CHAPTER

8 Rectification

1. Introduction

In this chapter we shall learn how the technique of integration can be applied to find the lengths of given plane curves. It is called **rectification.**

2. Length of the Arc of a Curve given by $y = f(x)$

Let the equation of the curve be $\quad y = f(x)$

Let A be a fixed point from which the length of the arc is measured. Let $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ be two neighbouring points on the curve. Let arc $AP = s$ and arc $AQ = s + \delta s$.

Draw PM, QN and PR perpendiculars as shown in the figure. When Q is very close to P *i.e.* when δs is very small, we can consider $\Delta\, PQR$ as a right angled triangle and write

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2 \qquad \therefore \left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

Taking the limits as $\delta x \to 0$, $\quad \dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

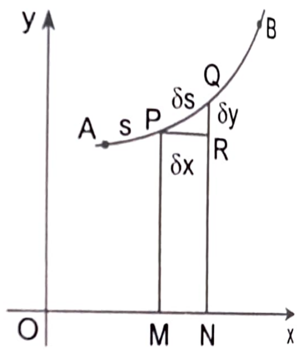

Fig. 8.1

If x coordinates of A and B are respectively x_1 and x_2 and arc $AB = s$ then by integration, we get,

$$\boxed{s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx} \qquad \text{.......... (1)}$$

Cor. 1 : If the curve is given as $x = f(y)$ then proceeding as above and dividing by $(\delta y)^2$, we get,

$$\boxed{s = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy} \qquad \text{.......... (2)}$$

Cor. 2 : If the curve is given in parametric form as $x = f_1(t)$ and $y = f_2(t)$ then dividing by $(\delta t)^2$,

$$\left(\frac{\delta s}{\delta t}\right)^2 = \left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2$$

From this, we get,

$$\boxed{s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt} \qquad \text{.......... (3)}$$

Solved Examples : Class (b) : 6 Marks

Type I : Using dy/dx

Example 1 (b) : Find the total length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

(M.U. 1988, 92, 2003, 07, 08)

Sol.: The curve is called **four cusped hypocycloid** or **astroid.** Its shape is shown in the figure.

Differentiating the given equation

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \qquad \therefore \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$

If s is the length of the arc AB from $A\,(a, 0)$ to $B\,(0, a)$ then

$$s = \int_a^0 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx = \int_a^0 \sqrt{1+\left(\frac{y^{2/3}}{x^{2/3}}\right)}\cdot dx$$

$$= \int_a^0 \sqrt{\left(\frac{x^{2/3}+y^{2/3}}{x^{2/3}}\right)}\cdot dx = \int_a^0 a^{1/3}\cdot x^{-1/3}\,dx$$

$$= a^{1/3}\cdot\frac{3}{2}\left[x^{2/3}\right]_a^0 = -\frac{3}{2}a^{1/3}\cdot a^{2/3} = -\frac{3}{2}a$$

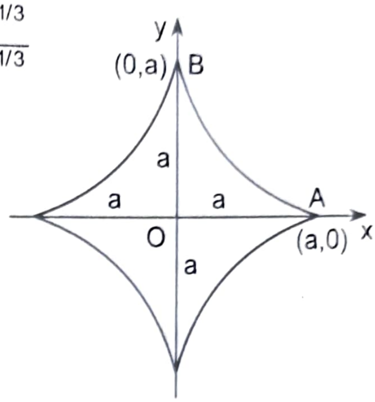

Fig. 8.2

$\therefore$ The total length of the curve $= 4s = 4\cdot\frac{3}{2}a = 6a$.

(For another method, see Ex. 2, page 8-7.)

Example 2 (b) : Show that the length of the arc of the curve $ay^2 = x^3$ from the origin to the point whose abscissa is b is $\frac{8a}{27}\left[\left(1+\frac{9b}{4a}\right)^{3/2}-1\right]$.

(M.U. 1989, 2002, 16)

Sol.: The curve is shown in the figure.

Differentiating w.r.t. x, $\quad 2ay\frac{dy}{dx} = 3x^2 \quad \therefore \frac{dy}{dx} = \frac{3x^2}{2ay}$

Now, O is (0, 0) and if the abscissa of P is b then its ordinate $= \sqrt{b^3/a}$.

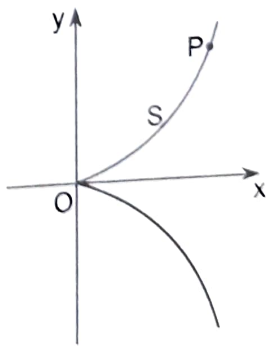

Fig. 8.3

If s is the length of the arc OP then,

$$s = \int_0^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx = \int_0^b \sqrt{1+\left(\frac{9x^4}{4a^2y^2}\right)}\cdot dx$$

$$= \int_0^b \sqrt{1+\frac{9x^4}{4a^2}\cdot\frac{a}{x^3}}\cdot dx = \int_0^b \sqrt{1+\frac{9}{4}\cdot\frac{x}{a}}\cdot dx$$

To integrate we put $1+\frac{9}{4}\cdot\frac{x}{a} = t$. When $x = 0$, $t = 1$ and when $x = b$, $t = 1+\frac{9}{4}\cdot\frac{b}{a}$ and $\frac{9}{4a}dx = dt$ i.e., $dx = \frac{4a}{9}dt$.

$$\therefore \quad s = \int_1^{1+(9b)/4a} \sqrt{t} \cdot \frac{4a}{9} dt = \frac{4a}{9}\left[\frac{t^{3/2}}{3/2}\right]_1^{1+(9b)/4a}$$

$$= \frac{4a}{9} \cdot \frac{2}{3}\left[\left(1+\frac{9b}{4a}\right)^{3/2} - 1\right] = \frac{8a}{27}\left[\left(1+\frac{9b}{4a}\right)^{3/2} - 1\right]$$

Example 3 (b) : Find the length of the arc of $y = e^x$ from (0, 1) to (1, e). **(M.U. 2003)**

Sol.: The required arc $= \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx$

$$\because \quad y = e^x \quad \therefore \quad \frac{dy}{dx} = e^x \quad \therefore \quad s = \int_0^1 \sqrt{1+e^{2x}} \cdot dx$$

To evaluate the integral, to remove the square root, we put

$$1 + e^{2x} = t^2 \quad \therefore \quad 2e^{2x}dx = 2t\,dt \quad \therefore \quad dx = \frac{t}{t^2-1} \cdot dt$$

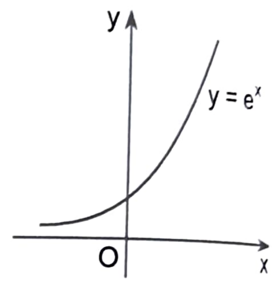

Fig. 8.4

Now, when $x = 0$, $t^2 = 2$ *i.e.*, $t = \sqrt{2}$, and

when $x = 1$, $t^2 = 1 + e^2$ *i.e.*, $t = \sqrt{1+e^2}$.

$$\therefore \quad s = \int_{\sqrt{2}}^{\sqrt{1+e^2}} t \cdot \frac{t}{t^2-1} dt = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{(t^2-1)+1}{t^2-1} dt$$

$$= \int_{\sqrt{2}}^{\sqrt{1+e^2}} dt + \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{dt}{t^2-1}$$

$$= [t]_{\sqrt{2}}^{\sqrt{1+e^2}} + \left[\frac{1}{2}\log\frac{(t-1)}{(t+1)}\right]_{\sqrt{2}}^{\sqrt{1+e^2}}$$

$$= \left[\sqrt{1+e^2} - \sqrt{2}\right] + \frac{1}{2}\left[\log\left(\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\right) - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$$

$$= \left[\sqrt{1+e^2} - \sqrt{2}\right] + \frac{1}{2}\left[\log\frac{(\sqrt{1+e^2}-1)^2}{(1+e^2)-1} - \log\frac{(\sqrt{2}-1)^2}{2-1}\right] \quad \text{[By rationalisation]}$$

$$= \left[\sqrt{1+e^2} - \sqrt{2}\right] + \log\left(\sqrt{1+e^2}-1\right) - \frac{1}{2}\log e^2 - \log(\sqrt{2}-1)$$

$$= \left[\sqrt{1+e^2} - \sqrt{2}\right] + \log\left(\sqrt{1+e^2}-1\right) - 1 - \log(\sqrt{2}-1) \quad [\because \log e^2 = 2\log e = 2]$$

Example 4 (b) : Find the length of the parabola $x^2 = 4y$ which lies inside the circle $x^2 + y^2 = 6y$. **(M.U. 1998, 2012)**

Sol.: The circle can be written as

$$x^2 + y^2 - 6y + 9 = 9 \quad i.e. \quad x^2 + (y-3)^2 = 3^2.$$

Hence, its centre is (0, 3) and radius 3. The parabola $x^2 = 4y$ is symmetrical about the y-axis.

The two curves intersect where $4y + y^2 = 6y$ $\quad [\because x^2 = 4y]$

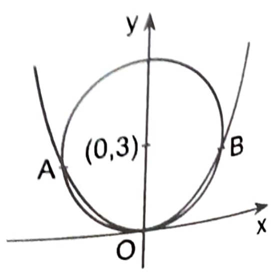

Fig. 8.5

$\therefore\ y^2 - 2y = 0 \quad \therefore\ y(y-2) = 0 \quad \therefore\ y = 0$ or 2.

When $y = 0$, $x = 0$ and when $y = 2$, $x = \pm 2\sqrt{2}$ $[\because\ x = \pm 2\sqrt{y}]$

Since, $y = \dfrac{x^2}{4}$, $\dfrac{dy}{dx} = \dfrac{x}{2}$.

$$\therefore \text{ Required length } = 2\int_0^{2\sqrt{2}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx = 2\int_0^{2\sqrt{2}} \left[1 + \frac{x^2}{4}\right] dx$$

$$= \int_0^{2\sqrt{2}} \sqrt{(x^2 + 4)}\, dx = \left[\frac{x}{2}\sqrt{x^2 + 4} + \frac{4}{2}\log\left(x + \sqrt{x^2 + 4}\right)\right]_0^{2\sqrt{2}}$$

$$= \sqrt{2} \cdot \sqrt{12} + 2\log\left(2\sqrt{2} + \sqrt{12}\right) - 2\log 2$$

$$= 2\sqrt{6} + 2\log\left(\frac{2\sqrt{2} + 2\sqrt{3}}{2}\right) = 2\left[\sqrt{6} + \log(\sqrt{2} + \sqrt{3})\right]$$

EXERCISE - I

Solve the following examples : Class (b) : 6 Marks

1. Find the lengths of the following curves as stated.

   (1) $y = \dfrac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$. **[Ans. :** 12 **]**

   (2) $y = 2x^{3/2}$ from $x = 0$ to 7. **[Ans. :** $\dfrac{1022}{27}$ **]**

   (3) $y = \dfrac{3}{4}x^{4/3} - \dfrac{3}{8}x^{2/3} + 5$ from $x = 1$ to $x = 8$. **[Ans. :** $\dfrac{99}{8}$ **]**

2. Find the length of the arc of the parabola $y^2 = 8x$ cut off by the latus rectum. **(M.U. 1995, 2205)**

   (See Fig. 8.6, page 8-5 with $a = 2$) **[Ans. :** $2\left[\sqrt{2} + \log(1 + \sqrt{2})\right]$

3. Find the length of the parabola $x^2 = 4by$ cut-off by its latus rectum. **(M.U. 23004)**

   (See Fig. 15.10(a), page 15-4) **[Ans. :** $2b[\sqrt{2} + \log(1 + \sqrt{2})]$ **]**

4. Prove that the length of the arc of the curve $y = \log\left(\dfrac{e^x - 1}{e^x + 1}\right)$ from $x = 1$ and $x = 2$ is $\log\left(e + \dfrac{1}{e}\right)$. **(M.U. 2001, 17)**

5. Find the length of the arc of the curve $y = \log\left(\tanh\dfrac{x}{2}\right)$ from $x = 1$ to $x = 2$. **(M.U. 2011)**

   [**Note :** Example 4 and 5 are similar.] **[Ans. :** $\log\left(e + \dfrac{1}{e}\right)$

6. Find the length of the arc of the curve $y = \log\sec x$ from $x = 0$ to $x = \pi/3$.

   [Ans. : $\log(2 + \sqrt{3})$ **]**

Type II : Using dx/dy

Example 1 (b) : Show that the length of the parabola $y^2 = 4ax$ from the vertex to the end of the latus rectum is $a\left[\sqrt{2}+\log(1+\sqrt{2})\right]$. **(M.U. 1996, 98, 99, 2006)**

Sol.: Let arc $OP = s$.

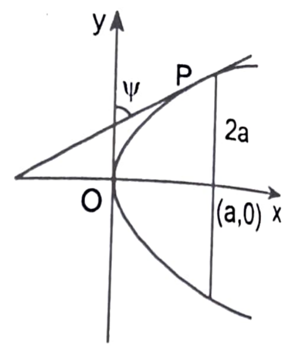

Fig. 8.6

$$\because \quad x = \frac{y^2}{4a} \qquad \therefore \quad \frac{dx}{dy} = \frac{y}{2a}$$

$$\therefore \quad s = \int_0^{2a} \sqrt{1+\left(\frac{dx}{dy}\right)^2} \cdot dy$$

$$= \int_0^{2a} \sqrt{1+\frac{y^2}{4a^2}} \cdot dy \qquad \text{[By (2), page 8-1]}$$

$$= \frac{1}{2a}\int_0^{2a} \sqrt{y^2+4a^2} \cdot dy$$

$$= \frac{1}{2a}\left[\frac{y}{2} \cdot \sqrt{y^2+4a^2} + \frac{4a^2}{2}\log\left(y+\sqrt{y^2+4a^2}\right)\right]_0^{2a}$$

$$= \frac{1}{2a}\left[a \cdot 2\sqrt{2} \cdot a + 2a^2\left\{\log(2a+2\sqrt{2}\cdot a) - \log 2a\right\}\right]$$

$$\therefore \quad s = a\left[\sqrt{2}+\log\left(\frac{2a+2\sqrt{2}\cdot a}{2a}\right)\right] = a\left[\sqrt{2}+\log(1+\sqrt{2})\right]$$

Example 2 (b) : Show that the length of the arc of the parabola $y^2 = 4ax$ cut-off by the line $3y = 8x$ is $a\left(\log 2 + \frac{15}{16}\right)$. **(M.U. 2013, 16)**

Sol. : The parabola and the line intersect at A where

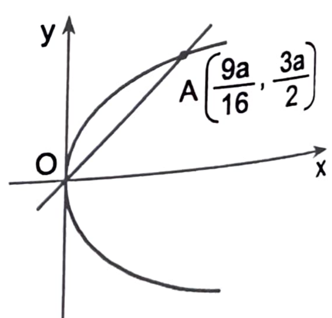

Fig. 8.6 (a)

$$\frac{64}{9}x^2 = 4ax \qquad \therefore \quad x = \frac{9a}{16}$$

$$\therefore \quad y^2 = 4a \cdot x = 4a \cdot \frac{9a}{16} = \frac{9}{4}a^2 \qquad \therefore \quad y = \frac{3a}{2}$$

i.e., $A\left(\frac{9a}{16}, \frac{3a}{2}\right)$.

Now, $x = \frac{y^2}{4a} \qquad \therefore \quad \frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}$.

$$\therefore \quad s = \int_0^{3a/2} \sqrt{1+\left(\frac{dx}{dy}\right)^2} \cdot dy = \int_0^{3a/2} \sqrt{1+\frac{y^2}{4a^2}} \cdot dy$$

$$= \frac{1}{2a}\int_0^{3a/2} \sqrt{y^2+4a^2} \cdot dy$$

$$= \frac{1}{2a}\left[\frac{y}{2} \cdot \sqrt{y^2+4a^2} + \frac{4a^2}{2}\log\left(y+\sqrt{y^2+4a^2}\right)\right]_0^{3a/2}$$

$$\therefore \quad s=\frac{1}{2}\left[\frac{3a}{4}\sqrt{\frac{9a^2}{4}+4a^2}+\frac{4a^2}{2}\log\left(\frac{3a}{2}+\sqrt{\frac{9a^2}{4}+4a^2}\right)-\frac{4a^2}{2}\log 2a\right]$$

$$=\frac{1}{2a}\left[\frac{3a}{8}\sqrt{25\,a^2}+2a^2\log\left(\frac{3a+\sqrt{25\,a^2}}{2}\right)-2a^2\log 2a\right]$$

$$=\frac{1}{2a}\left[\frac{3a}{8}\cdot 5a+2a^2\log\left(\frac{3a+5a}{4a}\right)\right]$$

$$=\frac{1}{2a}\left[\frac{15a^2}{8}+2a^2\log 2\right]=a\left(\frac{15}{16}+\log 2\right).$$

Example 3 (b) : Find the length of the curve $y=(x/2)^{2/3}$ from $x=0$ to $x=2$.

Sol. : We have, $y=(x/2)^{2/3}$

$$\therefore \quad \frac{dy}{dx}=\frac{2}{3}\cdot\left(\frac{x}{2}\right)^{-1/3}\cdot\frac{1}{2}=\frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

$$\therefore \quad s=\int_0^2\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx=\int_0^2\sqrt{1+\frac{4}{9}\cdot x^{-2/3}}\,dx$$

Since dy/dx is not defined at the lower limit $x=0$, we cannot use the above formula. To overcome the difficulty we consider the same curve in the form $x=\Phi(y)$ and use the other formula (2), page 8-1.

$$\because \quad \frac{x}{2}=y^{3/2} \qquad \therefore \quad x=2(y)^{3/2} \qquad \therefore \quad \frac{dx}{dy}=2\cdot\frac{3}{2}(y)^{1/2}=3\sqrt{y}$$

This derivative is defined at $x=0$ *i.e.* $y=0$.

Now, when $x=0$, $y=0$ and when $x=2$, $y=1$.

We now use the other formula (2), page 8-1.

$$\therefore \quad s=\int_0^1\sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy=\int_0^1\sqrt{1+9y}\cdot dy$$

$$=\frac{2}{3}\cdot\frac{1}{9}\left[(1+9y)^{3/2}\right]_0^1=\frac{2}{27}\left[10\sqrt{10}-1\right]$$

Example 4 (b) : Find the length of the curve $x=\frac{y^4}{4}+\frac{1}{8y^2}$ from $y=1$ to $y=2$.

Sol. : We have $\frac{dx}{dy}=y^3-\frac{1}{4y^3}$

$$1+\left(\frac{dx}{dy}\right)^2=1+\left(y^3-\frac{1}{4y^3}\right)^2=1+y^6-\frac{1}{2}+\frac{1}{16\,y^6}$$

$$=y^6+\frac{1}{2}+\frac{1}{16y^6}=\left(y^3+\frac{1}{4y^3}\right)^2$$

$$\therefore \quad s=\int_1^2\sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy=\int_1^2\left(y^3+\frac{1}{4y^3}\right)dy=\left[\frac{y^4}{4}-\frac{1}{8y^2}\right]_1^2=\frac{123}{32}.$$

Also show that the line $\theta = \pi/3$ divides the length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ in the first quadrant in the ratio 1 : 3.

(M.U. 2007, 08)

Sol.: The curve known as **four cusped hypocycloid** or **astroid** is shown in the figure.

(i) To rectify the curve it is convenient to use the parametric equations of the curve. They are

$$x = a\cos^3\theta, \quad y = b\sin^3\theta.$$

If s is the total length then

$$s = 4\int_0^{\pi/2}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\cdot d\theta$$

$$= 4\int_0^{\pi/2}\sqrt{(3a\cos^2\theta\sin\theta)^2 + (3b\sin^2\theta\cos\theta)^2}\cdot d\theta$$

$$= 4\int_0^{\pi/2} 3\cdot\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\cdot\sin\theta\cos\theta\,d\theta \quad \text{.......... (A)}$$

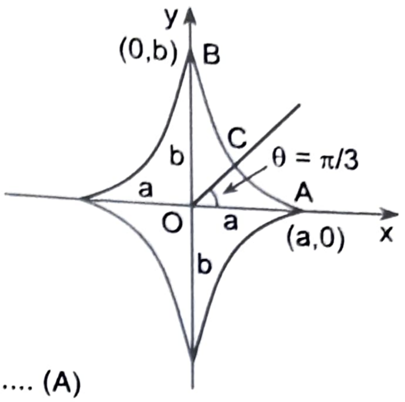

Fig. 8.8

To find the integral, we put $a^2\cos^2\theta + b^2\sin^2\theta = t^2$

$$\therefore \; 2(b^2 - a^2)\sin\theta\cos\theta\,d\theta = 2t\,dt$$

$$\therefore \; \sin\theta\cos\theta\,d\theta = \frac{t}{(b^2 - a^2)}dt$$

When $\theta = 0$, $t = a$ and when $\theta = \pi/2$, $t = b$

$$\therefore \; s = 12\int_a^b t\cdot\frac{t}{(b^2 - a^2)}dt = \frac{12}{(b^2 - a^2)}\int_a^b t^2\,dt = \frac{12}{b^2 - a^2}\left[\frac{t^3}{3}\right]_a^b$$

$$= \frac{4}{b^2 - a^2}\left[b^3 - a^3\right] = 4\frac{(a^2 + ab + b^2)}{(a + b)}$$

(ii) For deduction, we put $b = a$.

Hence, the total length of the second astroid

$$s = 4\cdot\frac{3a^2}{2a} = 6a.$$

(iii) For the third part, the length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ in the first quadrant is

$$AB = s = \frac{1}{4}\cdot 6a = \frac{3}{2}a$$

Now, the length cut off by $\theta = \pi/3$,

$$\text{arc } AC = \int_0^{\pi/3} 3a\sin\theta\cos\theta\,d\theta \qquad [\text{ Putting } b = a \text{ in (A) }]$$

$$\therefore \; \text{arc } AC = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/3} = \frac{3a}{2}\cdot\frac{3}{4} = \frac{9a}{8}$$

But arc $AB = 3a/2$,

$$\therefore \; \text{arc } BC = \frac{3a}{2} - \frac{9a}{8} = \frac{3a}{8}. \qquad \therefore \; \frac{\text{arc } BC}{\text{arc } AC} = \frac{1}{3}.$$

Example 4 (b) : For the curve

$$x = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\cdot\theta\right),\quad y = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\cdot\theta\right),$$

show that $s = \frac{4b}{a}(a+b)\cos\left(\frac{a\theta}{2b}\right)$ where s is measured from $\theta = \pi b/a$ to θ. **(M.U. 2005)**

Sol.: We have $\frac{dx}{d\theta} = -(a+b)\sin\theta + (a+b)\sin\left(\frac{a+b}{b}\cdot\theta\right)$

$$\frac{dy}{d\theta} = (a+b)\cos\theta - (a+b)\cos\left(\frac{a+b}{b}\cdot\theta\right)$$

$$\therefore \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (a+b)^2\left[2 - 2\sin\theta\,\sin\left(\frac{a+b}{b}\cdot\theta\right) - 2\cos\theta\,\cos\left(\frac{a+b}{b}\cdot\theta\right)\right]$$

$$= (a+b)^2\left[2 - 2\cos\left[\theta - \left(\frac{a+b}{b}\right)\cdot\theta\right]\right] = 2(a+b)^2\left[1 - \cos\left(-\frac{a\theta}{b}\right)\right]$$

$$= 2(a+b)^2\left[1 - \cos\frac{a\theta}{b}\right] = 4(a+b)^2\sin^2\left(\frac{a\theta}{2b}\right).$$

$$\therefore\ s = \int_{\pi b/a}^{\theta}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\cdot d\theta = \int_{\pi b/a}^{\theta} 2(a+b)\sin\left(\frac{a\theta}{2b}\right)\cdot d\theta$$

$$= 2(a+b)\left[-\frac{2b}{a}\cos\left(\frac{a\theta}{2b}\right)\right]_{\pi b/a}^{\theta} = -\frac{4b}{a}(a+b)\left[\cos\left(\frac{a\theta}{2b}\right) - \cos\frac{\pi}{2}\right]$$

$$\therefore\ s = \frac{4b}{a}(a+b)\cos\left(\frac{a\theta}{2b}\right) \qquad \text{[Numerically]}$$

Example 5 (b) : For the curve $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$, find the length of the arc of the curve measured from $t = 0$ to any point.

Sol.: We have $\frac{dx}{dt} = a(-2\sin t + 2\sin 2t)$, $\frac{dy}{dt} = a(2\cos t - 2\cos 2t)$

$$\therefore \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$= 4a^2\left[(\sin 2t - \sin t)^2 + (\cos t - \cos 2t)^2\right]$$

$$= 4a^2\left[2 - 2(\sin 2t\sin t + \cos t\cos 2t)\right]$$

$$= 8a^2\left[1 - \cos(2t - t)\right] = 8a^2\left[1 - \cos t\right]$$

$$= 16a^2\sin^2\frac{t}{2}.$$

$$\therefore \frac{ds}{dt} = 4a\sin\frac{t}{2}.$$

$$\therefore\ s = \int_0^t 4a\sin\frac{t}{2} = 8a\left[-\cos\frac{t}{2}\right]_0^t = 8a\left[1 - \cos\frac{t}{2}\right] = 16a\sin^2\frac{t}{4}.$$

Example 6 (b) : Find the length of one arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$.

(M.U. 1993, 98, 2002, 09, 14)

Sol.: The curve is shown on the right. For A, $\theta = 0$ and for B, $\theta = 2\pi$.

Now, $\dfrac{dx}{d\theta} = a(1 - \cos\theta)$ and $\dfrac{dy}{d\theta} = -a\sin\theta$

$$\therefore \quad s = \int_0^{2\pi} \sqrt{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]} \cdot d\theta$$

$$= \int_0^{2\pi} \sqrt{\left[a^2(1 - 2\cos\theta + \cos^2\theta) + a^2\sin^2\theta\right]} \cdot d\theta$$

$$= \int_0^{2\pi} a\sqrt{[2 - 2\cos\theta]} \cdot d\theta$$

$$= a\int_0^{2\pi} \sqrt{2 \cdot 2\sin^2(\theta/2)} \cdot d\theta$$

$$= 2a\int_0^{2\pi} \sin\frac{\theta}{2} \cdot d\theta = 2a\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= -4a[\cos\pi - \cos 0] = 8a.$$

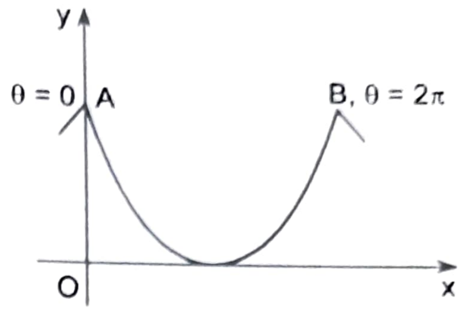

Fig. 8.9

Example 7 (b) : Find the length of the cycloid from one cusp to the next cusp $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

(M.U. 1990, 93, 97, 2003)

Sol.: The curve is shown on the next page. Let the arc be measured from the origin O. For A, $\theta = -\pi$ and for B, $\theta = \pi$, for O, $\theta = 0$.

Hence, the length of the arc $AB = 2$ arc $OB = 2\displaystyle\int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \cdot d\theta$.

But, $\dfrac{dx}{d\theta} = a(1 + \cos\theta)$, $\dfrac{dy}{d\theta} = a\sin\theta$.

$$\therefore \quad s = 2\int_0^{\pi} \sqrt{a^2(1 + \cos\theta)^2 + a^2\sin^2\theta} \cdot d\theta$$

$$= 2a\int_0^{\pi} \sqrt{2(1 + \cos\theta)} \cdot d\theta = 4a\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) \cdot d\theta = 4a\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8a.$$

Note ....

A cycloid can be given in four different forms :

(i) $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ (ii) $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$

(iii) $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ (iv) $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$

We have discussed these cycloids on pages 15-17 and 15-18.

As seen in the above two examples, whatever may be the form of the cycloid the length of one arc of the cycloid is always $8a$.

Example 8 (b) : Find the length of the above cycloid from one cusp to another cusp. If s is the length of the arc from the origin to a point $P(x, y)$ show that $s^2 = 8ay$. **(M.U. 1995, 2003)**

Sol. : We have proved the first part already.

For the second part, we have

$$s = \int_0^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \cdot d\theta$$

$$= 2a \cdot \left[2\sin\left(\frac{\theta}{2}\right)\right]_0^{\theta} \qquad \text{[As above]}$$

$$= 4a\sin\left(\frac{\theta}{2}\right)$$

$$\therefore \quad s^2 = 16\,a^2\sin^2\left(\frac{\theta}{2}\right) = 8a\left[a(1-\cos\theta)\right] = 8ay.$$

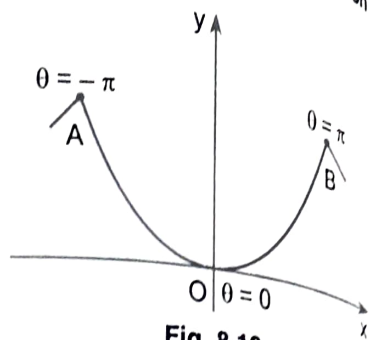

Fig. 8.10

Example 9 (b) : Prove that the length of the arc of the curve

$$x = a\sin 2\theta\,(1+\cos 2\theta), \quad y = a\cos 2\theta\,(1-\cos 2\theta)$$

measured from the origin to (x, y) is $\frac{4}{3}a\sin 3\theta$. **(M.U. 2005)**

Sol. : We have $\frac{dy}{d\theta} = -2a\sin 2\theta(1-\cos 2\theta) + 2a\cos 2\theta\sin 2\theta$

$$= -2a\sin 2\theta + 2a\sin 4\theta$$

$$\frac{dx}{d\theta} = 2a\cos 2\theta(1+\cos 2\theta) - 2a\sin^2 2\theta$$

$$= 2a\cos 2\theta + 2a\cos 4\theta$$

$$\therefore \quad s = \int_0^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \cdot d\theta$$

$$= \int_0^{\theta} \sqrt{\left[4a^2\cos^2 2\theta + 4a^2\cos^2 4\theta + 8a^2\cos 2\theta\cos 4\theta + 4a^2\sin^2 2\theta + 4a^2\sin^2 4\theta - 8a^2\sin 2\theta\sin 4\theta\right]} \cdot d\theta$$

$$= \int_0^{\theta} \sqrt{\left[4a^2 + 4a^2 + 8a^2\cos 6\theta\right]} \cdot d\theta$$

$$= \int_0^{\theta} 2\sqrt{2}\,a\sqrt{1+\cos 6\theta} \cdot d\theta = \int_0^{\theta} 2\sqrt{2}\,a \cdot \sqrt{2}\cos 3\theta\; d\theta$$

$$= 4a\int_0^{\theta}\cos 3\theta\; d\theta = 4a\left[\frac{\sin 3\theta}{3}\right]_0^{\theta} = \frac{4a}{3}\sin\theta.$$

Example 10 (b) : Show that the length of the tractrix $x = a\,[\,\cos t + \log\tan(t/2)\,]$, $y = a\sin t$ from $t = \pi/2$ to any point t is $a\log\sin t$.

Sol. : We have $\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan(t/2)}\sec^2\left(\frac{t}{2}\right) \cdot \frac{1}{2}\right)$

$$= a\left(-\sin t + \frac{1}{2\sin(t/2)\cos(t/2)}\right)$$

$$\therefore \quad \frac{dx}{dt} = a\left(-\sin t + \frac{1}{\sin t}\right) = a\frac{(1-\sin^2 t)}{\sin t} = a\frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dx} = a\cos t$$

Now, $$s = \int_{\pi/2}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt = \int_{\pi/2}^{t} \sqrt{a^2\frac{\cos^4 t}{\sin^2 t} + a^2\cos^2 t} \cdot dt$$

$$= a\int_{\pi/2}^{t} \frac{\cos t}{\sin t}\, dt = a[\log \sin t]_{\pi/2}^{t} = a\log\sin t.$$

Example 11 (b) : Prove that the length of the curve

$$x = e^{\theta}\left[\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right], \quad y = e^{\theta}\left[\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right]$$

measured from $\theta = 0$ to $\theta = \pi$ is $\frac{5}{2}\left[e^{\pi} - 1\right]$. **(M.U. 1999)**

Sol. : We have $x = e^{\theta}\left[\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right]$

$$\frac{dx}{d\theta} = e^{\theta}\left[\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right] + e^{\theta}\left[\frac{1}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right] = \frac{5}{2}e^{\theta}\cos\frac{\theta}{2}$$

$$\frac{dy}{d\theta} = e^{\theta}\left[\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right] + e^{\theta}\left[-\frac{1}{2}\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right] = -\frac{5}{2}e^{\theta}\sin\frac{\theta}{2}.$$

$$\therefore \quad \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \frac{25}{4}e^{2\theta}\cos^2\frac{\theta}{2} + \frac{25}{4}e^{2\theta}\sin^2\frac{\theta}{2} = \frac{25}{4}e^{2\theta}$$

$$\therefore \quad s = \int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \cdot d\theta = \int_0^{\pi} \frac{5}{2}e^{\theta}d\theta = \frac{5}{2}\left[e^{\theta}\right]_0^{\pi} = \frac{5}{2}\left[e^{\pi} - 1\right].$$

EXERCISE - III

Find the lengths of the following curves : Class (b) : 6 Marks

1. $x = a(2\cos\theta + \cos 2\theta)$, $y = a(2\sin\theta + \sin 2\theta)$ from $\theta = 0$ to any point θ.
   [**Ans. :** $8a\sin(\theta/2)$]
2. $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ from $\theta = 0$ to $\theta = 2\pi$. **(M.U. 1997)**
   (See Fig. 15.53, page 15-18) [**Ans. :** $8a$]
3. $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ from $\theta = 0$ to $\theta = 2\pi$. **(M.U. 1992)**
   [**Ans. :** $2\pi^2 a$]
4. $x = a\,e^{\theta}\sin\theta$, $y = a\,e^{\theta}\cos\theta$ from $\theta = 0$ to $\theta = \pi/2$. [**Ans. :** $\sqrt{2}\,(e^{\pi/2} - 1)\,a$]
5. $x = a(3\cos\theta - \cos 3\theta)$, $y = a(3\sin\theta - \sin 3\theta)$ from $\theta = \pi/2$ to any point θ.
   [**Ans. :** $6a\cos\theta$ numerically]
6. $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ between two consecutive cusps.
   (See Fig. 15.50, page 15-17) [**Ans. :** $8a$]

7. $x = e^{\theta}\,[\sin(\theta/2) + 2\cos(\theta/2)]$, $y = e^{\theta}\,[\cos(\theta/2) - 2\sin(\theta/2)]$ from $\theta = 0$ to $\theta = \pi$. [**Ans. :** $(5/2)\,[e^{\pi} - 1]$]

8. $x = \log(\sec\theta + \tan\theta) - \sin\theta$, $y = \cos\theta$ from $\theta = 0$ to any point θ. [**Ans. :** $\log\sec\theta$]

9. $x = a\,(t - \tan h\,t)$, $y = a\sec h\,t$ from $t = 0$ to any point t. [**Ans. :** $\log\cos h\,t$]

10. $x = 1 - \cos t + (3/5)\,t$, $y = (4/5)\sin t$ from $t = 0$ to $t = \pi$. [**Ans. :** $\pi + (6/5)$]

11. $x = a\cos t + a\,t\sin t$, $y = a\sin t - at\cos t$ from $t = 0$ to $t = \pi/2$. [**Ans. :** $\pi^2 a/8$]

Type IV : To find the length of a loop

Example 1 (b) : Show that if s is the arc of the curve $9y^2 = x(3-x)^2$ measured from the origin to the point $P(x, y)$ then $3s^2 = 3y^2 + 4x^2$. **(M.U. 1997)**

Sol.: Differentiating the given equation w.r.t. x,

$$18y\frac{dy}{dx} = (3-x)^2 + x \cdot 2(3-x)(-1)$$

$$= (3-x)(3-x-2x) = (3-x)(3-3x)$$

$$= 3(3-x)(1-x)$$

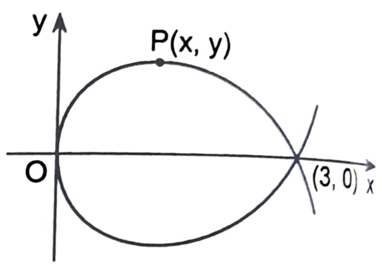

Fig. 8.11

$$\therefore \quad \frac{dy}{dx} = \frac{1}{6}\cdot\frac{1}{y}(3-x)(1-x)$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{1}{36}\cdot\frac{1}{y^2}(3-x)^2(1-x)^2$$

$$= \frac{1}{36}\cdot\frac{9}{x(3-x)^2}\cdot(3-x)^2(1-x)^2$$

$$= \frac{1}{4}\cdot\frac{(1-x)^2}{x}$$

$$\therefore \quad s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\cdot dx = \int_0^x \sqrt{1 + \frac{(1-x)^2}{4x}}\cdot dx$$

$$= \frac{1}{2}\int_0^x \sqrt{\frac{4x + 1 - 2x + x^2}{x}}\cdot dx = \frac{1}{2}\int_0^x \sqrt{\frac{(1+x)^2}{x}}\,dx = \frac{1}{2}\int_0^x \frac{1+x}{\sqrt{x}}\,dx$$

$$= \frac{1}{2}\int_0^x \left(x^{-1/2} + x^{1/2}\right)dx = \frac{1}{2}\left[2x^{1/2} + \frac{2}{3}x^{2/3}\right]_0^x$$

$$= x^{1/2} + \frac{1}{3}x^{3/2} = x^{1/2}\left(1 + \frac{x}{3}\right) = \frac{x^{1/2}}{3}(3+x)$$

$$\therefore \quad s^2 = \frac{x}{9}(3+x)^2$$

Now, $3y^2 + 4x^2 = \dfrac{x}{3}(3-x)^2 + 4x^2 = \dfrac{1}{3}\left[x(3-x)^2 + 12x^2\right]$

$$\therefore \quad 3y^2 + 4x^2 = \frac{1}{3}x\left[(3-x)^2 + 12x\right] = \frac{1}{3}x\left[9 - 6x + x^2 + 12x\right]$$

$$= \frac{1}{3}x\left[9 + 6x + x^2\right] = \frac{x}{3}(3+x)^2 = 3s^2$$

Example 2 (b) : Find the perimeter of the loop of the curve $9ay^2 = (x-2a)(x-5a)^2$.

Sol.: Differentiating the given equation w.r.t. x,

$$18ay\frac{dy}{dx} = (x-2a)\cdot 2(x-5a) + (x-5a)^2$$
$$= (x-5a)(2x-4a+x-5a)$$
$$= 3(x-5a)(x-3a)$$
$$= (x-5a)(3x-9a)$$

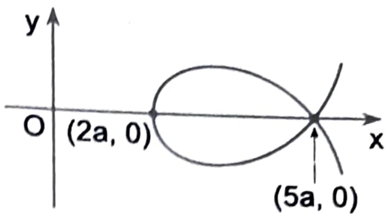

Fig. 8.12

$$\therefore \quad \frac{dy}{dx} = \frac{(x-5a)(x-3a)}{6ay}$$

$$\therefore \quad 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(x-5a)^2(x-3a)^2}{36a^2y^2} = 1+\frac{(x-5a)^2(x-3a)^2}{4a(x-2a)(x-5a)^2}$$
$$= 1+\frac{(x-3a)^2}{4a(x-2a)} = \frac{(x-a)^2}{4a(x-2a)}$$

$\therefore$ The perimeter of the loop of the curve

$$= 2\int_{2a}^{5a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx = 2\int_{2a}^{5a}\frac{x-a}{2\sqrt{a}\cdot\sqrt{x-2a}}dx$$
$$= \frac{1}{\sqrt{a}}\int_{2a}^{5a}\frac{(x-2a)+a}{\sqrt{x-2a}}dx = \frac{1}{\sqrt{a}}\int_{2a}^{5a}\left[\sqrt{x-2a}+a(x-2a)^{-1/2}\right]dx$$
$$= \frac{1}{\sqrt{a}}\left[\frac{2}{3}(x-2a)^{3/2}+a\cdot 2\cdot(x-2a)^{1/2}\right]_{2a}^{5a}$$
$$= \frac{1}{\sqrt{a}}\left[\frac{2}{3}(3a)^{3/2}+2a(3a)^{1/2}\right]$$
$$= \frac{1}{\sqrt{a}}\left[\frac{2}{3}\cdot 3\sqrt{3}\cdot a\sqrt{a}+2a\sqrt{3}\cdot\sqrt{a}\right]$$
$$= 2\sqrt{3}\,a+2a\cdot\sqrt{3} = 4\sqrt{3}\cdot a.$$

Example 3 (b) : Prove that the length of the arc of the curve $y^2 = x\left(1-\frac{1}{3}x\right)^2$ from the origin to the point $P(x, y)$ is given by $s^2 = y^2+\frac{4}{3}x^2$. Hence, rectify the loop. **(M.U. 1995, 97, 2002, 04)**

Sol.: The length of the arc OP is given by

$$s = \int_0^x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx$$

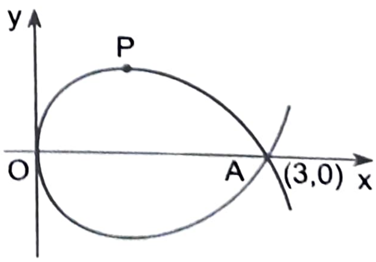

Fig. 8.13

Now, $\quad y = x^{1/2}\left(1-\frac{1}{3}x\right) = x^{1/2}-\frac{1}{3}x^{3/2}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}x^{-1/2}-\frac{1}{3}\cdot\frac{3}{2}x^{1/2} = \frac{1}{2}\frac{(1-x)}{\sqrt{x}}$$

$$\therefore \quad 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(1-x)^2}{4x} = \frac{(1+x)^2}{4x}$$

$$\therefore \quad s = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx = \int_0^x \frac{1+x}{2\sqrt{x}}\, dx = \frac{1}{2}\int_0^x \left(x^{-1/2} + x^{1/2}\right) dx$$

$$= \frac{1}{2}\left[\frac{x^{1/2}}{1/2} + \frac{x^{3/2}}{3/2}\right]_0^x = x^{1/2}\left(1+\frac{x}{3}\right)$$

$$\therefore \quad s^2 = x\left(1+\frac{x}{3}\right)^2 = x\left(1-\frac{x}{3}\right)^2 + \frac{4}{3}x^2 = y^2 + \frac{4}{3}x^2.$$

The length of half the loop i.e. the arc OA is obtained by putting $x = 3$ and $y = 0$ in the above result.

$$\therefore \quad s^2 = \frac{4}{3} \cdot 9 = 12 \qquad \therefore \quad s = 2\sqrt{3}$$

$\therefore$ The length of the complete loop $= 4\sqrt{3}$.

Example 4 (b) : Find the length of the loop of the curve $x = t^2,\ y = t\left(1-\dfrac{t^2}{3}\right)$.

(M.U. 2002, 06)

Sol.: Eliminating t, we get $y^2 = t^2\left(1-\dfrac{t^2}{3}\right)^2 = x\left(1-\dfrac{x}{3}\right)^2$

We get the same curve as above.

Example 5 (b) : Find the length of the loop of the curve $3ay^2 = x(x-a)^2$.

(M.U. 1987, 91, 2003, 07, 11)

Sol.: The length of the loop is given by $s = 2\displaystyle\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx$

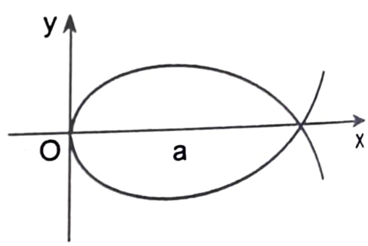

Fig. 8.14

Now differentiating the given function

$$6ay\frac{dy}{dx} = (x-a)^2 + x \cdot 2(x-a) = (x-a)(3x-a)$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{(x-a)^2(3x-a)^2}{36a^2y^2}$$

$$= \frac{(x-a)^2(3x-a)^2}{36a^2} \cdot \frac{3a}{x(x-a)^2} = \frac{(3x-a)^2}{12ax}$$

$$\therefore \quad 1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{(3x-a)^2}{12ax} = \frac{(3x+a)^2}{12ax}$$

$$\therefore \quad s = 2\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx = 2\int_0^a \frac{3x+a}{2\sqrt{3a}\cdot\sqrt{x}} \cdot dx = \frac{1}{\sqrt{3a}}\int_0^a \left(3\sqrt{x} + \frac{a}{\sqrt{x}}\right) dx$$

$$= \frac{1}{\sqrt{3a}}\left[3 \cdot \frac{x^{3/2}}{3/2} + a \cdot \frac{x^{1/2}}{1/2}\right]_0^a = \frac{1}{\sqrt{3a}}\left[2a^{3/2} + 2a^{3/2}\right] = \frac{4}{\sqrt{3}}a.$$

Example 6 (b) : Find the total length of the loop of the curve $9y^2 = (x+7)(x+4)^2$.

(M.U. 1997, 99, 2003, 14)

Sol.: If $y = 0$, $x = -7$ or $x = -4$, the loop intersects the x-axis at $x = -7$ and at $x = -4$.

If s is the total length of the loop

$$s = 2\int_{-7}^{-4} \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx$$

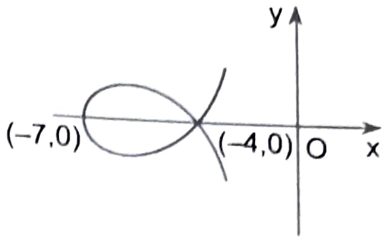

Fig. 8.15

Now, $$18y\frac{dy}{dx} = (x+7)\cdot 2(x+4) + (x+4)^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x+4)[2x+14+x+4]}{18y} = \frac{(x+4)(x+6)}{6y}$$

$$\therefore \quad 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(x+4)^2(x+6)^2}{36y^2} = 1+\frac{(x+4)^2(x+6)^2}{4(x+7)(x+4)^2}$$

$$= \frac{4x+28+x^2+12x+36}{4(x+7)} = \frac{(x+8)^2}{4(x+7)}$$

$$\therefore \quad s = 2\int_{-7}^{-4} \frac{x+8}{2\sqrt{x+7}}\,dx = 2\int_0^{\sqrt{3}} \frac{t^2+1}{2t}\cdot 2t\,dt \qquad [\text{Put } x+7 = t^2]$$

$$= 2\int_0^{\sqrt{3}} \left(t^2+1\right)\,dt = 2\left[\frac{t^3}{3}+t\right]_0^{\sqrt{3}} = 2\left(\sqrt{3}+\sqrt{3}\right) = 4\sqrt{3}.$$

EXERCISE - IV

Show that the length of the loop : Class (b) : 6 Marks

1. $3ay^2 = x^2(a-x)$ is $4a/\sqrt{3}$. (See Fig. 15.22, page 15-8)
2. $9ay^2 = x(x-3a)^2$ is $4\sqrt{3}\cdot a$. **(M.U. 1992)** (See Fig. 15.24, page 15-9)

3. Length of the Arc of a Curve given by $r = f(\theta)$

Let $$r = f(\theta) \qquad \text{.................... (1)}$$

be the polar equation of the given curve. Let A be the fixed point from which the length of the arc is measured. Let $P(r, \theta)$, $Q(r+\delta r, \theta+\delta\theta)$ be two neighbouring points on the curve. Let arc $AP = s$ and arc $AQ = s + \delta s$.

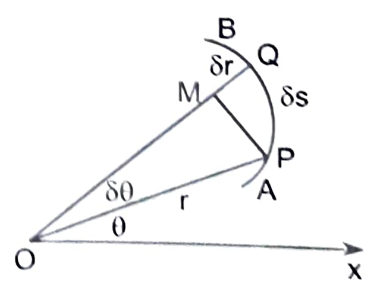

Fig. 8.16

Draw PM perpendicular to OQ. When $\delta\theta$ is small we can write $PM = r\,\delta\theta$, $MQ = \delta r$, and chord PQ = arc $PQ = \delta s$.

Hence, from the right angled $\Delta\, PQR$, we have

$$(\text{arc } PQ)^2 = (PM)^2 + (MQ)^2$$

$$\therefore \quad (\delta s)^2 = (r\,\delta\theta)^2 + (\delta r)^2 \qquad \text{.................... (2)}$$

$$\therefore \quad \left(\frac{\delta s}{\delta\theta}\right)^2 = r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2$$

Taking the limit as $\delta\theta \to 0$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

If for A and B, $\theta = \theta_1$ and $\theta = \theta_2$ respectively and arc $AB = s$ then

$$\boxed{s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta} \quad \text{.................... (3)}$$

Cor. If the curve is given by $\theta = f(r)$, then dividing the equation (2) by $(\delta r)^2$ and proceeding as above, we get

$$\boxed{s = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} \cdot dr} \quad \text{.................... (4)}$$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Find the perimeter of the cardioid $r = a(1 - \cos\theta)$ and prove that the line $\theta = 2\pi/3$ bisects the upper half of the cardioid. **(M.U. 1995, 98, 2002, 13, 17)**

Sol.: The shape of the curve is shown in the figure. We have, $O(0, 0)$ and $B(2a, \pi)$

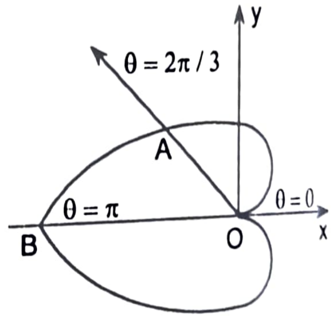

Fig. 8.17

$$\text{Arc } OB = s = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

$$\therefore \quad s = \int_0^{\pi} \sqrt{r^2 + a^2 \sin^2\theta} \cdot d\theta$$

$$= \int_0^{\pi} \sqrt{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta} \cdot d\theta$$

$$= a\int_0^{\pi} \sqrt{2(1 - \cos\theta)} \cdot d\theta = \int_0^{\pi} 2a\sin\left(\frac{\theta}{2}\right) d\theta$$

$$= 2a\left[-2\cos\left(\frac{\theta}{2}\right)\right]_0^{\pi} = 4a$$

$\therefore$ Perimeter of the cardioid $= 2s = 8a$.

Now, the arc where the line $\theta = 2\pi/3$, divides the cardioid is given by

$$\text{Arc } OA = \int_0^{2\pi/3} 2a\sin\frac{\theta}{2} \cdot d\theta = 2a\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi/3} = -4a\left[\frac{1}{2} - 1\right] = 2a.$$

Hence, the line $\theta = 2\pi/3$ bisects the upper half of the cardioid.

Example 1'(b) : Find the perimeter of the cycloid $r = a(1 + \cos\theta)$. **(M.U. 2015)** **[Ans. : 8a]**

Sol. : Left to you.

Note ....

As in the case of cycloids, the cardioids are also given in four forms :

(i) $r = a(1 + \cos\theta)$, (ii) $r = a(1 - \cos\theta)$, (iii) $r = a(1 + \sin\theta)$, (iv) $r = a(1 - \sin\theta)$.

There cardioids are studies on pages 15-14 and 15-15.

The lengths of the perimeters of the cardioid in each case is the same $8a$.

Example 2 (b) : Find the length of the cardioid $r = a(1 - \cos\theta)$ lying outside the circle $r = a\cos\theta$. **(M.U. 1997, 2000, 04, 08, 09, 10, 11, 13, 15)**

Sol.: The circle and the cardioid are shown in the figure. They intersect where $a(1 - \cos\theta) = a\cos\theta$ *i.e.* where $1 - \cos\theta = \cos\theta$ *i.e.* $1 = 2\cos\theta$ *i.e.* $\cos\theta = 1/2$ *i.e.* $\theta = 60° = \pi/3$.

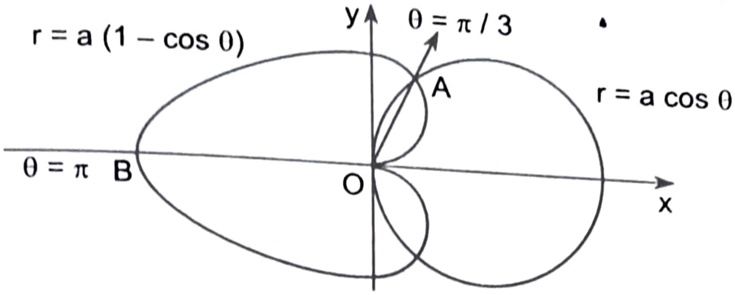

Fig. 8.18

The length of the cardioid outside the circle is 2 arc *AB* where for *B*, $\theta = \pi$ and for *A*, $\theta = \pi/3$.

$$\therefore \quad s = 2\int_{\pi/3}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

Now, as proved in the above Example No. 1,

$$s = 2\int_{\pi/3}^{\pi} 2a\sin\frac{\theta}{2} \cdot d\theta = 4a\left[-2\cos\frac{\theta}{2}\right]_{\pi/3}^{\pi}$$

$$= -8a\left[0 - \frac{\sqrt{3}}{2}\right] = 4a\sqrt{3}.$$

Example 3 (b) : Find the length of the cardioid $r = a(1 + \cos\theta)$ which lies outside the circle $r + a\cos\theta = 0$. **(M.U. 1999, 2004, 12)**

Sol.: The circle and the cardioid are shown in the figure.

They intersect where $a(1 + \cos\theta) = -a\cos\theta$ *i.e.* where $2\cos\theta = -1$ *i.e.* where $\cos\theta = -1/2$ *i.e.* $\theta = 2\pi/3$.

The length of the cardioid outside the circle is 2 arc *BA* where for *B*, $\theta = 0$ and for *A*, $\theta = 2\pi/3$.

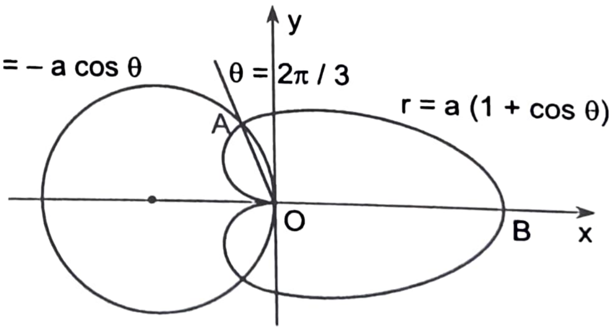

Fig. 8.19

Now, as in the above example,

$$\therefore \quad s = 2\int_0^{2\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

$$= 2\int_0^{2\pi/3} 2a\cos\left(\frac{\theta}{2}\right) \cdot d\theta$$

$$= 4a \cdot 2\left[\sin\left(\frac{\theta}{2}\right)\right]_0^{2\pi/3} = 4\sqrt{3}\,a.$$

Example 4 (b) : Find the perimeter of the cardioid $r = a(1 + \cos\theta)$. **(M.U. 2015)**

And show that the line $\theta = \pi/3$ divides the upper half of the cardioid into two equal to parts. **(M.U. 1993, 2002, 06)**

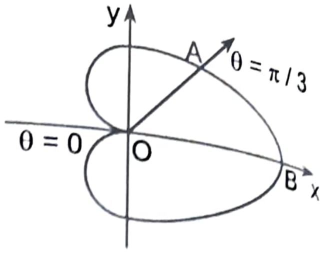

Fig. 8.20

Sol.: Arc $OB = s = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$

$$\therefore \quad s = \int_0^{\pi} \sqrt{r^2 + a^2 \sin^2 \theta} \cdot d\theta$$

$$= \int_0^{\pi} \sqrt{\left[a^2 (1 + \cos \theta)^2 + a^2 \sin^2 \theta\right]} \cdot d\theta$$

$$= \int_0^{\pi} a \sqrt{[2(1 + \cos \theta)]} \cdot d\theta = \int_0^{\pi} a \cdot \sqrt{4 \cos^2 (\theta/2)} \cdot d\theta$$

$$= \int_0^{\pi} 2a \cos\left(\frac{\theta}{2}\right) d\theta = 2a\left[2 \sin \frac{\theta}{2}\right]_0^{\pi} = 4a.$$

$\therefore$ Perimeter of the cardioid $= 2s = 8a$.

Now, the arc where the line $\theta = \pi / 3$ divides the cardioid is given by,

$$\text{Arc } OA = \int_0^{\pi/3} 2a \cdot \cos \frac{\theta}{2} \cdot d\theta = 2a\left[2 \sin \frac{\theta}{2}\right]_0^{\pi/3}$$

$$= 4a \sin \frac{\pi}{6} = 4a \cdot \frac{1}{2} = 2a.$$

$$\therefore \quad \text{Arc } OA = \frac{1}{2} \text{ Arc } OB.$$

Example 5 (b) : Find the length of the arc of the curve $r = a \sin^2\left(\frac{\theta}{2}\right)$ from $\theta = 0$ to any point $P(\theta)$.

Sol.: The required arc is given by

$$s = \int_0^{\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta = \int_0^{\theta} \sqrt{r^2 + \left(a \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right)^2} \cdot d\theta$$

$$= \int_0^{\theta} \sqrt{a^2 \sin^4 \frac{\theta}{2} + a^2 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}} \cdot d\theta$$

$$= \int_0^{\theta} \sqrt{a^2 \sin^2 \frac{\theta}{2}\left(\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}\right)} \cdot d\theta = a \int_0^{\theta} \sin \frac{\theta}{2}\, d\theta$$

$$\therefore \quad s = a\left[-2a \cos\left(\frac{\theta}{2}\right)\right]_0^{\theta} = 2a\left(1 - \cos \frac{\theta}{2}\right) = 4a \sin^2\left(\frac{\theta}{4}\right).$$

Example 6 (b) : Show that for the parabola $\frac{2a}{r} = 1 + \cos \theta$, the arc intercepted between the vertex and the extremity of the latus rectum is $a[\sqrt{2} + \log(1 + \sqrt{2})]$. **(M.U. 2002, 12)**

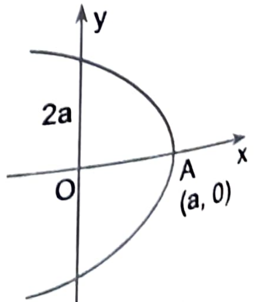

Fig. 8.21

Sol.: Since $r = \frac{2a}{1 + \cos \theta} = \frac{2a}{2 \cos^2 (\theta/2)} = a \sec^2\left(\frac{\theta}{2}\right)$.

$$\therefore \quad \frac{dr}{d\theta} = a \sec^2\left(\frac{\theta}{2}\right) \cdot \tan\left(\frac{\theta}{2}\right)$$

At the extremity L of the latus rectum $\theta = \frac{\pi}{2}$.

$$\therefore \quad s = \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

$$= \int_0^{\pi/2} \sqrt{a^2 \sec^4(\theta/2) + a^2 \sec^4(\theta/2) \cdot \tan^2(\theta/2)} \cdot d\theta$$

$$= \int_0^{\pi/2} a \sec^2(\theta/2) \sqrt{1 + \tan^2(\theta/2)} \cdot d\theta$$

Put $\tan\left(\frac{\theta}{2}\right) = t \quad \therefore \sec^2\left(\frac{\theta}{2}\right) \cdot d\theta = 2\,dt$

$$\therefore \quad s = \int_0^1 2a\sqrt{1+t^2} \cdot dt = 2a\left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\log\left(t + \sqrt{t^2+1}\right)\right]_0^1$$

$$= 2a\left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\log(1+\sqrt{2})\right] = a\left[\sqrt{2} + \log(1+\sqrt{2})\right]$$

Note ....

Putting $r\cos\theta = x$ and $r = \sqrt{x^2+y^2}$, we get

$$2a = r + r\cos\theta \qquad \therefore \quad (2a - x)^2 = x^2 + y^2$$

$$\therefore \quad y^2 = 4a^2 - 4ax = -4a(x-a)$$

This is a parabola with vertex at $(a, 0)$ and opening on the left.

Example 7 (b) : Show that the length of the arc of that part of cardioid $r = a(1+\cos\theta)$ which lies on the side of the line $4r = 3a\sec\theta$ away from the pole is $4a$. **(M.U. 1999, 2005, 10)**

Sol.: The cardioid is shown in the figure.

Now, $\quad 4r = 3a\sec\theta \quad$ *i.e.* $\quad 4r\cos\theta = 3a$.

i.e. $\quad 4x = 3a$ *i.e.* $x = \frac{3a}{4}$ is a line parallel to the y-axis.

Now, at the point of intersection A,

$$a(1+\cos\theta) = \frac{3a}{4}\sec\theta \quad \textit{i.e.} \quad 4(1+\cos\theta)\cos\theta = 3$$

i.e. $\quad 4\cos^2\theta + 4\cos\theta - 3 = 0 \quad \therefore \quad (2\cos\theta + 3)(2\cos\theta - 1) = 0$

$\therefore \quad \cos\theta = -\frac{3}{2}$ which is impossible, or $\cos\theta = \frac{1}{2} \quad \therefore \quad \theta = \frac{\pi}{3}$.

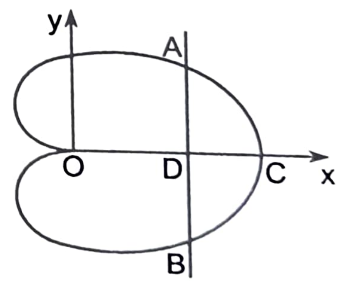

Fig. 8.22

$$\therefore \quad \text{Required arc } ACB = 2 \text{ Arc } AC = 2\int_0^{\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

$$= 2\int_0^{\pi/3} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta} \cdot d\theta$$

$$= 2\int_0^{\pi/3} a \cdot \sqrt{2(1+\cos\theta)} \cdot d\theta = 2a\int_0^{\pi/3} 2\cos\frac{\theta}{2}\, d\theta$$

$$= 4a\left[2\sin\frac{\theta}{2}\right]_0^{\pi/3} = 4a.$$

Example 8 (b) : Find the length of the Cissoid $r = 2a \tan\theta \sin\theta$ from $\theta = 0$ to $\theta = \pi/4$.

(M.U. 2002, 09)

Sol. : We have $\because\ r = 2a\tan\theta\sin\theta$

$$\therefore\ \frac{dr}{d\theta} = 2a\left[\sec^2\theta\sin\theta + \tan\theta\cos\theta\right] = 2a\left[\sec^2\theta\sin\theta + \sin\theta\right]$$

$$= 2a\sin\theta(\sec^2\theta + 1)$$

$$\therefore\ r^2 + \left(\frac{dr}{d\theta}\right)^2 = 4a^2\tan^2\theta\sin^2\theta + 4a^2\sin^2\theta(\sec^2\theta + 1)^2$$

$$= 4a^2\sin^2\theta(\sec^2\theta - 1 + \sec^4\theta + 2\sec^2\theta + 1)$$

$$= 4a^2\sin^2\theta(\sec^4\theta + 3\sec^2\theta)$$

$$= 4a^2\sin^2\theta\sec^2\theta(\sec^2\theta + 3)$$

$$= 4a^2\tan^2\theta(\sec^2\theta + 3)$$

$$= 4a^2\tan^2\theta(\tan^2\theta + 4)$$

$$\therefore\ s = \int_0^{\pi/4}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\cdot d\theta = \int_0^{\pi/4} 2a\tan\theta\sqrt{4 + \tan^2\theta}\cdot d\theta$$

For integration put $4 + \tan^2\theta = t^2$ $\quad\therefore\ 2\tan\theta\sec^2\theta\, d\theta = 2t\, dt$.

$$\therefore\ \tan\theta\, d\theta = \frac{t\,dt}{\sec^2\theta} = \frac{t\,dt}{1+\tan^2\theta} = \frac{t\,dt}{t^2 - 3}$$

When $\theta = 0,\ t = 2$; when $\theta = \frac{\pi}{4},\ t = \sqrt{5}$.

$$\therefore\ s = \int_2^{\sqrt{5}} 2a\cdot\frac{t^2}{t^2 - 3}\cdot dt = \int_2^{\sqrt{5}} 2a\cdot\left[1 + \frac{3}{t^2 - 3}\right] dt$$

$$= 2a\left[t + \frac{3}{2\sqrt{3}}\log\frac{t - \sqrt{3}}{t + \sqrt{3}}\right]_2^{\sqrt{5}}$$

$$= 2a\left[\sqrt{5} + \frac{\sqrt{3}}{2}\log\frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}} - 2 - \frac{\sqrt{3}}{2}\log\frac{2 - \sqrt{3}}{2 + \sqrt{3}}\right]$$

$$= 2a\left[\sqrt{5} - 2 + \frac{\sqrt{3}}{2}\left\{\log\frac{(\sqrt{5} - \sqrt{3})^2}{(5 - 3)} - \log\frac{(2 - \sqrt{3})^2}{4 - 3}\right\}\right]$$

$$= 2a\left[\sqrt{5} - 2 + \frac{\sqrt{3}}{2}\left\{\log(4 - \sqrt{15}) - \log(7 - 2\sqrt{3})\right\}\right]$$

Example 9 (b) : Find the total length of the curve $r = a\sin^3(\theta/3)$.

(M.U. 1997, 2003, 06)

Sol.: The curve is shown in the figure. For half the arc $OPABC$, θ varies from 0 to $3\pi/2$.

Since, $\quad r = a\sin^3\left(\frac{\theta}{3}\right)$

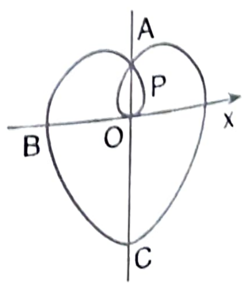

Fig. 8.23

$$\frac{dr}{d\theta} = a \cdot 3 \sin^2 \frac{\theta}{3} \cos \frac{\theta}{3} \cdot \frac{1}{3} = a \sin^2 \frac{\theta}{3} \cos \frac{\theta}{3}$$

$$\therefore \quad r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2 \sin^6 \frac{\theta}{3} + a^2 \sin^4 \frac{\theta}{3} \cos^2 \frac{\theta}{3} = a^2 \sin^4 \frac{\theta}{3}.$$

$$\therefore \quad s = 2\int_0^{3\pi/2} a \sin^2 \frac{\theta}{3}\, d\theta = a \int_0^{3\pi/2} \left(1 - \cos \frac{2\theta}{3}\right) d\theta$$

$$= a\left[\theta - \frac{3}{2} \sin \frac{2\theta}{3}\right]_0^{3\pi/2} = a \cdot \frac{3\pi}{2} = \frac{3}{2} \pi a.$$

Example 10 (b) : Find the length of the upper arc of one loop of Lemniscate $r^2 = a^2 \cos 2\theta$.

(M.U. 1990, 2002, 05, 07, 08)

Sol.: The curve is shown in the figure. It is clear that for upper half of one loop θ varies from 0 to $\pi / 4$.

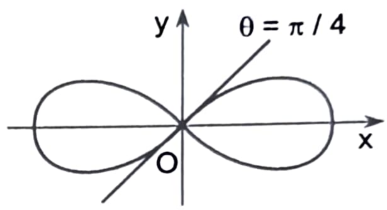

Fig. 8.24

$$\therefore \quad s = \int_0^{\pi/4} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

But $r = a\sqrt{\cos 2\theta}$

$$\therefore \quad \frac{dr}{d\theta} = a \cdot \frac{(-2 \sin 2\theta)}{2\sqrt{\cos 2\theta}}$$

$$\therefore \quad r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2 \cos 2\theta + \frac{a^2 \sin 2\theta}{\cos 2\theta} = \frac{a^2}{\cos 2\theta}$$

$$\therefore \quad s = \int_0^{\pi/4} \frac{a}{\sqrt{\cos 2\theta}}\, d\theta.$$

Put $t = 2\theta \qquad \therefore \quad dt = 2\, d\theta.$

When $\theta = 0$, $t = 0$; when $\theta = \dfrac{\pi}{4}$, $t = \dfrac{\pi}{2}$

$$\therefore \quad s = \frac{a}{2} \int_0^{\pi/2} \frac{dt}{\sqrt{\cos t}} = \frac{a}{2} \int_0^{\pi/2} \sin^0 t \cdot \cos^{-1/2} t\, dt$$

$$= \frac{a}{2} \cdot \frac{\overline{|(-1/2+1)/2} \cdot \overline{|1/2}}{\overline{|(-1/2+2)/2}} = \frac{a}{2} \cdot \frac{\overline{|1/4} \cdot \overline{|1/2}}{\overline{|3/4}}$$ [By (18), page 6-28]

But $\overline{|1/4} \cdot \overline{|3/4} = \pi\sqrt{2}$ [By (3), page 6-53]

$$\therefore \quad s = \frac{a}{4} \cdot \frac{\sqrt{\pi}\, \overline{|1/4}}{1} \cdot \frac{\overline{|1/4}}{\pi\sqrt{2}} = \frac{a}{4\sqrt{2}} \cdot \frac{\left(\overline{|1/4}\right)^2}{\sqrt{\pi}}.$$

Example 11 (b) : Show that the total perimeter of $r^2 = a^2 \cos 2\theta$ is $\dfrac{a}{\sqrt{2\pi}} \left(\overline{|1/4}\right)^2$.

(M.U. 1996, 2000)

Sol.: The perimeter $= 4s$ where s is as given above.

Example 12 (b) : Prove that the length of the spiral $r = a e^{\theta \cot \alpha}$ as r increases from r_1 to r_2 is given by $(r_2 - r_1) \sec \alpha$.

(M.U. 1997)

Sol.: Since, $r = a e^{\theta \cot \alpha}$. $\therefore \dfrac{dr}{d\theta} = a e^{\theta \cot \alpha} \cdot \cot \alpha = r \cot \alpha$

Since, the limits of integration are in terms of r we shall use the formula (4) given in the corollary. (See page 8-17)

$$\therefore \ s = \int_{r_1}^{r_2} \sqrt{\left[1 + r^2 \left(\frac{d\theta}{dr}\right)^2\right]} \cdot dr = \int_{r_1}^{r_2} \sqrt{\left[1 + r^2 \cdot \frac{1}{r^2 \cot^2 \alpha}\right]} \cdot dr$$

$$= \int_{r_1}^{r_2} \sqrt{\left[1 + \frac{1}{\cot^2 \alpha}\right]} \cdot dr = \int_{r_1}^{r_2} \sqrt{\left[1 + \tan^2 \alpha\right]} \cdot dr$$

$$= \sec \alpha \left[r \right]_{r_1}^{r_2} = (r_2 - r_1) \sec \alpha .$$

Example 13 (b) : Find the length of the arc of the parabola $r = \dfrac{6}{1 + \cos \theta}$ from $\theta = 0$ to $\theta = \pi / 2$.

Sol.: Since $r = \dfrac{6}{1 + \cos \theta} = \dfrac{6}{2 \cos^2 (\theta/2)} = 3 \sec^2 \left(\dfrac{\theta}{2}\right)$

$$\frac{dr}{d\theta} = 3 \cdot 2 \sec^2 \left(\frac{\theta}{2}\right) \tan \left(\frac{\theta}{2}\right) \cdot \frac{1}{2} = 3 \sec^2 \left(\frac{\theta}{2}\right) \tan \left(\frac{\theta}{2}\right)$$

$$\therefore \ r^2 + \left(\frac{dr}{d\theta}\right)^2 = 9 \sec^4 \left(\frac{\theta}{2}\right) + 9 \sec^4 \left(\frac{\theta}{2}\right) \cdot \tan^2 \left(\frac{\theta}{2}\right)$$

$$= 9 \sec^4 \left(\frac{\theta}{2}\right) \times \left[1 + \tan^2 \left(\frac{\theta}{2}\right)\right] = 9 \sec^6 \left(\frac{\theta}{2}\right)$$

$$\therefore \ r = \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta = \int_0^{\pi/2} 3 \cdot \sec^3 \left(\frac{\theta}{2}\right) d\theta$$

We shall find the integral by the method of integration by parts as follows.

$$\int \sec^3 x \, dx = \int \sec^2 x \cdot \sec x \cdot dx \qquad \textbf{[Note this]}$$

$$= \sec x \tan x - \int \tan x \sec x \tan x \cdot dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$= \sec x \tan x - \int \sec^3 x \cdot dx + \int \sec x \cdot dx$$

$$\therefore \ 2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \cdot dx$$

$$= \sec x \tan x + \log(\sec x + \tan x)$$

$$\therefore \ r = \frac{3}{2} \left[\sec \left(\frac{\theta}{2}\right) \tan \left(\frac{\theta}{2}\right) + \log \left[\sec \left(\frac{\theta}{2}\right) + \tan \left(\frac{\theta}{2}\right) \right] \right]_0^{\pi/2} \cdot 2$$

$$= 3 \left[\sqrt{2} + \log \left(\sqrt{2} + 1 \right) \right].$$

EXERCISE - V

Solve the following examples : Class (b) : 6 Marks

1. Find the circumferences of a circle of radius a by using the polar equation of the circle $r = a$. **[Ans. :** $2\pi a$ **]**
2. Find the length of the curve $r = \cos^3 (\theta / 3)$ from $\theta = 0$ to $\theta = \pi / 4$. **[Ans. :** $(\pi + 3) / 8$ **]**
3. Find the length of the curve $r = \sqrt{1 + \cos 2\theta}$ from $\theta = 0$ to $\theta = 2\pi\sqrt{2}$. **[Ans. :** 4π **]**
4. Find the length of the curve $r = \theta^2$ from $\theta = 0$ to $\theta = 2\sqrt{3}$. **[Ans. :** $56 / 3$ **]**
5. Find the length of the cardioid $r = a(1 - \cos\theta)$ lying inside the circle $r = a\cos\theta$. (See Fig. 8.18, page 8-18)
   (**Hint :** Point of intersection is given by $a(1 - \cos\theta) = a\cos\theta$
   $$\therefore \theta = \frac{\pi}{3} \quad \therefore s = 2\int_0^{\pi/3} 2a\sin\left(\frac{\theta}{2}\right) d\theta = 8a\left[1 - \frac{\sqrt{3}}{2}\right]$$)
6. Find the length of the spiral $r = a^{m\theta}$ lying inside the circle $r = a$.
   (**Hint :** The circle intersects the spiral at $(a, 0)$. The limits for r are 0 and a *i.e.* for θ are $-\infty$ and 0.)
   [Ans. : $\frac{a}{m}\sqrt{1 + m^2}$ **]**
7. Taking $s = 0$ at $\theta = 0$, find the length of the arc OP of the spiral $r = a e^{\theta \cot \alpha}$ from 0 to $P(\theta)$.
   [Ans. : $a \sec\alpha (e^{\theta\cot\alpha} - 1)$ **]**
8. Find the length of the cardiod $r = a(1 + \sin\theta)$. **[Ans. :** $8a$ **]**
   (**Hint :** Integrate from $-\pi / 2$ to $\pi / 2$) (See Fig. 15.43(a), page 15-15)

EXERCISE - VI

Solve the following examples : Class (a) : 3 or 4 Marks

1. Find the length of the following curve using integration $y = 2x + 5$ from $x = 1$ to 3.
   [Ans. : $2\sqrt{5}$ **]**
2. Find the length of the parabola $2y = x^2$ from $x = 0$ to $x = 1$. **[Ans. :** $\sqrt{2} + \frac{1}{2}\log(1 + \sqrt{2})$ **]**
3. Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 5$. **[Ans. :** $335 / 27$ **]**
4. Find the length of the curve $y = \log\cos x$ from $x = 0$ to $x = \pi / 3$. **[Ans. :** $\log(2 + \sqrt{3})$ **]**
5. Find the length of the curve $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$. **[Ans. :** π **]**
6. Find the length of the curve $y = x^2 - \frac{1}{8} \cdot \log x$ from $x = 1$ to $x = 2$. **[Ans. :** $3 + \frac{1}{8}\log 2$ **]**
7. Find the length of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ from $y = 1$ to $y = 3$. **[Ans. :** $\frac{53}{6}$ **]**
8. Find the length of the curve $x = \frac{y^{3/2}}{3} - y^{1/2}$ from $y = 1$ to $y = 9$. **[Ans. :** $\frac{32}{3}$ **]**

9. Find the length of the curve $x = t^3,\ y = \dfrac{3t^2}{2}$ from $t = 0$ to $t = \sqrt{3}$. [**Ans.** : 7]

10. Find the circumference of the circle $x^2 + y^2 = a^2$. [**Ans.** : $2\pi a$]

Summary

1. $s = \int_{x_1}^{x_2} \sqrt{1+\left(\dfrac{dy}{dx}\right)^2} \cdot dx$; $\quad s = \int_{y_1}^{y_2} \sqrt{1+\left(\dfrac{dx}{dy}\right)^2} \cdot dy$;

$s = \int_{t_1}^{t_2} \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2} \cdot dt$.

2. $s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2} \cdot d\theta$; $\quad s = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\dfrac{d\theta}{dr}\right)^2} \cdot dr$

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