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Electromagnetic Stress Tensor in Ponderable Media

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ABSTRACT: In this project we review the intricacies of defining the Electromagnetic Stress Tensor in ponderable medium, starting from a well motivated force expression we derive the Stress Energy tensor and discuss some applications through examples

Contents

1 Electromagnetic Stress Tensor in free space : Maxwell's Stress Tensor

First consider the conservation of energy, for a single charge q the rate of doing work by external electromagnetic fields E and B is $qv \cdot E$, where v is the velocity of the charge.The magnetic field does no work, since the magnetic force is perpendicular to the velocity, for a continuous distribution this becomes :

$$\int_V J \cdot E d^3x$$

This power represents a conversion of electromagnetic energy into mechanical or thermal energy.It must be balanced by a corresponding rate of decrease of electromagnetic energy within the volume V, this can be seen as using Maxwell's equations :

$$\int_V J \cdot E d^3x = \int_V \left[E \cdot (\nabla \times H) - E \cdot \frac{\partial D}{\partial t} \right] d^3x$$

Using the vector identity $\nabla \cdot (E \times H) = H \cdot (\nabla \times E) - E \cdot (\nabla \times H)$, we obtain :

$$\int_V J \cdot E d^3x = -\int_V \left[\nabla \cdot (E \times H) + E \cdot \frac{\partial D}{\partial t} + H \cdot \frac{\partial B}{\partial t} \right] d^3x$$

Now, to proceed further we make two assumptions :

- The macroscopic medium is linear in its electric and magnetic properties, with negligible dispersion and losses.
- The sum $u = \frac{1}{2}(E \cdot D + B \cdot H)$ represents the total electromagnetic energy density, even for time-varying fields.

Using the above two we can write :

$$-\int_V J \cdot E d^3x = \int_V \left(\frac{\partial u}{\partial t} + \nabla \cdot (E \times H) \right) d^3x$$

Since the volume we are considering is arbitrary, we can obtain a differential form of the above as :

$$\frac{\partial u}{\partial t} + \nabla \cdot S = -J \cdot E \tag{1.1}$$

Where the vector S is known as the Poynting Vector, it is given by :

$$S = E \times H \tag{1.2}$$

The physical meaning of the integral or differential form is that the time rate of change of electromagnetic energy within a certain volume, plus the energy flowing out through the boundary surfaces of the volume per unit time, si equal to the negative of the total work done by the fields on the sources within the volume. This is the statement of conservation of energy.

The emphasis so far has been on the energy of the electromagnetic fields. The work done per unit time per unit volume by the fields $J \cdot E$ is a conversion of electromagnetic energy into mechanical or heat energy. Since matter is ultimately composed of charged particles (electrons and atomic nuclei).

We can think of this rate of conversion of energy as the rate of increase in the energy of the matter or charged particles per unit volume. So, Poynting's Theorem for microscopic E and B fields is a statement of energy conservation of the combined system of charged particles and fields. Now, let's take the total energy of the charged particle in the volume V to be E_{mech} and assume that no particles are going out of this volume. Then, we have

$$\frac{dE_{mech}}{dt} = \int_V J \cdot E d^3x \tag{1.3}$$

For the combined system, using Poynting's Theorem, we can write,

$$\frac{dE}{dt} = \frac{d}{dt}(E_{mech} + E_{field}) = -\oint_S n \cdot S da \tag{1.4}$$

where

$$E_{field} = \int_V u d^3x = \frac{\epsilon_0}{2}\int_V (E^2 + c^2B^2) d^3x \tag{1.5}$$

2 Conservation of Momentum of a System of Charged Particles and Electromagnetic Field

Now, we are going to consider the conservation of momentum in a very similar way.
The total electromagnetic force on a charged particle can be given by the Lorentz force as

$$F = q(E + v \times B) \tag{2.1}$$

If the total momentum of the particles in volume V is P_{mech}, then the total force on all the particles in the volume V can be given as,

$$\frac{dP_{mech}}{dt} = \int_V (\rho E + J \times B) d^3x \tag{2.2}$$

where we have converted the sum over all the charged particles to an integral over all the charge and current densities ($J = \rho v$). Using Maxwell's equations, we can write

$$\rho = \epsilon_0 \nabla \cdot E \qquad and \qquad J = \frac{1}{\mu_0}\nabla \times B - \epsilon_0 \frac{dE}{dt} \tag{2.3}$$

Then,

$$\rho E + J \times B = \epsilon_0 [E(\nabla \cdot E) + B \times \frac{\partial E}{\partial t} - c^2 B \times (\nabla \times B)] \tag{2.4}$$

We can write

$$
\begin{aligned}
B \times \frac{\partial E}{\partial t} &= -\frac{\partial}{\partial t}(E \times B) \; + \; E \times \frac{\partial B}{\partial t} \\
&= -\frac{\partial}{\partial t}(E \times B) \; - \; E \times (\nabla \times E)
\end{aligned}
\tag{2.5}
$$

Now we will substitute for $B \times \frac{\partial E}{\partial t}$ and add $c^2 B(\nabla \cdot B) = 0$ to the square bracket to get

$$
\rho E + J \times B \; = \; \epsilon_0 [E(\nabla \cdot E) \; + \; c^2 B(\nabla \cdot B) \; - \; E \times (\nabla \times E) \; - \; c^2 B \times (\nabla \times B)] \; - \; \epsilon_0 \frac{\partial}{\partial t}(E \times B)
\tag{2.6}
$$

The rate of change of mechanical momentum can now be written as

$$
\begin{aligned}
\frac{dP_{mech}}{dt} \; + \; \frac{\partial}{\partial t} \int_V \epsilon_0 (E \times B) d^3x \; &= \; \epsilon_0 \int_V [E(\nabla \cdot E) \; - \; E \times (\nabla \times E) \; + \\
& \qquad c^2 B(\nabla \cdot B) \; - \; c^2 B \times (\nabla \times B)] d^3x
\end{aligned}
\tag{2.7}
$$

We can tentatively identify the integral on the LHS as the total electromagnetic momentum P_{field} in the volume V,

$$
P_{field} \; = \; \epsilon_0 \int_V (E \times B) d^3x \; = \; \mu_0 \epsilon_0 \int_V (E \times H) d^3x
\tag{2.8}
$$

The integrand can be represented as the density of electromagnetic momentum. This density is proportional to the energy-flux density S, the Poynting's vector.

We want to complete the identification of the volume integral of

$$
g \; = \; \frac{1}{c^2}(E \times H)
\tag{2.9}
$$

as electromagnetic momentum and establish equation with the rates of change of mechanical and field momentum as the conservation law for momentum. To do this, we need to convert the volume integral on the right into a surface integral of the normal component of some quantity that can be identified as momentum flow through the surface of the volume V. We denote the Cartesian co-ordinates by x_α, $\alpha = 1, 2, 3$. The $\alpha = 1$ component of the electric part of the integrand can be given by

$$
\begin{aligned}
[E(\nabla \cdot E) \; - \; E \times (\nabla \times E)]_1 = \; & E_1 \left(\frac{\partial E_1}{\partial x_1} \; + \; \frac{\partial E_2}{\partial x_2} \; + \; \frac{\partial E_3}{\partial x_3} \right) \; - \; E_2 \left(\nabla \times E \right)_3 \\
& + \; E_3 \left(\nabla \times E \right)_2
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
= \; & E_1 \left(\frac{\partial E_1}{\partial x_1} \; + \; \frac{\partial E_2}{\partial x_2} \; + \; \frac{\partial E_3}{\partial x_3} \right) - E_2 \left(\frac{\partial E_2}{\partial x_1} \; - \; \frac{\partial E_1}{\partial x_2} \right) \\
& + \; E_3 \left(\frac{\partial E_1}{\partial x_3} \; - \; \frac{\partial E_3}{\partial x_1} \right)
\end{aligned}
\tag{2.11}
$$

$$= \frac{\partial}{\partial x_1}(E_1^2) \; + \; \frac{\partial}{\partial x_2}(E_1 E_2) \; + \; \frac{\partial}{\partial x_3}(E_1 E_3) \\ - \; \frac{1}{2}\frac{\partial}{\partial x_1}(E_1^2 \; + \; E_2^2 \; + \; E_3^2) \quad (2.12)$$

By careful analysis of the above expression, we can write the αth component as

$$[E(\nabla \cdot E) \; - \; E \times (\nabla \times E)]_\alpha \; = \; \sum_\beta \frac{\partial}{\partial x_\beta}(E_\alpha E_\beta - \frac{1}{2} E \cdot E \delta_{\alpha\beta}) \qquad (2.13)$$

The divergence of a vector K is given as

$$\nabla \cdot K = \sum_\alpha \frac{\partial K_\alpha}{\partial x_\alpha} \qquad (2.14)$$

i.e., partial differentiation of αth component w.r.t. x_α and summation over α index which gives a scalar. In our case, we have a quantity with two indices α and β on RHS, which for a particular α, on partial differentiation w.r.t. x_β and summation over β index gives the αth component of a vector. The divergence of a second-rank tensor gives a vector. The terms for B are similar, so we have the divergence of a second-rank tensor, the Maxwell stress tensor $T_{\alpha\beta}$ on RHS, which by definition is given as

$$T_{\alpha\beta} \; = \; \epsilon_0[E_\alpha E_\beta \; + \; c^2 B_\alpha \; B_\beta \; - \; \frac{1}{2}(E \cdot E \; + \; c^2 B \cdot B)\delta_{\alpha\beta}] \qquad (2.15)$$

In component form, we can write

$$\frac{d}{dt}\,(P_{mech} \; + \; P_{field})_\alpha \; = \; \sum_\beta \int_V \frac{\partial}{\partial x_\beta} T_{\alpha\beta} d^3x \qquad (2.16)$$

Note that the divergence theorem for a vector K is given as

$$\int_V \nabla \cdot K d^3x = \oint_S K \cdot n da \qquad (2.17)$$

$$\int_V \sum_\beta \frac{\partial K_\beta}{\partial x_\beta} d^3x = \oint_S \sum_\beta K_\beta n_\beta da \qquad (2.18)$$

Now we apply the divergence theorem for αth component to get

$$\frac{d}{dt}\,(P_{mech} \; + \; P_{field})_\alpha \; = \; \oint_S \sum_\beta T_{\alpha\beta}\, n_\beta \, da \qquad (2.19)$$

where n is the outward normal to the closed surface S.

The equation just written above is a statement of momentum conservation. $\sum_\beta T_{\alpha\beta}\, n_\beta$ is the αth component per unit area of momentum flowing across the surface S into the volume V. In other words, we can also say that it is the force per unit area transmitted across the surface S and acting on the combined system of particles and fields inside the volume V.

3 Force on a ponderable medium in presence of Electromagnetic fields

[1]

The magnitude of the momentum of light in dielectric media has been the subject of debate and controversy for the past hundred years. There have been several arguments, from theory and experiment, as to why the photon momentum inside a dielectric material should or should not be expressed by either of the two competing formulas associated with the names of H. Minkowski and M. Abraham.

Treating the magnetization density $\boldsymbol{M}$ of a material medium as an Amperian current loop , we review the process to arrive at a specific expression for the force exerted by the electromagnetic field on $\boldsymbol{M}$. Our belief in the validity of this expression stems from our analysis of radiation pressure on semi-infinite slabs, the results of which turn out to be in complete agreement with the momentum conservation law.

Computing the total electromagnetic force on a rigid body requires, in addition to the force exerted on $\boldsymbol{M}$, the Lorentz force of the electromagnetic field on induced electrical charges and currents. The force density on induced currents is $\boldsymbol{F} = (\partial \boldsymbol{P}/\partial t) \times \boldsymbol{B}$, where $\boldsymbol{P}$ is the polarization density of the medium, and $\boldsymbol{B} = \mu_o(\boldsymbol{H} + \boldsymbol{M})$ is the magnetic induction. As for the induced (bound) charge density $\rho_b = -\nabla \cdot \boldsymbol{P}$, the corresponding force density may be written as $\boldsymbol{F} = -(\nabla \cdot \boldsymbol{P})\boldsymbol{E}$. For the LIH media discussed in the present paper, $\nabla \cdot \boldsymbol{P} = \varepsilon_0(\varepsilon - 1)\nabla \cdot \boldsymbol{E}$ vanishes everywhere within the bulk of the medium, thus confining the force of the E-field to surfaces and interfaces (where the induced charge density can be non-zero). An alternative formula for the E-field's contribution to the Lorentz force density, $\boldsymbol{F} = (\boldsymbol{P} \cdot \boldsymbol{\nabla})\boldsymbol{E}$, has been used extensively in the literature. The two formulations can be shown to yield the same total force (and torque) on rigid bodies , even though the force distribution obtained with $\boldsymbol{F} = -(\boldsymbol{\nabla} \cdot \boldsymbol{P})\boldsymbol{E}$ can differ substantially from that obtained using $\boldsymbol{F} = (\boldsymbol{P} \cdot \boldsymbol{\nabla})\boldsymbol{E}$. In what follows, whenever the total force exerted by the E-field happens to be non-zero, we will present two sets of results, one for each formulation.

2. Lorentz force of the electromagnetic field on the magnetization of a medium

The magnetization $\boldsymbol{M}(x, y, z, t) = M_x\hat{\boldsymbol{x}} + M_y\hat{\boldsymbol{y}} + M_z\hat{z}$ of a material at a given point in space and time is subject to the Lorentz force of the local magnetic field. The magnetic induction $\boldsymbol{B}$ exerts a force on electric currents, and since $\boldsymbol{M}$ is ultimately rooted in Amperian current loops on the atomic scale , it is natural to express the force of the B-field on $\boldsymbol{M}$ as the sum of contributions from all the various atomic currents that make up the M-field. A current I circulating around a small loop of area δ^2 produces a magnetic dipole moment $\boldsymbol{m} = I\delta^2\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to the loop's surface. Denoting the number density of the loops in the medium by N, we will have $\boldsymbol{M} = N\boldsymbol{m}$. Equivalently, one may assign a magnetic dipole moment $\boldsymbol{m} = \boldsymbol{M}\delta^3$ to each cubic region of volume δ^3; the three loop currents depicted in Fig. 1 will then be $M_x\delta, M_y\delta$, and $M_z\delta$, respectively.

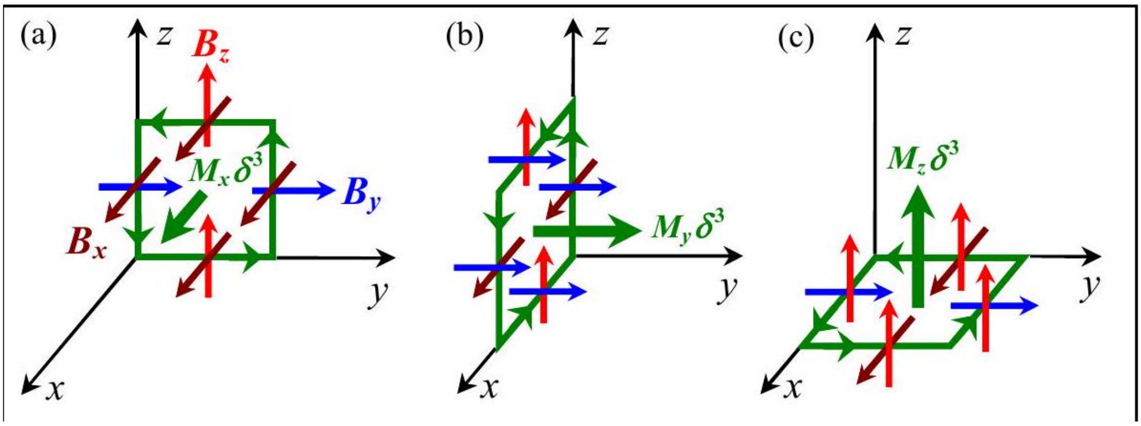


Fig. 1. The local magnetization $\boldsymbol{M} = M_x\hat{\boldsymbol{x}} + M_y\hat{\boldsymbol{y}} + M_z\hat{z}$ of the material is subject to various local B-field components: B_x (brown), B_y (blue), and B_z (red). A circulating current I around a loop of area δ^2 (green squares) produces a magnetic dipole $\boldsymbol{m} = I\delta^2\hat{\boldsymbol{n}}$ along the perpendicular unit vector $\hat{\boldsymbol{n}}$. The magnetization density is $\boldsymbol{M} = N\boldsymbol{m}$, where N is the number density of the loops.

Figure 1 shows three current loops representing the Cartesian components of $\boldsymbol{M}$ (green arrows) as well as the relevant components of $\boldsymbol{B}$ (brown, blue, and red arrows). According to the Lorentz law, the electromagnetic force on each leg of each loop is produced by the action of the local B-field. The various components of the force density (i.e., force per unit volume) for the loops of Fig. 1 may thus be written as follows:

$$\boldsymbol{F}_a = M_x\,(\partial B_x/\partial y)\,\hat{\boldsymbol{y}} + M_x\,(\partial B_x/\partial z)\,\hat{\boldsymbol{z}} - M_x\,(\partial B_y/\partial y)\,\hat{\boldsymbol{x}} - M_x\,(\partial B_z/\partial z)\,\hat{\boldsymbol{x}}, \tag{1a}$$

$$\boldsymbol{F}_b = M_y\,(\partial B_y/\partial x)\,\hat{\boldsymbol{x}} + M_y\,(\partial B_y/\partial z)\,\hat{\boldsymbol{z}} - M_y\,(\partial B_x/\partial x)\,\hat{\boldsymbol{y}} - M_y\,(\partial B_z/\partial z)\,\hat{\boldsymbol{y}}, \tag{1b}$$

$$\boldsymbol{F}_c = M_z\,(\partial B_z/\partial x)\,\hat{\boldsymbol{x}} + M_z\,(\partial B_z/\partial y)\,\hat{\boldsymbol{y}} - M_z\,(\partial B_x/\partial x)\,\hat{\boldsymbol{z}} - M_z\,(\partial B_y/\partial y)\,\hat{\boldsymbol{z}}. \tag{1c}$$

Adding the above forces together we find, after standard algebraic manipulations,

$$\boldsymbol{F}_m(x,y,z,t) = \boldsymbol{M}\times(\nabla\times\boldsymbol{B}) + (\boldsymbol{M}\cdot\nabla)\boldsymbol{B} - (\nabla\cdot\boldsymbol{B})\boldsymbol{M}. \tag{2}$$

The last term in Eq. (2) may be set to zero in accordance with Maxwell's equation $\nabla\cdot\boldsymbol{B} = 0$. As for the remaining terms, we note that the defining relation $\boldsymbol{B} = \mu_o(\boldsymbol{H}+\boldsymbol{M})$ indicates that a certain fraction of the B-field is produced by the local magnetization $\boldsymbol{M}$. If we exclude this part of $\boldsymbol{B}$ from exerting a force on its own progenitor, we are left with $\mu_0\boldsymbol{H}$ as the effective field that exerts a force on the current loops. We thus have

$$\boldsymbol{F}_m(x,y,z,t) = \mu_0[\boldsymbol{M}\times(\nabla\times\boldsymbol{H}) + (\boldsymbol{M}\cdot\nabla)\boldsymbol{H}]. \tag{3a}$$

Equation (3a) is our basic formula under "steady-state" conditions (i.e., in the absence of transient events) for the Lorentz force density on the magnetization $\boldsymbol{M}$ of magnetic (or magnetizable) materials. When integrated over the volume of interest, Eq. (3a) should yield the total force exerted by the H-field on the magnetic dipoles of the material. In

LIH materials where $\boldsymbol{M} = \chi \boldsymbol{H}$ and $\boldsymbol{B} = \mu_0(1+\chi)\boldsymbol{H} = \mu_0\mu\boldsymbol{H}$, the vector identity $\boldsymbol{A} \times (\nabla \times \boldsymbol{A}) + (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{A} = 1/2\nabla(\boldsymbol{A} \cdot \boldsymbol{A})$ further simplifies Eq. (3a) as follows:

$$\boldsymbol{F}_m(x, y, z, t) = 1/2\mu_0(\mu - 1)\nabla(\boldsymbol{H} \cdot \boldsymbol{H}). \tag{3b}$$

In deriving Eq. (3a), no assumptions were made about $\boldsymbol{B}, \boldsymbol{H}$, and $\boldsymbol{M}$ beyond the defining relation $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$ and the Maxwell equation $\nabla \cdot \boldsymbol{B} = 0$. In this formulation there is no natural way to introduce the magnetic charge density, which is usually defined as $\rho_m = -\nabla \cdot \boldsymbol{M}$ and considered analogous to the bound electric charge density $\rho_b = -\nabla \cdot \boldsymbol{P}$. Whereas the electric charges (free or bound) are acted upon by the E-field in accordance with the Lorentz law $\boldsymbol{F} = \rho_b \boldsymbol{E}$, in our formulation there is no corresponding interaction between the magnetic charge density ρ_m and the magnetic fields. Note, however, that when the field components whose derivatives appear in Eq. (3a) happen to be discontinuous at the boundaries and interfaces between adjacent media, one must be careful to account for the forces experienced by the magnetic dipoles at such boundaries.

The above expression for the Lorentz force on magnetization $\boldsymbol{M}$, combined with that pertaining to material polarization $\boldsymbol{P}$ discussed in Sec. 1, yields correct predictions for the radiation pressure in "steady-state" situations. In problems that involve transient events, such as the passage of the leading or trailing edge of a light pulse through a magnetic medium, Eq. (3) must be augmented by an additional term, $\partial\left(\boldsymbol{E} \times \boldsymbol{M}/c^2\right)/\partial t$, to account for the "hidden" or "intrinsic" mechanical momentum produced by the action of the E-field on magnetic dipoles. While proper derivation of this additional force term requires a foray into quantum electrodynamics, . The addition of $\partial\left(\boldsymbol{E} \times \boldsymbol{M}/c^2\right)/\partial t$ to Eq. (3), of course, does not modify the final results of steady-state calculations, as the new term generally vanishes upon time-averaging.

The final equation that emerges from the above discussion of the electromagnetic force exerted on the magnetization $\boldsymbol{M}$ is analogous to that of the Lorentz force experienced by the polarization $\boldsymbol{P}$, namely, $\boldsymbol{F}_e = (\boldsymbol{P} \cdot \nabla)\boldsymbol{E} + (\partial\boldsymbol{P}/\partial t) \times \boldsymbol{B}$, which can be equivalently written as $\boldsymbol{F}_e = (\boldsymbol{P} \cdot \nabla)\boldsymbol{E} + \boldsymbol{P} \times (\nabla \times \boldsymbol{E}) + \partial(\boldsymbol{P} \times \boldsymbol{B})/\partial t$.

2. Stress tensor in ponderable medium

The force expression defined for the Lorentz force density in a linear isotropic medium specified by its μ and ε parameters is given by : [2]

$$\boldsymbol{F}(\boldsymbol{r},t)=(\boldsymbol{P}\cdot\nabla)\boldsymbol{E}+(\boldsymbol{M}\cdot\boldsymbol{\nabla})\boldsymbol{H}+(\partial\boldsymbol{P}/\partial t)\times\mu_0\boldsymbol{H}-(\partial\boldsymbol{M}/\partial t)\times\varepsilon_0\boldsymbol{E}. \tag{1}$$

In conjunction with Eq. (1), Maxwell's equations in the MKSA system of units are:

$$\nabla\cdot\boldsymbol{D}=\rho_{\text{free}}\ , \tag{2a}$$

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}_{\text{free}}\ +\partial\boldsymbol{D}/\partial t, \tag{2b}$$

$$\nabla\times\boldsymbol{E}=-\partial\boldsymbol{B}/\partial t, \tag{2c}$$

$$\nabla\cdot\boldsymbol{B}=0. \tag{2 d}$$

In what follows, the medium will be assumed to have neither free charges nor free currents, that is, $\rho_{\text{free}}\ =0$ and $\boldsymbol{J}_{\text{free}}\ =0$. In the above equations, the electric displacement $\boldsymbol{D}$ and the magnetic induction $\boldsymbol{B}$ are related to the polarization density $\boldsymbol{P}$ and the magnetization density $\boldsymbol{M}$ as follows:

$$\boldsymbol{D}=\varepsilon_0\boldsymbol{E}+\boldsymbol{P}=\varepsilon_0\left(1+\chi_e\right)\boldsymbol{E}=\varepsilon_0\varepsilon\boldsymbol{E}, \tag{3a}$$

$$\boldsymbol{B}=\mu_0\boldsymbol{H}+\boldsymbol{M}=\mu_0\left(1+\chi_m\right)\boldsymbol{H}=\mu_0\mu\boldsymbol{H}. \tag{3b}$$

The first term on the right hand side of Eq. (1) may be rewritten using the identities

$$(\boldsymbol{P}\cdot\nabla)\boldsymbol{E}+(\nabla\cdot\boldsymbol{P})\boldsymbol{E}=\partial\left(P_x\boldsymbol{E}\right)/\partial x+\partial\left(P_y\boldsymbol{E}\right)/\partial y+\partial\left(P_z\boldsymbol{E}\right)/\partial z \tag{4a}$$

$$\nabla\cdot\boldsymbol{P}=\nabla\cdot(\boldsymbol{D}-\varepsilon_0\boldsymbol{E})=\rho_{\text{free}}\ -\varepsilon_{\text{o}}\nabla\cdot\boldsymbol{E}=-\varepsilon_{\text{o}}\nabla\cdot\boldsymbol{E}. \tag{4b}$$

A similar treatment can be applied to the second term in Eq. (1); here $\nabla\cdot\boldsymbol{B}$ is readily set to zero in accordance with Maxwell's 4^{th} equation.

$$(\boldsymbol{M}\cdot\nabla)\boldsymbol{H}+(\nabla\cdot\boldsymbol{M})\boldsymbol{H}=\partial\left(M_x\boldsymbol{H}\right)/\partial x+\partial\left(M_y\boldsymbol{H}\right)/\partial y+\partial\left(M_z\boldsymbol{H}\right)/\partial z. \tag{5a}$$

$$\nabla\cdot\boldsymbol{M}=\nabla\cdot(\boldsymbol{B}-\mu_0\boldsymbol{H})=\nabla\cdot\boldsymbol{B}-\mu_0\nabla\cdot\boldsymbol{H}=-\mu_0\nabla\cdot\boldsymbol{H}. \tag{5b}$$

The third term in Eq. (1) is rewritten by substituting for $\boldsymbol{P}$ in terms of $\boldsymbol{D}$ and $\boldsymbol{E}$, then invoking Maxwell's 2^{nd} equation. Similarly, the fourth term is rewritten by substituting for $\boldsymbol{M}$ in terms of $\boldsymbol{B}$ and $\boldsymbol{H}$, then invoking Maxwell's 3^{rd} equation. We find We find

$$(\partial\boldsymbol{P}/\partial t)\times\mu_0\boldsymbol{H}=(\partial\boldsymbol{D}/\partial t-\varepsilon_0\partial\boldsymbol{E}/\partial t)\times\mu_0\boldsymbol{H}=\mu_0(\nabla\times\boldsymbol{H})\times\boldsymbol{H}-\varepsilon_0\mu_0(\partial\boldsymbol{E}/\partial t)\times\boldsymbol{H}, \tag{6a}$$

$$(\partial\boldsymbol{M}/\partial t)\times\varepsilon_0\boldsymbol{E}=(\partial\boldsymbol{B}/\partial t-\mu_0\partial\boldsymbol{H}/\partial t)\times\varepsilon_0\boldsymbol{E}=-\varepsilon_0(\nabla\times\boldsymbol{E})\times\boldsymbol{E}-\varepsilon_0\mu_0(\partial\boldsymbol{H}/\partial t)\times\boldsymbol{E}. \tag{6b}$$

Substitution from Eqs.(4-6) into Eq.(1), followed by rearranging and combining the various terms yields,

$$
\begin{aligned}
\boldsymbol{F}(\boldsymbol{r},t) =& (\partial/\partial x)\left(P_x\boldsymbol{E}+M_x\boldsymbol{H}\right)+(\partial/\partial y)\left(P_y\boldsymbol{E}+M_y\boldsymbol{H}\right)+(\partial/\partial z)\left(P_z\boldsymbol{E}+M_z\boldsymbol{H}\right)\\
&+\varepsilon_0[(\nabla\cdot\boldsymbol{E})\boldsymbol{E}+(\nabla\times\boldsymbol{E})\times\boldsymbol{E}]+\mu_0[(\nabla\cdot\boldsymbol{H})\boldsymbol{H}+(\nabla\times\boldsymbol{H})\times\boldsymbol{H}]-\varepsilon_0\mu_0\partial(\boldsymbol{E}\times\boldsymbol{H})/\partial t
\end{aligned}
\tag{7}
$$

This equation can be further expanded and rearranged to yield,

$$
\begin{aligned}
\boldsymbol{F}(\boldsymbol{r},t)+(\partial/\partial t)\left(\boldsymbol{E}\times\boldsymbol{H}/c^2\right) =& (\partial/\partial x)\left\{P_x\boldsymbol{E}+M_x\boldsymbol{H}+\varepsilon_0\left[1/2\left(E_x^2-E_y^2-E_z^2\right)\hat{\boldsymbol{x}}+E_xE_y\hat{\boldsymbol{y}}+E_xE_z\hat{\boldsymbol{z}}\right]\right.\\
&\left.+\mu_0\left[1/2\left(H_x^2-H_y^2-H_z^2\right)\hat{\boldsymbol{x}}+H_xH_y\hat{\boldsymbol{y}}+H_xH_z\hat{z}\right]\right\}\\
&+(\partial/\partial y)\left\{P_y\boldsymbol{E}+M_y\boldsymbol{H}+\varepsilon_0\left[E_xE_y\hat{\boldsymbol{x}}+1/2\left(E_y^2-E_x^2-E_z^2\right)\hat{\boldsymbol{y}}+E_yE_z\hat{z}\right]\right.\\
&\left.+\mu_o\left[H_xH_y\hat{\boldsymbol{x}}+1/2\left(H_y^2-H_x^2-H_z^2\right)\hat{\boldsymbol{y}}+H_yH_z\hat{z}\right]\right\}\\
&+(\partial/\partial z)\left\{P_z\boldsymbol{E}+M_z\boldsymbol{H}+\varepsilon_0\left[E_xE_z\hat{\boldsymbol{x}}+E_yE_z\hat{\boldsymbol{y}}+1/2\left(E_z^2-E_x^2-E_y^2\right)\hat{\boldsymbol{z}}\right]\right.\\
&\left.+\mu_o\left[H_xH_z\hat{\boldsymbol{x}}+H_yH_z\hat{\boldsymbol{y}}+1/2\left(H_z^2-H_x^2-H_y^2\right)\hat{\boldsymbol{z}}\right]\right\}.
\end{aligned}
$$

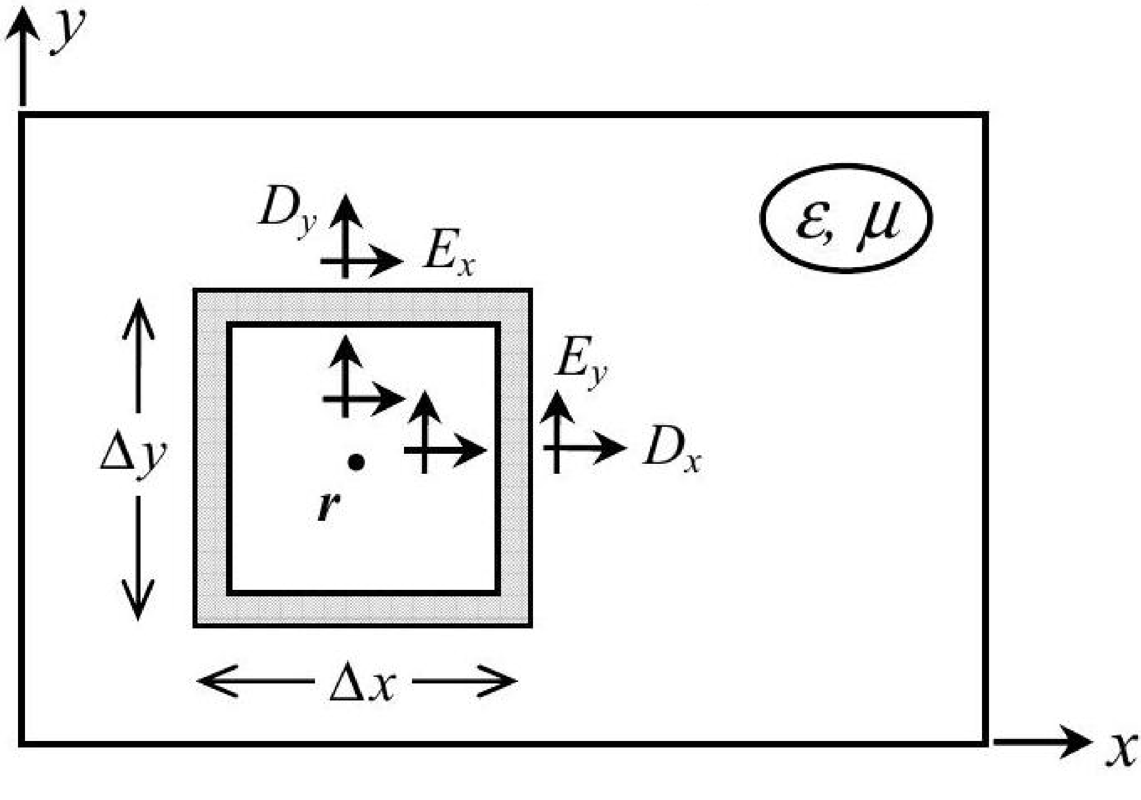


Fig. 1. A small cube of dimensions $\Delta x\times\Delta y\times\Delta z$ within a magnetic dielectric is separated from the surrounding medium by a fictitious vacuum-filled gap; the medium is specified by its (ε,μ) parameters. Assuming the gap is sufficiently narrow (compared to the wavelength of the electromagnetic field), its presence should not affect the distribution

of the fields throughout the medium. Within the gap, however, the various components of the electromagnetic field are determined by the standard boundary conditions derived from Maxwell's equations. In general, the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$ remain continuous across the gap, while, in the perpendicular direction, the components of $\boldsymbol{D}$ and $\boldsymbol{B}$ retain continuity.

Next, we integrate Eq. (8) over the small $\Delta x \times \Delta y \times \Delta z$ cube depicted in Fig. 1, normalize the resultant by the cube's volume, and consider the limit when $(\Delta x, \Delta y, \Delta z) \rightarrow 0$. The lefthand side of Eq. (8) thus remains intact, but several changes occur on the right-hand side. For instance, in the first term, integration over x yields the argument of $\partial / \partial x$, evaluated in the gaps on the left- and right-hand sides of the cube, then subtracted from each other. In these gaps, $P_x=0, M_x=0, \varepsilon_0 E_x=D_x$, and $\mu_0 H_x=B_x$, while the remaining components of $\boldsymbol{E}$ and $\boldsymbol{H}$ retain the values that they have in the adjacent material environment. (These gap fields are found by invoking standard boundary conditions, namely, the continuity of tangential $\boldsymbol{E}$ and $\boldsymbol{H}$, as well as perpendicular $\boldsymbol{D}$ and $\boldsymbol{B}$ components.) Similar arguments apply to the second and third terms on the right-hand side of Eq. (8), provided that, in the case of the 2^{nd} $\left(3^{\text{rd}}\right)$ term, the initial integration is carried over $y(z)$. When the integrals are fully evaluated and the result is normalized by the volume of the cube, we find, in the limit of a vanishing cube,

$$
\begin{aligned}
\boldsymbol{F}(\boldsymbol{r}, t)+\frac{\partial}{\partial t}\left(\boldsymbol{E} \times \boldsymbol{H} / c^2\right)= & \frac{\partial}{\partial x}\left(\left[1/2\left(\varepsilon_0^{-1} D_x^2-\varepsilon_0 E_y^2-\varepsilon_0 E_z^2\right) \hat{\boldsymbol{x}}+D_x E_y \hat{\boldsymbol{y}}+D_x E_z \hat{\boldsymbol{z}}\right]\right. \\
& \left.+\left[1/2\left(\mu_{\mathrm{o}}^{-1} B_x^2-\mu_0 H_y^2-\mu_0 H_z^2\right) \hat{\boldsymbol{x}}+B_x H_y \hat{\boldsymbol{y}}+B_x H_z \hat{\boldsymbol{z}}\right]\right) \\
& +\frac{\partial}{\partial y}\left(\left[E_x D_y \hat{\boldsymbol{x}}+1/2\left(\varepsilon_0^{-1} D_y^2-\varepsilon_0 E_x^2-\varepsilon_0 E_z^2\right) \hat{\boldsymbol{y}}+D_y E_z \hat{z}\right]\right. \\
& \left.+\left[H_x B_y \hat{\boldsymbol{x}}+1/2\left(\mu_0^{-1} B_y^2-\mu_0 H_x^2-\mu_0 H_z^2\right) \hat{\boldsymbol{y}}+B_y H_z \hat{z}\right]\right) \\
& +\frac{\partial}{\partial z}\left(\left[E_x D_z \hat{\boldsymbol{x}}+E_y D_z \hat{\boldsymbol{y}}+1/2\left(\varepsilon_0^{-1} D_z^2-\varepsilon_0 E_x^2-\varepsilon_0 E_y^2\right) \hat{z}\right]\right. \\
& \left.+\left[H_x B_z \hat{\boldsymbol{x}}+H_y B_z \hat{\boldsymbol{y}}+1/2\left(\mu_0^{-1} B_z^2-\mu_0 H_x^2-\mu_0 H_y^2\right) \hat{z}\right]\right)
\end{aligned} \tag{9}
$$

Equation (9) clearly identifies the Abraham momentum density $\boldsymbol{E} \times \boldsymbol{H} / c^2$ as the electromagnetic momentum density $\boldsymbol{G}(\boldsymbol{r}, t)$, and yields the following stress tensor T_{ij} (i.e., rate of flow of momentum per unit area per unit time) within the medium:

$$T_{xx}=1/2\left(\varepsilon_0 E_y^2+\varepsilon_0 E_z^2-\varepsilon_0^{-1} D_x^2\right)+1/2\left(\mu_0 H_y^2+\mu_0 H_z^2-\mu_0^{-1} B_x^2\right), \tag{10a}$$

$$T_{yx}=-D_x E_y-B_x H_y, \tag{10b}$$

$$T_{zx}=-D_x E_z-B_x H_z, \tag{10c}$$

$$T_{xy}=-E_x D_y-H_x B_y, \tag{10 d}$$

$$T_{yy}=1/2\left(\varepsilon_0 E_x^2+\varepsilon_0 E_z^2-\varepsilon_0^{-1} D_y^2\right)+1/2\left(\mu_0 H_x^2+\mu_0 H_z^2-\mu_{\mathrm{o}}^{-1} B_y^2\right), \tag{10e}$$

$$T_{zy}=-D_y E_z-B_y H_z \tag{10f}$$

$$T_{xz}=-E_x D_z-H_x B_z \tag{10g}$$

$$T_{yz}=-E_y D_z-H_y B_z, \tag{10 h}$$

$$T_{zz}=1/2\left(\varepsilon_0 E_x^2+\varepsilon_0 E_y^2-\varepsilon_0^{-1} D_z^2\right)+1/2\left(\mu_0 H_x^2+\mu_0 H_y^2-\mu_0^{-1} B_z^2\right). \tag{10i}$$

Equation (9) may thus be written as the following streamlined expression of momentum conservation:

$$\nabla \cdot \boldsymbol{T} + \boldsymbol{F}(\boldsymbol{r}, t) + \partial \boldsymbol{G}(\boldsymbol{r}, t)/\partial t = 0 \tag{11}$$

In its specific combination of the various components of the $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}$ and $\boldsymbol{B}$ fields, the stress tensor of Eq. (10) differs from both Abraham and Minkowski tensors

4 Example 1

We have a plane electromagnetic wave that propagates along the z-axis inside a medium with relative permittivity ϵ and relative permeability μ. The E-field of the linearly polarized plane-wave can be given by $E_0\hat{x}$ and H-field can be given as $H_0\hat{y} = \frac{1}{\mu\mu_0}\frac{E_0}{c/\sqrt{\epsilon\mu}}\hat{y} = Z_0^{-1}\sqrt{\epsilon/\mu}E_0\hat{y}$, where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of the free space. We assume our plane wave to be monochromatic and with angular frequency ω, the rate of flow of momentum per unit area per unit time along the z-axis can be given as

$$T_{zz} = \frac{1}{2}\left(\epsilon_0 E_x^2 + \mu_0 H_y^2\right) = \frac{1}{2}\epsilon_0\left(1 + \epsilon/\mu\right)E_0^2 cos^2(\omega t) \tag{4.1}$$

The rate of flow of energy per unit area per unit time can be given by the Poynting vector, $S = E \times H$.

$$S_z = E_x H_y = Z_0^{-1}\sqrt{\epsilon/\mu}E_0^2 cos^2(\omega t) \tag{4.2}$$

If a total of N photons, each having energy hf, cross the xy-plane at $z = 0$ during the time interval $[0, \tau]$, then

$$\begin{aligned}
\langle S_z\rangle\tau &= Nhf && (4.3)\\
\langle Z_0^{-1}\sqrt{\epsilon/\mu}E_0^2 cos^2(\omega t)\rangle &= Nhf && (4.4)\\
E_0^2 &= 2Z_0 Nhf\sqrt{\mu/\epsilon} && (4.5)\\
&= 2\sqrt{\mu_0/\epsilon_0}Nhf\sqrt{\mu/\epsilon} && (4.6)
\end{aligned}$$

The total momentum crossing the same plane during the same time interval can be given as

$$\begin{aligned}
\langle T_{zz}\rangle\tau &= \frac{1}{2}\epsilon_0\left(1 + \epsilon/\mu\right)E_0^2\langle cos^2(\omega t)\rangle && (4.7)\\
&= \frac{1}{2}\epsilon_0\left(1 + \epsilon/\mu\right) \times 2\sqrt{\mu_0/\epsilon_0}Nhf\sqrt{\mu/\epsilon} \times \frac{1}{2} && (4.8)\\
&= \frac{1}{2}N\left(\sqrt{\epsilon/\mu} + \sqrt{\mu/\epsilon}\right)hf/c && (4.9)
\end{aligned}$$

Therefore, the momentum of a single photon is $\frac{1}{2}\left(\sqrt{\epsilon/\mu} + \sqrt{\mu/\epsilon}\right)hf/c$ which agrees to our previous results. In non-magnetic dielectrics, i.e. $\mu = 1$, it will take the form

$\frac{1}{2}(\sqrt{\epsilon}+1/\sqrt{\epsilon})\,hf/c$ which is equal to the arithmetic average of the Minkowski and Abraham momenta, is always greater than the photon momentum in free space, hf/c.

In general, the photon momentum consists of an electromagnetic part and a mechanical part. In a non-dispersive medium where the group velocity of light equals its phase velocity, the electromagnetic momentum of a single photon is $hf/(\sqrt{\epsilon\mu}c)$, and the photon's mechanical momentum will be

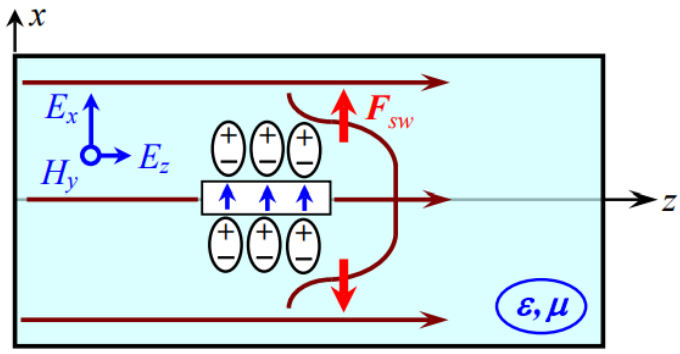
$$P_{mech} = \frac{1}{2}\left(\sqrt{\epsilon/\mu}+\sqrt{\mu/\epsilon}\right)\frac{hf}{c} - \frac{hf}{\sqrt{\epsilon\mu}c} \tag{4.10}$$

$$= \frac{(\epsilon+\mu-2)}{2\sqrt{\epsilon\mu}}\frac{hf}{c} \tag{4.11}$$

5 Example 2

Consider a collimated, monochromatic beam of light propagating along the z-axis in a linear, isotropic, and homogeneous medium characterized by its electric permittivity ε and magnetic permeability μ. The beam is linearly polarized with its electric field (E-field) amplitude E_o along the x-axis and magnetic field (H-field) amplitude H_o along the y-axis. The relationship between the E-field and H-field amplitudes is given by $H_o = Z_o^{-1}\sqrt{\varepsilon/\mu}E_o$, where Z_o is the impedance of free space.

Figure 1.



Additionally, there is a weak E_z component of the electric field that is an odd function of x and goes to zero at the center of the beam. The time-averaged rate of flow of x-momentum along the x-axis at the central yz-plane is given by:

$$\langle T_{xx}\rangle = \langle -\frac{1}{2}\varepsilon_0^{-1}D_x^2 + \frac{1}{2}\mu_0 H_y^2\rangle = \frac{1}{4}\varepsilon_0\left[\left(\frac{\varepsilon}{\mu}\right) - \varepsilon^2\right]E_o^2$$

To prove the given expression for $\langle T_{xx}\rangle$, we start with the expression for T_{xx}:

$$T_{xx} = \frac{1}{2}(\varepsilon_0 E_y^2 + \varepsilon_0 E_z^2 - \varepsilon_0^{-1}D_x^2) + \frac{1}{2}(\mu_0 H_y^2 + \mu_0 H_z^2 - \mu_0^{-1}B_x^2)$$

For a linear, isotropic, and homogeneous medium, the relationships between the electric and magnetic fields and the electric displacement and magnetic induction are given by:

$$D_x = \varepsilon_0 \varepsilon E_x, \quad B_x = \mu_0 \mu H_x$$

Since the beam is linearly polarized along the x-axis, the electric field components E_y and E_z are zero. Additionally, since the beam is collimated along the z-axis, the magnetic field component H_z is also zero and $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$. Therefore, the expression for T_{xx} simplifies to:

$$T_{xx} = -\frac{1}{2}\varepsilon_0^{-1}(\varepsilon_0 \varepsilon E_x)^2 + \frac{1}{2}\mu_0(\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\frac{\varepsilon}{\mu}}E_x)^2$$

Taking the time average of T_{xx}, denoted by $\langle T_{xx} \rangle$, and noting that the time-averaged square of the electric field amplitude is related to the intensity of the beam, we get:

$$\langle T_{xx} \rangle = \langle -\frac{1}{2}\varepsilon_0 \varepsilon^2 E_x^2 + \frac{1}{2}\varepsilon_0 \frac{\varepsilon}{\mu} E_x^2 \rangle$$

Simplifying, we find:

$$\langle T_{xx} \rangle = \langle \frac{1}{2}\varepsilon_0 \left(\frac{\varepsilon}{\mu} - \varepsilon^2 \right) E_x^2 \rangle$$

Since the beam is linearly polarized with its electric field amplitude along the x-axis denoted by E_o, we can replace E_x^2 with E_o^2, yielding:

$$\langle T_{xx} \rangle = \langle \frac{1}{2}\varepsilon_0 \left(\frac{\varepsilon}{\mu} - \varepsilon^2 \right) E_o^2 \rangle$$

Finally, by taking the time average over one cycle of oscillation, the factor of 1/2 becomes 1/4 due to the sinusoidal nature of the fields:

$$\langle T_{xx} \rangle = \frac{1}{4}\varepsilon_0 \left[\left(\frac{\varepsilon}{\mu} \right) - \varepsilon^2 \right] E_o^2$$

This completes the proof.

- **Momentum Flow Conversion:** The momentum flow, characterized by $\langle T_{xx} \rangle$, is entirely converted into two forces:
  - A force acting on the electric dipoles located just above the z-axis.
  - A second force, F_{sw}, exerted on the medium by the upper sidewall of the beam.
- **Force on Dipoles:** To understand the force on the dipoles immediately above the z-axis, a gap is introduced in the middle of the beam. The continuity of the perpendicular component of the electric displacement field ($D_\perp$) at this interface reveals that the electric field within the gap is equal to $\varepsilon E_o \hat{x}$. The average electric field at the interface is thus $\frac{1}{2}(\varepsilon + 1)E_o$, and the field gradient sensed by the interfacial dipole layer is proportional to $\frac{1}{2}(\varepsilon - 1)E_o$.

- **Dipole Density and Force Density:** The dipole density is given by $P = \varepsilon_o(\varepsilon - 1)E_o\hat{x}$. Using this, we find a force density at the interface given by $F_x = \frac{1}{2}\varepsilon_o(\varepsilon-1)^2E_o^2$. Hence, $\langle F_x \rangle = \frac{1}{4}\varepsilon_o(\varepsilon-1)^2E_o^2$.

- **Sidewall Force Density:** Adding this force density to $\langle T_{xx} \rangle$ yields $\langle F_x^{(sw)} \rangle = \frac{1}{4}\varepsilon_o\left[\left(\frac{\varepsilon}{\mu}\right) - 2\varepsilon + 1\right]E_o^2$, which is consistent with the sidewall force density of finite-diameter beams found in the referenced literature.

6 Example 3

Consider a collimated, monochromatic beam of finite width propagating in a homogeneous medium characterized by parameters ϵ (permittivity) and μ (permeability). The beam propagates in the xz-plane, making an angle θ with the z-axis, and has a finite diameter along the x-axis. The beam's footprint on the x-axis is of unit length, thus the beam width is equal to $\cos\theta$.

The electromagnetic field components of the beam are (E_x, E_z, H_y). A narrow gap opened parallel to the x-axis reveals the force exerted on the boundary layer electric dipoles due to the discontinuity in the E_z field.

The effective electric field gradient on the boundary dipole layer is proportional to

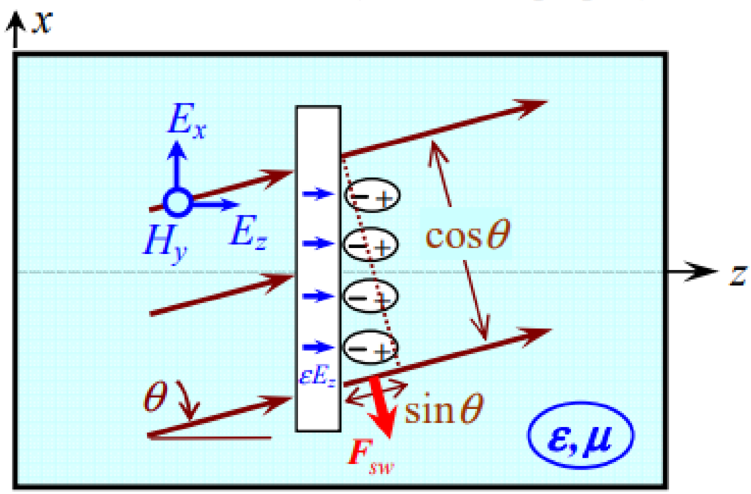
$$\frac{1}{2}\epsilon^{-1}E_0 \sin\theta z,$$

where E_0 is the amplitude of the electric field.

This gradient creates an imbalance between the forces acting at the upper and lower sidewalls of the beam.

Considering the forces on the boundary dipole layer and the imbalance of the sidewall forces, we can express the stress tensor components $T_{xz}\hat{x}+$ and $T_{zz}\hat{z}$, which yield the rate of flow of momentum along the propagation direction.

Figure 2.



6.1 Time-averaged rate of flow of momentum (per unit area per unit time) across the xy-plane

Using the derived stress tensor equations, we have:

$$T_{xz} = -E_x D_z - H_x B_z,$$

$$T_{zz} = \frac{1}{2}(\epsilon_0 E_x^2 + \epsilon_0 E_y^2 - \epsilon_0^{-1} D_z^2) + \frac{1}{2}(\mu_0 H_x^2 + \mu_0 H_y^2 - \mu_0^{-1} B_z^2).$$

Given that $H_x = 0$, $B_z = 0$, and $D_z = \epsilon_0 \epsilon E_z$, we can simplify the expressions:

$$\langle T_{xz}\hat{x} + +T_{zz}\hat{z}\rangle = -\langle E_x D_z\rangle \hat{x} + \frac{1}{2}\langle \epsilon_0 E_x^2 - \epsilon_0^{-1} D_z^2 + \mu_0 H_y^2\rangle \hat{z}$$

Using the relationships $D_z = \epsilon_0 \epsilon E_0$, $E_x = E_0 \cos\theta$, $E_z = E_0 \sin\theta$ and $H_y = \sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\frac{\varepsilon}{\mu}}E_0$,, we find:

$$\langle T_{xz}\hat{x} + +T_{zz}\hat{z}\rangle = \frac{1}{2}\epsilon_0 E_0^2 \left\{ \epsilon \cos\theta \sin\theta \hat{x} + \frac{1}{2}\left[\cos^2\theta - \epsilon^2 \sin^2\theta + \left(\frac{\epsilon}{\mu}\right)\right] \hat{z} \right\}.$$

6.2 Rate of flow of momentum

In our example of plane electromagnetic wave propagating along z-axis. We derived assuming a monochromatic plane wave with angular frequency ω, the rate of flow of momentum (per unit area per unit time) along the z-axis.

Thus, the rate of flow of momentum in a beam of cross-section $\cos\theta$:

$$\frac{d\mathbf{p}}{dt} = \frac{1}{4}\epsilon_0 \left[1 + \left(\frac{\epsilon}{\mu}\right)\right] E_0^2 \cos\theta \left(\sin\theta\hat{x} + \cos\theta\hat{z}\right).$$

6.3 Force on the boundary sidewalls

Momentum flow is entirely converted to a force on the electric dipoles located just above the z-axis and a second force, F_{sw}, exerted on the medium by the upper sidewall of the beam and we derived the sidewall force Thus, the Sidewall force is:

$$\langle F_{sw}\rangle = \frac{1}{4}\epsilon_o \left[\left(\frac{\epsilon}{\mu}\right)^{-2} \epsilon - 2\epsilon + 1\right] E_o^2 \sin\theta(-\cos\theta\hat{x} + \sin\theta\hat{z})$$

6.4 Force on the boundary electric dipoles

The strength of the dipole layer is given by $\mathbf{P} = \epsilon_0(\epsilon - 1)E_0 \sin\theta\hat{z}$, and the effective electric field gradient is proportional to $\frac{1}{2}(\epsilon - 1)E_0 \sin\theta\hat{z}$. Thus, the effective force on the dipole layer is:

$$\langle F\rangle = -\frac{1}{4}\epsilon_0(\epsilon - 1)^2 E_0^2 \sin^2\theta\hat{z}.$$

6.5 Combining the forces

By combining the forces acting on the sidewalls and the boundary dipole layer, we arrive at the same expression for the rate of momentum flow as derived earlier using the stress tensor:

$$\langle F\rangle + \langle F_{sw}\rangle + \frac{d\mathbf{p}}{dt} = \frac{1}{2}\epsilon_0 E_0^2 \left\{ \epsilon\cos\theta\sin\theta\hat{x} + \frac{1}{2}\left[\cos^2\theta - \epsilon^2\sin^2\theta + \left(\frac{\epsilon}{\mu}\right)\right]\hat{z} \right\}.$$

Thus, the stress tensor yields the correct rate of momentum flow along the propagation direction of the beam.

References

[1] M. Mansuripur, *Radiation pressure and the linear momentum of the electromagnetic field*, *Opt. Express* **12** (2004) 5375.

[2] M. Mansuripur, *Electromagnetic stress tensor in ponderable media*, *Opt. Express* **16** (2008) 5193.