MA2214 Final Exam Cheatsheet

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Permutations and Combinations

- If a natural number n has as its **prime factorization** $n = p_1^{k_1} \cdots p_r^{k_r}$, then a positive integer m is a divisor of n iff m is of the form $p_1^{a} \cdots p_r^{z}$ where every power of p_i in m is less than or equal to the power of p_i in n. So, the **number of divisors** of n is given by: $\prod_{i=1}^{r}(k_i + 1)$, inclusive of 1 and n.
- Note that 2 **circular permutations** are identical if any one of them can be obtained by a rotation of the other, i.e., we only care about the relative positions of the objects around the circle. Then, the number of circular permutations is: $Q_r^n = \frac{P_r^n}{r!}$.

Combinations

- $\binom{n}{r} = \binom{n}{n-r}$
- $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$
- Number of pairings of a set with n elements: $\frac{(2n!)}{n! \times 2^n}$.

Stirling Numbers of the First Kind Given $r, n \in Z$ with $0 \le n \le r$, let $s(r, n)$ denote the number of ways to arrange r distinct objects around n indistinguishable circles (i.e., we can move the circles around — the relative position of the circles don't matter! — AND we can rotate the circles since the positions ON the circle are indistinguishable too) such that each circle has at least one object. These numbers $s(r, n)$ are called the Stirling numbers of the first kind.

Results of Stirling Numbers:

- $s(r, 0) = 0$
- $s(r, r) = 1$
- $s(r, 1) = (r-1)!$
- $s(r, r-1) = \binom{r}{2}$
- $s(r, n) = s(r-1, n-1) + (r-1)s(r-1, n)$

Arrangements and Selections with Repetitions

- The number of r-permutations of a set with n distinct objects, with repetitions allowed, is given by n^r.
- If we have r_1 objects of type 1, r_2 objects of type 2, $\cdots$, r_k objects of type k, and all the objects of the same type are indistinguishable from each other, then the number of permutations of all the objects is given by: $\frac{n!}{r_1! r_2! \cdots r_k!}$ where $n = \sum_i r_i$ is the total number of objects.
- The multiset $M = \{r_1 \cdot a_1, r_2 \cdot a_2, \ldots, r_n \cdot a_n\}$ where r_i's are non-negative integers and a_i's are distinct objects, consists of r_1 a_1's r_2 a_2's, etc.
- Let H_r^n denote the number of r-element multi-subsets of a set with n distinct elements each of which can be repeated infinitely many times. And we have: $H_r^n = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$.
- H_r^n is also the solution to the "number of non-negative integer solutions of the equation $x_1 + \cdots + x_n = r$".
- H_r^n is also the solution to the "number of ways to distribute r indistinguishable objects into n distinct / labelled boxes where each box can hold any number of objects (incl. 0)" (observe that this is the same as assigning box numbers (from 1 to n) to each of the r objects).

Stirling Number of the Second Kind $S(r, n)$ is defined as the number of ways of distributing r distinct objects into n identical boxes such that no box is empty. Obvious results:

- $S(r, r) = 1$ for any $r \ge 0$
- $S(r, 1) = 1$ for any $r \ge 1$
- $S(r, 0) = S(0, n) = 0$ for all $r, n \in N$
- $S(r, r-1) = \binom{r}{2}$
- $S(r, n) = S(r-1, n-1) + nS(r-1, n)$

Binomial and Multinomial Coefficients

Binomial Theorem

For any integer $n \ge 0$,

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$$

Combinatorial Identities

- Number of Subsets:

$$\sum_{r=0}^{n}\binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

- Even and Odd Terms:

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r} = \binom{n}{r} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$$

and hence,

$$\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2k} = \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{2k+1} = 2^{n-1}$$

- $$\sum_{r=1}^{n} r\binom{n}{r} = \binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}$$
- Vandermonde's Identity:

$$\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0} = \binom{m+n}{r}$$

and a special case is:

$$\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots \binom{n}{n}^2 = \binom{2n}{n}$$

- Chu Shih-Chieh's Identity (Hockey Stick Identity):

$$\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$$

and,

$$\binom{r}{0} + \binom{r+1}{1} + \cdots \binom{r+k}{k} = \binom{r+k+1}{k}$$

Multinomial Theorem

Let $\binom{n}{n_1, n_2, \cdots, n_m}$ denote the number of ways to distribute n distinct objects into m distinct boxes such that n_1 of them are in box 1, n_2 in box 2, $\cdots$, and n_m in box m, where $\sum_{i=1}^{m} n_i = n$.

$$\binom{n}{n_1, n_2, \cdots, n_m} = \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-(n_1+n_2+\cdots+n_{m-1})}{n_m} = \frac{n!}{n_1! n_2! \cdots n_m!}$$

and this is precisely the coefficient of $x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$ in the expansion of $(x_1 + \cdots x_m)^n$.

Notice that the x_i's are symmetric, i.e., there is nothing special about x_1 vs. x_2, i.e., the coefficient of a term only depends on the set of frequencies of the powers NOT the ordering of the powers.

$$(x_1 + x_2 + \cdots + x_m)^n = \sum \binom{n}{n_1, n_2, \cdots, n_m} x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$$

where the sum is taken over all m-ary sequences $(n_1, n_2, \cdots, n_m)$ of non-negative integers with $\sum_{i=1}^{m} n_i = n$

Useful Identities:

- $$\binom{n}{n_1, n_2, \cdots, n_m} = \binom{n-1}{n_1-1, n_2, \cdots, n_m} + \binom{n-1}{n_1, n_2-1, \cdots, n_m} + \cdots + \binom{n-1}{n_1, n_2, \cdots, n_m-1}$$
- $$\sum \binom{n}{n_1, n_2, \cdots, n_m} = m^n$$

  by letting $x_1 = x_2 = \cdots = x_m = 1$ in the multinomial theorem.
- H_r^n is also the solution to "number of distinct terms in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$ (because there's a clear bijection from each distinct term to solution of $x_1 + \cdots x_m = n$.

Pigeonhole Principle and Ramsey Numbers

Pigeonhole Principle (PP): Let k and n be any two positive integers. If at least $kn + 1$ objects are distributed among n boxes, then (at least) 1 of the boxes must contain at least $k + 1$ objects.

A clique is a configuration considering of a finite set of vertices together with edges joining all pairs of vertices. A k-clique is a clique which has exactly k vertices.

Given $p, q \in \mathbb{N}$, let $R(p, q)$ denote the smallest natural number n such that for ANY coloring of the edges of an n-clique by 2 colors, there exists either a "blue p-clique" or a "red q-clique". The numbers $R(p, q)$ are called **Ramsey numbers**.

We have the following obvious results:

- $R(p, q) = R(q, p)$, i.e., blue and red are symmetric
- $R(1, q) = 1$
- $R(2, q) = q$ for $q \ge 2$

Ramsey's Theorem: For all integers $p, q \ge 2$, the number $R(p, q)$ always exists.

Bound: $R(p, q) \le R(p-1, q) + R(p, q-1)$

If both $R(p-1, q)$ and $R(p, q-1)$ are even, then the bound can be reduced by 1.

By induction, we get: $R(p, q) \le \binom{p+q-2}{p-1}$ for $p, q \ge 2$

Remember: $R(3, 3) = 6; R(3, 4) = 9; R(3, 3, 3) = 17$.

Generalized Pigeonhole Principle (GPP): Let $n, k_1, k_2, \cdots, k_n \in \mathbb{N}$. If $k_1 + k_2 + \cdots + k_n - (n-1)$ or more objects are put into n boxes, then either the first box contains at least k_1 objects, or the second box contains at least k_2 objects, ..., or the nth box contains at least k_n objects.

Generating Functions

Let $(a_r) = (a_0, a_1, \cdots, a_r, \cdots)$ be a sequence of numbers. The **ordinary generating function** for the sequence (a_r) is defined to be the power series:

$$A(x) = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots$$

For each $\alpha \in R$ (not necessarily an integer!) and each $r \in N$, define the generalized binomial coefficient $\binom{\alpha}{r}$ to be:

$$\binom{\alpha}{r} = \frac{P_r^\alpha}{r!}$$

where $P_r^\alpha = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-r+1)$.

Newton's Expansion:

$$(1 \pm x)^\alpha = 1 \pm \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 \pm \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots + (-1)^r\frac{\alpha\cdots(\alpha-r+1)}{r!}x^r$$

Then, we can derive the following common series:

- $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$
- $$\frac{1}{1-kx} = 1 + kx + k^2x^2 + k^3x^3 + \cdots$$
- $$\frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} H_r^n x^r$$
- $$1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

And some very useful techniques (assume that $A(x)$ is the ordinary generating function for (a_r)), then:

- To get the series of partial sums, we can multiply $A(x)$ by $\frac{1}{1-x}$. That is, $\frac{A(x)}{1-x}$ is the generating function for the sequence: $a_0, a_0 + 1, a_0 + a_1 + a_2, \cdots$ and in general $c_r = \sum_{i=0}^{r} a_i$.
- To get the difference between consecutive coefficients, you can use $(1-x)A(x)$. That is, $(1-x)A(x)$ is the generating function for the sequence $(a_0, a_1 - a_0, a_2 - a_1, \cdots)$
- $A'(x)$ is the generating function for the series $(0, a_1, 2a_2, 3a_3)$ (and in general, $c_r = (r+1)a_{r+1}$).
- $xA'(x)$ is the generating function for the sequence (c_r) where $c_r = ra_r$, i.e., $(0, a_1, 2a_2, 3a_r, \cdots)$
- - $\int_0^x A(t)dt$ is the generating function for the sequence (c_r) where:
  - $c_0 = 0$, and
  - $c_r = \frac{a_{r-1}}{r}$

  i.e., $(c_r) = (0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \cdots)$

Ordinary generating functions can be used when we want to count the number of ways of selecting r elements from an arbitrary multiset, *where the order does not matter.* If the order matters, we would have to use exponential generating functions instead.

And each product term corresponds to the ways we can select a particular element of the multiset.

Examples:

- Number of ways to select 4 members from $M = \{2 \cdot b, 1 \cdot c, 2 \cdot d, 1 \cdot e\}$. Generating function $= (1+x+x^2)(1+x)(1+x+x^2)(1+x)$.
- If $M = \{\infty \cdot x_1, \infty \cdot x_2, \cdots, \infty \cdot x_k\}$, then generating function $= (1+x+x^2+\cdots)^k = \frac{1}{(1-x)^k} = \sum_{r=0}^{\infty} H_r^n x^r$

Partitions

A partition of a positive integer n is a collection of positive integers whose sum is n, where the ordering is not taken into account. That is, a partition can be characterized by a multiset denoting the sizes of each part.
If $n = n_1 + n_2 + \cdots + n_k$ is a partition of n, we say that n is partitioned into k parts of sizes $n_1, n_2, \cdots, n_k$ respectively.
A partition of n is equivalent to a way of distributing n identical objects into n identical boxes.
Example: Let a_r be the number of partitions of an integer r into parts of sizes $1, 2, 3$. The generating function for (a_r) is:
$(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots) = \frac{1}{(1-x)(1-x^2)(1-x^3)}$.
Example: Let a_r be the number of partitions of r into distinct parts. Then, the generating function is $(1+x)(1+x^2)\cdots = \prod_{i=1}^{\infty}(1+x^i)$
A part in a partition is said to be odd if its size is odd. Let b_r denote the number of partitions of r into odd parts. Then, generating function for b_r is $\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$.
Useful properties:

- The number of partitions of r into distinct parts is equal to the number of partitions of r into odd parts.
- For each $n \in N$, the number of partitions of n into pars each of which appears at most twice, is equal to the number of partitions of n into parts of sizes which are not divisible by 3.
- For any $n, k \in N$, the number of partitions of n into parts, each of which appears at most k times, is equal to the number of partitions of n into parts the sizes of which are not divisible by $k+1$.
- Let $k, n \in N$ and $k \leq n$. Then, the number of partitions of n into k parts is equal to the number of partitions of n into parts the largest size of which is k (follows from Ferrers diagram).

Principle of Inclusion-Exclusion

$$|A_1 \cup \cdots \cup A_q| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{q+1}|A_1 \cap \cdots \cap A_q|$$

For an integer m with $0 \leq m \leq q$, let $E(m)$ denote the number of elements of S that possess exactly m of the q properties (here, E stands for exactly). And let $\omega(P_{i_1}, P_{i_2}, \cdots, P_{i_m})$ denote the number of elements of S that possess (at least) all of the properties $P_{i_1}, P_{i_2}, \cdots, P_{i_m}$. Also, let:

$$\omega(m) = \sum \omega(P_{i_1}, P_{i_2}, \cdots, P_{i_m})$$

where the summation is taken over all the m-combinations $\{i_1, i_2, \cdots, i_m\}$ of $\{1, 2, \cdots, q\}$.
We also define $\omega(0) = |S|$.
Let S be an n-element set, and let $\{P_1, \cdots, P_q\}$ be a set of q properties for elements of S. Then, for each $m = 0, 1, 2, \cdots, q$,

$$\begin{aligned} E(m) &= \omega(m) - \binom{m+1}{m}\omega(m+1) + \binom{m+2}{m}\omega(m+2) - \\ &\cdots + (-1)^{q-m}\binom{q}{m}\omega(q) \\ &= \sum_{k=m}^{q}(-1)^{k-m}\binom{k}{m}\omega(k) \end{aligned}$$

Corollary:

$$E(0) = \omega(0) - \omega(1) + \cdots + (-1)^q\omega(q) = \sum_{k=0}^{q}(-1)^k\omega(k)$$

Stirling's Number of Second Kind

Theorem: Let $F(n, m)$ where $n, m \in N$ denote the number of surjective mappings from $N_n \to N_m$. Then:

$$F(n, m) = \sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-k)^n$$

and $S(n, m) = \frac{1}{m!}F(n, m)$ where $S(n, m)$ is the number of ways to distribute n distinct objects into m identical boxes (Stirling's number of second kind).

Sieve of Eratosthenes

For every $n \in N, n > 1$, there exists primes $p_1 < p_2 < \cdots < p_k$ and positive integers $m_1, m_2, \cdots, m_k$ such that:

$$n = \prod_{i=1}^{k} p_i^{m_i}$$

and such a factorisation is unique, if we disregard the ordering of the primes.
Let $\phi(n)$ denote the number of integers between 1 and n which are coprime to n.
Let $n \in N$ and $n = p_1^{m_1} \cdots p_k^{m_k}$ be its prime factorization. Then:

$$\phi(n) = n\prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right)$$

Recurrence Relations

Linear Homogeneous Recurrence Relations

$$c_0a_n + c_1a_{n-1} + \cdots + c_ra_{n-r} = 0$$

is of the rth order, and homogeneous (since RHS = 0).
Case 1. If $\alpha_1, \alpha_2, \cdots, \alpha_r$ are the distinct characteristic roots of the recurrence relation (i.e., all the r roots are distinct), then

$$a_n = A_1(\alpha_1)^n + A_2(\alpha_2)^n + \cdots + A_r(\alpha_r)^n$$

where the A_i's are constants (that can be determined from the initial conditions), is the general solution.
Case 2. If $\alpha_1, \alpha_2, \cdots, \alpha_k$ where $1 \leq k \leq r$ are the distinct characteristic roots (i.e., we have k distinct roots, not necessarily all distinct) such that α_i is of multiplicity m_i for $i = 1, \cdots, k$, then the general solution of the recurrence relation is given by

$$a_n = \sum_{i=1}^{k}(A_{i1} + A_{i2}n + \cdots + A_{im_i}n^{m_i-1})(\alpha_i)^n$$

where the A_{ij}'s are constants.

Non-Homogeneous Linear Recurrence Relations

$$c_0a_n + c_1a_{n-1} + \cdots + c_ra_{n-r} = f(n)$$

Steps:

1. Find the general solution $a_n^{(h)}$ (h = homogeneous) of the linear homogeneous recurrence relation using previous section.
2. Find a particular solution $a_n^{(p)}$ of the given recurrence relation by "guessing" the form of the function $f(n)$ (and find the coefficients by plugging into the recurrence relation itself).
3. The general solution of the given recurrence relation is given by: $a_n = a_n^{(h)} + a_n^{(p)}$

Note: only use the initial conditions in the final step, not in step (2.)

$f(n)$	$a_n^{(p)}$
Ak^n where k is NOT a characteristic root of the homogeneous recurrence relation obtained in (1)	Bk^n
Ak^n where k IS a characteristic root with multiplicity m	Bn^mk^m
$\sum_{i=0}^{t} p_in^i$ where 1 is NOT a characteristic root	$\sum_{i=0}^{t} q_in^i$
$\sum_{i=0}^{t} p_in^i$ where 1 IS a characteristic root with multiplicity m	$n^m\sum_{i=0}^{t} q_in^i$
An^tk^n where k is NOT a characteristic root	$\left(\sum_{i=0}^{t} q_in^i\right)k^n$
An^tk^n where k IS a characteristic root with multiplicity m	$n^m\left(\sum_{i=0}^{t} q_in^i\right)k^n$

Graph Theory

Basic Terminology

- $G = (V, E)$ is a (n, m) graph.
- $v(G) = |V(G)| =$ order
- $e(G) = |E(G)| =$ size
- $N_G(v_i) =$ neighbors of vertex i
- $d_G(v_i) =$ degree of vertex i
- $\Delta(G) = \max(d_G(v_i))$
- $\delta(G) = \max(d_G(v_i))$
- $k-$regular graph = all vertices have degree k
- $O_n =$ empty graph of order n (size = 0)
- $K_n =$ complete graph of order n (size $= \binom{n}{2}$)
- subgraph $= V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$
- proper subgraph = either $V(H) \neq V(G)$ or $E(H) \neq E(G)$
- spanning subgraph $= V(H) = V(G)$
- induced subgraph $= E(H) = \{uv : u, v \in V(H), uv \in E(G)\}$
- $\bar{G} =$ complement of $G = V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{uv : uv \notin E(G)\}$
- adjacency matrix $A(G)$: $n \times n$, $s_{ij} = 1$ iff $v_iv_j \in E(G)$ else 0
- incidence matrix $M(G)$: $n \times m$, $m_{ij} = 1$ iff v_i is incident to edge j
- discrete laplacian (aka kirchoff matrix) $L(G)$: $n \times n$,
  $a_{ij} = \begin{cases} deg(v_i), & \text{if i = j,} \\ -1, & \text{if } v_iv_j \in E(G) \\ 0, & \text{otherwise} \end{cases}$

Walks, Paths and Trails

- walk: both vertices and edges can be repeated
- trail: vertices can be repeated, not edges
- path: all vertices distinct
- path $\implies$ trail (i.e., path is "stronger" than trail)
- open: start vertex $\neq$ end vertex
- length: number of edges
- circuit: closed trail of length ≥ 2
- cycle: circuit + no duplicate vertices (except start and end)
- P_n: path of order n
- C_n: cycle of order n
- u and v are "connected" if there is a path between them.
- disconnected graph: $\exists u, v$ s.t. u and v are not connected.
- component of u = all vertices reachable / connected from u
- $\omega(G) =$ number of connected components in G.
- $\omega(G) = 1 \iff$ G is connected.
- connectedness is an equivalence relation.

Bridges and Cut-Vertices

- $G \setminus V'$ (for some $V' \subseteq V(G)$) is the induced subgraph of G on the set $G \setminus V'$ of vertices.
- $G \setminus E'$ is the graph G with the edges in E deleted.
- cut vertex: removing the vertex increases the number of connected components
- bridge: removing this edge increases the number of connected components.

Spanning Trees

- spanning tree: spanning subgraph which is a tree
- $\tau(G)$: number of spanning trees of G
- $\tau(G) > 0$ iff G is connected
- $\tau(G) = 1$ iff it is a tree
- $\tau(C_n) = n$
- if G has a single cycle which is a copy of C_n, then $\tau(G) = n$
- if G has a bridge e, and $G \setminus \{e\}$ has components G_1, G_2 then $\tau(G) = \tau(G_1)\tau(G_2)$
- if G has cut-vertex v and $G \setminus \{v\}$ has components $G_1, \cdots, G_k$ then $\tau(G) = \prod_{i=1}^{k}\tau(G_i \cup \{v\})$
- if G is obtained by C_p and C_q sharing 1 edge, then $\tau(G) = (p+q-2) + (p-1)(q-1)$

Theorems

- Handshaking Lemma: $\sum_{i=1}^{n} d_G(v_i) = 2m$
- If there is a $u-v$ walk in G with length k, then there is a $u-v$ path in G with length at most k.
- If G is disconnected, then $\bar{G}$ is connected
- The following are equivalent characterizations of trees:
  1. connected graph with no cycle
  2. every two distinct vertices are joined by a unique path
  3. $m = n - 1$
- A vertex v is a cut-vertex iff $\exists a, b$ such that v is on every $a-b$ path
- If G is a tree, then every vertex in G is either an end-vertex (degree = 1) or a cut-vertex
- An edge e is a bridge iff e is not contained in any cycle of G
- In a tree, every edge is a bridge
- Cayley's formula: The number of spanning tree of K_n is n^{n-2}
- Kirchhoff's Matrix Tree theorem: Number of spanning trees in G is the cofactor of any entry in the discrete laplacian of G. (cofactor of ij $= (-1)^{i+j}\det A_{ij}$ where A_{ij} is obtained by removing the i-th row, and j-th column)
- Kirchhoff's Matrix Tree theorem $\implies$ Cayley's formula