GENERAL TOPOLOGY

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Abstract. We shall learn some general topology. Rest of the abstract is left as an exercise to the reader.

Definition 1. Induced metric: A metric which is derived from a norm. A normed space is a special metric space whose metric is derived from a norm.

Example 1. $\mathscr{C}[a,b]$: Set of all bounded continuous real function on a closed interval form the normed space with norm defined as

$$\|f\| = \int_a^b |f(x)|\ dx \quad \text{or,} \quad \|f\| = \sup|f(x)|$$

and the induced metric is

$$\|f-g\| = \int_a^b |f(x)-g(x)|\ dx \quad \text{or,} \quad \|f-g\| = \sup|f(x)-g(x)|$$

Definition 2. Distance of a point x from a set A:

$$d(x,A) = \inf\{d(x,a) \mid \forall a \in A\}$$

Diameter of the set:

$$d(A) = \sup\{d(a_1,a_2) \mid \forall a_1,a_2 \in A\}$$

Definition 3. Bounded mapping: A mapping f of a non-empty set into a metric space is said to be bounded if its range is bounded i.e. $\exists M \in \mathbb{R}$ such that $|f(x)| \leq M$

Example 2. A pseudo metric which is not a metric

$$f,g \in \mathbb{R}^2 \text{ and } d(f,g) := \text{difference between their } x \text{ coordinates}$$

Definition 4. Interval: A set $A \subset \mathbb{R}$ is an interval if

$$\forall x,y \in A \text{ and } \forall t \in \mathbb{R}\colon x \leq t \leq y \implies t \in A$$

Theorem 1. *Union of intervals with non empty intersection is an interval.*

Proof. Let $\{I_i\}$ be the set of interval and $a \in \cap_i I_i$.
Proof Idea: Take any two points in the union and show that they contains every point in between them (take general point and show that it will belong to the union).
Let $x,y \in \cup_i I_i$ and let $t \in \mathbb{R}\colon x \leq t \leq y$ then there are following possiblities:
$t < a$,
$t = a$ or,
$t > a$.
All are trivial to show that they lie in union. ■

1. Topological Spaces

Definition 5. Topology: A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

- ϕ and X are in $\mathcal{T}$.
- The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
- The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X with topology $\mathcal{T}$ is called an topological space $(X, \mathcal{T})$.

Definition 6. Open set of X: For the topological space $(X, \mathcal{T})$, a subset U of X is an open set of X if U belongs to the collection $\mathcal{T}$.

Example 3. Discrete Topology: If X is any set then collection of all subsets of X is a topology on X, called **discrete topology.**

Example 4. Indiscrete or trivial topology: The topology consisting of only ϕ and the whole set X is called **trivial topology.**

Example 5. Finite complement topology: Let X be a set and $\mathcal{T}$ be the collection of all subset U of X such that $X - U$ is either finite or X. Then $\mathcal{T}$ is called **finite complement topology**. (This topology is consists of subset of X whose complement is either finite or X.)

Proof. Let $\{U_i\}$ be the indexed family of subsets of X belongs to $\mathcal{T}$. ϕ and X are obviously there. Assume each $\bigcup_i U_i$ is non-empty (trivial for empty case):

$$X - \bigcup_i U_i = \bigcap_i (X - U_i)$$

Since each U_i is in $\mathcal{T}$, $X - U_i$ is finite. and $\bigcap \liminf_i X - U_i$ is contained in every $X - U_i$ hence it is finite.

To show $\bigcap_i^n X - U_i$ is in $\mathcal{T}$,

$$X - \bigcap_i^n U_i = \bigcup_i^n (X - U_i)$$

Rhs is finite union of finite sets hence it is finite. ∎

Example 6. Let X be set and $\mathcal{T}_c$ be the collection of all subsets U of X such that U^c is either countable or all of X. Then $\mathcal{T}_c$ is a topology of X.

Proof. ϕ and X are trivial inside $\mathcal{T}_c$. Let U_i be the indexed family of subsets of X. Assume $\bigcup_i U_i$ is non-empty (trivial for empty case). To show that $\bigcup_i U_i$ is in $\mathcal{T}_c$

$$X - \bigcup_i U_i = \bigcap_i (X - U_i)$$

Since, $X - U_i$ is countable for each i and $\bigcap_i (X - U_i)$ is in U_i for each i. Hence, $\bigcap_i (X - U_i)$ is countable.
To show that $\bigcap_i U_i$ is in $\mathcal{T}_c$, use the same argument as last example and the fact that finite union of countable sets is countable. ■

Definition 7. Finer or strictly finer topology: For a set X, if $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X such that $\mathcal{T} \subset \mathcal{T}'$ then we say $\mathcal{T}'$ is **finer** than $\mathcal{T}$ and if $\mathcal{T}'$ properly contains $\mathcal{T}$ then we say it's **strictly finer**. Then $\mathcal{T}$ is called **coarser** than $\mathcal{T}'$ or, **strictly coarser** if it is contained in $\mathcal{T}'$ properly.

Definition 8. Comparable: We say $\mathcal{T}$ is comparable with $\mathcal{T}'$ if either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$.

2. Basis for a Topology

Definition 9. If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

- For each $x \in X$, there is atleast one basis element $B \in \mathcal{B}$ such that $x \in B$.
- If x belongs to the intersection of two basis elements B_1 and B_2, then there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

We define a topology $\mathcal{T}$ generated by $\mathcal{B}$ as: A subset U of X is said to be open in X (e.g. an element of topology on X) if for all $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$.

Remark 1. Each element of the basis is an element of the topology.

Example 7. If X is any set then the collection of all one element subsets of X is a basis for the discrete topology on X.(Power set of X).

Proof. Trivial to see. (Caution: Do not take element of the topology on X. For basis, condition is on the elements of the set X hence take element of X and then check basis conditions.) ■

Lemma 1. *The collection $\mathcal{T}$ generated by the basis $\mathcal{B}$ is a topology.*

Proof. Let the collection $\mathcal{T} = \{U_i\}_{i \in I}$. Condition for the set U_i to belong to the collection is that for each $x \in U_i$ there exists an element $B \in \mathcal{B}$ and $x \in B \subset U_i$.

Membership of ϕ and X: For ϕ, it is vacuously true (true due to non-availability of elements in the set). For X, for each $x \in X$, there exists $B \in \mathcal{B}$ (by definition of basis) such that $x \in B$ and $B \subset X$.

Closure under arbitray union of elements. Now, assume that $\{U_i\}_{i \in I}$ is the indexed family of subsets of X which are elements of $\mathcal{T}$. We need to show that $\bigcup_{i \in I} U_i \in \mathcal{T}$. For each $x \in \bigcup_i U_i \implies x \in U_i$ for some i and $U_i \in \mathcal{T} \implies \exists B \in \mathcal{B}$ such that $x \in B \subset U_i$. This completes the argument.

Closure under finite intersection. We need to show that $\bigcap_{i=0}^{n} U_i \subset \mathcal{T}$. For each $x \in \bigcap_{i=0}^{n} U_i$

$$x \in U_i \ \forall i \implies \exists B_i \in \mathcal{B} \ \forall i \in \{0, 1, \dots n\}$$

Since, $x \in \bigcap_{i=0}^{n} B_i$ and B_i *'s* are basis elements hence by definition of basis, $\exists B' \in \mathcal{B}$ such that $x \in B' \subset \bigcap_{i=0}^{n} B_i$. Hence, $\bigcap_{i=0}^{n} U_i \subset \mathcal{T}$. ∎

Lemma 2. *Let X be a set; $\mathcal{B}$ is the set of all basis elements of the topology $\mathcal{T}$ on set X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. Since each element B of basis is in $\mathcal{T}$ and hence their union. For other way around, let $U \in \mathcal{T}$, then for each $x \in U \ \exists B_x \in \mathcal{B} \subset U$ hence, $U = \bigcup_{x \in U} B_x$. Therefore, each $U \in X$ is union of basis elements. ∎

Remark 2. Above lemma states that every set U in X can be expressed as union of basis elements of the topology, however this is **not unique.**

Lemma 3. *Let X be an topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element C of $\mathcal{C}$ such that $x \in C \subset \mathcal{C}$. Then $\mathcal{C}$ is a basis of the topology of X.*

Proof. First we will prove that $\mathcal{C}$ is the basis of the topology on X.

First condition of basis: Since X is a open set of itself hence hypothesis, by for each $x \in X$ there exists $C \in \mathcal{C}$ such that $x \in C \subset \mathcal{C}$.

Second condition of basis: Let $x \in C_1 \cap C_2$ for some open sets $C_1, C_2 \in \mathcal{C}$. Since C_1, C_2 are open in X then so is $C_1 \cap C_2$ hence by hypothesis for each $x \in C_1 \cap C_2$ there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

Topology generated by $\mathcal{C}$ equals topology of X. Let $\mathcal{T}_c$ be the topology generated by $\mathcal{C}$ and $\mathcal{T}$ be a topology on X. Let $U \in \mathcal{T}$. For each $x \in U$, by hypothesis, there exists $C_x \in \mathcal{C}$ such that $x \in C_x \subset U$ hence $U = \bigcup\limits_{x \in U} C_x$(union of elements of $\mathcal{C}$) $\implies \mathcal{T} \subset \mathcal{T}_c$.
Let $V \in \mathcal{T}_c \implies V = \bigcup\limits_{i \in I} C_i$ for each $C_i \in \mathcal{C}$ (by previous lemma). Since each C_i are open in X hence $C_i \in \mathcal{T}$ and $\mathcal{T}$ is a topology (their union will belong to $\mathcal{T}$). Hence, $V \in \mathcal{T} \implies \mathcal{T}_c \subset \mathcal{T}$. Therefore, $\mathcal{T}_c = \mathcal{T}$. ∎

Lemma 4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. TFAE*

(1) *$\mathcal{T}'$ is finer than $\mathcal{T}$.*
(2) *For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.*

Proof. (1) $\implies$ (2) (Idea is: Since $\mathcal{T} \subset \mathcal{T}'$, every set of $\mathcal{T}$ is a set in $\mathcal{T}'$. Hence, $B \in \mathcal{T}$ can be written in terms of basis of $\mathcal{T}'$)

We assume that $\mathcal{T} \subset \mathcal{T}'$. For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B \subset \mathcal{T}$. And $\mathcal{T} \subset \mathcal{T}' \implies B \subset \mathcal{T}'$. Therefore, there exists a $B' \in \mathcal{B}'$ such that $\forall x \in B, x \in B' \subset B$.

(2) $\implies$ (1). Assume (2) and let $U \in \mathcal{T}$. We need to show that $U \in \mathcal{T}'$. For each $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. From condition (2), there exists a $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset U \implies B' \subset U$. Therefore by definition of basis, $U \in \mathcal{T}' \implies \mathcal{T} \subset \mathcal{T}'$. ∎

Definition 10 (Standard Topology on $\mathbb{R}$)**.** If $\mathcal{B}$ is the collection of all open intervals in the real line

$$(a, b) = \{x \mid a < x < b\},$$

the topology generated by $\mathcal{B}$ is called the **standard topology** on the real line.

Definition 11 (Lower limit topology on $\mathbb{R}$)**.** If $\mathcal{B}'$ is the collection of all half-open interval of the form

$$[a, b) = \{x \mid a \leq x < b\},$$

where $a < b$, the topology generated by $\mathcal{B}'$ is called the **lower limit topology** on $\mathbb{R}$. $\mathbb{R}$ with this topology is denoted as $\mathbb{R}_l$.

Definition 12 (K-topology on $\mathbb{R}$)**.** Let K denote the set of all number of the form $1/n$, for $\mathbb{Z}_+$, and let $\mathcal{B}$ be the collection of all open intervals (a, b), along with all the set of the form $(a, b) - K$. Then the topology generated by $\mathcal{B}$ is called K**-topology** on $\mathbb{R}$. $\mathbb{R}$ with this topology is denoted as $\mathbb{R}_K$.

The open sets in K-topology are of the form $U \backslash C$ where U is open set in standard topology and $C \subset K$.

Exercise 1. Prove that the set $\mathcal{B} = \{(a, b) \mid a < b\} \cup \{(a, b) \backslash K \mid a < b\}$ is a basis of the topology on $\mathbb{R}$.

Solution. First condition of the basis is trivially satisfied since it contains the basis of standard topology.
For second condition: Let $x \in \mathbb{R}$ such that $x \in B_1 \cap B_2$, we need to show that there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Case 1. If $B_1 = (a, b)$ and $B_2 = (a, b) - K$ then $B_1 \cap B_2 = B_2$ (which can be taken as B_3.)

Case 2. If $B_1 \cap B_2$ are disjoint then $x \notin B_1 \cap B_2$.

Case 3. If $B_1 = (a, b)$ and $B_2 = (c, d) - K$ with $c \in (a, b)$ and $d > b$ then $B_1 \cap B_2 = (c, b) - K = B_3$.

Lemma 5. *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.*

Proof. Let $\mathcal{T}, \mathcal{T}', \mathcal{T}''$ are the topologies of $\mathbb{R}, \mathbb{R}_l, \mathbb{R}_K$. We want to show $\mathcal{T} \subsetneq \mathcal{T}'$ and $\mathcal{T} \subsetneq \mathcal{T}''$. For each $x \in \mathbb{R}$ and given a basis element $(a, b) \in \mathcal{B}_{\mathbb{R}}$ containing x, there exists $[x, b) \in \mathcal{B}_{\mathbb{R}_l}$ and $(a, b) \in \mathcal{B}_{\mathbb{R}_K}$ such that $x \in [x, b) \subset (a, b)$ and $x \in (a, b) \subset (a, b)$. By previous lemma $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{T} \subset \mathcal{T}''$.

Now, for each $x \in \mathbb{R}$ and given $[x, b) \in \mathcal{B}_{\mathbb{R}_l}$ and $(-1, 1) - K \in \mathcal{B}_{\mathbb{R}_K}$ there does not exists an open interval (a, c) where $c < b$ in $\mathcal{T}$ containing x but contained in $[x, b)$ and there does not exists an open interval $(c, d) \in \mathcal{T}$ where $c > a$ and $d < b$ containing 0 such that $x \in (c, d) \subset (a, b) - K$ ((c, d) will contain elements of K but later does not). Hence, $\mathcal{T} \subsetneq \mathcal{T}'$ and $\mathcal{T} \subsetneq \mathcal{T}''$.

For each $x \in \mathbb{R}_l$ and given a basis element $[x, b) \in \mathcal{B}_{\mathbb{R}_l}$ there does not exists $(a, b) - U$ in $\mathcal{T}''$ where $U \subset K$ (it can be ϕ for (a, b)) such that $x \in (a, b) - U \subset [x, b)$. Also for each $x \in \mathbb{R}_K$ and given $(a, b) - U$, where $U \subset K$, containing x there does not exists $[x, b)$ in $\mathcal{T}'$ such that $x \in [x, b) \subset (a, b) - U$. Hence $\mathcal{T}'$ and $\mathcal{T}''$ are not comparable. ∎

2.1. **Subbasis of a topology.**

Definition 13 (Subbasis)**.** A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The **topology generated by the subbasis** $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersection of elements of $\mathcal{S}$.

Theorem 2. *The collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$ is a topology.*

Proof. (Idea: We will prove that set of finite intersections of elements of $\mathcal{S}$ is a basis then by lemma 2, $\mathcal{T}$ is topology.) Let $\mathcal{B}$ be the set of finite intersections of elements of $\mathcal{S}$. For each $x \in X$, $x \in S_i \in \mathcal{S}$ for some i implies there exists $B \in \mathcal{B}$ such that $x \in B$. That is first condition for basis.

Let $x \in B_i \cap B_j$ for some $B_i, B_j \in \mathcal{B}$. Since $\mathcal{B}$ is a collection of all finite intersections hence there exists $B \in \mathcal{B}$ such that $B = B_i \cap B_j$ (intersection of two finite sets is finite.) Therefore, $x \in B \subset B_i \cap B_j$. ∎

Exercise 2. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?

Solution. No. For example take $\mathbb{R}$. Notice that $\{x\} \in \mathcal{T}_\infty$. Now, take $\mathbb{R} - \cup_{x \neq 0}\{x\}$ is $\{0\}$ which is not infinite. Hence arbitrary union of members of the collection is not in the topology.

Remark 3. $\{\mathcal{T}_\alpha\}$ is the family of topologies on the set X then $\cap \mathcal{T}_\alpha$ is the topology on the set X while $\cup \mathcal{T}_\alpha$ may not be the topology on X (union axiom fails).

$\cup \mathcal{T}_\alpha$ is the topology on X if they are contained into one another.

Exercise 3. Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.

Solution. Since $\cap\{\mathcal{T}_\alpha\}$ is the largest collection of sets contained in all $\mathcal{T}_\alpha \in \{\mathcal{T}_\alpha\}$ and it is a topology (basic application of axioms of topology). For **uniqueness**, let $\mathcal{T}'$ be another largest topology contained in all $\mathcal{T}_\alpha$ then for some $U \in \mathcal{T}'$ implies $U \in \mathcal{T}_\alpha$ for all $\mathcal{T}_\alpha \in \{\mathcal{T}_\alpha\}$ which implies $U \in \cap\{\mathcal{T}_\alpha\}$. $\mathcal{T}' \subset \cap\{\mathcal{T}_\alpha\}$. Other inclusion can also be shown in the similar way. If $U \in \cap\{\mathcal{T}_\alpha\}$ then it should be in $\mathcal{T}'$ otherwise $\mathcal{T}'$ can not be largest. Hence, $\cap\{\mathcal{T}_\alpha\} \subset \mathcal{T}'$. $\mathcal{T}' = \cap\{\mathcal{T}_\alpha\}$.

Let $\{\mathcal{T}_i\}$ be the indexed family of topologies such that for all $i \in I$, $\mathcal{T}_i$ contains $\{\mathcal{T}_\alpha\}$. Then $\cap\{\mathcal{T}_i\}$ will be the smallest topology containing $\{\mathcal{T}_\alpha\}$. For **uniqueness**, let $\mathcal{T}'$ be another such smallest topology then $\cap\{\mathcal{T}_\alpha\} \subset \mathcal{T}'$ since former is smallest and $\cap\{\mathcal{T}_\alpha\} \subset \mathcal{T}'$ by taking later as smallest. $\cap\{\mathcal{T}_\alpha\} = \mathcal{T}'$.

Exercise 4. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contains $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Solution. To show $\mathcal{T} = \cap\{\mathcal{T}_i\}$ in the following cases:

When $\mathcal{A}$ is a basis. Let $\mathcal{T}$ be the topology on X generated by $\mathcal{A}$ implies $\mathcal{T} = \cup\mathcal{A}$. And assume the family of *all* the topologies on X, each of one contains $\mathcal{A}$, to be $\{\mathcal{T}_i\}$.

($\Longleftarrow$). Since $\mathcal{T} \in \{\mathcal{T}_i\}$ because $\mathcal{T}$ is also a topology on X. Hence $\cap\{\mathcal{T}_i\}$ is a topology contains in every $\mathcal{T}_i$ particularly, $\cap\{\mathcal{T}_i\} \subset \mathcal{T}$.

($\Longrightarrow$). Let $U \in \mathcal{T}$ implies that $U = \cup\mathcal{A}$. Since $\mathcal{A} \subset \mathcal{T}_i$ for all i and each $\mathcal{T}_i$ is a topology hence $\cup\mathcal{A} \subset \mathcal{T}_i$ for all i. Since the union is arbitrary hence $U \subset \mathcal{T}_i$ for all i implies $U \subset \cap\{\mathcal{T}_i\}$. $\mathcal{T} \subset \cap\{\mathcal{T}_i\}$. Therefore, $\mathcal{T} = \cap\{\mathcal{T}_i\}$.

When $\mathcal{A}$ is a subbasis. Let $\mathcal{B}$ is the collection of all finite intersections of the elements of $\mathcal{A}$. Assume the topology generated by subbasis to be $\mathcal{T} = \cup\mathcal{B}$ and $\{\mathcal{T}_i\}$ to be *all* topologies on X each containing $\mathcal{A}$.

($\Longleftarrow$). This is same as above case.

($\Longrightarrow$). Let $\mathcal{T} = \cup\mathcal{B}$. Since each element B of $\mathcal{B}$ is finite intersection of elements of $\mathcal{A}$, each $B \subset A_i$ for A_i's whose intersection is equal to $B \implies \mathcal{B} \subset \mathcal{A}$. $\mathcal{A}$ is contained in each $\mathcal{T}_i \in \{\mathcal{T}_i\}$ implies that $\mathcal{B} \subset \cap\{\mathcal{T}_i\} \implies \cup\mathcal{B} \subset \cap\{\mathcal{T}_i\}$. This means $\mathcal{T} \subset \cap\{\mathcal{T}_i\}$.

Therefore, $\mathcal{T} = \cap\{\mathcal{T}_i\}$.

3. The Order Topology

Definition 14 (Order Topology)**.** Let X be a set with a simple order relation; assume X has more than one element. Let $\mathcal{B}$ be the collection of all the sets of the following types:

(1) All open interval (a, b) in X.
(2) All intervals of the form $[a_0, b)$ where a_0 is the smallest element (if any) of X.
(3) All intervals of the form $(a, b_0]$ where b_0 is the largest element (if any) of X.

The collection $\mathcal{B}$ is the basis for a topology on X which is called **order topology**.

Let's check if the set $\mathcal{B}$ satisfies the conditions for the basis: for any $x \in X$, there always exists atleast an element of (1) containing it. The smallest element (if any) lies in type (2) and similarly the largest element (if any) lies in type (3). For second condition of the basis: there are several cases that needed to be checked such as x lies in intersection of type (1) and (1), type (1) and (2), type (1) and (3), type (2) and (3) and so on. It can be showed that it satisfies the second condition.

Example 8 (Order topology on $\mathbb{Z}_+$ is discrete topology)**.** $\mathbb{Z}_+$ forms ordered set with smallest element. The order topology on $\mathbb{Z}_+$ is the discrete topology (Power set topology) because the basis of the discrete topology can be written in the above form. Basis for discrete topology is the set of singleton positive integers.

For every one-point set is open: for $n > 1$, the one-point set $\{n\} = (n-1, n+1)$ which is the basis element; and if $n = 1$, the one-point set $\{n\} = [1, 2)$ which is the element of the basis set.

Example 9. The set $X = \{1, 2\} \times \mathbb{Z}_+$ in the dictionary order is the ordered set with smallest element. The order topology on this set is *not* discrete topology. Proof: Let $a_n = 1 \times n$ and $b_n = 2 \times n$ then the element can be ordered as:

$$a_1, a_2, \ldots; b_1, b_2, \ldots$$

Here a_1 is the smallest element and $\{a_1\} = [a_1, a_2)$ which is a basis element. And other elements except b_1 can be written as $\{a_n\} = (a_{n-1}, a_{n+1})$ and same for $b_n, n \neq 1$. For b_1, it cannot be written as some element of the basis without containing the elements of sequence a_i.

Definition 15 (Rays)**.** If X is an ordered set, and a is an element of X, there are four subsets of X that are called the **rays** determined by a. They are the following:

(1) $(a, +\infty) = \{x \mid x > a\}$
(2) $(-\infty, a) = \{x \mid x < a\}$
(3) $[a, +\infty) = \{x \mid x \geq a\}$
(4) $(-\infty, a] = \{x \mid x \leq a\}$

Sets of first two types are called **open rays** and the last two types are called **closed rays**.

Open rays are open in order topology as it can be shown as: if there exists an largest element b_0 in X then (a, ∞) can be written as $(a, b_0]$ which is the basis element of order topology. (A set is open if it can be written as union of basis elements).

If there does not exists a largest then (a, ∞) can be written as union of (a, x) for $x > a$. In either case (a, ∞) is open. A similar argument applies for $(-\infty, b)$.

Theorem 3. *The open rays form a subbasis for the order topology* $(\mathcal{T})$ *on* X*.*

Proof. Let the collection of open rays to be $\mathcal{S}$. We need to show that $\mathcal{T} = \cup(\cap_{i=1}^{k} \mathcal{S})$ for all $k = 0, 1, \ldots$ and for $\mathcal{S}$ to be a subbasis.

($\Longleftarrow$) As we have seen above that open rays are open in order topology and so their finite intersections. Hence, $\cup(\cap_{i=1}^{k} \mathcal{S}) \subset \mathcal{T}$ because $\mathcal{T}$ is a topology.

($\Longrightarrow$) Every basis element of the order topology is the finite intersection of elements of $\mathcal{S}$. To see this, (a, b) is intersection of (a, ∞) and $(-\infty, b)$. And if there exists largest (a_0) and smallest (b_0) elements then $[a_0, b)$ and $(a, b_0]$ are themselves open rays. Hence basis of order topology is contained in the set of open rays and this implies $\mathcal{T} \subset \cup(\cap_{i=1}^{k} \mathcal{S})$. ∎

4. The Product Topology

Definition 16 (Product topology)**.** Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

The collection $\mathcal{B}$ forms the basis for product topology. For all $x \times y \in X \times Y$ is contained in the open set $X \times Y$ since $X \times Y$ belongs to $\mathcal{B}$. Also intersection of two open sets is open hence their intersection is the element of $\mathcal{B}$. Hence second condition of the basis is also satisfied.

Remark 4. The collection $\mathcal{B}$ is not the topology on X although it is collection of open sets. To see this, $(U_1 \times V_1) \cup (U_2 \times V_2)$ is open in $X \times Y$ when all individual sets are open but it is not contained in $\mathcal{B}$ since it cannot be written as product of two open sets. See figure 1.

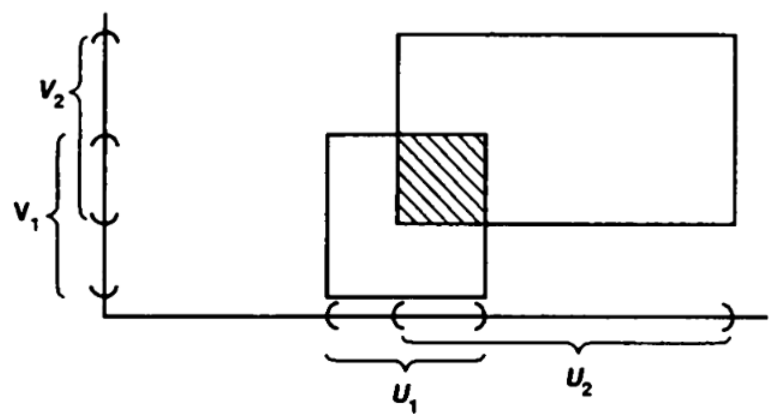

FIGURE 1

Theorem 4. *If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is the basis for the topology on Y, then the collection*

$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Proof. Let W be the open subset of $X \times Y$ which is the product topology hence by definition there exist $U \times V \in X \times Y$ such that U is open in X and V is open in Y. Since $\mathcal{B}$ and $\mathcal{C}$ are the bases for topology on X and Y respectively there exists $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that for each $x \in U \implies x \in B \subset U$ and for each $y \in V \implies y \in C \subset V$. Therefore, $x \times y \in B \times C \subset U \times V$. Hence by lemma 3, $\mathcal{D}$ is the basis for topology on $X \times Y$. ■

Exercise 5.

Definition 17 (Projections)**.** Let $\pi_1 : X \times Y \to Y$ be defined by the equation

$$\pi_1(x, y) = x;$$

let $\pi_2 : X \times Y \to Y$ be defined by the equation

$$\pi_2(x, y) = y.$$

The maps π_1 and π_2 are called the **projections** of $X \times Y$ onto X and Y respectively.

Remark 5. The maps π_1 and π_2 are surjective.

If U is an open set of X then $\pi_1^{-1}(U) = U \times Y$ which is open is $X \times Y$. Similarly, $\pi_2^{-1}(V) = X \times V$ which is open in $X \times Y$.

Theorem 5. *The collection*

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$*.*

Proof. Let $S_{ij} = \pi_1^{-1}(U_i) \cup \pi_2^{-1}(V_j)$. To show that $\mathcal{S}$ is subbasis we need to show that $\cup_j(\cup_i S_{ij}) = X$.

$(\implies)$ Let $x \in S_{ij}$ implies that $x \in U_i \times Y$ or $x \in X \in V_j$. In either case, $x \in X \times Y$ since $U_i \subset X$ and $V_j \subset Y$. $\cup_j(\cup_i S_{ij}) \subset X \times Y$.

$(\impliedby)$ Let $x \in X \times Y$. This case is trivial since $U_i = X$ and $V_j = Y$ and then $x \in (X \times Y) \cup (X \times Y)$. This implies that $x \in \cup_j(\cup_i S_{ij})$. Therefore, $\cup_j(\cup_i)S_{ij} = X \times Y$.

TODO ∎

Definition 18 (Open map)**.** A map $f : X \to Y$ is said to be an **open map** if for every open set U of X, the set $f(U)$ is open in Y.

Theorem 6. *The projection maps* $\pi_1 : X \times Y \to X$ *and* $\pi_2 : X \times Y \to Y$ *are open maps.*

Proof. Let $U \times V$ be the open set in $X \times Y$ where U is open is X and V is open in Y. Since, $\pi_1(U \times V) = U \subset X$ and U is open in X. Same argument can be applied for $\pi_2(U \times V)$. This shows that π_1 and π_2 are open maps. ∎

5. The Subspace topology

Definition 19. Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the **subspace topology**. With this topology, Y is called the **subspace** of X.

It is easy to check that $\mathcal{T}$ is the topology with the fact that

$$\bigcup_{i\in I} U_i \cap Y = (\cup_{i\in I} U_i) \cap Y$$

Lemma 6. *If $\mathcal{B}$ is a basis for the topology on X then the collection*

$$\mathcal{B}_y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is the basis for the subspace topology on Y.

Proof. Given an open set $U \subset X$. For each $y \in U \cap Y$, since $U \cap Y \subset U$, there exists $B \in \mathcal{B}$ such that $y \in B \subset U \cap Y$. Also, $y \in Y$ hence

$$\begin{aligned} y &\in B \cap Y \subset B \subset U \cap Y \\ y &\in B \cap Y \subset U \cap Y \end{aligned}$$

By lemma 3, the collection $\{B \cap Y \mid B \in \mathcal{B}\}$ is the basis for the subspace topology on Y. ■

Lemma 7. *Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.*

Proof. Since $U \in Y$. We have

$$\begin{aligned} U &= \bigcup (U_x \cap Y) \\ U &= (\bigcup U_x) \cap Y \end{aligned}$$

Since, U_x's are open in X so do their arbitrary union and Y is open in X. Finite intersection of open sets is open. Hence U is open in X. ■

Theorem 7 (Relationship between subspace and product topology)**.** *If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.*

Proof. Let $\mathcal{B}$ be the basis of the product topology on $A \times B$.

$$\mathcal{B} = \{U \times V \mid U \text{ in basis of } A; V \text{ in basis of } B\}$$

and $\mathcal{B}'$ be the basis of the topology $A \times B$ inherits as a subspace of $X \times Y$.

$$\mathcal{B}' = \{(A \times B) \cap (U_x \times V_y) \mid U_x \times V_y \text{ in basis of } X \times Y\}$$

We need to show that $\mathcal{B} = \mathcal{B}'$.

($\Longrightarrow$) Since, A is a subspace of X and B is a subspace of Y, implies $U = A \cap U_x$ where U_x is in the basis of X and $V = B \cap V_y$ where V_y is in the basis of Y. Hence.

$$U \times V = (A \cap U_x) \times (B \cap V_y)$$
$$U \times V = (A \times B) \cap (U_x \times V_y)$$

Last equation can be seen by drawing the figure. Hence, $\mathcal{B} \subset \mathcal{B}'$.

($\Longleftarrow$) For any element $(A \times B) \cap (U_x \cap V_y)$ in $\mathcal{B}'$, it can be written as $(A \cap U_x) \times (B \cap V_y)$ (can be seen from visualization). Since U_x is in the basis of X, $A \cap U_x$ is in the basis of A as a subspace of X. Same argument can be extended for $B \cap V_y$.

$$(A \times B) \cap (U_x \cap V_y) = U' \times V'$$
$$(A \times B) \cap (U_x \cap V_y) \in \mathcal{B}$$
$$\mathcal{B}' \subset \mathcal{B}$$

where U' and V' are some basis element of A and B respectively. Since, $\mathcal{B} = \mathcal{B}'$ hence the topology generated by them are equal. ∎

Remark 6. Let X be an ordered set in the order topology, and let Y be a subset of X. Then order relation on X, when restricted to Y, makes Y into an ordered set.

Remark 7. However, the resulting order topology on Y need not be the same as the topology that Y inherits as a subspace of X.

Definition 20 (Convex set)**.** Given an ordered set X, A subset Y of X is **convex** in X if for each pair of points $a < b$ of Y, the entire interval (a, b) of X is contained in Y.

Remark 8. The interval and rays in X are convex in X.

Theorem 8. *Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as topology Y inherits as a subspace of X.*

Proof. TODO ∎

6. Closed set and Limit points

Definition 21 (Closed set)**.** A subset A of a topological space X is said to be closed if the set $X - A$ is open.

Theorem 9. *Let X be a topological space. Then the following conditions hold:*

(1) *ϕ and X are closed.*

(2) *Arbitrary intersections of closed sets are closed.*
(3) *Finite union of closed sets are closed.*

Proof.

(1) This is obvious.
(2) Let $\{U_\alpha\}_{\alpha \in I}$ is the collection of closed subset of X. Let $J \subset I$. We write $(\cap_{\alpha \in J} U_\alpha)^c = \cup_{\alpha \in J} U_\alpha^c$. Since each U_α is closed implies U_α^c is open and so their arbitrary union. This shows that $\cap_{\alpha \in J} U_\alpha$ is closed.
(3) We write $(\cup_{i=1}^n U_i)^c = \cap_{i=1}^n U_i^c$. Since each U_i is closed hence U_i^c is open and so their finite intersection. This shows that $\cup_{i=1}^n U_i$ is closed.

■

Theorem 10. *Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.*

Proof. TODO ■

Theorem 11. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Proof. TODO ■

6.1. **Closure and Interior of a set.** Given a subset A of a topological space X, the **interior** of a A is defined as the union of all open sets contained in A, and the **closure** of A is defined as the intersection of all closed sets containing A.

$$\text{Int}A \subset A \subset \bar{A}$$

Remark 9. If A is open, then $\text{Int}A = A$ and if A is closed, then $\bar{A} = A$.

Theorem 12. *Let Y be a subspace of X, let A be a subset of Y, let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.*

Proof. Let denote the closure of A in Y as $cl_Y(A)$ and $\bar{A}$ as closure of A in X. Hence, $cl_Y(A) = \cap_{\alpha \in I}(U_X^\alpha \cap Y) = (\cap_{\alpha \in I} U_X^\alpha) \cap Y$ where each $U_X^\alpha \cap Y$ contains A and is closed in Y (which implies each U_X^α contains A and is closed in X). Since, arbitrary intersection of closed set is closed. $\cap_{\alpha \in I} U_X^\alpha$ is $\bar{A}$ (by definition of closure). ■

Theorem 13 (Closure in terms of basis)**.** *Let A be a subset of the topological space X.*

(1) *Then* $x \in \bar{A}$ *if and only if every open set containing* x *intersects* A*.*
(2) *Supposing the topology of* X *is given by a basis, then* $x \in \bar{A}$ *if and only if every basis element* B *containing* x *intersects* A*.*

Proof. TODO ∎

Remark 10. Consider the subspace $Y = (0, 1]$ of the real line. The set $A = (0, \frac{1}{2})$ is a subset of Y, its closure in $\mathbb{R}$ is the set $[0, \frac{1}{2}]$, and its closure in Y is the set $[0, \frac{1}{2}] \cap Y = (0, \frac{1}{2}]$.

Theorem 14. *Let* A *be a subset of the topological space* X*.*

(1) *Then* $x \in \bar{A}$ *if and only if every open set* U *containing* x *intersects* A*.*
(2) *Supposing the topology of* X *is given by a basis, then* $x \in \bar{A}$ *if and only if every basis element* B *containing* x *intersect* A*.*

Proof. (1) The contrapositive of the statement is "$x \notin \bar{A}$ if and only if there exists an open set U containing x which does not intersects A." This is easy to show. This implies $x \in X - \bar{A}$ and $\bar{A}$ is closed in X hence $U = X - \bar{A}$ is open in X containing x.
Conversely, let U be an open set containing x which does not intersect A. $X - U$ is closed set containing A hence by definition of $\bar{A}$, $\bar{A} \subset X - U$. This implies $x \notin \bar{A}$.

(2) From (1), we just have to show that every open set U containing x intersects A if and only if every basis element B containing x intersects A. ($\Longrightarrow$) If every open set containing x intersects A then so does every basis element B containing x since basis elements are open. ($\Longleftarrow$) Every open set containing x has to contain the basis elements containing x. Since these basis elements intersects A which implies the open set containing them intersects A. ∎

Definition 22 (Limit points)**.** If A is a subset of the topological space X and if x is a point of X, then x is a **limit point** of A if every neighbourhood of x intersects A in some point other than x itself.

$$x \in \overline{A - [x]}$$

Theorem 15. *Let* A *be a subset of the topological space* X*, let* A' *be the set of all limit points of* A*. Then*

$$\bar{A} = A \cup A'$$

Proof. TODO ∎

6.2. **Hausdorff Spaces.**

Definition 23 (Hausdorff Space)**.** A topological space X is called a **Hausdorff space** if for each pair x_1, x_2 of distinct points of X, there exists neighbourhood U_1 and U_2 of x_1 and x_2 respectively in X, that are disjoint.

This is also known as T-2 separartion axiom.

Theorem 16. *Every finite point set in a Hausdorff space X is closed.*

Proof. ■

7. Continuous functions

Definition 24 (Continuous function)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continous if for each open subset V of Y, the set $f^{-1}(V)$ is an open set of X.

If the topology on the range space Y is given by the basis elements $\mathcal{B}$ then it is enough to show that the **inverse image of the basis element is open**. Since open set V in Y is given by $V = B_1 \cup B_2 \cup \dots$ and

$$f^{-1}(V) = f^{-1}(B_1) \cup f^{-1}(B_2) \cup \dots$$

each element of RHS is open hence their arbitrary union is also open.

If the topology on Y is given by subbasis $\mathcal{S}$ then it is enough to show that **inverse image of each element of subbasis is open**. Since an element B of basis can be written as $S_1 \cap S_2 \cap \dots \cap S_n$.

$$f^{-1}(B) = f^{-1}(S_1) \cap f^{-1}(S_2) \cap \dots \cap f^{-1}(S_n)$$

each element of RHS is open hence $f^{-1}(B)$ is open then from above paragraph, it implies that f is continuous.

Exercise 6. Prove that the $\varepsilon - \delta$ defintion of continuity for real-valued functions is equivalent to the above definition of continuity.

Solution. ($\implies$) $\varepsilon - \delta$ implies the open ball definition.
Let $f : \mathbb{R} \to \mathbb{R}$ be the continuous function hence for given $\varepsilon > 0$ there exists a $\delta > 0$ such that for some $x_0 \in \mathbb{R}$, $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. Suppose $x \in \mathbb{R}$ such that $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. This implies

$$x \in f^{-1}(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$$

Then by $\varepsilon - \delta$ defintion of continuity, $x \in (x_0 - \delta, x_0 + \delta)$. This means $f^{-1}(f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subset (x_0 - \delta, x_0 + \delta)$. For other way if $x \in$

$(x_0 - \delta, x_0 + \delta)$ then by $\varepsilon - \delta$ defintion of continuity, $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. Hence, $(x_0 - \delta, x_0 + \delta) \subset f^{-1}(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

This equality implies that inverse image of the open set in range space $\mathbb{R}$ is open in domain space $\mathbb{R}$.

($\Longleftarrow$) If $f : \mathbb{R} \to \mathbb{R}$ is continuous then if V is open in range space $\mathbb{R}$ then $U = f^{-1}(V)$ is open in domain space $\mathbb{R}$. For some $x_0 \in \mathbb{R}$ and given $\varepsilon > 0$, $V = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ is open in $\mathbb{R}$. Since U contians x_0, there exists a basis element $(a, b) \in \mathbb{R}$ such that $x_0 \in (a, b) \subset U = f^{-1}(V)$. We choose $\delta > 0$ as $\min\{x_0 - a, x_0 - b\}$ then $(x_0 - \delta, x_0 + \delta) \subset U$ and whenever $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \varepsilon$.

Example 10. If $f : \mathbb{R} \to \mathbb{R}_l$ is identity function then f is not continuous since any open set $[a, b)$ (basis element of $\mathbb{R}_l$) containing x, its inverse image is $[a, b)$ which is not open in $\mathbb{R}$.

Example 11. If $g : \mathbb{R}_l \to \mathbb{R}$ is identity function then g is continuous since any open set (a, b) containing x, its inverse image is (a, b) which is open in $\mathbb{R}_l$. $((a, b) = \cup_{n=1}^{\infty}[a + \frac{1}{n}, b))$.

Theorem 17. *Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous.*
2. *For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$.*
3. *For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.*
4. *For each $x \in X$ and each neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subset V$.*

If the condition in (4) holds for the point x of X, we say that f is continous at the point x.

Proof. $(1 \implies 2)$ TODO ∎

7.1. **Homeomorphism.**

Definition 25 (Homeomorphism)**.** Let X and Y be the topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called homeomorphism.

Another way to define a homeomorphism is to say that it is a bijective correspondence $f : X \to Y$ such that $f(U)$ is open if and only if U is open.

Definition 26 (Topological properties)**.** Any properties of X that are entirely expressed in terms of the topology of X (i.e. in terms of open sets of X) yields, via the correspondence f, the corresponding property for the space Y, called topological properties.

Definition 27 (Topological imbedding)**.** Suppose that $f : X \to Y$ is an injective continuous map, where X and Y are topological spaces. Let Z be the image set $f(X)$, considered as a subspace of Y; then the function $f' : X \to Z$ obtained by restricting the range of f, is bijective. If f' is an homeomorphism of X with Z, then map $f : X \to Y$ is called a **topological imbedding** of X in Y.

Theorem 18 (Constructing continuous functions)**.** *TODO*

Theorem 19 (Pasting lemma)**.** *TODO*

7.2. **Quotient Topology.**

Definition 28 (Quotient map)**.** Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a **quotient map** provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

Definition 29 (Quotient topology)**.** If X is a space and A is a set and if $p : X \to A$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map; it is called the **quotient topology** induced by p.

The topology $\mathcal{T}$ is consists of those subsets U of A such that $p^{-1}(U)$ is open in X.

Definition 30 (Quotient space)**.** Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a **quotient space** of X.

A subset U of X^* is a collection of equivalence classes, and the set $p^{-1}(U)$ is the union of the equivalence classes belonging to U. Thus, open set of X^* is a collection of equivalence classes whose union is an open set of X.

Theorem 20. *Let $p : X \to Y$ be a quotient map; let A be a subspace of X that is saturated with respect to p; let $q : A \to p(A)$ be the map obtained by restricting p.*

(1) *If A is either open or closed in X, then q is a quotient map.*
(2) *If p is either an open map or a closed map, then q is a quotient map.*

Proof. TODO. ■

7.3. **Topological Groups.**

Definition 31 (Topological Group)**.** A *topological group* $(G, \cdot, \mathcal{T})$ consists of group $(G, \cdot)$ and a topology $\mathcal{T}$ on G for which the multiplication map

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto g \cdot h = gh \end{aligned}$$

and the inversion map

$$\begin{aligned} G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are continuous. We call $\mathcal{T}$ to be *group topology* on G. We can combine these two conditions as:

$$\begin{aligned} \kappa \colon G \times G &\to G \\ (g, h) &\mapsto g^{-1}h \end{aligned}$$

If G is a topological group then κ is continuous.

Conversely, if κ is continuous, then the maps $g \mapsto g^{-1} = \kappa(g, e)$ and $(g, h) \mapsto gh = \kappa(\kappa(g, e), h)$ are continous (composition of continuous maps).

Theorem 21. *Suppose that G is a topological group. For every $a \in G$, the right tranlation map*

$$\rho_a(g) = ga$$

the left tranlation map

$$\lambda_a(g) = ag$$

and the conjugation map

$$\gamma_a(g) = aga^{-1}$$

are homeomorphisms of G onto itself, with inverses $\lambda_{a^{-1}}$, $\rho_{a^{-1}}$, $\gamma_{a^{-1}}$, respectively.

Proof. We will show it from ρ_a rest all will follow the same.
Injectivity is trivial from group operation right cancellation. For surjectivity, let $g \in G$ then there exists $ga^{-1} \in G$ such that $\rho_a(ga^{-1}) = g$. Hence, it is surjective.

Let $U \in G$ be an open subset.

$$(\rho_a)^{-1}(U) = \{g \in G\colon \rho_a(g) \in U\} = Ua^{-1}$$

Since, $U = Ua^{-1}a$ is open in G and multiplication map $G \times G \to G$ is continuous implies that inverse image of $Ua^{-1}a \in G$ which is $(Ua^{-1}, a) \in G \times G$ is open. Hence, Ua^{-1} is open in G (product topology). This shows that ρ_a is continuous and since a is arbitrary, we can choose $(\rho_a)^{-1}$ as $\rho_{a^{-1}}$(inverse map is also continuous then). This also shows that ρ_a is homeomorphism.

Similarly, for other two maps, it can be shown that they are homeomorphism from $G \to G$. ■

Theorem 22. *A topological group G is Hausdorff if and only if some singleton $\{a\} \subset G$ is closed.*

Proof. ($\implies$) Assume G is Hausdorff, to show that $\{a\}^c$ is open. For all $b(\neq a) \in G$, there exists open nbd $U_b \in G$ such that $b \in U_b$. Therefore, $\bigcup\limits_{\substack{b \in G \\ b \neq a}} U_b = G \setminus \{a\} = \{a\}^c$ which is open.

($\impliedby$) Assume $\{a\} \in G$ is closed. Take the inverse image of $\{a\}$ under the continuous map $(g, h) \mapsto g^{-1}ha$. $g^{-1}ha = a \implies g = h$. Hence, the inverse image is $\Delta = \{(g, g)\colon g \in G\}$ which is closed in $G \times G$ ($\because$ map is continuous) which means G is Hausdorff. (Δ is closed in $G \times G$ implies G is Hausdorff; Δ^c is open then the open nbd for each point $(x, y), x \neq y$ is basis elements of $G \times G$ and intersection of those basis elements has to be empty otherwise $(z, z) \in \Delta^c$, z in intersection, which is not possible). ■

7.4. **Quotients.** Suppose H is a subgroup of a topological group G. We endow the set G/H of left cosets with the quotient topology w.r.t natural map

$$p\colon G \to G/H, \qquad g \mapsto gH.$$

Thus a subset of G/H is open if and only if its p-preimage is open.

Proposition 1. *Let G be a topological group and let H be a subgroup. Then the quotient map*

$$p\colon G \to G/H$$

is open. The quotient G/H is Hausdorff if and only if H is closed in G.

Proof. TODO ■

7.5. **Local Compactness.**

Definition 32 (Locally compact)**.** A space X is said to be locally compact at x if there is some compact subspace C_x of X and an open set U_x of X such that $x \in U_x \subset C_x$. If X is locally compact at each of its points, X is said simply to be locally compact.

Theorem 23. *Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y satisfying the following conditions:*

(1) *X is a subspace of Y.*
(2) *The set $Y - X$ consists of a single point.*
(3) *Y is a compact Hausdorff space.*

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

Proof. First, check the uniqueness. Let Y and Y' be two spaces satisfying these conditions. Define $h : Y \to Y'$. h is identity on X and it maps unique point of $Y - X$ to unique point of $Y' - X$. It is injective, surjective. For continuity, if $U \subset X \subset Y'$ then $h^{-1}(U)$ is open in $X \subset Y$(identity). For $U = \{\infty\} \subset Y'$ which is closed, $h^{-1}(U) = \{\infty'\} \subset Y$ which is closed in Y since Y is Hausdorff (singletons are closed in Hausdorff spaces). By symmetry, h^{-1} is also continuous.

Now, we assume that X is locally compact and Hausdorff and construct $Y = X \cup \{\infty\}$ where ∞ is a symbol for the extra element. The topology on Y is given by (1) open sets of X (2) all sets of the form $Y - C$ where C is a compact subspace of X. We need to check if this collection form a topology or not. (1) ϕ is in X implies it is in the collection. (2) since ϕ is compact in X hence $Y - \phi = Y$ is in the collection. ∎

Definition 33 (One-point compactification)**.** If Y is compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is said to be a *compactification* of X. If $Y - X$ equals a single point, then Y is called the *one-point compactification* of X.