MA2101 Test 1 Cheatsheet

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Important Facts

Vectors, Matrices

$$AB \neq BA$$
$$A(BC) = (AB)C$$
$$A(B+C) = AB + AC$$
$$\vec{u} \times \vec{v} = |\vec{u}||\vec{v}| \sin\theta \hat{n}$$
$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$
$$\det M^{-1} = \frac{1}{\det M}$$
$$\det M^T = \det M$$
$$\det AB = \det BA = (\det A) \times (\det B)$$
$$\det(A+B) \neq \det A + \det B$$
$$\det(cA) = c^n \det A$$
$$\text{Orthogonal} \iff MM^T = 1$$
$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(A^T)^{-1} = (A^{-1})^T$$
$$(AB)^T = B^T A^T$$
$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Trigonometry Facts

$$\sin^2 x + \cos^2 x = 1$$
$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\sin 2x = 2 \sin x \cos x = \frac{2\tan x}{1+\tan^2 x}$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$
$$\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

Taylor Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(2n+1)!} x^{2n+1}$$

Chapter 1

Linear Transformation

$$T(c\vec{u}) = cT(\vec{u})$$
$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$$

It maps straight lines to straight lines.
If you know what T does to $\hat{\imath}$ and $\hat{\jmath}$, then you know what T does to any vector.
The **matrix of a transformation** relative to $\hat{\imath}, \hat{\jmath}$ is given by putting what happens to each of $\hat{\imath}, \hat{\jmath}$ into the 2 columns. That is, the first column tells us what happened to $\hat{\imath}$ and the second column tells us what happened to $\hat{\jmath}$.

Important Transformations

Stretching along $\vec{u}$ by a factor α

$$S_{\vec{u}}^{\alpha} : \vec{v} \to \vec{v} + (\alpha - 1)(\vec{u} \cdot \vec{v})\vec{u}$$

For stretching along $\hat{\imath}$, the matrix is given by:

$$\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}$$

Reflection

Reflection along the line $x = y$ can be expressed in matrix form as:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

, i.e., $\hat{\imath}$ becomes $\hat{\jmath}$ and vice versa.

Shearing

Shearing parallel and along the direction of $\hat{\imath}$ by an angle θ is written as:

$$\begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}$$

where $\tan\theta$ is the distance moved by the top of the basic box. T
Note: Shearing by θ first, and then by ϕ again, also results in a shearing transformation BUT the shear angles don't add up.

Rotation

Rotating anti-clockwise by an angle θ is expressed as

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotations in 2D are commutative, but not in higher dimensions.
Note: "Differentiating a matrix" means differentiating each of the individual entries.
When composing linear transformations:

- Order of composition matters!
- Matrix relative to the composition is equal to the matrix product of the individual linear transformations.

Determinants and Inverses

The area of a parallelogram described by the 2 vectors $\vec{u}$ and $\vec{v}$ is given by $|\vec{u} \times \vec{v}|$.
Similarly, the volume of a parallelopiped described by $\vec{u}, \vec{v}, \vec{w}$ is given by $|\vec{u} \times \vec{v} \cdot \vec{w}|$.
Determinant of a transformation (in 2D) is defined to be the ratio of the final area of the basic box to the initial area of the basic box. So, $\det T = \pm 1$ means that the area ins unchanged (e.g. shearing, rotations, reflections) while $\det T = 0$ means the basic box is squashed flat, zero area.
We can use any basis vectors to define the determinant of a transformation.
A **singular** linear transformation:

- maps two different vectors to one vector
- destroys all of the vectors in at least one direction
- loses all information associated with these directions
- satisfies $\det T = 0$

Conversely, a non-singular transformation never maps 2 vectors to one, and has non-zero determinant. It is a "reversible" operation, i.e., an inverse exists.
We can find the inverse by finding the transpose of the cofactor matrix, and dividing by the determinant of the original matrix.
The **rank** of a matrix is the number of dimensions in the output space (image) of the transformation associated with it.

Solving Linear Systems

Let the system of equations be written as $\mathbf{M}\vec{r} = \vec{a}$

1. $\mathbf{M}$ is invertible (non-singular) $\implies$ unique solution exists.
2. $\mathbf{M}$ is singular
   (a) If $\vec{a}$ lies in the image vector space of $\mathbf{M}$, infinitely many solutions exist.
   (b) Else, no solution.

Eigenvectors and Eigenvalues

Eigenvectors are special vectors that don't have their direction changed when the transformation acts on them.
That is, $T\vec{u} = \lambda\vec{u}$, then $\vec{u}$ is called an eigenvector of $\mathbf{T}$, and the scalar λ is called the eigenvalue of $\vec{u}$.
You can find the eigenvalues by solving the equation $\det(T - \lambda I) = 0$.
There are infinitely many eigenvectors (all multiples of each other) for a given eigenvalue of a transformation.
A $n \times n$ matrix can have at most n eigenvalues (some of them may be repeated, or even complex). But if the matrix is real (all entries are real), then complex eigenvalues, if they exist, occur together in conjugate pairs.

Change of Basis

Let $\mathbf{P}$ be the transformation that takes $\hat{\imath} \to \vec{u}$ and $\hat{\jmath} \to \vec{v}$ where $\vec{u}, \vec{v}$ are not parallel (assume 2-D). That is, $\vec{u} = P\hat{\imath}$ and $\vec{v} = P\hat{\jmath}$.
Then, the components of ANY vector relative to $\vec{u}, \vec{v}$ are obtained by multiplying $\mathbf{P}^{-1}$ into the components relative to $\hat{\imath}, \hat{\jmath}$.
And the matrix of any transformation $\mathbf{T}$ relative to $\vec{u}, \vec{v}$ is obtained by multiplying $\mathbf{P}^{-1}$ on the left and $\mathbf{P}$ on the right into the matrix of $\mathbf{T}$ relative to $\hat{\imath}, \hat{\jmath}$.
The matrix of a transformation relative to its own eigenvectors (assuming that these form a basis, which is not always the case) is diagonal.

If the matrix $\mathbf{M}$ can be diagonalized, then $\mathbf{M} = \mathbf{PDP^{-1}}$, where $\mathbf{D}$ is the diagonal matrix consisting of the eigenvalues of $\mathbf{M}$, and $\mathbf{P}$ is the matrix that transforms the original (standard) basis to the eigenvectors
Not every matrix can be diagonalized.
Trace of $\mathbf{M}$ denoted by TrM is the sum of the diagonal entries.
In general, $TrMN = TrNM$ but $TrMN \neq TrM\ TrN$.
Hence, $Tr(P^{-1}DP) = Tr(DPP^{-1}) = TrD$, so the trace of a matrix is always the same, no matter which basis you use.
The trace is equal to the sum of the eigenvalues. The determinant is equal to the product of the eigenvalues.
The trace and determinant of a linear transformation are well-defined, they don't depend on the basis chosen to represent the transformation in matrix form.

Chapter 2

Vector

F^n is the set of all lists of length n consisting of elements of F ($\mathbb{R}$ or $\mathbb{C}$).
A **vector space** is a set V with an addition and scalar multiplicatin rule such that:

1. $\forall u, v \in V,\ u + v = v + u$ (commutativity of addition)
2. $\forall u, v, w \in V,\ (u + v) + w = u + (v + w)$ and $\forall a, b \in F, \forall v \in V,\ (ab)v = a(bv)$ (associativity of addition and scalar multiplication)
3. $\exists 0 \in V, \forall v,\ v + 0 = v$ (additive identity, called zero vector)
4. $\forall v \in V, \exists w \in V,\ v + w = 0$ (additive inverse)
5. $\forall v \in V,\ 1v = v$ where $1 \in F$ (multiplicative identity)
6. $\forall a, b, \in F, \forall u, v \in V,\ a(u + v) = au + av$ and $(a + b)u = au + bu$ (multiplication is distributive both ways)

Subspace

A subset U of a vector space V is called a **subspace** if U is a vector space with the same scalar multiplication and addition as in V.
Suppose U_1 and U_2 are sub-SETS of a vector space V. Then, the **sum** of these subsets, denoted by $U_1 + U_2$ is defined as the subset of V consisting of all vectors of the form $u_1 + u_2$ where $u_1 \in U_1, u_2 \in U_2$.
The sum of two *subspaces* is again a subspace.
Let U_1 and U_2 be *subspaces* of a vector space V. Suppose that the intersection $U_1 \cap U_2 = \{0\}$. Then, $U_1 + U_2$ is called the **direct sum** of U_1 and U_2, denoted by $U_1 \oplus U_2$.
Suppose that you've expressed a vector $v \in V$ as $u_1 + u_2 \in U_1 \oplus U_2$, then that expression is unique.
Consider F^n, and define the dot product of two vectors in the same way as in two or three dimensions. Let U be any subspace of F^n; since it has to contain the zero vector, it is a hyperplane passing through the origin. Let $U^\perp$ be the set of all vectors in F^n which are perpendicular to every vector in U. Then, $F^n = U \oplus U^\perp$ (given any subspace of F^n, we can find another another subspace of F^n whose direct sum gives us F^n). We can think of it as $U^\perp = F^n - U$.

Vector Space Isomorphisms

Let S and T be two sets. A mapping $F : S \to T$ is said to be:

- **surjective**: $\forall t \in T, \exists s \in S,\ F(s) = t$ (everything in T can be expressed in the form $F(s)$)
- **injective**: $\forall s_1, s_2 \in S,\ F(s_1) = F(s_2) \implies s_1 = s_2$ (there is at most one way to express any vector in T in the form $F(s)$)
- **bijective**: both surjective and injective.

F has an inverse $\iff$ F is a bijection.
Let U and V be vector spaces. A mapping $\Phi : U \to V$ is said to be a **vector space homomorphism** if $\forall u, v \in U, \forall a \in F$:

1. $\Phi(u + v) = \Phi(u) + \Phi(v)$
2. $\Phi(au) = a\Phi(u)$

If there exists a vector space homomorphism from $U \to V$ which is a bijection, then we call the homomorphism an **isomorphism** and we say that the two vector spaces are **isomorphic**.
F^{n+m} has two subspaces U and V isomorphic to F^n and F^m respectively such that $U \oplus V = F^{n+m}$ (sloppy version: $F^n \oplus F^m \leftrightarrow F^{n+m}$)
Vector space isomorphisms are not unique, i.e., if there is one isomorphism from one vector space to another, then there are (infinitely) many more.
A vector space is said to be **finite-dimensional** over F if it is isomorphic to F^n for some (finite) integer n.
No finite-dimensional vector space can be isomorphic to F^n for 2 different values of n, so this is a fixed property of the vector space. It is called the **dimension** of the vector space.

Spanning, Linear Independence, Basis

A **linear combination** of the elements of a list of vectors v_i is just any vector (which may or may not lie in the list) of the form $a^1v_1 + a^2v_2 + \cdots$, where the a^i are scalars.
The **span** of a list of vectors is defined to be the set of all linear combinations of the vectors in the list.
Given any list of vectors in V, the span of that list is a subspace of V.
A list of vectors v_i in V is said to be **linearly independent** if the equation $\sum a^i v_i = 0$ implies that $a^i = 0$ for *every* i.
Linear independence means that the elements of the span can be expressed as a linear combination in a *unique* way.
A **basis** of a vector space V is any list of vectors in V which both spans V and is linearly independent.
In other words, every vector is covered by this process (surjectivity), and it happens in a unique way (injectivity).
Every finite-dimensional vector space has a basis. In fact, all vector spaces (other than the one consisting of just the zero vector) have infinitely many basis.
When we write $v = a^i z_i$ for some basis z, the numbers a^i are called the **components** of v *relative to the basis*.
A basis for V is a vector space isomorphism (mapping) from $F^n \to V$, where n is the dimensionality of the vector space.
Suppose that $z : F^n \to V$ is a basis (in the new sense, i.e., it is a vector space isomorphism). Then, the list of vectors $z_i =\equiv z(e_i)$ is a basis for V (in the old-fashioned sense, i.e., it is a list of vectors).
So, the existence of a basis is just a matter of definition for the finite-dimensional case.
The number of vectors in an old-fashioned basis of a finite-dimensional vector space is always equal to the dimension.
Suppose U_1 and U_2 are subspaces of a finite-dimensional vector space V, with the intersection consisting only of the zero vector. Then, $\dim(U_1 \oplus U_2) = \dim(U_1) + \dim(U_2)$.
Suppose U is a subspace of a finite-dimensional vector space V. Then, V has another subspace W such that $V = U \oplus W$.

Results from Tutorials

Tutorial 1

1. Any 2×2 matrix can be expressed as:

$$a\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + c\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} + d\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$

2. Let $\mathbf{c}$ be an n-dimensional column matrix, and $\mathbf{r}$ be an n-dimensional row matrix. Then, $\mathbf{cr}$ is a $n \times n$ matrix, with rank one. In particular, it is singular.

Tutorial 2

1. To show that a vector w does *not* lie in the plane described by u, v, we can show that $(u \times v) \cdot w$ is non-zero. That is, the volume described by u, v, w is non-zero, and so w does not lie in the plane. Obviously, if the triple product $(u \times v) \cdot w = 0$, then w lies in the same plane as u and v.

2. Numbers are independent of basis! They cannot change when you change the basis.

3. Under a change of basis matrix $\mathbf{P}$, column vectors change from $\mathbf{c} \to \mathbf{P}^{-1}\mathrm{1c}$, row vectors change from $\mathbf{r} \to \mathbf{rP}$, and matrices change from $\mathbf{M} \to \mathbf{P^{-1}MP}$

4. A square matrix $\mathbf{A}$ is said to be **similar** to a matrix $\mathbf{B}$ if there exists a non-singular matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P^{-1}BP}$. This means that $\mathbf{A}$ and $\mathbf{B}$ are really just two different ways of representing a single linear transformation, since you get one from the other just by changing basis. Also, every diagonalisable matrix is similar to its transpose

Tutorial 3

1. A rotation matrix has no real eigenvalues except when $\theta = 0$ or π. But every 3x3 matrix has at least one real eigenvalue, so this is also true or 3-dimensional rotation matrices. A real matrix can have complex eigenvalues but these must occur in complex-conjugate pairs. And since the trace of a real matrix is real (and the number of eigenvalues is 3), there can either be zero or two complex eigenvalues, leaving at least one real eigenvalue.

2. A vector space cannot have two different zero vectors. A vector cannot have two different additive inverses (i.e., it is unique to a vector). $0v = 0$ where the 0 on the LHS is the number zero. We define the unique additive inverse of u to be $-u$. Then, $(-1)u = -u$.

3. It is a fact that $\int_0^\pi \sin(nx)\sin(mx) = 0$ for any $n \neq m$. Then, we can show that $\sin x, \sin 2x, \sin 3x$ are linearly independent functions. But they do not span the space of all smooth functions from $[0, \pi]$ because we cannot express $\sin 4x$ as the linear combination of these functions. In fact, the space has infinite dimensions (read: Fourier Series).

4. If U_1 and U_2 are subspaces of a finite-dimensional vector space V such that $V = U_1 \oplus U_2$, and ϕ is a vector space isomorphism from $V \to W$, then $W = \phi(U_1) \oplus \phi(U_2)$.

5. A vector space cannot be isomorphic to F^n for two different values of n. That is, F^m cannot be isomorphic to F^n for $m \neq n$.