) □

Lemma 1. *The collection $\mathcal{T}$ generated by the basis $\mathcal{B}$ is a topology.*

Proof. Let the collection $\mathcal{T} = \{U_i\}_{i\in I}$. Condition for the set U_i to belong to the collection is that for each $x \in U_i$ there exists an element $B \in \mathcal{B}$ and $x \in B \subset U_i$.

Membership of ϕ and X: For ϕ, it is vacuously true (true due to non-availability of elements in the set). For X, for each $x \in X$, there exists $B \in \mathcal{B}$ (by definition of basis) such that $x \in B$ and $B \subset X$.

Closure under arbitray union of elements. Now, assume that $\{U_i\}_{i\in I}$ is the indexed family of subsets of X which are elements of $\mathcal{T}$. We need to show that $\bigcup\limits_{i\in I} U_i \in \mathcal{T}$. For each $x \in \bigcup\limits_i U_i \implies x \in U_i$ for some i and $U_i \in \mathcal{T} \implies \exists B \in \mathcal{B}$ such that $x \in B \subset U_i$. This completes the argument.

Closure under finite intersection. We need to show that $\bigcap\limits_{i=0}^{n} U_i \subset \mathcal{T}$. For each $x \in \bigcap\limits_{i=0}^{n} U_i$

$$x \in U_i \ \forall i \implies \exists B_i \in \mathcal{B} \ \forall i \in \{0, 1, \dots n\}$$

Since, $x \in \bigcap\limits_{i=0}^{n} B_i$ and B_i *'s* are basis elements hence by definition of basis, $\exists B' \in \mathcal{B}$ such that $x \in B' \subset \bigcap\limits_{i=0}^{n} B_i$. Hence, $\bigcap\limits_{i=0}^{n} U_i \subset \mathcal{T}$. □

Lemma 2. *Let X be a set; $\mathcal{B}$ is the set of all basis elements of the topology $\mathcal{T}$ on set X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. Since each element B of basis is in $\mathcal{T}$ and hence their union. For other way around, let $U \in \mathcal{T}$, then for each $x \in U$ $\exists B_x \in \mathcal{B} \subset U$ hence, $U = \bigcup\limits_{x\in U} B_x$. Therefore, each $U \in X$ is union of basis elements. □

Remark 2. Above lemma states that every set U in X can be expressed as union of basis elements of the topology, however this is **not unique.**

Lemma 3. *Let X be an topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element C of $\mathcal{C}$ such that $x \in C \subset \mathcal{C}$. Then $\mathcal{C}$ is a basis of the topology of X.*

Proof. First we will prove that $\mathcal{C}$ is the basis of the topology on X.

First condition of basis: Since X is a open set of itself hence hypothesis, by for each $x \in X$ there exists $C \in \mathcal{C}$ such that $x \in C \subset \mathcal{C}$.
Second condition of basis: Let $x \in C_1 \cap C_2$ for some open sets $C_1, C_2 \in \mathcal{C}$. Since C_1, C_2 are open in X then so is $C_1 \cap C_2$ hence by hypothesis for each $x \in C_1 \cap C_2$ there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.
Topology generated by $\mathcal{C}$ equals topology of X. Let $\mathcal{T}_c$ be the topology generated by $\mathcal{C}$ and $\mathcal{T}$ be a topology on X. Let $U \in \mathcal{T}$. For each $x \in U$, by hypothesis, there exists $C_x \in \mathcal{C}$ such that $x \in C_x \subset U$ hence $U = \bigcup_{x \in U} C_x$(union of elements of $\mathcal{C}$) $\implies \mathcal{T} \subset \mathcal{T}_c$.
Let $V \in \mathcal{T}_c \implies V = \bigcup_{i \in I} C_i$ for each $C_i \in \mathcal{C}$ (by previous lemma). Since each C_i are open in X hence $C_i \in \mathcal{T}$ and $\mathcal{T}$ is a topology (their union will belong to $\mathcal{T}$). Hence, $V \in \mathcal{T} \implies \mathcal{T}_c \subset \mathcal{T}$. Therefore, $\mathcal{T}_c = \mathcal{T}$. □

Lemma 4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. TFAE*

(1) *$\mathcal{T}'$ is finer than $\mathcal{T}$.*
(2) *For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.*

Proof. (1) $\implies$ (2) (Idea is: Since $\mathcal{T} \subset \mathcal{T}'$, every set of $\mathcal{T}$ is a set in $\mathcal{T}'$. Hence, $B \in \mathcal{T}$ can be written in terms of basis of $\mathcal{T}'$)
We assume that $\mathcal{T} \subset \mathcal{T}'$. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subset \mathcal{T}$. And $\mathcal{T} \subset \mathcal{T}' \implies B \subset \mathcal{T}'$. Therefore, there exists a $B' \in \mathcal{B}'$ such that $\forall x \in B, x \in B' \subset B$.
(2) $\implies$ (1). Assume (2) and let $U \in \mathcal{T}$. We need to show that $U \in \mathcal{T}'$. For each $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. From condition (2), there exists a $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset U \implies B' \subset U$. Therefore by definition of basis, $U \in \mathcal{T}' \implies \mathcal{T} \subset \mathcal{T}'$. □

Definition 10 (Standard Topology on $\mathbb{R}$)**.** If $\mathcal{B}$ is the collection of all open intervals in the real line

$$(a, b) = \{x \mid a < x < b\},$$

the topology generated by $\mathcal{B}$ is called the **standard topology** on the real line.

Definition 11 (Lower limit topology on $\mathbb{R}$)**.** If $\mathcal{B}'$ is the collection of all half-open interval of the form

$$[a, b) = \{x \mid a \leq x < b\},$$

where $a < b$, the topology generated by $\mathcal{B}'$ is called the **lower limit topology** on $\mathbb{R}$. $\mathbb{R}$ with this topology is denoted as $\mathbb{R}_l$.

Definition 12 (K-topology on $\mathbb{R}$)**.** Let K denote the set of all number of the form $1/n$, for $\mathbb{Z}_+$, and let $\mathcal{B}$ be the collection of all open intervals (a, b), along with all the set of the form $(a, b) - K$. Then the topology generated by $\mathcal{B}$ is called **K-topology** on $\mathbb{R}$. $\mathbb{R}$ with this topology is denoted as $\mathbb{R}_K$.

The open sets in K-topology are of the form $U \backslash C$ where U is open set in standard topology and $C \subset K$.

Exercise 1. Prove that the set $\mathcal{B} = \{(a, b) \mid a < b\} \cup \{(a, b) \backslash K \mid a < b\}$ is a basis of the topology on $\mathbb{R}$.

Solution. First condition of the basis is trivially satisfied since it contains the basis of standard topology.
For second condition: Let $x \in \mathbb{R}$ such that $x \in B_1 \cap B_2$, we need to show that there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.
Case 1. If $B_1 = (a, b)$ and $B_2 = (a, b) - K$ then $B_1 \cap B_2 = B_2$ (which can be taken as B_3.)
Case 2. If $B_1 \cap B_2$ are disjoint then $x \notin B_1 \cap B_2$.
Case 3. If $B_1 = (a, b)$ and $B_2 = (c, d) - K$ with $c \in (a, b)$ and $d > b$ then $B_1 \cap B_2 = (c, b) - K = B_3$.

Lemma 5. *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.*

Proof. Let $\mathcal{T}, \mathcal{T}', \mathcal{T}''$ are the topologies of $\mathbb{R}, \mathbb{R}_l, \mathbb{R}_K$. We want to show $\mathcal{T} \subsetneq \mathcal{T}'$ and $\mathcal{T} \subsetneq \mathcal{T}''$. For each $x \in \mathbb{R}$ and given a basis element $(a, b) \in \mathcal{B}_{\mathbb{R}}$ containing x, there exists $[x, b) \in \mathcal{B}_{\mathbb{R}_l}$ and $(a, b) \in \mathcal{B}_{\mathbb{R}_K}$ such that $x \in [x, b) \subset (a, b)$ and $x \in (a, b) \subset (a, b)$. By previous lemma $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{T} \subset \mathcal{T}''$.

Now, for each $x \in \mathbb{R}$ and given $[x, b) \in \mathcal{B}_{\mathbb{R}_l}$ and $(-1, 1) - K \in \mathcal{B}_{\mathbb{R}_K}$ there does not exists an open interval (a, c) where $c < b$ in $\mathcal{T}$ containing x but contained in $[x, b)$ and there does not exists an open interval $(c, d) \in \mathcal{T}$ where $c > a$ and $d < b$ containing 0 such that $x \in (c, d) \subset (a, b) - K$ ((c, d) will contain elements of K but later does not). Hence, $\mathcal{T} \subsetneq \mathcal{T}'$ and $\mathcal{T} \subsetneq \mathcal{T}''$.

For each $x \in \mathbb{R}_l$ and given a basis element $[x, b) \in \mathcal{B}_{\mathbb{R}_l}$ there does not exists $(a, b) - U$ in $\mathcal{T}''$ where $U \subset K$ (it can be ϕ for (a, b)) such that $x \in (a, b) - U \subset [x, b)$. Also for each $x \in \mathbb{R}_K$ and given $(a, b) - U$, where $U \subset K$, containing x there does not exists $[x, b)$ in $\mathcal{T}'$ such that $x \in [x, b) \subset (a, b) - U$. Hence $\mathcal{T}'$ and $\mathcal{T}''$ are not comparable. $\square$

Definition 13 (Subbasis)**.** A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The **topology generated by the subbasis $\mathcal{S}$** is defined to be the collection $\mathcal{T}$ of all unions of finite intersection of elements of $\mathcal{S}$.

Theorem 2. *The collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$ is a topology.*

Proof. (Idea: We will prove that set of finite intersections of elements of $\mathcal{S}$ is a basis then by lemma 2, $\mathcal{T}$ is topology.) Let $\mathcal{B}$ be the set of finite intersections of elements of $\mathcal{S}$. For each $x \in X$, $x \in S_i \in \mathcal{S}$ for some i implies there exists $B \in \mathcal{B}$ such that $x \in B$. That is first condition for basis.

Let $x \in B_i \cap B_j$ for some $B_i, B_j \in \mathcal{B}$. Since $\mathcal{B}$ is a collection of all finite intersections hence there exists $B \in \mathcal{B}$ such that $B = B_i \cap B_j$ (intersection of two finite sets is finite.) Therefore, $x \in B \subset B_i \cap B_j$. $\square$

Exercise 2. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?

Solution. No. For example take $\mathbb{R}$. Notice that $\{x\} \in \mathcal{T}_\infty$. Now, take $\mathbb{R} - \cup_{x \neq 0}\{x\}$ is $\{0\}$ which is not infinite. Hence arbitrary union of members of the collection is not in the topology.

Remark 3. $\{\mathcal{T}_\alpha\}$ is the family of topologies on the set X then $\cap \mathcal{T}_\alpha$ is the topology on the set X while $\cup \mathcal{T}_\alpha$ may not be the topology on X (union axiom fails).

$\cup \mathcal{T}_\alpha$ is the topology on X if they are contained into one another.

Exercise 3. Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.

Solution. Since $\cap\{\mathcal{T}_\alpha\}$ is the largest collection of sets contained in all $\mathcal{T}_\alpha \in \{\mathcal{T}_\alpha\}$ and it is a topology (basis application of axioms of topology). For **uniqueness**, let $\mathcal{T}'$ be another largest topology contained in all $\mathcal{T}_\alpha$ then for some $U \in \mathcal{T}'$ implies $U \in \mathcal{T}_\alpha$ for all $\mathcal{T}_\alpha \in \{\mathcal{T}_\alpha\}$ which implies $U \in \cap\{\mathcal{T}_\alpha\}$. $\mathcal{T}' \subset \cap\{\mathcal{T}_\alpha\}$. Other inclusion can also be shown in the similar way. If $U \in \cap\{\mathcal{T}_\alpha\}$ then it should be in $\mathcal{T}'$ otherwise $\mathcal{T}'$ can not be largest. Hence, $\cap\{\mathcal{T}_\alpha\} \subset \mathcal{T}'$. $\mathcal{T}' = \cap\{\mathcal{T}_\alpha\}$.

Let $\{\mathcal{T}_i\}$ be the indexed family of topologies such that for all $i \in I$, $\mathcal{T}_i$ contains $\{\mathcal{T}_\alpha\}$. Then $\cap\{\mathcal{T}_i\}$ will be the smallest topology containing $\{\mathcal{T}_\alpha\}$. For **uniqueness**, let $\mathcal{T}'$ be another such smallest topology then $\cap\{\mathcal{T}_\alpha\} \subset \mathcal{T}'$ since former is smallest and $\mathcal{T}' \subset \cap\{\mathcal{T}_\alpha\}$ by taking later as smallest. $\cap\{\mathcal{T}_\alpha\} = \mathcal{T}'$.

Exercise 4. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contains $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Solution.
When $\mathcal{A}$ is a basis. Let $\mathcal{T}$ be the topology generated by $\mathcal{A}$ implies $\mathcal{T} = \cup\mathcal{A}$.