AN APPLICATION OF TOROIDAL FUNCTIONS IN ELECTROSTATICS

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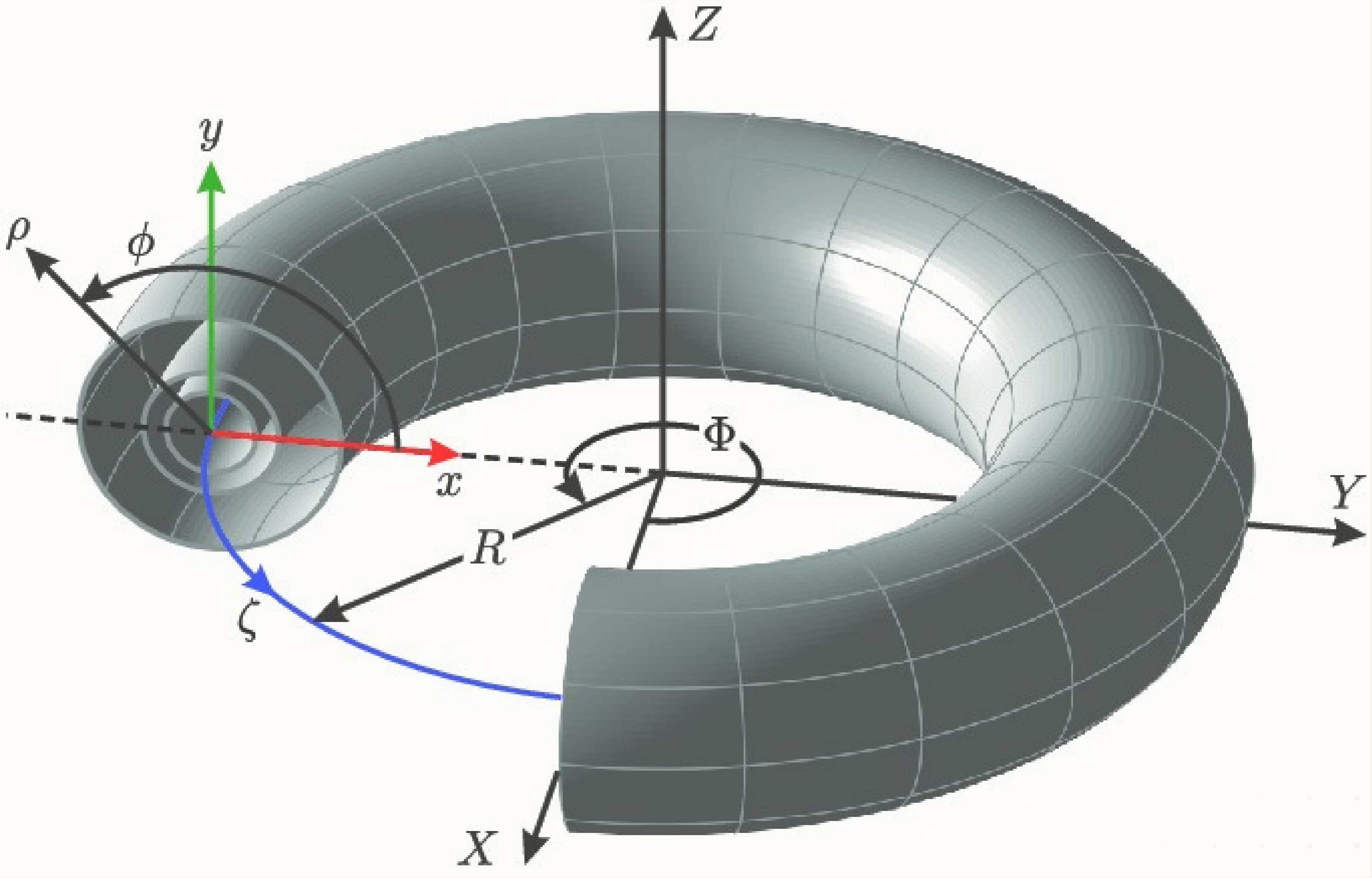
INTRODUCTION

- Traditional methods for **Calculating Electric Scalar Potential and Field** involve elliptic integrals.

Toroidal Functions
(alternative approach)

- Derived from Laplace's equation in toroidal coordinates.
- Special case - Legendre functions of the second kind, ideal for finite circular cylindrical geometries



HOW DOES THE METHOD OF TOROIDAL FUNCTION MAKE IT EASIER TO FORMULATE SOLUTIONS COMPARED TO THE ELLIPTIC INTEGRAL SOLUTION?

Benefits of Toroidal function

SINGLE SERIES SOLUTION

With the toroidal expansion method, only one series solution is needed, which is valid for observation points inside or outside the ring's radius. This simplifies the calculation process as only one set of equations needs to be considered, regardless of the observation point location

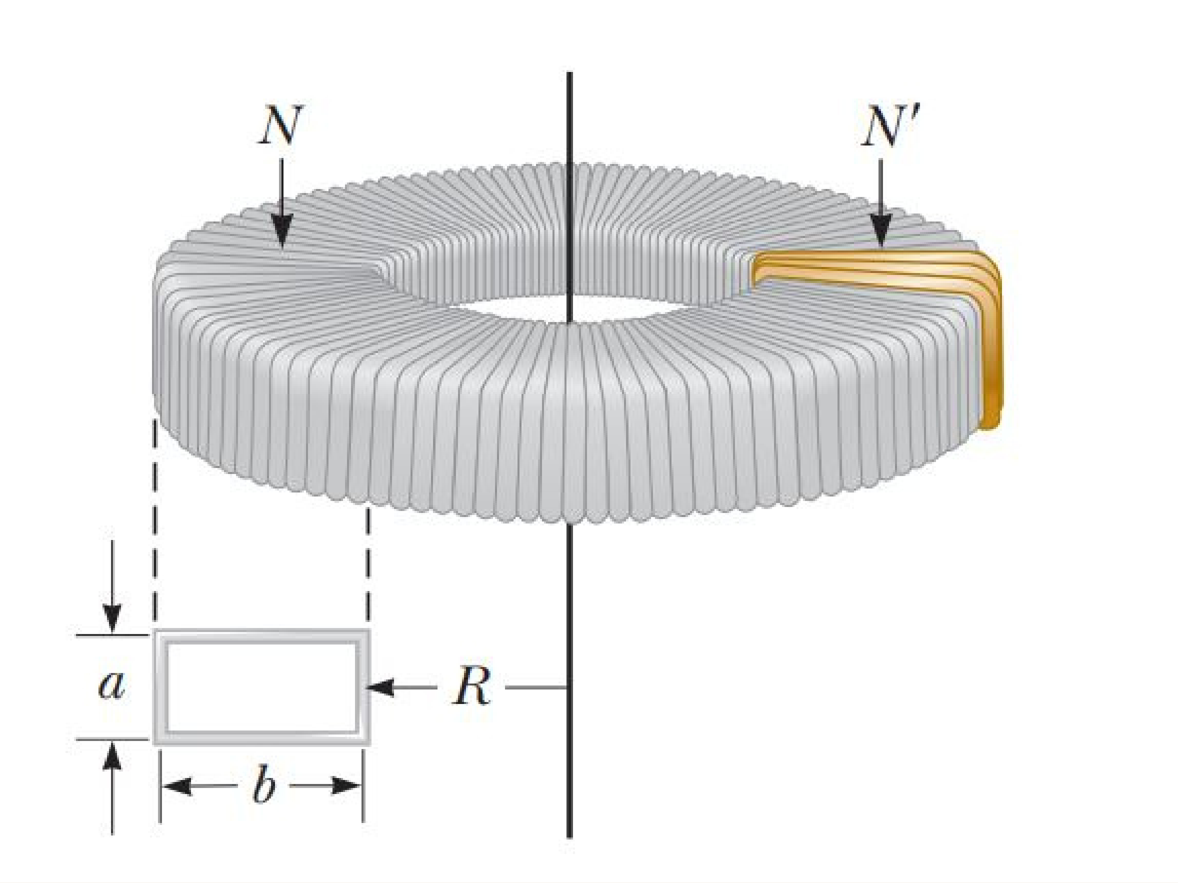
CONVERGENCE

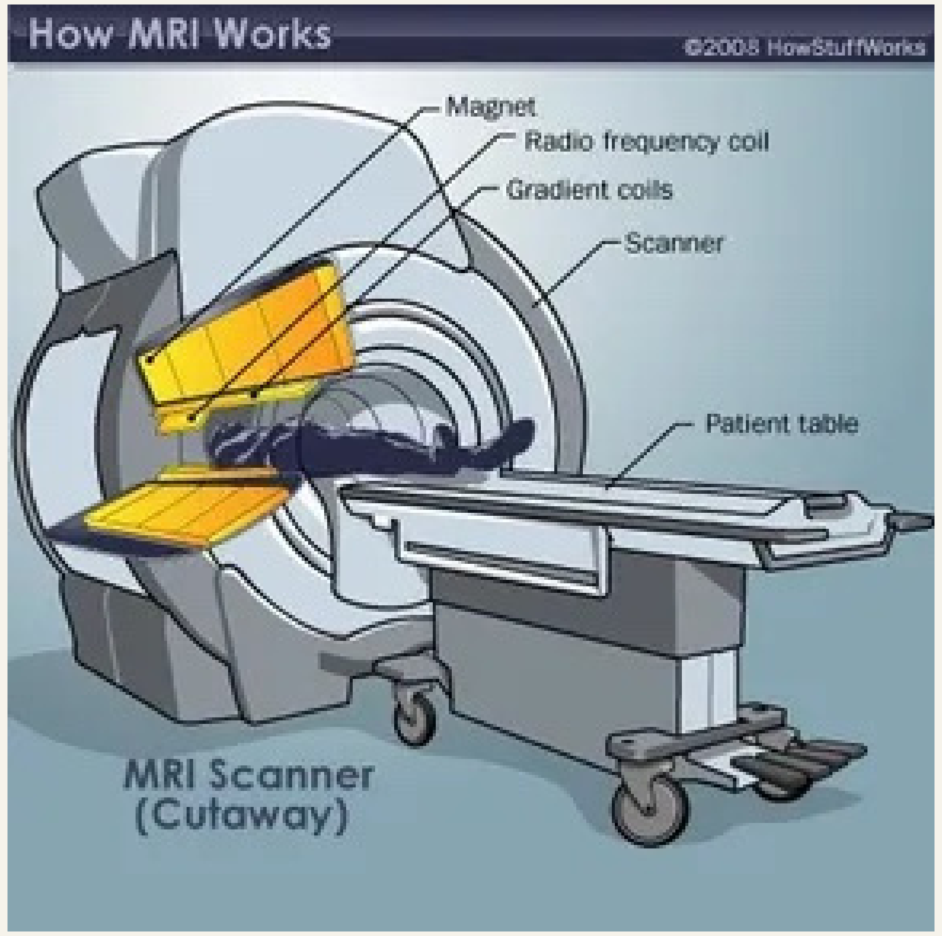
Toroidal functions exhibit high convergence rates, making them efficient for numerical computations. Unlike spherical harmonic solutions, toroidal functions provide accurate results even for observation points close to the source, ensuring reliable near-field solution

SIMPLICITY AND EASE OF APPLICATION

No need for complex mathematical manipulations typically involved in converting equations into elliptic integrals

Applications





- Widely applicable in coil design for MRI magnets, transformer coils, and cylindrical coils with rectangular cross-sections.
- Useful in calculating external field in permanent magnet motors.
- Recently explored for gravitational potential calculations.

This demonstrates the versatility and usefulness of toroidal functions in real-world scenarios

Scalar Potential Produced by a Charged Ring

Figure 1 shows how the problem's geometry is represented. We begin with the differential scalar potential, which is provided by eq 1, and use it to calculate the **electric scalar potential at any point in space P** that is not coincident with the charged ring.

$$d\Phi_P = \frac{1}{4\pi\epsilon_0}\frac{\lambda(\phi')a\,d\phi'}{|\mathbf{R}-\mathbf{a}|}, \quad (1)$$

where **R** is the **position vector of the observation point**, **a** is the **position vector of an element** of the circular filamentary line charge and **λ** is **charge density**.

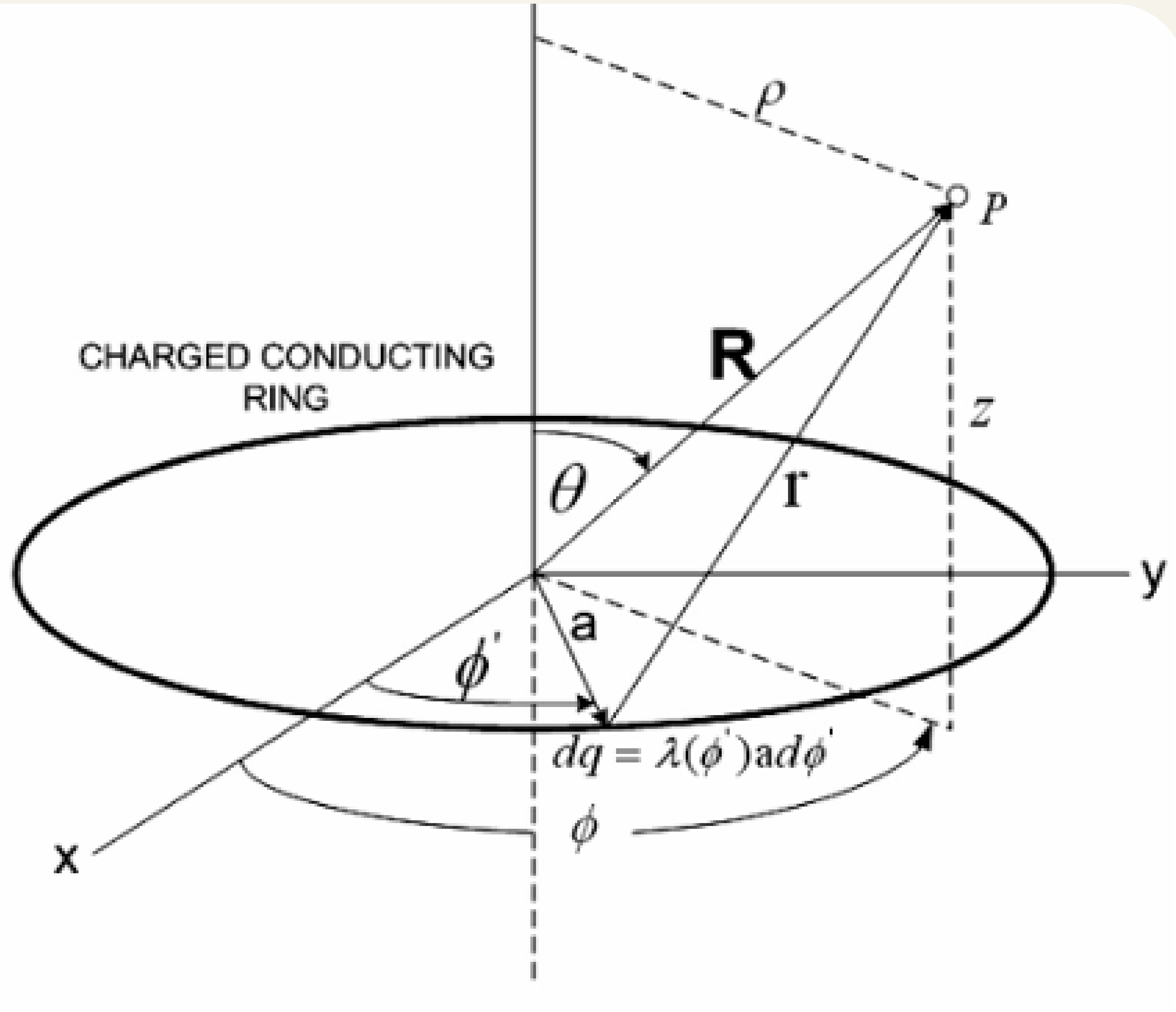


Fig. 1. Charged ring.

The **distance in circular cylindrical coordinates** between the source point (ρ’,φ’,z’) and an arbitrary observation point or field point (ρ,φ,z) is given by

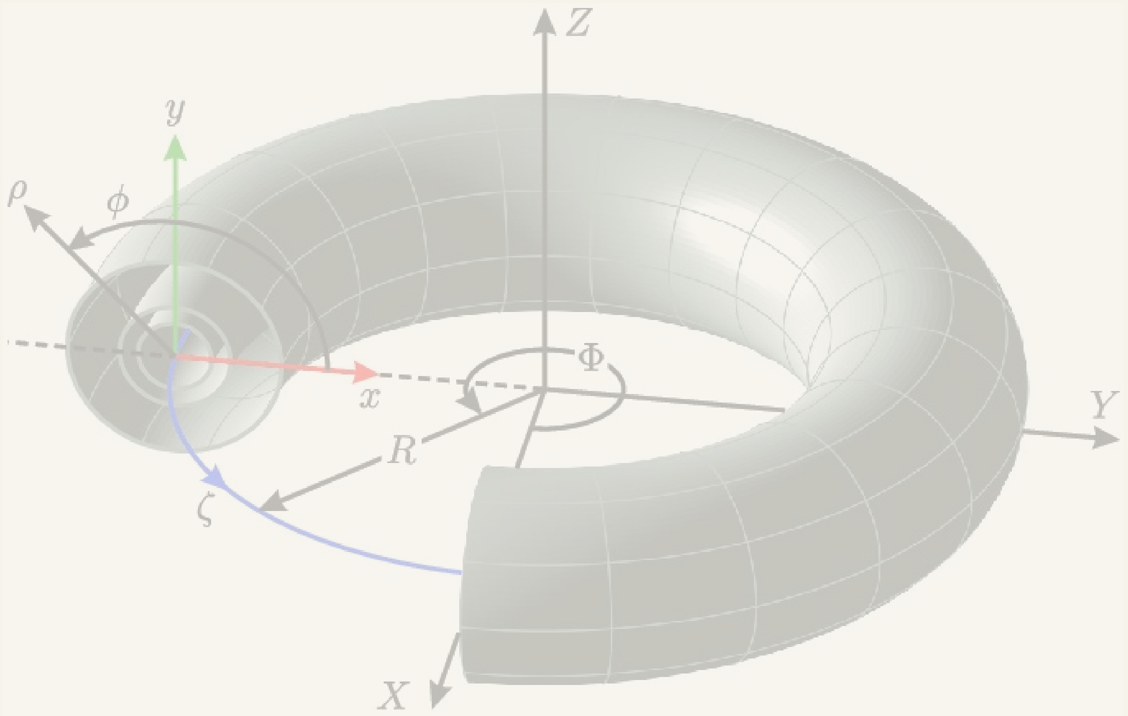
$$\frac{1}{|\mathbf{R}-\mathbf{r}'|} = \frac{1}{\sqrt{\rho^2+\rho'^2+(z-z')^2-2\rho\rho'\cos(\phi-\phi')}}. \quad (2)$$

In Fig. 1, r’=ρ’=|a|=a and z’=0. This coordinate system allows us to write Eq. 2 as

$$\frac{1}{|\mathbf{R}-\mathbf{a}|} = \frac{1}{\sqrt{\rho^2+a^2+z^2-2\rho a\cos(\phi-\phi')}}. \quad (3)$$

Using Equation (1) and (3), we write **scalar potential** as

$$\Phi_P = \frac{a}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\lambda(\phi')d\phi'}{\sqrt{\rho^2+a^2+z^2-2\rho a\cos(\phi-\phi')}}. \quad (4)$$



Equation (4) can be used directly and the **Green's function expansion** for Equation (2) is given by

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\pi\sqrt{\rho\rho'}} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-1/2}(\beta) \cos[m(\phi-\phi')], \qquad (5)$$

where **$\beta = (\rho^2+\rho'^2+(z-z')^2)/2\rho\rho'$, ε_m** is **Neumann's factor**, which is 1 for m=0 and 2 otherwise. **$Q_{m-1/2}(\beta)$** is the **Legendre function of the second kind** and of half-integral degree or a toroidal function of zeroth order. These functions are also referred to as **Q-functions** and can be represented by the formula.

$$Q_{m-1/2}(\beta) = \frac{\pi}{(2\beta)^{m+1/2} 2^m} \sum_{n=0}^{\infty} \frac{(4n+2m-1)\,!\,!}{2^{2n}(n+m)\,!\,n!} \frac{1}{(2\beta)^{2n}}. \quad (6)$$

Proof of Green's Expansion for 1/|R-r'|

In terms of the cylindrical coordinates (R,φ,z) the **Green's function** is written as (e.g., problem [3.14] of Jackson)

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \int_0^\infty dk\, J_m(kR) J_m(kR') e^{-k(z_>-z_<)} ,$$

where **Jm** is an **order m Bessel function of the first kind**. Using equation (13.22.2) in Watson (1944), we write

$$\int_0^\infty e^{-at} J_m(bt) J_m(ct)\, dt = \frac{1}{\pi\sqrt{bc}} Q_{m-1/2}\left(\frac{a^2+b^2+c^2}{2bc}\right) ,$$

where **Qm-1/2** is the **half-integer degree Legendre function of the second kind**.

We rewrite previous equation as

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{\pi\sqrt{RR'}}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}Q_{m-1/2}(\chi)$$

where **$\chi = (R^2+R'^2+(z-z')^2)/2RR'$**

Realizing that **$Q_{-1/2+m}(\chi) = Q_{-1/2-m}(\chi)$**, and that **$e^{i\theta}+e^{-i\theta} = 2\cos\theta$** , we can express in terms of all $m>=0$ as

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{\pi\sqrt{RR'}}\sum_{m=0}^{\infty}\epsilon_m\cos[m(\phi-\phi')]Q_{m-1/2}(\chi)$$

where **ε_m is Neumann's factor**, which is 1 for $m=0$ and 2 otherwise.

Using Equations (5) and (6), we can write Equation (4) as

$$\Phi_P = \frac{a}{4\pi^2\epsilon_0}\frac{1}{\sqrt{\rho a}}\sum_{m=0}^{\infty}\varepsilon_m Q_{m-1/2}(\beta)\int_0^{2\pi}\lambda(\phi')\cos[m(\phi - \phi')]d\phi', \tag{7}$$

where $\beta=(\rho^2+a^2+z^2)/2\rho a>1$.

For a **uniformly charged ring,** **λ(Φ')=q/2πa** and the **m=0 term** in Equation (7) is the only term that survives the integration. As a result, we obtain an expression for the **electric scalar potential** at an arbitrary point in space not coincident with the charged ring.

$$\Phi_P = \frac{q}{4\pi^2\epsilon_0\sqrt{\rho a}}Q_{-1/2}(\beta). \tag{8}$$

Using Equation (6),Q−1/2 is given by(m=0 here)

$$Q_{-1/2}(\beta) = \pi \sum_{n=0}^{\infty} \frac{(4n-1)\,!\,!}{2^{2n}(n\,!\,)^2} \left(\frac{\rho a}{\rho^2 + a^2 + z^2} \right)^{2n+1/2} . \qquad (9)$$

Using Equation (9),we can write Equation (8) as

$$\Phi_P = \frac{q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{(4n-1)\,!\,!}{2^{2n}(n\,!\,)^2} \frac{(\rho a)^{2n}}{(\rho^2 + a^2 + z^2)^{2n+1/2}} . \qquad (10)$$

Equation (10) is the **infinite series solution** for the scalar potential of a uniformly charged ring of radius a and is valid at an arbitrary point that is not coincident with the charged ring.

Formula:-

$$n!! \equiv \begin{cases} n\cdot(n-2)\ldots 5\cdot 3\cdot 1 & n>0 \text{ odd} \\ n\cdot(n-2)\ldots 6\cdot 4\cdot 2 & n>0 \text{ even} \\ 1 & n=-1,0. \end{cases}$$

Legendre function of the second kind

$$Q_n(x) = \begin{cases} U_n(1)\,V_n(x) & n = 0, 2, 4, \ldots \\ -V_n(1)\,U_n(x) & n = 1, 3, 5, \ldots \end{cases}$$

$$U_\nu(x) = 1 - \frac{\nu(\nu+1)}{2!}x^2 + \frac{\nu(\nu-2)(\nu+1)(\nu+3)}{4!}x^4 - \cdots$$

$$V_\nu(x) = x - \frac{(\nu-1)(\nu+2)}{3!}x^3 + \frac{(\nu-1)(\nu-3)(\nu+2)(\nu+4)}{5!}x^5 - \cdots$$

First several Legendre functions of the second kind

$$Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$Q_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

$$Q_2(x) = \frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}$$

$$Q_3(x) = \frac{5x^3-3x}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{5x^2}{2} + \frac{2}{3}$$

Harmonic charge density

$$\lambda(\phi') = \lambda_0 \cos(p\phi')$$

p=1,2,3.....

Now, we can observe some exciting simplification in the expression for scalar potential when we take the **harmonic charge density** on the ring where the total charge on the ring is **zero** instead of the **non-zero charge.**

We have an **infinite series solution** for uniform charge density by simplifying the equation given here.

Scalar Potential Equation

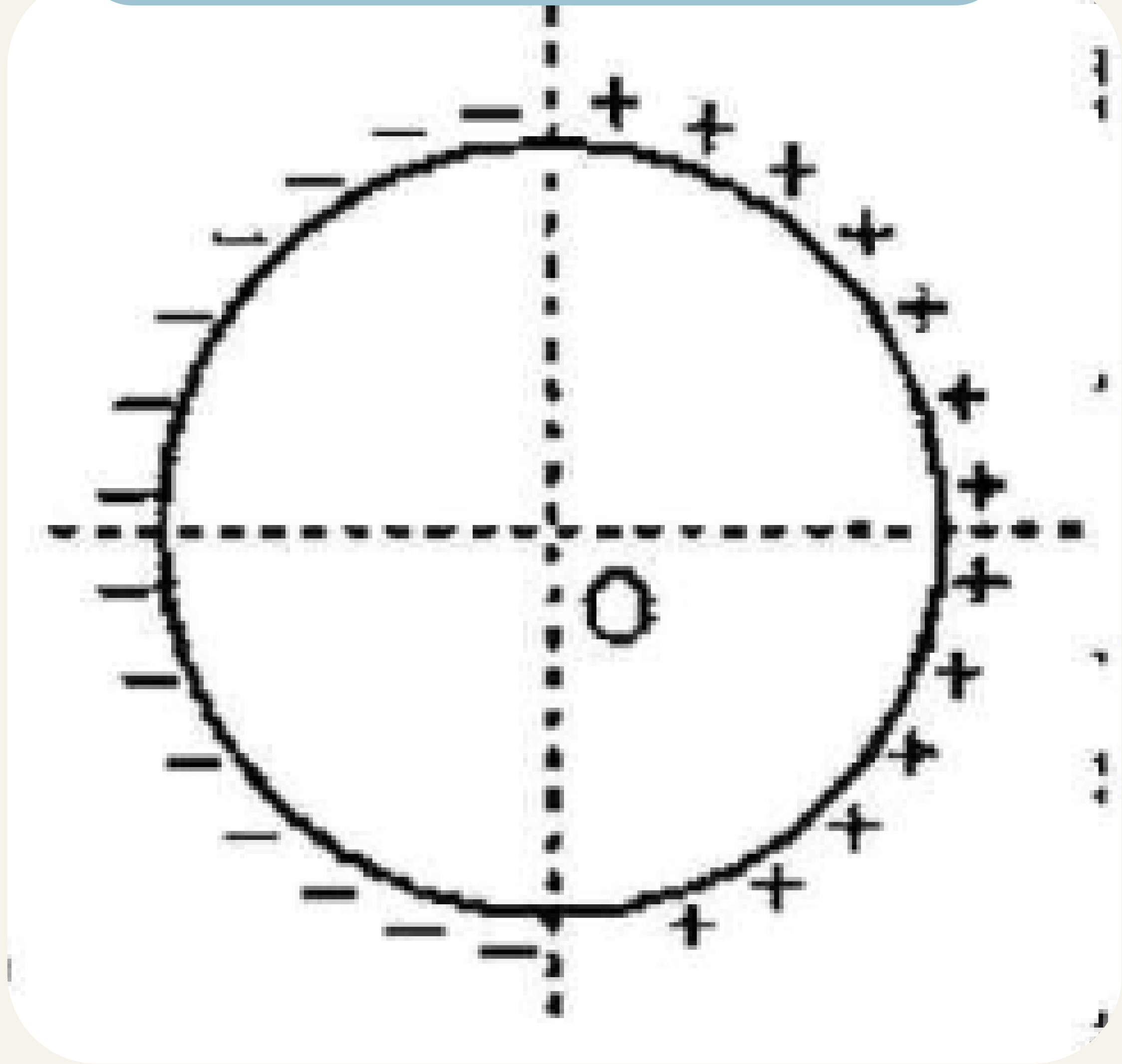
$$\Phi_P = \frac{a}{4\pi^2\epsilon_0}\frac{1}{\sqrt{\rho a}}\sum_{m=0} \varepsilon_m Q_{m-1/2}(\beta)\int_0^{2\pi}\lambda(\phi')\cos[m(\phi - \phi')]d\phi',$$

where $\beta=(\rho^2+a^2+z^2)/2\rho a>1.$

For a uniform charge density, the total charge on the source is nonzero; therefore, its **monopole contribution** dominates the total scalar potential in the far field.

$\lambda(\Phi') = \lambda_0 \cos(p\Phi')$ with p>=1, the **total charge on the ring is zero.**

Ring with charge density with p=1

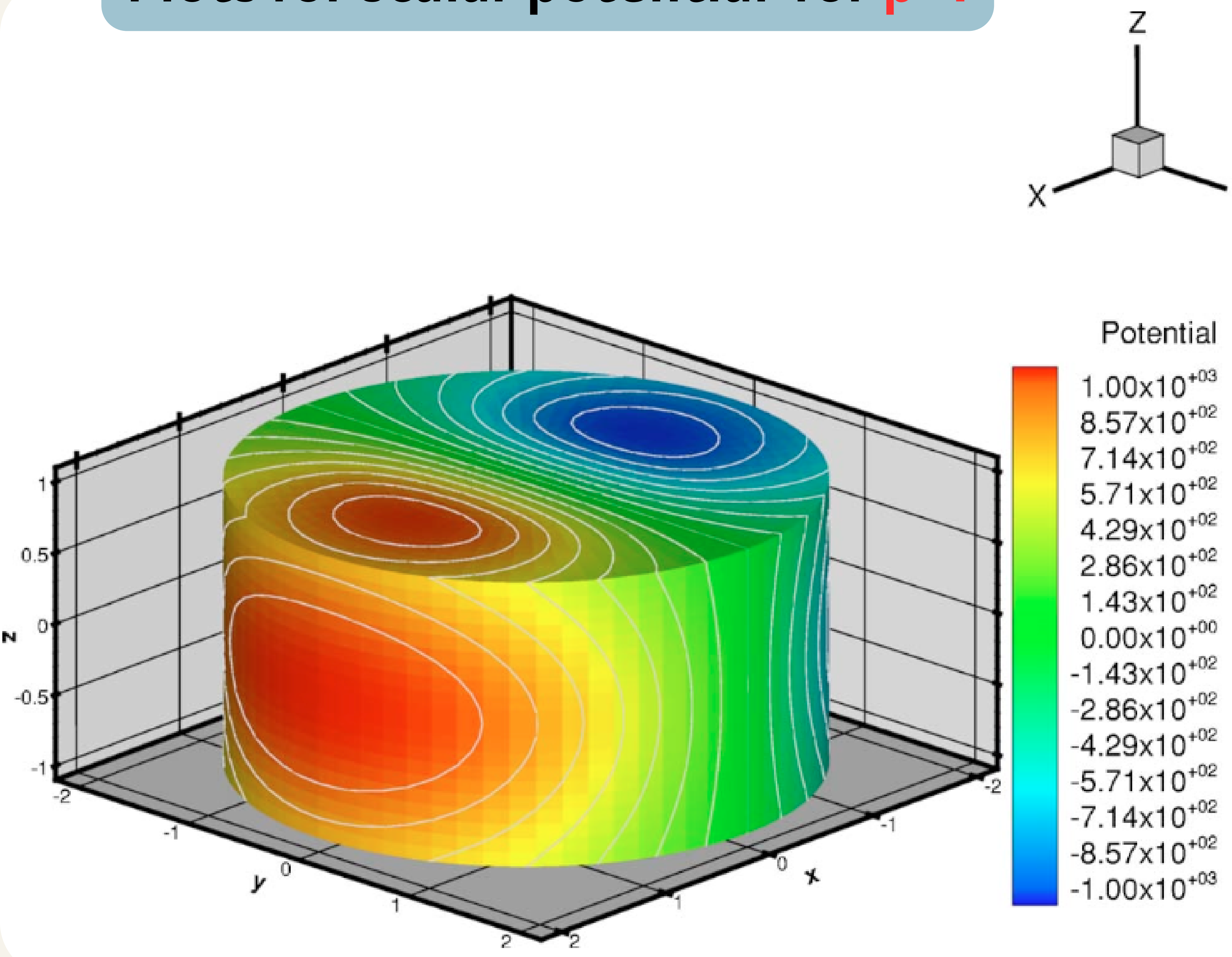


If the linear charge density of a charged ring is harmonic in cos(pΦ), then the scalar potential equation acts like a **filter.**

Filtering out only those specific harmonic components where **m=p**.

We will see the p=1 and p=2 as examples.

Plots for scalar potential for p=1

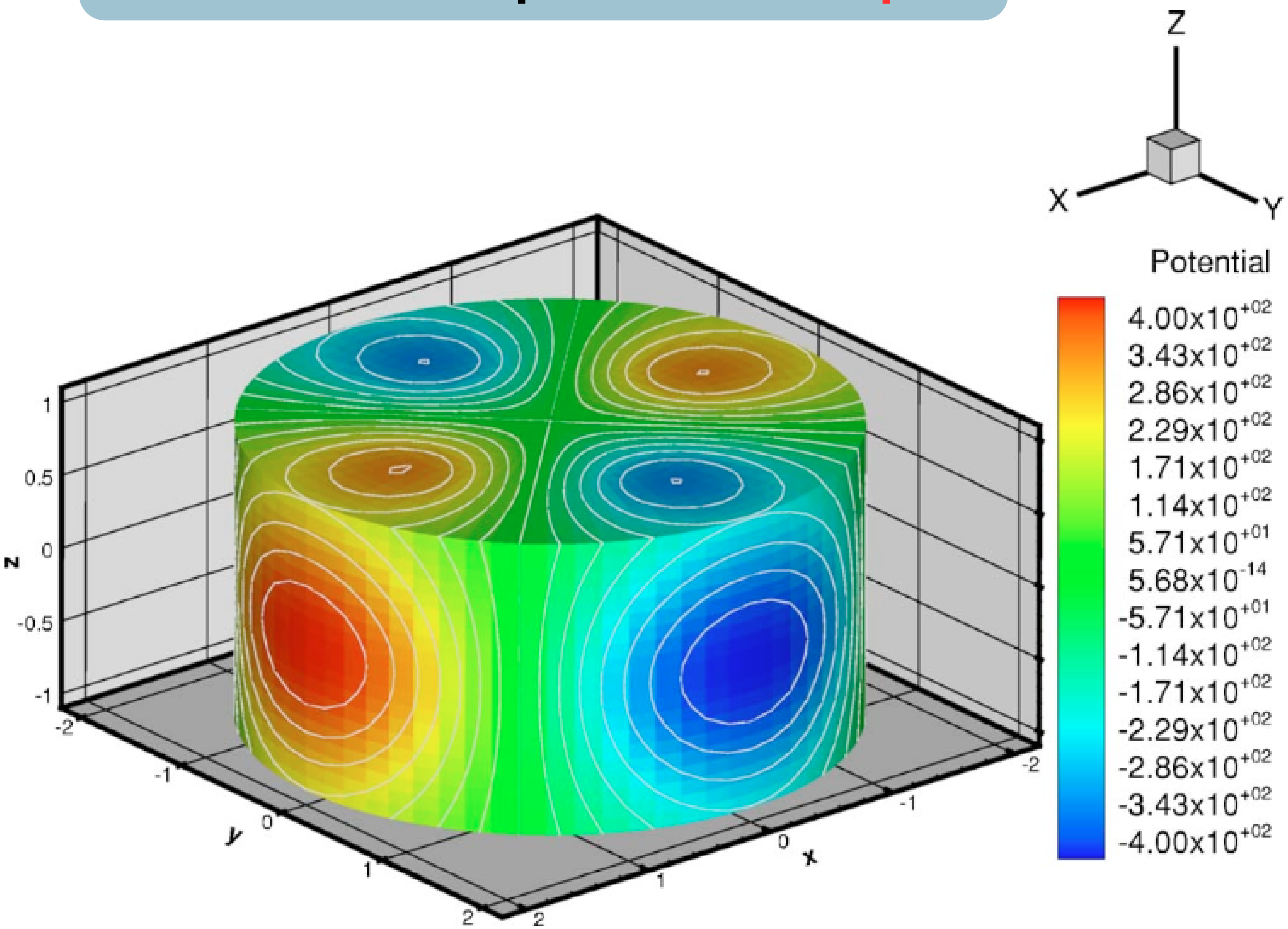


If **p=1**, then its **dipole contribution** dominates the total scalar potential in the far field. The reason is that if p=1, the m=1 term is the only term that survives the integration.

The (**Q 1/2**) term includes spherical dipole contribution, significantly contributing to the scalar potential.

$$\Phi_P = \frac{qQ_{1/2}(\beta)}{4\pi^2 \epsilon_0 \sqrt{\rho a}} \cos(\phi).$$

Plots for scalar potential for p=2



If **p=2**, then its **quadrupole contribution** dominates the total scalar potential in the far field. This is because the **m=2** term is the only term that survives the integration.

$$\Phi_P = \frac{qQ_{3/2}(\beta)}{4\pi^2\epsilon_0\sqrt{\rho a}}[2\cos^2(\phi) - 1].$$

Electric Field Produced by a Charged Ring

The electric field components due to a ring of charge with a harmonic charge density, **$\lambda(\Phi') = \lambda_0 \cos(p\Phi')$** with **$p = 1$**, can be calculated from: **$E = -\nabla \Phi p$**

For p = 1, the potential is given as :
$$\Phi_P = \frac{q Q_{1/2}(\beta)}{4\pi^2 \epsilon_0 \sqrt{\rho a}} \cos(\phi).$$

In cylindrical coordinates, Laplace's equation is written:
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

$$E_\rho = -\frac{\partial \Phi}{\partial \rho}$$

$$E_z = -\frac{\partial \Phi}{\partial z}$$

$$E_\phi = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi}$$

While taking the gradient of the Potential for the calculating the Electric field components, we need to compute the derivative of the Q-function.

$$Q_{m-1/2}(\beta) = \frac{\pi}{(2\beta)^{m+1/2} 2^m} \sum_{n=0}^{\infty} \frac{(4n+2m-1)!!}{2^{2n}(n+m)!\,n!} \frac{1}{(2\beta)^{2n}}.$$

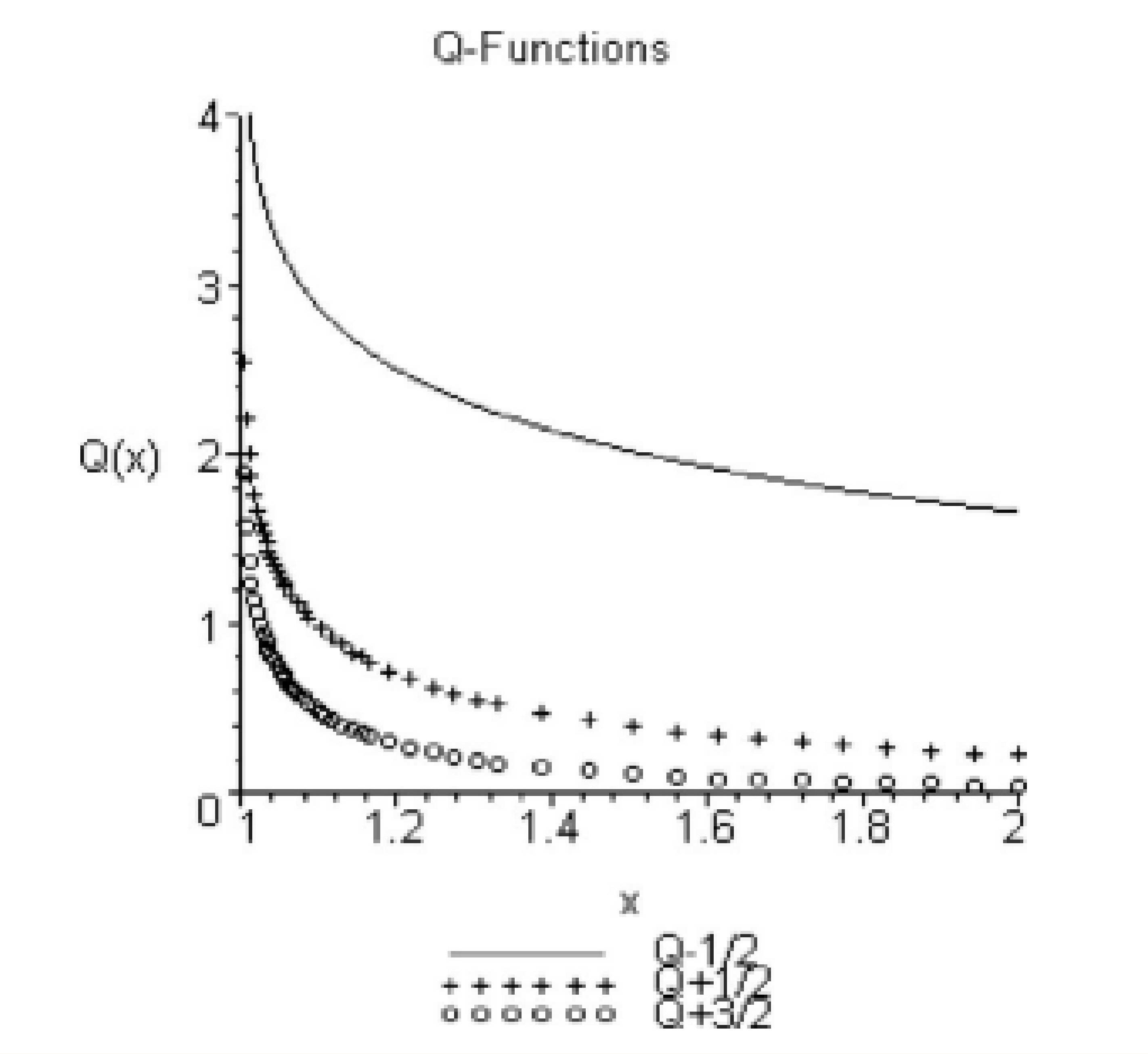
On taking the derivative of Q-function with respect to β, for m = 1, we again get similar series terms as the Q-function. This series can be written in terms of Q-function for m =0 and m = 1.

We get the equation for the derivative as :

$$\frac{dQ_{1/2}(\lambda)}{d\lambda} = \frac{\lambda Q_{1/2}(\lambda) - Q_{-1/2}(\lambda)}{2(\lambda^2 - 1)}.$$

The plots for Q-function and the derivative of Q-function are shown below :

(for m = 0, m = 1 and m = 2)



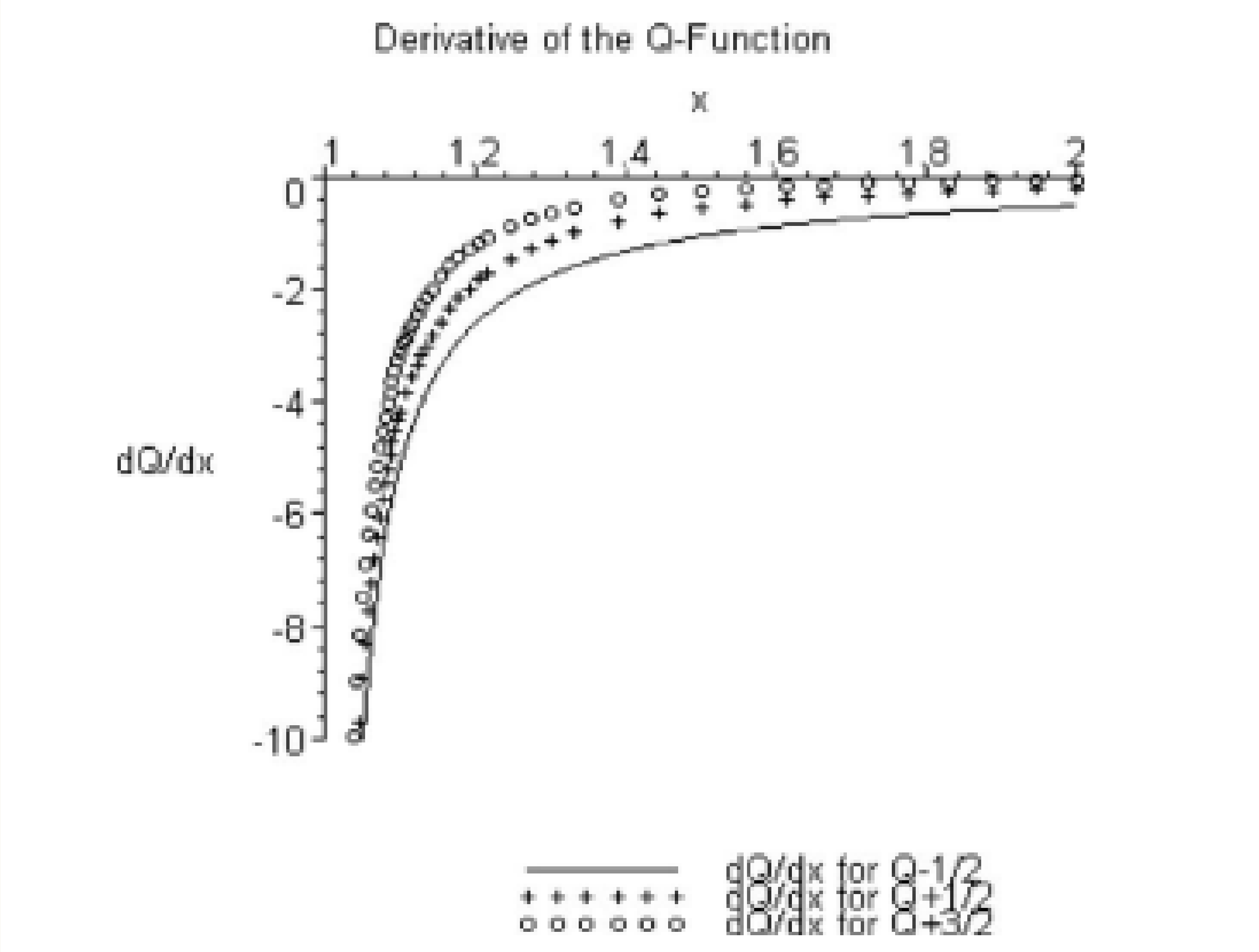


Fig. 6. A few Q-function derivatives.

On solving the equation for derivative of the Potential and using the previous expression for derivative of Q-function, we get the components of the Electric field as:

$$E_\rho = \frac{q\{g_1 Q_{-1/2}(\beta) + g_2 Q_{1/2}(\beta)\}}{4\pi^2 \epsilon_0 \rho \sqrt{\rho a D}} \cos(\phi)$$

$g1 = \rho a(\rho^2 - a^2 - z^2)$

$g2 = (-\rho^2 a^2 + \rho^2 z^2 + a^4 + 2a^2 z^2 + z^4)$

$$E_z = \frac{q\{g_3 Q_{-1/2}(\beta) + g_4 Q_{1/2}(\beta)\}}{4\pi^2 \epsilon_0 \sqrt{\rho a D}} \cos(\phi)$$

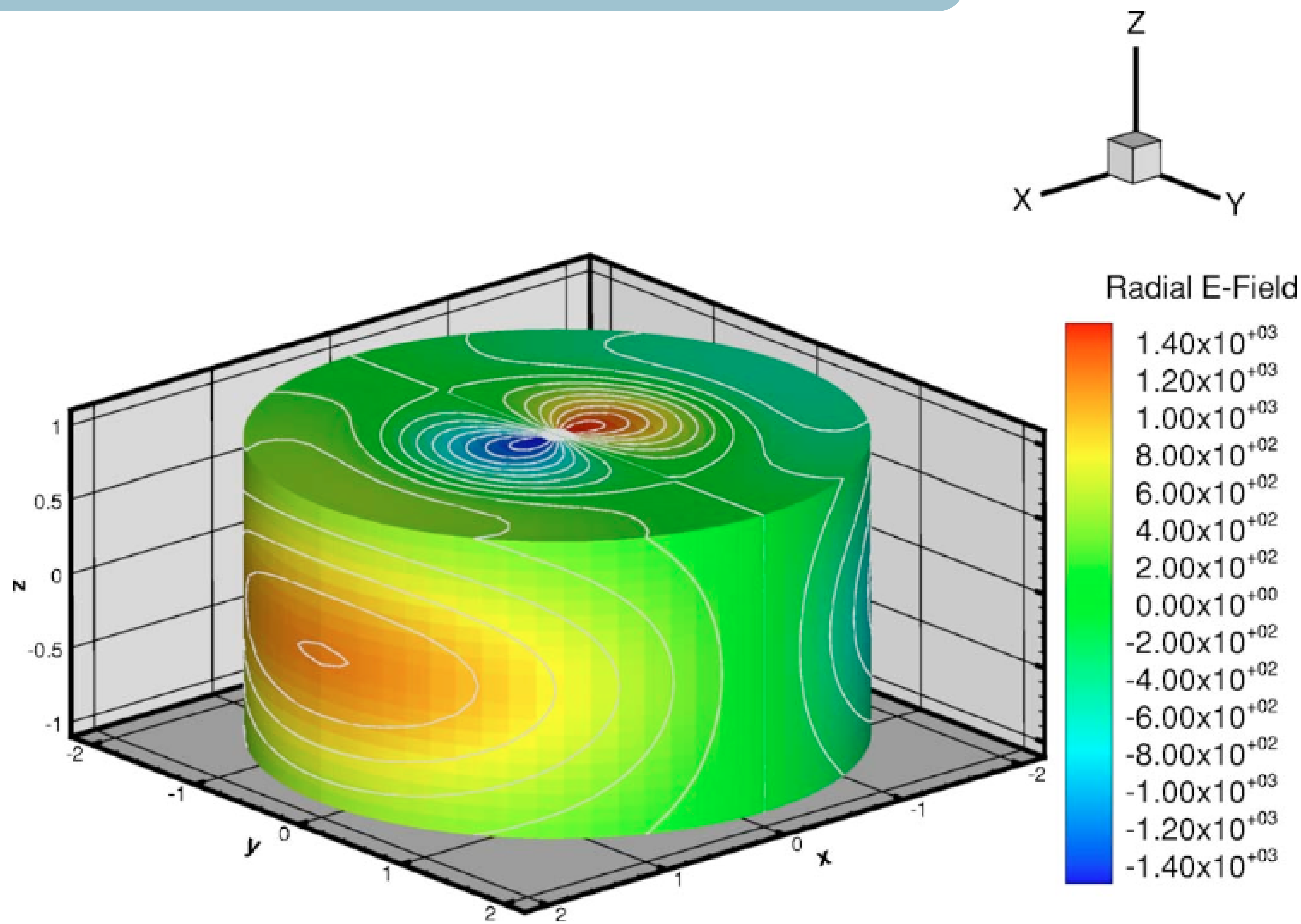
$g3 = 2\rho z a$

$g4 = -z(\rho^2 + a^2 + z^2)$

$$E_\phi = \frac{q Q_{1/2}(\beta)}{4\pi^2 \epsilon_0 \rho \sqrt{\rho a}} \sin(\phi)$$

$D = (\rho^2 + a^2 - 2\rho a + z^2) * (\rho^2 + a^2 + 2\rho a + z^2)$

The radial component of E-field for p=1



Discussion & Conclusion

- **Spherical Harmonic Analysis**

Spherical harmonic analysis is a mathematical method utilized to analyze functions on the surface of a sphere. It involves decomposing a function into a series of spherical harmonics, which are solutions to Laplace's equation in spherical coordinates.

In the context of calculating the electric field from a charged ring, spherical harmonic analysis requires two solutions—one valid for observation points inside the ring's radius ($R<a$) and one for points outside ($R>a$). However, these solutions suffer from slow convergence, particularly near the source, making them less effective for accurately determining near-field solutions.

- **Toroidal Expansion**

Toroidal expansion is another mathematical approach used for the same purpose. It involves expanding the electric field in terms of toroidal functions, which are solutions to Laplace's equation in toroidal coordinates.

Unlike spherical harmonics, toroidal functions exhibit faster convergence and a monotonic nature. This makes them more efficient and accurate for modeling electric fields inside or outside the ring's radius, requiring only one series solution valid for all observation points inside or outside the ring’s radius.

Consequently, toroidal expansion is often preferred over spherical harmonic analysis for calculating electric fields from charged rings due to its improved convergence and simplicity.

Toroidal functions have several properties :

1. **Monotonicity:** Toroidal functions are typically monotonic, meaning they continuously increase or decrease over their domain. This property makes them suitable for efficient numerical calculations and ensures stable convergence in series representations.
2. **Convergence**: Toroidal functions often exhibit rapid convergence, particularly in comparison to spherical harmonics. This property is advantageous for numerical methods and allows for accurate approximation of physical phenomena.
3. **Orthogonality**: Toroidal functions satisfy orthogonality relations, meaning that the integral of the product of two different toroidal functions over the toroidal domain is zero. This property simplifies the analysis of systems involving toroidal geometries and facilitates the decomposition of complex functions into simpler components.