MA2214 Midterm Cheatsheet

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Chapter 1 – Permutations and Combinations

Permutations

- If a natural number n has as its **prime factorization** $n = p_1^{k_1} \cdots p_r^{k_r}$, then a positive integer m is a divisor of n iff m is of the form $p_1^a \cdots p_r^z$ where every power of p_i in m is less than or equal to the power of p_i in n. So, the **number of divisors** of n is given by: $\prod_{i=1}^{r}(k_i + 1)$, inclusive of 1 and n.
- An r**-permutation** of a set of n distinct objects is a way of arranging any r of the objects in a row. And, $P_r^n = \frac{n!}{(n-r)!}$.
- Note that 2 **circular permutations** are identical if any one of them can be obtained by a rotation of the other, i.e., we only care about the relative positions of the objects around the circle. Then, the number of circular permutations is: $Q_r^n = \frac{P_r^n}{r!}$.

Combinations

- An r**-combination** of a set A of n distinct objects is simply an r-element subset of A. That is, the ordering of the elements in the subset is immaterial.
- By multiplication principle, we have: $P_r^n = C_r^n \cdot r!$ since any permutation can be obtained through a 2-step process: pick a combination of r objects out of n, and then arrange these r objects in a line.
- So, $C_r^n = \frac{n!}{(n-r)!r!}$.

Useful Identities:

- $\binom{n}{r} = \binom{n}{n-r}$
- $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

Useful Results

- Number of pairings of a set with n elements: $\frac{(2n!)}{n! \times 2^n}$.

Stirling Numbers of the First Kind Given $r, n \in Z$ with $0 \le n \le r$, let $s(r, n)$ denote the number of ways to arrange r distinct objects around n indistinguishable circles (i.e., we can move the circles around — the relative position of the circles don't matter! — AND we can rotate the circles since the positions ON the circle are indistinguishable too) such that each circle has at least one object. These numbers $s(r, n)$ are called the Stirling numbers of the first kind.

Results of Stirling Numbers:

- $s(r, 0) = 0$
- $s(r, r) = 1$
- $s(r, 1) = (r-1)!$
- $s(r, r-1) = \binom{r}{2}$
- $s(r, n) = s(r-1, n-1) + (r-1)s(r-1, n)$

Injection and Bijection Principles

- A mapping $f : A \to B$ is **injective** (one-one) if $a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$
- A mapping $f : A \to B$ is **surjective** (onto) if $\forall b \in B, \exists a \in A, f(a) = b$
- A mapping $f : A \to B$ is **bijective** if it is both injective and surjective.
- **Injection Principle (IP)**: For finite sets A and B, if $f : A \to B$ is an injection, then $|A| \le |B|$.
- **Bijection Principle (BP)**: For finite sets A and B, if $f : A \to B$ is a bijection, then $|A| = |B|$.

Arrangements and Selections with Repetitions

- The number of r-permutations of a set with n distinct objects, with repetitions allowed, is given by n^r.
- If we have r_1 objects of type 1, r_2 objects of type 2, $\cdots$, r_k objects of type k, and all the objects of the same type are indistinguishable from each other, then the number of permutations of all the objects is given by: $\frac{n!}{r_1!r_2!\cdots r_k!}$ where $n = \sum_i r_i$ is the total number of objects.
- The multiset $M = \{r_1 \cdot a_1, r_2 \cdot a_2, \ldots, r_n \cdot a_n\}$ where r_i's are non-negative integers and a_i's are distinct objects, consists of r_1 a_1's r_2 a_2's, etc.
- Let H_r^n denote the number of r-element multi-subsets of a set with n distinct elements each of which can be repeated infinitely many times. And we have: $H_r^n = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$.
- H_r^n is also the solution to the "number of non-negative integer solutions of the equation $x_1 + \cdots + x_n = r$".
- H_r^n is also the solution to the "number of ways to distribute r indistinguishable objects into n distinct / labelled boxes where each box can hold any number of objects (incl. 0)" (observe that this is the same as assigning box numbers (from 1 to n) to each of the r objects).

Stirling Number of the Second Kind $S(r, n)$ is defined as the number of ways of distributing r distinct objects into n identical boxes such that no box is empty. Obvious results:

- $S(r, r) = 1$ for any $r \ge 0$
- $S(r, 1) = 1$ for any $r \ge 1$
- $S(r, 0) = S(0, n) = 0$ for all $r, n \in N$
- $S(r, r-1) = \binom{r}{2}$
- $S(r, n) = S(r-1, n-1) + nS(r-1, n)$

Chapter 2 - Binomial and Multinomial Coefficients

Binomial Theorem

For any integer $n \ge 0$,

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n \\ &= \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r\end{aligned}$$

Combinatorial Identities

- Number of Subsets:

$$\sum_{r=0}^{n}\binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

- Even and Odd Terms:

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r} = \binom{n}{r} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$$

  and hence,

$$\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2k} = \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{2k+1} = 2^{n-1}$$

- $$\sum_{r=1}^{n} r\binom{n}{r} = \binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}$$

- Vandermonde's Identity:

$$\begin{aligned}\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i} &= \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0} \\ &= \binom{m+n}{r}\end{aligned}$$

  and a special case is:

$$\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots \binom{n}{n}^2 = \binom{2n}{n}$$

- Chu Shih-Chieh's Identity (Hockey Stick Identity):

$$\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$$

  and,

$$\binom{r}{0} + \binom{r+1}{1} + \cdots \binom{r+k}{k} = \binom{r+k+1}{k}$$

Multinomial Theorem

Let $\binom{n}{n_1, n_2, \cdots, n_m}$ denote the number of ways to distribute n distinct objects into m distinct boxes such that n_1 of them are in box 1, n_2 in box 2, $\cdots$, and n_m in box m, where $\sum_{i=1}^{m} n_i = n$.

$$\begin{aligned}\binom{n}{n_1, n_2, \cdots, n_m} &= \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-(n_1+n_2+\cdots+n_{m-1})}{n_m} \\ &= \frac{n!}{n_1!n_2!\cdots n_m!}\end{aligned}$$

and this is precisely the coefficient of $x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$ in the expansion of $(x_1 + \cdots x_m)^n$.
Notice that the x_i's are symmetric, i.e., there is nothing special about x_1 vs. x_2, i.e., the coefficient of a term only depends on the set of frequencies of the powers NOT the ordering of the powers.

$$(x_1 + x_2 + \cdots + x_m)^n = \sum\binom{n}{n_1, n_2, \cdots, n_m}x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$$

where the sum is taken over all m-ary sequences $(n_1, n_2, \cdots, n_m)$ of non-negative integers with $\sum_{i=1}^{m} n_i = n$

Useful Identities:

- $$\begin{aligned}\binom{n}{n_1, n_2, \cdots, n_m} =& \binom{n-1}{n_1-1, n_2, \cdots, n_m} + \binom{n-1}{n_1, n_2-1, \cdots, n_m} \\ &+ \cdots + \binom{n-1}{n_1, n_2, \cdots, n_m-1}\end{aligned}$$

- $$\sum \binom{n}{n_1, n_2, \cdots, n_m} = m^n$$
by letting $x_1 = x_2 = \cdots = x_m = 1$ in the multinomial theorem.
- H^n_r is also the solution to "number of distinct terms in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$ (because there's a clear bijection from each distinct term to solution of $x_1 + \cdots x_m = n$.

Chapter 3 - Pigeonhole Principle and Ramsey Numbers

Pigeonhole Principle (PP): Let k and n be any two positive integers. If at least $kn+1$ objects are distributed among n boxes, then (at least) 1 of the boxes must contain at least $k+1$ objects.

A clique is a configuration considering of a finite set of vertices together with edges joining all pairs of vertices. A k-clique is a clique which has exactly k vertices.

A 1-clique is just a vertex, a 2-clique is an edge joining 2 vertices, a 3-clique is a triangle, etc.

Given $p, q \in \mathbb{N}$, let $R(p,q)$ denote the smallest natural number n such that for ANY coloring of the edges of an n-clique by 2 colors, there exists either a "blue p-clique" or a "red q-clique". The numbers $R(p,q)$ are called **Ramsey numbers**.

We have the following obvious results:
- $R(p,q) = R(q,p)$, i.e., blue and red are symmetric
- $R(1,q) = 1$
- $R(2,q) = q$ for $q \geq 2$

Ramsey's Theorem: For all integers $p, q \geq 2$, the number $R(p,q)$ always exists.
Bound:
$$R(p,q) \leq R(p-1,q) + R(p,q-1)$$

Generalized Pigeonhole Principle (GPP): Let $n, k_1, k_2, \cdots, k_n \in \mathbb{N}$. If $k_1 + k_2 + \cdots + k_n - (n-1)$ or more objects are put into n boxes, then either the first box contains at least k_1 objects, or the second box contains at least k_2 objects, ..., or the nth box contains at least k_n objects.

Chapter 5 - Generating Functions

Ordinary Generating Functions

Let $(a_r) = (a_0, a_1, \cdots, a_r, \cdots)$ be a sequence of numbers. The **ordinary generating function** for the sequence (a_r) is defined to be the power series:
$$A(x) = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots$$
If we have two generating functions, $A(x)$ and $B(x)$, then:
$$C(x) = A(x) + B(x), \text{ where } c_r = a_r + b_r$$
and
$$D(x) = A(x) \cdot B(x), \text{ where } d_r = \sum_{k=0}^{r} a_k b_{r-k}$$
For each $\alpha \in R$ (not necessarily an integer!) and each $r \in N$, define the generalized binomial coefficient $\binom{\alpha}{r}$ to be:
$$\binom{\alpha}{r} = \frac{P^\alpha_r}{r!}$$
where $P^\alpha_r = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-r+1)$.
Newton's Expansion:
$$(1 \pm x)^\alpha = 1 \pm \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 \pm \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \cdots + (-1)^r \frac{\alpha \cdots (\alpha-r+1)}{r!} x^r$$
Then, we can derive the following common series:
- $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$
- $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$
- $$\frac{1}{1-kx} = 1 + kx + k^2x^2 + k^3x^3 + \cdots$$
- $$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots (r-1)x^r + \cdots$$
- $$\begin{aligned}\frac{1}{(1-x)^n} &= 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots \\ &= 1 + \binom{n}{1}x + \binom{n+1}{2}x^2 + \cdots \binom{n+r-1}{r}x^r \\ &= \sum_{r=0}^{\infty} H^n_r x^r\end{aligned}$$
- $$1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

And some very useful techniques (assume that $A(x)$ is the ordinary generating function for (a_r)), then:
- To get the series of partial sums, we can multiply $A(x)$ by $\frac{1}{1-x}$. That is, $\frac{A(x)}{1-x}$ is the generating function for the sequence: $a_0, a_0+1, a_0+a_1+a_2, \cdots$ and in general $c_r = \sum_{i=0}^{r} a_i$.
- To get the difference between consecutive coefficients, you can use $(1-x)A(x)$. That is, $(1-x)A(x)$ is the generating function for the sequence $(a_0, a_1 - a_0, a_2 - a_1, \cdots)$
- $A'(x)$ is the generating function for the series $(0, a_1, 2a_2, 3a_3)$ (and in general, $c_r = (r+1)a_{r+1}$).
- $xA'(x)$ is the generating function for the sequence (c_r) where $c_r = ra_r$, i.e., $(0, a_1, 2a_2, 3a_r, \cdots)$
- - $\int_0^x A(t)dt$ is the generating function for the sequence (c_r) where:
  - $c_0 = 0$, and
  - $c_r = \frac{a_{r-1}}{r}$

  i.e., $(c_r) = (0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \cdots)$

Ordinary generating functions can be used when we want to count the number of ways of selecting r elements from an arbitrary multiset, *where the order does not matter*. If the order matters, we would have to use exponential generating functions instead.
And each product term corresponds to the ways we can select a particular element of the multiset.
Examples:
- Number of ways to select 4 members from $M = \{2 \cdot b, 1 \cdot c, 2 \cdot d, 1 \cdot e\}$. Generating function = $(1+x+x^2)(1+x)(1+x+x^2)(1+x)$.
- If $M = \{\infty x_1, \infty x_2, \cdots, \infty x_k\}$, then generating function $= (1+x+x^2+\cdots)^k = \frac{1}{(1-x)^k} = \sum_{r=0}^{\infty} H^n_r x^r$

Partitions

A partition of a positive integer n is a collection of positive integers whose sum is n, where the ordering is not taken into account. That is, a partition can be characterized by a multiset denoting the numbers.
If $n = n_1 + n_2 + \cdots + n_k$ is a partition of n, we say that n is partitioned into k parts of sizes $n_1, n_2, \cdots, n_k$ respectively.
A partition of n is equivalent to a way of distributing n identical objects into n identical boxes.
Example: Let a_r be the number of partitions of an integer r into parts of sizes $1, 2, 3$. The generating function for (a_r) is: $(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots) = \frac{1}{(1-x)(1-x^2)(1-x^3)}$.
Example: Let a_r be the number of partitions of r into distinct parts. Then, the generating function is $(1+x)(1+x^2)\cdots = \prod_{i=1}^{\infty}(1+x^i)$
A part in a partition is said to be odd if its size is odd. Let b_r denote the number of partitions of r into odd parts. Then, generating function for b_r is $\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$.
Useful properties:
- The number of partitions of r into distinct parts is equal to the number of partitions of r into odd parts.
- For each $n \in N$, the number of partitions of n into pars each of which appears at most twice, is equal to the number of partitions of n into parts of sizes which are not divisible by 3.
- For any $n, k \in N$, the number of partitions of n into parts, each of which appears at most k times, is equal to the number of partitions of n into parts the sizes of which are not divisible by $k+1$.
- Let $k, n \in N$ and $k \leq n$. Then, the number of partitions of n into k parts is equal to the number of partitions of n into parts the largest size of which is k (follows from Ferrers diagram).

Tutorials

- $P^n_r = nP^{n-1}_{r-1}$
- $P^n_r = \frac{n}{n-r} P^{n-1}_r$
- $P^{n+1}_r = P^n_r + rP^n_{r-1}$
- $P^{n+1}_r = r! + r(P^n_{r-1} + P^{n-1}_{r-1} + \cdots + P^r_{r-1}$
- $(n+1)(n+2)\cdots(2n)$ is divisible by 2^n since $\frac{(2n)!}{n!2^n}$ is the number of pairings of a set with n elements.
- For any function $f(k)$, we can separate / decompose the even and odd terms of f as follows:
$$\sum_{k=0, k \ even}^{n} f(k) = f(0) + f(2) + f(4) + \cdots = \sum_{k=0}^{n} \frac{1^k + (-1)^k}{2} f(k)$$
and
$$\sum_{k=1, k \ odd}^{n} f(k) = f(1) + f(3) + f(5) + \cdots = \sum_{k=0}^{n} \frac{1^k - (-1)^k}{2} f(k)$$