Probability

Probability theory a branch of mathematics concerned with the ~~analysis~~ numerical descriptions of how likely an event is to occur, or how likely it is that a preposition is true.

Probability of an event is a number between 0 and 1, where roughly speaking,
0 — indicates impossibility of the event
and 1 — indicates certainty.

Experiment: An experiment is a procedure that yields one of a given set of possible outcomes.

Ex: ① Experiment of rolling a dice gives a set of possible outcomes as number on top face can be from 1 to 6.

② Tossing a coin is an experiment with two possible outcomes namely Heads and Tails.

Sample: The sample space of the experiment is the set of all possible outcomes.

Ex: In case of an experiment of rolling a dice, the sample space is $\{1,2,3,4,5,6$

② In the experiment of rolling two dice the sample space is

$$\{ (1,1), (1,2), \ldots (1,6) \\ (2,1), (2,2) \ldots (2,6) \\ \vdots \\ (6,1)\ (6,2) \ldots (6,6) \}.$$

<u>Event</u>: An Event is a subset of the sample space.

Ex: Getting an even number on the top face of a rolled dice is an event. say A

$$A = \{2, 4, 6\}$$

<u>Equi-Probable Sample Space</u>:

Consider a random experiment, which has n mutually exclusive outcomes. The sample space S is said to be an equi-probable sample space if all the outcomes are equally likely.

Laplace's Definitions of the Probability:

Let E be an event with finitly many possible outcomes, the probability is defined as —.

$$P(E) = \frac{|E|}{|S|}$$, E is subset of a finite sample space S of equally likely outcomes!

Ex: ① In an experiment of rolling dice, the sample space $S = \{1, 2, 3, 4, 5, 6\}$

$P(1) = \frac{1}{6}$, $P(2) = \frac{1}{6}$, $P(3) = \frac{1}{6}$

... $P(6) = \frac{1}{6}$

Another Defn//

If there are n mutually exclusive, collectively exhaustive and equally likely outcomes of an experiment and if m of them are favourable to an event E, then the probability of occurrence of E, denoted by P(E) is defined as

$$P(E) = \frac{m}{n},$$ where $0 \leq m \leq n$,

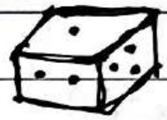
Thus $P(E) \leq 1$.

Ex: ① What is the probability ~~f~~ of dog will fly? $(\underline{\underline{0}})$ $\underline{\underline{0\%}}$

Impossible event

Ex: ② What is the probability of getting a natural number in an experiment of rolling a dice.

Dice has possibilities.
$\{1, 2, 3, 4, 5, 6\}$



P(Getting a natural number) = 1 (100%)

certain event (100% chance of occuring event)

$$P = \frac{\text{No. of ways an event can occur}}{\text{Total possible outcomes}}$$

$$P(3) = \frac{1}{6}$$

$$P(\text{odd number}) = \frac{3}{6}$$

(odd numbers 1, 3, 5)

③ In a pack of 52 cards,

$$P(\text{Jack}) = \frac{4}{52}$$

$$P(\text{face}) = \frac{12}{52}$$

$$(J-4,\ Q-4,\ K-4)$$

(4) In an experiment of tossing a coin,

$$P(H) = \frac{1}{2}, \qquad P(T) = \frac{1}{2}$$

Note ① $^nC_0 = 1$

② $^nC_r = {}^nC_{n-r}$

③ $^nP_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!$

④ $^nP_0 = 1$

⑤ $^nP_1 = n$.

⑥ $^nP_{n-1} = \dfrac{n!}{(n-n+1)!} = \dfrac{n!}{1!} = n!$

⑦ $^nP_r = {}^nC_r \times r!$

Collectively exhaustive events:

Events A & B are said to be collectively exhaustive events if $A \cup B = S$, S - sample space.

Mutually exclusive events (or disjoint)

Two events are said to be mutually exclusive events if they cannot both occur at the same time.

$\boxed{P(A \text{ and } B) = 0}$, A, B are events

Ex: In an experiment of tossing a single coin, which can result in either heads or tails, but not both.

Axioms of Probability:

Axiom-I : $0 \leq P(E) \leq 1$

Axiom-II : $P(S) = 1$, S - Sample space.

or $\sum_{i=1}^{\infty} P(E_i) = 1$, where E_i's are mutually exclusive and collectively exhaustive events.

Axiom-III : For any sequence of mutually exclusive events $E_1, E_2, \cdots$ etc.
(i.e $E_i \cap E_j = \phi$, if $i \neq j$)

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

This axiom can also be written as
$P(E_1 \text{ or } E_2) = P(E_1) + P(E_2)$ where

E_1 and E_2 are mutually exclusive events.

Properties of Probability:

1. $\boxed{P(E) = 1 - P(E^c)}$
where E^c is a complement of E
i.e $E \cap E^c = \phi$ and $E \cup E^c = S$

2. $\boxed{\text{If } E \subset F \text{ then } P(E) \leq P(F)}$

3. $\boxed{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$

4. $\boxed{P(\phi) = P(\text{Impossible event}) = 0}$

5. $\boxed{P(E^c \cup F) \leq P(E) + P(F)}$

6. $\boxed{P(E^c \cap F) = P(F) - P(E \cap F)}$

Ex-1 What is the probability that 13-card bridge hand contains:
(a) All 13 hearts
(b) 13 cards of the same suit
(c) 7 spade cards and 6 club cards
(d) 7 cards of one suit and 6 cards of another.
(e) 4 diamonds, 6 hearts, 2 spades and 1 club
(f) 4 cards of one suit, 6 cards of second suit, 2 cards of third suit and 1 card of fourth suit.

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bridge hand - the cards held in a card game by a player.

(a)

$$P(13\,H) = \frac{{}^{13}C_{13}}{{}^{52}C_{13}} = \frac{1}{{}^{52}C_{13}} = 0.1575 \times 10^{-12}$$

(b) P(13 cards of the same suit)

$$= \frac{{}^{4}C_{1}\,{}^{13}C_{13}}{{}^{52}C_{13}} = 6.229 \times 10^{-12}$$

(c) P(7 spade cards, 6 club cards)

$$= \frac{{}^{13}C_{7} \cdot {}^{13}C_{6}}{{}^{52}C_{13}} = 4.637 \times 10^{-6}$$

(d) P(7 cards of one suit and 6 cards of another)

$$= \frac{{}^{4}C_{1}\,{}^{13}C_{7} \cdot {}^{3}C_{1} \cdot {}^{13}C_{6}}{{}^{52}C_{13}}$$

$$= 0.5564 \times 10^{-4}.$$

(e) P(4 diamonds, 6 hearts, 2 spades and 1 club)

$$= \frac{{}^{13}C_{4}\,{}^{13}C_{6}\,{}^{13}C_{2}\,{}^{13}C_{1}}{{}^{52}C_{13}} = 0.1959 \times 10^{-2}$$

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(f) P(4 cards of one suit, 6 cards of second suit, 2 cards of third suit and 1 card of fourth suit)

$$= \frac{{}^4C_1 \, {}^{13}C_4 \, {}^3C_1 \, {}^{13}C_6 \, {}^2C_1 \, {}^{13}C_2 \, {}^1C_1 \, {}^{13}C_1}{{}^{52}C_{13}}$$

Ex: (2) What is the probability of getting a 2 or a 5 when a die is rolled?

Soln: $P(\text{Getting } 2) = \frac{1}{6}$

$P(\text{Getting } 5) = \frac{1}{6}$

As $|S| = 6$, $S = \{1, 2, 3, 4, 5, 6\}$

Now,

$$P(2 \text{ or } 5) = P(2) + P(5) - P(2 \text{ and } 5).$$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

Ex: (3) Consider the example of finding the probability of selecting a black card or a 6 from a deck of 52 cards.

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Q. Find : P (Black card or 6)

P(Selecting a black card) = $\frac{26}{52}$

P(Selecting a 6) = $\frac{4}{52}$

P(Selecting both a black and a 6) = $\frac{2}{52}$

∴ P(Black card or 6)

= P(Black card) + P(6) − P(Black and 6)

$= \frac{26}{52} + \frac{4}{52} - \frac{2}{52}$

$= \frac{28}{52} = \underline{\underline{7/13}}$.

Independent and Dependent Events :

When multiple events occur, if the outcome of one event does not affect the outcome of the other events, they are called Independent events.

Say, a die is rolled twice. The outcome of the first roll doesn't affect the second outcome. These two are independent events.

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Ex: ① Say, a coin is tossed twice. What is the probability of getting two consecutive tails?

Soln: When a coin is tossed twice, the sample space S is -

$$S = \{(H,H), (H,T), (T,H), (T,T)\}$$

$$P(\text{two consecutive tails}) = \frac{1}{4}$$

OR

$$P(\text{getting a tail in one toss}) = \frac{1}{2}$$

The coin is tossed twice

so $1/2 \times 1/2 = 1/4$ is the answer.

Ex: A box contains 4 blue, 2 red and 3 black pens. If a pen is drawn at random from the pack, replaced and the process repeated 2 more times, What is the probability of drawing 2 blue pens and 1 black pen?

Soln: Here, total number of pens = 9

$P(\text{drawing } 1^{st} \text{ blue pen}) = 4/9$

$P(\text{drawing } 2^{nd} \text{ blue pen}) = 4/9$

P(drawing 1 black pen) = $3/9$

$\therefore$ P (drawing 2 blue pens and 1 black pen) $= 4/9 \cdot 4/9 \cdot 3/9$

$= 48/729$

$= 16/243$

Dependent Events:

When two events occur, if the outcome of one event affects the outcome of the other, they are called dependent events.

Ex(1) A box contains 4 blue, 2 red and 3 black pens. If a pen is drawn at random from the pack, NOT replaced and then another pen is drawn. What is the probability of drawing 2 blue pens and 1 black pen?

Soln: P(drawing 1st blue pen) $= 4/9$

P(drawing 2nd blue pen) $= 3/8$

P(drawing 1 black pen) $= 3/7$

Hence, probability of drawing 2 blue pens and 1 black pen $= 4/9 \times 3/8 \times 3/7$

$= 1/14$

Ex: ② What is the probability of drawing a king and a queen consecutively from a deck of 52 cards, without replacement.

Sol:

$P(\text{drawing a king}) = 4/52 = 1/13$

After drawing 1 card, no. of cards remained $= 51$

$\therefore P(\text{drawing a queen}) = 4/51$

Hence,

$P(\text{drawing a king and a queen})$

$= 1/13 \times 4/51$

$= 4/663$

More Examples:

Ex: 1 A coin is thrown 3 times. What is the probability that atleast one head is obtained?

Sol: Sample space $S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

$\therefore$ Total no. of ways $= |S| = 8$

Favourable cases for atleast one head $= 7$

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$\therefore$ p(of getting atleast one head) $= 7/8$

OR:
p(of getting atleast one head)
$= 1 - p(\text{no head})$

$= 1 - 1/8 = \frac{8-1}{8}$

$= 7/8$

Ex: ② Find the probability of getting a numbered card when a card is drawn from the pack of 52 cards.

soln: Total cards $= 52$
Numbered cards $= \{2,3,4,5,6,7,8,9,10\}$

For 4 suits, total ~~ea~~ numbered cards $= 4 \times 9$
$= 36$

$$p(\text{getting a numbered card}) = \frac{36}{52} = \frac{9}{13}$$

Ex: ② What is the probability of getting a sum of 7 when two dice are thrown?

soln:

$$S = \{ (1,1),(1,2),(1,3) \cdots (1,6)$$
$$(2,1),(2,2),(2,3) \cdots (2,6)$$
$$\vdots$$
$$(3,1),(3,2),(3,3), \cdots (3,6) \}$$

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$\therefore$ A = ~~set~~ getting a sum of 7
= { (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3) }

$\therefore P(A) = \frac{6}{36} = 1/6$

Ex: 3 A card is drawn at random from the pack of 52 cards.
(i) find the probability that it is an honor card
(ii) It is a face card.

soln: (i) Honor cards are (A, J, Q, K)
4 cards from 4 suits $= 4 \times 4 = 16$

A = getting a honor card.

$\therefore P(A) = 16/52 = 4/13$

(ii) face cards are (J, Q, K).
3 cards from each suit $= 3 \times 4 = 12$

$\therefore$ B = getting a face card.

$P(B) = 12/52 = 3/13$

Ex (4) A problem is given to three persons P, Q, R whose respective chances of solving it are $2/7$, $4/7$, $4/9$ respectively. What is the probability that the problem is solved ?

Soln. P = problem is solved by P
Q = — " — by Q
R = — " — by R

$\therefore P(\bar{P}) = 1 - P(P)$, $P(P) = 2/7$
$= 1 - 2/7 = 5/7$

$P(\bar{Q}) = 1 - P(Q)$, $P(Q) = 4/7$
$= 1 - 4/7$
$= 3/7$

$P(\bar{R}) = 1 - P(R)$, $P(R) = 4/9$
$= 1 - 4/9$
$= 5/9$

$\therefore$ Probability of problem getting solved $= 1 - 5/7 \times 3/7 \times 5/9$

$= 122/147$.

Ex (5) Two dice are thrown together. What is the probability that the number obtained on one of the dice is multiple of number obtained on the other dice?

Sol^n: $|S| = 36$

$$S = \{ (1,1)\ (1,2)\ \cdots\ (1,6)$$
$$(2,1)\ (2,2)\ \cdots\ (2,6)$$
$$\vdots$$
$$(6,1)\ (6,2)\ \cdots\ (6,6) \}$$

Now A = the number obtained on one dice is multiple of number on the other dice.

$$A = \{ (1,1), (1,2)\ (1,3), (1,4), (1,5), (1,6)$$
$$(2,1), (2,2), (2,4), (2,6)$$
$$(3,1), (3,3), (3,6)$$
$$(4,1), (4,2), (4,4),$$
$$(5,1), (5,5)$$
$$(6,1), (6,2), (6,3), (6,6) \}$$

$\therefore\ |A| = 22.$

Hence, $P(A) = \frac{22}{36} = 11/18$

Conditional Probability:

Conditional Probability is calculating the probability of an event given that another event has already occured.

Formula:

$$P(A|B) = \frac{P(A\cap B)}{P(B)}, \quad P(B) \neq 0$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)}, \quad P(A) \neq 0$$

$A|B$, Probability of A w.r.t B

OR Probability of A taking B as sample space.

OR probability of A when B has already occurred.

$B|A$, probability of B w.r.t A

OR Probability of B taking A as sample space

OR Probability of B when A has already occurred.

Now,

Note: (i) $P(A|B) = \frac{P(A\cap B)}{P(B)}$

$$= \frac{n(A\cap B)/\cancel{n(S)}}{n(B)/\cancel{n(S)}}$$

$$= \frac{n(A\cap B)}{n(B)}$$

$$= \frac{|A\cap B|}{|B|}$$

$$\therefore P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{|A\cap B|}{|B|}, \ |B| \neq 0$$

$$|||^{ly} \ P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{|A\cap B|}{|A|}, \ |A| \neq 0$$

Ex: ① Suppose three friends X, Y, Z line up at random in a queue
Find
$P(A|B)$ for A = Z on one end
B = X in middle

Soln: $S = \{XYZ, XZY, YXZ, YZX, ZXY, ZYX\}$

$$P(A|B) = \frac{n(A\cap B)}{n(B)} = \frac{2}{2} = 1.$$

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Q.2. Dice is thrown twice and sum of numbers is observed to be 4. What is probability that number 2 has appears atleast once?

Soln: A = sum of numbers is 4

B = No. 2 has appeared atleast once.

$$S = \{(1,1), (1,2) \cdots (1,6)$$
$$(2,1), (2,2) \cdots (2,6)$$
$$\vdots$$
$$(6,1), (6,2) \cdots (6,6)\}$$

$\therefore |S| = 36$

$A = \{(2,2), (1,3), (3,1)\}$, $|A| = 3$

$B = \{(1,2), (2,1), (2,2), (2,3), (2,4)$
$(2,5), (2,6), (3,2), (4,2), (5,2)$
$(6,2)\}$

$\therefore |B| = 11$

$$P(B|A) = \frac{|A \cap B|}{|A|} = \frac{1}{3}.$$

($\because$ B|A, as event A is already occurred in question).

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Ex. 3 In a class, 40% of the students study maths and science. 60% of the students study maths.
What is the probability of a student studying science given he/she is already studying math?

Soln:
~~A~~ M = students studying maths

S = students studying science

$$|M \cap S| = \frac{40}{100} = 0.40$$

$$|M| = \frac{60}{100} = 0.60$$

$$P(S|M) = \frac{|M \cap S|}{|M|} = \frac{0.40}{0.60} = 2/3$$
$$= 0.67$$

Ex. 4. Two standard dice with 6 sides are thrown and the faces are recorded. Given that the sum of two faces equals to 10, what is the probability that the first throw equals to 5?

Soln: Let A = be the event for which the two faces sums is equal to 10.
B = the event for which the first throw equals to 5.

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$$S = \{ (1,1), (1,2) \cdots (1,6) \\ (2,1)\ (2,2) \cdots (2,6) \\ \vdots \\ (3,1)\ (3,2) \cdots (3,6) \}$$

$$A = \{ (5,5), (4,6), (6,4) \}$$

$$B = \{ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6) \}$$

$$\therefore \quad A \cap B = \{ (5,5) \}$$

Now, we have to find

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{|A\cap B|}{|A|} = \frac{1}{3}$$

Ex: 5 : Two cards are drawn at random from a pack of 52 cards.
Find the probability that both are diamond.
(i) Without replacement → (Conditional prob. problem)
(ii) With replacement.

Soln : A = first card is diamond.

B = second card is diamond.

$$P(A) = \frac{13}{52}$$

$$P(B|A) = \frac{12}{51}$$

Hence, probability of selecting two diamond cards without replacement

$$= P(A) \cdot P(B|A)$$

$$= \frac{\cancel{13}}{\cancel{52}_{4}} \times \frac{\cancel{12}^{3}}{51} = \frac{1}{17}$$

(ii) With replacement (Independent events) [NOT conditional Problem]

$$P(A) = \frac{13}{52}, \quad P(B) = \frac{13}{52}$$

Hence, probability of selecting two diamond cards with replacement is

$$= P(A) \cdot P(B)$$

$$= \frac{13}{52} \times \frac{13}{52}$$

$$= \frac{1}{16}$$

Ex: 6 A pair of dice is rolled. Find the probabilities of the following events,

(i) The sum of two numbers is even
(ii) The sum of two numbers is atleast 8
(iii) The product of two numbers is less than or equal to 9.

(i) 0.5 (ii) 15/36 (iii) 17/36

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7. A bag contains 4 white and 2 black balls. Another bag contains 3 white and 5 red balls. One ball is drawn from each bag.
What is the probability that they are of different colours?

$$4/6 \times 5/8 + 2/6 \times 3/8 = 13/48$$

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Ex: 1 In how many different ways, the letters of the word 'CONCERNING' can be arranged in a line.

~~Sol~~

Ex: 2 How many arrangements of 'MANAGEMENT' are there, in which the two M'S are separated?

Ex: 3 There are 6 boys and 3 girls in a class. An entertaiment committee of 5 is to selected such that there are 3 boys and 2 girls in the committee. In how many ways can be committee be selected? Determine the number of ways, if there is atleast one girl in the committee?

Solutions :

(1) $$\frac{10!}{2!\,1!\,3!\,1!\,1!\,1!\,1!} = \frac{10!}{12} = 302400$$

(2) n = total number of letters = 10

number of M'S = 2

—"— A'S = 2

—"— N'S = 2

G'S = 1

E'S = 2

T'S = 1

∴ Total no. of arrangements

$$= \frac{10!}{2!\,2!\,2!\,2!} = 226800$$

If two M's remain together then this happens in 9 ways.

i.e. When two M's occupy places (1,2) (2,3) (3,4), (4,5), (5,6), (6,7), (7,8), (8,9) and (9,10).

Remaining letters are ANAGEENT

$n = 8$

~~(A)~~ number of A'S = 2

N'S = 2

G'S = 1

E'S = 2

T'S = 1

If two M's together then number of arrangements is

$$9 \times \frac{8!}{2!2!2!} = 9 \times 7! = 45360$$

Hence if two M's are separated, number of arrangements is

$$= 226800 - 45360$$
$$= 181440$$

Soln ③ Number of boys $= 6$, Number of girls $= 3$

Hence, Number of ways in which a committee of 5 can be selected containing 3 boys and 2 girls is

$$= {}^6C_3 \times {}^3C_2$$

$$= \frac{6!}{3! \cdot (6-3)!} \times \frac{3!}{2!(3-2)!}$$

$$= \frac{2\,\cancel{6} \times 5 \times \cancel{4}^{2} \times \cancel{3!}}{\cancel{3!} \cdot \cancel{3} \times 2!} \times \frac{3 \times \cancel{2!}}{\cancel{2!}\,1!}$$

$$= 2 \times 5 \times 2 \times 3$$

$$= 60 \text{ ways.}$$

Soln ③ (ii) To find number of ways, if there is at least one girl in the committee.

	Boy(6)	Girls(3)	(Total) committee
(i)	4	1	5
(ii)	3	2	5
(iii)	2	3	5

(i) ${}^6C_4 \times {}^3C_1 =$

(ii) ${}^6C_3 \times {}^3C_2 =$

(iii) ${}^6C_2 \times {}^3C_3 =$

∴ Required number of ways

$= \quad + \quad +$

$= 120$ ways.

Combination with Repetition:

If a collection of objects consists of
r_1 identical objects of type-1
r_2 — " ——— type-2
⋮
r_k — " ——— type-k.

then the total number of unordered samples of size n, chosen from given collection is $\binom{n+k-1}{n} = {}^{n+k-1}C_n$

Ex: (1) A library has atleast 6 copies of each of the same book on Algebra, Geometry and Calculus.
In how many ways can we select 6 books?

sol^n: k = number of types = 3
$n = 6$

Total number of selection is
$$= \binom{n+k-1}{n} = \binom{8}{6} = 28$$
$$= {}^8C_6 = 28$$

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Ex: 2 How many ways are there to select 8 balls from 50 red, 25 blue, and 30 green balls? How many selections include ~~atleast~~ 3 green balls?

Soln: number of types $= k = 3$

$n = 8$

Number of selections is $^{n+k-1}C_n$

$= {}^{8+3-1}C_8 = {}^{10}C_8 = 45$

$${}^{10}C_8 = \frac{10!}{8!(10-8)!} = \frac{10!}{8!\,2!}$$

$$= \frac{\cancel{10}^{5} \times 9 \times \cancel{8!}}{\cancel{8!} \times \cancel{2!}}$$

$= 45$

(ii) Suppose selection includes ~~atleast~~ 3 green balls, then it remains to select 5 more balls.

$k = 3$

$n = 5$

Number of selections is $^{n+k-1}C_n$

$= {}^{7}C_5 = 21$

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Ex: 4 How many ways are there to pick a 5 - person basket ball team from 10 possible players? How many teams, if the weakest player and the strongest player must be on the team?

Soln 1 Number of ways to pick a 5-person basket ball team from 10 possible players.

This can be done in $^{10}C_5$ ways

$$^{10}C_5 = \frac{10!}{5!(10-5)!} = \frac{10\times9\times8\times7\times6\times\cancel{5!}}{5!\,\cancel{5!}}$$

$$= \frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1}$$

$$= 252 \text{ ways.}$$

2 If the weakest player and the strongest player is on the team, this can be done in. $^{10-2}C_{5-2}$ ways

$$^{8}C_3 = 56 \text{ ways.}$$

Ex: 5 If a coin is flipped 10 times, what is the probability of 8 or more heads?

Soln: Total number of outcomes $= 2^{10}$.

$|S| = 2^{10}$.

A = Event contains heads 8 or 9 or 10

$= {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}$.

$=$

$= 56$ ways.

~~Prob.~~ $P(A) = \frac{|A|}{|S|} = \frac{56}{2^{10}}$

Ex: 6. There are six different French books, eight different Russian books and five different spanish books.
How many ways are there to arrange the books in a row on a shelf with all books of the same language grouped together?

Sol^n: ① 6 different French books can be arranged among themselves in

$$^6P_6 = 6! \text{ ways.}$$

② 8 different Russian books can be arranged among themselves in

$$^8P_8 = 8! \text{ ways.}$$

③ 5 different spanish books can be arranged among themselves in

$$^5P_5 = 5! \text{ ways.}$$

Since there are three groups, these groups can be arranged among themselves in

$$^3P_3 = 3! \text{ ways.}$$

∴ The required number of arranged $= (3!)(6!)(8!)(5!)$

Ex-7 How many four digit numbers can be formed from the digits 1, 2, 3, 4, 5 with repetition possible, which are divisible by 5?

sol^n:

Th	H	T	U
			5
↑ 5	↑ 5	↑ 5	

Four digit number is divisible by 5, iff unit's place has 0 or 5

But Here ~~digits~~ available numbers are 1, 2, 3, 4, 5, so unit's place has only one choice i.e 5.

Thousand's place can be filled in 5 ways
Hundred's — " ——— 5 ways
Ten's — " ——— 5 ways

Hence, total four digit numbers which are divisible by 5

$= 5 \times 5 \times 5 \times 1$
$= 125$ ways.

Ex: 1. A throw is made with two dice. Find the probability of getting a score of (i) 10 points (ii) Atleast 10 points (iii) Atmost 10 points.

Soln. Sample space $S = \begin{Bmatrix} (1,1), (1,2), (1,3), \cdots (1,6) \\ (2,1), (2,2) \cdots (2,6) \\ \vdots \\ (6,1), (6,2), \cdots (6,6) \end{Bmatrix}$

(i) A : score is 10 points
$= \{(4,6), (5,5), (6,4)\}$

$$P(A) = \frac{3}{36} = \frac{1}{12}$$

(ii) B : Atleast 10 points. (7/ 10)
$B = \{(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)\}$

$$P(B) = \frac{6}{36} = \frac{1}{6}$$

(iii) C : Atmost 10 points. (≤ 10)
$C = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$
$(5,1), (5,2), (5,3), (5,4), (5,5)$
$(6,1), (6,2), (6,3), (6,4)\}$

$$P(C) = \frac{33}{36} = \frac{11}{12}$$

Assigment - 6.

1. From a well shuffled pack of cards, three cards are drawn at random. Find the probability that they are king, queen and jack combination. $\left(\frac{16}{5525}\right.$

2. Among six books, there are two volumes of one book. These books are arranged in a random order on a shelf. Find the probability that two volumes are always together.

3. Four cards are drawn at random from well shuffled pack of 52 cards.
Find the probability that
(i) two cards are red and two are black
(ii) all cards are of different suit
(iii) all are of same suit.
(iv) One is king

4. A committee consisting of 3 gents and 2 ladies wants to sit in a row for a photograph. What is the probability that the ladies will occupy extreme positions of two ends?

Soln: $n(S) = m = 5!$

A = Ladies occupy extreme positions.

$\therefore n(A) = 2! \times 3!$

L_1 L_2

or L_2 L_1

$$P(A) = \frac{2! \times 3!}{5!} = 0.1$$

Ex: 5. A committee of four is to be formed from 3 engineers, 4 economists, 2 statisticians and 1 chartered accountant.

(i) What is the probability that each of four categories of profession is included.

(ii) What is the probability that the committee consist of CA and atleast one engineer.

Soln: $n(S) = {}^{10}C_4 = 210$.

(i) A : each of four categories of profession is included.

$$n(A) = {}^3C_1 \cdot {}^4C_1 \cdot {}^2C_1 \cdot {}^1C_1 = 24$$

$$\therefore P(A) = \frac{24}{210} =$$

(ii) B : committee consist of CA and atleast one engineer.

	(3) Engineer	(4) Economist (2) statistician	(1) CA
①	${}^3C_1$	${}^6C_2$	${}^1C_1$
②	${}^3C_2$	${}^6C_1$	${}^1C_1$
③	${}^3C_3$	${}^6C_0$	${}^1C_1$

$$\therefore n(B) = {}^3C_1 \cdot {}^6C_2 \cdot {}^1C_1 + {}^3C_2 \cdot {}^6C_1 \cdot {}^1C_1 + {}^3C_3 \cdot {}^6C_0 \, {}^1C_1$$

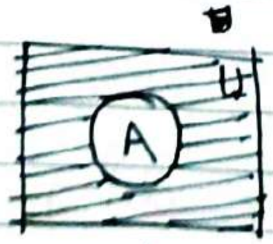
$$= 45 + 18 + 1$$

$$= 64.$$

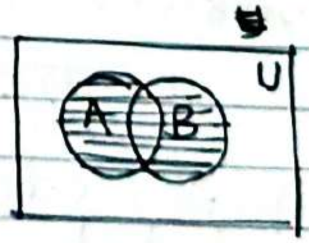
$$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{64}{210}.$$

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
2. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

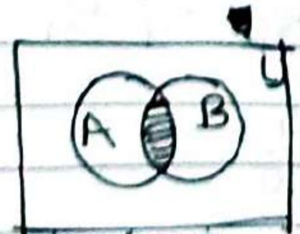
~~Ex:1~~ +



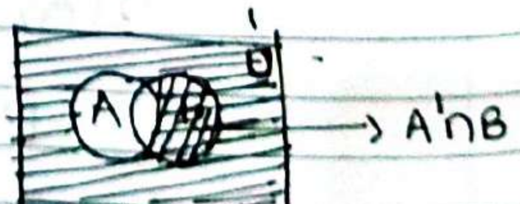
A'



A∪B



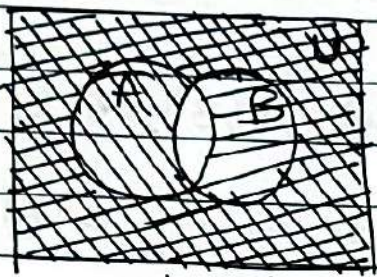
A∩B



A'∩B

Hence $|A' \cap B| = |B| - |A \cap B|$

$|||^{ly}$ $|A \cap B'| = |A| - |A \cap B|$

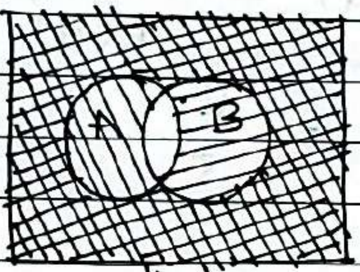


$A' \cap B'$

$A' \cap B' = U - A \cup B$

$\therefore P(A' \cap B') = P(U) - P(A \cup B)$

$P(A' \cap B') = 1 - P(A \cup B)$



$A' \cup B'$

$A' \cup B' = U - A \cap B$

$\therefore P(A' \cup B') = P(U) - P(A \cap B)$

$\therefore P(A' \cup B') = 1 - P(A \cap B)$

Ex. 6 If $P(A) = 0.6$, $P(B) = 0.5$, $P(A \cap B) = 0.3$
Find $P(A')$, $P(A \cup B)$, $P(A' \cap B)$, $P(A' \cap B')$
and $P(A' \cup B')$

Ex. 7. $P(A) = 0.3$, $P(B) = 0.2$, $P(C) = 0.5$,
$P(A \cap B) = 0.17$, $P(A \cap C) = 0.25$, $P(B \cap C) = 0.15$
$P(A \cap B \cap C) = 0.1$
Find (i) $P(A \cup B \cup C)$ (ii) $P(A \cup B)$ (iii) $P(A' \cap B \cap C)$
(iv) $P(A' \cap B' \cap C)$ (v) $P(A' \cap B' \cap C')$

Note: Independence of two events :

Two events A and B are said to be independent if $P(A \cap B) = P(A) \cdot P(B)$

Conditional Probability :

Ex: 1. A pair of fair dice is rolled. If the sum of 8 has appeared.
Find the probability that one of dice shows three.

Soln: $B = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$

$\rightarrow A = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)$
$(1,3),(2,3),(4,3),(5,3),(6,3)\}$

$$P(A \text{ when } B \text{ has occurred}) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{n(A\cap B)}{n(S)}}{\frac{n(B)}{n(S)}}$$

$$= \frac{2}{5}$$

Ex: 2. The personnel department of a company has 100 engineers when distribution is as given below.

Age(years)	BE	ME	Total
20 - 30	20	5	25
30 - 40	25	10	35
>40	10	30	40
	55	45	100

If one engineer is selected at random
Find (i) Probability that he is only BE
(ii) Probability that he has masters degree given age beyond 40.
(iii) Probability that he is under 30 given he is M.E.

Solⁿ: (i) $P(\text{He is only BE}) = \frac{55}{100}$

(ii) A: He has masters degree
B: He has age > 40

$\therefore P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{30/\cancel{100}}{40/\cancel{100}} = \frac{3}{4}$

(iii) C: He is under 30
D: He is ME

$\therefore P(C|D) = \frac{P(C\cap D)}{P(D)} = \frac{5/\cancel{100}}{45/\cancel{100}} = \frac{1}{9}$

Theorem: (i) $0 \le P(A|B) \le 1$, for any $A \subset U$ (or Ω)
$U \equiv$ Universal set

(ii) $P(A\cup C|B) = P(A|B) + P(C|B) - P(A\cap C|B)$

(iii) $P(A'|B) = 1 - P(A|B)$
(iv) $P(A|A') = 0$
(v) If A and B are independent then
$P(A|B) = P(A)$ As $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)\cdot \cancel{P(B)}}{\cancel{P(B)}} = P(A)$

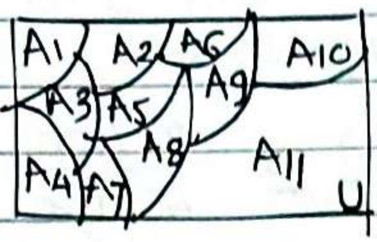
~~(vi)~~

$\|^{ly}$ $P(B|A) = P(B)$

(vi) Multiplication thm:
$P(A\cap B) = P(A|B)\cdot P(B) = P(B|A)\cdot P(A)$

Partition of sample space

[collection of mutually exclusive and exhaustive events]



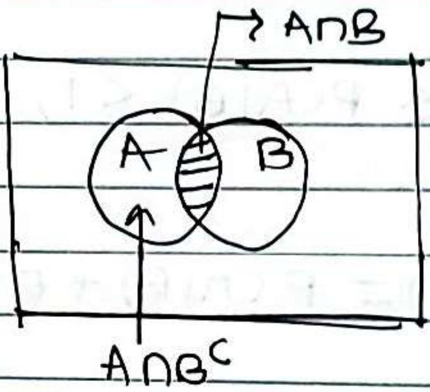
$$A_i \cap A_j = \phi \ , \ i, j = 1, 2, \cdots 11$$

$$\& \bigcup_{i=1}^{11} A_i = U (= \Omega)$$

Bay's theorem :

For any two events A and B

$$P(A) = P(A \cap B^c) + P(A \cap B)$$

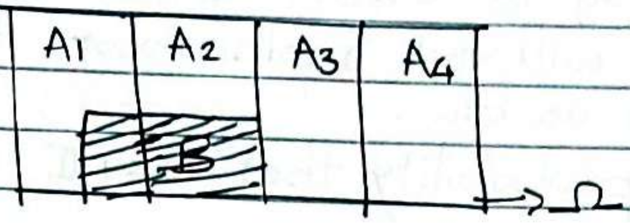


$$\therefore P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B) \cdot P(B)}{P(A \cap B^c) + P(A \cap B)}$$

$$= \frac{P(A|B) \cdot P(B)}{P(A|B^c) \cdot P(B^c) + P(A|B) \cdot P(B)}$$

is called Bay's formula.

Bay's theorem :



Suppose $A_1, A_2, \cdots A_n$ form a partition of sample space Ω of a random experiment. Suppose B is any other event with $P(B) > 0$ then

$$P(A_i|B) = \frac{P(A_i) \cdot P(B|A_i)}{\sum_{j=1}^{n} P(A_j) \cdot P(B|A_j)}, \quad i = 1, 2, \cdots$$

For above venn diagram ;

A_1, A_2, A_3, A_4 – Partition for Ω

Let B is any other event then by Bay's thm ;

$$P(A_2|B) = \frac{P(A_2) \cdot P(B|A_2)}{P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + P(A_3) P(B|A_3) + P(A_4) \cdot P(B|A_4)}$$

$$= \frac{P(A_2 \cap B)}{P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) + P(A_4 \cap B)}$$

Ex 1. If Bag I contains 6 blue and 4 red balls.
Bag II contains 2 blue and 6 red balls.
Bag III contains 1 blue and 8 red balls.

(i) A bag is chosen at random, a ball is drawn randomly from this bag.
It turns out to be blue. Find the probability that bag I was chosen.

(ii) A bag is chosen at random, two balls are drawn without replacement from this bag, both balls are blue.
Find the probability that bag II was chosen.

Soln: A_1: bag I is chosen,
A_2: bag II is chosen,
A_3: bag III is chosen

$$P(A_1) = P(A_2) = P(A_3) = 1/3$$

(i) B: the ball drawn is blue

To find: $P(A_1|B)$

By Bay's thm;

$$P(A_1|B) = \frac{P(A_1)\cdot P(B|A_1)}{P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2) + P(A_3)\cdot P(B|A_3)}$$

Note that: $P(B|A_1) = 6/10$, $P(B|A_2) = 2/8 = 1/4$

$$P(B|A_3) = 1/9$$

$$\therefore P(A_1|B) = \frac{1/3 \cdot 6/10}{1/3 \cdot 6/10 + 1/3 \cdot 1/4 + 1/3 \cdot 1/9}$$

$$= 0.6243$$

(ii) C: both balls are blue.

By Bay's thm ;

$$P(A_2|C) = \frac{P(A_2) \cdot P(C|A_2)}{P(A_1) \cdot P(C|A_1) + P(A_2) \cdot P(C|A_2) + P(A_3) \cdot P(C|A_3)}$$

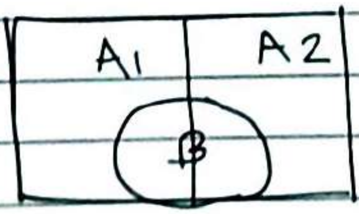
$$P(A_1) = P(A_2) = P(A_3) = 1/3$$

$$P(C|A_1) = \frac{6C_2}{10C_2}, \quad P(C|A_2) = \frac{2C_2}{8C_2}, \quad P(C|A_3) = 0$$

$\therefore$ ~~P(C|A~~ $P(A_2|C) =$

Ex: 2. Two urn's identical in appearance, contain respectively 3 white and 2 black balls ; and 2 white and 5 black balls.
One urn is selected at random and ball is drawn from it. what is the probability that it is black?

Soln :



A_1 : Urn 1 is selected.
A_2 : Urn 2 is selected.

B : black ball is drawn.

$$P(B) = P(B \cap A_1) + P(B \cap A_2)$$
$$= P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2)$$
$$= \frac{1}{2} \cdot \frac{2}{5} + \frac{1}{2} \cdot \frac{5}{7} = 0.5571$$

Discrete Random Variable :

[discrete ≡ ~~continuous~~ distinct]

A discrete random variable is a variable that may take on either a finite sequence of values or an infinite sequence of values such as 0, 1, 2, 3, ...

Ex: (i) Flipping a coin has two outcomes say head or tail (finite)

(ii) Rolling a die has six outcomes (1, 2, 3, 4, 5, 6)

(iii) Bank wants to keeps a track of number of people that ~~who~~ are waiting or ~~who~~ arrived at an ATM during any given fifteen minute period. so 0-person could ~~people~~ do that or 1-person, 2-person .....

This is an example of infinite sequence.

Ex: 1. Let's assume a test has five parts. Define a discrete random variable as x ≡ # parts passed.

so possible outcomes are —

$$x = 0, 1, 2, 3, 4 \text{ or } 5.$$

Ex: 2. Let 3 coins are tossed simultaneously.

$$\Omega = \{ HHH, HTH, HHT, THH, TTH, THT, HTT, TTT \}$$

Define: $X : \Omega \longrightarrow \mathbb{R}$

$X(x) =$ no. of heads.

Range set $= \{ 0, 1, 2, 3 \}$

$X(HHH) = 3,\ X(HTH) = X(HHT) = X(THH) = 2$
$X(TTH) = X(THT) = X(HTT) = 1,$
$X(TTT) = 0.$

$\therefore P(X=0) = 1/8,\ P(X=1) = 3/8,\ P(X=2) = 3/8$

$P(X=3) = 1/8.$

x	0	1	2	3
$P(x)$	$1/8$	$3/8$	$3/8$	$1/8$

This is probability distribution of tossing 3 coins simultaneously.

<u>Note</u>: (i) $P_i \geq 0$.

(ii) $\sum P_i = 1$

[In above e.g. $\sum P_i = 1/8 + 3/8 + 3/8 + 1/8 = 1$]

<u>Continous random variable</u>: A random variable is called continous if it takes uncountably infinite values.

for
Discrete random variable,

<u>Probability mass function</u>: Let X be a discrete random variable defined on sample space Ω. Suppose $\{x_1, x_2, \cdots, x_n\}$ is the range set of X.
We assign, $P_i = P(X = x_i)$ called probability of x_i such that

(i) $p_i \geq 0$, for all i

(ii) $\sum_{i=1}^{n} p_i = 1$

then p defined above is called probability mass function (p.m.f)

Table :

x_1	x_2	$\cdots$	x_n
p_1	p_2	$\cdots$	p_n

is called distribution for r.v. X.

Ex: A pair of fair dice is thrown.
Let $X =$ sum of uppermost faces.
find the probability distribution.

Range set $= \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

X	subset of Ω	$P(X=x_i)$
2	$\{(1,1)\}$	$1/36$
3	$\{(1,2),(2,1)\}$	$2/36$
4	$\{(1,3),(2,2),(3,1)\}$	$3/36$
5	$\{(2,3),(1,4),(3,2),(4,1)\}$	$4/36$
6	$\{(1,5),(2,4),(3,3),(4,2),(5,1)\}$	$5/36$
7		$6/36$
8		$5/36$
9		$4/36$
10		$3/36$
11		$2/36$
12		$1/36$

Ex:2. For the following probability distribution of X

X	0	1	2	3	4
$P(X=x)$	k	3k	5k	2k	k

(i) Find the value of k

(ii) Find $P(X \geq 2)$, $P(X<3)$, $P(X \leq 1)$

$\sum_{i=1}^{4} P_i = 1$, $\quad k+3k+5k+2k+k=1$

$12k=1$

$\Rightarrow k = 1/12$

$$P(X \geq 2) = 1 - P(X<2)$$
$$= 1 - (P(X=0)+P(X=1))$$
$$= 1 - 1/12 - 3/12$$
$$= 2/3$$

$$P(X<3) = P(X=0)+P(X=1)+P(X=2)$$
$$= 1/12 + 3/12 + 5/12$$
$$= 3/4$$

$$P(X\le 1) = P(X=0)+P(X=1)$$
$$= 1/3$$

Ex: Verify whether the following function can be p.m.f

①

x	1	2	3	4
$P(x)$	0.2	0.4	0.3	0.5

<u>Note</u>: $p_i \geq 0$

$\sum p_i = 1.4 \neq 1$

Hence, not p.m.f

② $P(x) = \frac{x-1}{2}$, if $x = 0, 1, 2)$

$= 0$, o.w.

As $P(0) = -1/2 \ngeq 0$

$\Rightarrow$ not p.m.f.

③

x	-1	2	3	4
$P(x)$	0.5	-0.3	0.3	0.5

As $P(x=2) < 0$

$\Rightarrow$ not p.m.f.

Cumulative distribution function (c.d.f) or distribution function (d.f)

Let x be a discrete r.v taking values $x_1, x_2, \cdots x_n$ with probabilities $p_1, p_2, \cdots p_n$ respectively.
The c.d.f or d.f is denoted by $F(x)$ and is defined as -

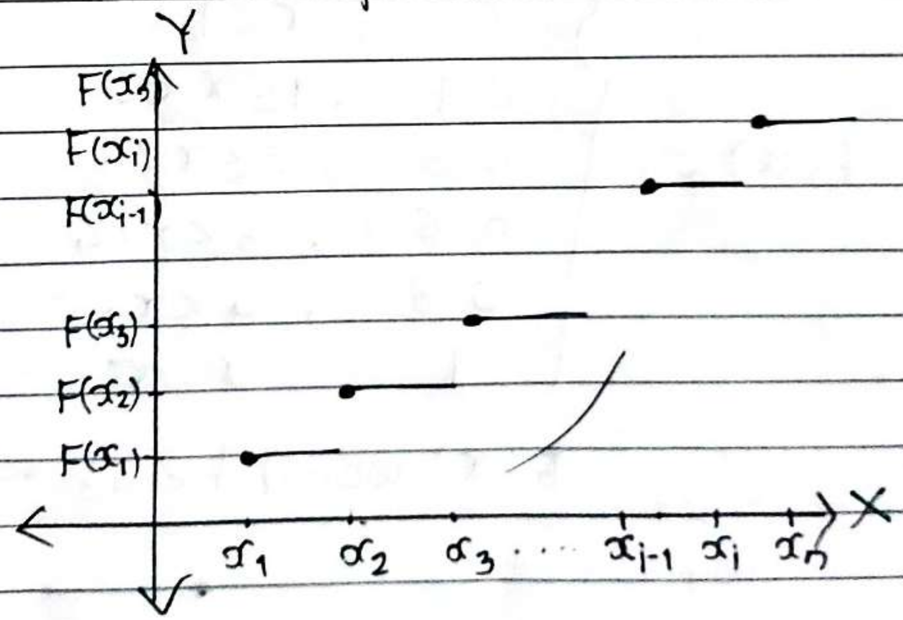
$$F(x_i) = P(X \le x_i) = \sum_{k=1}^{i} p_k$$

Remark for c.d.f (Cumulative distribution function)

The c.d.f is defined for all values of $x \in \mathbb{R}$

Note: 1. Since r.v. takes only isolated values, the function is constant ~~bet~~ in between two successive values of X and have jump at the points x_i, $i = 1, 2, \cdots n$.

2. Hence, for a d.r.v, distribution function is a step function.



Ex: 1. Consider the following probability distribution of a discrete r.v. X

(i) Obtain the c.d.f of X

(ii) Draw the graph of c.d.f

x_i	1	2	3	4	5
p_i	0.1	0.2	0.3	0.2	0.2

Soln: Since $F(x_i) = \sum_{j=1}^{i} p_j$

We obtain $F(x_i)$ by taking cumulative sum of probabilities p_i as follows:

x_i	1	2	3	4	5
$F(x_i)$	0.1	$0.1+0.2 = 0.3$	$0.3+0.3 = 0.6$	$0.6+0.2 = 0.8$	$0.8+0.2 = 1$

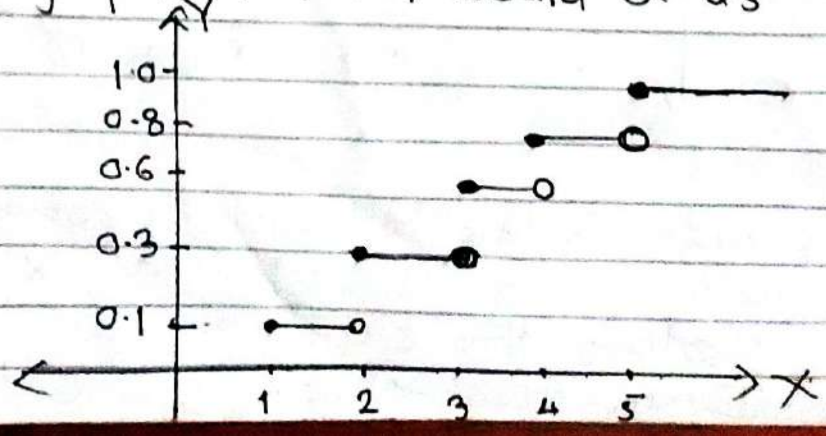
Note that: $F(x_n) = F(5) = 1$

Since there is no value of x beyond 5 with positive probability,

While describing the function $F(x)$, we write.

$$F(x) = \begin{cases} 0 & , x < 1 \\ 0.1 & , 1 \le x < 2 \\ 0.3 & , 2 \le x < 3 \\ 0.6 & , 3 \le x < 4 \\ 0.8 & , 4 \le x < 5 \\ 1 & , x \geq 5 \end{cases}$$

The graph of $F(x)$ would be as —



- the point is included.
- o the point is excluded.

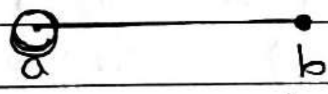
Properties of distribution function F :

It is used extensively in statistical Inference, the main branch of statistics.

(i) $F(x)$ is defined for all $x \in R$, real line
(ii) $0 \leq F(x) < 1$;
(iii) $F(x)$ is non-decreasing function of x
i.e if $a < b$, then $F(a) \leq F(b)$
(iv) Size of jump at x_i is $P(X = x_i)$
(v) $F(-\infty) = 0$ and $F(\infty) = 1$

(vi) Let a and b be two real numbers where $a < b$, then using d.f, we can compute probabilities of different event as —

1. $P(a < X \leq b) = P[X \leq b] - P[X \leq a]$
$= F(b) - F(a)$



$a < x \leq b$

2. $P(a \leq X \leq b) = P[X \leq b] - P[X \leq a] + P[X = a]$
$= F(b) - F(a) + P(a)$

$a \leq x \leq b$

3. $P(a \leq x < b) = P[X \leq b] - P[X \leq a] - P[X = b] + P[X = a]$

$= F(b) - F(a) - P(b) + P(a)$

$$P(a < X < b) = P[X \le b] - P[X \le a] - P[X = b]$$
$$= F(b) - F(a) - P(b)$$

$$P(X > a) = 1 - P(X \le a)$$
$$= 1 - F(a)$$

$$P(X \geq a) = 1 - P[X \le a] + P[X = a]$$
$$= 1 - F(a) + P(a)$$

Ex: 1. The following is the cumulative distribution function of a discrete r.v.

X	-3	-1	0	1	2	3	5	8
$F(X)$	0.1	0.3	0.45	0.65	0.75	0.90	0.95	1.00

Find (i) p.m.f of X (ii) $P(0 < X < 2)$
(iii) $P(1 \le X \le 3)$ (iv) $P(-3 < X \le 2)$
(v) $P(-1 \le X < 1)$ (vi) $P(X = \text{even})$
(vii) $P(X > 2)$ (viii) $P(X \geq 3)$
(ix) $P(X = -3 \mid X < 0)$ (x) $P(X \le 3 \mid X > 0)$

Soln. (i) Since

$$F(x_i) = \sum_{j=1}^{i} P_j$$

$$F(x_{i-1}) = \sum_{j=1}^{i-1} P_j$$

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$$P_i = F(x_i) - F(x_{i-1})$$

$\therefore$ The p.m.f of x is given by -

X	-3	-1	0	1	2	3	5	8
P(X)	0.1	0.2	0.15	0.2	0.1	0.15	0.05	0.05

(ii) $P(0 < X < 2) = F(2) - F(0) - P(2)$
$= 0.75 - 0.45 - 0.1$
$= 0.2$

(iii) $P(1 \leq X \leq 3) = F(3) - F(1) + P(1)$
$= 0.9 - 0.65 + 0.2$
$= 0.45$

(iv) $P(-3 < X \leq 2) = F(2) - F(-3)$
$= 0.75 - 0.1$
$= 0.65$

(V) $P(-1 \leq X < 1) = F(1) - F(-1) - P(1) + P(-1)$
$= 0.65 - 0.3 - 0.2 + 0.2$
$= 0.35$

(vi) $P(X = \text{even}) = P(X=0) + P(X=2) + P(X=8)$
$= 0.15 + 0.1 + 0.05$
$= 0.3$

(vii) $P(X > 2) = 1 - F(2)$
$= 1 - 0.75$
$= 0.25$

(viii) $F(X \geq 3) = 1 - F(3) + P(3)$
$= 1 - 0.9 + 0.15 = 0.25$

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(ix) To compute $P(X=-3 \mid X<0)$,
Let us define A: event $(X=-3)$
and B: event $(X<0)$

$\therefore$ We find $P(A|B)$

Now, $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$

and $A\cap B = (X=-3)\cap(X<0)$
$= X=-3$

$\therefore P(X=-3|X<0) = \dfrac{P(X=-3)}{P(X<0)}$

$= \dfrac{\cancel{P(0.3)}\ 0.1}{P(-3)+P(-1)}$

$= \dfrac{0.1}{0.1+0.2}$

$= 0.33$

(x) $P(X\leq 3 \mid X>0) = \dfrac{P(X\leq 3, X>0)}{P(X>0)}$

$= \dfrac{P(0<X\leq 3)}{P(X>0)}$

$= \dfrac{0.45}{1-0.45}$

$= \dfrac{0.45}{0.55}$

$= 0.8182$.

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Discrete Uniform distribution :

Let x be discrete r.v. taking values $1, 2, \cdots n$. X is said to follow a discrete uniform distribution if it's p.m.f is given by –

$$P(X=x) = \frac{1}{n}, \quad x = 1, 2, \cdots n$$
$$= 0, \quad \text{otherwise}$$

n is called parameter of the distribution.

Ex: 1 If the class representative is selected at random from 50 students, therefore a roll number is selected from 1 to 50.
Thus, if x denotes the roll number selected, then since all numbers are equally likely, the p.m.f of x is given by –

$$P(X) = \frac{1}{50}, \quad x = 1, 2, \cdots 50$$
$$= 0, \quad \text{o.w.}$$

Note : The above distribution is called uniform distribution because it treats all the values of the variable uniformly.
Thus D.U.D (discrete uniform distributio... is applied whenever all values of r.v are equally likely.

Ex: 2. Let X denote the number on the face of an unbiased die, when it is rolled.

$$\therefore P(X) = \frac{1}{6}, \quad x = 1, 2, \cdots 6$$
$$= 0, \quad \text{o.w.}$$

3. A computer generates a digit randomly from 0 to 9

$$\therefore P(X) = \frac{1}{10}, \quad X = 0, 1, \dots 9$$
$$= 0, \quad o \cdot w.$$

Mean and Variance:

$$P(X) = \frac{1}{n}, \quad X = 1, 2, \dots n$$
$$= 0, \quad o \cdot w.$$

$$\mu_1 = \text{Mean} = E(X) = \sum_{x=1}^{n} x P(x)$$
$$= \frac{1}{n} \sum_{1}^{n} x$$
$$= \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

$$\boxed{\text{Mean} = E(X) = \frac{n+1}{2} = \mu_1}$$

$$\left[\text{As } \sum_{1}^{n} x = \frac{n(n+1)}{2} \right]$$

$$\boxed{\mu_2 = \text{Variance} = E(X^2) - [E(X)]^2}$$

~~Now~~,

$$E(X^2) = \sum_{1}^{n} x^2 P(X)$$
$$= \frac{1}{n} \sum_{1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6n}$$
$$= \frac{(n+1)(2n+1)}{6}$$

As $\sum x^2 = \frac{n(n+1)(2n+1)}{6}$

$$\therefore \boxed{\mu_2 = E(x^2) = \frac{(n+1)(2n+1)}{6}}$$

$$\therefore Var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$\boxed{Var(x) = \frac{n^2-1}{12}}$$

$$\therefore \boxed{S.D(x) = \sqrt{Var(x)} = \sqrt{\frac{n^2-1}{12}}}$$

Bionomial distribution:

A discrete r.v X taking values 0, 1, 2, ... n is said to follow a bionomial distribution with parameters n and p if it's p.m.f is given by

$$P[X = x] = P(x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1 \cdots n$$
$$0 < p < 1$$
$$q = 1 - p$$
$$= 0 \qquad o.\omega$$

$$\left[\binom{n}{x} = {}^nC_x \right]$$

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Notation: $X \sim B(n, p)$

Remark: Note that: $\sum_{x=0}^{n} p(x) = \sum_{n=0}^{n} {}^nC_x p^x q^{n-x}$

$$= (p+q)^n$$
$$= 1$$

Note: The probabilities are terms in the binomial expression of $(p+q)^n$, hence the name binomial distribution is given

Real life Examples:

1. Number of defective items in a lot of n items produced by a machine.
2. Number of male births out of n births in a hospital.
3. Number of correct answers in MCQ test
4. Number of rainy days in a month

Ex: 1. A coin with $p = 1/3$ as the probability of head is tossed 6 times.
Find the probability of getting
(i) 4 heads
(ii) at least 2 heads
(iii) at most 1 head
(iv) 4 tails
(v) at least one tail
(vi) all tails.

Soln: Let X denote number of heads obtained

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in 6 tosses of the coin.

$\therefore X \longrightarrow B(n=6, p=1/3)$ and it's p.m.f is given by —

$$p(x) = {}^6C_x\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{6-x}, \quad x = 0,1,2,\cdots 6$$

$$= 0 \quad , \text{o.w}$$

(i) $P[X=4] = P(4) = {}^6C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^{6-4}$

$= 0.0823$

(ii) $P(X \geq 2) = 1 - P(X<2)$

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - \left[{}^6C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{6-0} + {}^6C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{6-1}\right]$

$= 1 - \left(\frac{2}{3}\right)^5$

$= 0.8683$

(iii) $P(X \leq 1) = P(0) + P(1)$

$= {}^6C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{6-0} + {}^6C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{6-1}$

$= 0.1317$

(iv) Getting 4 tails is equivalent to getting 2 ~~tails~~ heads

$\therefore$ Required probability $P(X=2)$

$= {}^6C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{6-2} = 0.3292$

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(v) Getting atleast one tail means that only event of getting all 6 heads should not happen.

$\therefore$ Required probability $= 1 - P(X=6)$

$= 1 - \left(\frac{1}{3}\right)^6$

$= 0.99998$

(vi) Getting 'all tails' is same as 'no head'

$\therefore P(X=0) = {}^6C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{6-0} = \left(\frac{2}{3}\right)^6$

$= 0.0878$

Ex: 2. In a quality control department of a rubber tube manufacturing factory, 10 rubber tubes are randomly selected from each day's production for inspection. If not more than 1 of the tubes is found to be defective, the production lot is approved. Otherwise it is rejected. Find the probability of the rejection of a day's production lot if the true proportion of defectives in the lot is 0.3.

Soln : Suppose X denotes the number of defectives tubes in 10 randomly selected tubes

$$\therefore X \longrightarrow B(n=10, p=0.3)$$

The production lot is accepted if not more than one tube (i.e at the most one tube) is found defective.

$$\therefore P(\text{Accepting the lot}) = P(X=0) + P(X=1)$$

$$= q^n + npq^{n-1}$$

$$= (0.7)^{10} + 10 \times (0.3)(0.7)^9$$

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$= 0.1493$

$\therefore P(\text{Rejection of the lot}) = 1 - 0.1493$
$= 0.8507.$

Ex: 3. A r.v. $X \longrightarrow B(n=6, p)$.
Find p if $9\,P(X=4) = P(X=2)$

Soln: $P(X=x) = {}^nC_x\, p^x q^{n-x}$; $x = 0, 1 \cdots n$

Here $n = 6$.

$\therefore 9\,P(4) = P(2)$

$\therefore 9\; {}^6C_4\, p^4 q^{6-4} = {}^6C_2\, p^2 q^{6-2}$

$\therefore 9p^2 = q^2$ $\quad \because {}^6C_4 = {}^6C_2$

$\therefore 9p^2 = (1-p)^2$ $\quad ({}^nC_r = {}^nC_{n-r})$
$= 1 - 2p + p^2$

$\therefore 8p^2 + 2p - 1 = 0$
$(4p-1)(2p+1) = 0$

$\therefore p = \frac{1}{4}$ or $p = -\frac{1}{2}$

$p = -1/2$ inadmissible
Hence $p = \frac{1}{4}$.

Mean and Variance :

Let $X \longrightarrow B(n, p)$.

The p.m.f is given by —

$$p(x) = {}^nC_x \, p^x q^{n-x}, \quad x = 0, 1, 2, \cdots n$$
$$= 0 \quad , \quad o.w$$

Note: for binomial distribution, computation of factorial moments is easier than raw or central moments.

consider,

$$\text{mean} = \sum_{x=0}^{n} x \, p(x)$$
$$= \sum_{x=0}^{n} x \, {}^nC_x \, p^x \cdot q^{n-x}$$
$$= \sum_{x=0}^{n} \frac{x \, n!}{x!(n-x)!} p^x \cdot q^{n-x}$$
$$= 0 + \sum_{1}^{n} \frac{\cancel{x} \cdot n!}{\cancel{x}(x-1)!(n-x)!} p^x \cdot q^{n-x}$$
$$= \sum_{1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x \cdot q^{n-x}$$
$$= \sum_{1}^{n} \frac{n(n-1)!}{(x-1)!((n-1)-(x-1))!} p^{x-1} q^{n-x}$$
$$= np \sum_{1}^{n} \frac{(n-1)!}{(x-1)![(n-1)-(x-1)]!} p^{x-1} q^{n-x}$$
$$= np \sum_{1}^{n} {}^{n-1}C_{x-1} \, p^{x-1} \cdot q^{n-1-(x-1)}$$
$$= np \, (p+q)^{n-1} \quad (\text{using binomial expansion})$$

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mean = np , As $p+q=1$

Hence, $\boxed{E(X) = \text{mean} = np}$

$$\mu_2 = E[X(X-1)] = \sum x(x-1)\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum_0^n \frac{x(x-1)\, n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= n(n-1) \sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} q^{n-2-(x-2)}$$

$$= (n)(n-1)\, p^2 (p+q)^{n-2}$$

$$\mu_2 = n(n-1)\, p^2$$

$$\mu_2' = E(X^2) = E[X(X-1)] + E(X)$$

$$= \mu_2 + \mu_1'$$

$$= n(n-1)p^2 + np$$

$$Var(X) = \mu_2' - \mu_1'^2 = n(n-1)p^2 + np - n^2p^2$$

$$= npq$$

$$\boxed{Var(x) = npq}$$

$$\boxed{S \cdot D(X) = \sqrt{npq}}$$

Ex. 4. Let $X \longrightarrow B(n,p)$

(i) Comment on the following

$E(X)=7$ and $var(X)=12$

(ii) $E(X)=4$ and $S.D(X)=\sqrt{3}$

What are the values of n, p and q?

Soln : (i) $np=7,\ npq=12$

$\Rightarrow 7q=12$

$q=12/7>1$

$\therefore$ The statement is false.

(ii) $np=4,\ \sqrt{npq}=\sqrt{3}$

$\Rightarrow npq=3$

$\Rightarrow 4q=3$

$q=\frac{3}{4} \Rightarrow p=1-q$

$=1-\frac{3}{4}=\frac{1}{4}$

and $npq=3$

$n \times \frac{1}{4} \cdot \frac{3}{4}=3$

$\Rightarrow \underline{n=16}$.

Additive property :

Let $X \longrightarrow B(n_1, p)$

$Y \longrightarrow B(n_2, p)$

and X and Y are independent.

Then $Z=X+Y \longrightarrow B(n_1+n_2, p)$

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Examples on Binomial distribution.

Ex: 1. During war, 1 ship out of 9 was sunk on an average in making a certain voyage. What was the probability that exactly 3 out of a convoy of 6 ships would arrive safely?

soln: p = the probability of a ship arriving safely.

$p = 8/9$ (since 1 ship out of 9 was sunk hence, we say that eight ships out of 9 are safely arrive)

$\therefore q = 1 - p = 1 - 8/9 = 1/9$

$n = 6.$

$\therefore$ The probability that exactly 3 ships arrive safely $= {}^6C_3 \left(\frac{8}{9}\right)^3 \left(\frac{1}{9}\right)^{6-3}$

$= \frac{10240}{(9)^6}$ $(\because r = 3$

Ex: 2. If the probability of a defective bolt is 0.1 find (a) mean (b) the S.D for the distribution bolts in a total of 400 (c) variance

soln: Given $n = 400$, $p = 0.1$, $q = 0.9$

mean $= np = 400 \times 0.1 = 40$

$S.D = \sqrt{npq} = \sqrt{400 \times 0.1 \times 0.9}$
$= 6.0$

Variance $= \sigma^2 = npq = 36.00$
$= 400 \times 0.1 \times 0.9$

Ex: 3. A dice is tossed thrice. A success is getting 1 or 6 on a toss.
Find the mean and variance of the number of successes.

Soln: Given, $n=3$, $p=\frac{2}{6}=\frac{1}{3}$, $q=\frac{2}{3}$

$$\text{Mean} = np = 3 \times \frac{1}{3} = 1$$

$$\text{Variance} = \sigma^2 = npq = 3 \times 1/3 \times 2/3 = 2/3$$

Ex: 4. 20% of bolts produced by a machine are defective. Determine the probability that out of 4 bolts chosen at random.
(i) One bolt is defective
(ii) at most 2 bolts are defective.

Soln: Given 20% of bolts are defective.
i.e $p = 0.2$
$q = 1-p = 1-0.2 = 0.8$
$n=4$.
By binomial distribution,
$$P(x=r) = {}^nC_r\, p^r q^{n-r}$$

(i) probability of 1 bolt defective is,
$$P(1) = 4C_1 p^1 q^3$$
$$= 4C_1 (0.2)(0.8)^3$$
$$= 0.4096$$

(ii) Probability of at most two bolts are defective is,
$$P(\leq 2) = P(0) + P(1) + P(2)$$

$$= {}^4C_1 p^0 q^4 + {}^4C_1 p^1 q^3 + {}^4C_2 p^2 q^2$$

$$= 0.4096 + 0.4096 + 0.1536$$

$$= 0.9728.$$

Ex. 6.2.5 : If 10% of rivets produced by a machine are defective. Find the probability that out of 5 rivets chosen at random :

(i) None will be defective. (ii) One will be defective

(iii) At least two will be defectives

Soln. :

Step I : Given : 10% of rivets are defective

$$\therefore \quad p = \frac{10}{100} = 0.1$$

$$\therefore \quad q = 1 - p = 0.9 \qquad (\because p + q = 1)$$

Here, $n = 5$

Step II : By Binomial distribution,

$$P(x = r) = {}^nC_r \, p^r \, q^{n-r}$$

(i) Probability of none rivets will be defective is,

$$P(x = r = 0) = {}^5C_0 \, p^0 \, q^5 = {}^5C_0 \, (0.1)^0 \, (0.9)^5 = 0.5905$$

(ii) Probability of one rivet will be defective is,

$$P(r = 1) = {}^5C_1 \, p \, q^4 = {}^5C_1 \, (0.1) \, (0.9)^4 = 0.3281$$

Step III :

(iii) Probability of atleast two rivets will be defective is,

$$P(r \geq 2) = P(2) + P(3) + P(4) + P(5) = 1 - [\, P(0) + P(1) \,]$$

$$= 1 - [\, 0.5905 + 0.3281 \,] \qquad \text{[by result of (i), (ii)]}$$

$$= \mathbf{0.0814}$$

Ex. 6.2.6 : On an average a box containing 10 articles is likely to have 2 defective. If we consider a consignment of 100 boxes, how many of them are expected to have three or less defectives ?

Soln. :

Step I :

Given : On an average a box containing 10 articles is likely to have 2 defective i.e. out of 10 articles 2 are defective.

$$\therefore \quad p = \frac{2}{10} = 0.2$$

$$\therefore \quad q = 1 - p = 0.8 \qquad (\because p + q = 1)$$

Here, $n = 10$

Step II : By Binomial distribution,

$$\mathbf{P(x = r) = {}^nC_r\, p^r\, q^{n-r}}$$

$\therefore$ Probability of three or less defective is,

$$\begin{aligned}
\mathbf{P(r \le 3)} &= \mathbf{P(r = 0) + P(r = 1) + P(r = 2) + P(r = 3)} \\
&= {}^{10}C_0\, p^0 q^{10} + {}^{10}C_1\, p\, q^9 + {}^{10}C_2\, p^2 q^8 + {}^{10}C_3\, p^3 q^7 \\
&= {}^{10}C_0\, (0.2)^0 (0.8)^{10} + {}^{10}C_1\, (0.2)^1 (0.8)^9 + {}^{10}C_2\, (0.2)^2 (0.8)^8 + {}^{10}C_3\, (0.2)^3 (0.8)^7 \\
&= 0.1074 + 0.2684 + 0.3020 + 0.2013 \\
P(r \le 3) &= 0.8791
\end{aligned}$$

Step III : There are 100 boxes.

Expected boxes having three or less defective articles.

$$= 0.8791 \times 100 = 87.91 = \mathbf{88\ boxes}$$

Ex. 6.2.7 : Six dices are thrown 729 times. How many times do you expect at least three dice to show a five or six.

Soln. :

Step I : Here, $n = 6$

The chance of getting 5 or 6 with one dice is

$$p = \frac{2}{6} = \frac{1}{3} \quad \text{and} \quad q = \frac{2}{3} \quad \text{and} \quad N = 729, \quad r = 3$$

Step II : $\mathbf{N (p + q)^n \Rightarrow N \sum P(r) = N \sum P(r = 3)}$

$\therefore$ The expected number of times of at least three dice showing five or six

$$= \mathbf{N \sum P(r \ge 3)}$$

Step III :

$$\begin{aligned}
\therefore P(r) &= \mathbf{N [P(3) + P(4) + P(5) + P(6)]} \\
&= 729 \left[{}^6C_3 (1/3)^3 (2/3)^3 + {}^6C_4 (1/3)^4 (2/3)^2 + {}^6C_5 (1/3)^5 (2/3)^1 + {}^6C_6 (1/3)^6 (2/3)^0 \right] \\
&= \frac{729}{(3)^6} \left[\frac{6!}{3!\,3!} \times 2^3 + \frac{6!}{4!\,2!} \times (2)^2 + \frac{6!}{5!\,1!} \times (2)^1 + \frac{6!}{6!\,0!} \times (2)^0 \right] \\
&= 160 + 60 + 12 + 1 = 233
\end{aligned}$$

Ex. 6.2.8 : Out of 800 families with 4 children each, how many families would be expected to have (i) 2 boys and 2 girls (ii) at least one boy (iii) no girl (iv) at most two girls.

Soln.:

Step I : Given : $n = 4 ; N = 800, \ P = \frac{1}{2} ; q = \frac{1}{2}$.

By Binomial distribution, $\mathbf{NP(r) = N\, {}^nC_r\, p^r\, q^{n-r}}$

Step II :

(i) The expected number of families having 2 boys and 2 girls

$$= \mathbf{N\,P\,(r)} = 800 \times \left[{}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2\right] = 800 \times 6 \times \frac{1}{16} = \mathbf{300}$$

Step III :

(ii) The expected number of families having at least one boy = P(1) + P(2) + P(3) + P(4)

$$= 800\left[{}^4C_2\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + {}^4C_2\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + {}^4C_3\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right) + {}^4C_4\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^0\right]$$

$$= 750 \;\textbf{ or }\; \mathrm{N}\,[1 - \mathrm{P}\,(x = r = 0)] = \mathrm{N}\left[1 - {}^4C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{4}\right)^4\right] = 800\left[1 - \frac{1}{2^4}\right]$$

Step IV :

(iii) The expected number of families having no girls i.e. having 4 boys

$$\mathrm{P}\,(r = 4) = 800 \times {}^4C_4\left(\frac{1}{2}\right)^4 = \mathbf{50}$$

Step V :

(iv) The expected number of families having at most two girls i.e. having at least two boys

$$= \mathbf{N\,[\,P\,(r = 2) + P\,(r = 3) + P\,(r = 4)]}$$

$$= 800\left[{}^4C_2\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + {}^4C_3\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^1 + {}^4C_4\left(\frac{1}{2}\right)^4\right] = \mathbf{550}$$

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Poisson Distribution.

Note: 1. In this section, we deal with discrete random variable defined on countably infinite sample space.

2. In real life situations, we experience number of situations where chances of occurrence of an event in a short time interval is very small. However, there are ~~no~~ infinitly many opportunities to occur.
The number of occurrences of such events follows poisson distribution.

Ex: 1. Number of defective items found in a ~~good~~ lot of large size
2. Number of persons standing in a queue
3. Number of deaths due to snake bite in a certain village.
4. Number of persons affected due to radiation, residing near a nuclear power plant.

Note: 1. P.D is also useful in theory of reliability
2. It is found to be useful in theory of queues.

* 3. Poisson distribution is a particular limiting form of the binomial distribution.

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Def^n//. A discrete random variable X taking values 0, 1, 2, ⋯ is said to follow poisson distribution with parameter m if it's probability mass function (p.m.f) is given by -

$$P[X=x] = \frac{e^{-m} m^{x}}{x!}, \quad x = 0, 1, 2, \cdots, \; m > 0$$
$$= 0, \quad \text{o.w.}$$

Note: 1. X follows poisson distribution with parameter m is symbollically written as -

$$X \longrightarrow P(m)$$

Its p.m.f is denoted by $p(x)$.

2. $$e^{m} = 1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots$$

$$\log_e(1-a) = -\left(a + \frac{a^2}{2} + \frac{a^3}{3} + \frac{a^4}{4} + \cdots\right), \text{if } |a| < 1$$

3. Note that: $p(x)$ is a p.m.f

(i) $p(x) \geq 0, \; \forall x$, since $e^{-m} > 0$

(ii) $$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{\infty} \frac{e^{-m} m^{x}}{x!}$$
$$= e^{-m} \sum_{x=0}^{\infty} \frac{m^{x}}{x!}$$
$$= e^{-m}\left[1 + m + \frac{m^2}{2!} + \cdots\right]$$
$$= e^{-m} \cdot e^{m}$$
$$= 1$$

Mean and Variance :

$$\text{Mean} = E(X) = \sum_{x=0}^{\infty} x P(x)$$

$$= \sum_{x=1}^{\infty} \frac{x e^{-m} \cdot m^x}{x!}$$

$$\boxed{\text{Mean} = m \cdot e^{-m} \cdot e^{m} = m}$$

Now,

$$E(X^2) = \sum_{x=0}^{\infty} x^2 P(x)$$

$$= \sum_{x=0}^{\infty} [x(x-1) + x] \left(\frac{e^{-m} \cdot m^x}{x!} \right)$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-m} \cdot m^x}{x!} + \sum_{x=0}^{\infty} \frac{x e^{-m} m^x}{x!}$$

$$= e^{-m} m^2 \sum_{x=2}^{\infty} \frac{m^{x-2}}{(x-2)!} + E(X)$$

$$= e^{-m} \cdot m^2 \cdot e^{m} + m$$

$$E(X^2) = m^2 + m$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2$$

$$= m^2 + m - m^2$$

$$\boxed{Var(X) = m}$$

Hence, $\boxed{\text{mean}(X) = Var(X) = m}$

That is mean and variance of poisson distribution are equal and each is equal to the parameter of the distribution.

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Ex: 1. The average number of misprints per page of a book is 1.5. Assuming the distribution of number of misprints to be poisson; Find

(i) the probability that a particular book is free from misprints.

(ii) number of pages containing more than one misprint. if the book contains 900 pages.

sol^n: Let X: Number of misprints on a page in the book.

Given: $X \longrightarrow P(m=1.5)$

$\text{mean} = E(X) = m = 1.5$

Here the p.m.f is given by,

$$P[X=x] = \frac{e^{-1.5}(1.5)^x}{x!}$$

(i) $$P[X=0] = \frac{e^{-1.5}(1.5)^0}{0!}$$

$$= e^{-1.5}$$

$$= 0.22130$$

(ii) $$P[X>1] = 1 - P[X \leq 1]$$

$$= 1 - \{P[X=0] + P[X=1]\}$$

$$= 1 - \left\{ e^{-1.5} + \frac{e^{-1.5} \cdot (1.5)^1}{1!} \right\}$$

$$= 1 - \{0.223130 + 0.334695\}$$

$$= 0.442175$$

$\therefore$ Number of pages in the book containing more than one misprint

$$= 900\, P[X > 1]$$
$$= 900 \times (0.442175)$$
$$= 397.9575$$
$$\simeq 398$$

Ex: 2 Number of road accidents on a high way during a month follows a poisson distribution with mean 5. Find the probability that in a certain month, number of accidents on the highway will be

(i) Less than 3 (ii) Between 3 and 5 (iii) more than 3

Soln : Let X : number of road accidents on highway during a month.

Given : $X \longrightarrow P(m = 5)$

$\therefore$ The p.m.f is given by,

$$P[X = x] = \frac{e^{-5} \cdot 5^{x}}{x!}, \quad x = 0, 1, 2, \cdots$$

(i) $P[X < 3] = P[X \leq 2] = P[X = 0] + P[X = 1] + P[X = 2]$

$$= \frac{e^{-5} \cdot 5^{0}}{0!} + \frac{e^{-5} \cdot 5^{1}}{1!} + \frac{e^{-5} \cdot 5^{2}}{2!}$$
$$= 0.006738 + 0.033690 + 0.084224$$
$$= 0.124652$$

(ii) $P[3 \leq X \leq 5] = P(3) + P(4) + P(5)$

$$= 0.140374 + 0.175467 + 0.175467$$
$$= 0.491308$$

(iii) $P[X > 3] = 1 - P[X \leq 3]$

$$= 1 - [P(0) + P(1) + P(2) + P(3)]$$
$$= 0.734974$$

Ex. 3 Suppose $X_i \longrightarrow P(m_i = 1.2i)$, $i = 1, 2, 3, 4$ and X_i are independent of each other. If $S = \sum_{i=1}^{4} X_i$.

Find (i) $P(S=12)$ (ii) $P(S>3)$ (iii) $P(S-2<2)$

Soln : $X_i \longrightarrow \text{Poisson}(m_i = 1.2\,i)$ and X_i are independent r.v.

By additive property

$$S = \sum_{i=1}^{4} X_i \longrightarrow \text{Poisson}\left(m = \sum_{i=1}^{4} m_i\right)$$

$$\therefore S \longrightarrow \text{poisson}\left(m = 1.2 \sum_{i=1}^{4} i\right)$$

$$\text{i.e. } S \longrightarrow \text{poisson}\left[m = 1.2 \times 10 = 12\right]$$

(i) $\therefore P[S=12] = \dfrac{e^{-12}(12)^{12}}{x!}$

$= 0.10557$

(ii) $P(S>3) = 1 - P(S \leq 3)$

$= 1 - [P(S=0) + P(S=1) + P(S=2) + P(S=3)]$

$$= 1 - \left[\frac{e^{-12} \cdot 12^0}{0!} + \frac{e^{-12} \cdot 12^1}{1!} + \frac{e^{-12} \cdot 12^2}{2!} + \frac{e^{-12} \cdot 12^3}{3!}\right]$$

$$= 1 - [0.000006 + 0.000074 + 0.000442 + 0.001770]$$

$$= 0.997708$$

(iii) $P[S-2<2] = P[S<4]$

$= P[S \leq 3]$

$= 0.000006 + 0.000074 + 0.000442 + 0.001770$

$= 0.002292$

Ex: 4 If X and Y are indepent poisson random variables with means 2 and 4 respectively find (i) $P\left[\frac{X+Y}{2} < 1\right]$

(ii) $P[3(X+Y) \geq 9]$

soln: Let $Z = X+Y \longrightarrow P(m = 2+4 = 6)$

(i) $P\left[\frac{Z}{2} < 1\right] = P[Z<2]$

$= P[Z=0] + P(Z=1)$

$= 0.017352.$

(ii) $P[3Z \geq 9] = P(Z \geq 3)$

$= 1 - P(Z<3)$

$= 0.93803$

Continous random variables:

In this chapter, the random variables defined on continous sample space

A sample space which is finite or countably infinite is called as denumerable or countable.

If the sample space is not countable then it is called continous.

Continous random variable:

A random variable $X(\omega)$ as a real valued function on domain Ω.

If the range set of $X(\omega)$ is continous, the r.v is continous.

Ex: 1. Weight of a person in kg.

2. Consumption of electricity of a town in a specific month.

3. Daily rainfall in mm at a particular place.

Probability Density function:

A function $f(x)$ is called probability density function if

(i) $f(x) \geq 0, \quad \forall x, \quad -\infty < x < \infty$

(ii) $\int_{-\infty}^{\infty} f(x) = 1$

Ex 1. Given the density function,

$$f(x) = k \cdot e^{-\alpha x}, \quad x \geq 0, \alpha > 0$$
$$= 0, \quad o.w.$$

Find k

Soln: $f(x) \geq 0$, $\int_{-\infty}^{\infty} f(x) = 1$

$$\Rightarrow \int_{0}^{\infty} k \cdot e^{-\alpha x} dx = 1$$

$$\Rightarrow k \cdot \left[\frac{e^{-\alpha x}}{-\alpha}\right]_{0}^{\infty} = 1$$

$$\Rightarrow -\frac{k}{\alpha}[0-1] = 1$$

$$\therefore k = \alpha$$

Ex: 2. Verify whether the following function $f(x)$ is probability density function.

a) $f(x) = \frac{1}{5}$, $-2 \leq x \leq 2$

$= 0$, o.w.

Soln: (i) $f(x) \geq 0$.

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-2}^{2} \frac{1}{5}\,dx$

$$= \frac{1}{5}[x]_{-2}^{2}$$

$$= \frac{4}{5} \neq 1$$

Hence $f(x)$ is not p.d.f.

(b) $f(x) = \frac{200}{3x^3}$, $5 \leq x \leq 10$

$= 0$, o.w.

Soln: (i) $f(x) \geq 0, \forall x$.

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{5}^{10} \frac{200}{3x^3}\,dx$

$= \frac{200}{3}\left[\frac{x^{-2}}{-2}\right]_5^{10}$

$= \frac{100}{3}\left[\frac{-1}{x^2}\right]_5^{10}$

$= \frac{100}{3}\left[-\frac{1}{100}+\frac{1}{25}\right]$

$= \frac{\cancel{100}}{3}\left(\frac{-1+4}{\cancel{100}}\right)$

$= 1$

Hence $f(x)$ is probability density function.

Probability: For a continous r.v, probability is assigned to an interval and it is given by -

$$P(a \leq x \leq b) = \int_a^b f(x)\,dx$$, where $f(x)$ is prob. density function

Cumulative distribution function or distribution function for continous random variable with probability density function is denoted by -

$$F(x_1) = P(X \leq x_1) = \int_{-\infty}^{x_1} f(x)\,dx$$

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Ex: If the probability density function $f(x)$ is given by

$$f(x) = \frac{3}{2}x^2 , \quad -1 \le x \le 1$$
$$= 0 \quad , \quad o.w$$

Find (i) $F(x)$ (ii) $P(-1/3 \le x \le 1/3)$
(iii) $P(x \geq 0)$ (iv) $P(-1/2 \le x \le 3/2)$

Soln: (i) $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-1}^{x} 3/2\, t^2\,dt$

$$= 3/2 \left[t^3/3\right]_{-1}^{x}$$
$$= 3/2 \left(x^3/3 + 1/3\right)$$
$$= \frac{1}{2}(x^3+1)$$

$\therefore F(x) = \frac{1}{2}(x^3+1) \quad x \in (-1,1)$

$= 1 \quad , \quad x \geq 1$

(ii) $P(-\frac{1}{3} \le x \le 1/3) = \int_{-1/3}^{1/3} \frac{3}{2}x^2\,dx$

$$= 3/2 \left[x^3/3\right]_{-1/3}^{1/3}$$
$$= 1/2 \left(\frac{1}{27} + 1/27\right)$$
$$= 1/27$$

(iii) $P(x \geq 0) = \int_{0}^{\infty} f(x)\,dx = \int_{0}^{1} 3/2\, x^2\,dx$

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$$= \frac{3}{2}\left(\frac{x^3}{3}\right)_0^1$$

$$P(x \geq 0) = 1/2$$

(iv) $P\left(-\frac{1}{2} \leq x \leq \frac{3}{2}\right) = \int_{-1/2}^{1} f(x)\,dx + \int_{1}^{3/2} f(x)\,dx$

$$= \int_{-1/2}^{1} \frac{3}{2}x^2\,dx + \int_{1}^{3/2} 0\,dx$$

$$= \frac{3}{2}\left[\frac{x^3}{3}\right]_{-1/2}^{1}$$

$$= \frac{1}{2}\left(1+\frac{1}{8}\right)$$

$$= 9/16$$

Mean and Variance for continous r.v :

Let X be a continous r.v with p.d.f $f(x)$, then mean or expectation of X is denoted by $E(X)$ and is defined as –

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \text{, provided an integral exists.}$$

Note : If the p.d.f of X is defined over an interval $[a, b]$, then

$$E(X) = \int_{a}^{b} x f(x)\,dx$$

$Var(X) = E(X^2) - [E(X)]^2$

Ex 1. If X is a r.v with p.d.f

$$f(x) = 6x(1-x) \; ; \quad 0 < x < 1$$
$$= 0 \quad , \text{ o.w.}$$

Find (i) Mean (ii) Variance of X

Soln: Mean $= E(X) = \int_0^1 x f(x)\, dx$

$$= \int_0^1 x \cdot 6x(1-x)\, dx$$

$$= 6\int_0^1 x^2\, dx - 6\int_0^1 x^3\, dx$$

$$= 6\left(\frac{x^3}{3}\right)_0^1 - 6\left(\frac{x^4}{4}\right)_0^1$$

$$= 0.5$$

Now, $E(X^2) = \int_0^1 x^2 f(x)\, dx$

$$= \int_0^1 x^2\, 6x(1-x)\, dx$$

$$= 6\int_0^1 x^3(1-x)\, dx$$

$$= 6\int_0^1 x^3\, dx - 6\int_0^1 x^4\, dx$$

$$= 6\left(\frac{x^4}{4}\right)_0^1 - 6\left(\frac{x^5}{5}\right)_0^1$$

$$= 0.3$$

Hence, $Var(X) = 0.3 - (0.5)^2$
$= 0.05$

Exponential Distribution :

A continous r.v taking non-negative values is said to follow exponential distribution with mean θ if it's probability density function (p.d.f) is given by —

$$f(x) = \frac{1}{\theta} e^{-x/\theta} \quad , \quad x \geq 0, \theta > 0$$
$$= 0 \quad , \quad o.w.$$

Notation : $X \longrightarrow Exp(\theta)$

Note : (#) We shall verify that $f(x)$ is p.d.f

(i) obivously $f(x) \geq 0$, $\because x \geq 0, \theta > 0$ and $e^{-x/\theta} > 0$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \frac{1}{\theta} e^{-x/\theta}\,dx$

$$= \frac{1}{\theta}\left[\frac{e^{-x/\theta}}{-1/\theta}\right]_0^{\infty}$$

$$= \left[-e^{-x/\theta}\right]_0^{\infty}$$

$$= 1$$

from ① & ②, it is clear that $f(x)$ is a p.d.f.

Note: If $\theta = 1$, then the distribution is called standard exponential distribution. Hence it's p.d.f is given by

$$f(x) = e^{-x} \quad ; \quad x \geq 0, \theta > 0$$
$$= 0 \quad , \quad o.w.$$

Mean and Variance:

If $X \longrightarrow Exp(\theta)$, then it can be shown that

$$Mean = E(X) = \theta$$
$$Var(X) = \theta^2$$
$$S.D(X) = \theta$$

Thus for exponential distribution, mean and S.D are same.

Distribution Function of Exp.

Let $X \longrightarrow Exp(\theta)$.

Then distribution function is given by

$$F(x) = P[X \leq x]$$
$$= \int_0^x f(t)\,dt = \int_0^x \frac{1}{\theta} e^{-t/\theta}\,dt$$
$$= \frac{1}{\theta} \int_0^x e^{-t/\theta}\,dt$$
$$= \frac{1}{\theta} \left[-\frac{e^{-t/\theta}}{1/\theta} \right]_0^x$$
$$= \left[e^{-t/\theta} \right]_0^x = 1 - e^{-x/\theta}$$

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$\therefore F(x) = 1 - e^{-x/\theta}, \quad x > 0, \theta > 0$

Note: When X is life time of a component, $P[X > x]$ is taken as reliability function or survival function.

Here, $P[X > x] = 1 - P[X \le x]$
$= 1 - (1 - e^{-x/\theta})$

$$\boxed{P[X > x] = e^{-x/\theta}, \quad x > 0, \theta > 0}$$

Ex: 1. Suppose that the life time of a certain make of T.V tube is exponentially distributed with a mean life 1600 hrs. What is the probability that:
(i) the tube will work upto 2400 hrs?
(ii) the tube will survive after 1000 hrs?

S/n: Let X: Number of hours that the T.V tube work.

Given: $X \longrightarrow \exp(\theta)$, where $\theta = 1600$

We know that if $X \longrightarrow \exp(\theta)$ then

$$P[X \le x] = 1 - e^{-x/\theta}, \quad x > 0, \theta > 0$$

(i) $P[X \le 2400] = 1 - e^{-\frac{2400}{1600}}$

$= 1 - e^{-1.5}$

$= 1 - 0.223130$

$= 0.77687$

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(ii) $P[X > 1000] = e^{-\frac{1000}{1600}}$

$= e^{-0.625}$

$= 0.5353319$

Ex 2. The life time in hours of a certain electric component follows exponential distribution with distribution function.

$$F(x) = 1 - e^{-0.004x} \quad ; \quad x \geq 0.$$

What is (i) the probability that the component will survive 200 hrs?
(ii) the probability that it will fail during 250 to 350 hrs?
(iii) expected life time of the component?

Soln: Let X: life time (in hrs) of the electric component.

Given: $X \longrightarrow Exp(\theta)$

and $F(x) = 1 - e^{-x/\theta} = 1 - e^{-0.004x}$

$$\theta = \frac{1}{0.004} = 250 \text{ hrs}$$

(i) $P[X > 200] = e^{-200/250} = e^{-0.8}$

$= 0.449329$

(ii) $P[250 < X < 350] = P[X > 250] - P[X > 350]$

$= e^{-\frac{250}{250}} - e^{-\frac{350}{250}}$

$= e^{-1} - e^{-1.4}$

$= 0.367879 - 0.246597$

$= 0.121282$

(iii) $E(X) = \theta = 250$ hrs.

Normal Distribution:

Note: Normal distribution is one of the most commonly used distribution.
The variables such as intelligent quotient, height of a person, weight of a person, errors in measurement of physical quantities follow normal distribution.

Def^n // A continous random variable X is said to follow normal distribution with parameters μ and σ^2 if it's p.d.f is -

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2},$$

$$-\infty < x < \infty, \ -\infty < \mu < \infty, \ \sigma > 0$$

Note: 1. A r.v X follows normal distribution with parameters μ and σ^2 is symbollically written as $X \longrightarrow N(\mu, \sigma^2)$

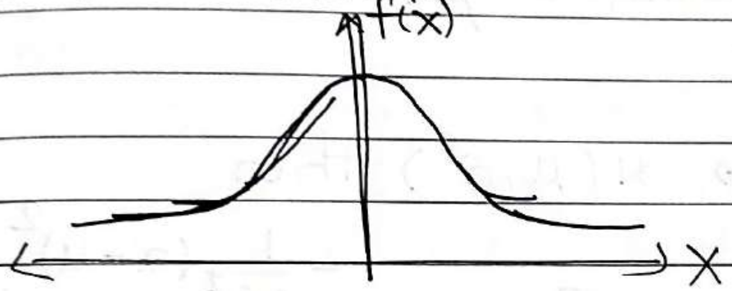
2. If $\mu = 0$, $\sigma^2 = 1$, then the normal variable is called as standard normal variable. i.e $N(0, 1)$
Generally, it is denoted by Z.

The p.d.f of Z is

$$f(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}} \quad , \quad -\infty < Z < \infty$$

with $\pi = 3.141159$ and $e \approx 2.71828$

3. The probability density curve of $N(\mu, \sigma^2)$ is bell shaped, symmetric about μ and mesokurtic.



Naturally, the curve of standard normal distribution, is symmetric around zero.

4. The maximum height of probability density curve is $= \frac{1}{\sigma\sqrt{2\pi}}$

5. As the curve is symmetric about μ, the mean, median and mode coincide and all are equal to μ.

6. The parameter σ^2 is also the variance of X. Hence $s.d(X) = \sigma$

<u>Relation between $N(\mu, \sigma^2)$ and $N(0,1)$</u> :

If $X \longrightarrow N(\mu, \sigma^2)$ then $Z = \frac{x - \mu}{\sigma} \longrightarrow N(0,1)$

This result is useful while computing probabilities of a $N(\mu, \sigma^2)$ variable.

Computation of probabilities:

If $f(x)$ is a p.d.f of X then

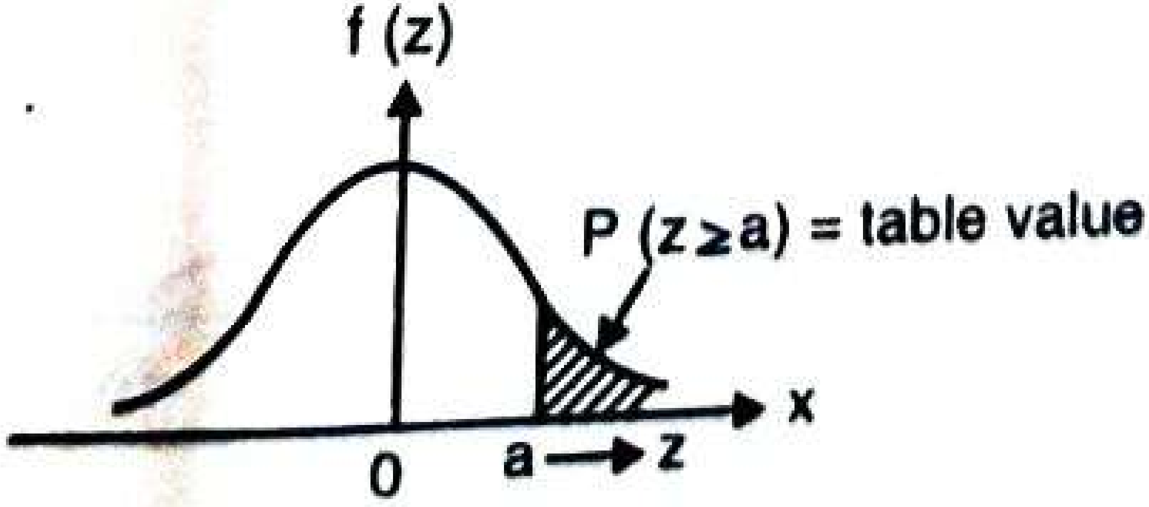
$$P(a < X < b) = \int_a^b f(x)\,dx .$$

It is equivalent to the area under the curve $f(x)$, bounded by x-axis and ordinates $x=a$, $x=b$.

If $X \longrightarrow N(\mu, \sigma^2)$ then

$$P(X > a) = \int_a^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\,dx \quad \text{--- (1)}$$

Note: To evaluate (1), we need to convert it in terms of $N(0,1)$ by using transformation $z = \dfrac{x-\mu}{\sigma}$



f (z)
P (z ≥ a) = table value
x
0
a
z

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In order to obtain probability of any region, we use following rules.

1. The total area under the probability density curve is 1

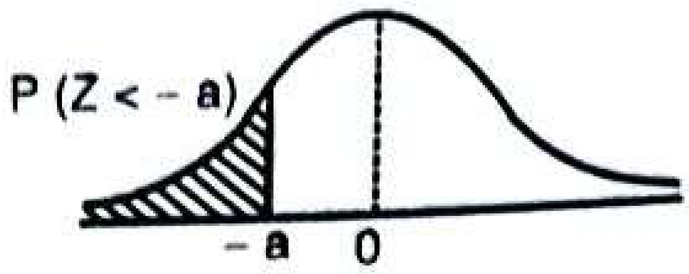
i.e $$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

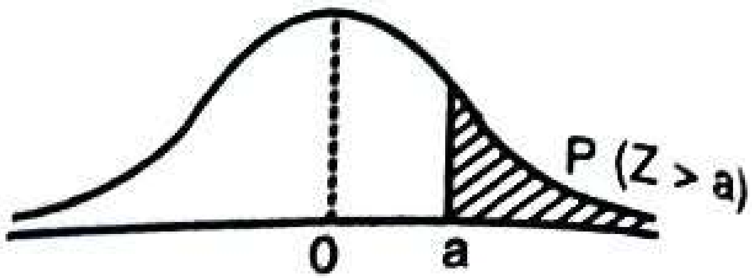
2. The p.d.f curve of $N(0,1)$ is symmetric about '0'.

In other words,

$$Z \longrightarrow N(0,1).$$

$$P(Z>a) = P(Z<-a)$$


P (Z < – a)
– a
0


P (Z > a)
0
a

3. Area below the ordinate
$= 1 -$ Area above the ordinate

$\therefore P(Z < a) = 1 - P(Z \geq a)$
$= 1 -$ table value.

4. If $X \longrightarrow N(\mu, \sigma^2)$ and suppose we want to find $P[X \geq a]$, then we transform the variable X to Z, and obtain probability as -

$$P[X \geq a] = P\left[\frac{x-\mu}{\sigma} \geq \frac{a-\mu}{\sigma}\right]$$
$$= P\left[Z \geq \frac{a-\mu}{\sigma}\right]$$
$$= \text{table value}$$

Ex: 1. Let $X \longrightarrow N(3,4)$. (μ, σ^2)

Find (i) $P(X>5)$ (ii) $P(X<1)$ (iii) $P(X>0)$

(iv) $P(X<6)$ (v) $P(2<X<6)$ (vi) $P(4<X<6)$

(vii) $P(|X|>4)$ (viii) $P(|X-3|<3.92)$

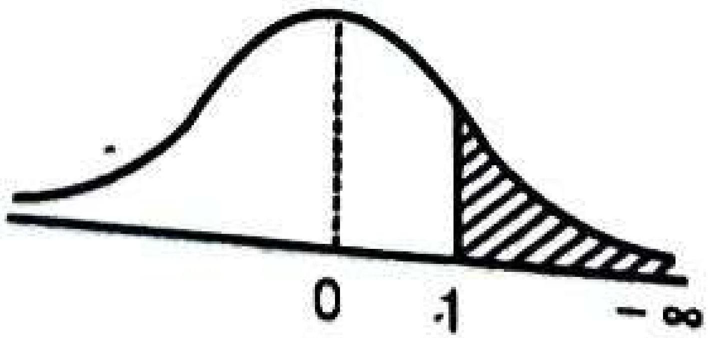
Soln: We need to express first of all the probabilities in terms of standard normal variable Z.

(i) $P(X>5) = P\left(\frac{X-\mu}{\sigma} > \frac{5-3}{2}\right)$, As $\mu=3$, $\sigma=2$

$= P(Z>1)$, $Z = \frac{X-\mu}{\sigma}$

From normal prob. integral table, we get shaded region of $P(Z>1)$ as

$P(Z>1) = 0.15866$


0
1
– ∞

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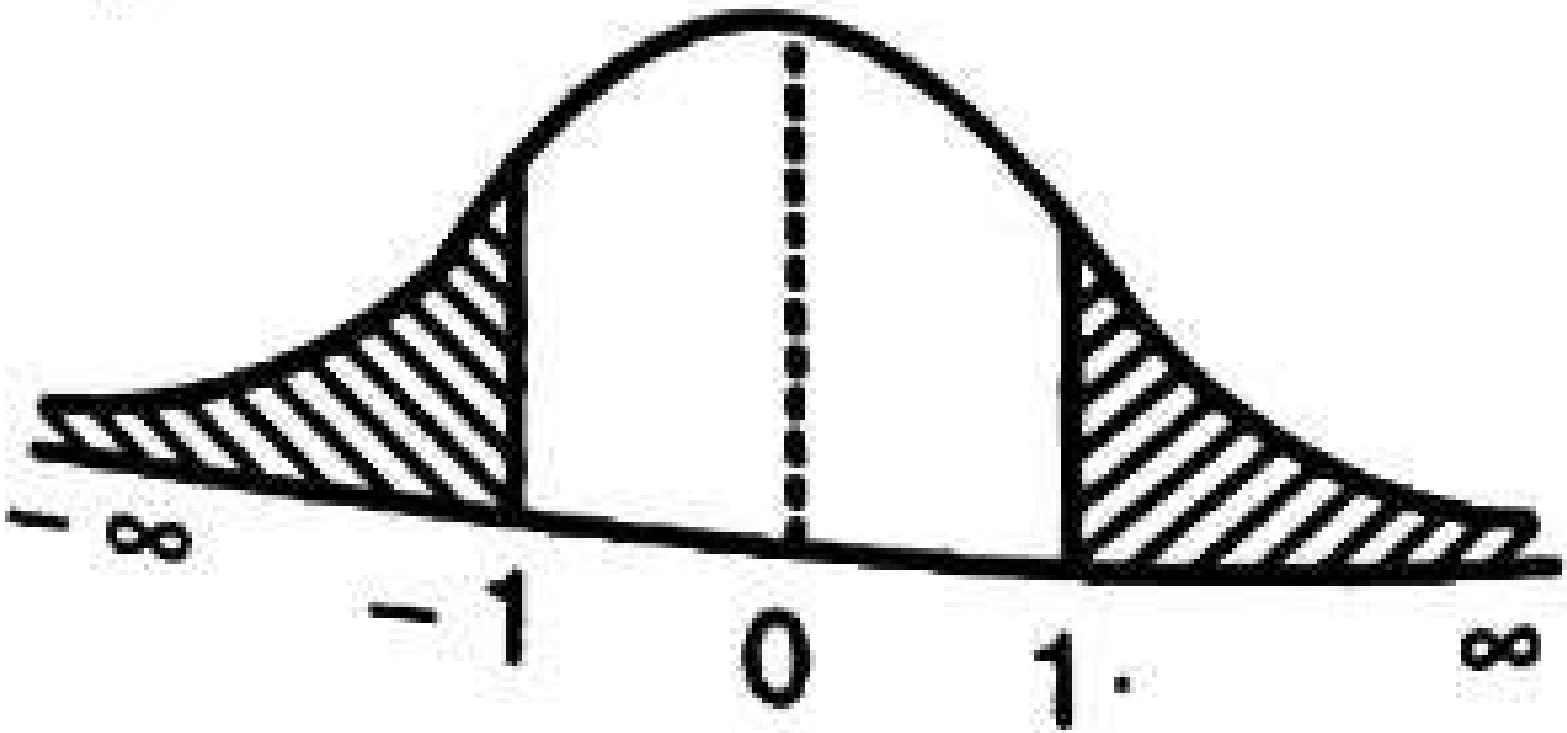
(ii) $P(X < 1) = P\left(\frac{X-\mu}{\sigma} < \frac{1-3}{2}\right)$.

$= P(Z < -1)$

$= P(Z > 1)$ (due to symmetry)

$= 0.15866$ (From the table)

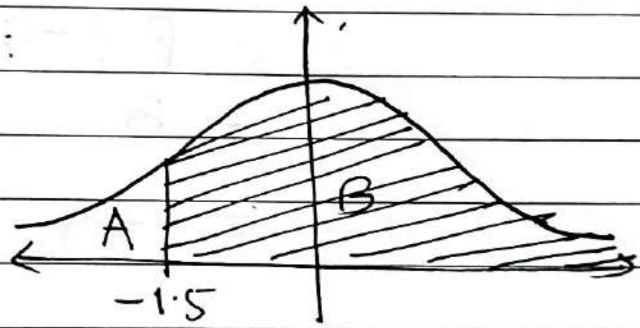


− ∞
−1
0
1
∞

(iii) $P(X > 0) = P\left(\frac{X-\mu}{\sigma} > \frac{0-3}{2}\right)$

$= P(Z > -1.5)$

Diagram:



Since only tail area is given in table, we use the fact that $A+B=1$.

$\therefore P(Z > -1.5) = 1 - A$

$= 1 - P(Z < -1.5)$

$= 1 - P(Z > 1.5)$

$= 1 - 0.66087$

$= 0.933913$

(iv) $P(X < 6) = P\left(\frac{X-\mu}{\sigma} < \frac{6-3}{2}\right)$

$= P(Z < 1.5)$

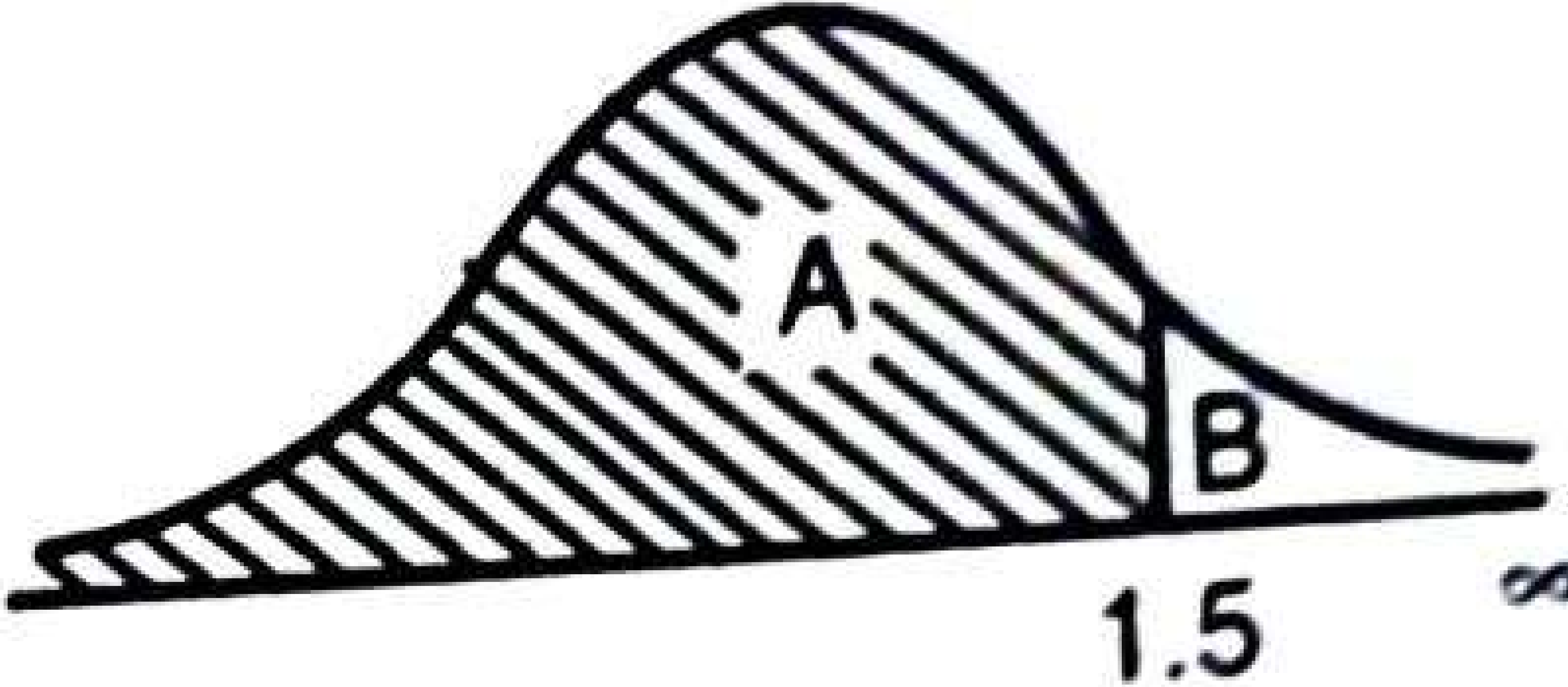
$= A$

$$= 1 - \beta$$

$$= 1 - P(X > 1.5)$$

$$= 1 - 066087$$

$$P(X < 6) = 0.933913$$


A
B
1.5
∞

(V) $P(2 < X < 6) = P\left(\frac{2-3}{2} < \frac{X-\mu}{\sigma} < \frac{6-3}{2}\right)$

$= P(-0.5 < Z < 1.5)$

$= B$

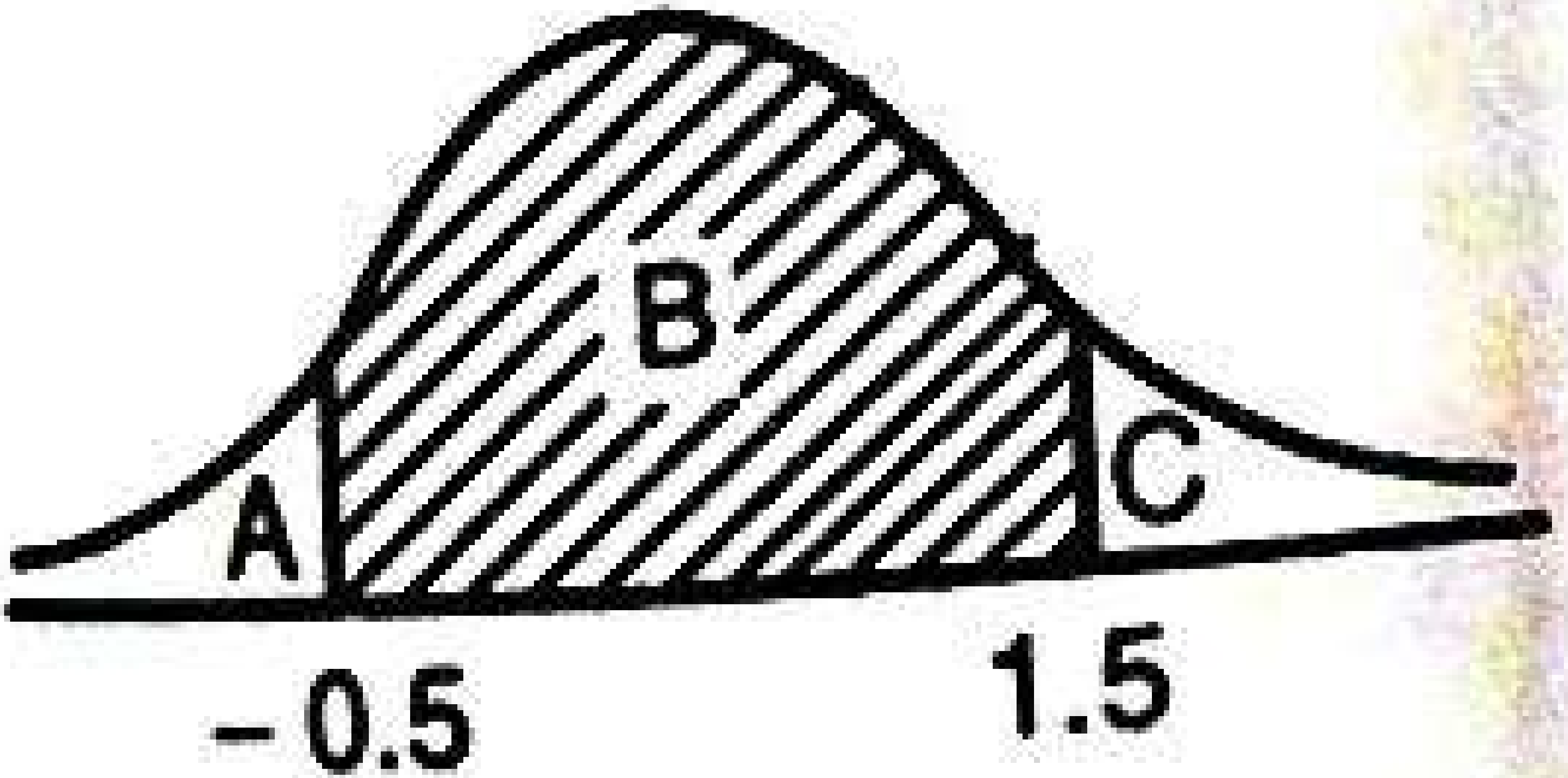
$= 1 - A - C \quad (\because A + B + C = 1)$

$= 1 - P(Z < 0.5) - P(Z > 1.5)$

$= 1 - P(Z > 0.5) - P(Z > 1.5)$

$= 1 - 0.30854 - 0.066087$

$= 0.625375$


A
B
C
−0.5
1.5

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(vi) $P(4 < X < 6) = P\left(\frac{4-3}{2} < \frac{X-\mu}{6} < \frac{6-3}{2}\right)$

$= P(0.5 < Z < 1.5)$

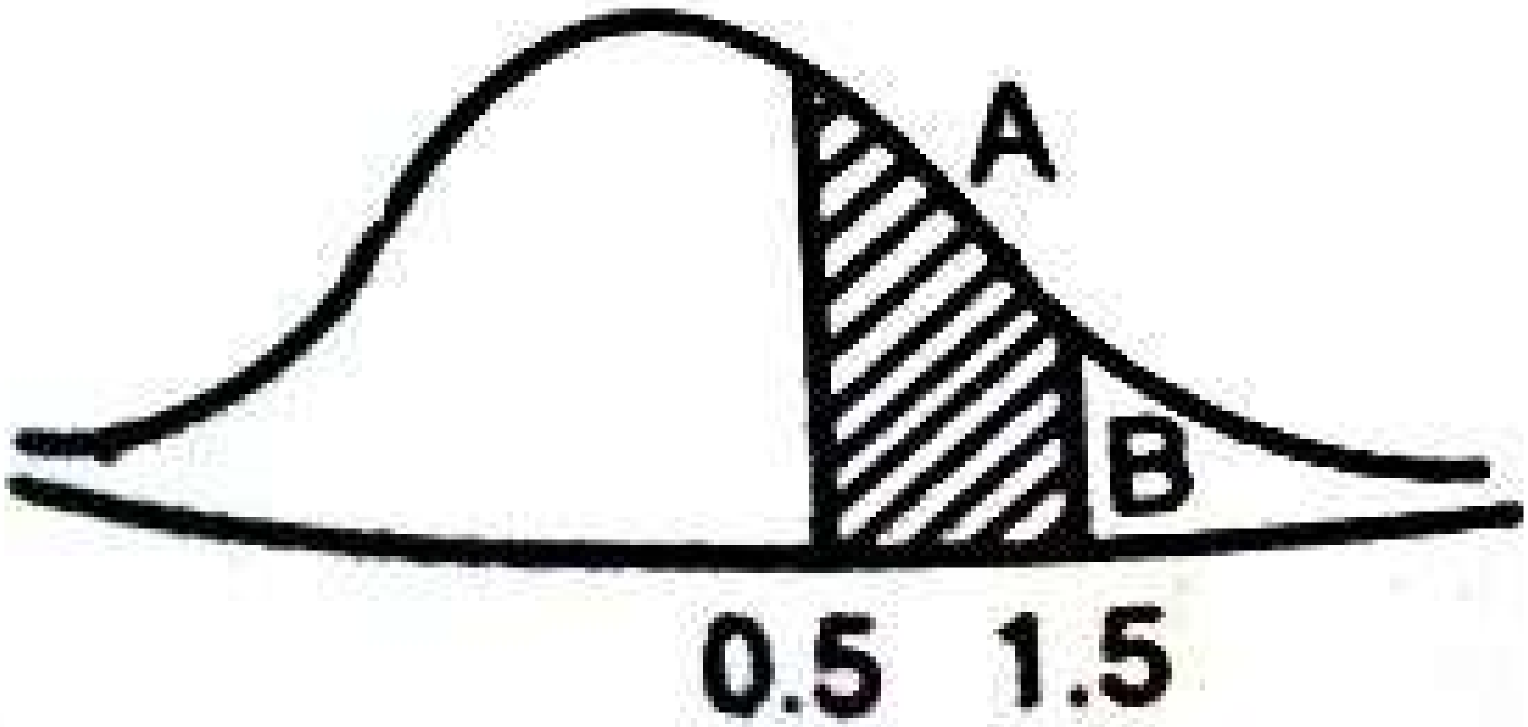
$= A$

$= (A+B) - B$

$= P(Z > 0.5) - P(Z > 1.5)$

$= 0.30854 - 0.066807$

$\cong 0.241533$


A
B
0.5
1.5

(viii) $P(|X| > 4) = P(X > 4) + P(X < -4)$

$$= P\left(\frac{X-\mu}{\sigma} > \frac{4-3}{2}\right) + P\left(\frac{X-\mu}{\sigma} < \frac{-4-3}{2}\right)$$

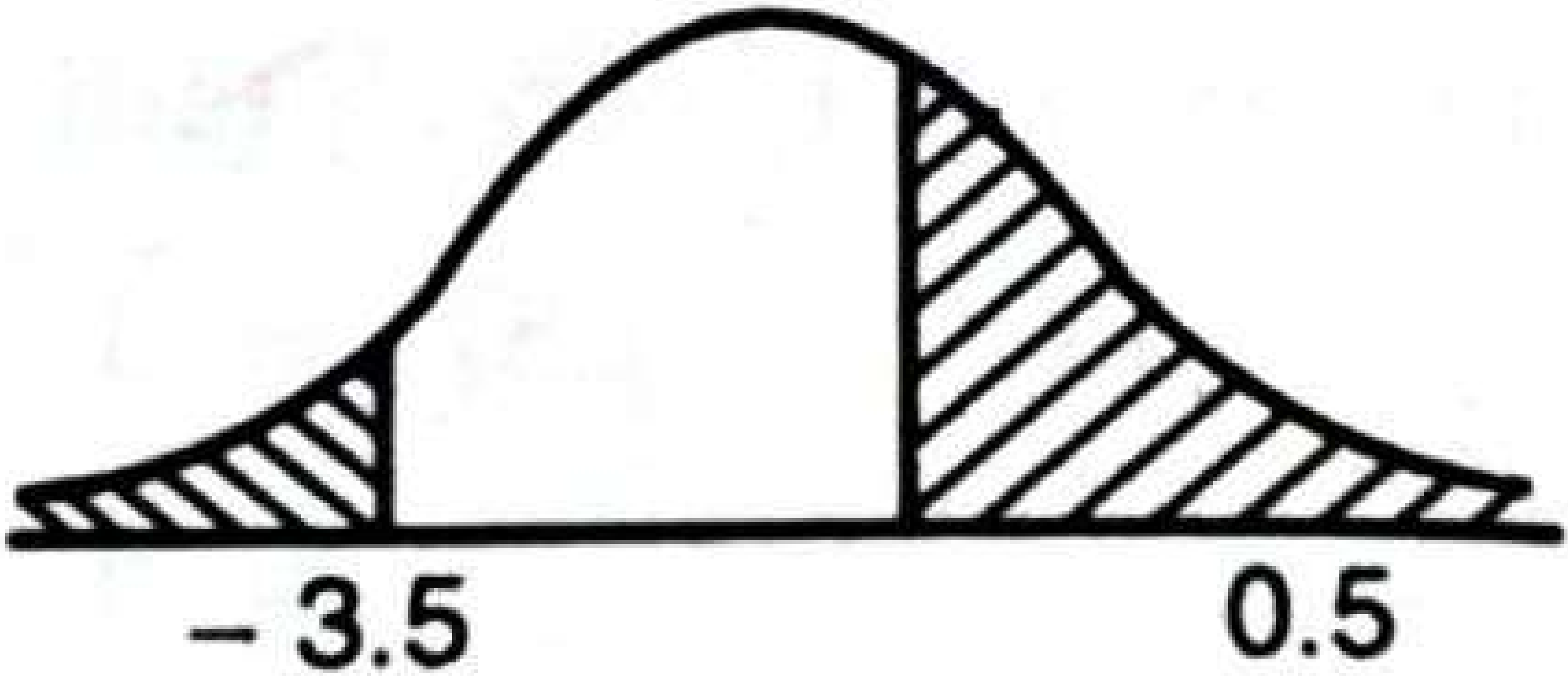
$$= P(Z > 0.5) + P(Z < -3.5)$$

$$= P(Z > 0.5) + P(Z > 3.5)$$

due to symmetry.

$$= 0.30854 + 0.00023263$$

$$= 0.30877263.$$



− 3.5
0.5

(viii) $P(|X-3| < 3.92)$

$$= P\left(\frac{|X-3|}{2} < \frac{3.92}{2}\right)$$

$$= P(|Z| < 1.96)$$

$$= P(-1.96 < Z < 1.96)$$

$$= B$$

$$= 1 - A - C$$

$$= 1 - 2C$$

$$= 1 - 2 \times 0.024998$$

$$= 0.950004$$

<u>Note</u>: $P(X \geq \mu) = P(Z \geq 0) = \frac{1}{2}$

and $P(X \leq \mu) = P(Z \leq 0) = \frac{1}{2}$

Hence μ is a median of X.

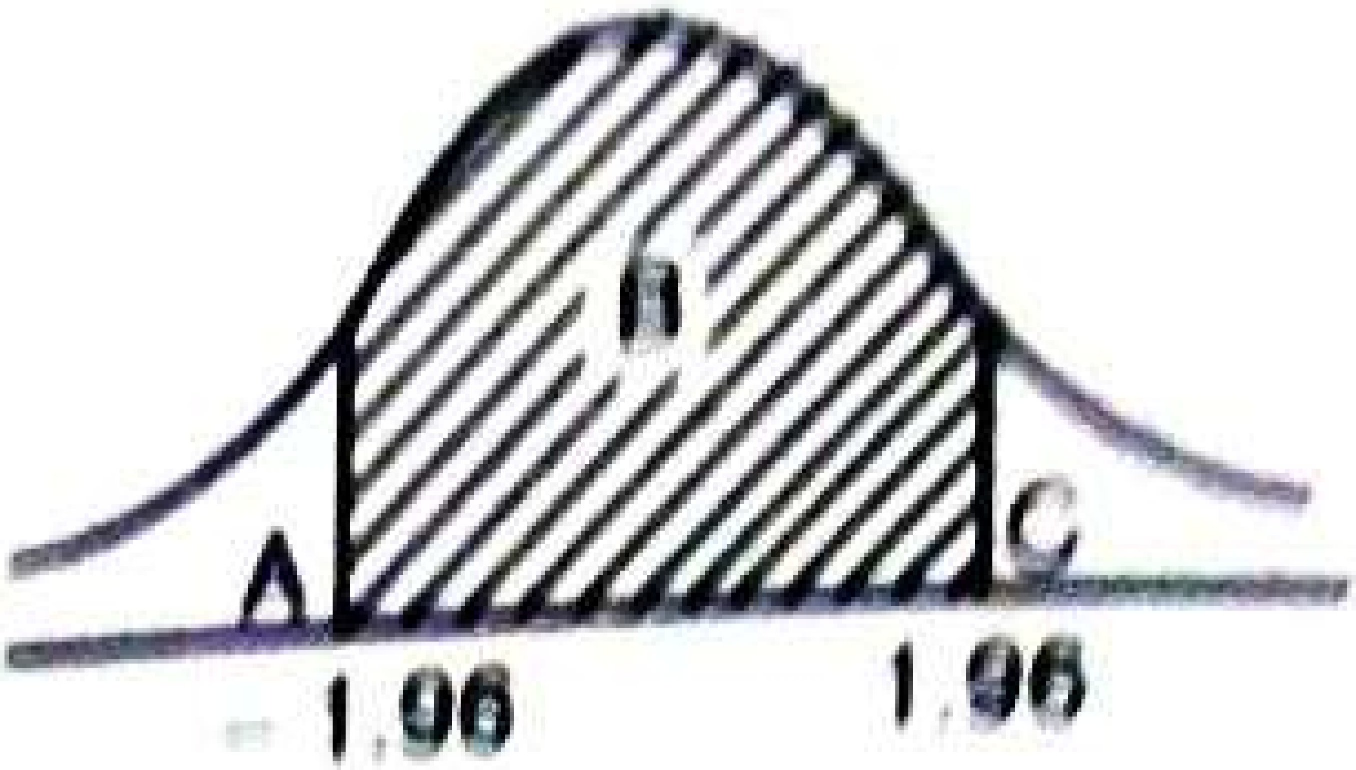


Fig. 11.11

EXERCISE 8 (A)

1. Define continuous random variable and explain the terms (i) probability density function (ii) distribution function of continuous r. v.
2. State the four characteristic properties of distribution function of a continuous r. v.
3. Describe how distribution function of a r. v. X is used in obtaining $P(a < X < b)$
4. Verify which of the following can be looked upon as p. d. f. of r. v. X.

   (a) $f(x) = 2 \quad ; \quad 0 \le x \le \frac{1}{2}$

   (b) $f(x) = x e^{-x} \quad ; \quad x > 0$

   (c) $f(x) = x \quad ; \quad 0 \le x \le 1$

   $ = 2 - x \quad ; \quad 1 \le x \le 2$

   (d) $f(x) = \dfrac{1}{(x+1)^2} \quad ; \quad x \ge 0$

   (e) $f(x) = \frac{1}{2} e^{-|x-m|} \quad ; \quad -\infty < x < \infty, \; -\infty < m < \infty$

   (f) $f(x) = \sin x \quad ; \quad 0 \le x \le \pi/2.$

5. Find the value of constant c in each of the following p. d. f.s.

   (a) $f(x) = c x^3 \quad ; \quad 0 \le x \le 1$

   (b) $f(x) = c \quad ; \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

   (c) $f(x) = c x (2 - x) \quad ; \quad 0 \le x \le 2$

   (d) $f(x) = c e^{-mx} \quad ; \quad x > 0, \; m > 0$

   (e) $f(x) = c \sin^3 x \cos x \quad ; \quad 0 < x < \pi/2.$

6. Suppose X is a r. v. with p. d. f.

   $$f(x) = \frac{1}{(1+x)^2}, \quad x \ge 0$$

   If $A = \{1 < x < 9\}$ and $B = \{0 < x < 4\}$, find $P(A)$, $P(B)$, $P(A')$, $P(A \cap B)$, $P(A \cup B)$, $P(A' \cap B)$, $P(A' \cup B')$, $P(A' \cap B')$.

7. If Y is a r. v. with p. d.f.

   $$f(y) = \frac{k}{\sqrt{y}} \text{ if } 0 < y < 4$$
   $$= 0 \quad \text{elsewhere}$$

find (i) the value of k

(ii) distribution function of Y

(iii) $P(1 < Y < 2)$

8. If X is a r. v. with distribution function

$$F(x) = \begin{cases} 0 & ; \quad x < -1 \\ (x+1)/2 & ; \quad -1 \le x < 1 \\ 1 & ; \quad x \ge 1. \end{cases}$$

find (i) $P(2 < X < 3)$

(ii) p. d. f. of X

(iii) $E(X)$, $E(X^2)$, var (X).

9. A r. v. X has distribution function

$$F(x) = \begin{cases} 0, & x < 5 \\ 1 - \dfrac{25}{x^2}, & x \ge 5 \end{cases}$$

find $P(X > 10)$,

10. If a r. v. X has p. d. f.

$$f(x) = \frac{c}{x} \qquad 1 < x < 3$$

then find (i) c (ii) E (X) (iii) var (X).