THEORY OF ODE

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Abstract. We shall learn some theory related to ordinary differential equations.

1. Preliminaries from Real Analysis

Definition 1 (Pointwise convergence)**.** Let I be any interval in $\mathbb{R}$. Let $f_n : I \to \mathbb{R}, n = 1, 2, \dots,$ be a sequence of functions. We say that the $\{f_n\}$ converges pointwise to a function $f : I \subset \mathbb{R} \to \mathbb{R}$ if the sequence $\{f_n(x)\}$ converges to $f(x)$ for every $x \in I$.

Remark 1. Uniform convergence preserves continuity, interchange of limit and integral.

Theorem 1. *Uniform limit of the sequence of continuous function is continuous.*

Remark 2. Converse of the above theorem is not always true.

Theorem 2 (Cauchy Criterion)**.** *Let $\{f_n\}_{n\geq 1}$ a sequence of function defined on a metric space (X, d_X) with values in a complete metric space (Y, d_Y). Then there exists a function $f : X \to Y$ such that*

$$f_n \to f \text{ uniformly on } X$$

if and only if the following condition is satisfied: For every $\varepsilon > 0$, there exists an integer n_0 such that

$$m, n \geq n_0 \text{ implies } d_Y(f_m(x), f_n(x)) < \varepsilon$$

for every $x \in X$.

Proof. ($\implies$) Assume that the sequence $\{f_n\}$ converges uniformly on X. For given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\forall m > n_0$ and for all $x \in X$

$$d_Y(f_m, f) \leq \frac{\varepsilon}{2}$$

and there exists $n_1 \in \mathbb{N}$ such that $\forall n \geq n_1$

$$d_Y(f_n, f) \leq \frac{\varepsilon}{2}$$

Take $n_3 = \min\{n_0, n_1\}$ then for all $m, n \geq n_3$

$$\begin{aligned} d_Y(f_m, f_n) &\leq d_Y(f_m, f) + d_Y(f, f_n) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &\leq \varepsilon \text{ for all } x \in X \end{aligned}$$

($\Longleftarrow$) Conversely, suppose that $m, n \geq n_0$ implies that $d_Y(f_m, f_n) < \varepsilon$ for all $x \in X$. Then for each $x \in X$, the sequence $\{f_n(x)\}_{n\geq 1}$ is cauchy in a complete space Y and therefore converges in Y. Let $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in X$. For $k > 0$ then

$$d_Y(f_n(x), f_{n+k}(x)) < \frac{\varepsilon}{2}$$

for every $k = 0, 1, \ldots$ and every $x \in X$.

$$d_Y(f_n(x), f(x)) = \lim_{n\to\infty} d_Y(f_n(x), f_{n+k}(x)) \leq \frac{\varepsilon}{2} < \varepsilon$$

$$d_Y(f_n(x), f(x)) < \varepsilon$$

for every $x \in X$ and $f_n \to f$ uniformly over X. □

Theorem 3 (Weierstrass M-test)**.** *Let $f_1, f_2, \ldots$ be a sequence of real valued functions defined on a set X and suppose that*

$$|f_n(x)| \leq M_n$$

for all $x \in X$ and all $n = 1, 2, \ldots$ If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

Proof. If $\sum_{n=1}^{\infty} M_n$ converges, then for given $\varepsilon > 0$,

$$\left|\sum_{k=n}^{m} f_k(x)\right| \leq \sum_{k=n}^{m} |f_k(x)| \leq \sum_{k=n}^{m} M_k < \varepsilon$$

for every $x \in X$ and provided that m, n are sufficiently large. Now, by cauchy criterion, uniform convergence follows. □

Remark 3. For Uniform boundedness and equicontinuity, the underline space is assumed to be compact hence equicontinuity here is same as uniform equicontinuity.

Remark 4. Some authors define pointwise and uniform equicontinuity separately and then on compact space they prove the equivalence.

Definition 2 (Uniform Boundedness)**.** A sequence of functions $\{f_n\}$ defined on I(compact) is said to be uniformly bounded if there exists a constant $M > 0$ such that $|f_k(x)| \leq M$, for all $x \in I$, for all $k \in \mathbb{N}$.

Definition 3 (Equicontinuity)**.** A sequence of functions $\{f_k\}$ defined on I(compact) is said to be equicontinuity on I, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f_k(x) - f_k(y)| < \varepsilon$ whenever $x, y \in I$ and $|x - y| < \delta$, and for all k.

Remark 5. If the family of functions is equicontinuous, then each member in the family is uniformly continuous. Converse may not be true always. For e.g. $f_k(x) = x^k, 0 \leq x \leq 1, k = 1, 2, \dots$ is not an equicontinuous family; however each member is uniformly continuous function (continuous on closed and bounded interval).

Example 1. Each finite set of functions defined on compact set is equicontinuous. If the set is singleton then its trivial. For two element set, take minimum of the δ's and then for finite set take minimum of all δ's .

Theorem 4. *For the sequence of functions f_n and f defined on compact set E and $f_n \to f$ uniformly then the family of functions $A = \{f_n \colon n \in \mathbb{N}\}$ is equicontinuous.*

Proof. Since $f_n \to f$ uniformly, by cauchy criterion of uniform continuity $\exists n_0 \in \mathbb{N}$ such that $\forall m, n \geq n_0$

$$|f_m(x) - f_n(x)| < \frac{\varepsilon}{3}$$

Look at the family of functions $B = \{f_1, f_2, \dots, f_{n_0}\}$, since it is finite, it is equicontinuous. For $k \in \mathbb{N}$

$$|f_k(x) - f_k(y)| < \frac{\varepsilon}{3}$$

whenever $|x - y| < \delta$ for all $f_k \in B$ and for all $x, y \in E$. Now, for each $f_n \in A$

$$\begin{aligned}
|f_n(x) - f_n(y)| &\leq |f_n(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(y)| + |f_{n_0}(y) - f_n(y)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&< \varepsilon \text{ whenever } |x - y| < \delta
\end{aligned}$$

□

Theorem 5 (Arzela-Ascoli)**.** *Let $\{f_k\}$ be a sequence of functions in $C[a, b]$ which is uniformly bounded and equicontinuous. Then, there exists a subsequence $\{f_{k_n}\}$ of $\{f_k\}$ such that $\{f_{k_n}\}$ converges uniformly to a function $f \in C[a, b]$.*

Proof. TODO □

Definition 4 (Lipschitz continuity)**.** A function $f\colon D \subset \mathbb{R} \to \mathbb{R}$, is said to be *locally* Lipschitz in D if for any $x_0 \in D$, there exists a neighbourhood N_{x_0} of x_0 and an $\alpha = \alpha(x_0) > 0$ such that

$$|f(x) - f(y)| \leq \alpha(x_0)|x - y|, \text{ for all } x, y \in N_{x_0}$$

The function $f\colon D \subset \mathbb{R} \to \mathbb{R}$ is said to be *globally* Lipschitz in D if there exists $\alpha > 0$ such that

$$|f(x) - f(y)| \leq \alpha|x - y|, \text{ for all } x, y \in D$$

Remark 6. The smallest α satisfying the above equation is called *Lipschitz constant* of f. It should be finite.

Remark 7. If f is globally Lipschitz, then it is uniformly continuous.(Take $\delta = \frac{\varepsilon}{M}$).

Theorem 6 (Sufficient condition for Lipschitz continuity)**.** *Suppose D is an open interval in $\mathbb{R}$ and $f\colon D \to \mathbb{R}$ is differentiable on D and $\alpha = \sup\limits_{x \in D} |f'(x)| < \infty$. Then, f is Lipschitz with a Lipschitz constant less than or equal to α.*

Proof. Use mean value theorem:

$$\frac{|f(b) - f(a)|}{|b - a|} = f'(c) \leq \alpha$$

And Lipschitz constant is by definition the smallest α satisfying above inequality. Therefore, Lipschitz constant $\leq \alpha$. $\square$

Example 2. Example of Lipschitz continuous functions: polynomials, polynomials of sine and cosine, exponential functions etc.

Definition 5 (Lipschitz continuity for vector valued map)**.** A function $\mathbf{f}(t, \mathbf{y})\colon (a, b) \times D \to \mathbb{R}^n$ is said to be Lipschitz continuous (globally) with respect to $\mathbf{y}$ if there exists $\alpha > 0$ such that

$$|\mathbf{f}(t, \mathbf{y_1}) - \mathbf{f}(t, \mathbf{y_1})| \leq \alpha|\mathbf{y_1} - \mathbf{y_2}|$$

for all $(t, \mathbf{y_1})$ and $(t, \mathbf{y_2})$ in $(a, b) \times D$. α should be finite.

Theorem 7 (Sufficient condition for Lipschitz continuity of $\mathbf{f}(t, \mathbf{y})$)**.** *Let $\mathbf{f}\colon (a, b) \times D \to \mathbb{R}^n$ be a C^1 vector valued function, where D is a convex domain in $\mathbb{R}^n$ such that*

$$\sup_{(t,\mathbf{y}) \in (a,b) \times D} \left| \frac{\partial f_i}{\partial y_j}(t, \mathbf{y}) \right| = \alpha < \infty,$$

for $i, j = 1, 2, \ldots, n$. Then , $\mathbf{f}(t, \mathbf{y})$ is Lipschitz continuous on $(a, b) \times D$ with respect to $\mathbf{y}$ having a Lipschitz constant less than or equal to a multiple of α.

Remark 8 (Convex Domain)**.** A set is called a convex domain if it contains all the line segments between any two points of the set.

Remark 9. Lipschitz continuity is a smoothness property stronger than continuity, but weaker than differentiability, locally. There can be functions which are not differentiable but still they can be lipschitz continuous.

Theorem 8 (Calculus Lemma)**.** *Let (a,b) be a finite or infinite interval and $h\colon (a,b) \to \mathbb{R}$ satisfy either*

(i) h is bounded above and non-decreasing or

(ii) h is bounded below and non-increasing,

then, $\lim\limits_{t\to b} h(t)$ exists.

Proof. Let $\alpha = \sup\limits_{t\in(a,b)} h(t)$. For some $\varepsilon > 0$, $\alpha - \varepsilon$ is not a supremum. Hence there exists t_0 such that $\alpha - \varepsilon < h(t_0) < \alpha$. For some $t \in (a,b)$ such that $t \geq t_0$ then

$$\alpha - \varepsilon < h(t_0) \leq h(t) < \alpha$$

Then $\alpha - h(t) < \varepsilon$. Also above can be written as $\alpha < h(t_0) + \varepsilon \leq h(t) + \varepsilon < \alpha + \varepsilon$. Then $h(t) + \varepsilon < \alpha + \varepsilon \implies h(t) < \alpha + \varepsilon \implies h(t) - \alpha < \varepsilon$. Therefore

$$|h(t) - \alpha| < \varepsilon \ \forall t \geq t_0 \implies \lim_{t\to b} h(t) = \alpha$$

□

Theorem 9 (Change of Variable Formula)**.** *Let $g\colon [c,d] \to \mathbb{R}$ be a C^1 function and let $[a,b]$ be any interval containing the image of g, that is $g[c,d] \subset g[a,b]$. If $f\colon [a,b] \to \mathbb{R}$ is a continuous function, then*

$$\int_c^d f(g(t))g'(t)dt = \int_{g(c)}^{g(d)} f(x)dx.$$

Theorem 10 (Generalized Leibnitz Formula)**.** *Let $\alpha, \beta\colon [a,b] \to \mathbb{R}$ be differentiable functions and c, d be real numbers satisfying*

$$c \leq \alpha(t), \beta(t) \leq d, \text{ for all } t \in [a,b].$$

Let $f\colon [a,b] \times [c,d] \to \mathbb{R}$ be a continuous function such that $\frac{\partial f}{\partial t}(t,s)$ is also continuous. Define

$$F(t) = \int_{\alpha(t)}^{\beta(t)} f(t,s)ds.$$

Then, F is differentiable and

$$\frac{dF}{dt} = \int_{\alpha(t)}^{\beta(t)} \frac{\partial f}{\partial t}(t,s)ds + f(t,\beta(t))\frac{d\beta}{dt} - f(t,\alpha(t))\frac{d\alpha}{dt}.$$

Definition 6 (Banach space)**.** A normed linear space which is a complete metric space (metric space by the norm) is called a Banach space.

Example 3 (Examples of Banach space)**.** $\mathbb{R}^n$ under supremum norm and p-norm. Function space $C[a,b]$ with supremum norm. However, it is *not* a complete space with repect to $\|\cdot\|_1$ norm.

Remark 10. A sequence $\{f_n\} \subset C[a,b]$, $f_n \to f$ in supremum norm is equivalent to saying that f_n converges uniformly to f ($\exists \delta > 0$ which works in supremum norm for f, will work for all f in the sequence.)

Theorem 11 (Banach Fixed Point theorem)**.** *Suppose (X,d) is a complete metric space and $T\colon X \to X$ is a contraction, that is, there exists an $\alpha \in (0,1)$ such that*

$$d(Tx,Ty) \leq \alpha d(x,y)$$

for all $x,y \in X$. Then, T has a unique fixed point $x^* \in X$. Further, the sequence $\{x_k\}$ defined by $x_k = Tx_{k-1}, x_0 \in X$ is arbitrary and $k = 1,2,\dots,$ converges to x^*.

Remark 11. If we omit the contraction condition then $f(x) = x+1$ show there does not exists any fixed point. If completion condition is droped then $f\colon (0,1) \to (0,1)$ defined by $f(x) = mx, 0 < m < 1$ is a contraction with $\alpha = m$ but no fixed point.

Proof. Choose any $x_0 \in X$ and define the sequence $x_1 = Tx_0, x_2 = T^2x_0, \cdots, x_k = T^kx_0, \cdots$. We need to show that sequence $\{x_k\}$ is a cauchy sequence. To see this

$$d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1}) \leq \alpha d(x_n,x_{n-1})$$

By induction,

$$d(x_{n+1},x_n) \leq \alpha^n d(x_1,x_0)$$

Consider, for $m < n$ and using triangle inequality

$$d(x_n,x_m) \leq d(x_n,x_{n-1}) + \cdots + d(x_{m+1},x_m) \leq \alpha^m \frac{1-\alpha^{n-m-1}}{1-\alpha} d(x_1,x_0)$$

and the rhs $\to 0$ as $n,m \to \infty$. Hence the sequence is cauchy and by completeness, there exists $x^* \in X$ such that $x_k \to x^*$. By continuity of T, we get $Tx_k \to Tx^*$. By continuity of T, we get $Tx_k \to Tx^*$. But $Tx_k = x_{k+1} \to x^*$. Thus, $Tx^* = x^*$. If there exists y^* with same properties as x^* then by uniqueness of limits $x^* = y^*$. □

Remark 12. Here, fixed point can be constructed with any desired accuracy as limiting value is fixed point. And secondly, any point can be taken as an initial guess.

Corollary 1. *Let $T: X \to X$ where X is a complete metric space, be such that T^k is a contraction for some $k \geq 1$. Then, T has a unique fixed point.*

Proof. Since, T^k is contraction map hence it has a unique fixed point (by Banach fixed point theorem). Let x^* be that point i.e. $T^k x^* = x^*$. Applying T, we get $T^k(Tx^*) = Tx^*$ implies Tx^* is the fixed point for $T^k \implies Tx^* = x^*$. Therefore, x^* is the fixed point of T. For uniqueness, assume x_1 is another fixed point for $T \implies Tx_1 = x_1$. Applying T repeatedly, we get $T^k x_1 = x_1$ but fixed point for T^k is $x^*(unique) \implies x^* = x_1$. □

2. Preliminaries from Linear Algebra

Definition 7 (Normed Linear Space)**.** A norm, denoted by $\|\cdot\|$ on a vector space or a linear space X is a mapping from $X \to \mathbb{R}$ that satisfies

- $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$,
- $\|ax\| = |a|\|x\|$,
- (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$,

for all $x, y \in X$ and scalar a.

Example 4. $\mathbb{R}^n, \mathbb{C}^n$

Remark 13. Every normed linear space is a metric space with metric induced by the norm. A metric space can be equipped with different norms which are fundamentally different. $C[0,1]$ with sup norm is complete while with the integral norm $\left(\int_0^1 |f(x)|\,dx\right)$ is not complete.

Definition 8 (Vector Field)**.** A vector field on a space(most commonly Euclidean space) is a function $\vec{F}$ that assigns a vector to each point of the space.

Definition 9 (Matrix norm)**.** $|A|$ should satisfy the following criterion

- $|A| \geq 0$; $|A| = 0$ if and only if $A = 0$,
- $|aA| = |a||A|$,
- (Triangle inequality) $|A + B| \leq |A| + |B|$,
- $|AB| \leq |A||B|$

for all $A, B \in M_n(\mathbb{R})$ and scalars a.

Remark 14. $M_n(\mathbb{R})$ is a complete metric space.

2.1. **Matrix Exponential e^A.** Let $A \in \mathbb{M}_n\mathbb{R}$, define the sequence of matrices

$$S_k = I + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!}.$$

For $k > l$,

$$|S_k - S_l| \le \sum_{j=l+1}^{k} \frac{|A|^j}{j!} \to 0 \text{ as } l, k \to \infty.$$

Thus $\{S_k\}$ is a Cauchy sequence and converges to some $S \in M_n(\mathbb{R})$.

Definition 10. Given $A \in M_n(\mathbb{R})$, e^A is defined as

$$e^A = S$$

where $S = \lim\limits_{k\to\infty} \sum\limits_{j=0}^{k} \frac{A^j}{j!}$.

Remark 15. $e^A \in M_n(\mathbb{R})$. Also, $|e^A| \le e^{|A|}$ (substitute the values to see this.)

Remark 16. For diagonal matrix $A = diag(\lambda_1, \ldots, \lambda_n) \implies A^j = diag(\lambda_1^j, \ldots, \lambda_n^j)$ and $e^A = \sum\limits_{j=0}^{\infty} \frac{A^j}{j!}$ then

$$e^A = \sum_{j=0}^{\infty} \frac{1}{j!}\, diag(\lambda_1^j, \ldots, \lambda_n^j) \implies e^A = diag(\sum_{j=0}^{\infty} \frac{\lambda_1^j}{j!}, \ldots, \sum_{j=0}^{\infty} \frac{\lambda_n^j}{j!}),$$

therefore,

$$e^A = diag(e^{\lambda_1}, \ldots, e^{\lambda_n}).$$

Here are important observations:

(1) If $A \sim B \implies \exists$ non-singular matrix P such that $B = PAP^{-1}$. Then $e^A \sim e^B$ ($\because B^j = PA^jP^{-1} \implies e^B = P\left(\sum\limits_{j=0}^{\infty} \frac{A^j}{j!}\right)P^{-1} \implies e^B = Pe^AP^{-1}$).

(2)

3. Doubts

Ques 1. How to prove theorem 7? Is there a mean value theorem for vector valued functions that I can apply here?
Ques 2. Proof for change of varible formula in Theorem 9?
Ques 2. Proof for generalized Leibnitz formula in Theorem 10?