Tutorial Sheet (Q-1)

2. Let X denote the number of times a photocopy machine will malfunction: 0, 1, 2, or 3 times, on any given month. Let Y denote the number of times a technician is called on an emergency call. The joint p.m.f. $p(x, y)$ is presented in the table below:

	x				
y	0	1	2	3	$p_Y(y)$
0	0.15	0.30	0.05	0	0.50
1	0.05	0.15	0.05	0.05	0.30
2	0	0.05	0.10	0.05	0.20
$p_X(x)$	0.20	0.50	0.20	0.10	1.00

a) Find the probability $P(Y > X)$.

$P(Y > X) = p(0, 1) + p(1, 2) = 0.05 + 0.05 = \mathbf{0.10}$.

b) Find $p_X(x)$, the marginal p.m.f. of X. ↑

c) Find $p_Y(y)$, the marginal p.m.f. of Y. ↑

d) Are X and Y independent? If not, find $\text{Cov}(X, Y)$.

Tutorial Sheet (Q-1)

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X and Y are **NOT independent**.

$$E(X) = 0 \times 0.20 + 1 \times 0.50 + 2 \times 0.20 + 3 \times 0.10 = 1.2.$$

$$E(Y) = 0 \times 0.50 + 1 \times 0.30 + 2 \times 0.20 = 0.7.$$

$$E(XY) = 1 \times 0.15 + 2 \times 0.05 + 3 \times 0.05 + 2 \times 0.05 + 4 \times 0.10 + 6 \times 0.05 = 1.2.$$

$$Cov(X, Y) = E(XY) - E(X) \times E(Y) = 1.2 - 1.2 \times 0.70 = \mathbf{0.36}.$$

If $p_{XY(}x,y) = p_X(x)\, p_Y(y)$ then X and Y are independent.
If X and Y independent joint random variable then $E[XY] = E[X]E[Y]$ vice versa is not true.

Tutorial Sheet (Q-2)

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1. Consider two continuous random variables X and Y with joint p.d.f.

$$f(x,y) = \begin{cases} \frac{2}{81}x^2 y & 0 < x < K,\ 0 < y < K \\ 0 & \text{otherwise} \end{cases}$$

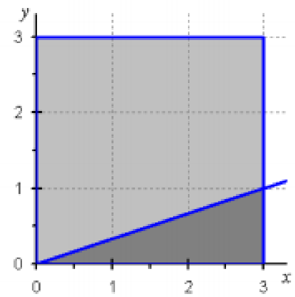
a) Find the value of K so that $f(x,y)$ is a valid joint p.d.f.

$$1 = \int_0^K\int_0^K \frac{2}{81}x^2\, y\, dx\, dy = \frac{K^5}{243}. \qquad \Rightarrow \qquad K = \mathbf{3}.$$

b) Find $P(X > 3Y)$.

$$P(X > 3Y) = \int_0^3 \left(\int_0^{x/3} \frac{2}{81}x^2\, y\, dy \right) dx$$

$$= \int_0^3 \frac{1}{729}x^4\, dx = \mathbf{\frac{1}{15}}.$$

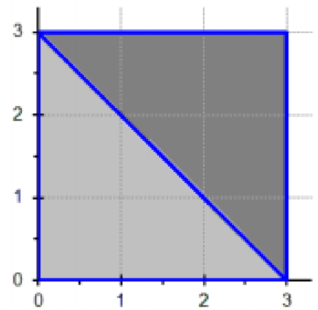


OR

$$P(X > 3Y) = \int_0^1 \left(\int_{3y}^{3} \frac{2}{81}x^2\, y\, dx \right) dy = \ldots = \mathbf{\frac{1}{15}}.$$

Tutorial Sheet (Q-2)

c) Find $P(X+Y>3)$.



$$P(X+Y>3) = \int_0^3\left(\int_{3-x}^{3} \frac{2}{81}x^2\, y\, dy\right)dx$$

$$= \int_0^3 \frac{1}{81}x^2\left[9-(3-x)^2\right]dx$$

$$= \frac{1}{81}\cdot\int_0^3\left(6x^3 - x^4\right)dx$$

$$= \frac{1}{81}\cdot\left(\frac{3}{2}x^4 - \frac{1}{5}x^5\right)\Big|_0^3 = \frac{1}{81}\cdot\left(\frac{243}{2}-\frac{243}{5}\right) = \mathbf{0.90}.$$

OR

$$P(X+Y>3) = 1-\int_0^3\left(\int_0^{3-x} \frac{2}{81}x^2\, y\, dy\right)dx = 1-\int_0^3 \frac{1}{81}x^2(3-x)^2\, dx$$

$$= 1-\frac{1}{81}\cdot\int_0^3\left(9x^2-6x^3+x^4\right)dx = 1-\frac{1}{81}\cdot\left(3x^3-\frac{3}{2}x^4+\frac{1}{5}x^5\right)\Big|_0^3$$

$$= 1-\frac{1}{81}\cdot\left(81-\frac{243}{2}+\frac{243}{5}\right) = \mathbf{0.90}.$$

Tutorial Sheet (Q-2)

d) Are X and Y independent?

$$f_X(x) = \int_0^3 \frac{2}{81} x^2\, y\, dy = \frac{1}{9} x^2, \quad 0 < x < 3,$$

$$f_Y(y) = \int_0^3 \frac{2}{81} x^2\, y\, dx = \frac{2}{9} y, \quad 0 < y < 3.$$

$f(x, y) = f_X(x) \cdot f_Y(y)$. $\Rightarrow$ X and Y are **independent**. $\mathrm{Cov}(X, Y) = 0.$

Tutorial Sheet (Q-3)

Let $f(x,y)=\begin{cases} k(x+y); & x>0, y>0, 3x+y<3 \\ 0 & ; \text{e.w.} \end{cases}$ be a pdf,

Find

(i) k (Ans: 1/2)

(ii) $P(X<Y)$

(iii) marginal pdfs and conditional pdfs; are X and Y independent?

(iv) $\text{Cov}(X+2, Y-3), \rho_{XY}, \rho(-2X+3, 2Y+7)$

(v) $\text{Cov}(-2X+3Y-4, 4X+7Y+5)$

(Solution is partly available)

Tutorial Sheet (Q-3)

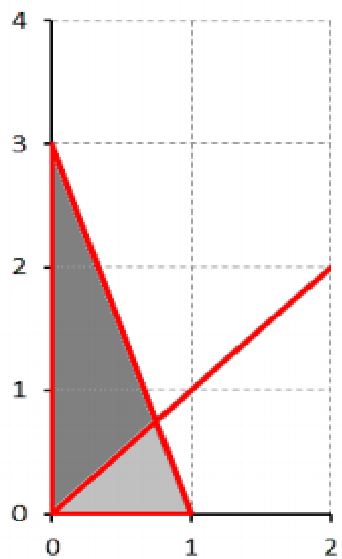
3. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{x+y}{2}, \qquad x > 0, \quad y > 0,$$

$$3x + y < 3,$$

zero otherwise.

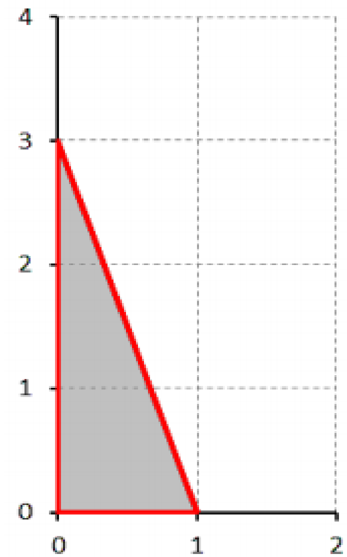
a) Find the probability P (X < Y).



intersection point:

$y = x$ and $x + 3y = 3$

$x = \frac{3}{4}$ and $y = \frac{3}{4}$



$$\mathrm{P}(\mathrm{X} < \mathrm{Y}) = \int_0^{3/4} \left(\int_x^{3-3x} \frac{x+y}{2}\, dy \right) dx$$

$$= \int_0^{3/4} \left(\frac{9}{4} - 3x \right) dx = \mathbf{\frac{27}{32}}.$$

OR

$$\mathrm{P}(\mathrm{X} < \mathrm{Y}) = 1 - \int_0^{3/4} \left(\int_y^{1-(y/3)} \frac{x+y}{2}\, dx \right) dy = 1 - \int_0^{3/4} \left(\frac{1}{4} + \frac{1}{3}y - \frac{8}{9}y^2 \right) dy = \mathbf{\frac{27}{32}}.$$

Tutorial Sheet (Q-3)

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b) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^{3-3x} \frac{x+y}{2} dy = \frac{9}{4} - 3x + \frac{3}{4}x^2, \qquad 0 < x < 1.$$

c) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_0^{1-(y/3)} \frac{x+y}{2} dx = \frac{1}{4} + \frac{1}{3}y - \frac{5}{36}y^2, \qquad 0 < y < 3.$$

d) Are X and Y independent? If not, find $\operatorname{Cov}(X, Y)$.

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independe**

OR

$f_{X,Y}(x, y) \neq f_X(x) \times f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

Tutorial Sheet (Q-3)

innovate achieve lead

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^1 x \cdot \left(\frac{9}{4} - 3x + \frac{3}{4}x^2\right)dx$$

$$= \int_0^1 \left(\frac{9}{4}x - 3x^2 + \frac{3}{4}x^3\right)dx = \frac{9}{8} - 1 + \frac{3}{16} = \frac{5}{16}.$$

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_0^3 y \cdot \left(\frac{1}{4} + \frac{1}{3}y - \frac{5}{36}y^2\right)dy$$

$$= \int_0^3 \left(\frac{1}{4}y + \frac{1}{3}y^2 - \frac{5}{36}y^3\right)dy = \frac{9}{8} + 3 - \frac{405}{144} = \frac{21}{16}.$$

$$E(XY) = \int_0^1 \left(\int_0^{3-3x} xy \cdot \frac{x+y}{2}\,dy\right)dx = \int_0^1 \left(\frac{x^2}{4}(3-3x)^2 + \frac{x}{6}(3-3x)^3\right)dx$$

$$= = \int_0^1 \left(\frac{9}{2}x - \frac{45}{4}x^2 + 9x^3 - \frac{9}{4}x^4\right)dx = \frac{9}{4} - \frac{15}{4} + \frac{9}{4} - \frac{9}{20} = \frac{6}{20} = \frac{3}{10}.$$

$$\mathrm{Cov}(X, Y) = E(XY) - E(X) \times E(Y) = \frac{3}{10} - \frac{5}{16} \times \frac{21}{16} = -\frac{\mathbf{141}}{\mathbf{1280}} \approx -0.11016.$$

Tutorial Sheet (Q-3)

Use

1. $\text{Cov}(aX+b, cY+d) = ac\,\text{Cov}(X,Y)$

2. $\rho(aX+b, cY+d) = \dfrac{ac}{|a||c|}\rho_{XY};\ a \neq 0, c \neq 0$

3. $\text{Cov}(aX+bY+h, cX+dY+k) = ac\sigma_x^2 + (ad+bc)Cov(X,Y) + bd\sigma_y^2$

Tutorial Sheet (Q-4)

Let $f(x,y)=\begin{cases} ke^{-y}; & 0<y<\infty, -y<x<y \\ 0 & ;\text{ e.w.} \end{cases}$ be a pdf,

(i) Find k (Ans: 1/2)

(ii) Find marginal pdfs and conditional pdfs.

(iii) Compute $\text{Cov}(X,Y)$ and hence decide whether X and Y independent.

(Solution is partly available)

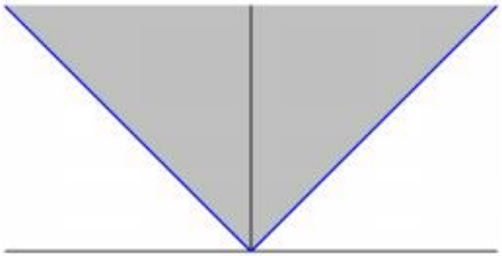
Tutorial Sheet (Q-4)

innovate achieve lead

Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{1}{2} e^{-y}, \quad 0 < y < \infty, \quad -y < x < y,$$

zero otherwise.



a) Find the marginal probability density function of X, $f_X(x)$.

If $x < 0$, $\quad f_X(x) = \int_{-x}^{\infty} \frac{1}{2} e^{-y}\, dy = \frac{1}{2} e^{x}, \quad x < 0.$

If $x > 0$, $\quad f_X(x) = \int_{x}^{\infty} \frac{1}{2} e^{-y}\, dy = \frac{1}{2} e^{-x}, \quad x > 0.$

$$f_X(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty. \quad \text{(double exponential)}$$

b) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_{-y}^{y} \frac{1}{2} e^{-y}\, dx = y\, e^{-y}, \quad 0 < y < \infty.$$

(Gamma dist.)

Tutorial Sheet (Q-4)

$\mathrm{E}(\mathrm{X}) = 0,$ since the distribution of X is symmetric about 0.

$\mathrm{E}(\mathrm{Y}) = 2,$ since Y has a Gamma distribution

$$\mathrm{E}(\mathrm{XY}) = \int_0^\infty \left(\int_{-y}^{y} \frac{1}{2} e^{-y}\, dx \right) dy = \int_0^\infty \frac{1}{2} e^{-y} \left(\int_{-y}^{y} dx \right) dy = 0.$$

$$\mathrm{Cov}(\mathrm{X}, \mathrm{Y}) = \mathrm{E}(\mathrm{XY}) - \mathrm{E}(\mathrm{X}) \times \mathrm{E}(\mathrm{Y}) = \mathbf{0}.$$

Recall: Independent $\Rightarrow$ Cov = 0

Cov = 0 $\not\Rightarrow$ Independent

Tutorial Sheet (Q-7)

(a) Assuming all 300 restaurants are equally likely, the joint pdf table is:

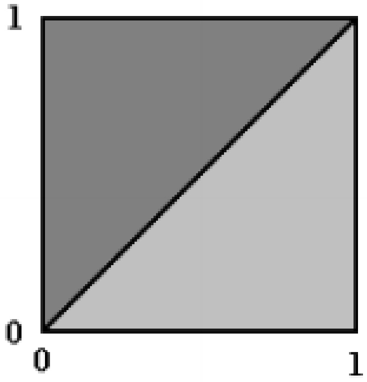
Quality (x)	Meal Price (y) 1	2	3	Total
1	0.14	0.13	0.01	**0.28**
2	0.11	0.21	0.18	**0.50**
3	0.01	0.05	0.16	**0.22**
Total	**0.26**	**0.39**	**0.35**	

(b) E(X) = 1.94, Var (X) = 0.4964
(c) E(Y) = 2.09, Var (Y) = 0.6019
(d) Cov(X,Y) = 0.2854, Cor(X,Y) = 0.5221

(e) Using above data, we get a moderately positive relationship between food quality and meal price. Therefore, it is not very likely to find a low-cost restaurant that is also high quality, although it is possible. There are 3 of them in NCR Delhi out of 300 selected restaurants, leading to f(3,1) = 0.01.

Tutorial Sheet (Q-5)

$$f(x,y) = 6x^2 y, \qquad 0 \le x \le 1,\ \ 0 \le y \le 1.$$



$$P(X < Y) = \int_0^1 \left(\int_0^y 6x^2 y \, dx \right) dy = \int_0^1 y \left(\int_0^y 6x^2 \, dx \right) dy$$

$$= \int_0^1 y \left(2x^3\right)_0^y dy = \int_0^1 2y^4 \, dy = \left(\frac{2}{5} y^5\right)_0^1 = \mathbf{\frac{2}{5}}.$$

OR

$$P(X < Y) = \int_0^1 \left(\int_x^1 6x^2 y \, dy \right) dx = \int_0^1 x^2 \left(\int_x^1 6y \, dy \right) dx = \int_0^1 x^2 \left(3y^2\right)_x^1 dx$$

$$= \int_0^1 \left(3x^2 - 3x^4\right) dx = \left(x^3 - \frac{3}{5} x^5\right)_0^1 = \mathbf{\frac{2}{5}}.$$

Tutorial Sheet (Q-5)

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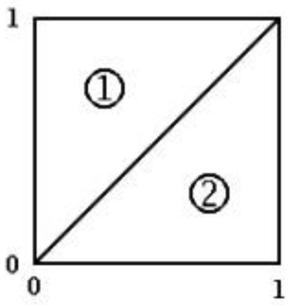
Hint 1: If Dick arrives first (that is, if X > Y), then Jane's waiting time is zero.
If Jane arrives first (that is, if X < Y), then her waiting time is Y – X.

Hint 2: $E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\cdot f(x,y)\,dx\,dy$

$f(x,y) = 6x^2 y, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$



① $y > x$ Jane is waiting for Dick.
Jane's waiting time = $y - x$

② $x > y$ Dick is waiting for Jane.
Jane's waiting time = 0

$$\int_0^1\left(\int_0^y (y-x)\cdot 6x^2 y\,dx\right)dy + \int_0^1\left(\int_0^x 0\cdot 6x^2 y\,dy\right)dx$$

$$= \int_0^1\left(\int_0^y 6x^2y^2\,dx\right)dy - \int_0^1\left(\int_0^y 6x^3 y\,dx\right)dy$$

$$= \int_0^1 2y^5\,dy - \int_0^1 1.5y^5\,dy = \frac{1}{3} - \frac{1}{4} = \mathbf{\frac{1}{12}}\text{ hour} = \mathbf{5}\text{ minutes.}$$

Tutorial Sheet (Q-6)

Let $T_1, T_2, \ldots, T_k$ be independent Exponential random variables.

Suppose $E(T_i) = \dfrac{1}{\lambda_i}$, $\quad i = 1, 2, \ldots, k.$

That is, $f_{T_i}(t) = \lambda_i e^{-\lambda_i t}, \quad t > 0, \quad i = 1, 2, \ldots, k.$

Denote $T_{\min} = \min(T_1, T_2, \ldots, T_k)$.

Show that $T_{\min}$ also has an Exponential distribution. What is the mean of $T_{\min}$?

Hint: Consider $P(T_{\min} > t) = P(T_1 > t \text{ AND } T_2 > t \text{ AND } \ldots \text{ AND } T_k > t)$.

Tutorial Sheet (Q-6)

innovate achieve lead

Since $T_1, T_2, \ldots, T_k$ are independent,

$$P(T_{min} > t) = P(T_1 > t \text{ AND } T_2 > t \text{ AND } \ldots \text{ AND } T_k > t)$$

$$= P(T_1 > t) \times P(T_2 > t) \times \ldots \times P(T_k > t)$$

$$= e^{-\lambda_1 t} \times e^{-\lambda_2 t} \times \ldots \times e^{-\lambda_k t}$$

$$= e^{-(\lambda_1 + \lambda_2 + \ldots + \lambda_k) t}, \qquad t > 0.$$

$$F_{T_{min}}(t) = P(T_{min} \le t) = 1 - P(T_{min} > t) = 1 - e^{-(\lambda_1 + \lambda_2 + \ldots + \lambda_k) t}, \qquad t > 0.$$

$$f_{T_{min}}(t) = (\lambda_1 + \lambda_2 + \ldots + \lambda_k) e^{-(\lambda_1 + \lambda_2 + \ldots + \lambda_k) t}, \qquad t > 0.$$

$\Rightarrow$ T_{min} has an Exponential distribution with mean $\dfrac{1}{\lambda_1 + \lambda_2 + \ldots + \lambda_k}$.

Tutorial Sheet (Q-7)

innovate achieve lead

Food Safety and Standards Authority of India (FSSAI) has conducted an evaluation of 300 restaurants in the National Capital Region (NCR) Delhi. Each restaurant received a rating on a 3-point scale on typical meal price (1 least expensive to 3 most expensive) and quality (1 lowest quality to 3 greatest quality). A cross-tabulation of the rating data is provided below.

	Meal Price (y)			
Quality (x)	**1**	**2**	**3**	**Total**
1	42	39	3	**84**
2	33	63	54	**150**
3	3	15	48	**66**
Total	**78**	**117**	**105**	**300**

(a) Develop a bivariate probability distribution for quality and meal price of a randomly selected restaurant in the NCR Delhi. Assume that X and Y are the respective random variables corresponding to quality rating and meal price. Assumptions, made if any, should be stated clearly.

(b) Compute the expected value and variance for quality rating, X.

(c) Compute the expected value and variance for meal price, Y.

(d) Compute the covariance and the correlation coefficient between quality and meal price.

(e) Using above results, do you suppose it is very likely to find a low cost restaurant in the NCR Delhi that is also high quality?

Tutorial Sheet (Q-5)

Dick and Jane have agreed to meet for lunch between noon (0:00 p.m.) and 1:00 p.m. Denote Jane's arrival time by X, Dick's by Y, and suppose X and Y are independent with probability density functions

$$f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that Jane arrives before Dick. That is, find $P(X < Y)$.

Find the expected amount of time Jane would have to wait for Dick to arrive.