GENERAL TOPOLOGY

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Abstract. We shall learn some general topology.

Definition 1. Induced metric: A metric which is derived from a norm. A normed space is a special metric space whose metric is derived from a norm.

Example 1. $\mathscr{C}[a,b]$: Set of all bounded continuous real function on a closed interval form the normed space with norm defined as

$$\|f\| = \int_a^b |f(x)|\, dx \quad \text{or,} \quad \|f\| = \sup |f(x)|$$

and the induced metric is

$$\|f-g\| = \int_a^b |f(x)-g(x)|\, dx \quad \text{or,} \quad \|f-g\| = \sup |f(x)-g(x)|$$

Definition 2. Distance of a point x from a set A:

$$d(x, A) = \inf\{d(x,a) \mid \forall a \in A\}$$

Diameter of the set:

$$d(A) = \sup\{d(a_1, a_2) \mid \forall a_1, a_2 \in A\}$$

Definition 3. Bounded mapping: A mapping f of a non-empty set into a metric space is said to be bounded if its range is bounded i.e. $\exists M \in \mathbb{R}$ such that $|f(x)| \leq M$

Example 2. A pseudo metric which is not a metric

$$f, g \in \mathbb{R}^2 \text{ and } d(f,g) := \text{difference between their } x \text{ coordinates}$$

Definition 4. Interval: A set $A \subset \mathbb{R}$ is an interval if

$$\forall x, y \in A \text{ and } \forall t \in \mathbb{R} \colon x \leq t \leq y \implies t \in A$$

Theorem 1. *Union of intervals with non empty intersection is an interval.*

Proof. Let $\{I_i\}$ be the set of interval and $a \in \cap_i I_i$.
Proof Idea: Take any two points in the union and show that they contains every point in between them (take general point and show

that it will belong to the union).
Let $x, y \in \cup_i I_i$ and let $t \in \mathbb{R}$: $x \leq t \leq y$ then there are following possiblities:
$t < a$,
$t = a$ or,
$t > a$.
All are trivial to show that they lie in union. □

1. Topological Spaces

Definition 5. Topology: A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

- ϕ and X are in $\mathcal{T}$.
- The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
- The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X with topology $\mathcal{T}$ is called an topological space $(X, \mathcal{T})$.

Definition 6. Open set of X: For the topological space $(X, \mathcal{T})$, a subset U of X is an open set of X if U belongs to the collection $\mathcal{T}$.

Example 3. Discrete Topology: If X is any set then collection of all subsets of X is a topology on X, called **discrete topology.**

Example 4. Indiscrete or trivial topology: The topology consisting of only ϕ and the whole set X is called **trivial topology.**

Example 5. Finite complement topology: Let X be a set and $\mathcal{T}$ be the collection of all subset U of X such that $X - U$ is either finite or X. Then $\mathcal{T}$ is called **finite complement topology**. (This topology is consists of subset of X whose complement is either finite or X.)

Proof. Let $\{U_i\}$ be the indexed family of subsets of X belongs to $\mathcal{T}$. ϕ and X are obviously there. Assume each $\bigcup\limits_i U_i$ is non-empty (trivial for empty case):

$$X - \bigcup_i U_i = \bigcap_i (X - U_i)$$

Since each U_i is in $\mathcal{T}$, $X - U_i$ is finite. and $\bigcap \liminf_i X - U_i$ is contained in every $X - U_i$ hence it is finite.

To show $\bigcap\limits_i^n X - U_i$ is in $\mathcal{T}$,

$$X - \bigcap_i^n U_i = \bigcup_i^n (X - U_i)$$

Rhs is finite union of finite sets hence it is finite. □

Example 6. Let X be set and $\mathcal{T}_c$ be the collection of all subsets U of X such that U^c is either countable or all of X. Then $\mathcal{T}_c$ is a topology of X.

Proof. ϕ and X are trivial inside $\mathcal{T}_c$. Let U_i be the indexed family of subsets of X. Assume $\bigcup\limits_i U_i$ is non-empty (trivial for empty case). To show that $\bigcup\limits_i U_i$ is in $\mathcal{T}_c$

$$X - \bigcup_i U_i = \bigcap_i (X - U_i)$$

Since, $X - U_i$ is countable for each i and $\bigcap\limits_i (X - U_i)$ is in U_i for each i. Hence, $\bigcap\limits_i (X - U_i)$ is countable.

To show that $\bigcap\limits_i U_i$ is in $\mathcal{T}_c$, use the same argument as last example and the fact that finite union of countable sets is countable. □

Definition 7. Finer or strictly finer topology: For a set X, if $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X such that $\mathcal{T} \subset \mathcal{T}'$ then we say $\mathcal{T}'$ is **finer** than $\mathcal{T}$ and if $\mathcal{T}'$ properly contains $\mathcal{T}$ then we say it's **strictly finer**. Then $\mathcal{T}$ is called **coarser** than $\mathcal{T}'$ or, **strictly coarser** if it is contained in $\mathcal{T}'$ properly.

Lemma 1. *Let $\mathcal{B}$ and $\mathcal{B}'$ be the bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then following statements are equivalent:*

- *$\mathcal{T}'$ is finer than $\mathcal{T}$.*
- *For each*

Definition 8. Comparable: We say $\mathcal{T}$ is comparable with $\mathcal{T}'$ if either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$.

2. Basis for a Topology

Definition 9. If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

- For each $x \in X$, there is atleast one basis element $B \in \mathcal{B}$ such that $x \in B$.
- If x belongs to the intersection of two basis elements B_1 and B_2, then there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

We define a topology $\mathcal{T}$ generated by $\mathcal{B}$ as: A subset U of X is said to be open in X (e.g. an element of topology on X) if for all $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$.

Remark 1. Each element of the basis is an element of the topology.

Example 7. If X is any set then the collection of all one element subsets of X is a basis for the discrete topology on X.(Power set of X).

Proof. Trivial to see. (Caution: Do not take element of the topology on X. For basis, condition is on the elements of the set X hence take element of X and then check basis conditions.) $\square$

Lemma 2. *The collection $\mathcal{T}$ generated by the basis $\mathcal{B}$ is a topology.*

Proof. Let the collection $\mathcal{T} = \{U_i\}_{i \in I}$. Condition for the set U_i to belong to the collection is that for each $x \in U_i$ there exists an element $B \in \mathcal{B}$ and $x \in B \subset U_i$.
Membership of ϕ and X: For ϕ, it is vacuously true (true due to non-availability of elements in the set). For X, for each $x \in X$, there exists $B \in \mathcal{B}$ (by definition of basis) such that $x \in B$ and $B \subset X$.
Closure under arbitray union of elements. Now, assume that $\{U_i\}_{i \in I}$ is the indexed family of subsets of X which are elements of $\mathcal{T}$. We need to show that $\bigcup_{i \in I} U_i \in \mathcal{T}$. For each $x \in \bigcup_i U_i \implies x \in U_i$ for some i and $U_i \in \mathcal{T} \implies \exists B \in \mathcal{B}$ such that $x \in B \subset U_i$. This completes the argument.

Closure under finite intersection. We need to show that $\bigcap_{i=0}^{n} U_i \subset \mathcal{T}$. For each $x \in \bigcap_{i=0}^{n} U_i$

$$x \in U_i \ \forall i \implies \exists B_i \in \mathcal{B} \ \forall i \in \{0, 1, \dots n\}$$

Since, $x \in \bigcap_{i=0}^{n} B_i$ and B_i *'s* are basis elements hence by definition of basis, $\exists B' \in \mathcal{B}$ such that $x \in B' \subset \bigcap_{i=0}^{n} B_i$. Hence, $\bigcap_{i=0}^{n} U_i \subset \mathcal{T}$. $\square$

Lemma 3. *Let X be a set; $\mathcal{B}$ is the set of all basis elements of the topology $\mathcal{T}$ on set X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. Since each element B of basis is in $\mathcal{T}$ and hence their union. For other way around, let $U \in \mathcal{T}$, then for each $x \in U$ $\exists B_x \in \mathcal{B} \subset U$ hence, $U = \bigcup_{x \in U} B_x$. Therefore, each $U \in X$ is union of basis elements. $\square$

Remark 2. Above lemma states that every set U in X can be expressed as union of basis elements of the topology, however this is **not unique.**

Lemma 4. *Let X be an topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$,* *there is an element C of $\mathcal{C}$ such that $x \in C \subset \mathcal{C}$. Then $\mathcal{C}$ is a basis of the topology of X.*

Proof. First we will prove that $\mathcal{C}$ is the basis of the topology on X.
First condition of basis: Since X is a open set of itself hence hypothesis, by for each $x \in X$ there exists $C \in \mathcal{C}$ such that $x \in C \subset \mathcal{C}$.
Second condition of basis: Let $x \in C_1 \cap C_2$ for some open sets $C_1, C_2 \in \mathcal{C}$. Since C_1, C_2 are open in X then so is $C_1 \cap C_2$ hence by hypothesis for each $x \in C_1 \cap C_2$ there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.
Topology generated by $\mathcal{C}$ equals topology of X. Let $\mathcal{T}_c$ be the topology generated by $\mathcal{C}$ and $\mathcal{T}$ be a topology on X. Let $U \in \mathcal{T}$. For each $x \in U$, by hypothesis, there exists $C_x \in \mathcal{C}$ such that $x \in C_x \subset U$ hence $U = \bigcup\limits_{x \in U} C_x$(union of elements of $\mathcal{C}$) $\implies \mathcal{T} \subset \mathcal{T}_c$.
Let $V \in \mathcal{T}_c \implies V = \bigcup\limits_{i \in I} C_i$ for each $C_i \in \mathcal{C}$ (by previous lemma). Since each C_i are open in X hence $C_i \in \mathcal{T}$ and $\mathcal{T}$ is a topology (their union will belong to $\mathcal{T}$). Hence, $V \in \mathcal{T} \implies \mathcal{T}_c \subset \mathcal{T}$. Therefore, $\mathcal{T}_c = \mathcal{T}$. □

Lemma 5. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. TFAE*

(1) *$\mathcal{T}'$ is finer than $\mathcal{T}$.*
(2) *For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.*

Proof. (1) $\implies$ (2) We assume that $\mathcal{T} \subset \mathcal{T}'$. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subset \mathcal{T}$. And $\mathcal{T} \subset \mathcal{T}' \implies B \subset \mathcal{T}'$. Therefore, there exists a $B' \in \mathcal{B}'$ such that $\forall x \in B, x \in B' \subset B$.
(2) $\implies$ (1). Assume (2) and let $U \in \mathcal{T}$. We need to show that $U \in \mathcal{T}'$. For each $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. From condition (2), there exists a $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset U \implies B' \subset U$. Therefore by definition of basis, $U \in \mathcal{T}' \implies \mathcal{T} \subset \mathcal{T}'$. □

Definition 10 (Standard Topology on $\mathbb{R}$)**.** If $\mathcal{B}$ is the collection of all open intervals in the real line

$$(a, b) = \{x \mid a < x < b\},$$

the topology generated by $\mathcal{B}$ is called the **standard topology** on the real line.

Definition 11 (Lower limit topology on $\mathbb{R}$)**.** If $\mathcal{B}'$ is the collection of all half-open interval of the form

$$[a, b) = \{x \mid a \leq x < b\},$$

where $a < b$, the topology generated by $\mathcal{B}'$ is called the **lower limit topology** on $\mathbb{R}$. $\mathbb{R}$ with this topology is denoted as $\mathbb{R}_l$.

Definition 12 (K-topology on $\mathbb{R}$)**.** Let K denote the set of all number of the form $1/n$, for $\mathbb{Z}_+$, and let $\mathcal{B}$ be the collection of all open intervals (a, b), along with all the set of the form $(a, b) - K$. Then the topology generated by $\mathcal{B}$ is called K**-topology** on $\mathbb{R}$. $\mathbb{R}$ with this topology is denoted as $\mathbb{R}_K$.

Exercise 1. Prove that the set $\mathcal{B} = \{(a, b) \mid a < b\} \cup \{(a, b) \backslash K \mid a < b\}$ is a basis of the topology on $\mathbb{R}$.

Solution. First condition of the basis is trivially satisfied since it contains the basis of standard topology.
For second condition: Let $x \in \mathbb{R}$ such that $x \in B_1 \cap B_2$, we need to show that there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.
Case 1. If $B_1 = (a, b)$ and $B_2 = (a, b) - K$ then $B_1 \cap B_2 = B_2$ (which can be taken as B_3.)
Case 2. If $B_1 \cap B_2$ are disjoint then $x \notin B_1 \cap B_2$.
Case 3. If $B_1 = (a, b)$ and $B_2 = (c, d) - K$ with $c \in (a, b)$ and $d > b$ then $B_1 \cap B_2 = (c, b) - K = B_3$.