MA2101 Final Exam Cheatsheet

by Devansh Shah

Useful Mathematical Facts

- $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(2n+1)!} x^{2n+1}$$
$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

Chapter 4

- The set of all eigenvectors associated with λ is a subspace of V, and is called the **eigenspace** associated with λ.
- The eigenvectors and eigenvalues of a transformation on a real vector space need not be real.
- In general, if any collection of eigenvectors have eigenvalues that are all different, then those eigenvectors are all linearly independent.
- The inverse is not true: it's possible to have a set of linearly independent vectors with the same eigenvalue.
- The number of distinct eigenvalues for an operator on a finite dimensional vector space cannot be larger than dim(V).
- If v is an eigenvector with eigenvalue λ, any scalar multiple of v is also an eigenvector with eigenvalue λ.
- **Fundamental Theorem of Algebra**: Any polynomial with complex coefficients can be completely factorized, i.e., it can be written in the form $a(x-a_1)(x-a_2)\cdots$, where the factorization is unique (up to the ordering).
- Every operator on a finite-dimensional vector space over the complex numbers has at least one eigenvalue, a complex number (which may be real).
- An operator is diagonalizable $\iff$ there exists a basis consisting of eigenvectors $\iff$ every eigenvalue has an eigenspace with dimensionality equal to its multiplicity.

Upper Triangular Matrices

- A matrix is upper-triangular if all entries below the diagonal are zero.
- **Triangularizability**: Let T be an operator on a complex finite-dimensional vector space. Then there always exists a basis t for V such that, for all i, Tt_i is expressible as a linear combination of the t_j with $j \leq i$.
- Let T be any operator on a finite-dimensional vector space V, let z be any basis for V, and let T_z^* be the operator on F^n defined by T and z as $T_z^* = z^{-1} \circ T \circ z$. Then for any z, T_z^* has the same eigenvalues as T.
- So, we can find the eigenvalues of T by studying (*any of*) its matrices.
- Square matrices A and B are said to be **similar** if there exists another (non-singular) matrix P such that $A = P^{-1}BP$.
- Similar matrices have the same eigenvalues. If A satisfies some polynomial equation, B also satisfies that same polynomial equation.
- The matrix of T relative to the special basis t (above) is upper-triangular. So, every matrix is similar to an upper-triangular matrix.
- A complex operator is invertible if, and only if, every entry down the diagonal is non-zero when it is expressed as an upper-triangular matrix. (Note that this says nothing about diagonalisability).
- The diagonal entries of an upper-triangular matrix are its eigenvalues.
- If $dim(V) = n$, and an operator on V has n distinct eigenvalues, then that operator is diagonalizable. But the inverse is not true: it's possible for an operator to have fewer than n distinct eigenvalues, and still be diagonalizable.

Jordan Canonical Form (JCF)

- Every matrix has a Jordan Canonical Form.
- Every Jordan Block (as a matrix itself) has only 1 eigenspace (corresponding to eigenvalue λ) which is one-dimensional. That is, each Jordan Block only contributes one (linearly independent from the rest) eigenvector.
- So, the presence of Jordan Block of size > 1 means you don't have enough eigenvectors to make a basis - and that's why you can't diagonalize it.
- Every operator on a complex vector space has a Jordan basis. This matrix is unique up to the ordering of the basis vectors.
- The **multiplicity** of λ is the number of times the eigenvalue λ appears down the diagonal (in the JCF).
- Let $\lambda_1, \lambda_2, \cdots$ be the eigenvalues of a linear operator T, and let m_i be the multiplicity of eigenvalue λ_i. Then the **characteristic polynomial** of T is the polynomial:
$$\chi_T(x) = (x-\lambda_1)^{m_1} \times (x-\lambda_2)^{m_2} \times \cdots$$
Clearly, for all i, $\chi_T(\lambda_i) = 0$.
- **Cayley-Hamilton Theorem**:
$$\chi_T(T) = 0$$
The transformation (its matrix relative to any basis) satisfies its own characteristic equation.
- If J is a Jordan Block of size m and having eigenvalue λ_i along its diagonal, then $(J - \lambda_i I)^m = 0$, i.e., it will map everything to the zero vector after applying it m times (hence, **nilpotent**).
- We can also use the Cayley-Hamilton theorem to find the inverse of a matrix. For example, if M has 3 distinct eigenvalues 1, 2, 3. Then, the characteristic polynomial is: $(x-1)(x-2)(x-3) = x^2 - 6x^2 + 11x - 6$. By CH, we have, $M^3 - 6M^2 + 11M + 6I = 0$. We can multiply both sides by M^{-1} (if it exists) and get $M^2 - 6M + 11 + 6M^{-1} = 0$ and solve for M^{-1}.

What the JCF Means

- We make use of the facts of complex numbers:
  - $\overline{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$
  - $\overline{c+d} = \overline{c} + \overline{d}$, and $\overline{c \times d} = \overline{c} \times \overline{d}$ where $^-$ represents the conjugate operator
- Also, we know that the eigenvalues of the rotation matrix are $e^{i\theta}$ and $e^{-i\theta}$.
- Then, we can classify all linear transformations using their JCFs as follows:
  - If a linear operator T on R^n has a real, diagonal JCF, then T has a geometric interpretation as either a sequence of stretches or reflections (if negative), or as a crushing (if zero) transformation.
  - If a linear operator T on R^n has a diagonal, not necessarily real, JCF, then T has a geometric interpretation as either a sequence of stretches and reflections (when all eigenvalues are real), or as a crushing transformation (when one of the eigenvalue is zero), or as a sequence of stretches and rotations (when the eigenvalues are complex)
  - If the JCF contains non-trivial Jordan blocks, but still with real numbers down the diagonal, then each Jordan block corresponds to a stretching (in all those dimensions by the same amount λ) and shearing (by the same amount, in each of the (i,i+1) planes).
  - If the JCF contains non-trivial Jordan blocks with complex eigenvalues, it corresponds to a mixture of stretches, rotations, and shears.
- Every linear operator T on R^n has a geometric interpretation as a combination of stretches, reflections, crushings, rotations, and shears.

Chapter 5

Bilinear Forms

- A **bilinear form** on a finite-dimensional vector space V is a mapping b from $V \times V \to \mathbb{R}$ which is linear in both "slots". That is, $\forall u, v, w \in V,\ c, d, \in \mathbb{R}$:
$$b(cu + dv, w) = cb(u, w) + db(v, w)$$
and similarly for the other slot.
- We don't require bilinear forms to be symmetric, i.e., $b(u, v)$ does not have to be equal to $b(v, u)$
- *dot* is a bilinear form from $\mathbb{R}^n \times \mathbb{R}^n \to R$
- The set of bilinear forms on V, $\mathbf{B}(V)$ is a vector space (with natural definitions for addition and scalar multiplication)
- Let $\alpha, \beta \in \hat{V}$, and let $u, v \in V$. Then the **tensor product**:
$$\alpha \otimes \beta(u, v) = \alpha(u)\beta(v)$$
- If z_i is a basis for V, then the full set $\zeta^i \otimes \zeta^j$ is a basis for $\mathbf{B}(V)$. We can obtain the components of any bilinear form $b = b_{ij}\zeta^i \otimes \zeta^j$ by letting b act on (z_i, z_j). That is, $b_{ij} = b(z_i, z_j)$.
- In matrix notation, b_{ij} is the entry in the ith row, and jth column.
- Every bilinear form b defines a linear map from $V \to \hat{V}$ as:
$$b^{\#}(v)(u) \equiv b(u, v)$$
and vice versa. So, in this way of thinking, bilinear forms map $V \to \hat{V}$.
- Under a change of basis (with matrix P), we have the following:
$$v \to P^{-1}v;\ \alpha \to \alpha P;\ T \to P^{-1}TP;\ B \to P^T BP;\ S \to P^T S \bar{P}$$

Inner Products

- An **inner product** g is a bilinear form that satisfies:
  - **positivity**: $g(u, u) \geq 0\ \forall u \in V$
  - **definiteness**: $g(u, u) = 0 \iff u = 0$
  - **symmetry**: $g(u, v) = g(v, u)\ \forall u, v \in V$
- The set of inner products on V is NOT a subspace (because the zero bilinear form is not definite)
- The matrix of an inner product is symmetric and **non-singular**.
- In matrix notation, if $u = a^i z_i$ and $v = b^i z_i$, then
$$g(u, v) = g_{ij}a^i b^j = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Lengths, Angles, Geometry

- **Length**: $|v| = \sqrt{g(v, v)}$
- **Orthogonality**: Vectors $u, v \in V$ are said to be orthogonal if $g(u, v) = 0$.
- **Pythagoras Theorem**: for all pairs of orthogonal vectors $u, v \in V$, $|u+v|^2 = |u|^2 + |v|^2$
- **Orthogonal Decomposition**: Given any vector u and any non-zero vector v, then u can be expressed as the sum of two vectors $u_{||v}$ and $u_{\perp v}$ such that $u_{||v}$ is parallel to v and $u_{\perp v}$ is orthogonal to v. We have:
$$u_{||v} \equiv \frac{g(u, v)}{|v|^2} v,\ u_{\perp v} = u - u_{||v}$$
- **Cauchy-Schwarz Inequality**: $|g(u, v)| \leq |u||v|$
- **Triangle Inequality**: $|u + v| \leq |u| + |v|$
- **Angle**: The angle between u and v is defined as the number θ between 0 and π (inclusive) that satisfies:
$$\cos\theta = \frac{g(u, v)}{|u||v|}$$

Orthonormal Bases

- **Orthonormal Basis**: A basis z_i is said to be orthonormal iff $g(z_i, z_j) = I_{ij}$
- The matrix of g relative to *any* orthonormal basis is the identity matrix.
- Then, the change of basis matrix P from one orthonormal basis to another must satisfy $P^T P = I$, and is said to be **orthogonal**. Thus, the columns of P, when regarded as vectors in $\mathbb{R}^n$, are orthonormal to each other.
- Suppose y_i is a list of orthonormal vectors (not necessarily a basis), then,
$$|a^i y_i|^2 = (a^1)^2 + (a^2)^2 + (a^3)^2 + \cdots$$
from Pythagoras.
- Any list of orthonormal vectors is linearly independent (from above).
- Any list of $n = dim(V)$ orthonormal vectors is a basis.

Gram-Schmidt Orthogonalization

- We can always convert any basis to an orthonormal basis by performing GSO. Hence, every finite-dimensional inner product space has an orthonormal basis.
- The change of basis matrix for GSO is upper-triangular.

Reisz Representation

- Let (V, g) be an inner product space, and let $u \in V$. We can define an isomorphism $\Gamma : V \to \hat{V}$ by:
$$\Gamma(u)(v) \equiv g(v, u),\ \forall v \in V$$
- So, any dual vector α can be expressed as $\Gamma(u)$ for some unique $u \in V$; and for any $v \in V$, the action of α on v can be written as $\alpha(v) = g(v, u)$.
- For an orthonormal basis z_i, the dual basis is just $\Gamma(z_i)$.
- If $v = a^i z_i$ and $\alpha = \Gamma(v) = p_j \zeta^j$, then $p_i = g_{ij}a^j$ (since $\Gamma(v)(z_i) = g(z_i, a^j z_j)$). In terms of components arranged as a matrix: $\alpha = Gv$. So, $v = G^{-1}\alpha$ (and since g is non-singular, this always exists).
- Let T be an operator on the inner product space (V, g). Then, $\Gamma \circ T$ is a linear mapping from $V \to \hat{V}$ (which can be thought of as a bilinear form). Explicitly, $\tau(u, v) = g(u, Tv)$ is the (reisz-equivalent) bilinear form defined by T, and in matrix notation (where entries are the components): $\tau = GT$, since $\tau_{ij} = g_{ik}T_j^k$. And since G is invertible, we have: $T = G^{-1}\tau$.
- Relative to any orthonormal basis for V, the matrix of τ is exactly the matrix of T (since $G = I$ relative to any orthonormal basis).

The Complex Case

- Define a **scalar product** on $\mathbb{C}^n$ by $u \cdot v \equiv u^T \bar{v}$.
- A **sesquilinear form** s on a complex vector space V is a mapping from $V \times V \to \mathbb{C}$ which is linear in the first slot but conjugate -linear in the second slot. That is,
$$s(cu, v) = cs(u, v); \quad s(u, cv) = \overline{c}s(u, v)$$

- Given any s, we can define $s^{\#} : V \to \hat{V}$ by:
$$s^{\#}(v)(u) \equiv s(u, v)$$
where $s(v)$ is indeed a dual vector (since the first slot of s is linear) but $s^{\#}$ is no longer linear; it is conjugate-linear.
- A **conjugate-linear map** $F : V \to W$ is a map that satisfies:
$$F(au + bv) = \overline{a}F(u) + \overline{b}F(v)$$
- The matrix of s relative to any basis z_i is (similar to bilinear forms) given by $s_{ij} = s(z_i, z_j)$
- Under a change of basis, the matrix of s changes as: $S \to P^T S \bar{P}$.
- A **complex inner-product space** (V, g) (where g is a sesquilinear form) satisfies:
  - positivity: $g(u, u) \geq 0, \ \forall u \in V$
  - definiteness: $g(u, u) = 0 \iff u = 0$
  - **conjugate-symmetry**: $g(u, v) = \overline{g(v, u)}$ (in matrix form: $g_{ij} = \overline{g_{ji}}$)
- The change of basis matrix between orthonormal bases in the complex case always satisfies $P^T \overline{P} = I$ (or equivalently, $\overline{P}^T P = I$) and such a matrix is said to be **unitary**.
- The Reisz map $\Gamma(u)(v) = g(v, u)$ where $\Gamma : V \to \hat{V}$ still holds, but Γ is no longer linear; it is conjugate-linear.
- Similarly, $\tau(u, v) = g(u, Tv)$ is the Reisz-equivalent sesquilinear form defined by the operator T. In matrix notation, $\tau = G\overline{T}$, so if the basis is orthonormal, $\tau = \overline{T}$)

Schur Triangularization Theorem

- Given any orthonormal basis y_i (which can always find through GSO), we can always find another orthonormal basis z_i such that the matrix of T relative to z_i is upper-triangular. And the change of basis matrix P from $y_i \to z_i$ is unitary (since both y_i and z_i are orthonormal).
- Moreover, since the matrix of $\tau = \overline{T}$ (as z_i is orthonormal), the diagonal entries of τ are the complex-conjugate of the eigenvalues of T.

Spectral Theorem

- **Hermitian**: A sesquilinear form s is said to be Hermitian if, for all $u, v \in V$, we have:
$$s(u, v) = \overline{s(v, u)}$$
and is said to be **Anti-Hermitian** if you get a minus sign on the right side. For matrices, it means $s_{ij} = \pm\overline{s_{ji}}$ respectively.
- In the real case (i.e., for bilinear forms), we have symmetric and anti-symmetric instead.
- When we say "T is Hermitian", we mean that the Reisz-equivalent sesquilinear form $\tau(u, v) = g(u, Tv)$ is Hermitian. And this means that: $g(u, Tv) = g(Tu, v)$, since g is Hermitian (by conjugate-symmetry).
- If M is a complex matrix, then it is said to be Hermitian if $M^T = \overline{M}$ (and similarly Anti-Hermitian). So, (anti)Hermitian sesquilinear forms have (anti)Hermitian matrices. Rewritten as: $\overline{M}^T = M$
- **Spectral Theorem**: Any <u>Hermitian</u> sesquilinear form on a complex inner product space has a <u>real</u>, <u>diagonal</u> matrix relative to some <u>orthonormal</u> basis. The relevant change of basis can be unitary (if we start with an orthonormal basis, it will be).
- So, if M is the matrix of a Hermitian sesquilinear form relative to any orthonormal basis, then, we can find a unitary matrix P such that $D = P^T M \overline{P}$ where D is diagonal with all entries real.
- Proof: Use Schur, to find a unitary P such that $U = P^T M \overline{P}$ where U is upper-triangular. Then, taking the transpose and complex-conjugate, $\overline{U}^T = P^T \overline{M}^T \overline{P}$. But since M is Hermitian, $\overline{M}^T = M$, and so RHS of both equations are equal, meaning $U = \overline{U}^T$. This implies U is diagonal, with all entries real.
- Implication: all eigenvalues of an operator which is Reisz-equivalent to a Hermitian sesquilinear form are real numbers. And the columns of P give us an <u>orthonormal basis of eigenvectors</u>. (This does not mean that *every* basis of eigenvectors is orthonormal. Here, we get this result because P is unitary *and* it diagonalises M, hence it is both orthonormal *and* a basis of eigenvectors.
- For a real symmetric bilinear form with matrix M:
  - M is Hermitian, so it can be diagonalised to a real, diagonal matrix D.
  - Additionally, we can find a set of <u>real</u> eigenvectors that form an orthonormal basis, given by the unitary P.
  - That is, we can find P that satisfies: $D = P^T M P$ (so, we want $\overline{P}^T = P^T$, i.e., P to be real)
- All complex anti-Hermitian and real anti-symmetric matrices are also diagonalisable (with their Reisz equivalent operator having pure imaginary values) - so the entries of the diagonal matrix will be purely imaginary. And it is impossible to find a full set of real eigenvectors that forms an orthonormal basis, and so, P must be a complex unitary matrix (not orthogonal).

Chapter 6

Multilinear Forms

- A **multilinear form** of degree m on an n-dimensional vector space is a mapping from $V^m \to F$ which is linear in every slot.
- The set of multilinear forms of degree m on a vector space V is a vector space defined in the natural way.
- A bilinear form ψ is on a real finite-dimensional vector space V is called a **two-form** if it is antisymmetric, i.e., $\psi(u, v) = -\psi(v, u)$ for any $u, v \in V$.
- The space of two-forms in a subspace of the space of bilinear forms.
- **Wedge Product**:
$$\alpha \wedge \beta \equiv \alpha \otimes \beta - \beta \otimes \alpha$$
and clearly, it is a two-form. Also, $\alpha \wedge \alpha = 0$ for any α.
- Any two form ψ can be written as:
$$\psi = \frac{1}{2}\psi_{ij}\zeta^i \wedge \zeta^j$$
where $\psi_{ij} = \psi(z_i, z_j)$
- The full set $\zeta^i \wedge \zeta^j$ contains many repetitions and many of them are also zero.
- A **three-form** is just a trilinear form ψ that is antisymmetric in every slot, i.e., $\psi(u, v, w) = -\psi(v, u, w) = -\psi(u, w, v)$.
- Then, we can define:
$$\begin{aligned}\alpha \wedge \beta \wedge \gamma =& \alpha \otimes \beta \otimes \gamma - \beta \otimes \alpha \otimes \gamma - \alpha \otimes \gamma \otimes \beta \\ & -\gamma \otimes \beta \otimes \alpha + \gamma \otimes \alpha \otimes \beta + \beta \otimes \gamma \otimes \alpha\end{aligned}$$
- Any three-form can be written as:
$$\psi = \frac{1}{6}\psi_{ijk}\zeta^i \wedge \zeta^j \wedge \zeta^k$$
- The space of m-forms on an n-dimensional vector space is a vector space of dimension $\binom{n}{m}$.
- Hence, the space of n-forms on an n-dimensional vector space is one-dimensional.
- Note that whenever you feed the same 2 vectors (aka repetition) to any m-form, it evaluates to zero (since it is anti-symmetric).

Linear Algebra in 1 Dimension

- If T is a linear operator on V, define an linear operator called $\hat{T}$ on m-forms on V by:
$$\hat{T}(\psi)(u, v, w, \cdots) \equiv \psi(Tu, Tv, Tw, \cdots)$$
- For any operator T on V, we define a scalar $\triangle(T)$ by:
$$\hat{T}(\Omega) = \triangle(T)\Omega$$
where Ω is any n-form on V (and V itself has n dimensions).
- $\triangle(T)$ is the eigenvalue of $\hat{T}$.
- Let t be a basis for V such that the matrix of T relative to t is upper-triangular. Then,
$$\begin{aligned}\hat{T}\Omega(t_1, t_2, t_3, \cdots) &= \Omega(Tt_1, Tt_2, Tt_3, \cdots) \\ &= \lambda_1\lambda_2\lambda_3 \cdots \Omega(t_1, t_2, t_3, \cdots)\end{aligned}$$
where the λ_i's are the eigenvalues of T.
- So, $\triangle(T) = \lambda_1^{m_1} \times \lambda_2^{m_2} \times \cdots$ where m_i is the multiplicity of λ_i.
- $\triangle(T_z^*) = \triangle(T)$ for any z (the choice of basis doesn't matter, we can work with any of the matrices of T)
- Let $S, T \in L(V, V)$, for any finite-dimensional vector space V. Then, $\triangle(TS) = \triangle(T)\triangle(S) = \triangle(ST)$
- $\triangle(T) \neq 0 \iff T$ is bijective (i.e., has no zero eigenvalue).
- $\triangle(T - \lambda I) = 0$ for every eigenvalue λ.

Properties of Determinant

- $\triangle(PQ) = \triangle(QP)$
- $\triangle(P^{-1}) = \frac{1}{\triangle(P)}$
- $\triangle(P^T) = \triangle(P)$
- $\triangle(cP) = c^n\triangle(P)$ (where n is the dimensionality of P)

Volume

- Let $u, v, w, \cdots$ be any ordered set of n vectors in a n-dimensional space V. Then they define a **parallelotope** with these vectors as edges.
- Define $\Theta(u, v, w, \cdots)$ to be the volume mapping from $V^n \to \mathbb{R}$ which should satisfy
  1. The volume should be non-zero iff if the vectors $u, v, w, \cdots$ are linearly independent.
  2. If you stretch any *one* vector in $u, v, w, \cdots$ by a scalar factor of c, the volume should also change by c.
  3. If you obtain a new parallelotope by adding any multiple of one of the $u, v, w, \cdots$ to a different one, then the two parallelotopes should have the same volume (related to shearing).
  4. Volume cannot be negative.
  5. The volume of the parallelotope generated by *any* orthonormal basis should be 1.
- Θ is multilinear, and it is also antisymmetric. Hence, it is an n-form.
- Let z be an orthonormal basis, with dual basis ζ, then define:
$$\omega_z \equiv \zeta^1 \wedge \zeta^2 \wedge \zeta^3 \wedge \cdots$$
then clearly $\omega_z(z_1, z_2, z_3, \cdots) = 1$. And also, $\omega_z(y_1, y_2, y_3, \cdots) = \pm 1$ for any other orthonormal basis y too because the change of basis matrix is orthonormal, i.e., $P^T P = I$ and $\triangle(P^T P) = \triangle(P)^2 = 1 \implies \triangle(P) = \pm 1$.
- We can define $\Theta = |\omega_z|$ to be the **volume form**, where the choice of z doesn't matter.
- **Orientated Inner Product Space**: (V, g, ω_z), i.e, an inner product space with a definite choice of ω_z. Then, a basis x (not necessarily orthonormal) is said to be *positively oriented* (w.r.t. ω_z) if $\omega_z(x_1, x_2, \cdots) > 0$, otherwise it is called *negatively oriented*.
- The Klein bottle is non-orientable, but still has a well-defined volume.
- In $\mathbb{R}3$, we have: $\Theta(u, v, w) = |(u \times v) \cdot w|$.
- Under a linear transformation T, volumes of parallelotopes in inner product spaces change by a factor of $|\triangle(T)|$.
- Expressed relative to a non-orthonormal basis, whose dual basis is η, the volume form is:
$$\Theta = \sqrt{\triangle(G)}|\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge \cdots|$$
where G is the matrix of the inner product relative to the new basis. Relative to an orthonormal basis, $\triangle(G) = 1$.

Tutorials

Tutorial 9

- Given (V, g), we can define $h(\alpha, \beta) = g(\Gamma^{-1}(\alpha), \Gamma^{-1}(\beta))$ to be an inner product on $\hat{V}$. We can show that $\Gamma(z_i) = g_{ij}\zeta^j$ and so, $\Gamma^{-1}(\zeta^j) = g^{ij}z_i$ where g^{ij} is the inverse of the matrix of g. So, the matrix of h is the inverse of the matrix of g (recall that $h_{ij} = h(\zeta^i, \zeta^j)$).
- In fact, $h = g^{ij}K(z_i) \otimes K(z_j)$ where $K(v)(\alpha) = \alpha(v)$ is the natural isomorphism from $V \to \hat{\hat{V}}$.
- $g = \alpha \otimes \beta + \beta \otimes \alpha$ can never be an inner product on V with dim > 1. Prove by contradiction. To show that it violates definiteness, let v be such that $\Gamma(v) = \alpha$ (by Reisz, such a v exists). Then, find u such that $g(u, v) = 0$, i.e., u, v are orthogonal. But then, $g(u, u) = 0$ for $u \neq 0$. Hence, g cannot be an inner product.
- The change of basis matrix in GSO is upper-triangular. The product of two upper-triangular matrices is also upper-triangular. So is the inverse of an upper-triangular matrix.
- If z_i is an orthonormal basis, then $\Gamma(z_i)$ gives us a dual basis.
- The Reisz correspondence between operators and bilinear forms on a real inner product space is a vector space isomorphism. (Corollary: the dimensionality of $L(V, V)$ is the same as $L(V \times V, F)$, unsurprising since they both have matrices of the same dimension). In matrix form: $\tau = GT$ and so, $T = G^{-1}\tau$.

Tutorial 10

- If T is a Reisz-equivalent to a symmetric bilinear/sesquilinear form, and u, v are eigenvectors of T with distinct eigenvalues, then u, v are orthogonal to each other, i.e,. $g(u, v) = 0$. Because, $g(u, Tv) = g(v, Tu)$ and so, $(\lambda_u - \lambda_v)g(u, v) = 0$, which implies $g(u, v) = 0$.
- If $S, T \in L(V, V)$, then the transpose of ST is $\widehat{ST} = \hat{T}\hat{S}$ (we know this is true for matrices). Also, this can be used to show that $(P^T)^{-1} = (P^{-1})^T$.
- Reflection around a mirror placed at an angle θ to the x-axis can be written as:
$$\begin{pmatrix}\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta)\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$$
- Any antisymmetric matrix A satisfies $D = P^T A\overline{P}$ where P is unitary, and D is a diagonal matrix consisting of pure imaginary matrix (either zero or occurring in $\pm$ pairs, since trace is zero).
- Every unitary matrix N satisfies $D = P^T N\overline{P}$ where P is unitary, and D is a diagonal matrix with entries equal to unimodular complex numbers (aka $|c| = 1$).
- A symmetric matrix is positive and definite iff all of its eigenvalues are positive.

Tutorial 11

- Any Jordan matrix is similar to its transpose (consider the change of basis by just reversing the order of the basis vectors). Hence, every matrix is similar to its own transpose (since every matrix can be written in JCF).
- Given any T such that for any u, v, $g(Tu, Tv) = g(u, v)$, i.e., T doesn't change length of any vector or the angle between two vectors, then T is called an isometry. Clearly, T is non-singular, and has an inverse, which is also an isometry. An isometry has $\triangle(T) = \pm 1$. If $\triangle(T) = 1$, then T is a rotation, else reflection. If H is the matrix of the fixed inner product relative to the canonical basis, M is the matrix of an isometry, then $M^T HM = H$.
- The rotation matrix (for any $\mathbb{R}^n$ is orthogonal, and so must have unimodular (possibly complex) eigenvalues (and it can always be diagonalized).
- If P is orthogonal ($P^T P = I$), then $\triangle(P) = \pm 1$.
- Let G be a set and $\cdot$ be a map from $G \times G \to G$ such that:
  - $(f \cdot g) \cdot h = f \cdot (g \cdot h), \ \forall f, g, h \in G$
  - $\exists e \ \forall g, \ e \cdot g = g \cdot e = g$
  - $\forall g \ \exists g^{-1}, \ g \cdot g^{-1} = g^{-1} \cdot g = e$

  then G is called a **group**, with group product $\cdot$.