ASSIGNMENT 6

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Abstract. This document contains solution for assignment 6 of General Topology course.

Sol. 1.

Let $\bigcup A_n = C \cup D$ be a separation. For some $i \in \mathbb{N}$, $A_i \subset C$ or D. WLOG, let $A_i \subset C$, Since $A_i \cap A_{i+1} \neq \phi$ and A_{i+1} is connected, we have $A_{i+1} \subset C$. $i \in \mathbb{N}$ is arbitrary implies we have A_i's in C for all $i \in \mathbb{N}$. Therefore, $D = \phi$ which implies $\bigcup A_n$ is connected.

Sol. 2.

Let $X = C \cup D$ be a separartion. $p^{-1}(y \in Y) = \{x \in X \mid y = p(x)\}$ is connected implies $p^{-1}(y \in Y) \subset C$ or D. WLOG, assume it is C. For $y' \in Y$, $p^{-1}(y')$ can lie based on two cases:
Case 1: $p^{-1}(y') \subset C$.
Since, $y' \in Y$ is arbitrary implies it is true for any $y' \in Y$ implies $D = \phi$. Then X is connected.
Case 2: $p^{-1}(y') \subset D$.
Let A be the set of all $y \in Y$ such that $p^{-1}(y) \subset D$ implies $(A \subset p(D))$. Also, for all $x \in D$, we have $p(x) \in A$ implies $p(D) \subset A$ (otherwise if $p(x) \notin A$ then there exists $y = p(x)$ in $Y \setminus A$ such that $p^{-1}(y) \subset D$ (why? since $x \in D$ and $x \in p^{-1}(y)$ and latter is connected), which contradicts the definition of A). Hence, $p^{-1}(A) = D$ which is clopen implies open. This implies A is open (since p is quotient map).

Similarly, for all $y \in Y \setminus A$, $p^{-1}(y) \subset C$ (otherwise, if it in D then y has to be in A which is not the case). Hence, $p^{-1}(Y \setminus A) = C$ (for $C \subset p^{-1}(Y \setminus A)$ observe that for all $x \in C$, $p(x) \subset Y \setminus A$ otherwise, if $p(x) \subset A$ then x has to be in D which is not the case). Hence, $p^{-1}(Y \setminus A)$ is clopen and $Y \setminus A$ is open (since p is quotient map).

Both A and $Y \setminus A$ are open in Y and their union is Y implies that they are clopen in Y and forms a separation of Y which is a contradiction since Y is connected. Hence, X is connected.

Sol. 3.