PH 444: Electromagnetic Theory
Course Project Report

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Acknowledgment

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Introduction

In this project, our aim is to explore an alternative method for **calculating the electric scalar potential and electric field** at points in space that are not coincident with a charged filamentary ring, without resorting to elliptic integrals. Typically, this problem involves complex mathematical formulations, often relying on elliptic integrals. However, we propose a novel approach utilizing **toroidal functions**, which are derived from the solution of **Laplace's equation in toroidal coordinates**. This method, although not commonly discussed in physics curriculum, offers simplicity and ease of application.

Toroidal functions, particularly a special class known as the **Legendre functions of the second kind**, prove to be uniquely suited for handling finite circular cylindrical geometries. This alternative approach not only simplifies the calculation process but also extends its applicability beyond just the scenario of a charged ring. Indeed, the toroidal expansion technique finds use in diverse applications such as coil design for MRI magnets, transformer coils, and circular cylindrical coils with rectangular cross-sections. Moreover, it has been successfully employed in determining external fields generated by permanent magnet motors.

Beyond electromagnetic phenomena, the toroidal expansion method has recently found utility in fields beyond physics, including gravitational potential calculations.

In cylindrical coordinates, Laplace's equation becomes,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}+\frac{\partial^2 \phi}{\partial z^2}=0$$

Now, in toroidal coordinates, Laplace's equation becomes,

$$\operatorname{csch} v\,(\cos u-\cosh v)^3\left[\frac{\partial}{\partial u}\left(\frac{\sinh v}{\cosh v-\cos u}\,\frac{\partial}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{\sinh v}{\cosh v-\cos u}\,\frac{\partial}{\partial v}\right)+\frac{\partial}{\partial \phi}\left(\frac{\operatorname{csch} v}{\cosh v-\cos u}\,\frac{\partial}{\partial \phi}\right)\right]f=0.$$

Scalar Potential

1. **Scalar potential with constant charged ring**

our analysis of the scalar potential generated by a charged ring with a harmonic charge density distribution, we have discovered intriguing simplifications in the expression when considering scenarios w

Figure 1 shows how the problem's geometry is represented. We begin with the differential scalar potential, which is provided by, and use it to calculate the electric scalar potential at any point in space P that is not coincident with the charged ring.

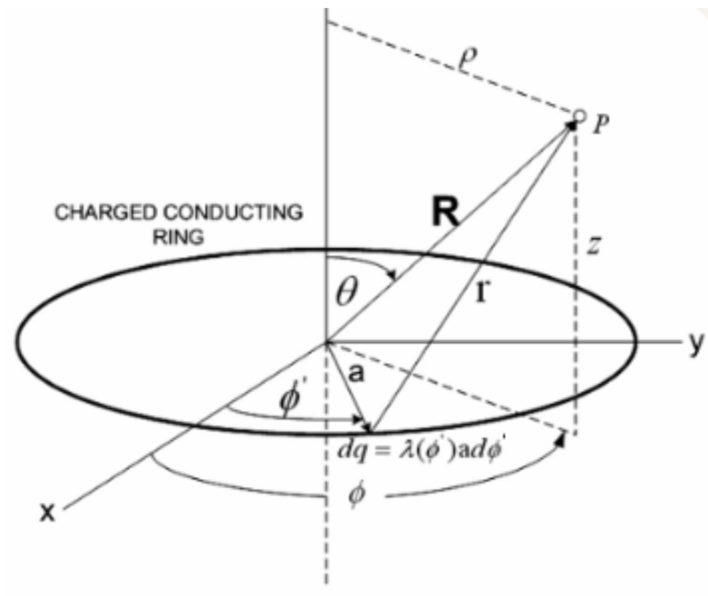


Fig. 1. Charged ring.

$$d\Phi_P = \frac{1}{4\pi\epsilon_0}\frac{\lambda(\phi')ad\phi'}{|\mathbf{R}-\mathbf{a}|}, \tag{1}$$

where R is the position vector of the observation point, a is the position vector of an element of the circular filamentary line charge and λ is charge density.

The distance in circular cylindrical coordinates between the source point (ρ’,φ’,z’) and an arbitrary observation point or field point (ρ,φ,z) is given by

$$\frac{1}{|\mathbf{R}-\mathbf{r}'|} = \frac{1}{\sqrt{\rho^2+\rho'^2+(z-z')^2-2\rho\rho'\cos(\phi-\phi')}}. \qquad (2)$$

In Fig. 1, r’ = ρ’ = |a| = a and z’ = 0.This coordinate system allows us to write Eq. 2 as

$$\frac{1}{|\mathbf{R}-\mathbf{a}|} = \frac{1}{\sqrt{\rho^2+a^2+z^2-2\rho a\cos(\phi-\phi')}}. \qquad (3)$$

Using Equation (1) and (3), we write scalar potential as

$$\Phi_P = \frac{a}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\lambda(\phi')d\phi'}{\sqrt{\rho^2+a^2+z^2-2\rho a\cos(\phi-\phi')}}. \qquad (4)$$

Equation (4) can be used directly and the Green’s function expansion for Equation (2) is given by

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\pi\sqrt{\rho\rho'}}\sum_{m=0}^{\infty}\varepsilon_m Q_{m-1/2}(\beta)\cos[m(\phi-\phi')], \qquad (5)$$

where $\beta = (\rho^2+\rho'^2+(z-z')^2)/2\rho\rho'$, ε_m is Neumann’s factor, which is 1 for m=0 and 2 otherwise. $Q_{m-1/2}(\beta)$ is the Legendre function of the second kind and of half-integral degree or a toroidal function of zeroth order. These functions are also referred to as Q-functions and can be represented by the formula.

$$Q_{m-1/2}(\beta) = \frac{\pi}{(2\beta)^{m+1/2}2^m}\sum_{n=0}^{\infty}\frac{(4n+2m-1)\,!\,!}{2^{2n}(n+m)\,!\,n!}\frac{1}{(2\beta)^{2n}}. \qquad (6)$$

Using Equations (5) and (6), we can write Equation (4) as

$$\Phi_P = \frac{a}{4\pi^2\epsilon_0}\frac{1}{\sqrt{\rho a}}\sum_{m=0}^{\infty} \varepsilon_m Q_{m-1/2}(\beta)\int_0^{2\pi} \lambda(\phi')\cos[m(\phi - \phi')]d\phi', \tag{7}$$

where $\beta=(\rho^2+a^2+z^2)/2\rho a>1$.

For a uniformly charged ring, λ(Φ')=q/2πa and the m=0 term in Equation (7) is the only term that survives the integration. As a result, we obtain an expression for the electric scalar potential at an arbitrary point in space not coincident with the charged ring.

$$\Phi_P = \frac{q}{4\pi^2\epsilon_0\sqrt{\rho a}}Q_{-1/2}(\beta). \tag{8}$$

Using Equation (6),Q−1/2 is given by(m=0 here)

$$Q_{-1/2}(\beta) = \pi\sum_{n=0}^{\infty} \frac{(4n-1)!!}{2^{2n}(n!)^2}\left(\frac{\rho a}{\rho^2+a^2+z^2}\right)^{2n+1/2}. \tag{9}$$

Using Equation (9),we can write Equation (8) as

$$\Phi_P = \frac{q}{4\pi\epsilon_0}\sum_{n=0}^{\infty} \frac{(4n-1)!!}{2^{2n}(n!)^2}\frac{(\rho a)^{2n}}{(\rho^2+a^2+z^2)^{2n+1/2}}. \tag{10}$$

Equation (10) is the infinite series solution for the scalar potential of a uniformly charged ring of radius a and is valid at an arbitrary point that is not coincident with the charged ring.

Proof of Green's Expansion for 1/|R-r'|

In terms of cylindrical coordinates (R,φ,z) the Green's function is written as (Chapter 3 Jackson)

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\int_0^{\infty} dk\, J_m(kR)J_m(kR')e^{-k(z_> - z_<)},$$

where Jm is an order m Bessel function of the first kind. Using equation (13.22.2) in Watson (1944), we write

$$\int_0^\infty e^{-at} J_m(bt) J_m(ct)\, dt = \frac{1}{\pi\sqrt{bc}}\, Q_{m-1/2}\left(\frac{a^2+b^2+c^2}{2bc}\right),$$

where Qm-1/2 is the half-integer degree Legendre function of the second kind. We rewrite previous equation as

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{\pi\sqrt{RR'}} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} Q_{m-1/2}(\chi)$$

where $\chi = (R^2+R'^2+(z-z')^2)/2RR'$ Realizing that $Q_{-1/2+m}(\chi) = Q_{-1/2-m}(\chi)$, and that $e^{i\theta}+e^{-i\theta} = 2\cos\theta$, we can express in terms of all $m>=0$ as

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{\pi\sqrt{RR'}} \sum_{m=0}^{\infty} \epsilon_m \cos\,[m(\phi-\phi')] Q_{m-1/2}(\chi)$$

where ε_m is Neumann's factor, which is 1 for m=0 and 2 otherwise

2. **Scalar potential with harmonic charged ring**

In our analysis of the scalar potential generated by a charged ring with a harmonic charge density distribution, we have discovered intriguing simplifications in the expression when considering scenarios where the total charge on the ring is zero. This contrasts with cases where the total charge is non-zero, leading to distinct behaviors in the scalar potential field.

$$\Phi_P = \frac{a}{4\pi^2\epsilon_0}\frac{1}{\sqrt{\rho a}}\sum_{m=0}^{\infty}\varepsilon_m Q_{m-1/2}(\beta)\int_0^{2\pi}\lambda(\phi')\cos[m(\phi - \phi')]d\phi',$$

where $\beta=(\rho^2+a^2+z^2)/2\rho a>1$.

When the linear charge density on the ring follows a harmonic function $\lambda(\Phi')=\lambda_0\cos(p\Phi')$ with $p\geq 1$ and the total charge on the ring is zero, we observe a significant simplification in the scalar potential equation. In this context, the scalar potential equation acts as a filter, selectively emphasizing specific harmonic components where m=p.

By examining specific cases such as p=1 and p=2, we can further elucidate the impact of harmonic charge density on the scalar potential.

- For p=1, the dipole contribution dominates the total scalar potential in the far field. This dominance arises from the survival of the m=1 term during integration, with the Q1/2 term encapsulating a significant spherical dipole contribution.

$$\Phi_P = \frac{qQ_{1/2}(\beta)}{4\pi^2\epsilon_0\sqrt{\rho a}}\cos(\phi).$$

- In the case of p=2, the quadrupole contribution takes precedence in shaping the total scalar potential in the far field. Here, the survival of the m=2 term in the integration process highlights the importance of quadrupole effects on the scalar potential distribution.

$$\Phi_P = \frac{qQ_{3/2}(\beta)}{4\pi^2\epsilon_0\sqrt{\rho a}}[2\cos^2(\phi) - 1].$$

Electric Field Produced by a Charged Ring

The electric field components due to a ring of charge with a harmonic charge density, $\lambda(\Phi') = \lambda_0 \cos(p\Phi')$ with p = 1, can be calculated from: $E = -\nabla \Phi p$

For p = 1, the potential is given as :

$$\Phi_P = \frac{q Q_{1/2}(\beta)}{4\pi^2 \epsilon_0 \sqrt{\rho a}} \cos(\phi).$$

In cylindrical coordinates, Laplace's equation is written:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

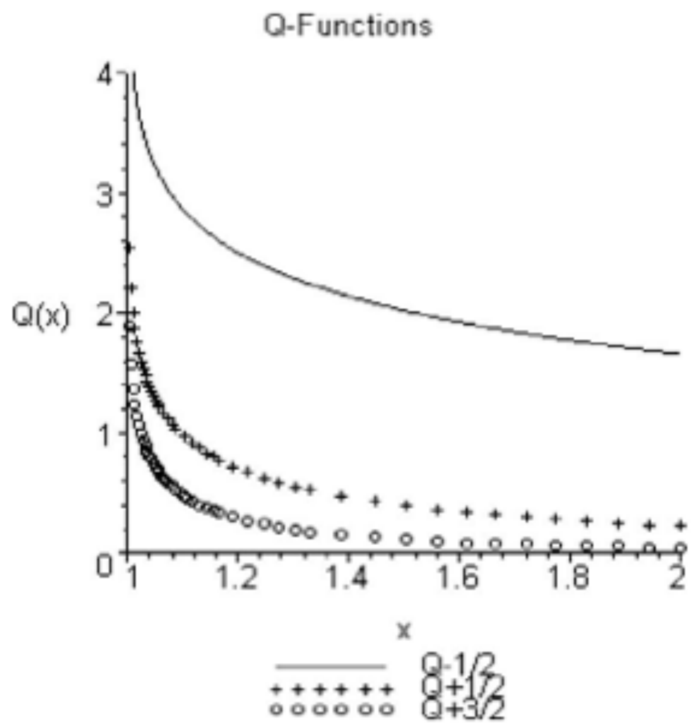
$$E_\rho = -\frac{\partial \Phi}{\partial \rho} \qquad E_\phi = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi} \qquad E_z = -\frac{\partial \Phi}{\partial z}$$

While taking the gradient of the potential for calculating the electric field components, we need to compute the derivative of the Q-function. On taking the derivative of Q-function with respect to β, for m = 1, we again get similar series terms as the Q-function. This series can be written in terms of Q-function for m=0 and m =1. We get the equation for the derivative as :

$$\frac{dQ_{1/2}(\lambda)}{d\lambda} = \frac{\lambda Q_{1/2}(\lambda) - Q_{-1/2}(\lambda)}{2(\lambda^2 - 1)}.$$

The plots for Q-function and the derivative of Q-function are shown below :

(for m=0, m=1 and m=2)



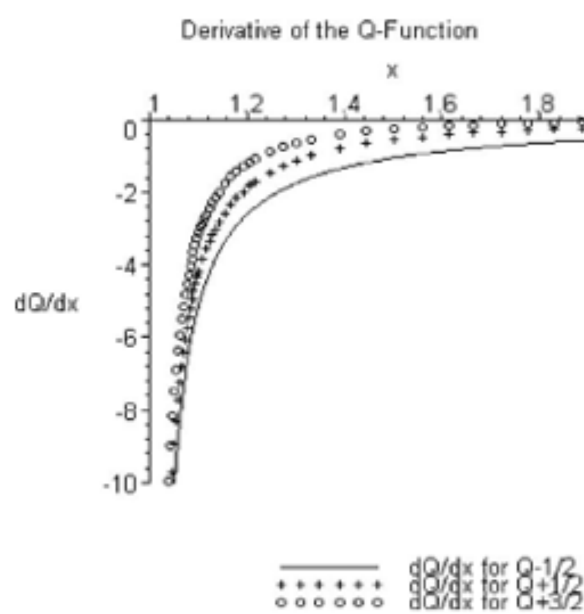
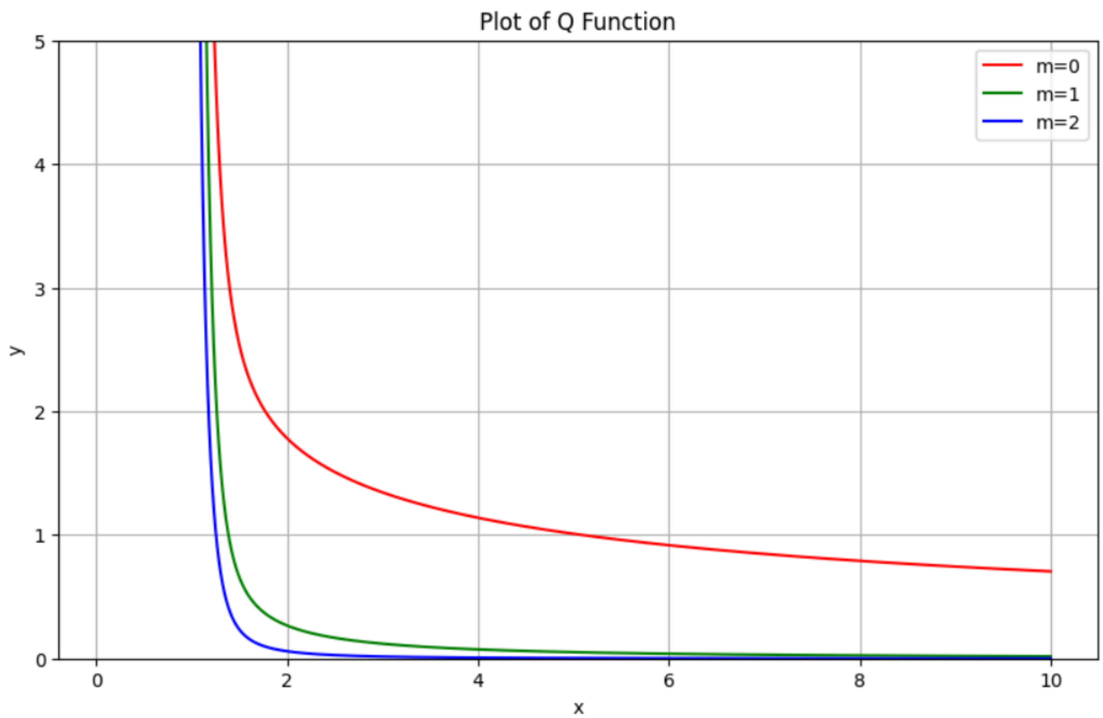
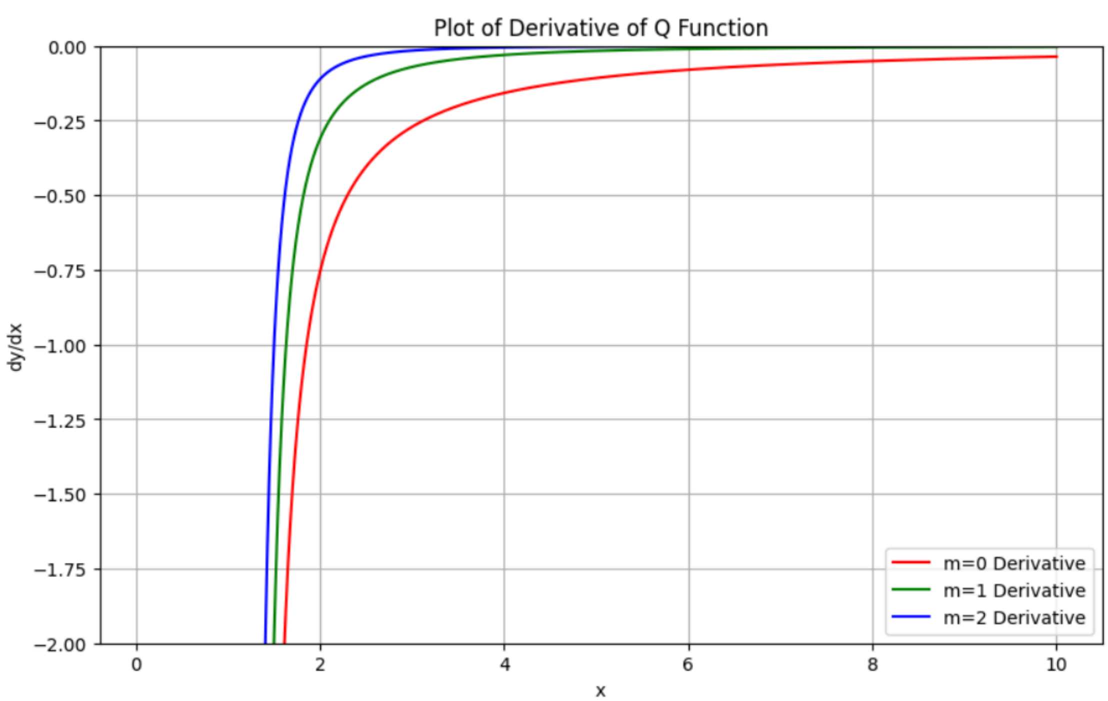


Fig. 6. A few Q-function derivatives.

Here are the plots we plotted using Python.





On solving the equation for derivative of the Potential and using the previous expression for derivative of Q-function, we get the components of the Electric field as:

$$E_\rho = \frac{q\{g_1 Q_{-1/2}(\beta) + g_2 Q_{1/2}(\beta)\}}{4\pi^2 \epsilon_0 \rho \sqrt{\rho a} D} \cos(\phi)$$

$$E_z = \frac{q\{g_3 Q_{-1/2}(\beta) + g_4 Q_{1/2}(\beta)\}}{4\pi^2 \epsilon_0 \sqrt{\rho a} D} \cos(\phi)$$

$$E_\phi = \frac{q Q_{1/2}(\beta)}{4\pi^2 \epsilon_0 \rho \sqrt{\rho a}} \sin(\phi)$$

where

$g1=\rho a(\rho^2-a^2-z^2)$

$g2=(-\rho^2 a^2+\rho^2 z^2+a^4+2a^2 z^2+z^4)$

$g3=2\rho za$

$g4=-z(\rho^2+a^2+ z^2)$

$D=(\rho^2+a^2-2\rho a+z^2)*(\rho^2+a^2+2\rho a+ z^2)$

Conclusions

- Spherical Harmonic Analysis

Spherical harmonic analysis is a mathematical method utilized to analyze functions on the surface of a sphere. It involves decomposing a function into a series of spherical harmonics, which are solutions to Laplace's equation in spherical coordinates. In the context of calculating the electric field from a charged ring, spherical harmonic analysis requires two solutions—one valid for observation points inside the ring's radius (Ra). However, these solutions suffer from slow convergence, particularly near the source, making them less effective for accurately determining near-field solutions.

- Toroidal Expansion

Toroidal expansion is another mathematical approach used for the same purpose. It involves expanding the electric field in terms of toroidal functions, which are solutions to Laplace's equation in toroidal coordinates. Unlike spherical harmonics, toroidal functions exhibit faster convergence and a monotonic nature. This makes them more efficient and accurate for modeling electric fields inside or outside the ring's radius, requiring only one series solution valid for all observation points inside or outside the ring's radius. Consequently, toroidal expansion is often preferred over spherical harmonic analysis for calculating electric fields from charged rings due to its improved convergence and simplicity.

Toroidal functions have several properties :

1. Monotonicity: Toroidal functions are typically monotonic, meaning they continuously increase or decrease over their domain. This property makes them suitable for efficient numerical calculations and ensures stable convergence in series representations.
2. Convergence: Toroidal functions often exhibit rapid convergence, particularly in comparison to spherical harmonics. This property is advantageous for numerical methods and allows for accurate approximation of physical phenomena.
3. Orthogonality: Toroidal functions satisfy orthogonality relations, meaning that the integral of the product of two different toroidal functions over the toroidal domain is zero. This property simplifies the analysis of systems involving toroidal geometries and facilitates the decomposition of complex functions into simpler components.