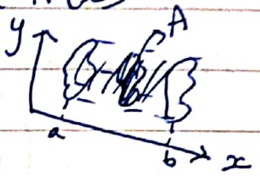
MA1521 - Test 3 Notes.

Applications of Integrals.

1. General formula for volume: Area of cross-section × height

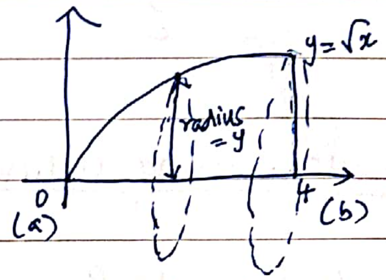
∴ Volume of a solid of integrable cross sectional area $A(x)$ from ~~a = x~~ $x = a$ to $x = b$ is $\int_a^b A(x)\,dx$



2. Solid of Revolution: The Disk Method.

(i) Rotation about x-axis : $V = \int_a^b \pi [R(x)]^2\,dx$

where $R(x)$: radius (y coordinate in terms of x)

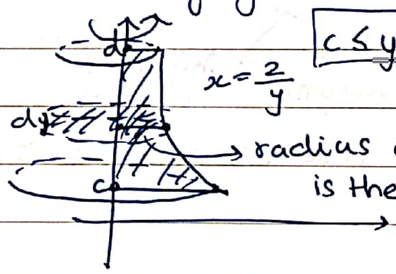


(ii) Rotation about horizontal axis ($y = c$) : $\int_a^b \pi\,(f(x) - c)^2\,dx$

(because radius $= f(x) - c$)

(iii) Rotation about y-axis : Just replace x with y ie, express the radius (x coordinate) in terms of y.

$\therefore\ V = \int_c^d \pi [R(y)]^2\,dy$



$\boxed{c \le y \le d}$

$x = \frac{2}{y}$

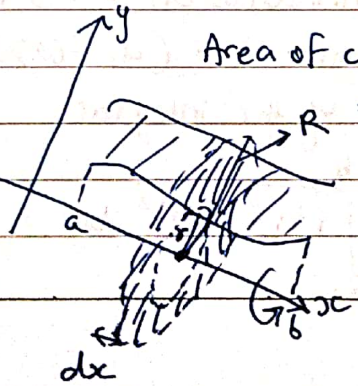
radius at any value y is the x-coordinate.

3. Solid of Revolution: The Washer Method.

If the region to revolve to generate a solid does not border or cross the axis of revolution, then the solid has a hole in it. The cross-sections perpendicular to the axis of revolution are washers instead of disks. A washer has outer radius $R(x)$ and inner radius $r(x)$

(i) Rotation about x-axis.

$V = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right) dx$



Area of cross-section $= \pi(R^2 - r^2)$

Thickness $= dx$

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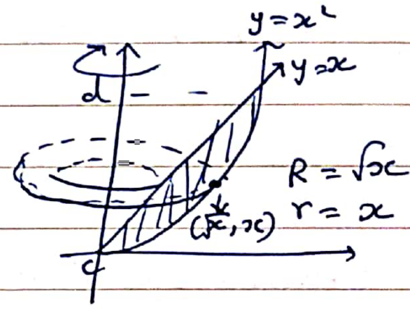
(ii) Rotation about y-axis

Just express x as a function of y (as x is the radius now that changes depending on y)

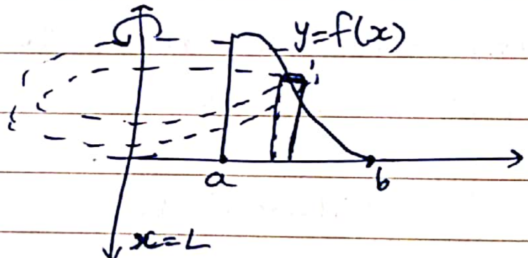
$$\therefore V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy$$



3. Volume using cylindrical Shells.

Shell formula for revolution about a vertical line: The volume of a solid generated by revolving the region between the x-axis and the graph of a continuous function $y=f(x) \geq 0$, $L \leq a \leq x \leq b$ about a vertical line $x=L$ is

$$V = \int_a^b 2\pi \underbrace{(x-L)}_{\text{shell radius}} \underbrace{f(x)}_{\text{shell height}} \underbrace{dx}_{\text{thickness variable}}$$



(Use this when washer method is awkward ie, difficult to write $x = R(y)$)

Similarly, for revolution about horizontal line ($y=c$) : $\int_a^b 2\pi(y-c)F(y)dy$

4. Arc Length.

(i) If f' is continuous on $[a,b]$ then the length (arclength) of $y=f(x)$ from the point $A=(a,f(a))$ to the point $B=(b,f(b))$ is the value of the integral,

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

↳ (as x varies from a to b)

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If $\frac{dy}{dx}$ does not exist at some pt. on the curve, it is possible that $\frac{dx}{dy}$ could exist (eg. vertical tangent). $\underset{\Downarrow}{dy/dx = \infty}$ but $\frac{dx}{dy} = 0$ So, convert it into a fn of y:

(ii)

Formula for length of $x = g(y)$, $c \leq y \leq d$.

If g' is cont. on $[c,d]$, arclength of curve $x = g(y)$ from the point $A(g(c), c)$ to $B(g(d), d)$ is

$$L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1+[g'(y)]^2}\, dy$$

y varies from c to d $\Rightarrow$ limits depend on variable of integration

Differential form, $ds = \sqrt{dx^2 + dy^2}$
(s → arc length)

5. Areas of Surface of Revolution.

(i) If $f(x) \geq 0$ is continuously differentiable ($\to f'$ is continuous) on $[a,b]$, the area of the surface generated by revolving the graph of $y = f(x)$ about the x-axis is,

$$S = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x) \sqrt{1+[f'(x)]^2}\, dx$$

(ii) If $x = g(y) \geq 0$ is continuously differentiable on $[c,d]$ the area of the surface generated by revolving $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y) \cdot \sqrt{1+[g'(y)]^2}\, dy.$$

Sequences and Series.

1. The sequence (a_n) converges to the number L if for every $\varepsilon > 0$ there exists an integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$.
We write $\lim_{n\to\infty} a_n = L$ or simply $a_n \to L$ and L is called the limit of the sequence. Otherwise, it diverges.
The sequence (a_n) diverges to infinity if $\forall M \; \exists N$ such that $\forall n > N$, $a_n > M$

→ The convergence/divergence of a sequence is not affected by the values of any number of its ~~any~~ initial terms. Only the part of the sequence that remains after discarding some initial number of terms determines whether the sequence has a limit and the value of the limit when it does exist

2. For sequences (a_n) and (b_n) of real numbers, if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then
   (i) Sum rule : $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$ (and Difference Rule)

   (ii) Constant Multiple Rule : $\lim_{n\to\infty} (K \cdot b_n) = K \cdot B$

   (iii) Product Rule : $\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$

   (iv) Quotient Rule : $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}$ provided $B \neq 0$

3. Sandwich Theorem (Squeeze Theorem for Sequences.
Let $(a_n), (b_n), (c_n)$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N and if $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$

then $\lim_{n\to\infty} b_n = L$

(The terms of b_n are sandwiched b/w a_n and c_n, forcing the limit)

Then, if $|b_n| \leq c_n$ and $c_n \to 0$, then $b_n \to 0$ because $-c_n \leq b_n \leq c_n$ ☆

4. Suppose that $f(x)$ is defined for all $x \geq n_0$ and (a_n) is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then,

if $\lim_{x\to\infty} f(x) = L$ then, $\lim_{n\to\infty} a_n = L$ $\left\{\begin{array}{l} n: \text{integer} \\ x: \text{real number} \end{array}\right\}$

↳ can apply L'Hôpital's rule

eg. Show $\lim_{n\to\infty} \frac{\ln n}{n} = 0$. Take $f(x) = \frac{\ln x}{x}$. Apply L'Hôpital.

5. Common Sequences limits.

(i) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ (ii) $\lim_{n\to\infty} \sqrt[n]{n} = 1$ ($n^{\frac{1}{n}}$) (iii) $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$ $(x > 0)$

(iv) $\lim_{n\to\infty} x^n = 0$ $(|x|<1)$ (v) $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ (vi) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

6. A sequence (a_n) is bounded from above if $\exists M$ such that $a_n \leq M \; \forall n$. The number M is an upper bound for (a_n). If M is an upper bound for (a_n) but no number less than M is an upper bound for (a_n), then M is the <u>least upper bound</u>. Similarly for bounded from below. If (a_n) is bounded from above and below, then (a_n) is bounded. Otherwise (a_n) is an unbounded sequence.
A sequence (a_n) is nondecreasing if $a_n \leq a_{n+1} \; \forall n$ ie, $a_1 \leq a_2 \leq a_3 \leq \ldots$
It is nonincreasing if $a_n \geq a_{n+1} \; \forall n$. It is <u>monotonic</u> if it is either non-decreasing or non-increasing.

7. <u>Monotonic Sequence Theorem.</u>
(i) If (a_n) is both bounded and monotonic, then the sequence converges.
(ii) If (a_n) is bounded from above and nondecreasing, it converges.
(iii) If (a_n) is bounded from below and nonincreasing, it converges.
<u>Note</u>: This does not mean that all convergent sequences are monotonic.
eg. $a_n = \frac{(-1)^{n+1}}{n}$

8. Given a sequence (a_n), define (S_n) where $S_n = \sum_{i=1}^{n} a_i$ called the sequence of partial sums of the series.
If the sequence of partial sums converges to a limit L, we say the series converges and that its sum is L ie, $\sum_{n=1}^{\infty} a_n = L$.

9. <u>Geometric Series.</u>
If $|r|<1$, then $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ (convergent)

If $|r| \geq 1$, then $\sum_{n=1}^{\infty} ar^{n-1}$ diverges

<u>Note</u>: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$ (Telescoping series)

10. <u>n^{th}-term Test for Divergent Series.</u>
If $\sum a_n$ converges then $a_n \to 0$.

Contrapositive: If $\lim_{n\to\infty} a_n \neq 0$, $\sum_{n=1}^{\infty} a_n$ diverges.

11. If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then
(i) $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n = A \pm B$
(ii) $\sum k\cdot a_n = k\cdot \sum a_n = kA$

<u>Note</u>: ① every non-zero scalar multiple of a divergent series diverges
② if either <u>one</u> of $\sum a_n$ or $\sum b_n$ diverges, $\sum a_n \pm b_n$ diverges.

<u>Caution</u>: $\sum(a_n + b_n)$ can converge even if both $\sum a_n$ and $\sum b_n$ diverge
<u>eg.</u> $\sum a_n = 1+1+1+1+\dots$ $\quad \sum b_n = (-1)+(-1)+(-1)\dots$

<u>Note</u>: If $\sum_{n=k}^{\infty} a_n$ converges for any $k>1$, $\sum_{n=1}^{\infty} a_n$ converges.

Adding/deleting a finite number of terms will not alter the series' convergence or divergence.

12. A series of Σa_n ~~converges~~ of nonnegative terms converges if and only if its partial sums are bounded from above (from monotonic sequence theorem)

13. Integral Test for convergence/divergence.

Let (a_n) be a sequence of positive terms. Suppose that $a_n = f(n)$ where f is a continuous, positive, decreasing function of x for all $x \geq N$ $(N > 0)$. Then the series Σa_n and the integral $\int_N^\infty f(x)\,dx$ both converge or both diverge.

eg. $$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges if } p > 1 & \text{eg. } \frac{1}{n^2}, \frac{1}{n^{3/2}} \dots \\ \text{diverges if } p \leq 1 & \text{eg. } \frac{1}{n}, \frac{1}{\sqrt{n}} \end{cases}$$

↳ p-series.

14. Comparison Tests. (Only for positive terms)

(i) Direct Comparison Test.

Let Σa_n and Σb_n be 2 series with $0 \leq a_n \leq b_n \ \forall n$. Then,

1. If Σb_n converges, then Σa_n also converges.
2. If Σa_n diverges, then Σb_n also diverges.

(ii) Limit Comparison Test

Let $a_n > 0$ and $b_n > 0 \ \forall n \geq N$.

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$ then Σa_n and Σb_n both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and Σb_n converges, then Σa_n converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and Σb_n diverges, then Σa_n diverges.

15. A series Σa_n converges absolutely (or is absolutely convergent) if the corresponding series of absolute values $\Sigma|a_n|$ converges.

Absolute Convergence Test : If $\Sigma|a_n|$ converges, Σa_n converges.
(But the inverse is not necessarily true)

16. The Ratio Test. (works also for negative terms)
Let Σa_n be any series and suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

(a) If $L < 1$, the series converges absolutely.
(b) If $L > 1$, the series diverges.
(c) If $L = 1$, the test is inconclusive.

17. The Root Test (works also for negative terms)
Let Σa_n be any series and suppose that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$

(a) If $L < 1$, the series converges absolutely
(b) If $L > 1$ (or L is ∞), the series diverges.
(c) If $L = 1$, the test is inconclusive

18. The Alternating Series Test
The series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$ converges if the following

Conditions are satisfied:
(i) The u_n's are all positive
(ii) The u_n's are eventually non-increasing : $u_n \geq u_{n+1}$ for all $n \geq N$, for some n.
(iii) $u_n \to 0$ as $n \to \infty$

Caution: This test is for convergence of an alternating series and cannot be used to conclude that a series diverges

19. A series that is convergent but not absolutely convergent is called <u>conditionally convergent</u>. eg. alternating harmonic series $\sum \frac{(-1)^{n+1}}{n}$

ie, $\sum |a_n|$ converges $\Rightarrow \sum a_n$ and $\sum |a_n|$ converge : Absolutely convergent
$\sum a_n$ converges but $\sum |a_n|$ diverges : Conditionally convergent
$\sum a_n$ diverges : Divergent.

<u>Note</u>: You can rearrange the terms of a series <u>only</u> if it is absolutely convergent (Rearrangement Thm)

20. Power series about 0 (ie, $x=0$): $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots c_n x^n + \ldots$

Power series about $x=a$: $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$

Here, a is called the 'center' of the series and the coefficients $c_0, c_1, c_2 \ldots$ are constants.

☆ <u>Note</u>: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots x^n + \ldots$, $\boxed{-1 < x < 1}$

↳ geometric series with $a:1$, $r:x$ converges iff $|r|<1$ ie, $|x|<1$

21. <u>Convergence Theorem for Power Series</u>

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x=d$, then it diverges for all x with $|x| > d$

∴ The convergence of $\sum c_n (x-a)^n$ is described by one of the following:

(i) There is a positive number R (called the Radius of convergence) such that the series diverges for x with $|x-a| > R$ but converges absolutely for $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a-R$ and $x = a+R$.

(ii) Series converges absolutely $\forall x$ (ie, $R = \infty$)

(iii) Series converges at $x=a$ and diverges elsewhere ($R=0$)

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Then the interval of radius R centered at $x=a$ is called the interval of convergence

How to test a power series for convergence:

① Use ratio test / root test to find R such that series converges for all x with $|x-a|<R$ and diverges elsewhere

② Test convergence at $x=a-R$ and $x=a+R$ using comparison test, integral test, etc (Make sure to test endpoints separately)

Note: Only values of x in the region $a-R<x<a+R$ can be substituted in the power series for it to remain convergent. Don't just apply anyhow.

22. Term by Term Differentiation

If $\sum c_n (x-a)^n$ has radius of convergence $R>0$, it defines a function, $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ on the interval $a-R<x<a+R$.

This function f has derivatives of all orders inside the interval and we obtain the derivatives by differentiating the original series term (ie, you can pull the derivative inside the Σ)

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \quad \left(\text{Notice it is } n=1 \because \frac{d(c_0)}{dx}=0\right)$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2} \quad \text{and so on.}$$

Each of these derived series converges at every point of the interval $a-R<x<a+R$

Caution: This is only for power series ($\sum c_n (x-a)^n$) and might not work for other series eg. $\sum \frac{\sin(n!x)}{n^2}$ converges for all x but its derivative $\sum \frac{n!\cos(n!x)}{n^2}$ diverges for all x. This is because it is not a power series (not a sum of positive integer powers of x)

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23. Term-by-Term Integration.

Suppose that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges on $a-R < x < a+R$ for $R > 0$. Then,

$$\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C \text{ converges for } a-R < x < a+R.$$

24. Taylor and Maclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then, the taylor series generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

The Maclaurin series of f is the taylor series generated by f at $x=0$

or $$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \dots$$

(i) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$

(ii) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$

(iii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$

(vi) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \le 1$

(vii) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots = \sum_{n=1}^{\infty} -\frac{x^n}{n}, \quad \cancel{|x| < 1} \; -1 \le x < 1$

(viii) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \le 1$

↳ useful to evaluate limits that cannot be solved using L'Hôpital's rule

Put $x=1$ to estimate $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

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Note: Cauchy-Schwarz Inequality.

In 2 dimensional vector space $\mathbb{R}^2$, $(u \cdot v)^2 \leq \|u\|^2 \|v\|^2$

where u and v are vectors

In general, $\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leq \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right)$

This is useful to test for convergence and divergence (Comparison test)

eg. To show that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges whenever $\sum a_n$ converges.

Functions of 2 variables.

1. A point (x_0, y_0) in a region R in the xy-plane is an interior point of R if it is the center of a disk of positive radius that lies entirely in R.
A point (x_0, y_0) is a boundary point of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. The boundary point itself need not belong to R.
A region is open if it contains entirely of interior points. A region is closed if it contains all its boundary points.
A region is bounded if it lies inside a disk of finite. Else, it is unbounded.

2. We say that a function $f(x,y)$ approaches the limit L as (x,y) approaches (x_0, y_0), and write $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x,y) in the domain of f,

$$|f(x,y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

(ie, for all (x,y) in the disk of radius δ centered at (x_0, y_0))

All the limit properties hold : Sum, difference, product, quotient, power, root etc

To show that a limit does not exist, find 2 paths, approaching along which gives different limits (or a certain path does not give a finite limit) → Two-path test for non-existence of limit

Caution: Having the same limit along all straight lines $\not\Rightarrow$ limit exists

3. A function $f(x,y)$ is continuous at the point (x_0,y_0) if
(i) f is defined at (x_0,y_0)
(ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
(iii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$ (limiting value = functional value)

4. Continuity of Compositions
If f is continuous at (x_0,y_0) and g is a single variable function ~~at~~ continuous at $f(x_0,y_0)$ then the composition $h = g\circ f$ defined by $h(x,y) = g(f(x,y))$ is continuous at (x_0,y_0)
eg. e^{x-y}, $\cos\frac{xy}{x^2+1}$, $\ln(1+x^2y^2)$ are all continuous functions

5. Partial derivatives.
$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} = \left.\frac{d}{dx}f(x,y_0)\right|_{x=x_0}$$
$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h} = \left.\frac{d}{dy}f(x_0,y)\right|_{y=y_0}$$

Note: The mere existence of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at a point (say, x_0,y_0) is not sufficient to guarantee the differentiability of f at (x_0,y_0)

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$: first differentiate w.r.t. y, then w.r.t. x

$$\frac{\partial^2 f}{\partial x\,\partial y} = f_{yx}, \quad \frac{\partial^2 f}{\partial y\,\partial x} = f_{xy}, \quad \frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

Mixed derivative Theorem: Whenever f_x, f_y, f_{xy}, f_{yx} are all continuous at (a,b), $f_{xy}(a,b) = f_{yx}(a,b)$ ie, $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$
(Also called Clairaut's theorem)

6. Differentiability.

(i) A function $z = f(x,y)$ is differentiable at (x_0, y_0) if $f_x(x_0,y_0)$ exists and $f_y(x_0,y_0)$ exists, and $\Delta z = f(x+x_0, y+y_0) - f(x_0, y_0)$ satisfies an equation of the form

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.

(ii) if the partial derivatives f_x and f_y of a function $f(x,y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

(iii) If $f(x,y)$ is differentiable at (x_0,y_0) then f is continuous at (x_0,y_0)

Note: To show f is not differentiable at (x_0,y_0), it suffices to show that f is not continuous at (x_0,y_0)

: The existence of f_x, f_y does not guarantee differentiability but the continuity of f_x, f_y guarantees differentiability

7. Chain Rule. (for one independent variable)

If $f(x,y) = f(u(t), v(t))$ where x and y are functions of t, then $\frac{d}{dt} f(u(t),v(t)) = f_x(u(t),v(t)) \cdot u'(t) + f_y(u(t),v(t)) \cdot v'(t)$

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

~~In case of 2 independent variables and~~

8. Extreme values and Saddle points.

Recall, critical points must be interior

$\hookrightarrow$ both $f_x = 0$ and $f_y = 0$, or one or both of f_x and f_y do not exist at that point

If $f(x,y)$ has local max/min value at an interior point (a,b) and f_x and f_y exist there then $f_x(a,b) = 0$, $f_y(a,b) = 0$

Saddle point $\approx$ Point of Inflection

9. <u>Second Derivative Test for Local Extreme Values.</u>

Suppose that $f_x(a,b) = f_y(a,b) = 0$ then

(i) f has local maximum at (a,b) if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b)

(ii) f has local minimum at (a,b) if $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b)

(iii) f has a saddle point at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a,b)

(iv) The test is inconclusive if $f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a,b)

The expression $f_{xx} f_{yy} - f_{xy}^2$ is called the <u>discriminant</u> or <u>Hessian</u> of f.

* For absolute maxima/minima, check boundary points also (one boundary at a time) → because the second derivative test does not apply to boundary points of a function's domain, where it is possible for a function to have extreme values along with non-zero derivatives. Also, it does not apply to points where f_x or f_y fail to exist

∴ To get extreme values : check all boundary points for extrema
: all interior points where $f_x = 0 = f_y$
: all interior points where f_x / f_y DNE

10. <u>Lagrange Multipliers.</u>

Let $f(x,y)$ and $g(x,y)$ be continuously differentiable functions. If the maximum/minimum ~~of~~ value of $f(x,y)$ subject to the constraint $g(x,y) = 0$ occurs at a point (a,b) where

$$(g_x(a,b),\ g_y(a,b)) \neq (0,0) \quad \text{(ie, both cannot be 0)}$$

then, $(f_x(a,b),\ f_y(a,b)) = \lambda\,(g_x(a,b),\ g_y(a,b))$ for some λ.

ie, to get extrema values of a fn subject to constraints, just solve the equations : $f_x = \lambda g_x$ and $f_y = \lambda g_y$ and $\underbrace{g(x,y) = 0}_{\text{The point must satisfy the constraint}}$

$\lambda \neq 0$

A'ZONE

Multiple Integrals.

1. Fubini's Theorem. (for rectangular regions)

If $f(x,y)$ is continuous throughout the rectangular region $R: a \le x \le b,\ c \le y \le d$ then,

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Fubini's Theorem (Stronger Form)

~~If R~~ Let $f(x,y)$ be continuous on a region R.

(i) If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$ with g_1 and g_2 continuous on $[a,b]$ then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(ii) If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$ with h_1, h_2 continuous on $[c,d]$ then,

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

You can either sweep horizontally first, then vertically or reverse. Just ensure that your limits are correct.

2. Properties of Double Integrals.

(i) Sum, difference, constant multiple

(ii) Domination: $\iint_R f(x,y)\,dA \ge \iint_R g(x,y)\,dA$ if $f(x,y) \ge g(x,y)$ on R.

(iii) Additivity: If R is the union of 2 non-overlapping regions R_1 and R_2 then,

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

3. Double Integral can also be used to give area (just take $f(x,y) = 1$)
Area of a closed, bounded plane region R is $A = \iint_R dA$

4. Average value of f over R $= \dfrac{1}{\text{area of } R} \iint_R f \, dA$

5. Double Integrals in polar form:

$$\iint_R f(r,\theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

$r^2 = x^2 + y^2$

$\theta = \tan^{-1} \dfrac{y}{x}$

Area in polar coordinates: $\iint_R r\, dr\, d\theta$ (Recall area of sector $= \frac{1}{2} r^2 \theta$)

$$\iint_R f(x,y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

6. Substitution in Double Integrals.

(a) Jacobian determinant $J(u,v)$ of transformation $x = g(u,v)$ & $y = h(u,v)$

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \cdot \frac{\partial x}{\partial v}$$

→ Writing x, y as functions of u, v

∴ differential change in area is given by,

$$dx\, dy = \left(\frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \cdot \frac{\partial x}{\partial v} \right) du \cdot dv$$

$|J|$ gives the scaling factor.

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Substitution.

Suppose that $f(x,y)$ is continuous over the region R. Let G be the preimage of R under the transformation $x = g(u,v)$, $y = h(u,v)$, which is assumed to be one-to-one on the interior of G.

Then,

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

Note: Binomial Series.

For $-1 < x < 1$, $\quad (1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$

where we define $\binom{m}{1} = m$, $\binom{m}{2} = \frac{m(m-1)}{2!}$

$$\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!} \quad \text{for } k \geq 3$$

eg. $(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2} x^2 + \cdots$

Note:

$$(1+x)^n = 1 + nx + \frac{(n)(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$$

A formula for Implicit Differentiation.

Suppose that $F(x,y)$ is differentiable and that the equation $F(x,y) = 0$ defines y as a differentiable function of x. Then,

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$