Course: *Mathematical Foundation for Electrical Stream- 2* ***(23MA2BSMES)***

Mathematical Foundation for Computer science Stream- 2 ***(23MA2BSMCS)***

Unit 1: INTEGRAL CALCULUS

Fubini's Theorem (First form)

If $f(x,y)$ is continuous on the rectangular region$R: a \leq x \leq b, c \leq y \leq d$, then

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

Fubini's Theorem (Second form)

If $f(x,y)$ is continuous on the rectangular regionR.

(i) If R is defined by$a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on$[a,b]$, then $\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$

(ii) If R is defined by$c \leq y \leq d, h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on$[c,d]$, then $\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$

The area of a closed, bounded region R in polar coordinate plane is $A = \iint_R r dr d\theta$

The volume of a closed, bounded region D in space is $V = \iiint_D dV$

Mass of the lamina corresponding to the region R with variable density $\rho(x,y)$ is given by $m = \iint_R \rho(x,y)dxdy$.

1. DOUBLE INTEGRALS

I. Evaluate the following: -

1. $\int_0^1 \int_0^1 \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}}$. $\quad Ans: \frac{\pi^2}{4}$.
2. $\int_1^2 \int_3^4 (xy + e^y)dydx$. $\quad Ans: \frac{21+4(e^4-e^3)}{4}$.
3. $\int_3^4 \int_1^2 \frac{dydx}{(x+y)^2}$. $\quad Ans: \log\left(\frac{25}{24}\right)$.
4. $\int_1^2 \int_0^x \frac{dydx}{x^2+y^2}$. $\quad Ans: \frac{\pi}{2}\log(2)$.
5. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$ $\quad Ans: \frac{\pi}{4}\log(1+\sqrt{2})$.
6. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)dydx$. $\quad Ans: \frac{3}{35}$.

7. $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}dxdy$ $Ans: \frac{\ln(8)^2}{2} - \ln(8) + \frac{1}{2}$..

II. Evaluate the following over the specified region:-

1. $\iint_R xydxdy$, where R is the region bounded by the circle $x^2 + y^2 = a^2$ in the first quadrant.

   $Ans: \frac{a^4}{8}$

2. $\iint_A xy(x+y)dA$, where A is the area bounded by the parabola $y = x^2$ and the line$y = x$.

   $Ans: 0.0536$

3. $\iint_R xydxdy$, where R is the domain bounded by x-axis, ordinate $x = 2a$ and the curve$x^2 = 4ay$.

   $Ans: \frac{a^4}{3}$

4. $\iint_D x^2dxdy$, where D is the domain in the first quadrant bounded by the hyperbola$xy = 16$, and the lines $y = x, y = 0$ and$x = 8$.

   $Ans: \frac{3020}{3}$

III. Change the order of integration and hence evaluate the following:-

1. $\int_0^a\int_y^a \frac{xdxdy}{x^2+y^2}$ $Ans: \frac{\pi a}{4}$.
2. $\int_0^a\int_{\sqrt{ax}}^a \frac{y^2dydx}{\sqrt{y^4-a^2x^2}}$ $Ans: \frac{\pi a^2}{6}$.
3. $\int_0^1\int_x^1 \frac{1}{1+y^4}dydx$ $Ans: \frac{\pi}{8}$
4. $\int_0^1\int_x^{\sqrt{x}} xydydx$ $Ans: \frac{1}{24}$.
5. $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}dydx$
6. $\int_0^\infty\int_0^x xe^{-x^2/y}dydx$ $Ans: \frac{1}{2}$.
7. $\int_0^3\int_0^{\sqrt{4-y}} (x+y)dxdy$ $Ans: 601/60$.
8. $\int_0^1\int_{e^x}^e \frac{dydx}{\log y}$ $Ans: e - 1$.
9. $\int_0^a\int_{x/a}^{\sqrt{x/a}} (x^2+y^2)dxdy$ $Ans: \frac{a^3}{28} + \frac{a}{20}$.
10. $\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}dydx$ $Ans: \frac{\pi a^3}{6}$.
11. $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx$ $Ans: \frac{16}{3}a(2\sqrt{a^2} - a)$.

12. $\int_0^1 \int_{x^2}^{2-x} xy dx dy$ $Ans: \frac{3}{8}.$

13. $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dy dx$ $Ans: 1 - 1/\sqrt{2}.$

IV. Evaluate the following by transforming into polar coordinates: -

1. $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2) dy dx$ $Ans: \frac{\pi a^4}{8}.$
2. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$ $Ans: \frac{\pi}{4}.$
3. $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2} dy dx$ $Ans: 9\pi.$
4. $\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2} dx dy$ $Ans: \frac{\pi}{4}(e-1)..$
5. $\int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2-y^2}} \log_e (x^2+y^2) dx dy$, where $(a > 0)$ $Ans: \frac{\pi a^2}{8}(2log2 - 1)$

V. Area of the region using double integral

1. Find the area lying inside the cardioid $r = a(1+\cos\,\theta)$ and outside the circler $r = a.$

   $Ans: \frac{3\pi a^2}{2}$

2. Find the area of Lemniscate $r^2 = a^2 \cos\, 2\theta.$

3. Find the area of the cardioid $r = a(1+\cos\,\theta).$

4. Find the area which is inside the circle $r = 3a\cos\,\theta$ and outside the cardioid $r = a(1 + \cos\,\theta)$

   $Ans: \pi a^2$

5. Find the area lying inside the circle $r = a\sin\,\theta$ and outside the cardioid $r = a(1-\cos\,\theta).$

   $Ans: a^2(1-\frac{\pi}{4})$

2. TRIPLE INTEGRALS

VII. Evaluate the following:-

1. $\int_0^1 \int_0^2 \int_1^2 x^2 yz dx dy dz$ $Ans: \frac{7}{3}$
2. $\int_{-c}^c \int_{-b}^b \int_{-a}^a (x^2+y^2+z^2) dz dy dx$ $Ans: \frac{4}{3}abc(a^2+b^2+c^2).$
3. $\int_0^2 \int_1^z \int_0^{yz} xyz dx dy dz$ $Ans: 0$
4. $\int_0^6 \int_0^{6-x} \int_1^{6-x-z} dy dz dx$ $Ans: 18$

5. $\int_0^4 \int_0^{\frac{1}{2}} \int_0^{x^2} \frac{1}{\sqrt{x^2-y^2}} dydxdz$ $Ans: 0.5113.$

6. $\int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{\sqrt{a^2-y^2-z^2}} xdxdydz.$

7. $\int_0^1 \int_0^x \int_0^{x+y} (x+y+z)dzdydx$ $Ans: \frac{7}{8}.$

8. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyzdzdydx$ $Ans: \frac{1}{48}$

9. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}}$ $Ans: \frac{\pi^2}{8}$

10. $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z)dydxdz$ $Ans: 0$

11. $\int_0^1 \int_0^1 \int_0^{2-x^2-y^2} xye^z dzdxdy$ $Ans: \frac{1}{4}(e-2).$

12. $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4-x^2}} dydxdz$

VIII. Evaluate the following over the region R bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$.

1. $\iiint_R (x+y+z)dxdydz.$
2. $\iiint_R xyzdxdydz$ $Ans: \frac{1}{720}.$
3. $\iiint_R \frac{dxdydz}{(1+x+y+z)^3}$ $Ans: \frac{1}{2}(log2 - \frac{5}{8}).$

IX. Volume of the region using triple integral

1. Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the co-ordinate planes. $Ans: \frac{abc}{6}$
2. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$ $Ans: \frac{4}{3}\pi abc$
3. Find the volume of the sphere $x^2+y^2+z^2=a^2.$ $Ans: 4(\frac{\pi}{12}+(\frac{\sqrt{3}}{2}-1))$

3. BETA AND GAMMA FUNCTIONS

I. Express the following in terms of Gamma functions: -

1. $\int_0^\infty e^{-kx}x^{p-1}dx, k>0$ $Ans: \frac{\gamma(p)}{k^p}.$
2. $\int_0^\infty \frac{x^4}{4^x}dx.$ $Ans: \frac{3}{4(log2)^5}.$
3. $\int_0^\infty 3^{-4x^2}dx.$
4. $\int_0^1 x^m[\log_e x]^n dx$, where n is an integer and $m>-1$. $Ans: (-1)^n n!/(m+1)^{n+1}$.
5. $\int_0^1 x^{q-1}\left[\log_e\left(\frac{1}{x}\right)\right]^{p-1} dx\ (p>0, q>0).$
6. $\int_0^\infty e^{-t^2}t^{2n-1}dt.$ $Ans: \frac{1}{2}\gamma(n)$

II. Express the following in terms of Gamma functions: -

1. $\int_0^{\pi/2} \sqrt{\cot\,\theta}\,d\theta.$
2. $\int_0^{\pi/2} (\sqrt{\tan\,\theta} + \sqrt{\sec\,\theta})\,d\theta.$
3. $\int_0^1 x^m(1-x^n)^p\,dx.$
4. $\int_0^1 \frac{dx}{\sqrt{1-x^4}}.$

III. Prove the following: -

1. $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin\,n\pi}$, given $\int_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin\,n\pi}$ and hence deduce the value of $\int_0^\infty \frac{dy}{1+y^4}$.
2. $\int_0^\infty xe^{-x^8}dx \times \int_0^\infty x^2e^{-x^4}dx = \frac{\pi}{16\sqrt{2}}.$
3. $\int_0^{\pi/2} \sqrt{\sin\,\theta}\,d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\,\theta}} = \pi.$
4. $\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}.$
5. $\frac{\beta(m+1,n)}{m} = \frac{\beta(m,n)}{m+n} = \frac{\beta(m,n+1)}{n}.$