ASSIGNMENT 5

DEVANSH TRIPATHI IMS22090

Abstract. This document contains solution for assignment 5 of General Topology course.

Sol. 1.

($\Longleftarrow$) Assume for any $x \in X$ and any open set U_x containing x, there exists open set V containing x such that $\overline{V}$ is compact and $\overline{V} \subset U_x$. Let $C = \overline{V}$ be the compact set containing x and a neighbourhood V around x. Since x is arbitrary, it is true for all x. Therefore, X is locally compact.

($\Longrightarrow$) Assume X is locally compact and Hausdorff then there exists a compact Hausdorff space Y such that X is subspace of Y and $Y - X = \{\infty\}$. For each $x \in X$ and a neighbourhood U of x, since U is open in X, it is open in Y which implies $C = Y - U$ is closed and compact (closed subset of compact space).

Since Y is Hausdorff then for any $x \in X \subset Y$ and a compact $C \subset Y$ there exists two disjoint open sets V and W in Y such that $x \in V$ and $C \subset W$. Since, $V \cap W = \phi$ implies $\overline{V} \cap W = \phi$ (if $\overline{V} \cap W \neq \phi$ then $x \in W$ and either $x \in V$ or $x \in V'$ (V' is limit point set of V). $x \in V$ will contradict $V \cap W$ trivially. For $x \in V'$, then for any neighbourhood S around x we have $S \cap V \setminus \{x\} \neq \phi$ and since W is open set containing x implies it contains S that will again contradict $V \cap W = \phi$).

$\overline{V}$ is closed in Y therefore it is compact and $\overline{V} \cap W = \phi$ implies $\overline{V} \cap C = \phi$. Since, $C = Y - U$ hence we have $\overline{V} \subset U$.

Sol. 2.

(In this question, we have to assume X is Hausdorff.)

Let X be a locally compact Hausdorff space. For any $x \in X$ and an open neighbourhood U of x, there exists an open neighbourhood V of x such that $\overline{V} \subset U$ and $\overline{V}$ is compact (from question 1). Let $A \subset X$ be **open** in X. Take arbitrary $x \in A$ and an open neighbourhood $U \cap A$ of x which is open in A (since U is open in X). Also, $U \cap A$ is open in X (both U and A are open in X). This implies $U \cap A$ is open in Y which is one point compactification of X. Let $C = Y - (U \cap A)$, it is closed in Y hence compact. Since Y is Hausdorff, there exists an open set V and W in Y containing x and C, respectively such that $V \cap W = \phi$ which implies $\overline{V} \cap C = \phi$ (argument for this is in question 1). Therefore, $\overline{V} \subset U \cap A$ and since $\overline{V}$ is closed in Y hence compact in Y implies compact in X (since X is subspace of Y). Since, $x \in A$ is arbitrary, from question 1 we have A is locally compact.

Let $A \in X$ be closed. Since, X is locally compact. For each $x \in X$ there exists a compact set C_x containing x and its neighbourhood U_x. Since, $C_x \cap A$ is closed in C_x, its compact in C_x which implies it is compact in X (in subspace topology). Also, $U_x \cap A$ is open in A (subspace topology) and contained in $C_x \cap A$ (since $U_x \subset C$). Hence, A is locally compact.

Sol. 3.

(i). Suppose for contradiction that $(\mathbb{R}, \mathcal{T}_{cof})$ is metric space and $d \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a metric. For some $x \in \mathbb{R}$, open balls will be $B(x, r) \colon = \{y \mid d(x, y) < r, r \in \mathbb{R}^+\}$. Open balls are open sets in metric spaces. Hence, $\mathbb{R} - B(x, r)$ should be finite.

$$\mathbb{R} - B(x, r) = \{z \mid d(x, z) \geq r, r \in \mathbb{R}^+\}$$

which is not fin¡ite. This shows contradiction. Hence, $(\mathbb{R}, \mathcal{T}_{cof})$ is not a metric space.

(ii). Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $(\mathbb{R}, \mathcal{T}_{cof})$. This means $\mathbb{R} - U_\beta$, $\beta \in I$, is finite, say $\{x_1, \ldots, x_n\}$. Take U_1 containing x_1, U_2 containing x_2 and so on (they exists since $\{U_\alpha\}_{\alpha \in I}$ is an open cover). Then $\{U_\beta, U_1, \ldots, U_n\}$ is a finite subcover. Hence, $(\mathbb{R}, \mathcal{T}_{cof})$ is compact.

Remark 1. In fact, in above argument there is nothing specific to $\mathbb{R}$. This also shows that any set having cofinite topology is compact.

(iii). Since compactness implies limit point compactness implies that $(\mathbb{R}, \mathcal{T}_{cof})$ is also limit point compact.

(iv). Let $(x_n) \in X$ be a sequence. Let $A = \{x_n \mid n \in \mathbb{Z}_+\}$ be a set.

Case 1: If A is finite then there exists $N \in \mathbb{Z}_+$ and $x \in A$ such that $x_n = x$ for all $n > N$. Hence, there exists a constant subsequence that is trivially convergent.

Case 2: If A is infinite then there exists a limit point of A since $(\mathbb{R}, \mathcal{T}_{cof})$ is limit point compact. Since every convergent sequence is bounded implies A is bounded and then by Bolzano-Weierstrass, we have a convergent subsequence in A.

Therefore, $(\mathbb{R}, \mathcal{T}_{cof})$ is sequentially compact.