Operations Research: MATH 5810-001
1.3.1. Largest Ball in a Polyhedron

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Executive Summary

A polyhedron is a solid convex region in space with flat faces or sides. By "convex", we mean that any line segment connecting two points in the polyhedron is itself contained in the polyhedron. Examples of polyhedra in two dimensions include squares, triangles, pentagons, etc. Examples of polyhedra in three dimensions include cubes, tetrahedrons (pyramids), octahedrons, etc. Note that in mathematics, a polyhedron is not required to be bounded. For example, an infinitely tall prism with no top or bottom is a polyhedron.

In this report, we describe a solution to the following problem: Find the largest ball that can fit inside a given polyhedron. In particular, find the maximum radius that a ball inside the polyhedron can have, and find the center of one such largest ball.

Our solution is a linear program that can be solved for any given polyhedron. We implemented our linear program in the modeling language GLPK. We tested our program for a tetrahedron (a "pyramid" with four triangular faces) and obtained reasonable results.

1. Problem Description

Let $\mathbb{E}^n$ be n-dimensional Euclidean space (see Section 2. for an explanation). Given vectors $a_1, \ldots, a_m \in \mathbb{E}^n$ and scalars $b_1, \ldots, b_m \in \mathbb{R}$, define the polyhedron

$$P = \{x \in \mathbb{E}^n : a_i^\top x \leq b_i \text{ for } i = 1, \ldots, m\}.$$

Find the largest ball

$$B_r(p) = \{x \in \mathbb{E}^n : \|x - p\| \leq r\}$$

that is contained in P. In other words, find the largest radius r such that $B_r(p) \subset P$ for some center p. Apply the general solution to the three-dimensional tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(2, 5, 0)$, and $(1, 6, 1)$.

Our general solution is a linear program that returns the radius r and center p of the largest ball contained in a given polyhedron P. We solve our linear program for the tetrahedron example described above.

2. Background and Resources

Euclidean space $\mathbb{E}^n$ is the real vector space $\mathbb{R}^n$ equipped with the Euclidean norm $\|\cdot\|$ induced by the ordinary dot product.

A linear program is an optimization problem of the form

$$\text{Minimize } z = c_1 x_1 + \cdots + c_n x_n \text{ subject to}$$

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &\leq b_2 \\ \vdots \quad \vdots \quad \vdots &\leq \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\leq b_m. \end{aligned}$$

3. Description of General Solution

Let $P \subset \mathbb{E}^n$ be a polyhedron as defined in the problem statement. For every $i = 1, \ldots, m$, define the hyperplane

$$P_i = \{x \in \mathbb{E}^n : a_i^\top x = b_i\}.$$

Observe that

$$a_i^\top \left(\frac{b_i}{\|a_i\|^2} a_i \right) = \frac{b_i (a_i^\top a_i)}{\|a_i\|^2} = \frac{b_i \|a_i\|^2}{\|a_i\|^2} = b_i.$$

Thus, $\frac{b_i}{\|a_i\|^2}a_i \in P_i$. Note $\frac{a_i}{\|a_i\|}$ is a unit vector orthogonal to P_i. Since $\frac{b_i}{\|a_i\|^2}a_i = \frac{b_i}{\|a_i\|}\left(\frac{a_i}{\|a_i\|}\right)$, then the component of $\frac{b_i}{\|a_i\|^2}a_i$ orthogonal to P_i is $\frac{b_i}{\|a_i\|}$.

Let $p \in P$. The component of p orthogonal to P_i is $\frac{a_i^\top p}{\|a_i\|}$. Therefore, the distance from p to P_i is given by

$$|\frac{b_i}{\|a_i\|} - \frac{a_i^\top p}{\|a_i\|}|.$$

But since $p \in P$, then $a_i^\top p \leq b_i$. Thus, $b_i - a_i^\top p \geq 0$, and hence the distance from p to P_i is simply

$$\frac{b_i}{\|a_i\|} - \frac{a_i^\top p}{\|a_i\|}.$$

For a ball $B_r(p)$ to be contained in P, we require r to satisfy

$$0 \leq r \leq \frac{b_i}{\|a_i\|} - \frac{a_i^\top p}{\|a_i\|}, \text{ for all } i = 1, \ldots, m.$$

To find the maximum feasible value of r, we therefore have the following linear program:

Maximize r subject to

$$\begin{aligned} \|a_i\|r + a_i^\top p \leq b_i \quad & (i = 1, \ldots, m) \\ r \geq 0. & \end{aligned}$$

Note that $r \geq 0$ guarantees that $a_i^\top p \leq b_i$ and hence $p \in P$.

See Section A1. for the implementation of the above linear program in GLPK.

4. Application to the Specific Context

Consider the tetrahedron $T \subset \mathbb{E}^3$ defined by the vertices $(0, 0, 0)$, $(1, 0, 0)$, $(2, 5, 0)$, and $(1, 6, 1)$. This tetrahedron is therefore defined by four planes P_i, $i = 1, 2, 3, 4$. To apply the general solution from Section 3. to T, we must first express each plane P_i by an equation of the form $a_i^\top x = b_i$, where a_i is a vector normal to P_i and b_i is the scalar off-set of P_i from the origin.

Define the planes P_i, $i = 1, 2, 3, 4$, such that

$$\begin{aligned} (0,0,0), (1,6,1), (2,5,0) &\in P_1 \\ (0,0,0), (2,5,0), (1,0,0) &\in P_2 \\ (0,0,0), (1,0,0), (1,6,1) &\in P_3 \\ (2,5,0), (1,0,0), (1,6,1) &\in P_4. \end{aligned}$$

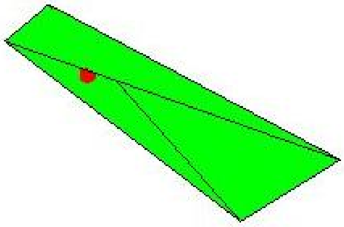

Figure 1: Image of the largest ball (red) contained in the tetrahedron (green) defined in Section 4. . The front face of the tetrahedron has been removed to show the ball inside.

The normal vector a_i is obtained as a cross product of two vectors in P_i. The order of the cross product is chosen such that the tetrahedron T has an outward orientation; i.e., a_i points "out of" T. We therefore have:

$$\begin{aligned}
a_1^\top &= (1,6,1) \times (2,5,0) = (-5,2,-7) \\
a_2^\top &= (2,5,0) \times (1,0,0) = (0,0,-5) \\
a_3^\top &= (1,0,0) \times (1,6,1) = (0,-1,6) \\
a_4^\top &= [(2,5,0) - (1,0,0)] \times [(1,6,1) - (1,0,0)] = (5,-1,6).
\end{aligned}$$

Since the planes P_1, P_2, and P_3 contain the origin, then $b_1 = b_2 = b_3 = 0$. Since P_4 contains $(1,0,0)$, then

$$b_4 = a_4^\top (1,0,0)^\top = 5.$$

We applied our general solution to the tetrahedron T with associated normal vectors a_i and scalar off-sets b_i as computed above. We found that the maximum radius for a ball contained in T is $r \approx 0.11965$. We found that the center of one such largest ball is $p \approx (0.95714, 1.44570, 0.11965)$.

5. Interpretations and Conclusions

Visually, the answer obtained in Section 4. for the largest ball in a particular tetrahedron appears reasonable (Figure 1). We therefore conclude that our algorithm (i.e., our general solution) for finding the largest ball inside a given polyhedron is effective. Future work will include testing the algorithm for other examples of polyhedra.

A1. Code for the General Solution

```
# Largest Ball in a Polyhedron

/* Model  Section */

param m;  # number of hyperplanes defining
   polyhedron
set M := 1..m;

param n;  # dimension of Euclidean space
set N := 1..n;

param A{i in M, j in N};  # rows are normal vectors
   to hyperplanes
param b{i in M};  # off-sets of the hyperplanes

var p{i in N};  # center of ball
var r >= 0;  # radius of ball


maximize Z:  # maximize radius
r;

s.t. C{i in M}:  # radius r is in polyhedron
sqrt(sum{j in N} A[i,j]**2)*r + sum{j in N} A[i,j]*p
   [j] <= b[i];


solve;

printf "Center␣p␣=␣␣(";
printf{j in N}: "%5g␣␣", p[j];
printf ")\n";
printf "\n";
printf "Radius␣r␣=␣␣%5g", r;

data;

param m := 4;
```

```
param n := 3;

param A:  # the ith row is a_i^T
  1 2 3 :=
1 -5 2 7
2 0 0 -5
3 -0 -1 6
4 5 -1 6;

param b :=  # the ith entry is b_i
1 0
2 0
3 0
4 5;

end;
```