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MIT WORLD PEACE UNIVERSITY | PUNE
TECHNOLOGY, RESEARCH, SOCIAL INNOVATION & PARTNERSHIPS

CET2001B Advanced Data Structure

School of Computer Engineering and Technology

S. Y. B. Tech

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Heaps

- Heap as a Data Structure
- Types of heap – Min heap and Max heap
- Operations on Heap
- Heap Sort
- Applications of Heap

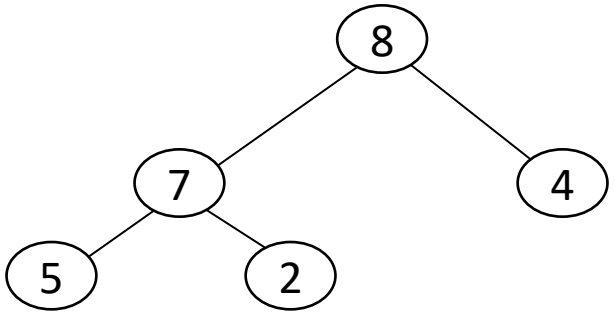
The Heap Data Structure

Def: A **max** (min) heap is a tree in which the **key value** in each node is **no smaller** (larger) than the key values in its **children** (if any).

- A max heap is a complete binary tree that is also a max tree.
- A min heap is a complete binary tree that is also a min tree.

Order (heap) property: for any node x (for Max heap)

Parent(x) ≥ x



Heap

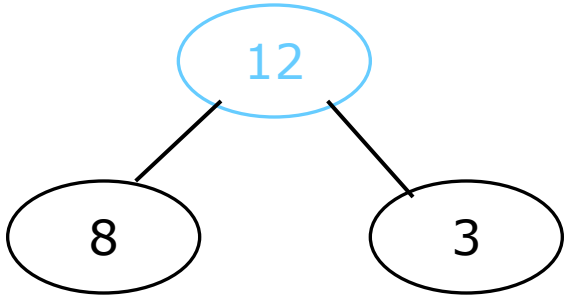
Example of Max heap

"The root is the maximum element of the heap!"

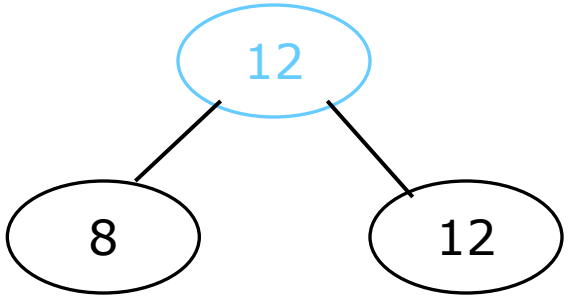
A heap is a binary tree that is filled in order

The heap property **(for Max heap)**

A node has the heap property if the value in the node is as large as or larger than the values in its children



Blue node has heap property



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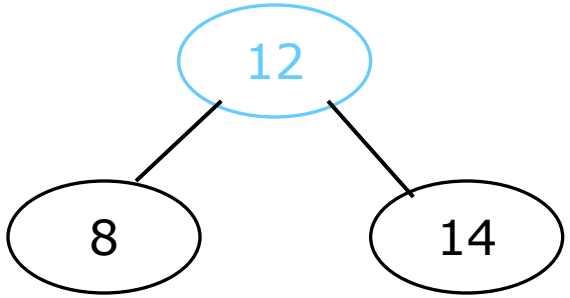

Blue node does not have heap property

All leaf nodes automatically have the heap property

A binary tree is a heap if *all* nodes in it have the heap property

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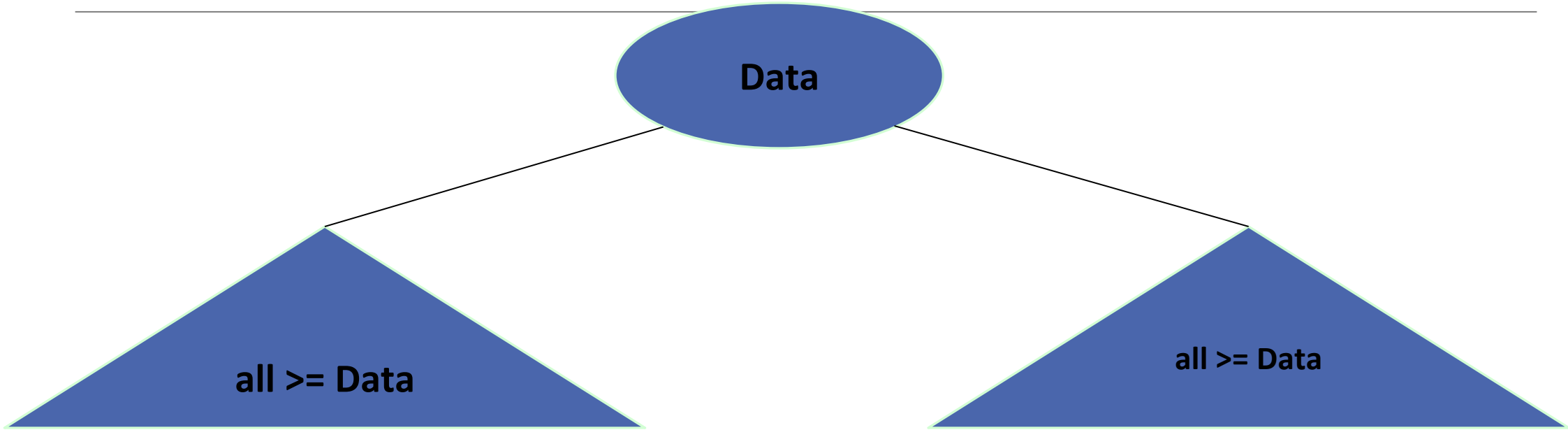
Types of Heap

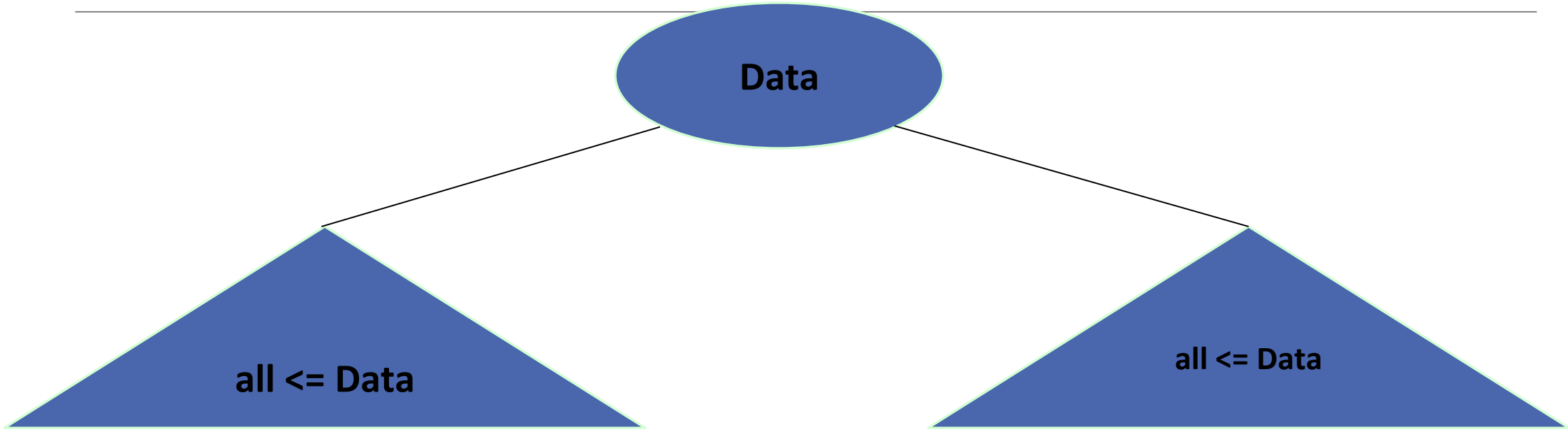
- **Min-heap**
- **Max-heap**

Min Heap



- In min-heap, the key value of each node is lesser than or equal to the key value of its children
- In addition, every path from root to leaf should be sorted in ascending order

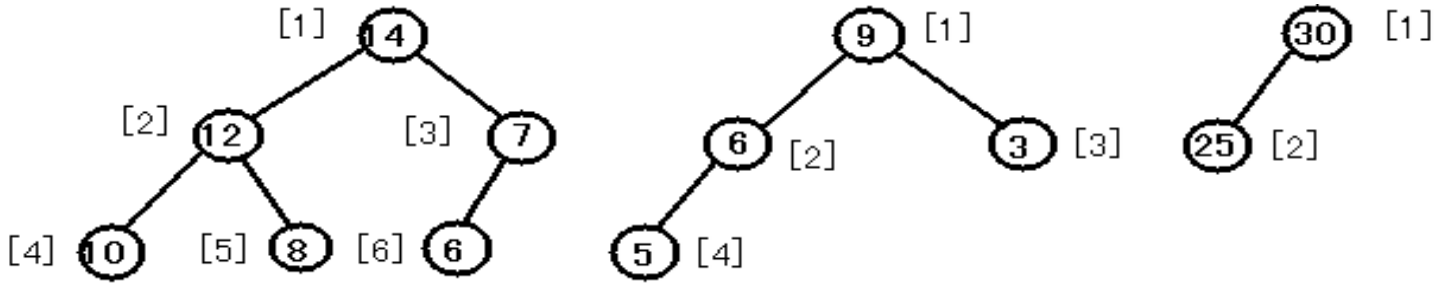
Max Heap



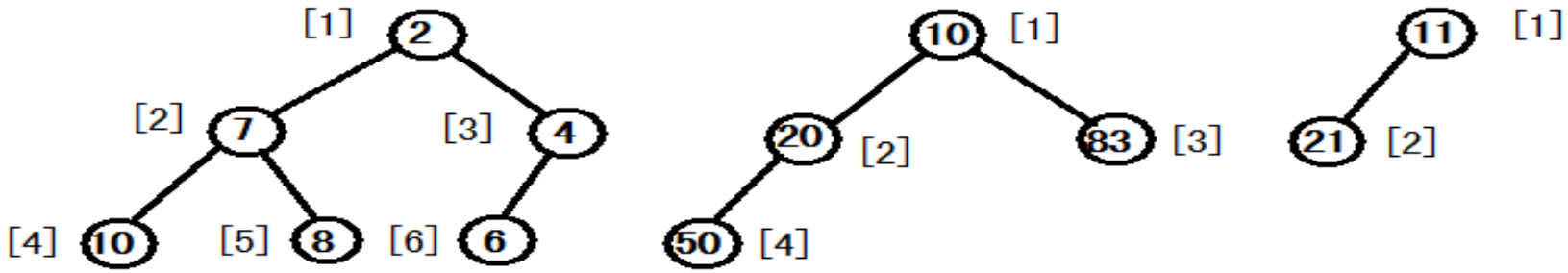
- A max-heap is where the key value of a node is greater than the key values in of its children

Example of Heap

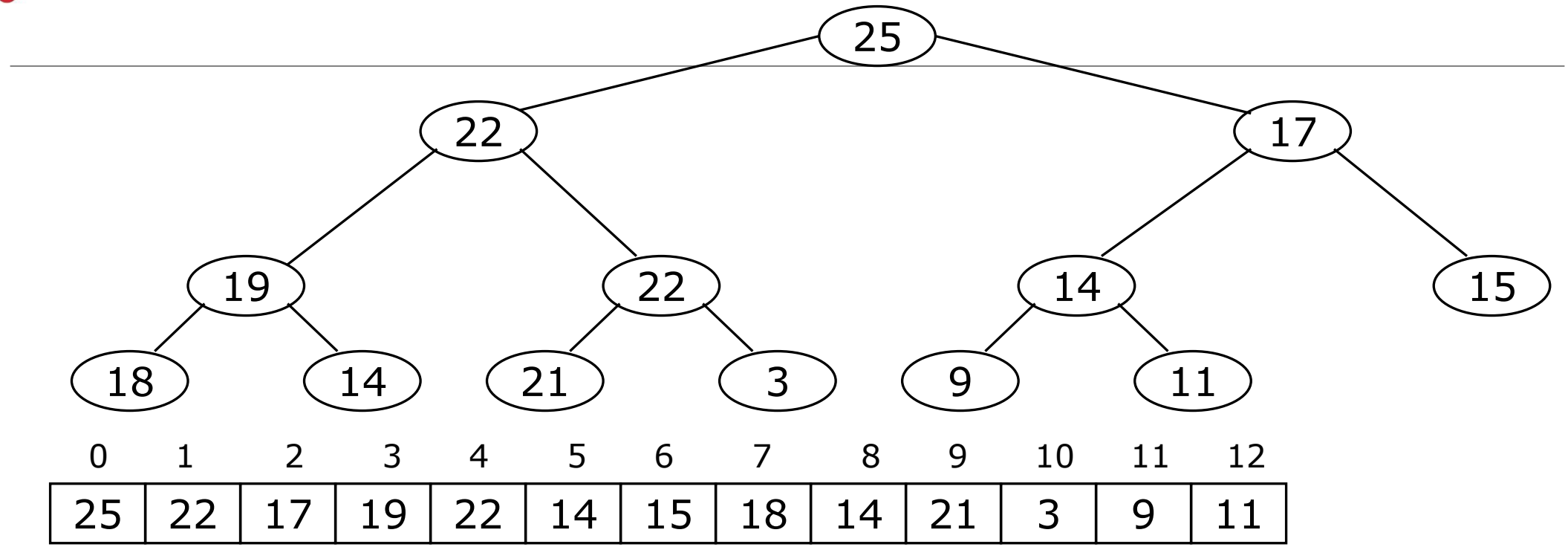
Max-Heap



Min-Heap



Mapping into an array



Notice:

- The left child of index i is at index 2*i+1(index starting from 0)
- The right child of index i is at index 2*i+2
- Example: the children of node value (19) [index 3] are at [index 7] (18) and [index 8] (14)

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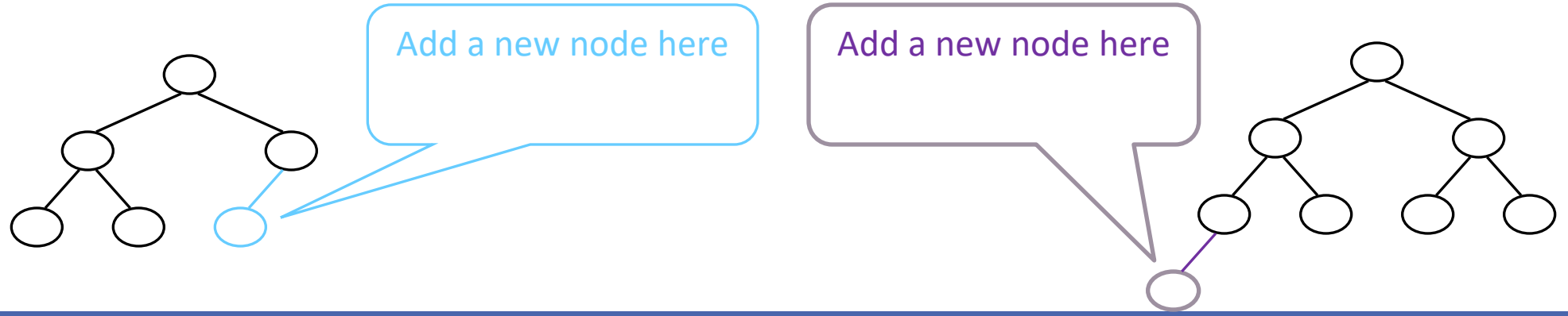
Operations on Heaps

- Create—To create an empty heap to which ‘root’ points
- Insert—To insert an element into the heap
- Delete—To delete max (or min) element from the heap
- ReheapUp—To rebuild heap when we use the insert() function
- ReheapDown—To build heap when we use the delete() function

Constructing a (max) heap

- A tree consisting of a single node is automatically a heap
- We construct a heap by adding nodes one at a time:
  - Add the node just to the right of the rightmost node in the deepest level
  - If the deepest level is full, start a new level
- Examples:



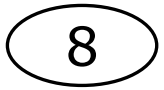
Constructing a (max)heap

contd…

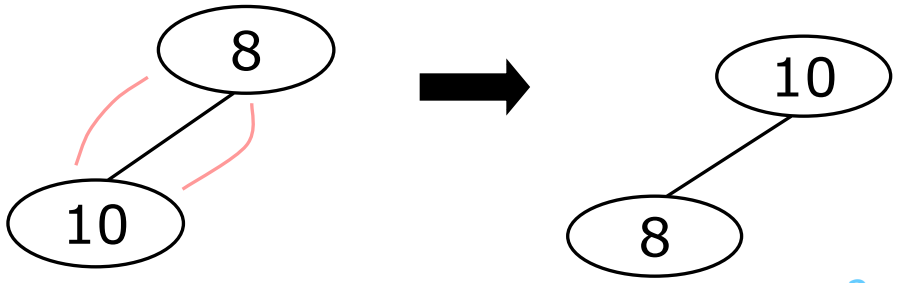
- Each time we add a node, we may destroy the heap property of its parent node
- To fix this, we shift up
- But each time we shift up, the value of the topmost node in the shift may increase, and this may destroy the heap property of *its* parent node
- We repeat the shifting up process, moving up in the tree, until either
  - We reach nodes whose values don't need to be swapped (because the parent is *still* larger than both children), or
  - We reach the root

Constructing a heap

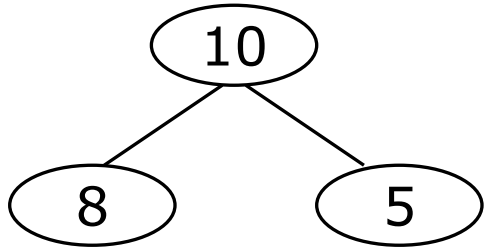
contd…



1



2



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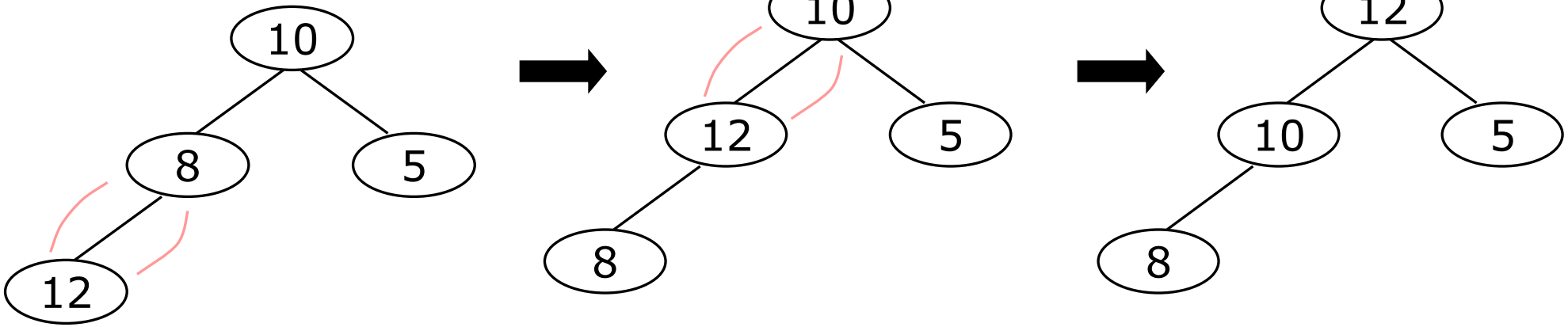
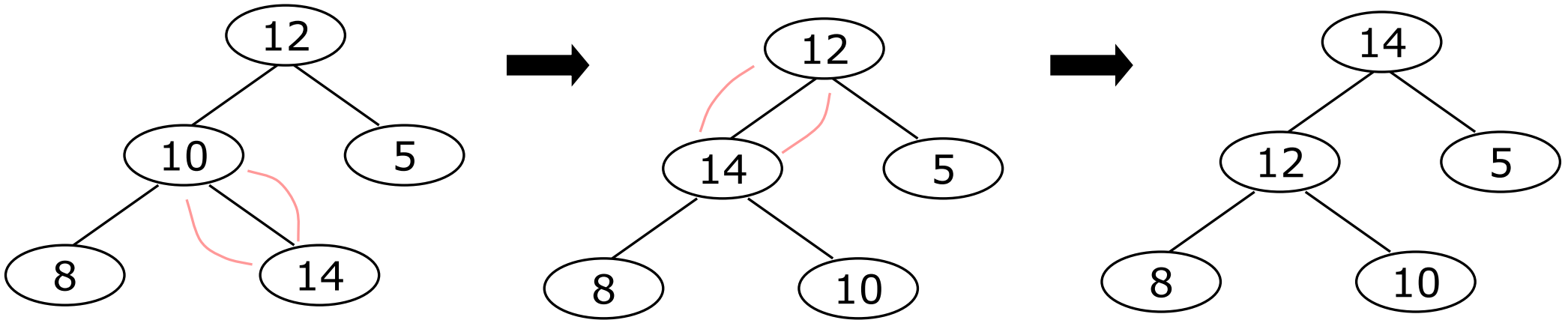


4

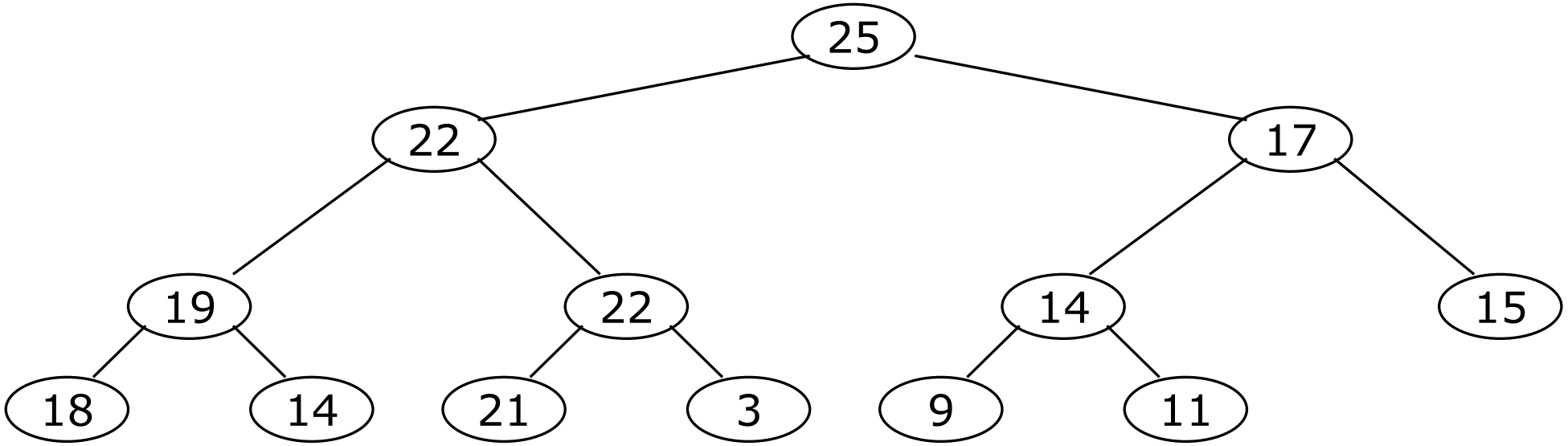
Other children are not affected



- The node containing 8 is not affected because its parent gets larger, not smaller
- The node containing 5 is not affected because its parent gets larger, not smaller
- The node containing 8 is still not affected because, although its parent got smaller, its parent is still greater than it was originally

A sample heap

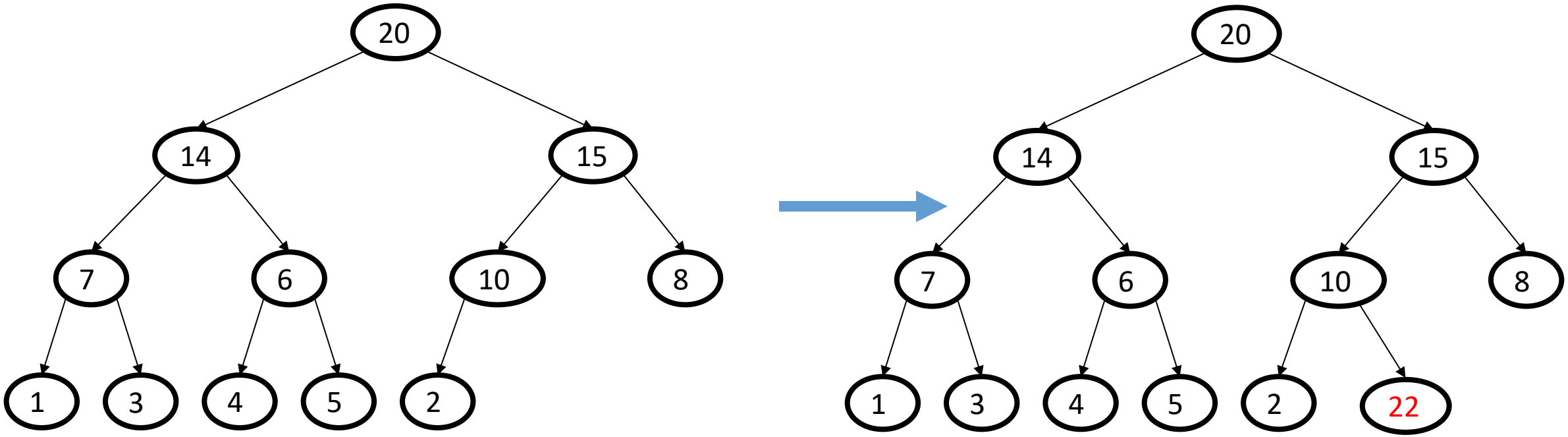
- Here's a sample binary tree after it has been heapified



- Notice that heapified does *not* mean sorted
- Heapifying does *not* change the shape of the binary tree; this binary tree is balanced and left-justified because it started out that way

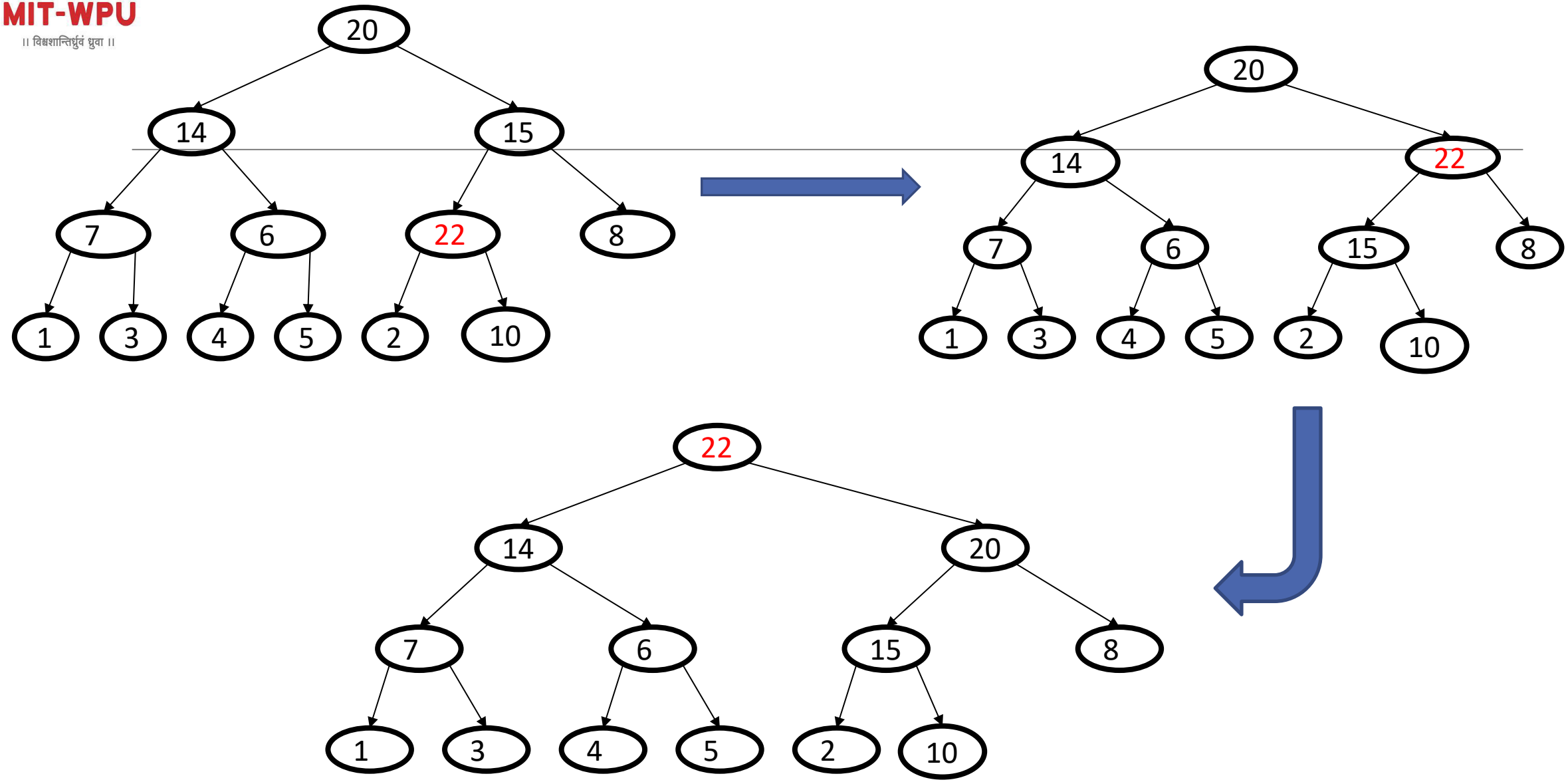
Insert (max heap)

```
insert(22)
```



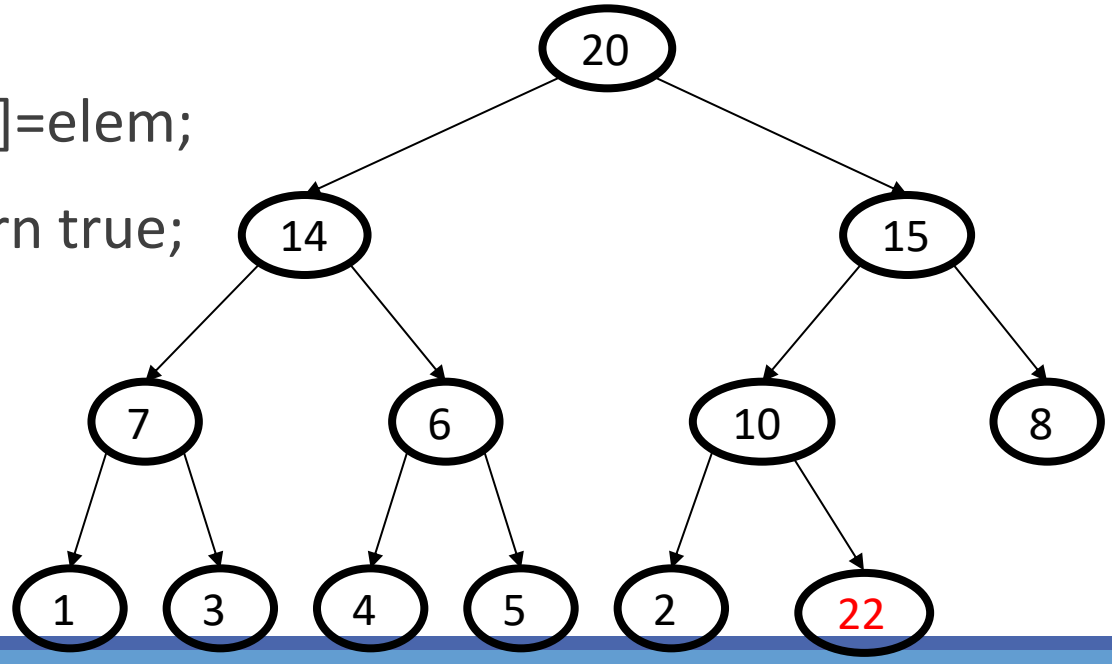
Insert (max heap)

contd…



```
Algorithm add()
{     //index of a starting from 0
      j=0 ; a[20];
     Get Choice
    Repeat
    {
     a[j]=elem;
       insert(a,j+1);
       j++;
     } until(choice==‘n’);
}
```

```
Algorithm Insert(a,n)
{
      i=n;  elem=a[n-1];
      if(i!=1){
          while((i>0)&&(a[(i/2)-1]<elem){
                    a[i-1]=a[(i/2)-1];
                    i=(i/2);
          }
          a[i-1]=elem;
          return true;
      }
}
```



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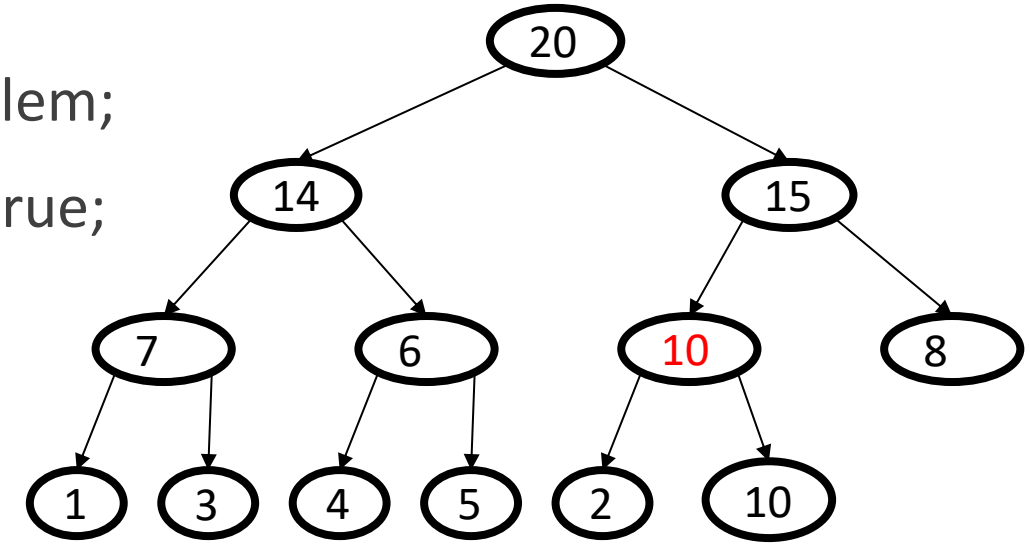

elem = 22

```
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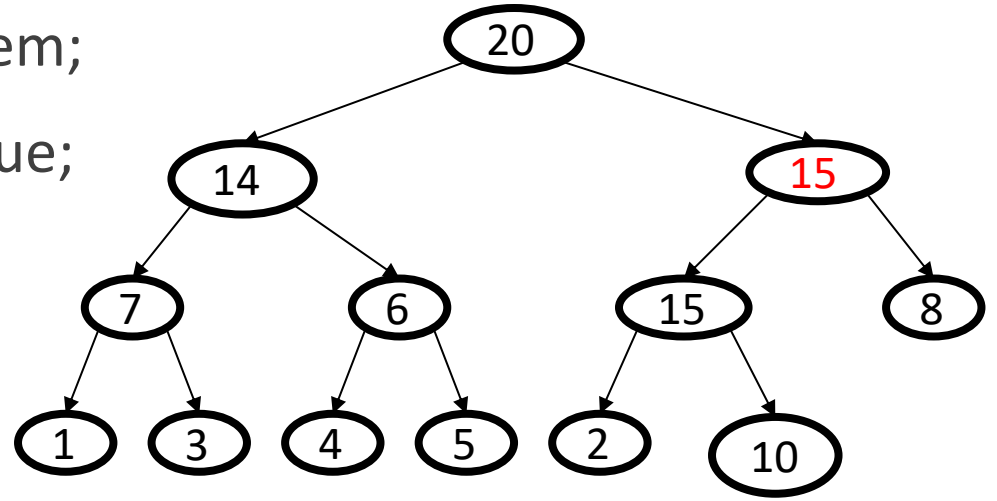

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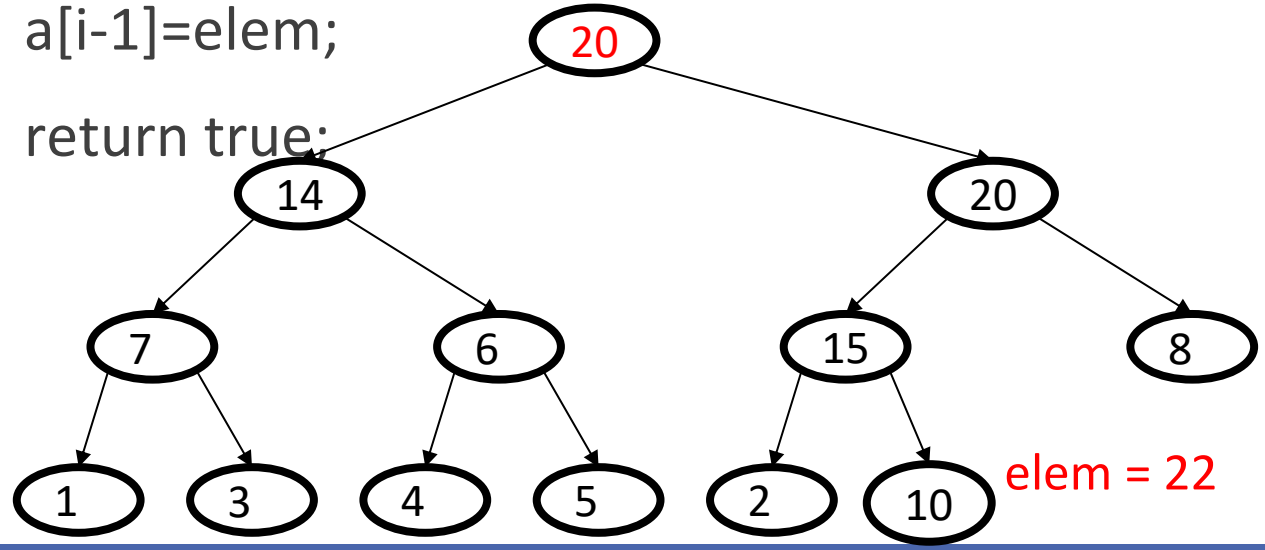

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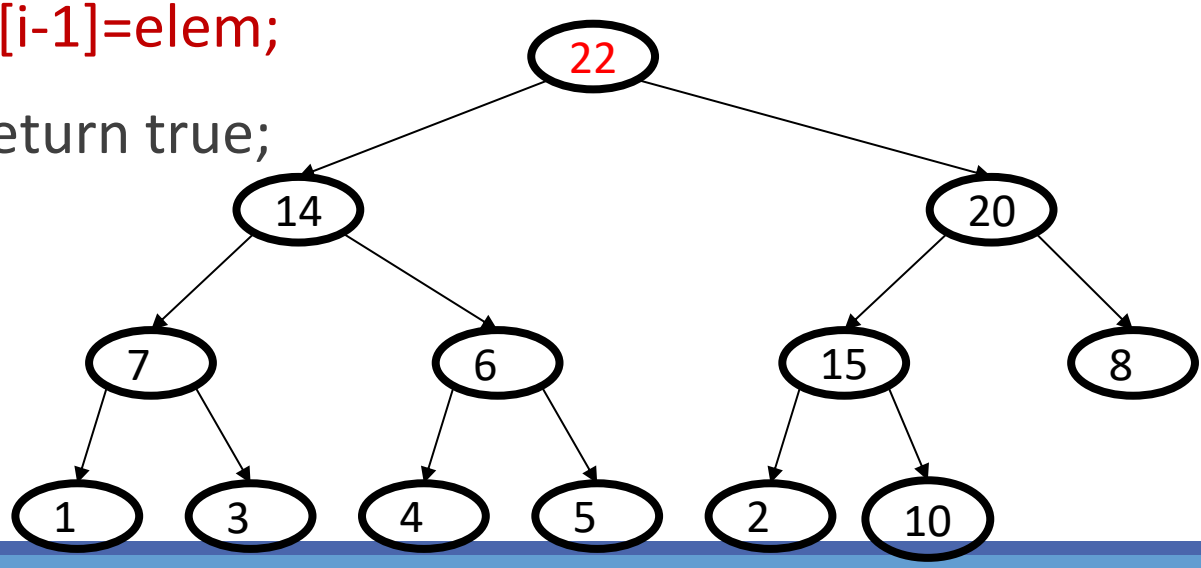
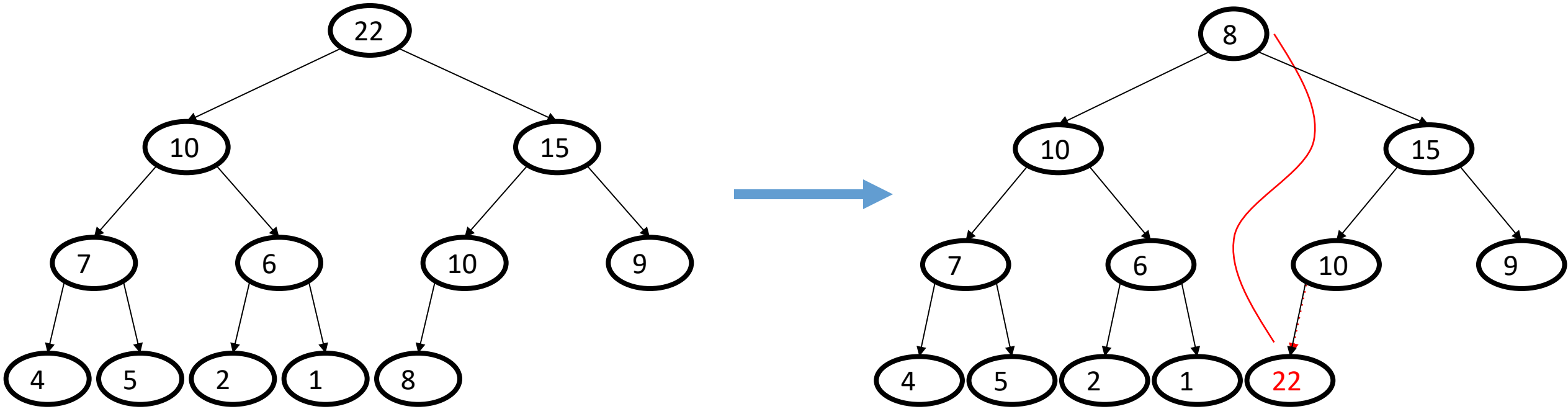


DeleteMax

`pqueue.deleteMax()`



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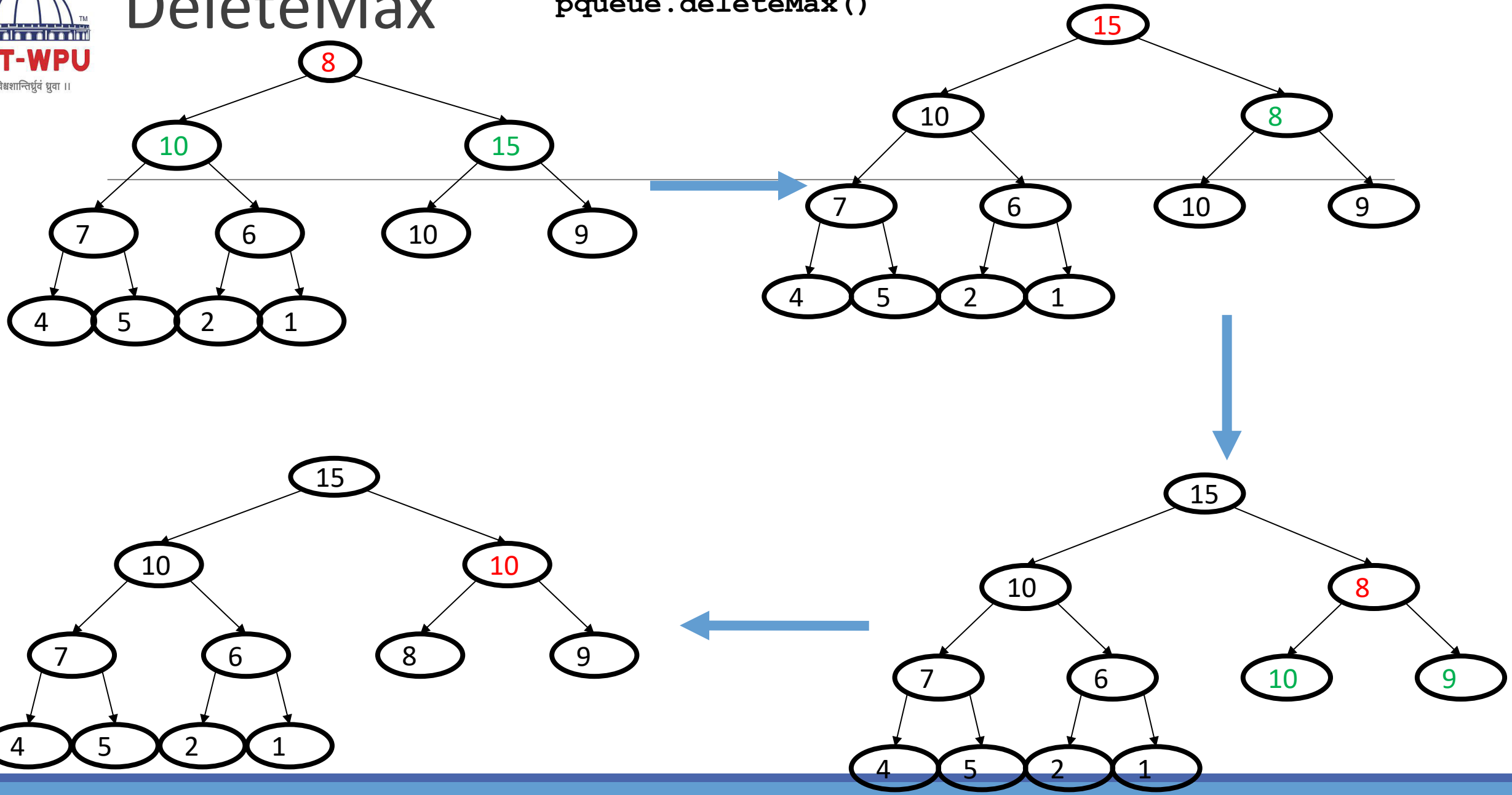
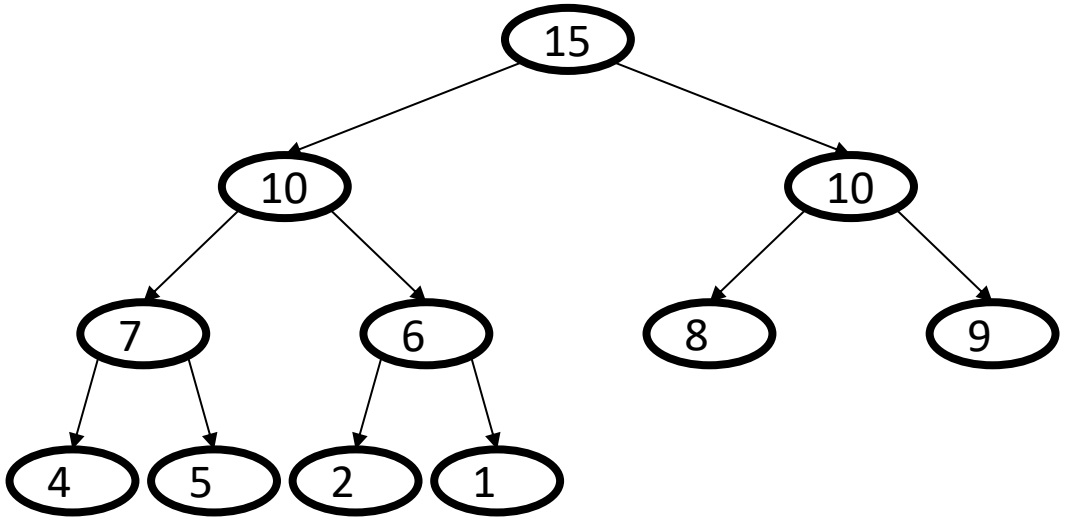


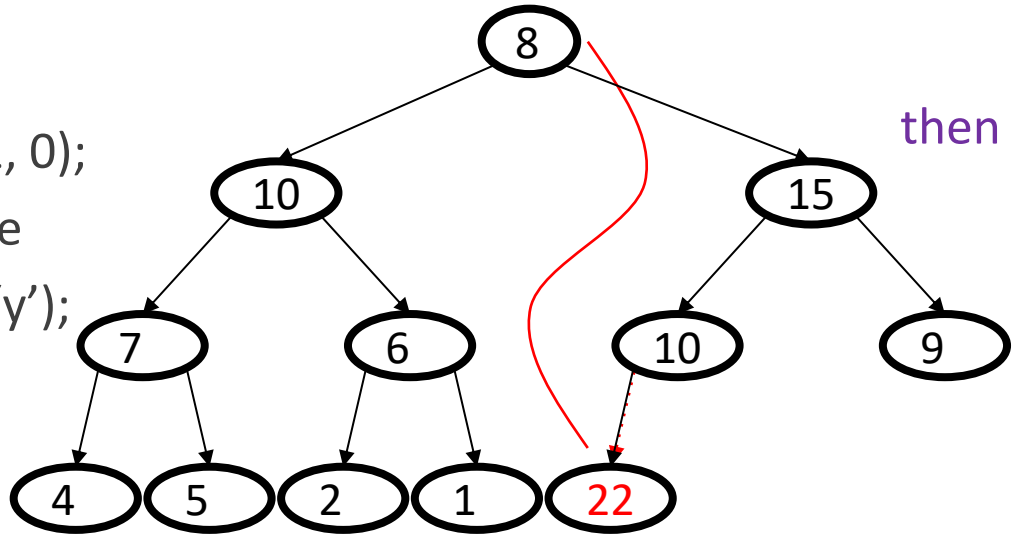
DeleteMax (Final)

`pqueue.deleteMax()`



Delete (Max) element Algorithm(index starting from 0)

```
Algorithm  Deleteheap(a,n) //n are the no. of elements
 in array
{
      // Interchange the maximum with the element at
 the end of array
 repeat{
       t=a[0];
       a[0]=a[n-1] ;
       a[n-1]=t;
       n --;
      Adjust(a, n-1, 0);
      accept choice
    }until(choice='y');
}
```




```
Algorithm Adjust(a,n,i)
{
        while(2*i+1<=n) do
        {
                j=2*i+1;       //index of left child
// compare left and right child and let j be the
 larger child
          if((j+1 <= n) and (a[j+1] > a[j]))
                    j=j+1
           If a[i] >= a[j])
              then break;  //if parent > children
 then break
          else
          {
              //swap a[i] with a[j]
             temp=a[i]; a[i]=a[j];  a[j]=temp;
              i=j;
          }
      } //end of while
}
```

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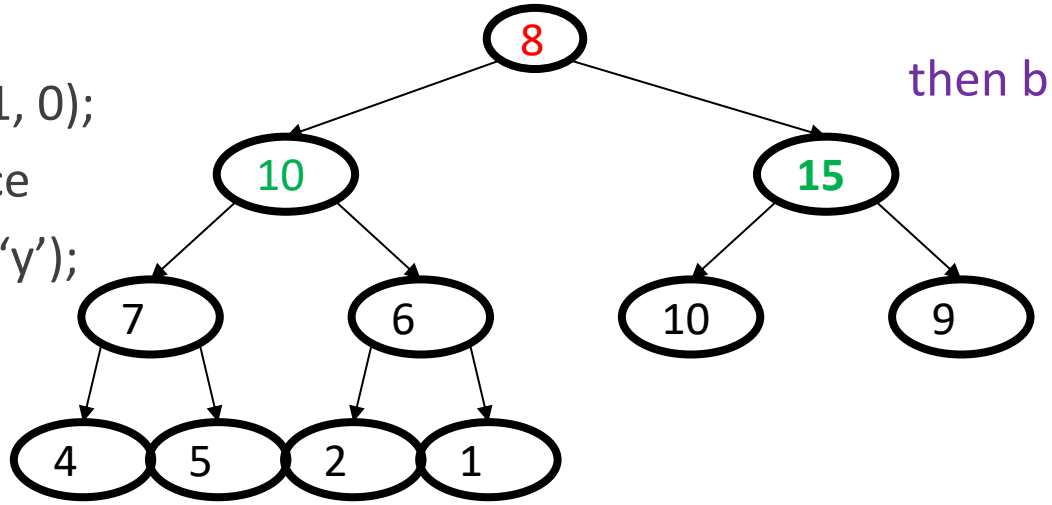


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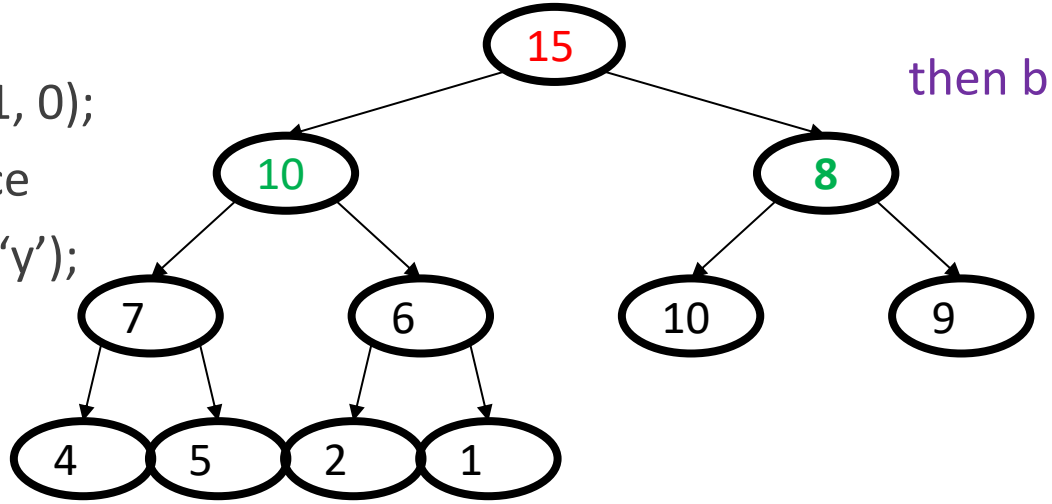


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Delete (Max) element Algorithm(index starting from 0)

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 larger child
       if((j+1 <= n) and (a[j+1] > a[j]))
                  j=j+1
          If a[i] >= a[j])
             then break;  //if parent > children
 then break
            else
            {
              //swap a[i] with a[j]
             temp=a[i]; a[i]=a[j];  a[j]=temp;
              i=j;
            }
      } //end of while
}
```
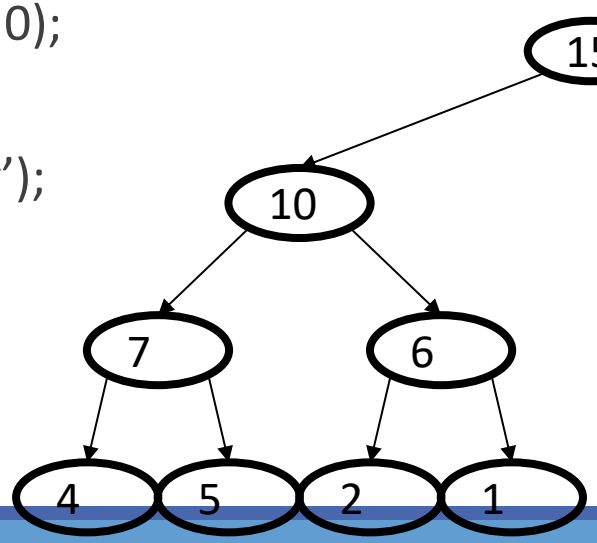
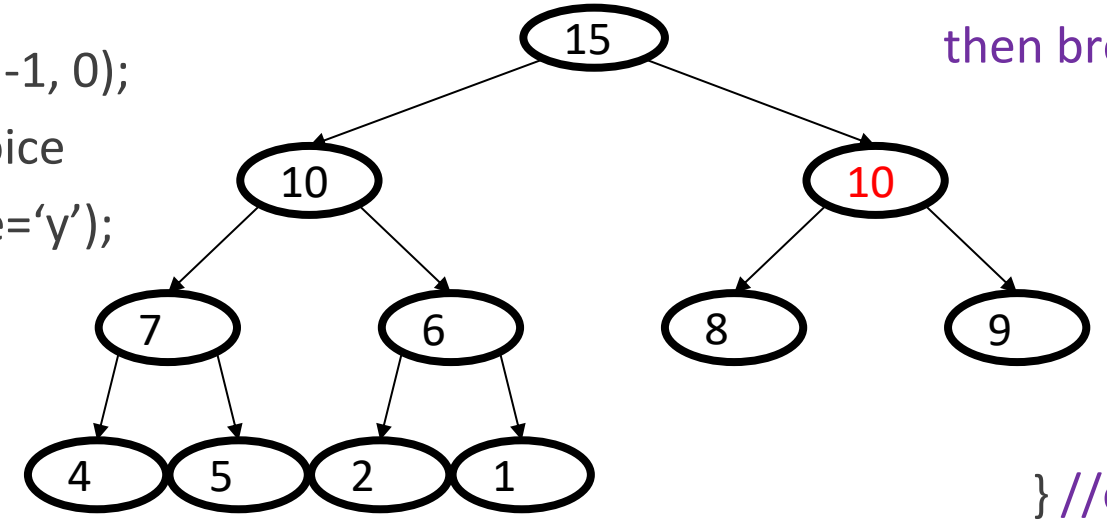



Delete (Max) element Algorithm(index starting from 0)

```
Algorithm  Deleteheap(a,n) //n are the no. of elements
 in array
{
      // Interchange the maximum with the element at
 the end of array
 repeat{
        t=a[0];
       a[0]=a[n-1] ;
       a[n-1]=t;
       n --;
      Adjust(a, n-1, 0);
      accept choice
    }until(choice='y');
}
```




```
Algorithm Adjust(a,n,i)
{
        while(2*i+1<=n) do
        {
                j=2*i+1;       //index of left child
// compare left and right child and let j be the
 larger child
        if((j+1 <= n) and (a[j+1] > a[j]))
                j=j+1
        If a[i] >= a[j])
           then break;  //if parent > children
 then break
        else
        {
            //swap a[i] with a[j]
           temp=a[i]; a[i]=a[j];  a[j]=temp;
            i=j;
        }
    } //end of while
}
```

Sorting

- Other than as a priority queue, the heap has one other important usage: heap sort
- Heap sort is one of the fastest sorting algorithms, achieving speed as that of the quicksort and merge sort algorithms
- The advantages of heap sort are that it does not use recursion and it is efficient for any data order
- There is no worse-case scenario in the case of heap sort

Heap Sort

- Steps for heap sort (ascending order) are as follows:
  1. Build the heap tree(for given array as it may not be in heap tree form)
  2. Start Delete Heap operations, storing each deleted element at the end of the heap array
  3. After performing step 2, again adjust the heap tree (ReHeapDown)
  4. Repeat step 2 and 3 for n-1 times to sort the complete array

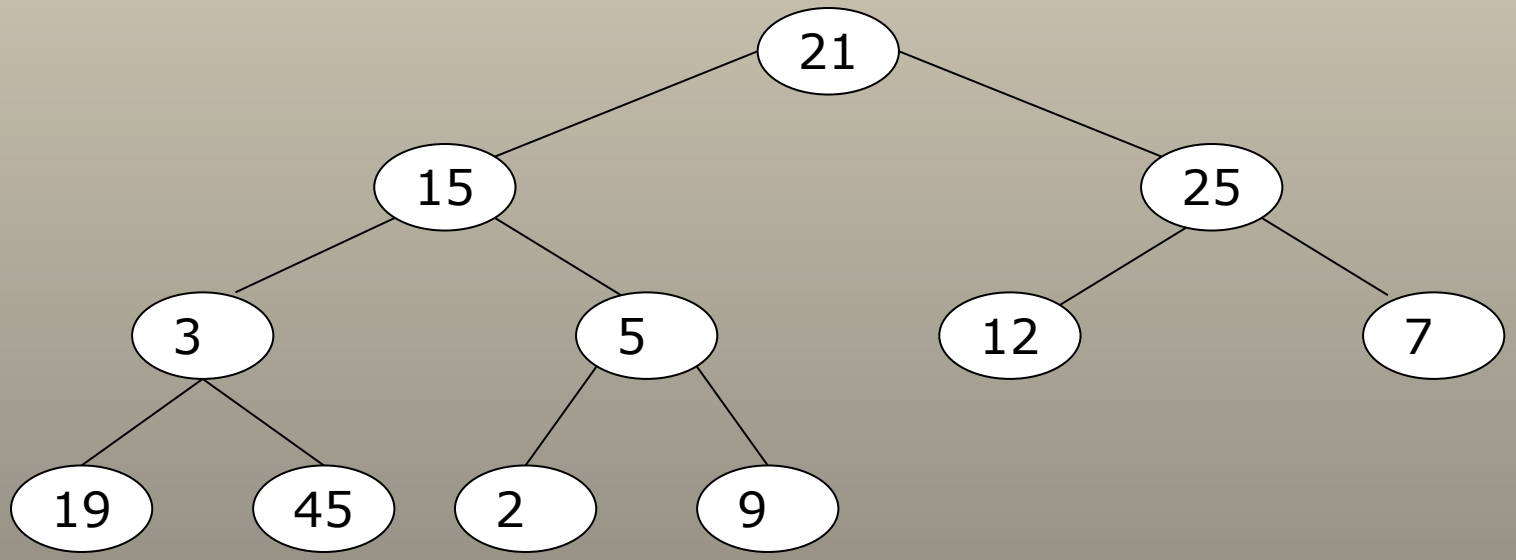
Sample Run

- Start with unordered array of data

Array representation:

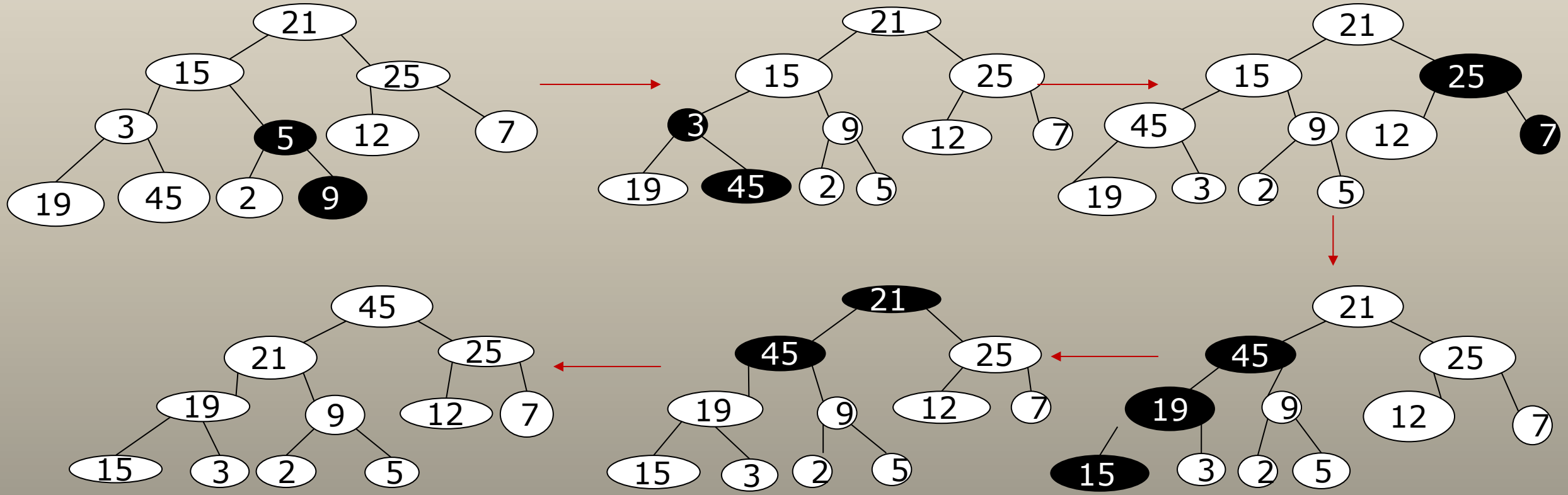
21	15	25	3	5	12	7	19	45	2	9

Binary tree representation:

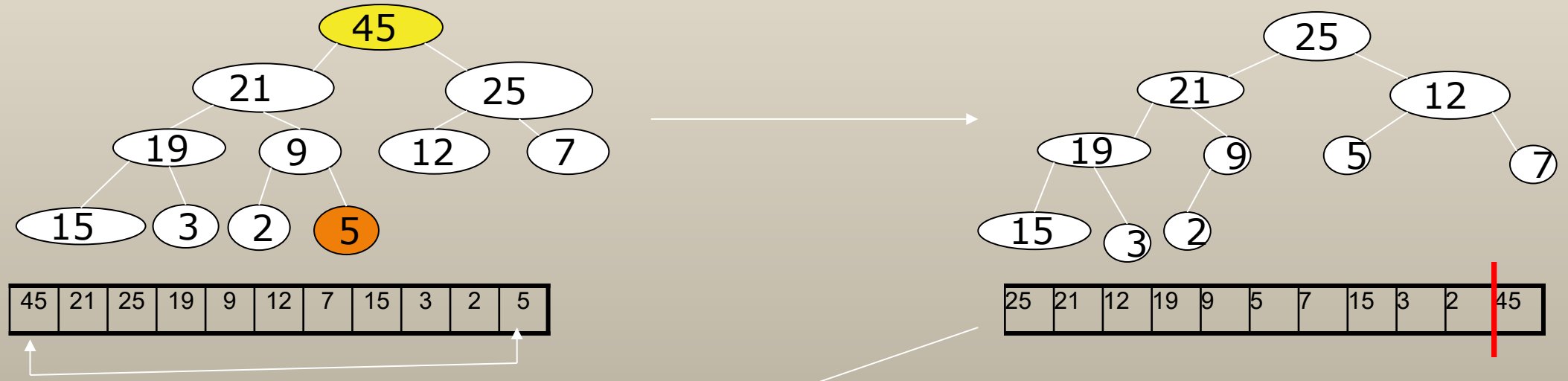


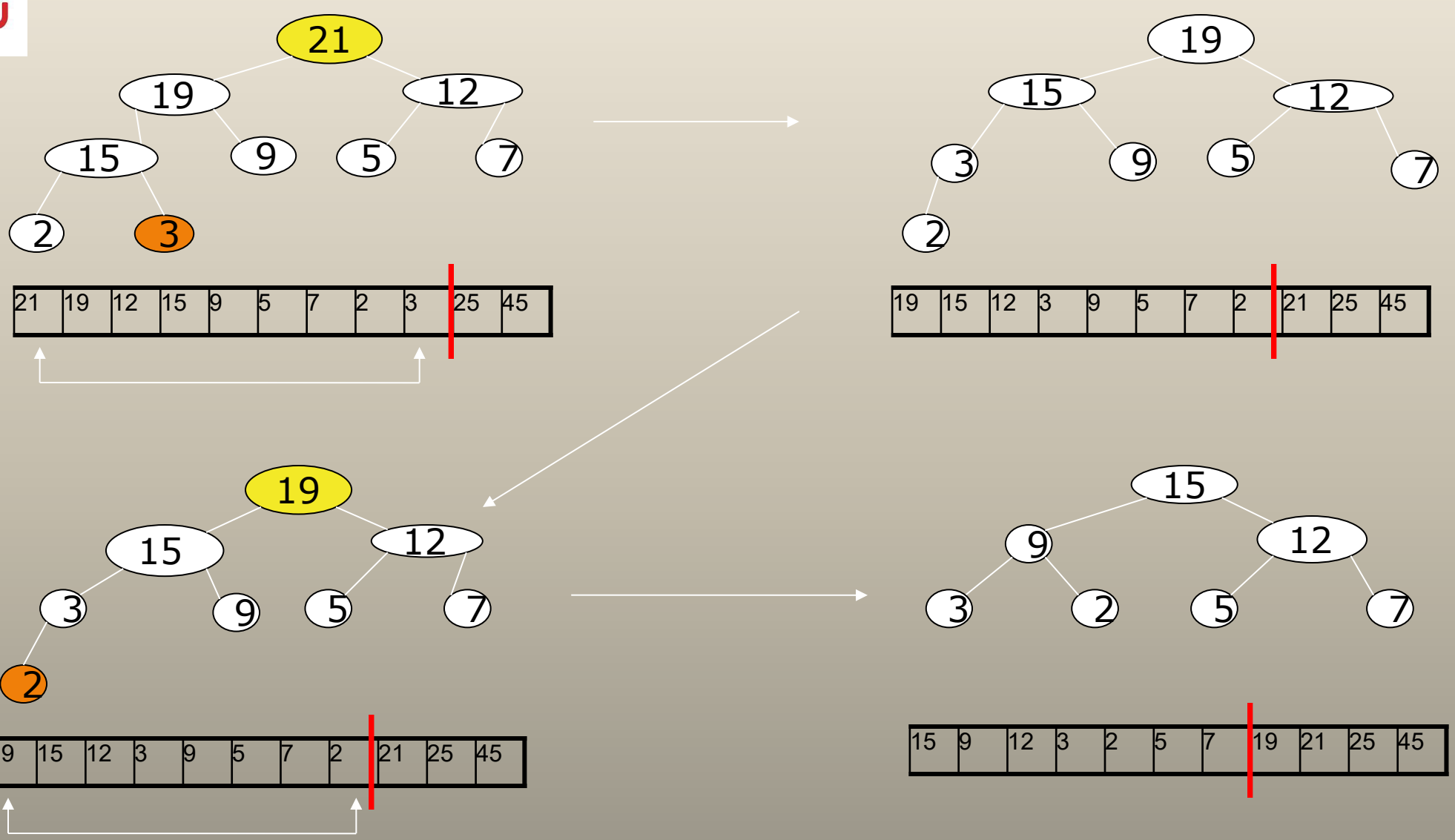
Sample Run

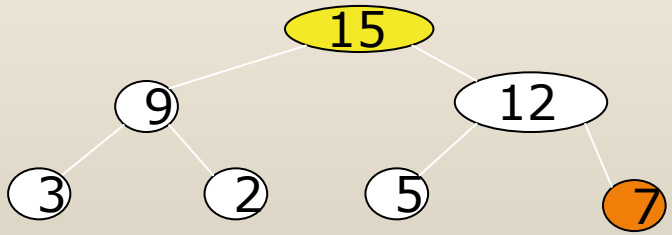
- Heapify the binary tree –(from (n/2)-1 to 0)

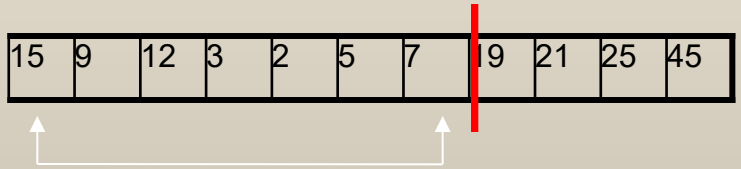


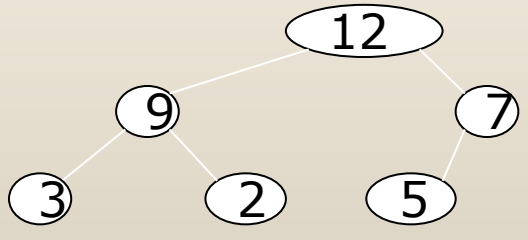
Step 2 – perform n – 1 deleteMax(es) and replace last element in heap with first, then re-heapify. Place deleted element in the last node's position.

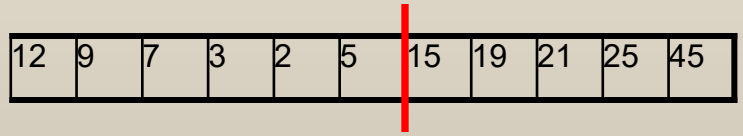


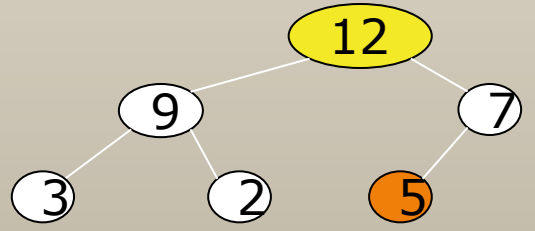


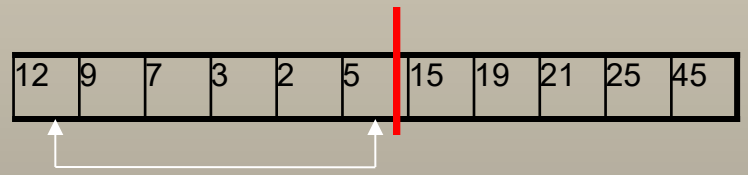


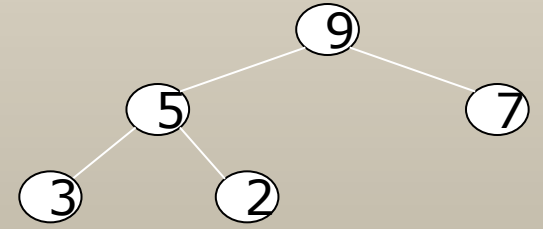


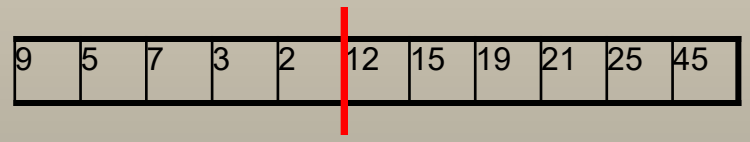






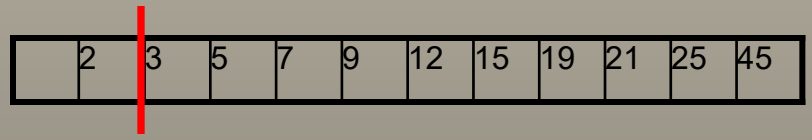






This continues till only one is left

...and finally

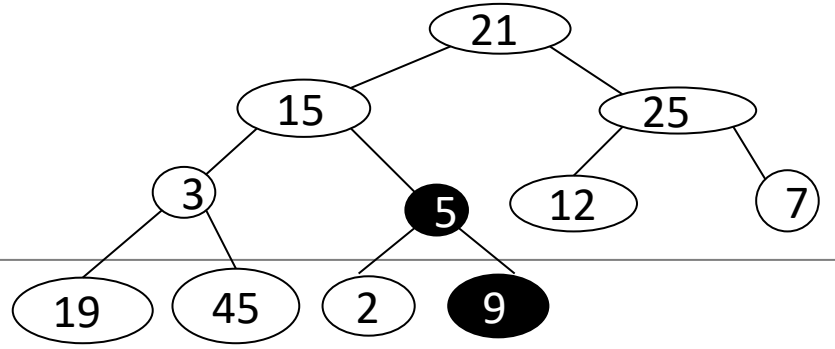


Heap sort Algorithm (index starting from 0)

```
Algorithm HeapSort(a,n)
//here n is the total no. of elements in the array
{
    for i = (n/2)-1  to 0  step -1 do
        Adjust(a,n-1,i)
   // Interchange the new maximum with the element at the end of array
    while(n>0)
    {
       t=a[0];
       a[0]=a[n-1] ;
       a[n-1]=t;
       n --;
      Adjust(a, n-1, 0);
    }
}
```

```
Algorithm Adjust(a,n,i)
{
    while(2*i+1<=n) do
    {
        j=2*i+1;        //index of left child
        if((j+1 <= n) and (a[j+1] > a[j]))
                j=j+1;    // compare left and right child and let j be the larger child
        If a[i] >= a[j])
           then break;  //if root >children then break
        else
        {
            //swap a[i] with a[j]
            temp=a[i]; a[i]=a[j];  a[j]=temp;
             i=j;
        }
    } //end of while
}
```

```
Algorithm Adjust(a,n,i)
{
        while(2*i+1<=n) do
        {
                j=2*i+1;        //index of left child
                if((j+1 <= n) and (a[j+1] > a[j]))
                        j=j+1;    // compare left and right child and let j be the larger child
                If a[i] >= a[j])
                   then break;  //if root >children then break
                else
                {
                     //swap a[i] with a[j]
                     temp=a[i]; a[i]=a[j];  a[j]=temp;
                      i=j;
                }
        } //end of while
}
```



Heap Applications

- Selection algorithm
- Scheduling and prioritizing (priority queue)
- Sorting

Selection Problem

- For the solution to the problem of determining the kth element, we can create the heap and delete k − 1 elements from it, leaving the desired element at the root.
- So the selection of the k^{th} element will be very easy as it is the root of the heap
- For this, we can easily implement the algorithm of the selection problem using heap creation and heap deletion operations
- This problem can also be solved in O(nlogn) time using priority queues

Scheduling and prioritizing (priority queue)

- The heap is usually defined so that only the largest element (that is, the root) is removed at a time.
- This makes the heap useful for scheduling and prioritizing
- In fact, one of the two main uses of the heap is as a prioriy queue, which helps systems decide what to do next

Applications of priority queues where heaps are implemented

- CPU scheduling
- I/O scheduling
- Process scheduling.

Symbol Table

Symbol Table

- Introduction to Symbol Tables
- Static tree table- Optimal Binary Search Tree (OBST)
- Dynamic tree table-AVL tree
- Multiway search tree- B-Tree

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Symbol Table

- Symbol table is defined as a name-value pair

- Symbol tables are data structures that are used by compilers to hold information about source-program constructs.

Symbol Tables

Associates **attributes with identifiers used in a program**

- For instance, a **type attribute is usually associated with each identifier**
- A symbol table is a necessary component
  - Definition (declaration) of identifiers appears once in a program
  - Use of identifiers may appear in many places of the program text
- Identifiers and attributes are entered by the analysis phases
  - When processing a definition (declaration) of an identifier
  - In simple languages with only global variables and implicit declarations: **The scanner can enter an identifier into a symbol table if it is not already there**
  - In block-structured languages with scopes and explicit declarations: **The parser and/or semantic analyzer enter identifiers and corresponding attributes**

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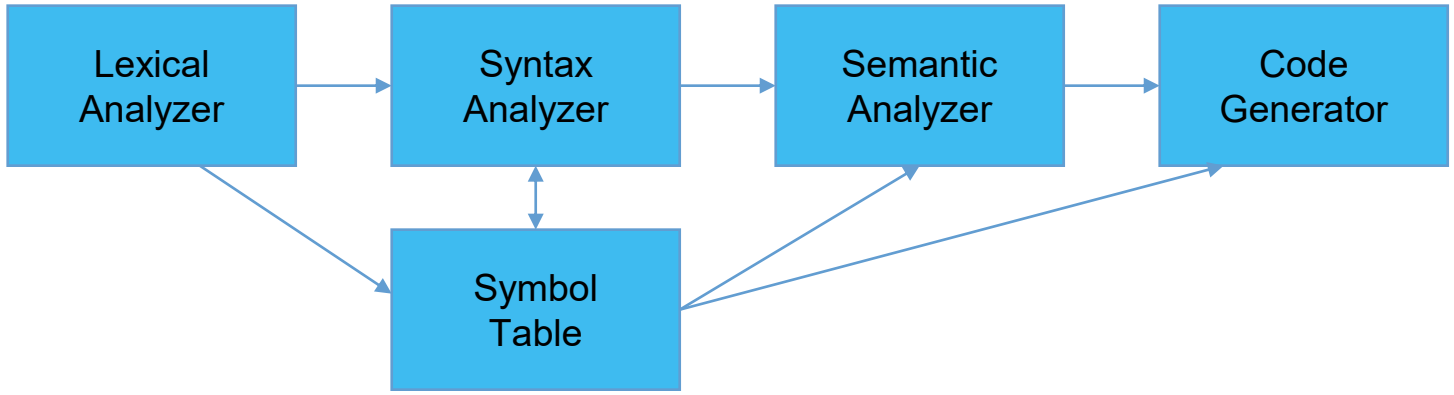
Symbol Tables

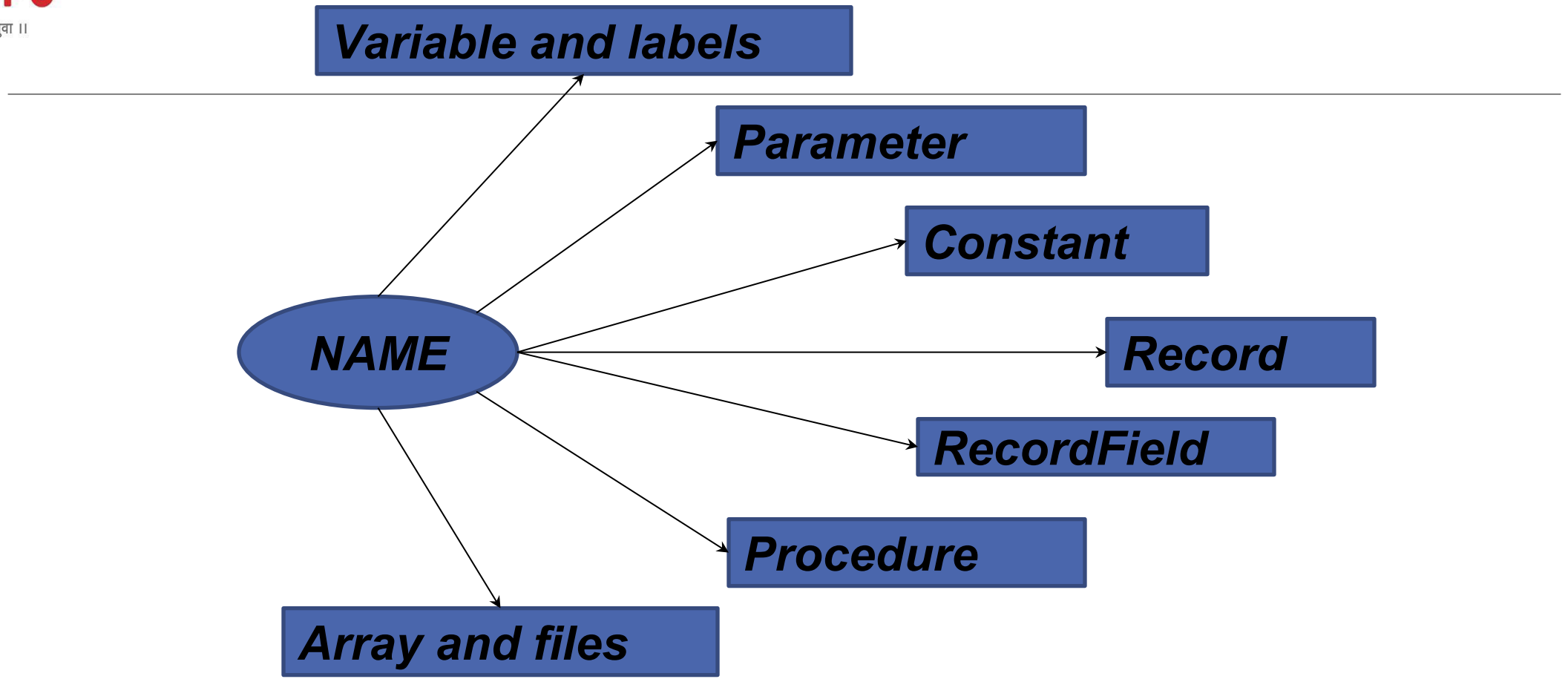
- Symbol table information is used by the analysis and synthesis phases
  - To verify that used identifiers have been defined (declared)
  - To verify that expressions and assignments are semantically correct – **type checking**
  - To generate intermediate or target code

The Symbol Table

- When identifiers are found, they will be entered into a symbol table, which will hold all relevant information about identifiers.
- This information will be used later by the semantic analyzer and the code generator.



SYMBOL TABLE - NAMES



SYMBOL TABLE-ATTRIBUTES

- Each piece of information associated with a name is called an ***attribute.***
- Attributes are language dependent.
- Different classes of Symbols have different Attributes

Variable, Constants	Procedure or function	Array
• Type, Line number where declared, Lines where referenced, Scope	• Number of parameters, parameters themselves, result type.	• # of Dimensions, Array bounds.

Symbol Table

- While compilers and assemblers are scanning a program, each identifier must be examined to determine if it is a keyword
- This information concerning the keywords in a programming language is stored in a symbol table
- Symbol table is a kind of ‘keyed table’
- The keyed table stores <key, information> pairs with no additional logical structure

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Symbol Table (Continued)

- The operations performed on symbol tables are the following:
  - Insert the pairs <key, information> into the collection
  - Remove the pairs <key, information> by specifying the key
  - Search for a particular key
  - Retrieve the information associated with a key

Possible implementations

- Unordered list: for a very small set of variables.
  - Simplest to implement
  - Implemented as an array or a linked list
  - Linked list can grow dynamically – alleviates problem of a fixed size array
  - Insertion is fast O(1), but lookup is slow for large tables – O(n) on average.
- Ordered linear list:
  - If an array is sorted, it can be searched using binary search – $O(log_2 n)$
  - Insertion into a sorted array is expensive – *O(n) on average*
  - Useful when set of names is known in advance – table of reserved words

Possible implementations

- Binary search tree:
  - Can grow dynamically
  - Insertion and lookup are *O(log$_2$n) on average*

Representation of Symbol Table

There are two different techniques for implementing a keyed table:

- Static Tree Tables
- Dynamic Tree Tables

Static Tree Tables

- When symbols are known in advance and no insertion and deletion is allowed, it is called a static tree table

- An example of this type of table is a reserved word table in a compiler

Dynamic Tree Table

- A dynamic tree table is used when symbols are not known in advance but are inserted as they come and deleted if not required
- Dynamic keyed tables are those that are built on-the-fly
- The keys have no history associated with their use

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Optimal Binary Search Trees

- To optimize a table knowing what keys are in the table and what the probable distribution is of those that are not in the table, we build an optimal binary search tree (OBST)
- Optimal binary search tree is a binary search tree having an average search time of all keys optimal
- An OBST is a BST with the minimum cost

Optimal binary search trees

A full binary tree may not be an optimal binary search tree if the identifiers are searched for with different frequency

Consider these two search trees,

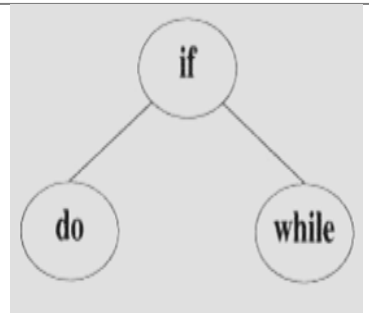
If we search for each identifier with equal Probability

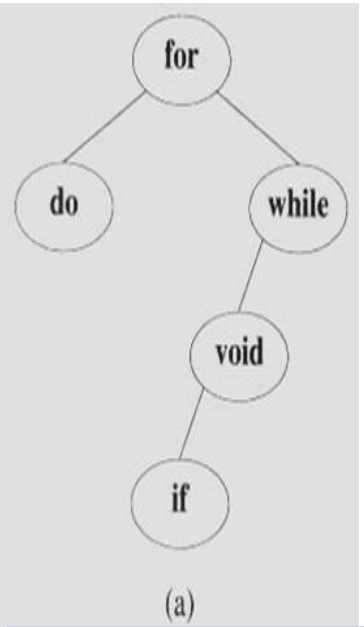
In first tree (a)

- the average number of comparisons for successful search is 2.4.
- Comparisons for second tree is 2.2.

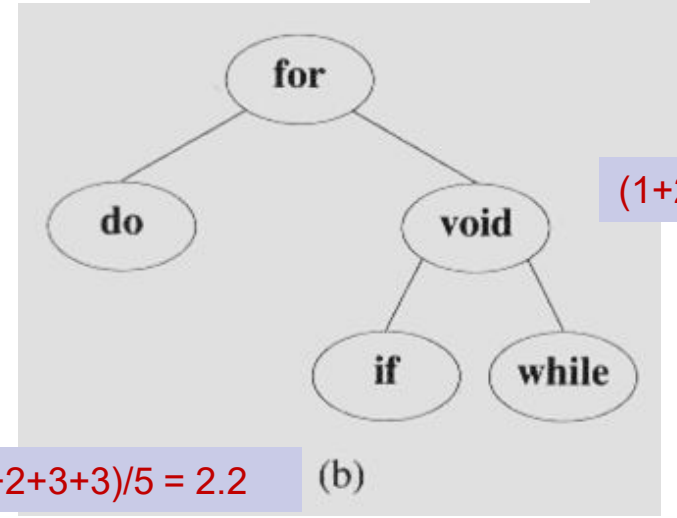
The second tree (b) has

- a better worst case search time than the first tree.
- a better average behavior.





(1+2+2+3+4)/5 = 2.4

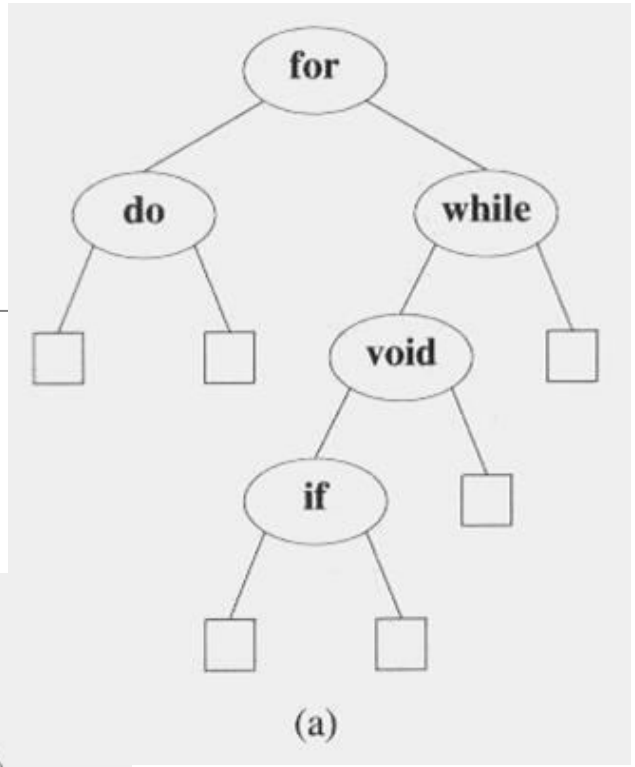


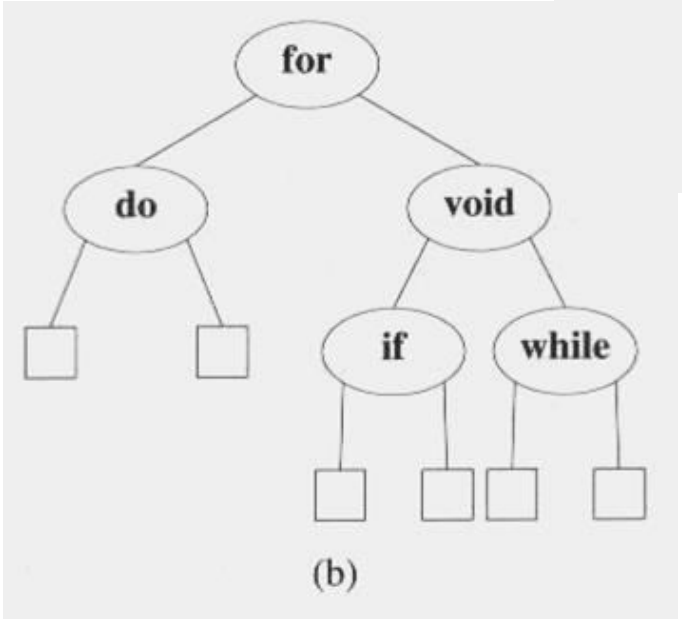
(1+2+2+3+3)/5 = 2.2

Optimal binary search trees

In evaluating binary search trees, it is useful to add a special square node at every place there is a null links.

- We call these nodes *external nodes*.
- We also refer to the external nodes as *failure* nodes.
- The remaining nodes are *internal nodes*.
- A binary tree with external nodes added is an *extended binary tree*





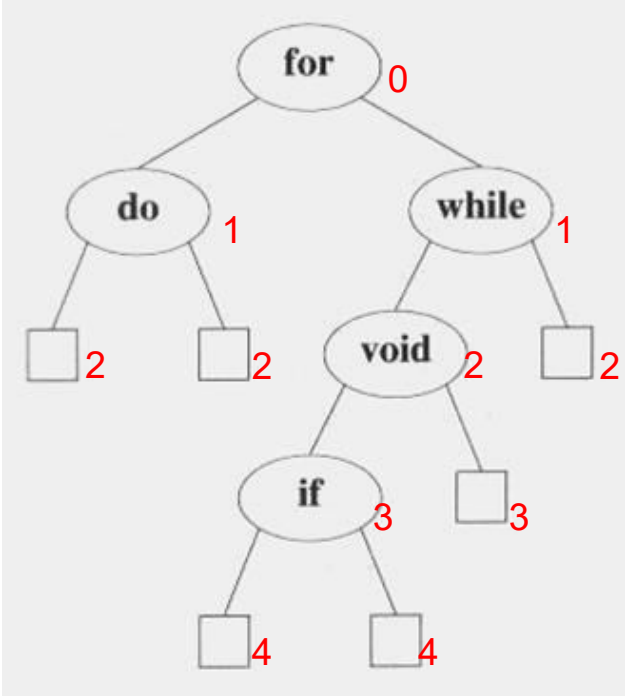
Optimal binary search trees

- *External / internal path length*

  - The sum of all external / internal nodes' levels.

- For example
  - Internal path length, *I*, is:
    $$I = 0 + 1 + 1 + 2 + 3 = 7$$
  - External path length, *E*, is :
    $$E = 2 + 2 + 4 + 4 + 3 + 2 = 17$$

- A binary tree with *n* internal nodes are related
- by the formula : $E = I + 2n$



Optimal binary search trees

In the binary search tree:

- The identifiers $a_1, a_2, \ldots, a_n$ with $a_1 < a_2 < \ldots < a_n$
- The probability of searching for each a_i is p_i
- The total cost (when only successful searches are made) is:

$$\sum_{i=1}^{n} p_i \cdot \text{level}(a_i)$$

Optimal binary search trees

- If we replace the null subtree by a failure node, we may partition the identifiers that are not in the binary search tree into n+1 classes E_i, $0 \leq i \leq n$
  - E_i contains all identifiers x such that $a_i < x < a_{i+1}$
  - For all identifiers in a particular class, E_i, the search terminates at the same failure node

Optimal binary search trees

We number the failure nodes from 0 to *n* with *i* being for class E_i, $0 \le i \le n$.

- If q_i is the probability that the identifier we are searching for is in E_i, then the cost of the failure node is:

$$\sum_{i=0}^{n} q_i \cdot (\text{level(failure node } i) - 1)$$

Therefore, the total cost of a binary search tree is:

$$\sum_{i=1}^{n} p_i \cdot \text{level}(a_i) + \sum_{i=0}^{n} q_i \cdot (\text{level (failure node } i) - 1)$$

- An optimal binary search tree for the identifier set $a_1, \ldots, a_n$ is one that minimizes
- Since all searches must terminate either successfully or unsuccessfully, we have

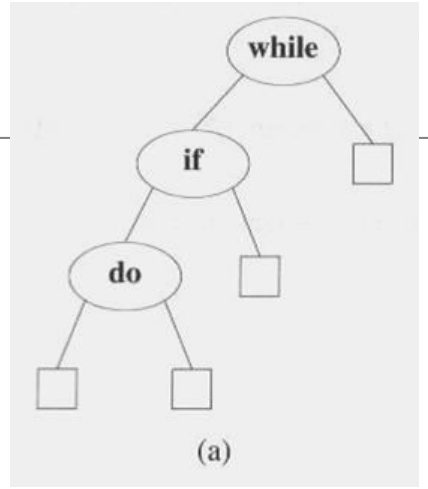
$$\sum_{i=1}^{n} p_i + \sum_{i=0}^{n} q_i = 1$$

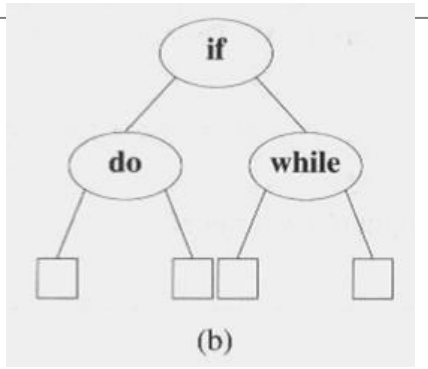
Optimal binary search trees

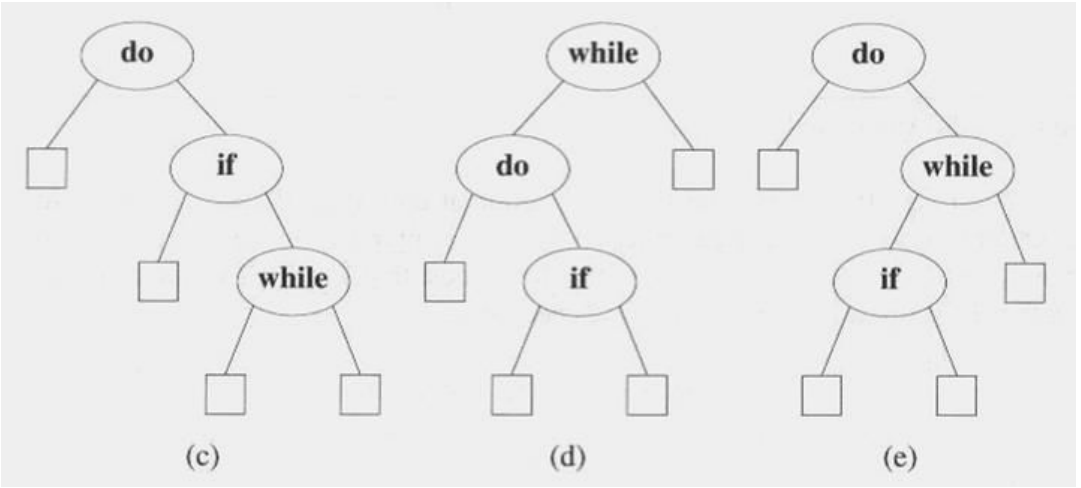
The possible binary search trees for the identifier set (a_1, a_2, a_3) = (**do**, **if**, **while**)

- The identifiers with equal probabilities, $p_i=q_j=1/7$ for all i, j,
  - cost(tree a) = 15/7;
  - cost(tree b) = 13/7; (optimal)
  - cost(tree c) = 15/7;
  - cost(tree d) = 15/7;
  - cost(tree e) = 15/7;

- $p_1 = 0.5$, $p_2 = 0.1$, $p_3 = 0.05$, $q_0 = 0.15$, $q_1 = 0.1$, $q_2 = 0.05$, $q_3 = 0.05$
  - cost(tree a) = 2.65;
  - cost(tree b) = 1.9;
  - cost(tree c) = 1.5; (optimal)
  - cost(tree d) = 2.05;
  - cost(tree e) = 1.6;







OPTIMAL BINARY SEARCH TREES (Continued)

With equal probability P(i)=Q(i)=1/7.

Let us find an OBST out of these.

Cost(tree a)=$\sum_{1\le i\le n} P(i)*\text{level a}(i) + \sum_{0\le i\le n} Q(i)*\text{level (Ei)} - 1$

(2-1) (3-1) (4-1) (4-1)

=1/7[1+2+3 + 1 + 2 + 3 + 3] = 15/7

Cost (tree b) =17[1+2+2+2+2+2+2] =13/7

Cost (tree c) =cost (tree d) =cost (tree e) =15/7

∴ tree b is optimal.

OPTIMAL BINARY SEARCH TREES (Continued)

If P(1) =0.5 ,P(2) =0.1, P(3) =0.005 , Q(0) =.15 , Q(1) =.1, Q(2) =.05 and Q(3) =.05 find the OBST.

Cost (tree a) = .5 x 3 +.1 x 2 +.05 x 3 +.15x3 +.1x3 +.05x2 +.05x1 = 2.65

Cost (tree b) =1.9 , Cost (tree c) =1.5 ,Cost (tree d) =2.05 ,

Cost (tree e) =1.6 Hence tree C is optimal.

Binary Search Tree - Worst Time

Worst case running time is O(N)

- What happens when you Insert elements in ascending order?
  - Insert: 2, 4, 6, 8, 10, 12 into an empty BST
- Problem: Lack of “balance”:
  - compare depths of left and right subtree
- Unbalanced degenerate tree

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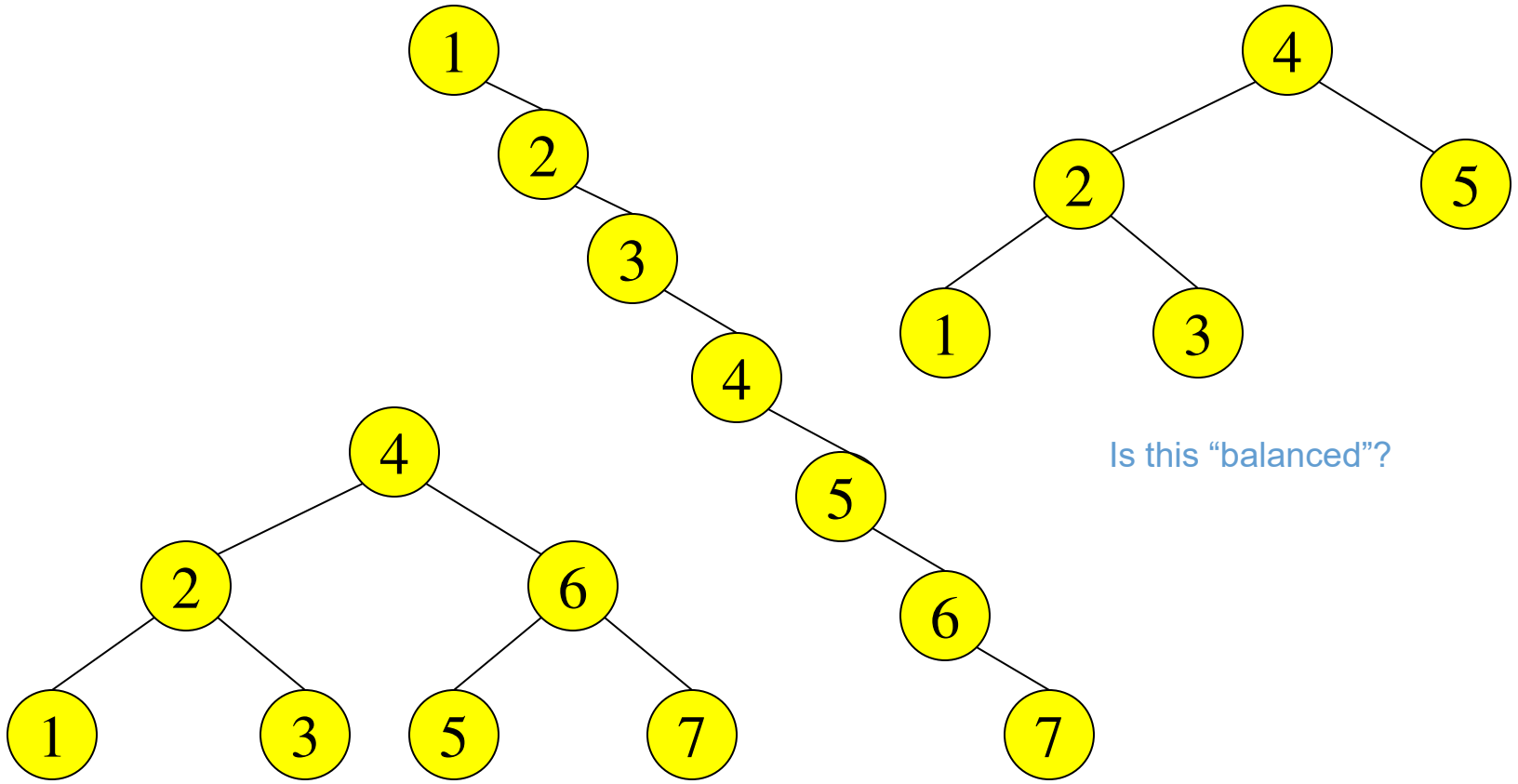
Balanced and unbalanced BST



Is this “balanced”?

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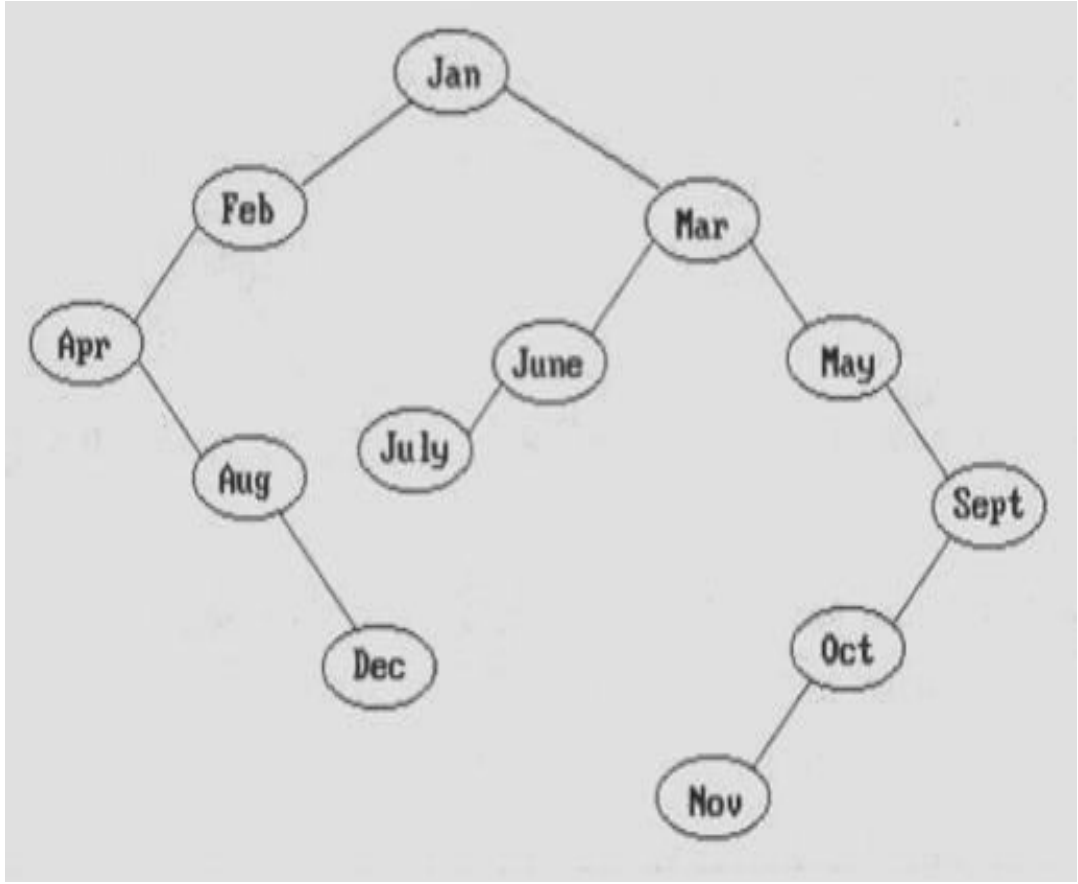
Balancing Binary Search Trees

- Many algorithms exist for keeping binary search trees balanced
  - Adelson-Velskii and Landis (AVL) trees (height-balanced trees)
  - Splay trees and other self-adjusting trees
  - B-trees and other Multiway search trees

AVL Trees

We also may maintain dynamic tables as binary search trees.

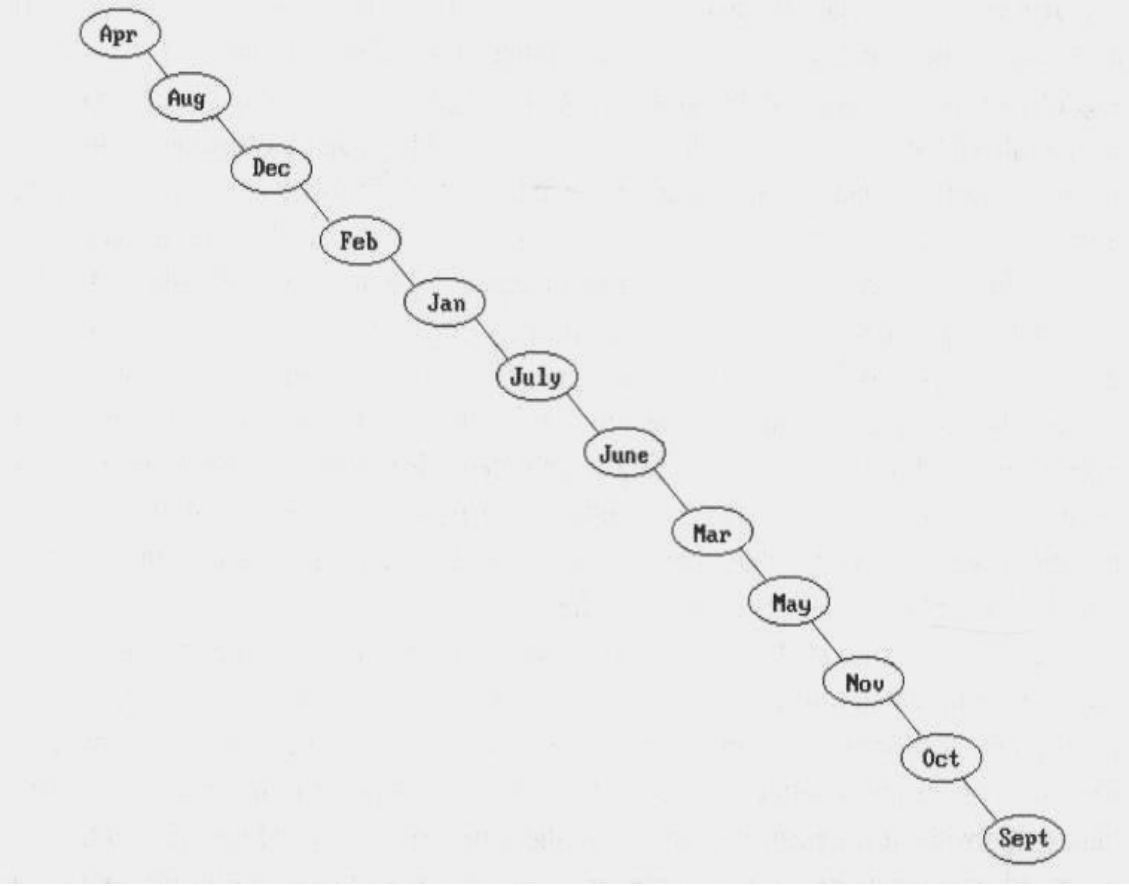
- The binary search tree obtained by entering the months *January* to *December*, in that order, into an initially empty binary search tree
- The maximum number of comparisons needed to search for any identifier in the tree (for *November*).
- Average number of comparisons is 42/12 = 3.5



AVL Trees

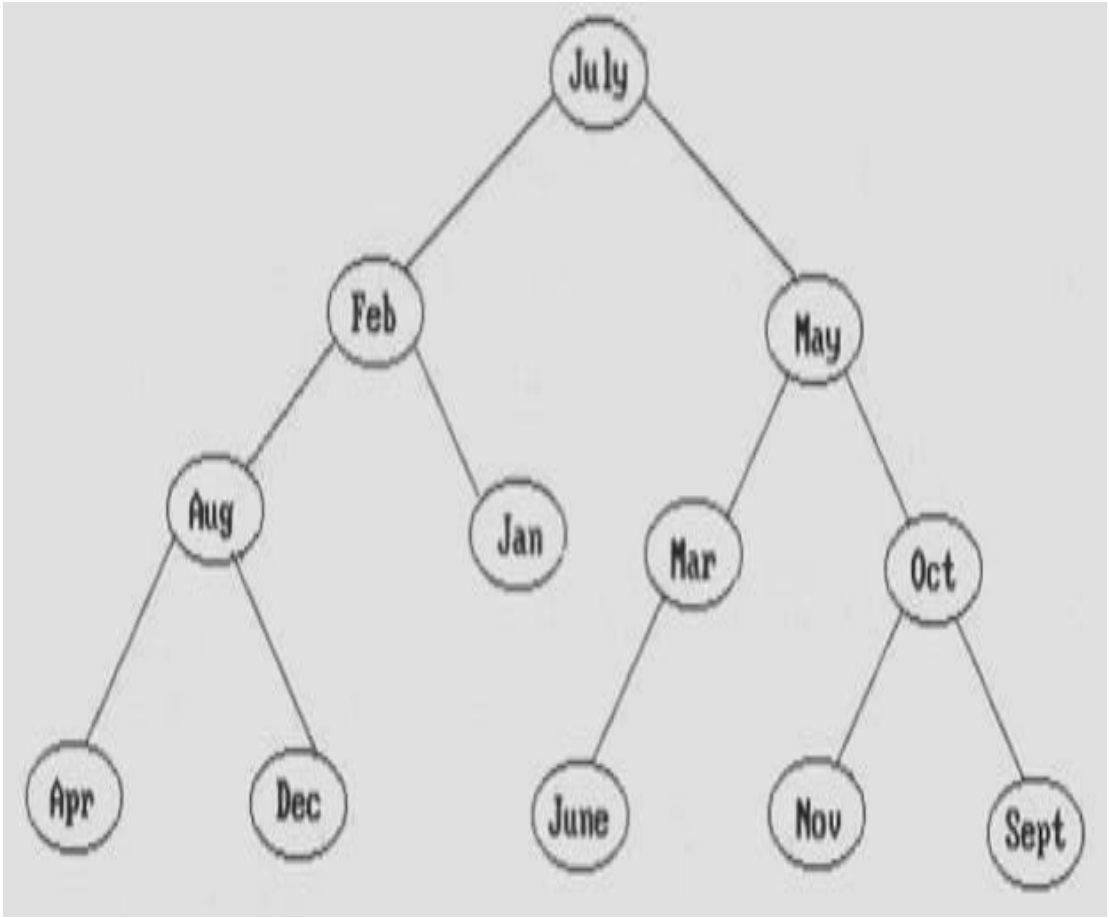
Suppose that we now enter the months into an initially empty tree in alphabetical order

- The tree degenerates into the chain number of comparisons: maximum: 12 and average: 6.5
- In the worst case, binary search trees correspond to sequential searching in an ordered list



Another insert sequence

- In the order *Jul*, *Feb*, *May*, *Aug*, *Jan*, *Mar*, *Oct*, *Apr*, *Dec*, *Jun*, *Nov*, and *Sep*
- Well balanced and does not have any paths to leaf nodes that are much longer than others.
- Number of comparisons: maximum: 4, and average: 37/12 ≈ 3.1.
- All intermediate trees created during the construction of Figure are also well balanced



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AVL Trees

- Adelson-Velskii and Landis introduced a binary tree structure (*AVL trees*):
  - Balanced with respect to the heights of the subtrees.
  - We can perform dynamic retrievals in O(log*n*) time for a tree with n nodes.
  - We can enter an element into the tree, or delete an element from it, in O(log*n*) time. The resulting tree remain height balanced.
  - As with binary trees, we may define AVL tree recursively

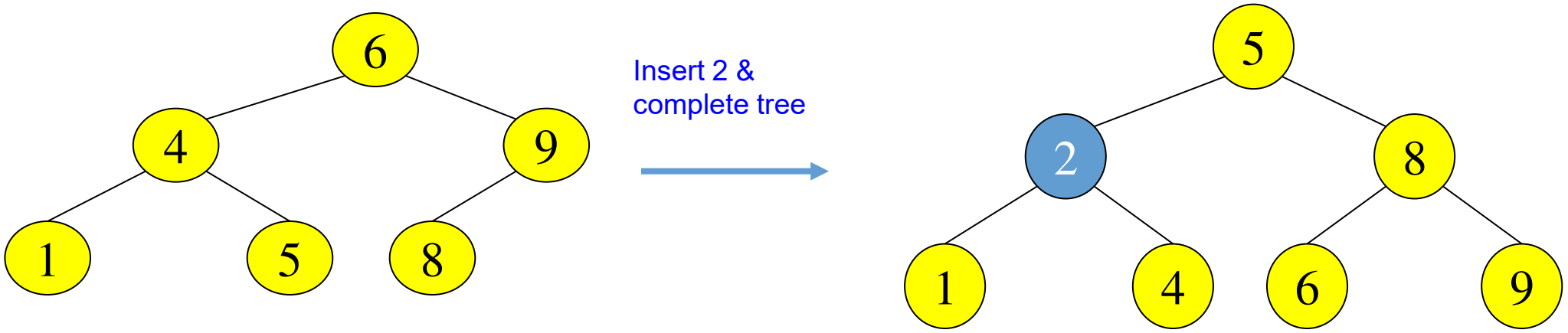
Perfect Balance

Want a complete tree after every operation

- tree is full except possibly in the lower right

This is expensive

- For example, insert 2 in the tree on the left and then rebuild as a complete tree



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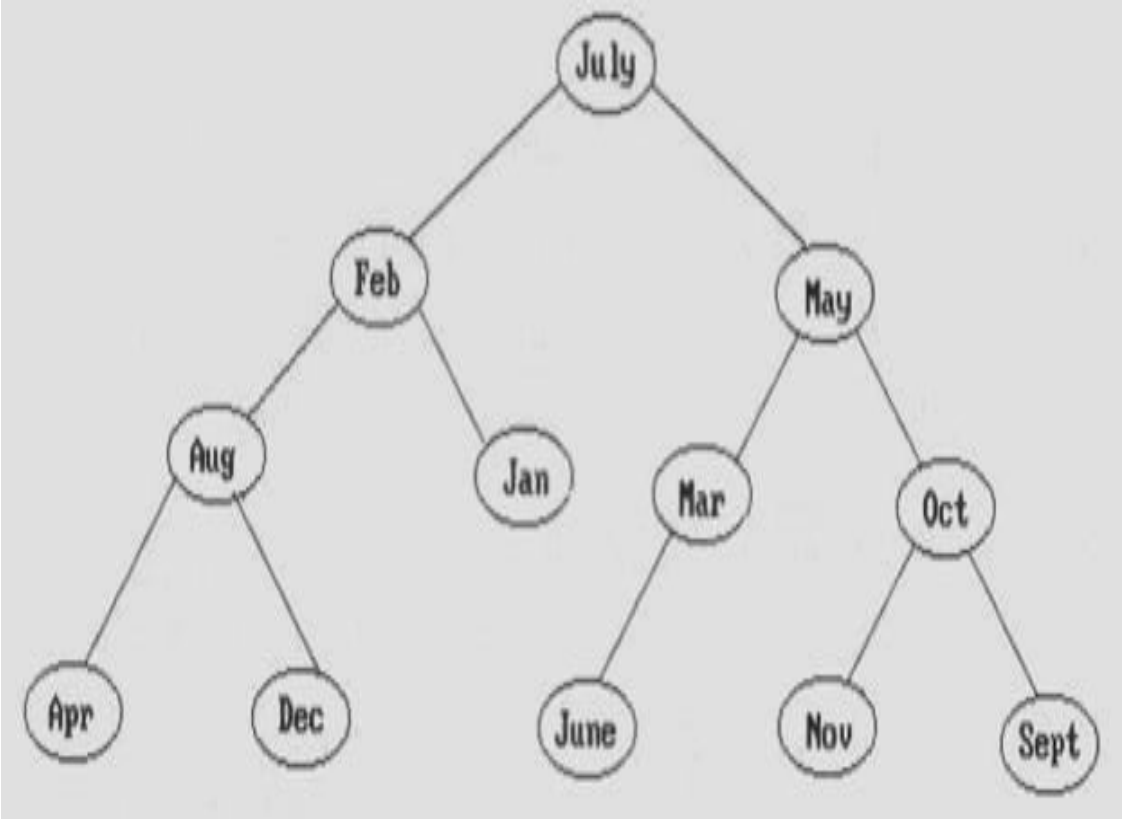
AVL - Good but not Perfect Balance

- AVL trees are height-balanced binary search trees
- Balance factor of a node
  - height(left subtree) - height(right subtree)
- An AVL tree has balance factor calculated at every node
  - For every node, heights of left and right subtree can differ by no more than 1
  - Store current heights in each node

AVL Trees Definition

- An empty binary tree is height balanced.
- If T is a nonempty binary tree with T_L and T_R as its left and right subtrees, then T is *height balanced iff*
  - T_L and T_R are height balanced, and
  - $|h_L - h_R| \leq 1$ where h_L and h_R are the heights of T_L and T_R, respectively.
- The definition of a height balanced binary tree requires that every subtree also be height balanced



AVL Trees

- The numbers by each node represent the difference in heights between the left and right subtrees of that node
- We refer to this as the balance factor of the node
- **Definition:**
  - The balance factor, *BF(T)*, of a node, *T*, in a binary tree is defined as h_L - h_R, where h_L/h_R are the heights of the left/right subtrees of *T*.
  - For any node *T* in an AVL tree *BF(T)* = -1, 0, or 1.

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AVL Trees

- We carried out the rebalancing using four different kinds of rotations: *LL*, *RR*, *LR*, and *RL*
  - *LL* and *RR* are symmetric as are *LR* and *RL*
  - These rotations are characterized by the nearest ancestor, *A*, of the inserted node, *Y*, whose balance factor becomes ±2.
    - *LL*: *Y* is inserted in the left subtree of the left subtree of *A*.
    - *LR*: *Y* is inserted in the right subtree of the left subtree of *A*
    - *RR*: *Y* is inserted in the right subtree of the right subtree of *A*
    - *RL*: *Y* is inserted in the left subtree of the right subtree of *A*

Insertions in AVL Trees

Let the node that needs rebalancing be α.

There are 4 cases:

Outside Cases (require single rotation) :

1. Insertion into left subtree of left child of α.
2. Insertion into right subtree of right child of α.

Inside Cases (require double rotation) :

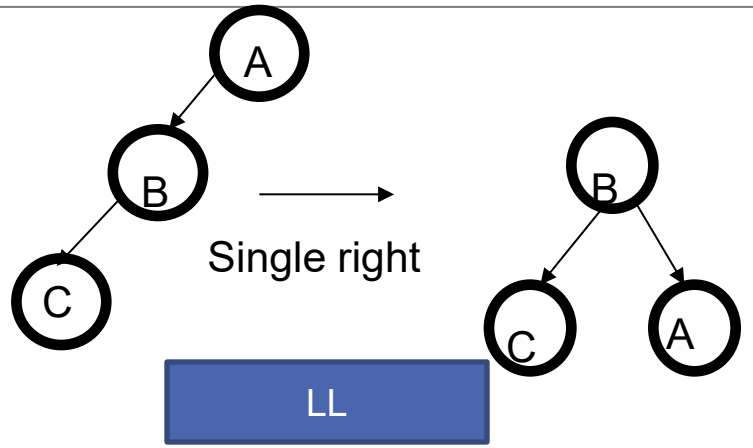
3. Insertion into right subtree of left child of α.
4. Insertion into left subtree of right child of α.

The rebalancing is performed through four separate rotation algorithms.

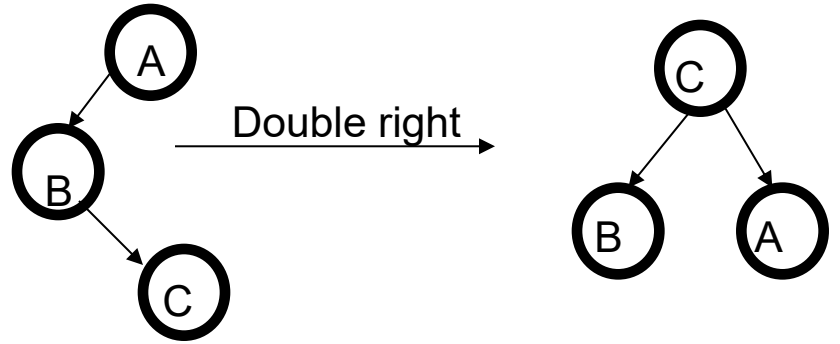
AVL Rotations

- To perform rotations identify specific node **A** whose **BF(A) is neither** (0, 1, -1).
- **LL** : Inserted node is in the **left subtree** of **left subtree** of node A. ***(SINGLE RIGHT ROTATION)***
- **RR** : Inserted node is in the **right subtree** of **right subtree** of node A. ***(SINGLE LEFT ROTATION)***
- **LR** : Inserted node is in the **right subtree** of **left subtree** of node A. ***(RR + LL)***
- **RL** : Inserted node is in the **left subtree** of **right subtree** of node A. ***(LL + RR)***

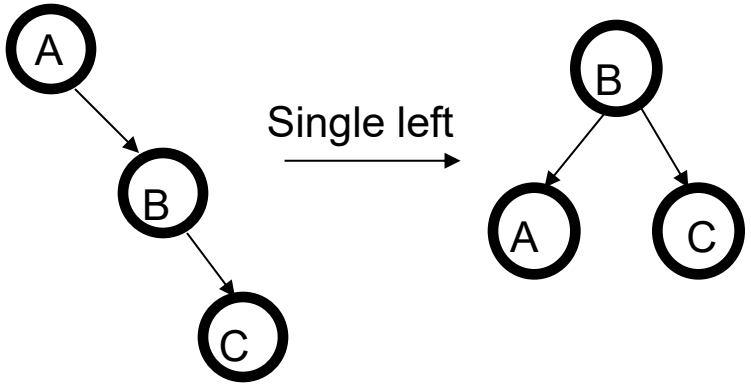
AVL Balancing : Four Rotations



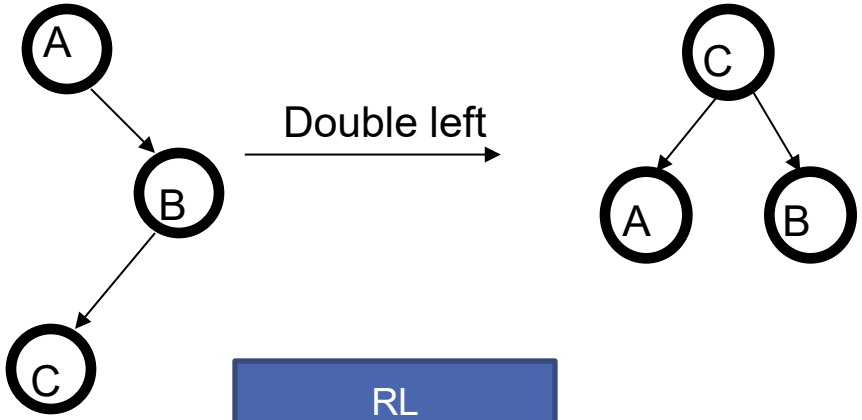
LL



LR



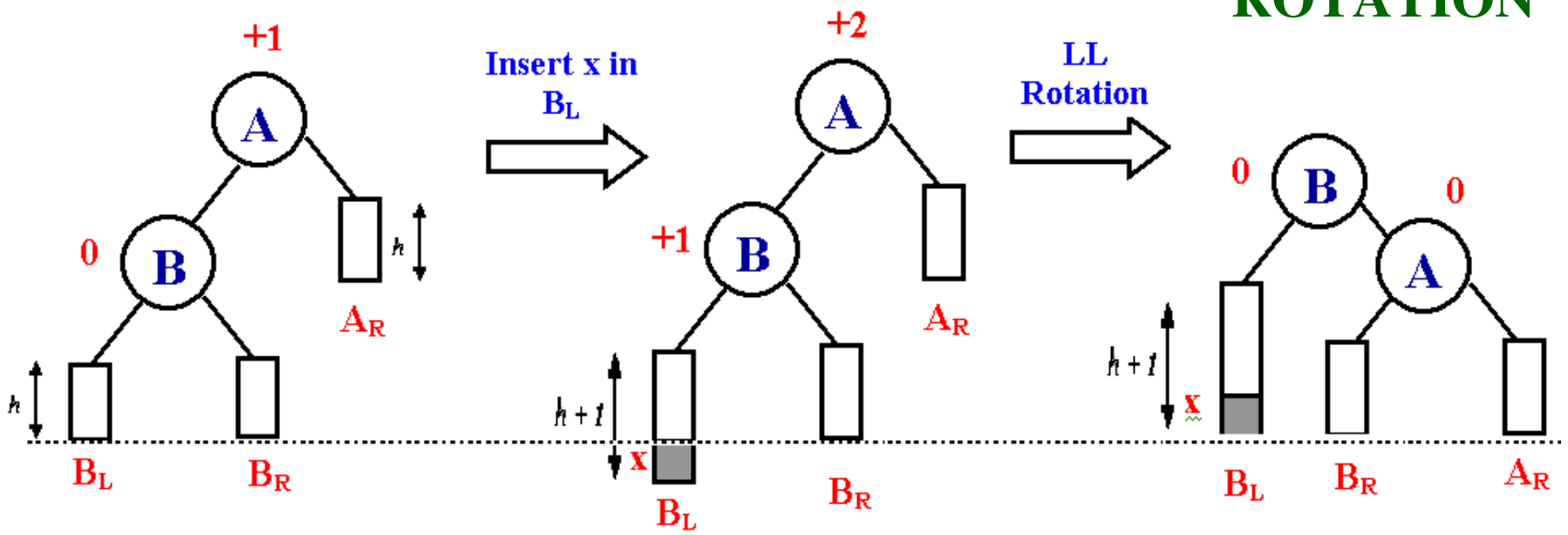
RR



RL

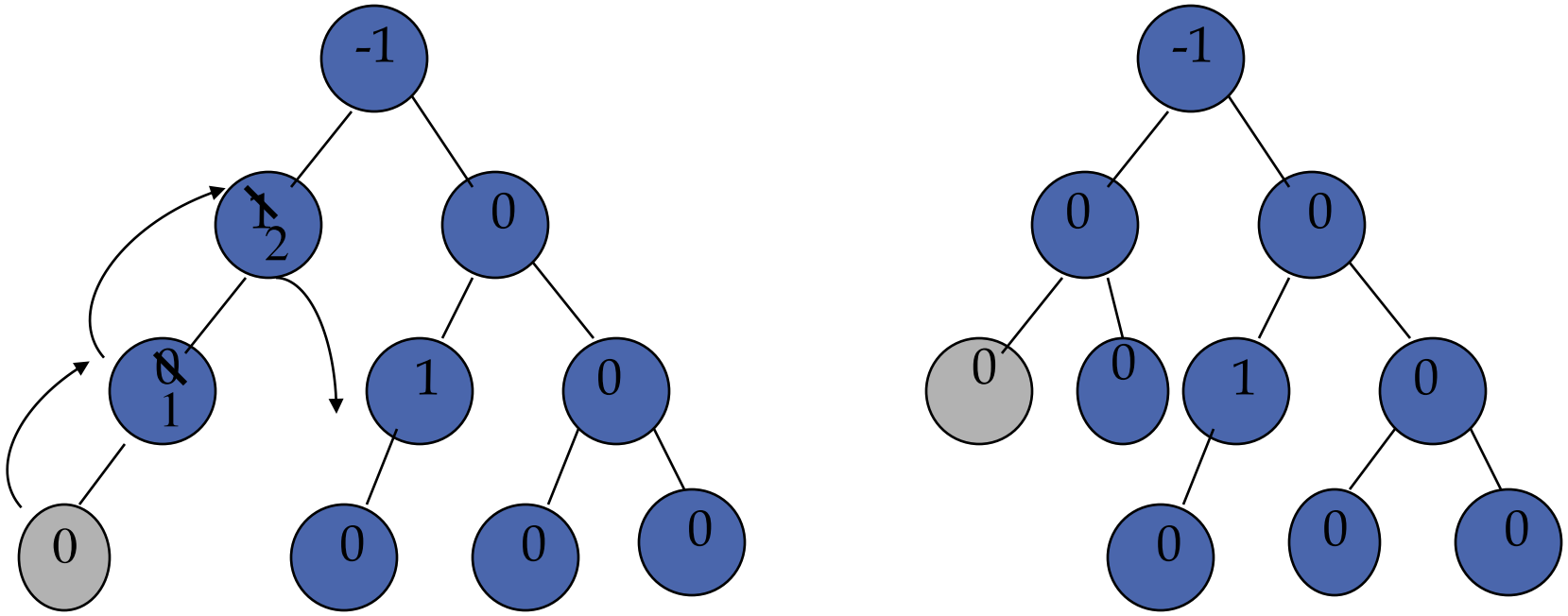
LL Rotations

Balanced AVL tree | Unbalanced AVL tree | Balanced AVL Tree after **ROTATION**



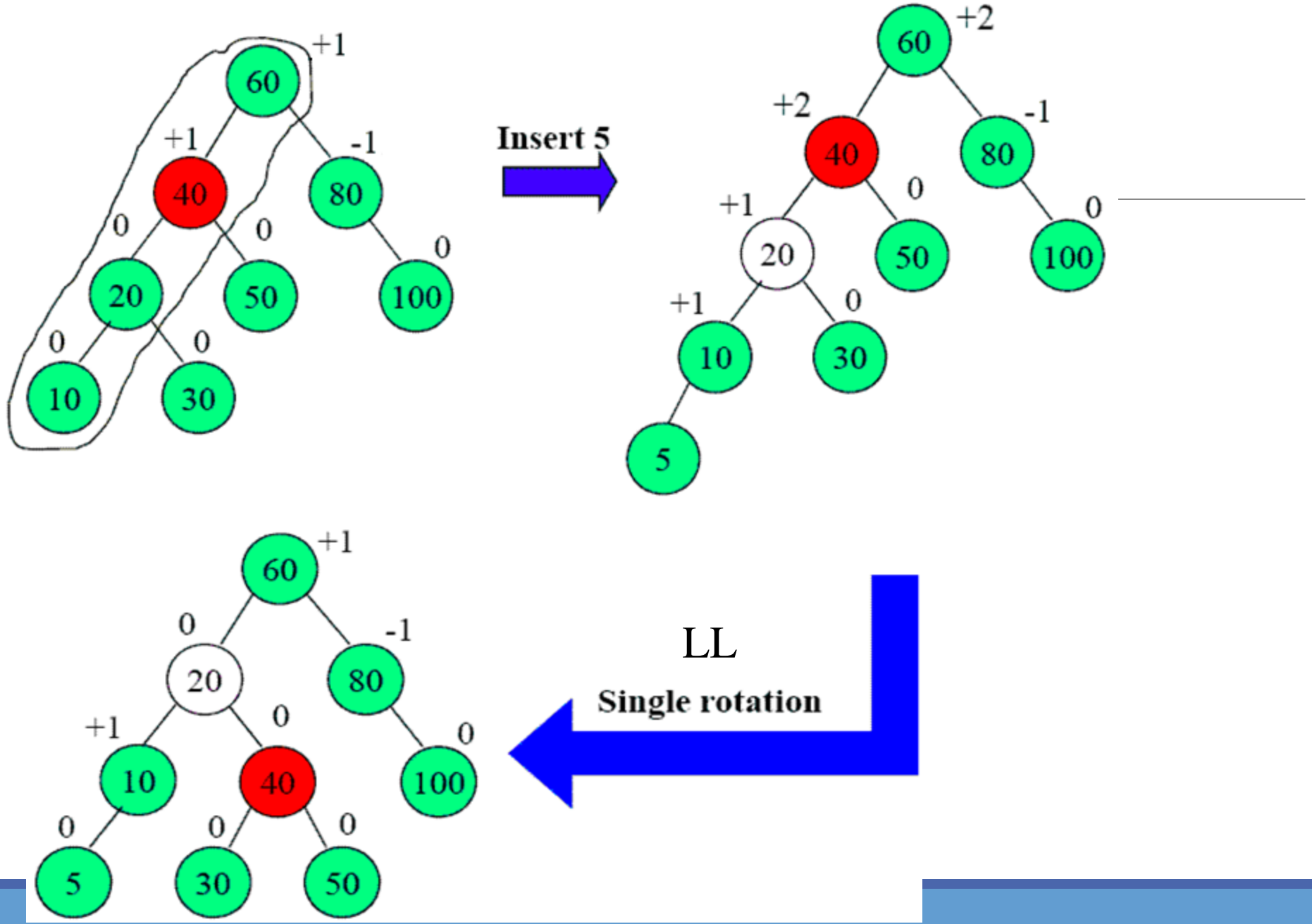
B_L : Left subtree of B
B_R : Right subtree of B
A_R : Right subtree of A
h : Height

LL Rotation



simple right rotation required

LL Rotation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+1
40
+1
80
-1
20
0
50
0
100
0
10
0
30
0
Insert 5
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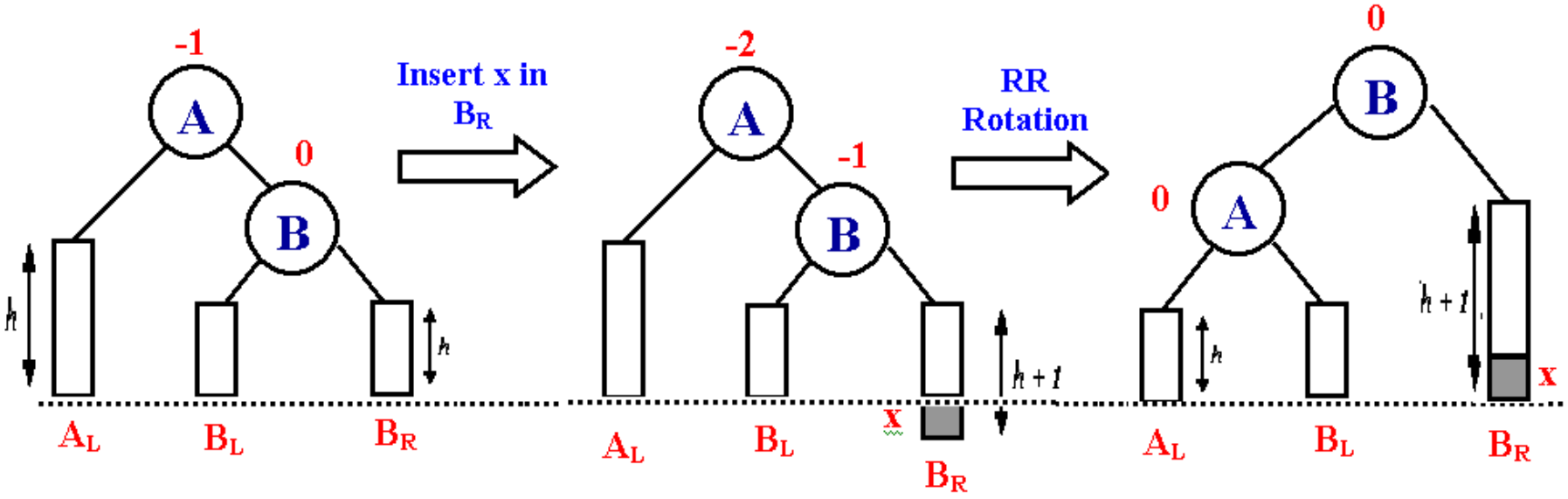
+2
80
-1
20
+1
50
0
100
0
10
+1
30
0
5
60
+1
20
0
80
-1
10
+1
40
0
100
0
5
0
30
0
50
0
LL
Single rotation

RR Rotations

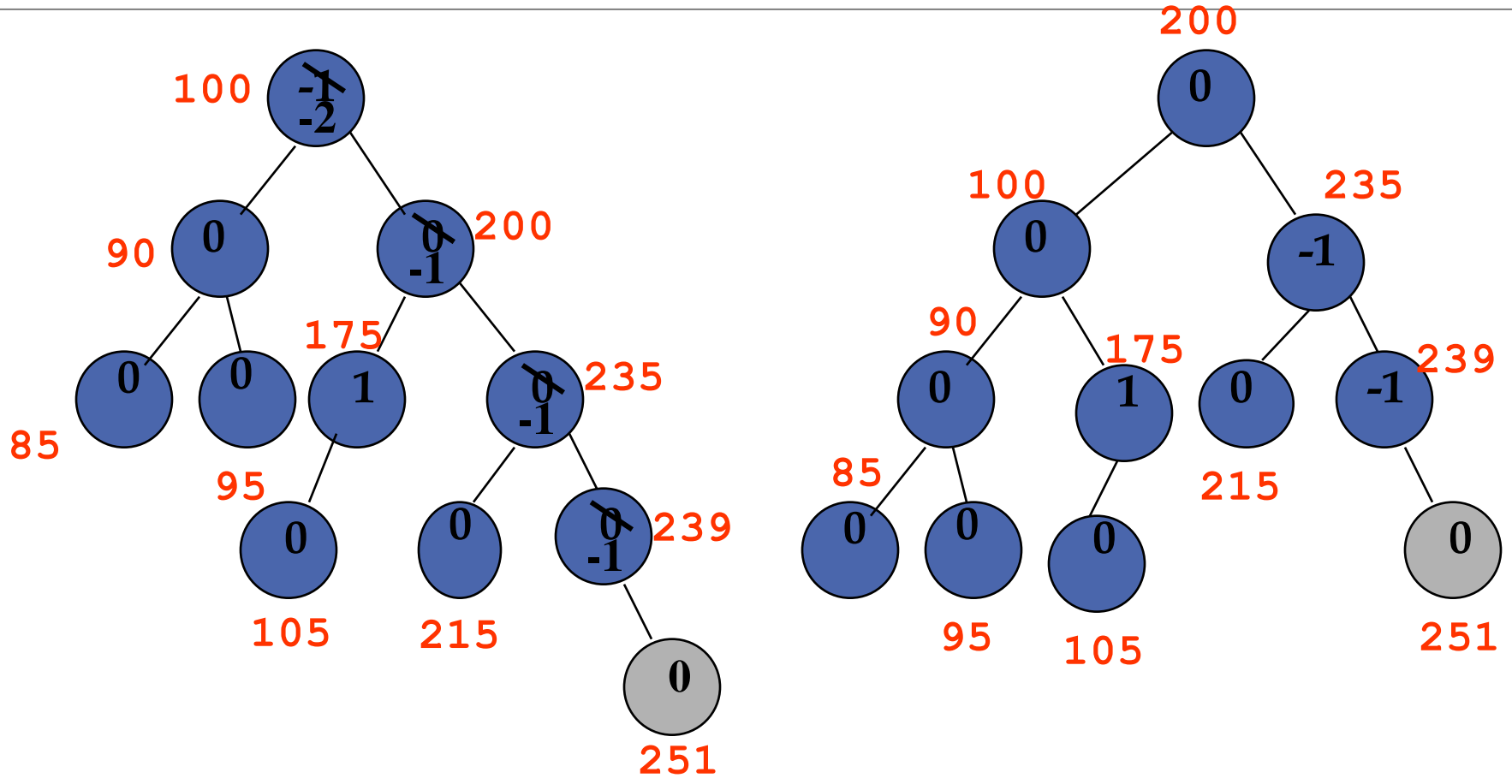
Balanced AVL tree

Unbalanced AVL tree

Balanced AVL Tree after **ROTATION**

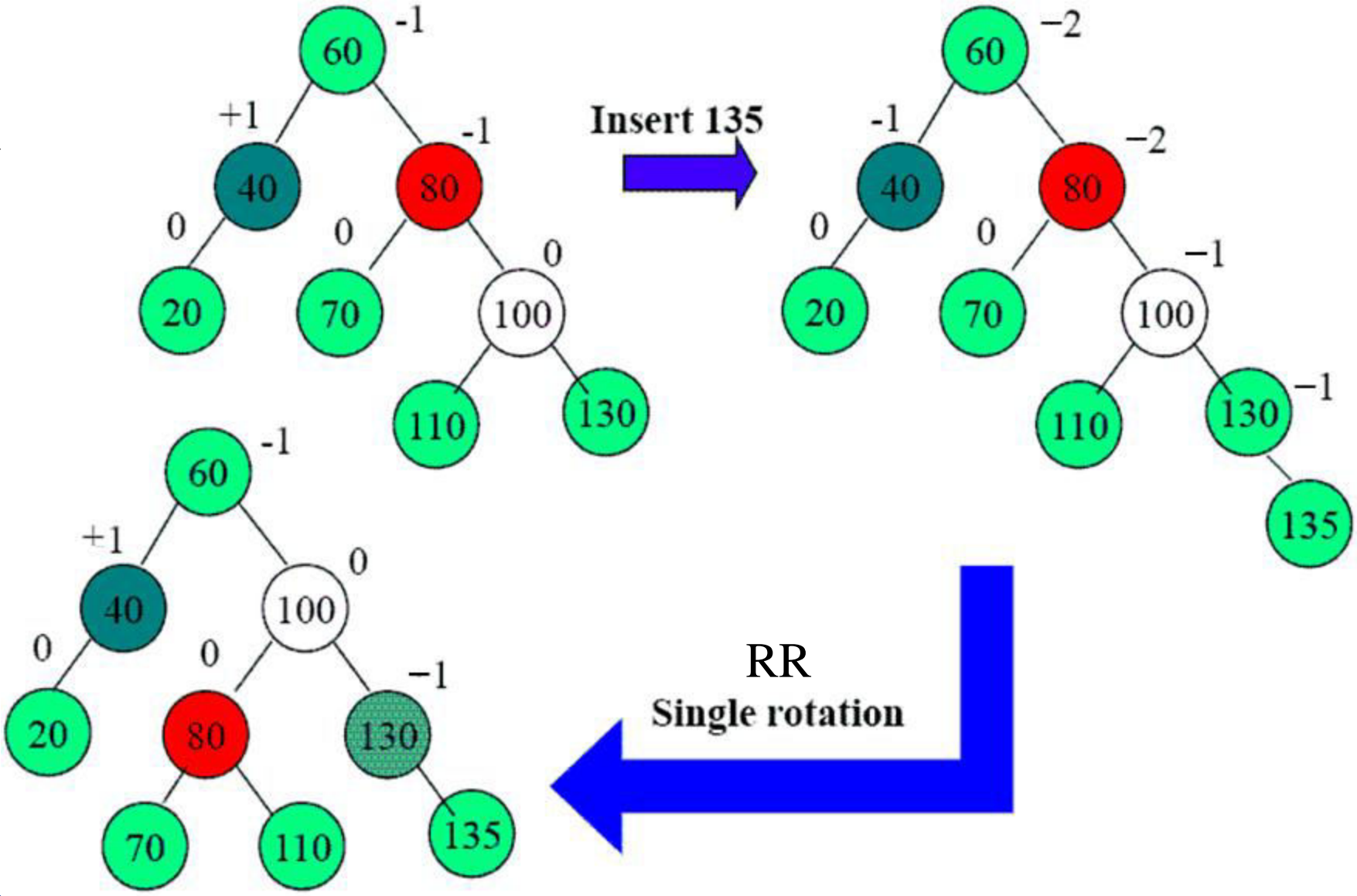


RR Rotation



simple left rotation required

RR Rotation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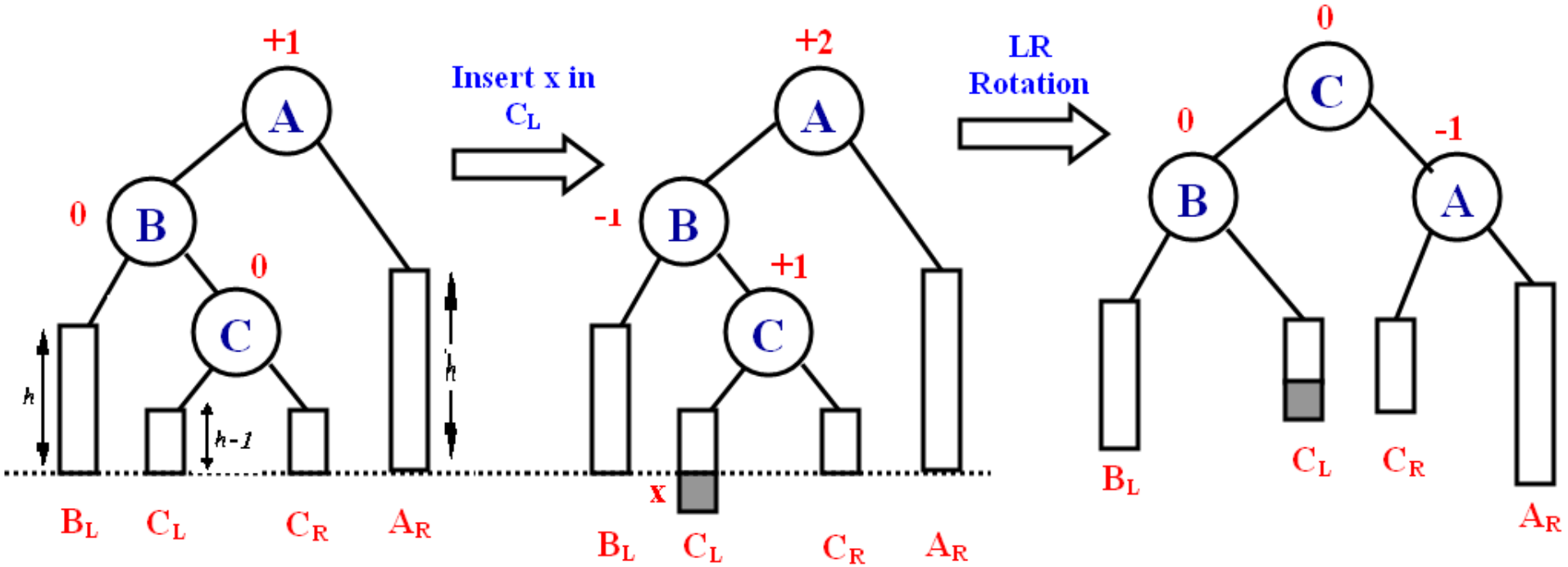
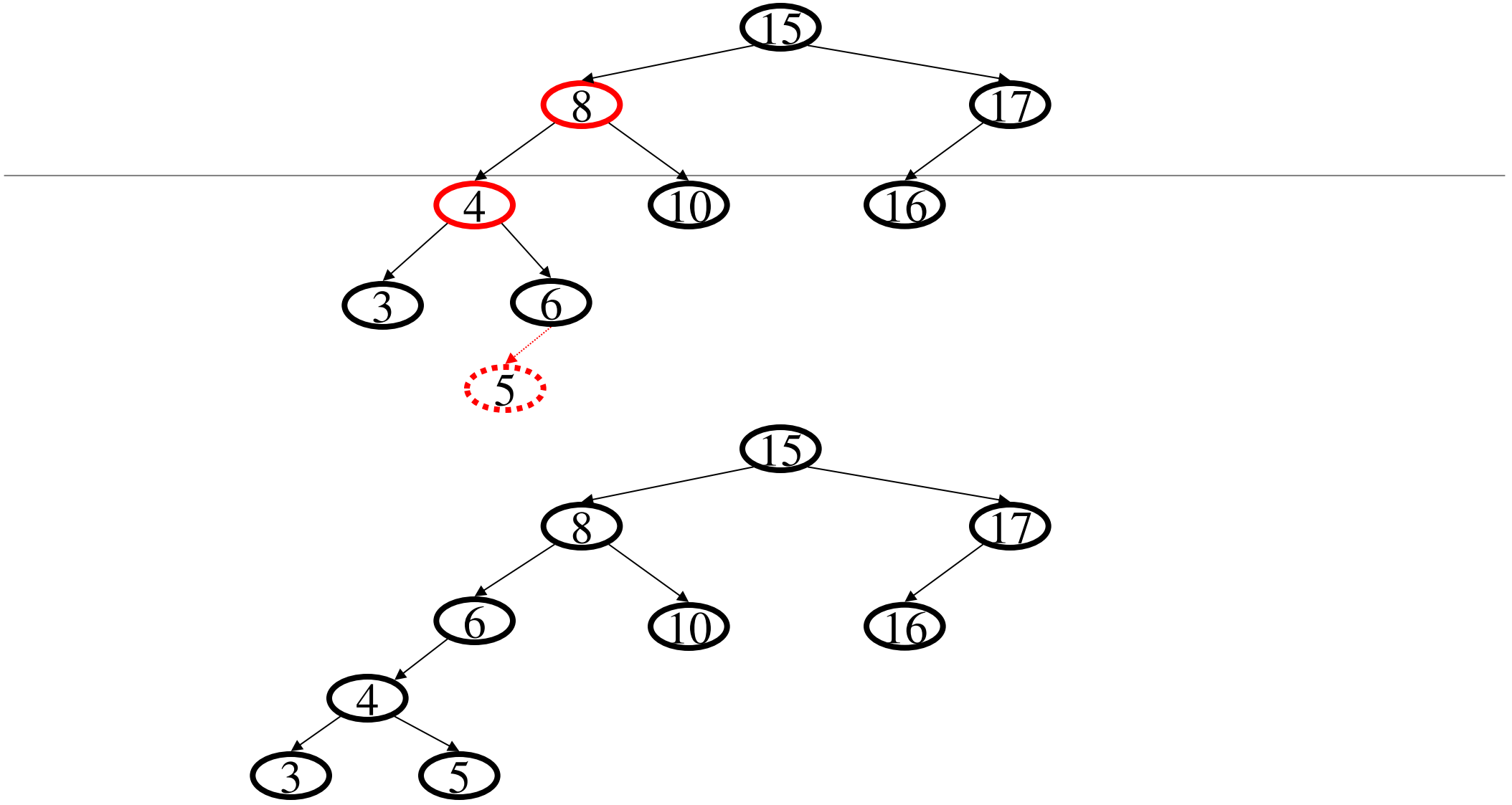
-1
+1
40
-1
80
0
0
20
70
0
100
110
130
Insert 135
60
−2
-1
40
−2
80
0
0
20
70
−1
100
110
130
−1
135
60
-1
+1
40
0
100
0
0
−1
20
80
130
70
110
135
RR
Single rotation

LR Rotations

Balanced AVL tree	Unbalanced AVL tree	Balanced AVL Tree after ROTATION

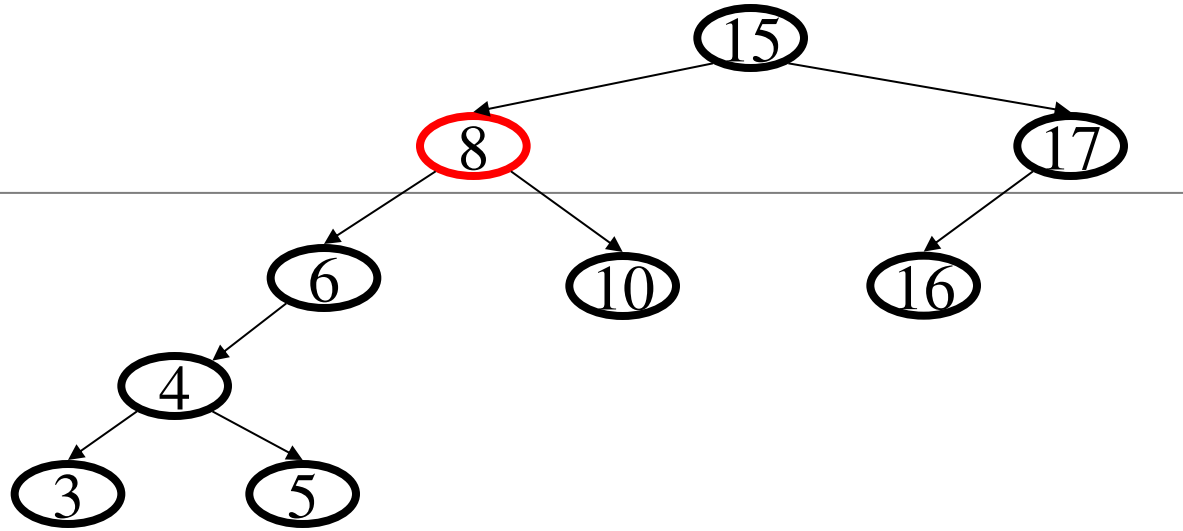


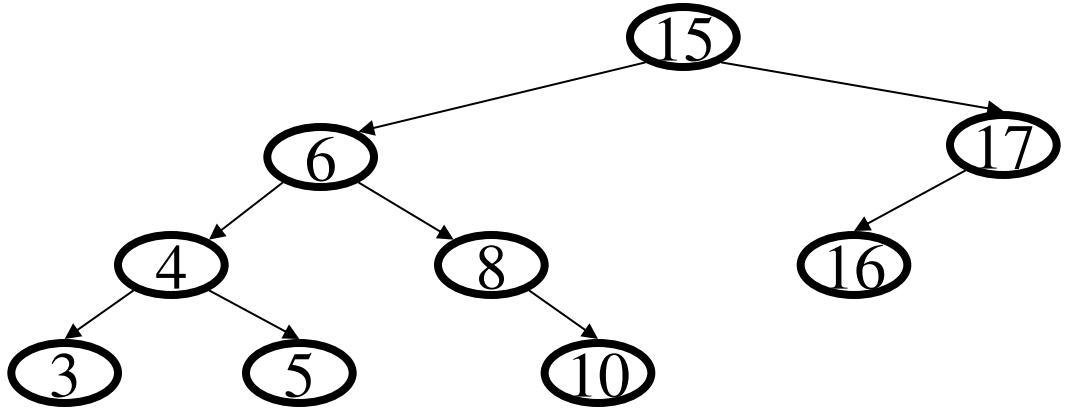
Examples



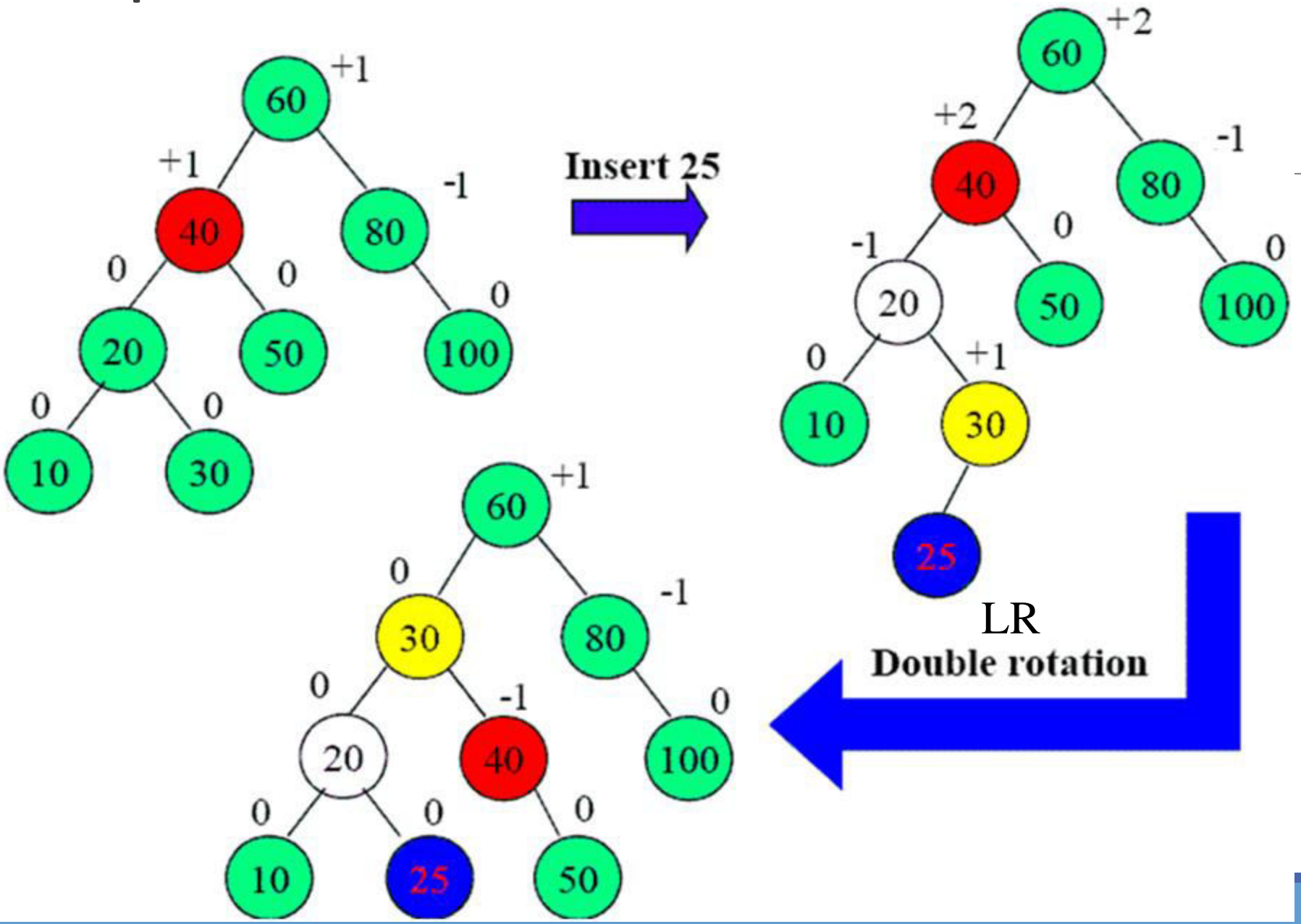
Double rotation, step 1

Double rotation, step 2





Example 1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+1
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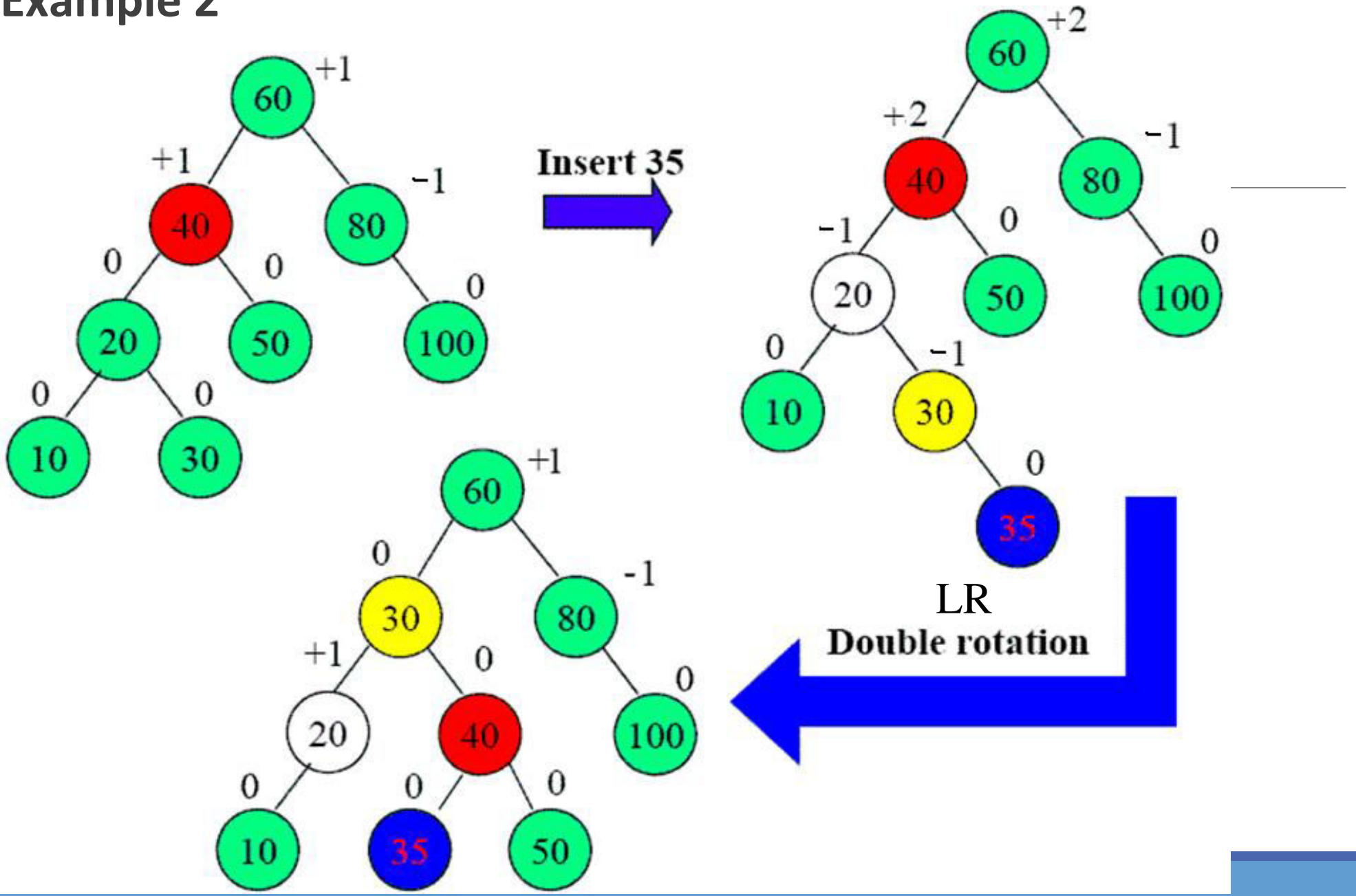
+1
80
-1
20
0
50
0
100
0
10
0
30
0
Insert 25
60
+2
40
+2
80
-1
20
-1
50
0
100
0
10
0
30
+1
25
LR
Double rotation
60
+1
30
0
80
-1
20
0
40
-1
100
0
10
0
25
0
50
0

Example 2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+1
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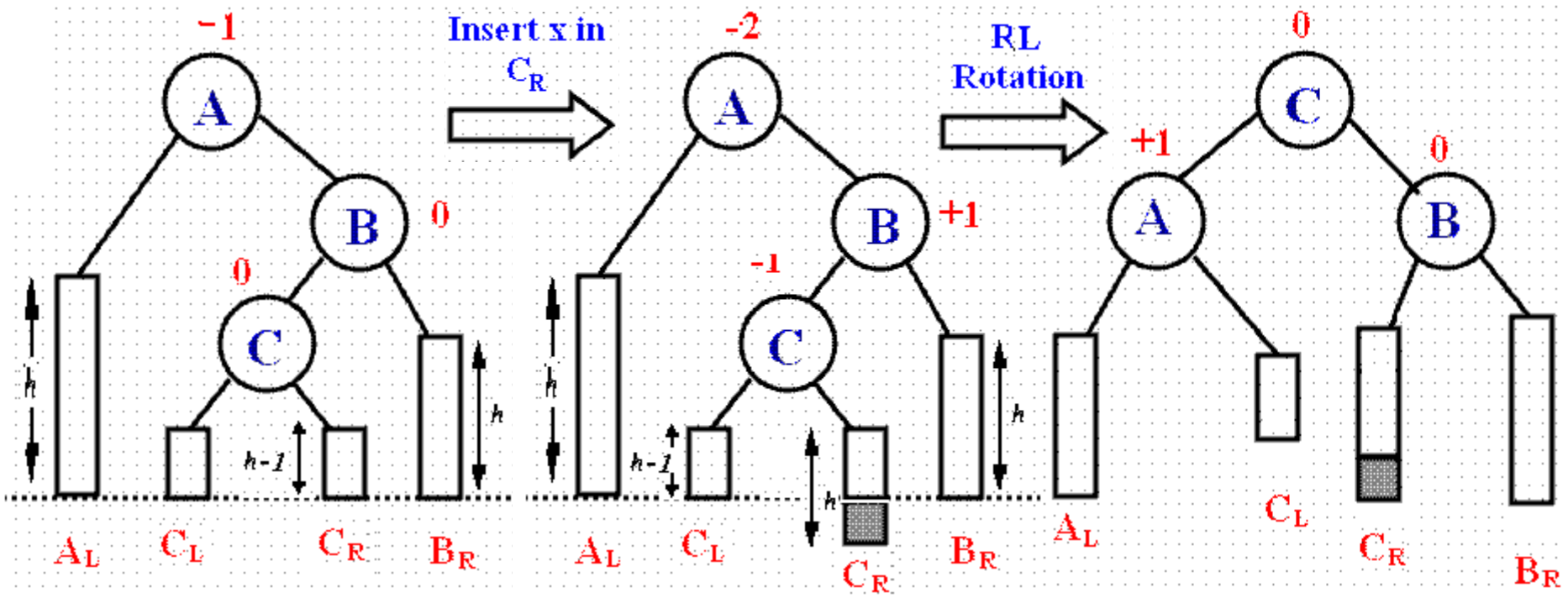
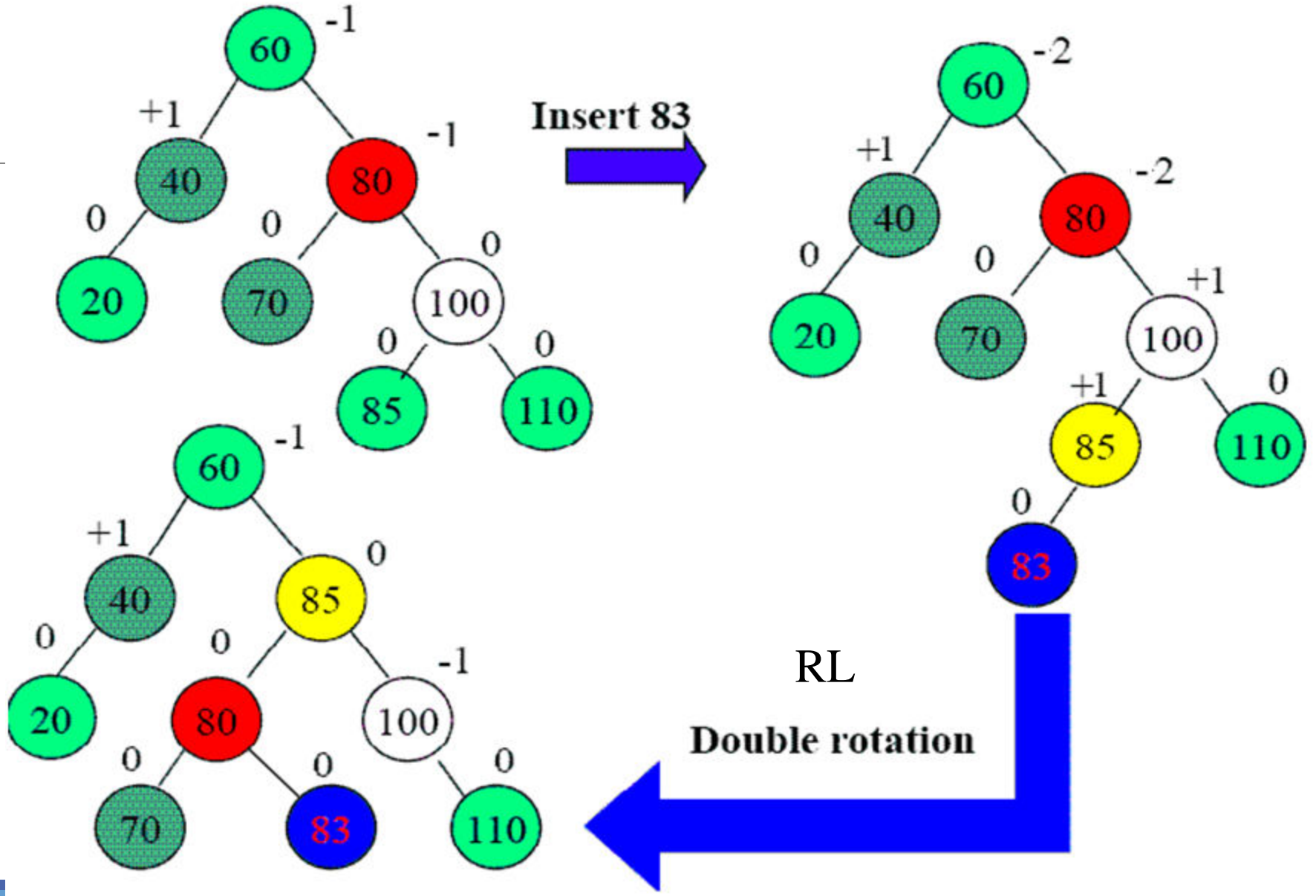
+1
80
-1
20
0
50
0
100
0
10
0
30
0
Insert 35
60
+2
40
+2
80
-1
20
-1
50
0
100
0
10
0
30
-1
35
0
LR
Double rotation
60
+1
30
0
80
-1
20
+1
40
0
100
0
10
0
35
0
50
0

RL Rotations

Balanced AVL tree	Unbalanced AVL tree	Balanced AVL Tree after **ROTATION**



Example 1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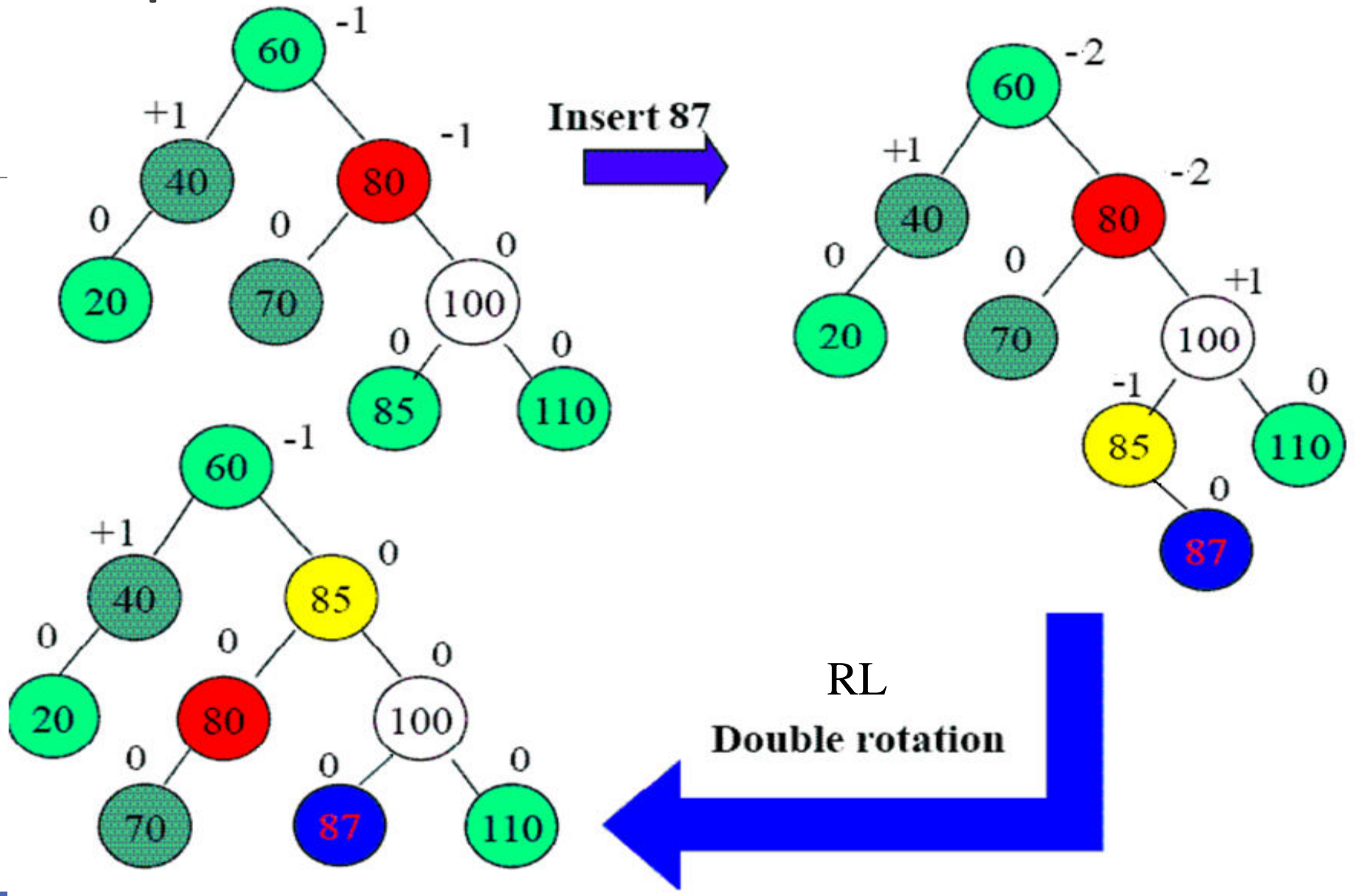
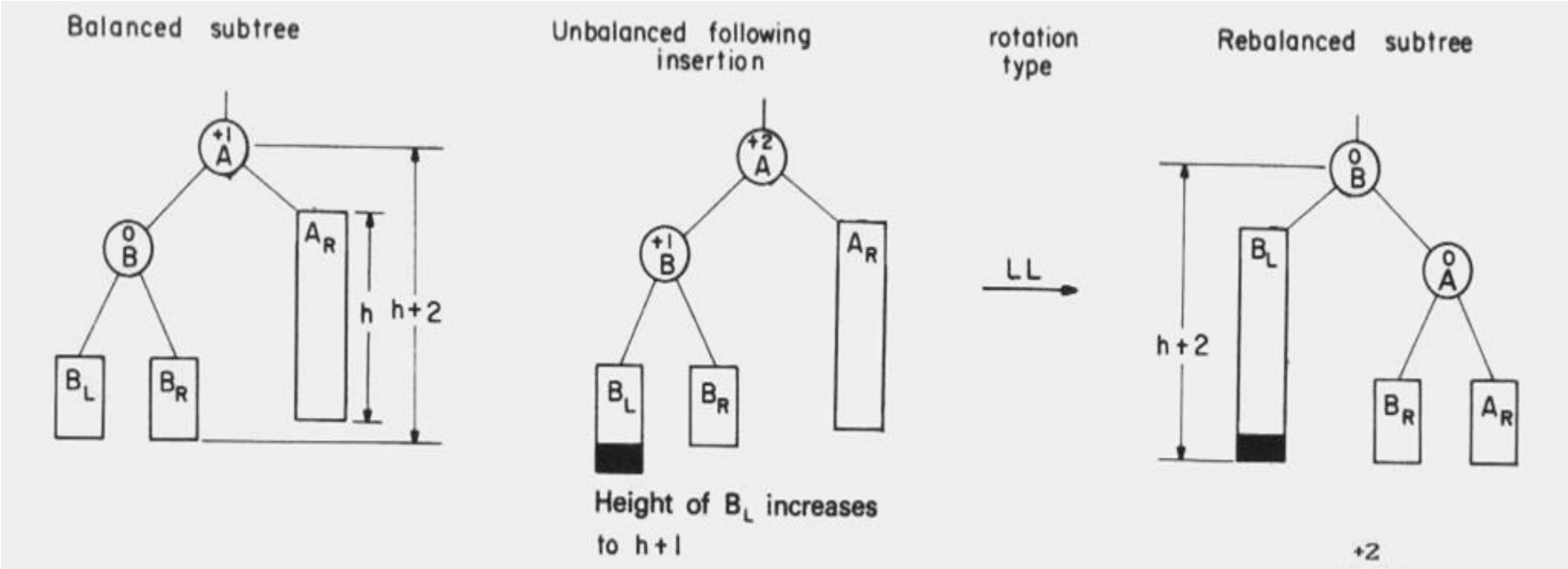
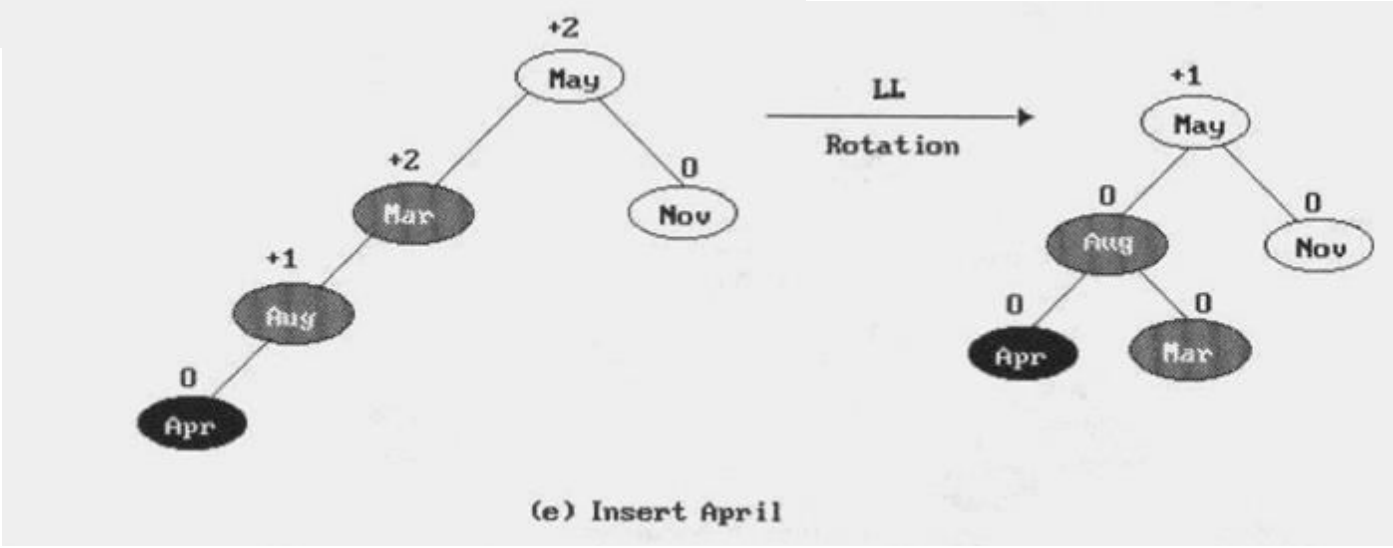
-1
+1
40
80
-1
0
0
20
70
0
100
0
0
85
110
Insert 83
60
-2
+1
40
80
-2
0
0
20
70
+1
100
+1
85
0
110
0
83
RL
Double rotation
60
-1
+1
40
85
0
0
0
20
80
100
-1
0
0
0
70
83
110

Example 2


60
-1
+1
40
-1
80
0
0
20
70
0
100
0
0
85
110
Insert 87
60
-2
+1
40
-2
80
0
0
20
70
+1
100
-1
0
85
110
0
87
RL
Double rotation
60
-1
+1
40
0
85
0
0
0
20
80
100
0
0
0
70
87
110

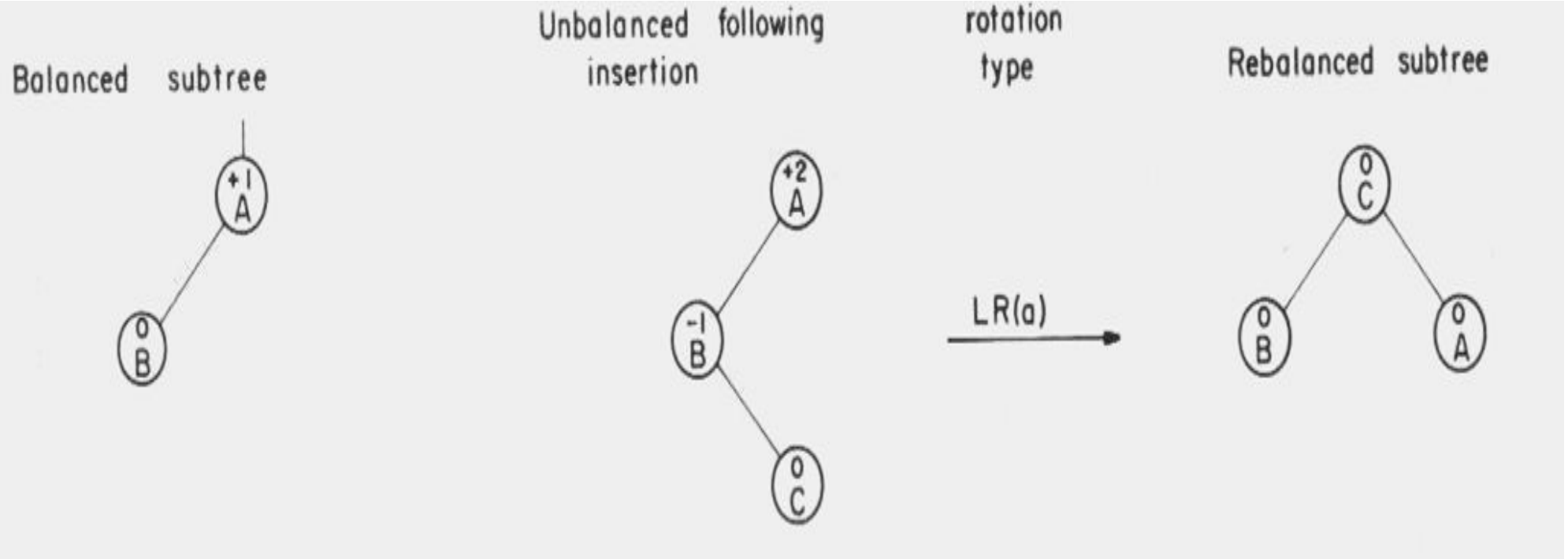
Rebalancing rotations



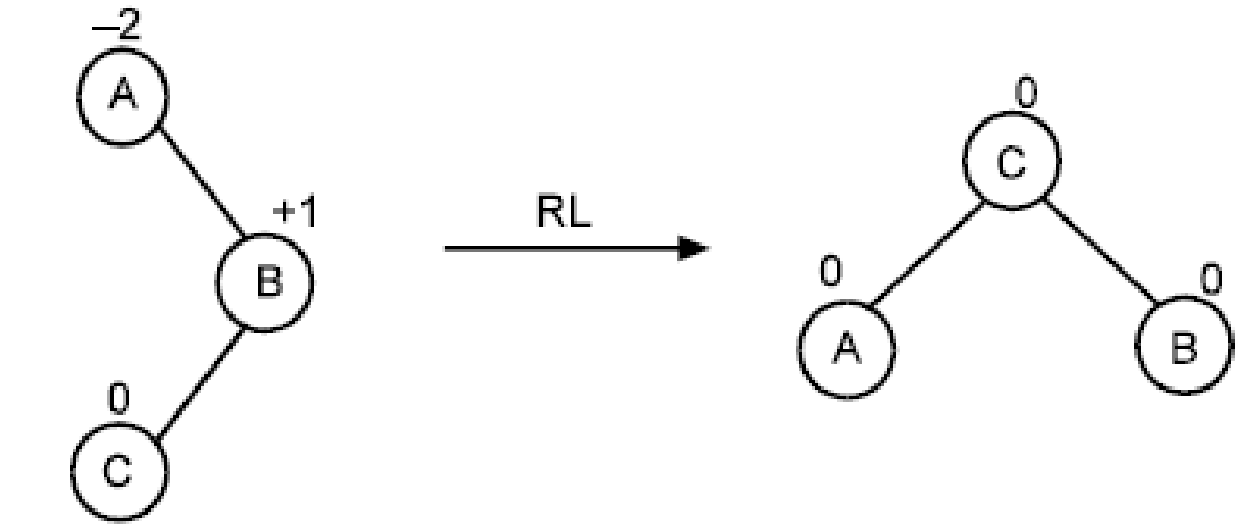


AVL Trees

Rebalancing rotations (cont'd)



Case 4: RL (Left of Right)



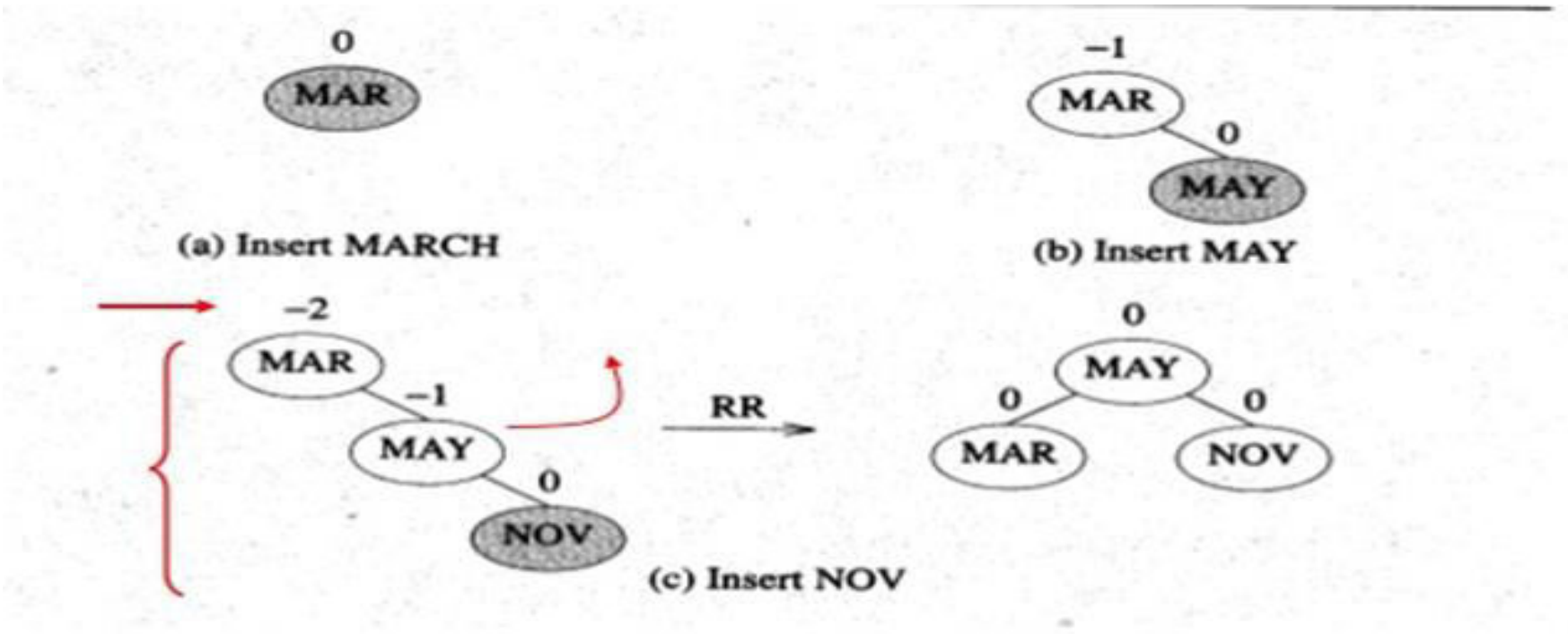
Unbalanced tree due to insertion

Balanced tree

(a)

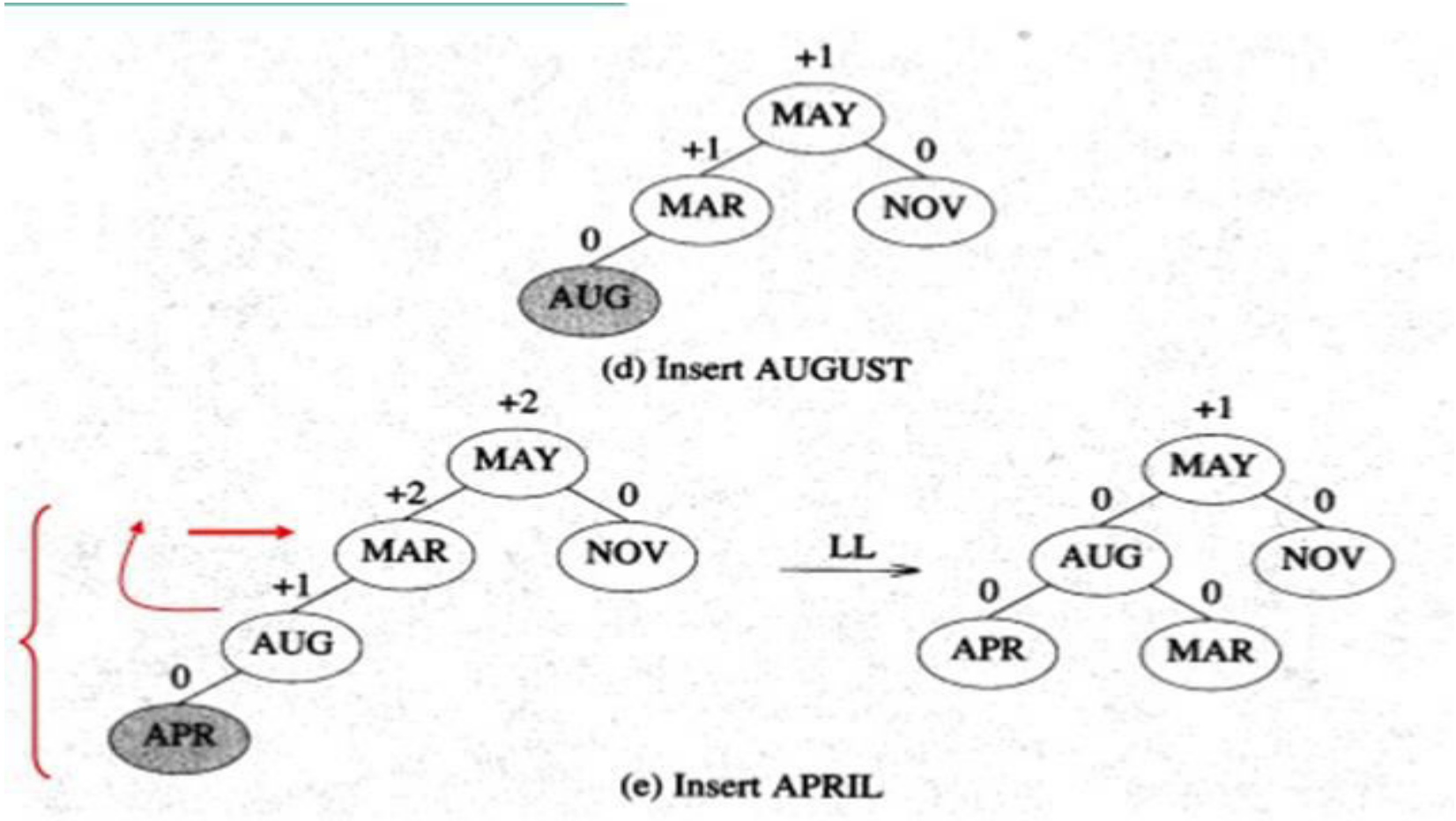
- This time we will insert the months into the tree in the order
- *Mar*, *May*, *Nov*, *Aug*, *Apr*, *Jan*, *Dec*, *Jul*, *Feb*, *Jun*, *Oct*, *Sep*

- It shows the tree as it grows, and the restructuring involved in keeping it balanced.

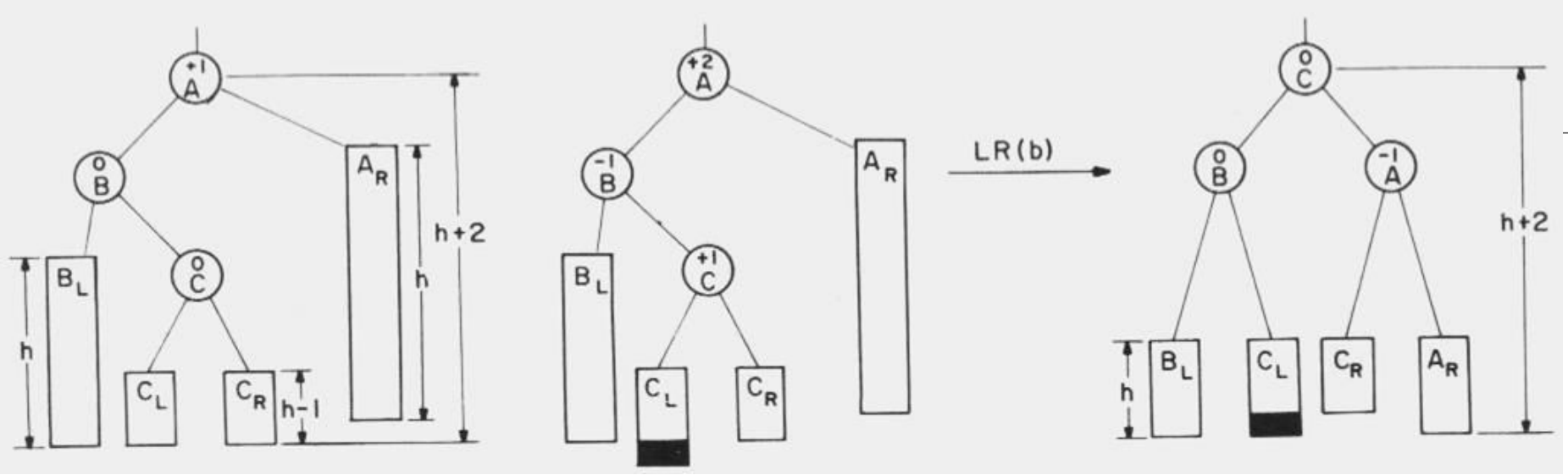


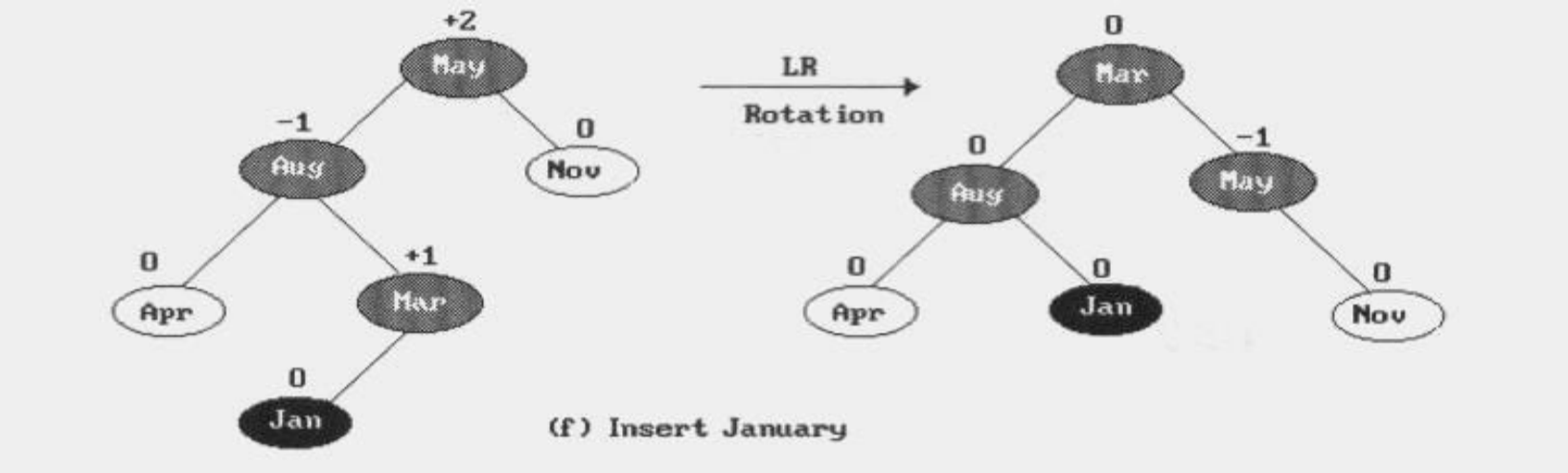
- This time we will insert the months into the tree in the order
- *Mar*, *May*, *Nov*, *Aug*, *Apr*, *Jan*, *Dec*, *Jul*, *Feb*, *Jun*, *Oct*, *Sep*

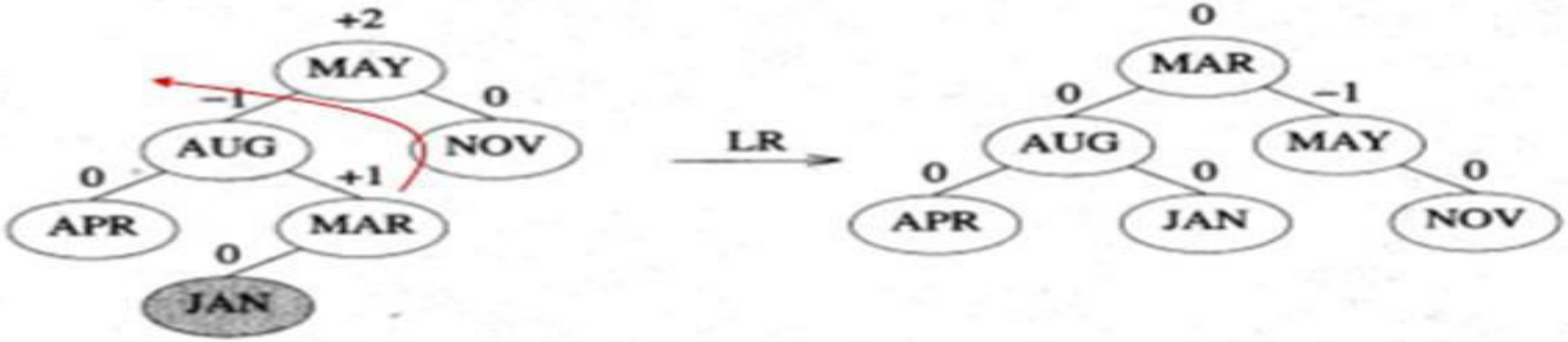
- It shows the tree as it grows, and the restructuring involved in keeping it balanced.



Rebalancing rotations (cont'd)





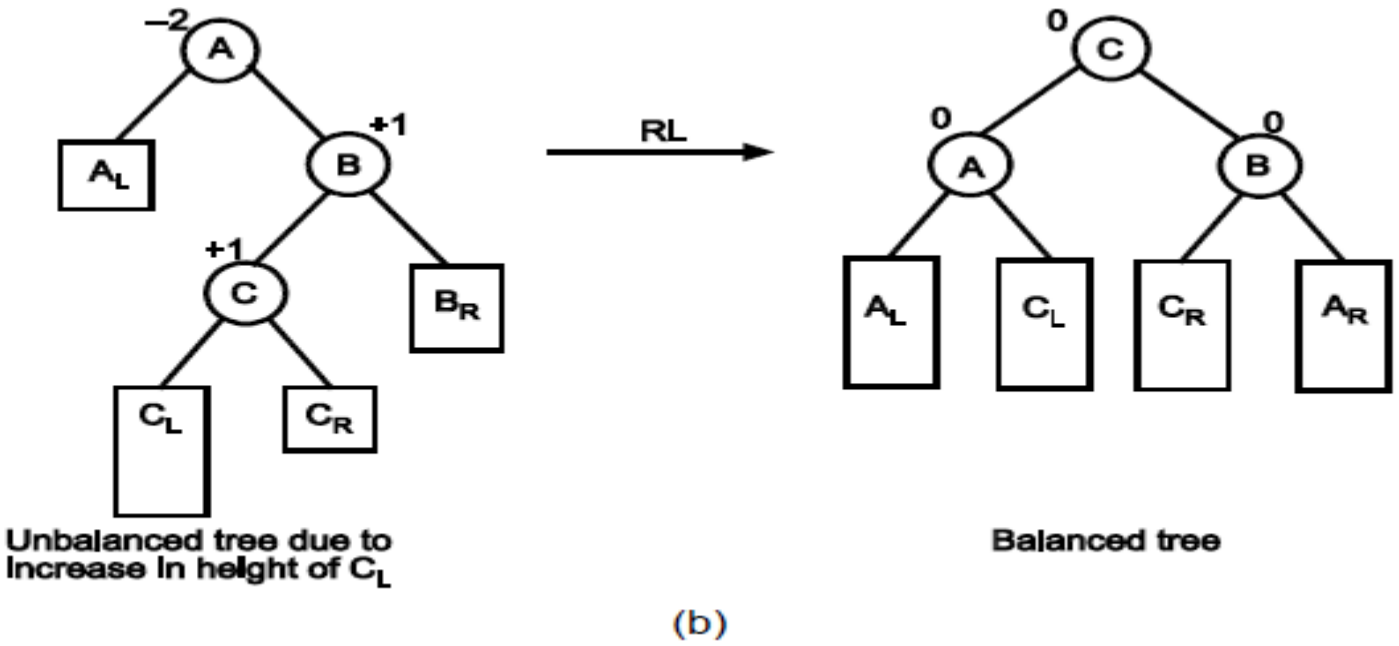


(f) Insert JANUARY

(g) Insert DECEMBER

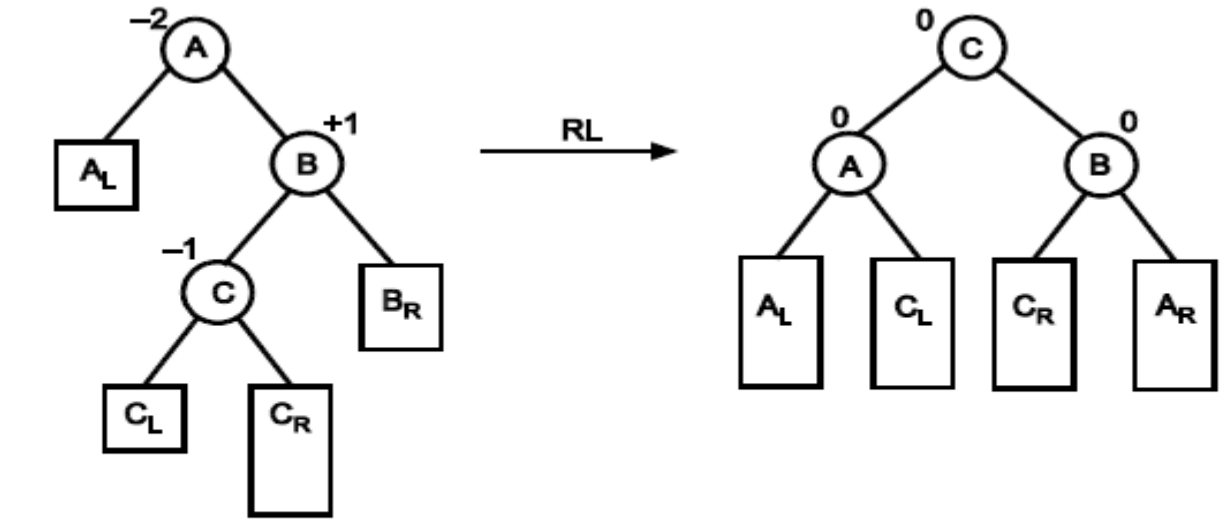
(h) Insert JULY

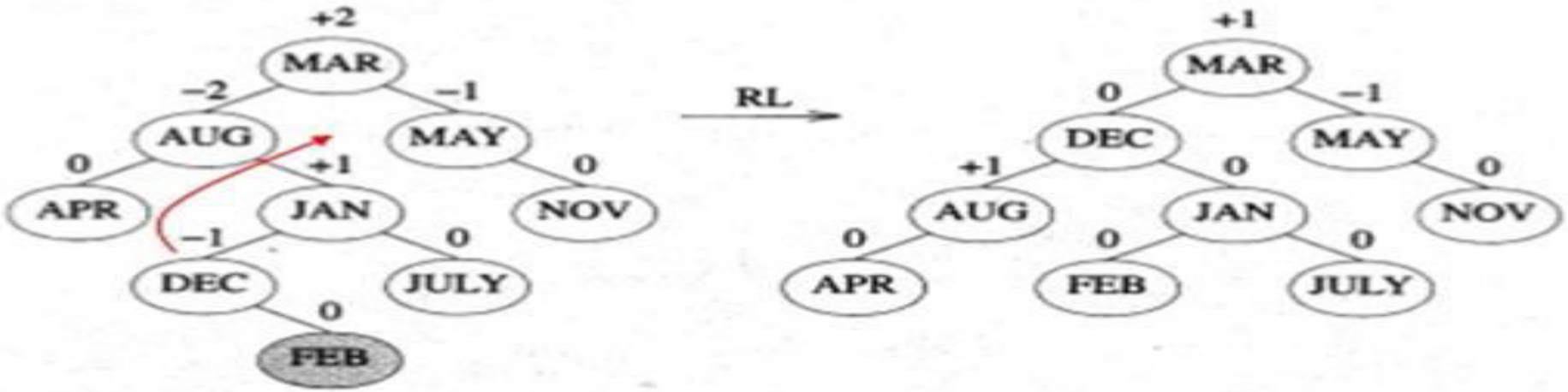
Case 4: RL(Left of Right)



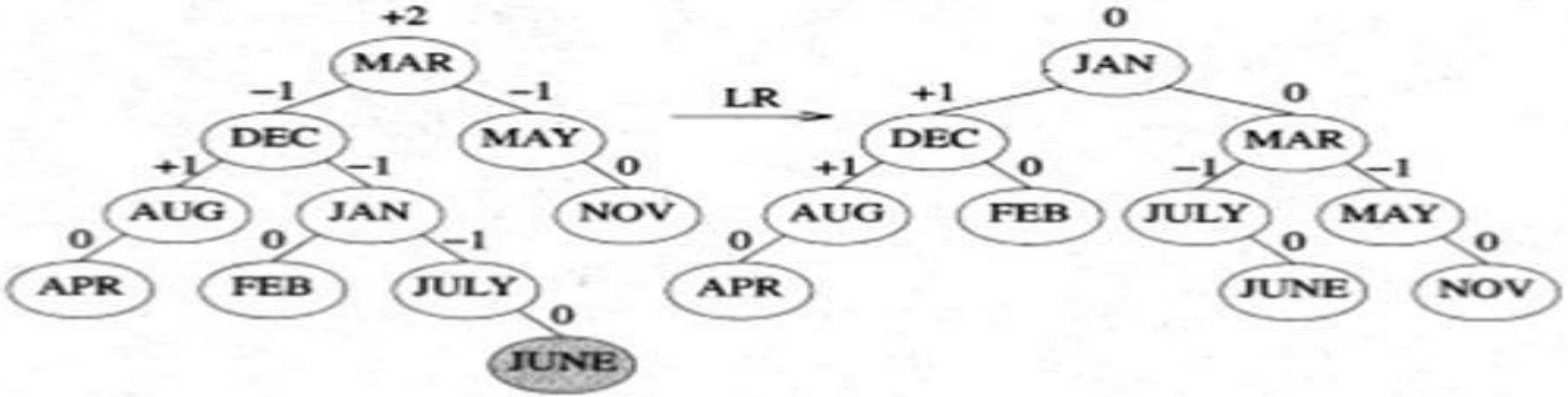
(b)

Case 4: RL (Left of Right)



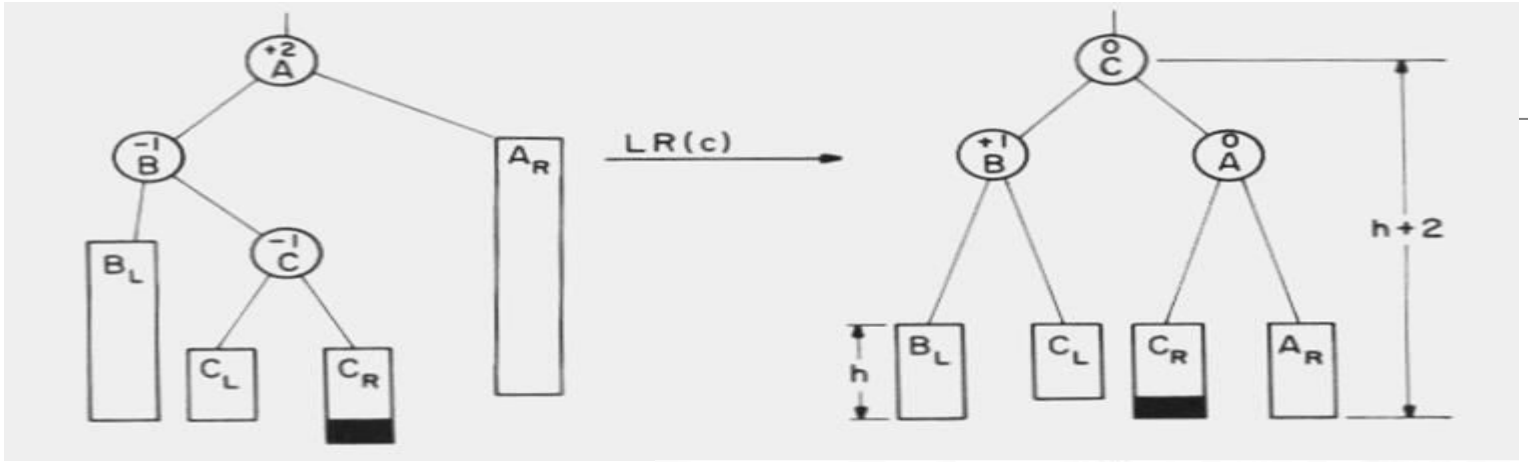


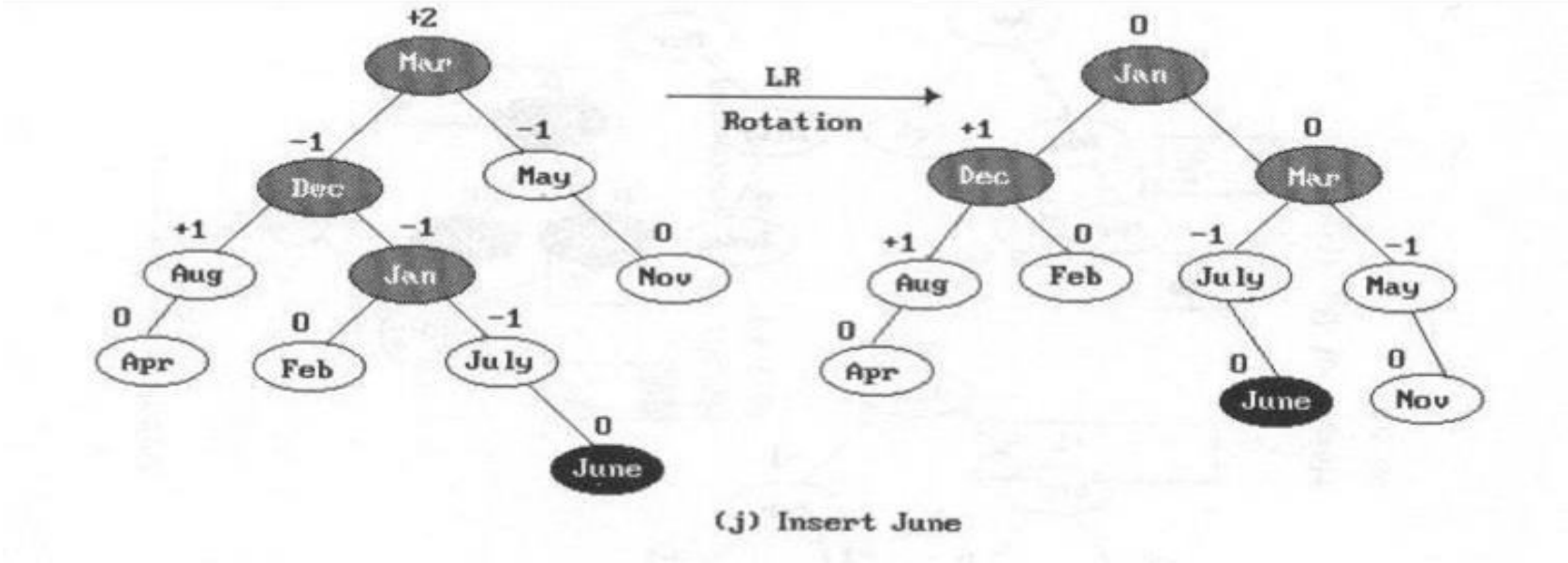
(i) Insert FEBRUARY



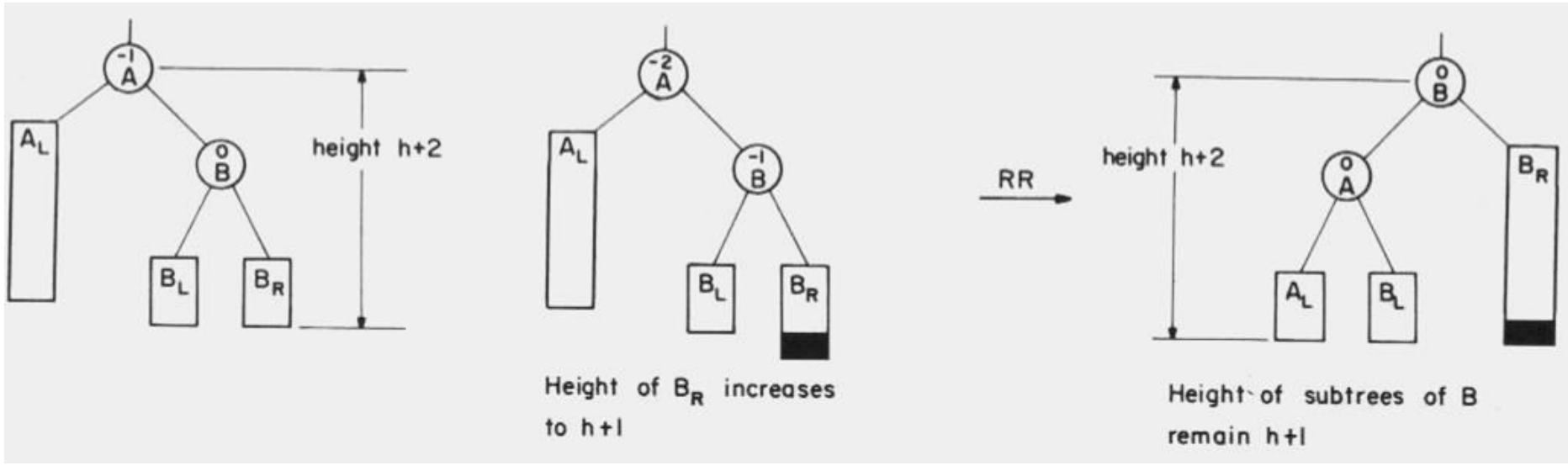
(j) Insert JUNE

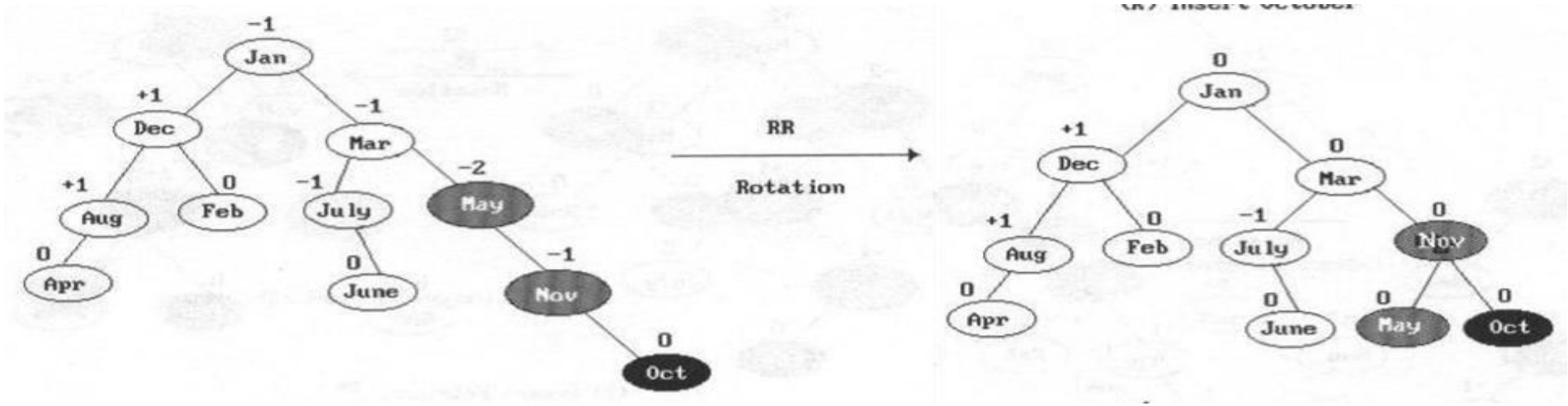
Case 3: LR (Left of Right)



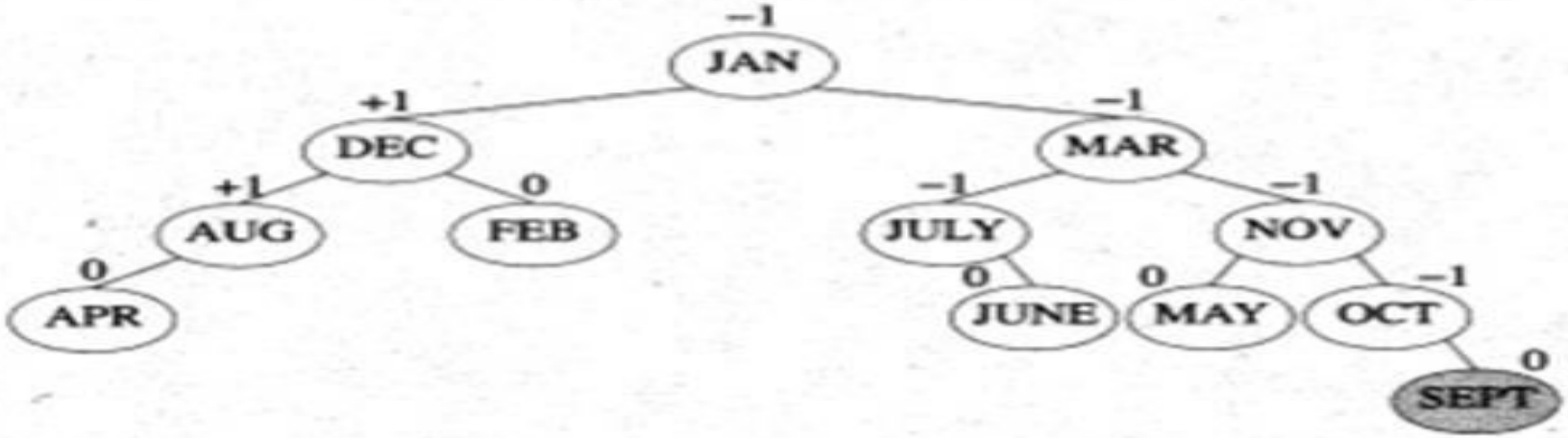


AVL Trees


-1
A
A_L
0
B
height h+2
B_L
B_R
-2
A
A_L
-1
B
B_L
B_R
Height of B_R increases to h+1
RR
0
B
height h+2
0
A
B_R
A_L
B_L
Height of subtrees of B remain h+1


-1
Jan
+1
Dec
-1
Mar
+1
Aug
0
Feb
-1
July
-2
May
0
Apr
0
June
-1
Nov
0
Oct
RR
Rotation
0
Jan
+1
Dec
0
Mar
+1
Aug
0
Feb
-1
July
0
Nov
0
Apr
0
June
0
May
0
Oct

(k) Insert OCTOBER



(l) Insert SEPTEMBER

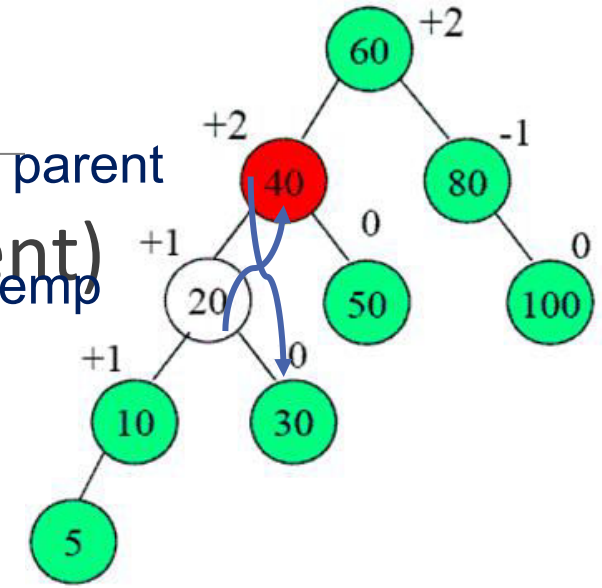
Node Structure for AVL Tree

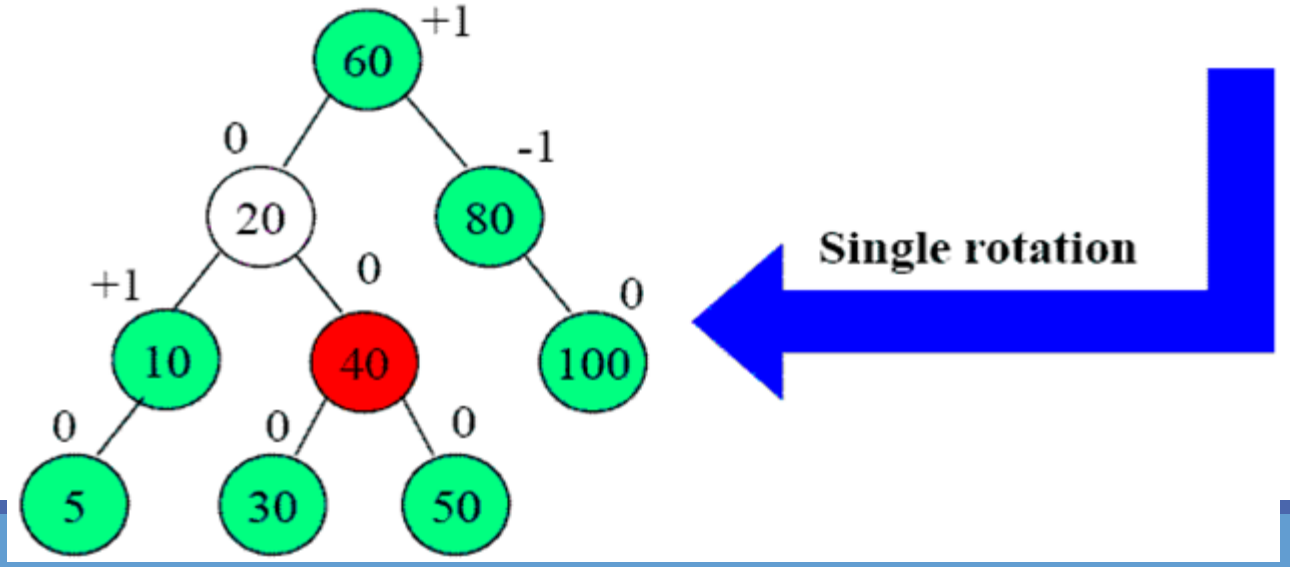
```
class avl_node
{
    int value;
     string word, meaning;
    node *left, *right;
};
```

```
class avlTree
{
    public:
        int height(avl_node *);
        int diff(avl_node *);
        avl_node *rr_rotation(avl_node *);
        avl_node *ll_rotation(avl_node *);
        avl_node *lr_rotation(avl_node *);
        avl_node *rl_rotation(avl_node *);
        avl_node* balance(avl_node *);
        avl_node* insert(avl_node *, int );
        void display(avl_node *, int);
        void inorder(avl_node *);
        void preorder(avl_node *);
        void postorder(avl_node *);
        avlTree()
        {
            root = NULL;
        }
};
```

LL Rotation

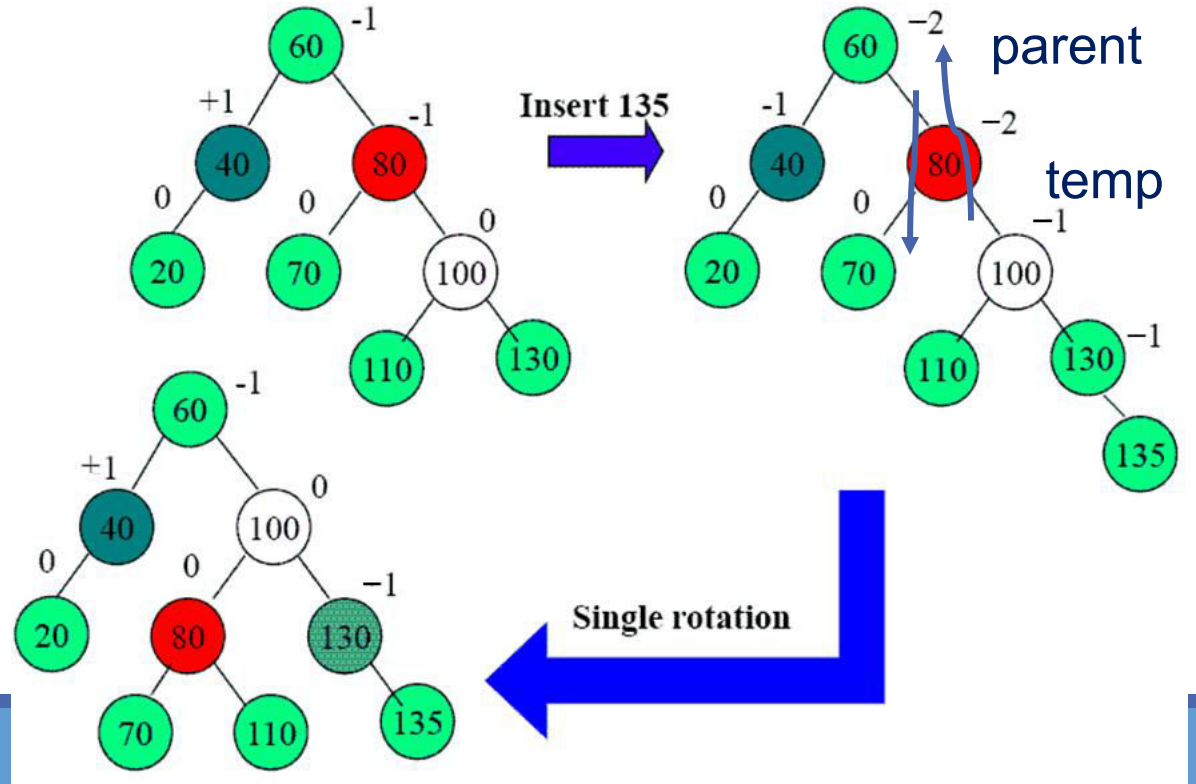
```
avl_node* avlTree:: LL_rotation (avl_node *parent)
{
    avl_node * temp = parent->left;
    parent->left = temp->right;
    temp->right = parent;
    return temp;
}
```



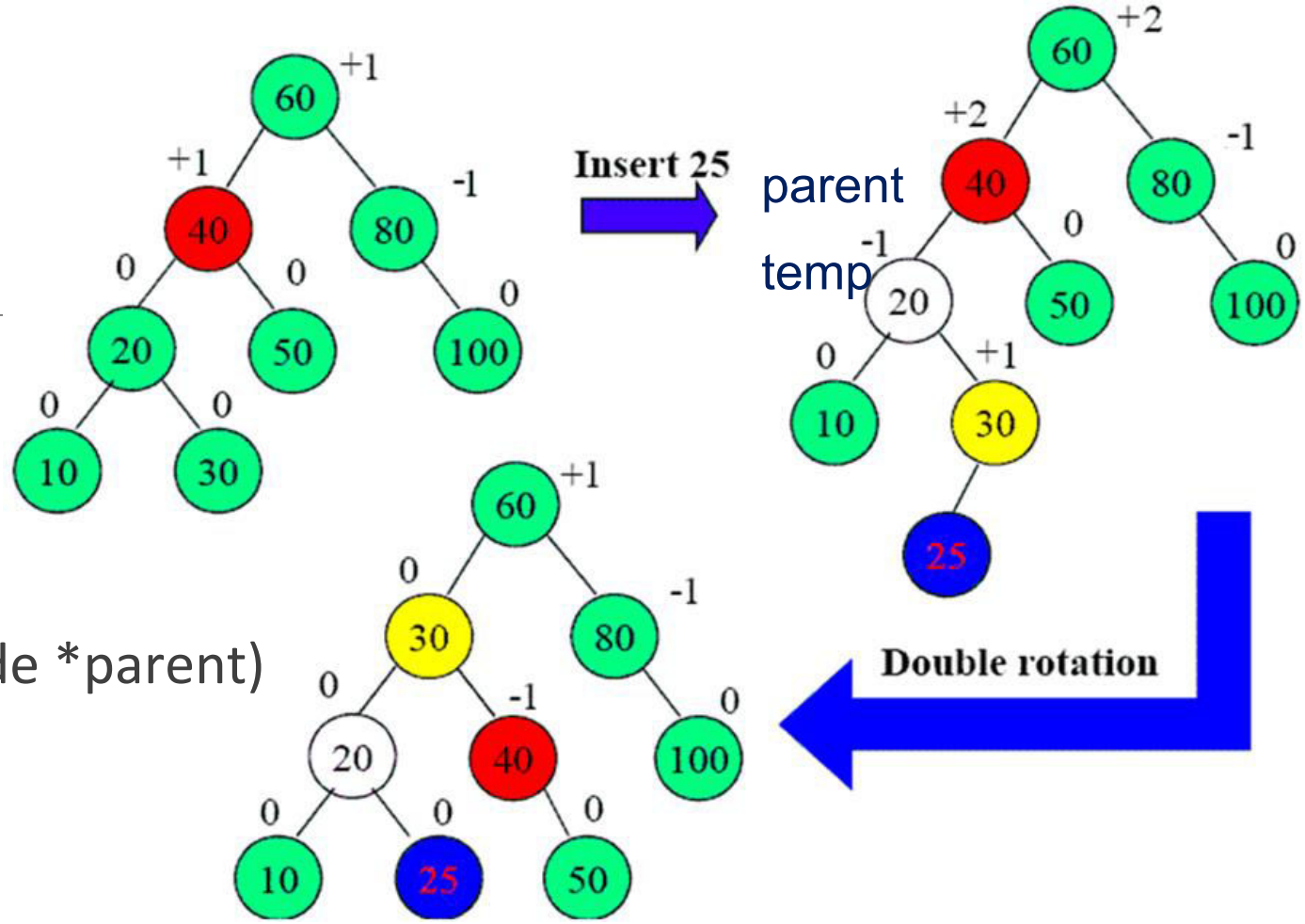


RR Rotation

```
avl_node* avlTree:: RR_rotation (avl_node *parent)
{
    avl_node* temp = parent->right;
    parent->right = temp->left;
    temp->left = parent;
    return temp;
}
```

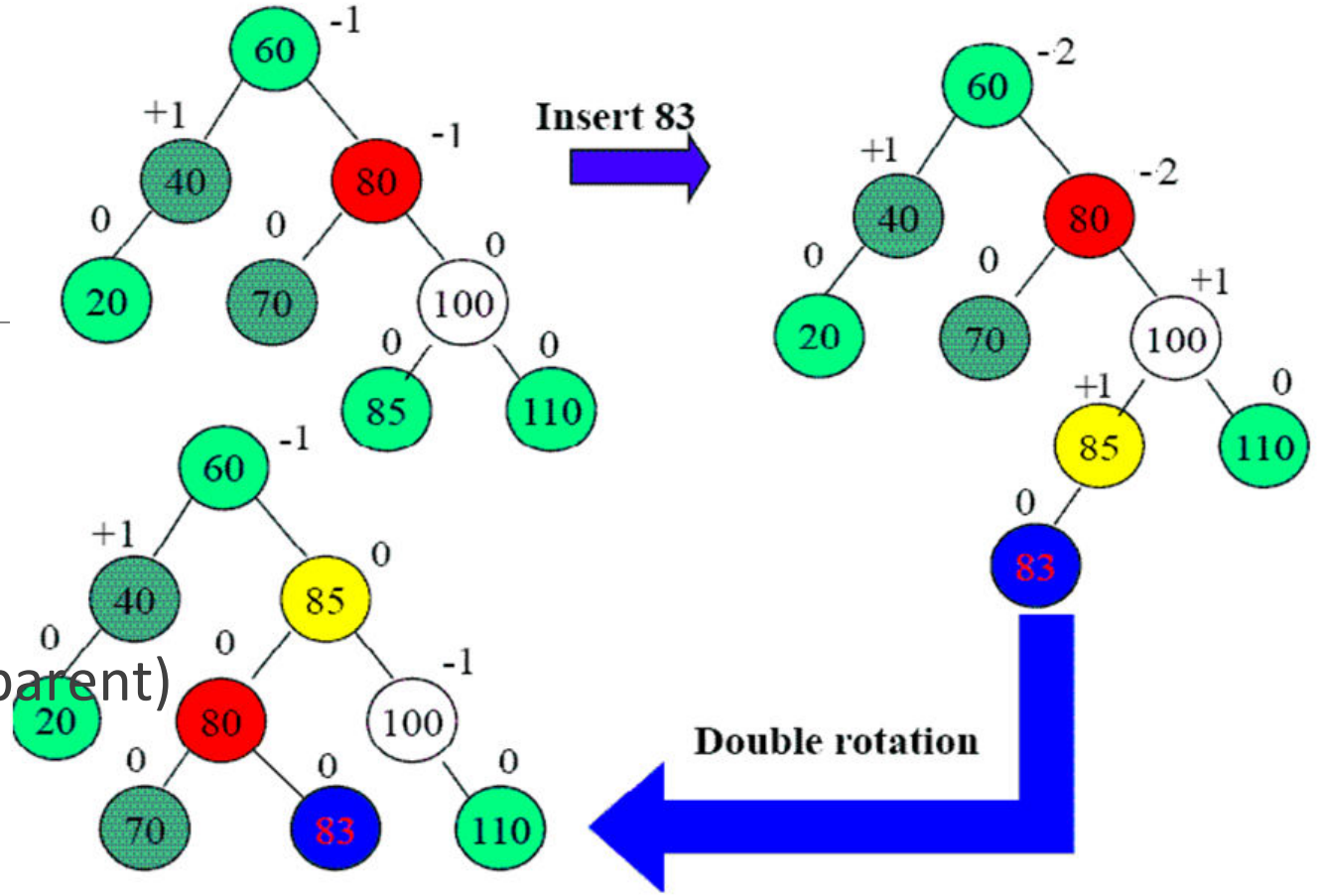


LR Rotation



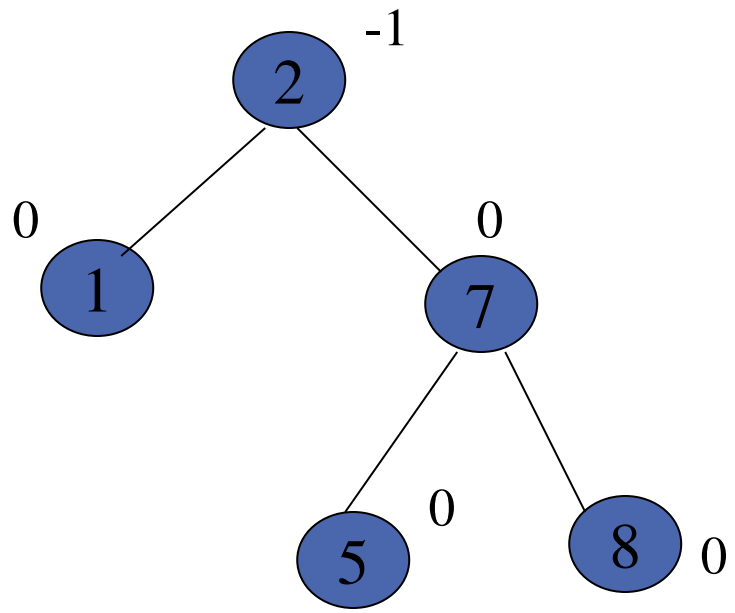

```
avl_node* avlTree:: LR_rotation (avl_node *parent)
{
    avl_node *temp = parent->left;
    parent->left = RR_rotation (temp); //calling RR rotation
    return LL_rotation (parent);
    // return root after LL rotation
}
```

RL Rotation



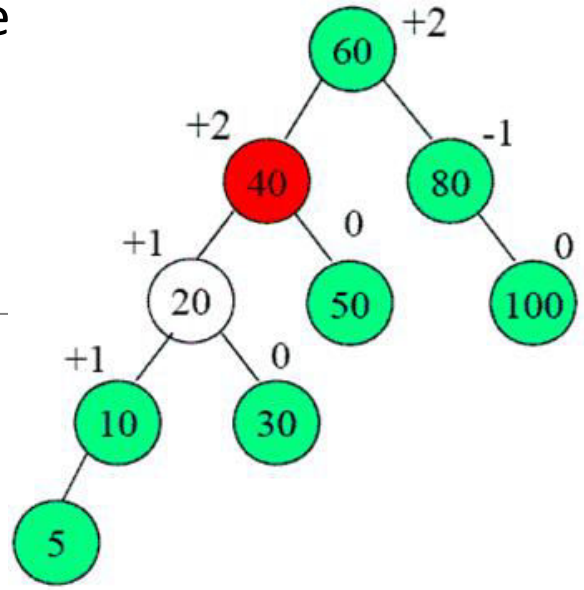

```
avl_node* avlTree:: LR_rotation (avl_node *parent)
{
    avl_node *temp = parent->right;
    parent->right = LL_rotation (temp); // calling LL rotation
    return RR_rotation (parent);
    // returen root after RR rotation
}
```

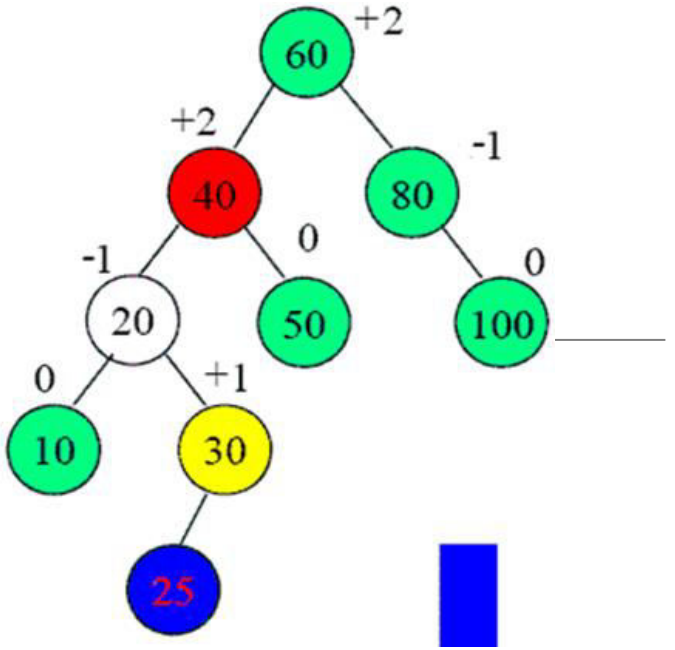
```
int avlTree::diff(avl_node *temp)
{
   int l_height = height (temp->left);
   int r_height = height (temp->right);
   int b_factor= l_height - r_height;
   return b_factor;
}
```

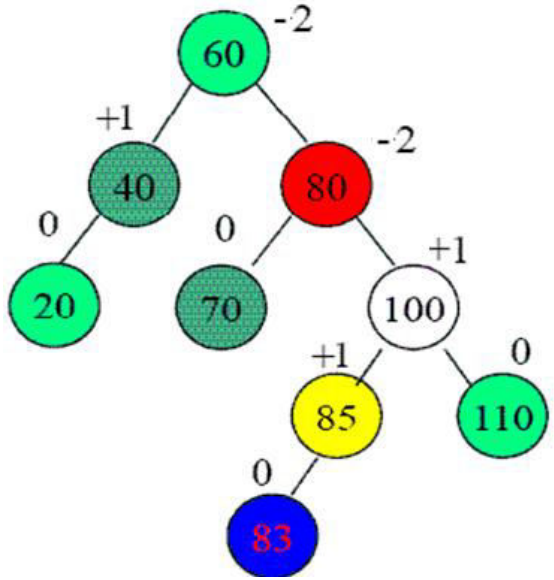


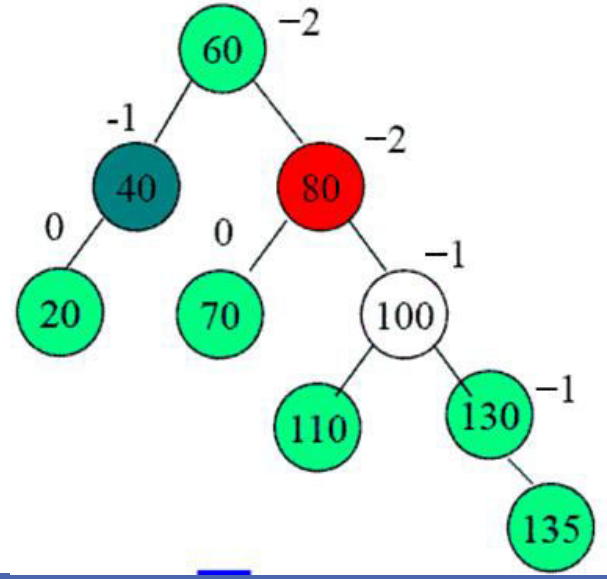

```
int avlTree::height(avl_node *temp)
{
   int h = 0;
   if (temp != NULL)
   {
      int l_height = height (temp->left);
      int r_height = height (temp->right);
      int max_height = max (l_height, r_height);
      h = max_height + 1;
   }
   return h;
}
```

```
avl_node* avlTree::: balance(avl_node
{
   int bal_factor = diff (temp);
   if (bal_factor > 1)
   {
      if (diff (temp->left) > 0)
         temp = LL_rotation (temp);
      else
         temp = LR_rotation (temp);
   }
   else if (bal_factor < -1)
   {
      if (diff (temp->right) > 0)
         temp = RL_rotation (temp);
      else
         temp = RR_rotation (temp);
   }
   return temp;
}
```





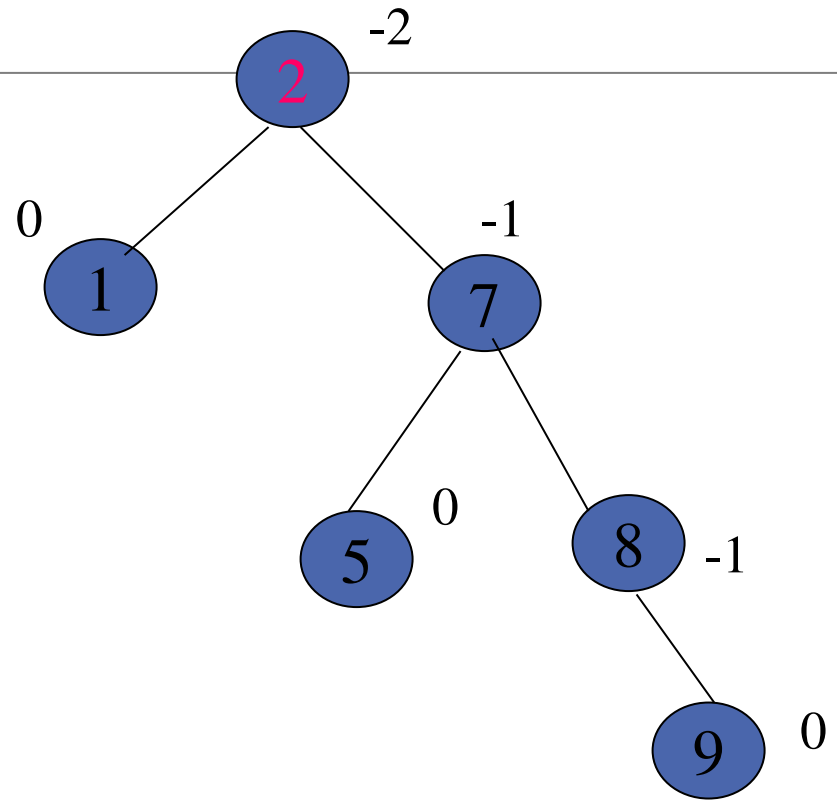




```
void avlTree::insert()          //workhorse to insert
{
 char ch;
  do
   {
avl_node *temp;
        cout<<“Enter word and it’s meaning”;
        cin>>temp->word>>temp->meaning;
 root=insert(root, temp);
        cout<<“Enter your choice”;
        cin>>ch;
        }    while(ch == ‘Y’);
 }
```

```
avl_node* avlTree:: insert( avl_node *root, avl_node *temp)
{
    if (root == NULL)
    {
        root = new avl_node;
        root->word= temp->word;
        root->meaning= temp->meaning;
        root->left = NULL;
        root->right = NULL;
        return root;
    }
     else if (temp->word< root->data)
    {
        root->left = insert(root->left, temp);
        root = balance (root);
    }
    else if (temp->word >= root->data)
    {
        root->right = insert(root->right, temp);
        root = balance (root);
    }
    return root;
}
```

Insert 9



```
Algorithm display (node *ptr, int level)  // consider level =1
{
    if (ptr!=NULL)
    {       display(ptr->right, level + 1);
        printf("\n");
        if (ptr == root)
        cout<<"Root -> ";
        for (i = 0; i < level && ptr != root; i++)
            cout<<"        ";
        cout<<ptr->data;
        display(ptr->left, level + 1);
    }
}
```

Splay Tree

Introduction

- Splay tree is another variant of binary search tree. In a splay tree, the recently accessed element is placed at the root of the tree. A splay tree is defined as follows..
- Splay Tree is a self - adjusted Binary Search Tree in which every operation on an element rearrange the tree so that the element is placed at the root position of the tree.
- In a splay tree, every operation is performed at root of the tree. All the operations on a splay tree are involved with a common operation called "Splaying".
- Splaying an element is the process of bringing it to the root position by performing suitable rotation operations.
- In a splay tree, splaying an element rearrange all the elements in the tree so that splayed element is placed at root of the tree.
- With the help of splaying an element we can bring most frequently used element closer to the root of the tree so that any operation on those element performed quickly. That means the splaying operation automatically brings more frequently used elements closer to the root of the tree.

Splay Trees

In balanced tree schemes, explicit rules are followed to ensure balance.

In splay trees, there are no such rules.

Search, insert, and delete operations are like in binary search trees, except at the end of each operation a special step called **splaying is done.**

Splaying ensures that all operations take O(lgn) amortized time.

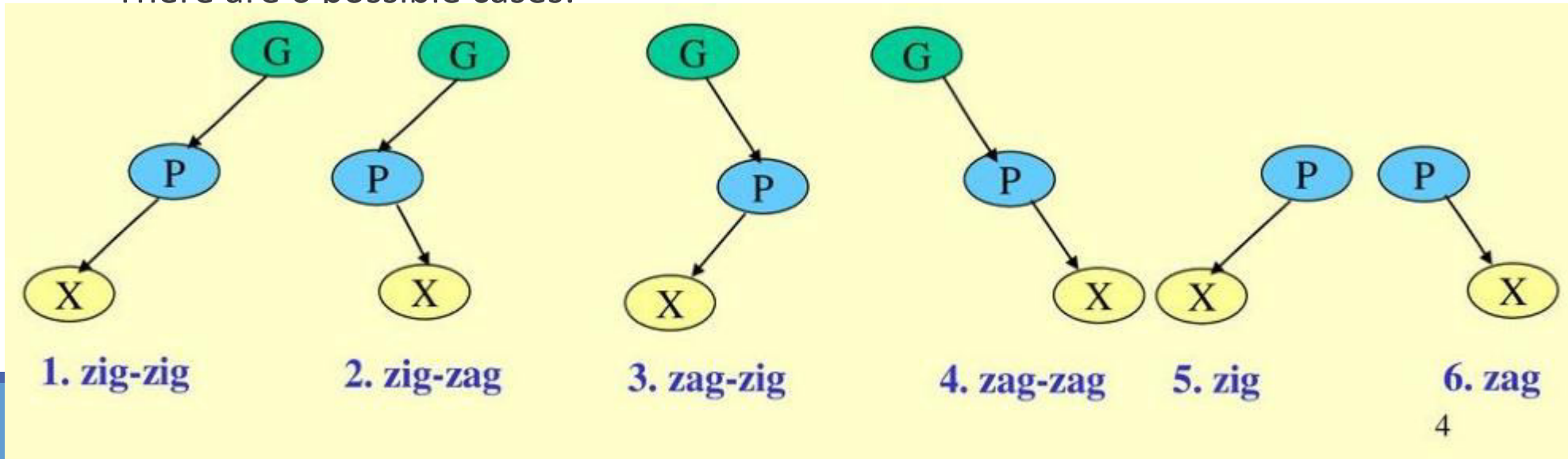
Splay Tree Terminology

- Let X be a non-root node, i.e., has at least 1 ancestor.

- Let P be its parent node.
- Let G be its grandparent node (if it exists)
- Consider a path from G to X: -
- Each time we go left, we say that we "zig“
- Each time we go right, we say that we "zag"
- There are 6 possible cases:

When node X is accessed, apply one of six rotat operations:

- Single Rotations (X has a P but no G)
  zig, zag
- Double Rotations (X has both a P and a G)
  zig-zig, zig-zag zag-zig, zag-zag



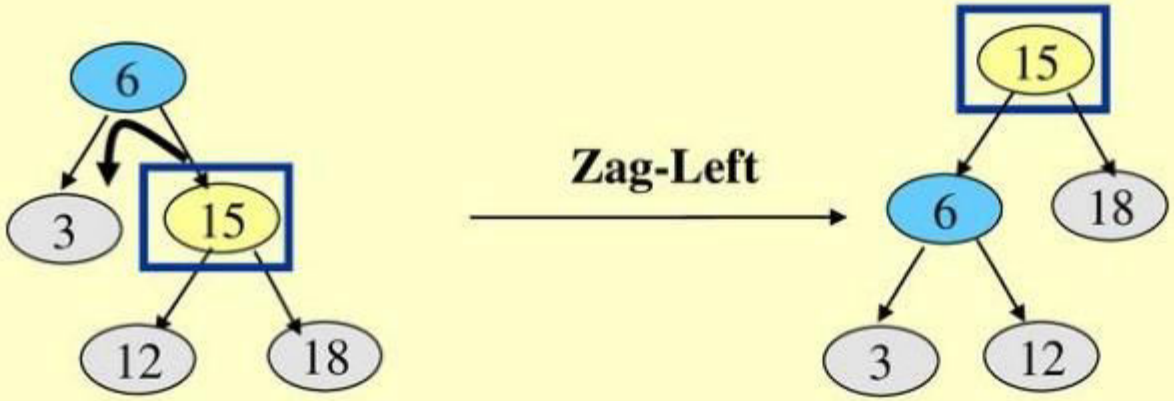
Splay Trees: Zig Operation

- "Zig" is just a single rotation, as in an AVL tree
- Suppose 6 was the node that was accessed (e.g. using Search)

- "Zig-Right" moves 6 to the root.
- Can access 6 faster next time: O(1)
- Notice that this is simply a right rotation in AVL tree terminology.

Splay Trees: Zag Operation

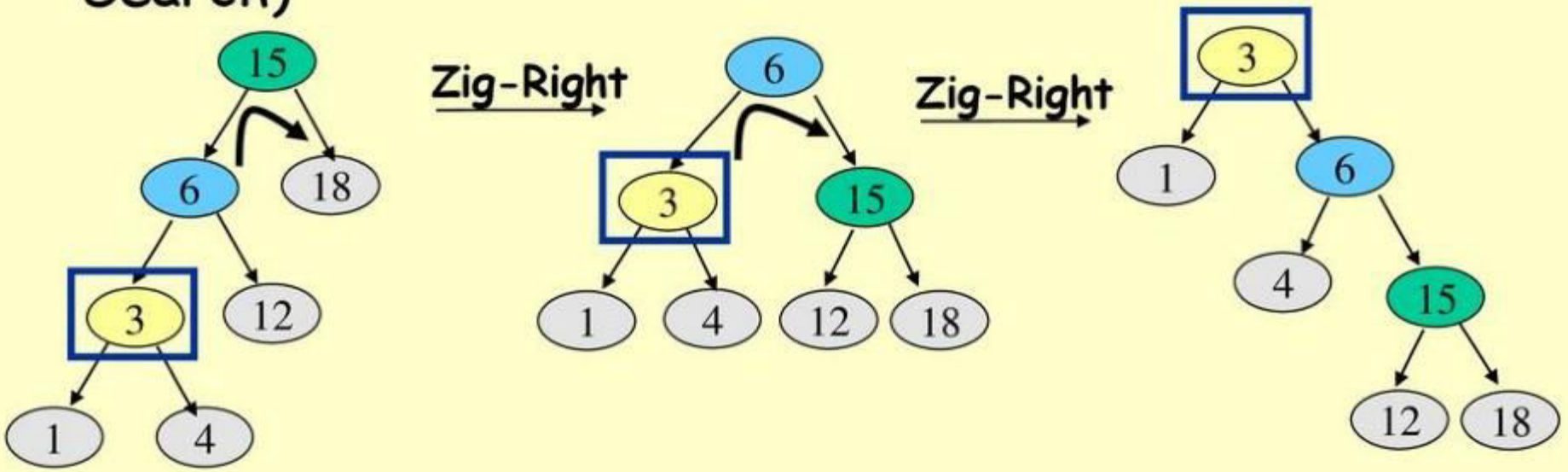
- "Zag" is just a single rotation, as in an AVL tree
- Suppose 15 was the node that was accessed (e.g., using Search)



- "Zag-Left"moves 15 to the root.
- Can access 15 faster next time: O(1)
- Notice that this is simply a left rotation in AVL tree terminology

Splay Trees: Zig-Zig Operation

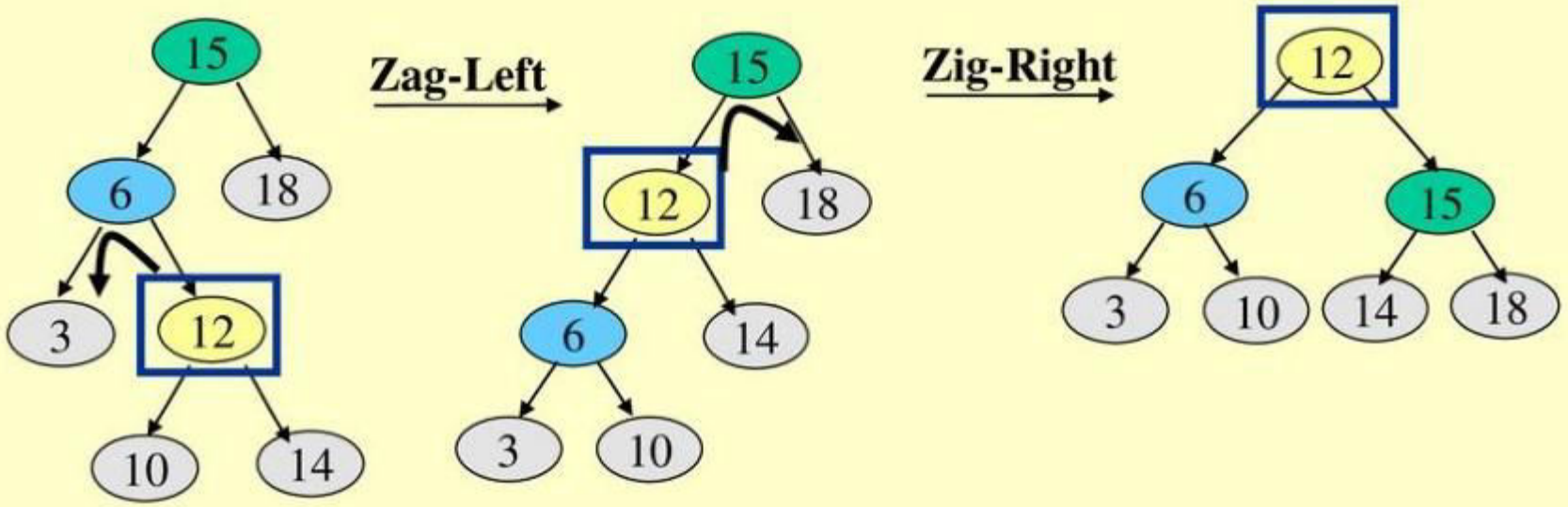
- "Zig-Zig" consists of two single rotations of the same type
- Suppose 3 was the node that was accessed (e.g., using Search)



- Due to "zig-zig" splaying, 3 has bubbled to the top!
- Note: Parent-Grandparent is rotated first.

Splay Trees: Zig-Zag Operation

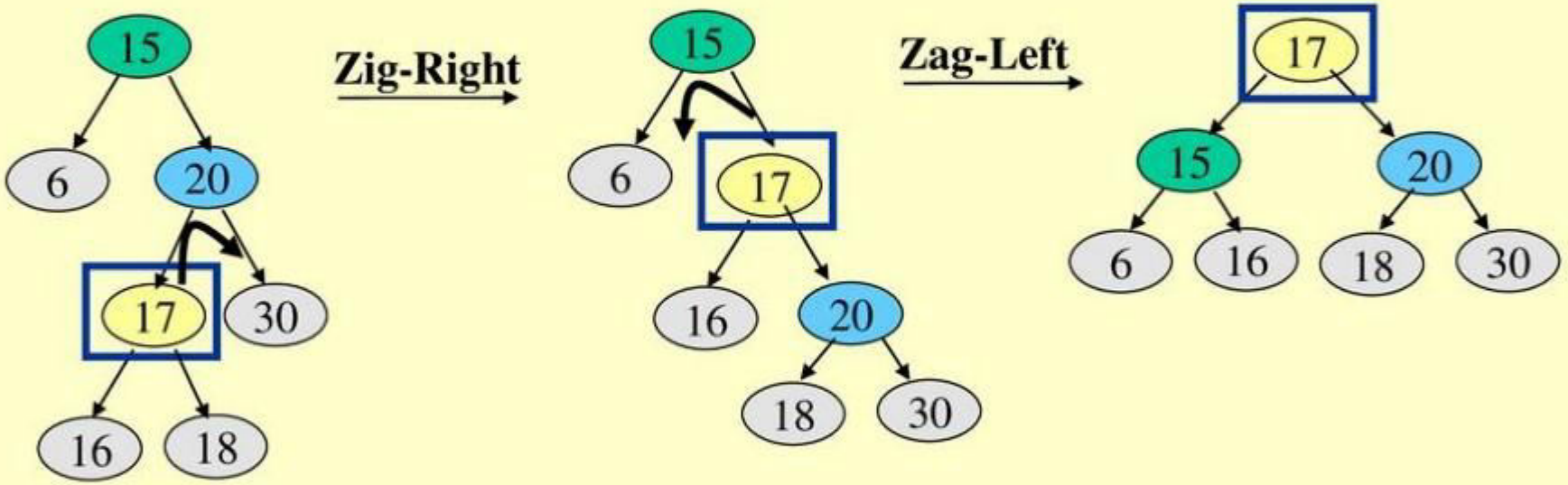
- "Zig-Zag" consists of two rotations of the opposite type
- Suppose 12 was the node that was accessed (e.g., using Search)



- Due to "zig-zag" splaying, 12 has bubbled to the top!
- Notice that this is simply an LR imbalance correction in AVL tree terminology (first a left rotation, then a right rotation)

Splay Trees: Zag-Zig Operation

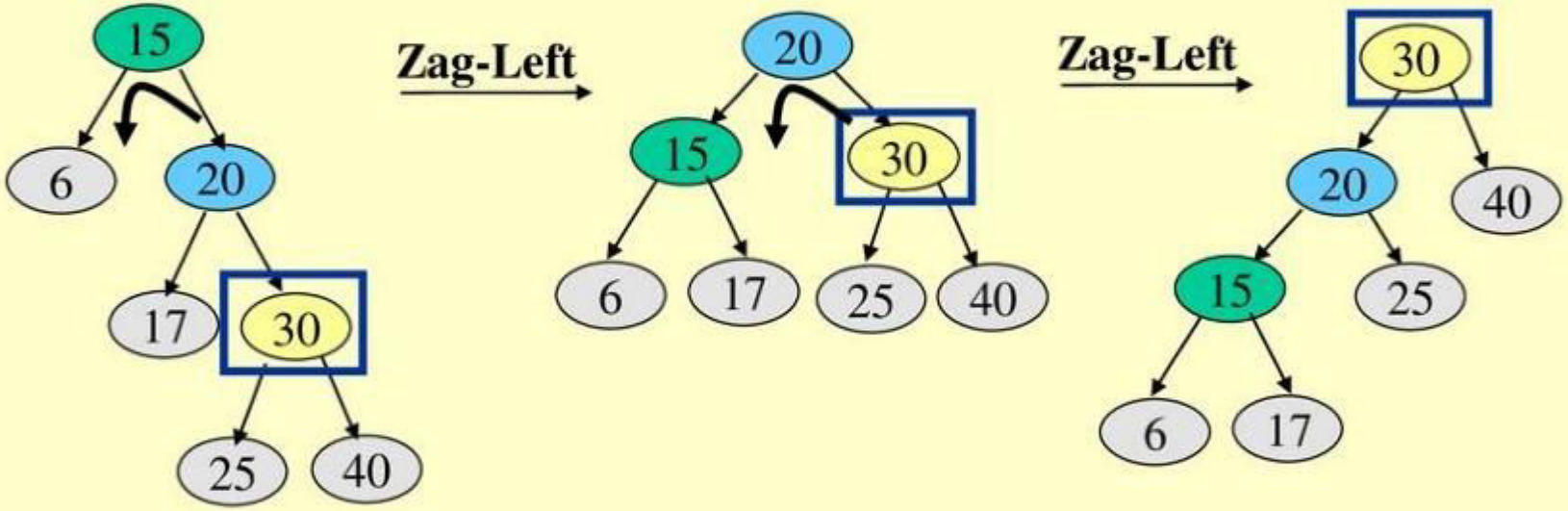
- "Zag-Zig" consists of two rotations of the opposite type
- Suppose 17 was the node that was accessed (e.g., using Search)



- Due to "zag-zig" splaying, 17 has bubbled to the top!
- Notice that this is simply an RL imbalance correction in AVL tree terminology (first a right rotation, then a left rotation)

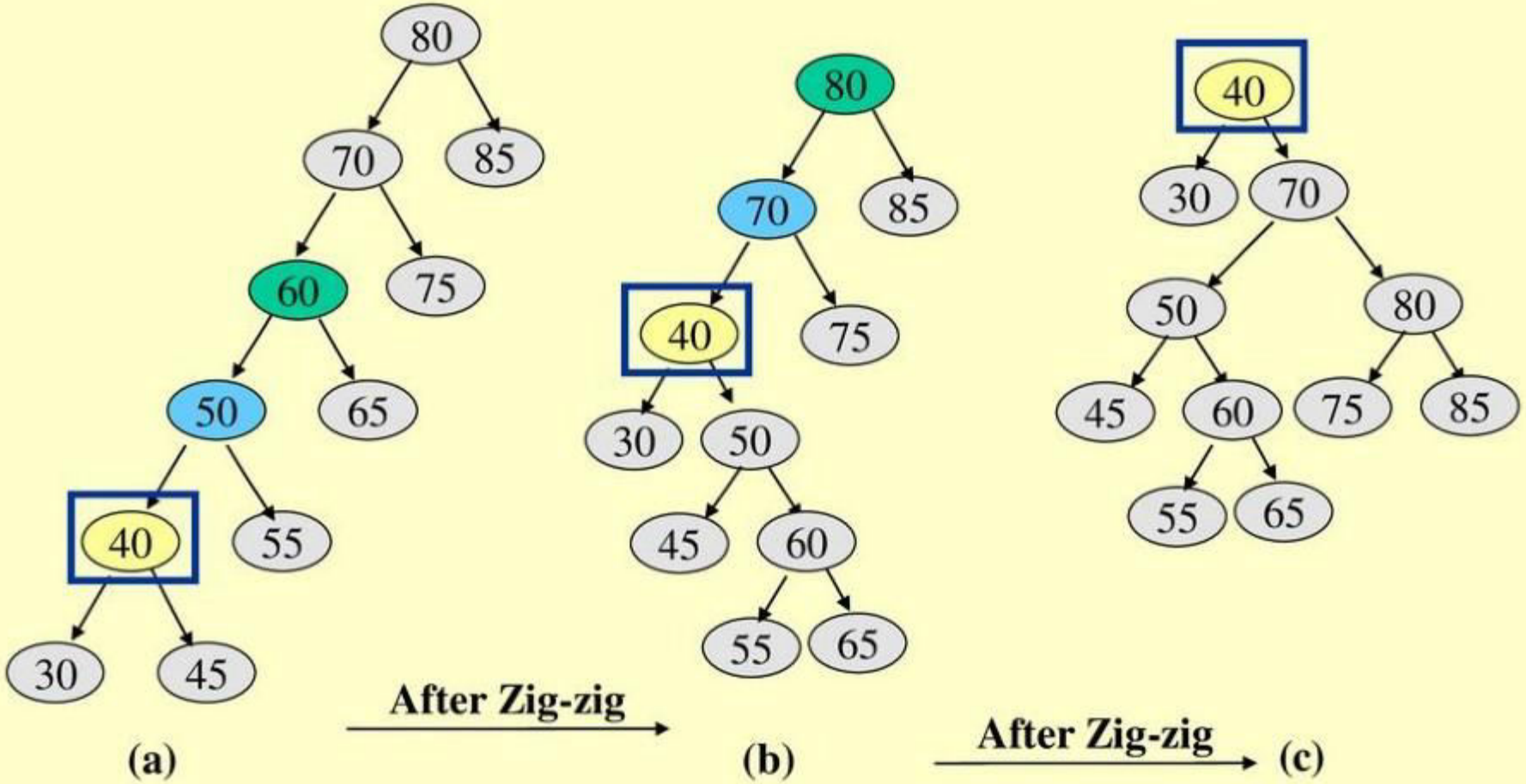
Splay Trees: Zag-Zag Operation

- "Zag-Zag" consists of two single rotations of the same type
- Suppose 30 was the node that was accessed (e.g., using Search)

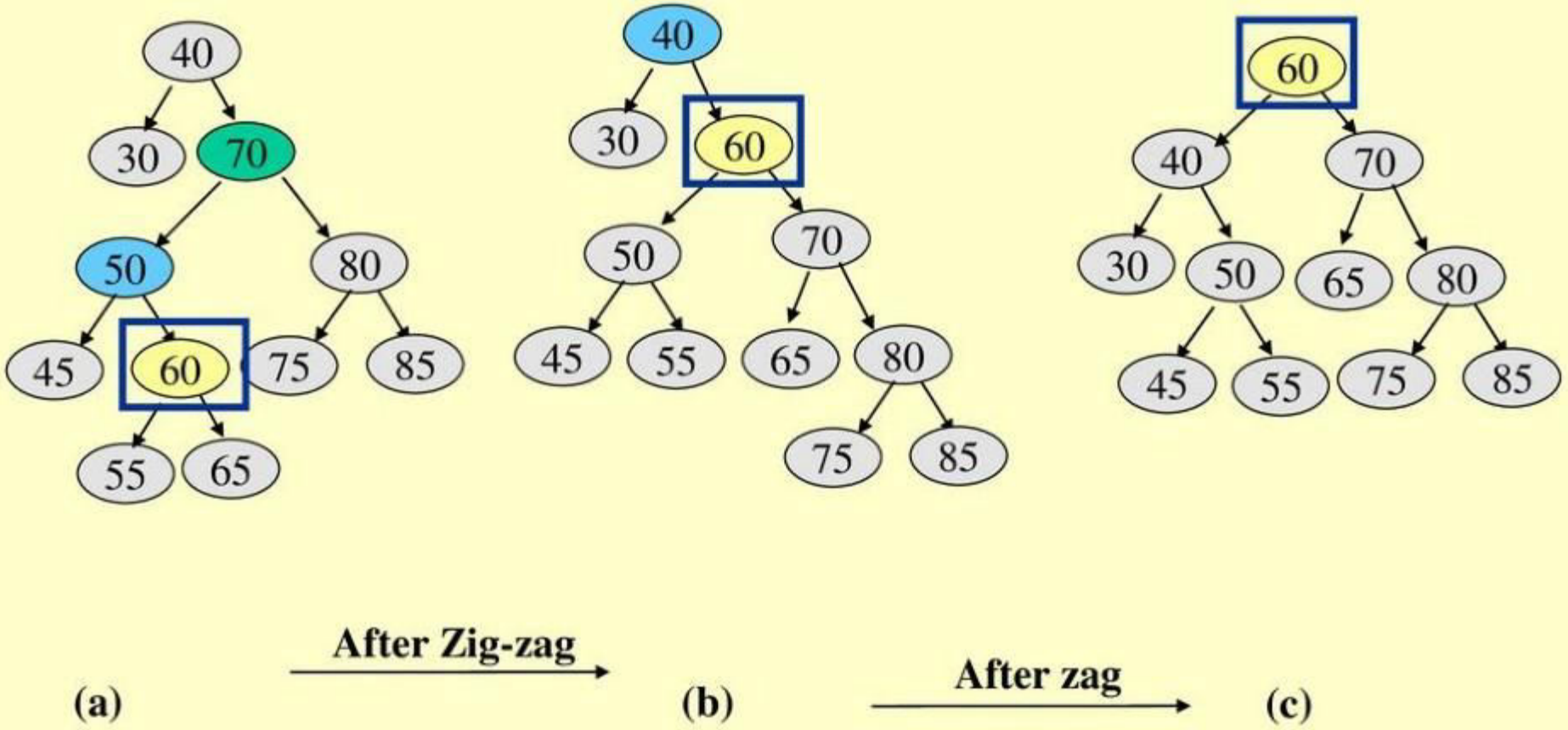


- Due to "zag-zag" splaying, 30 has bubbled to the top!
- Note: Parent-Grandparent is rotated first.

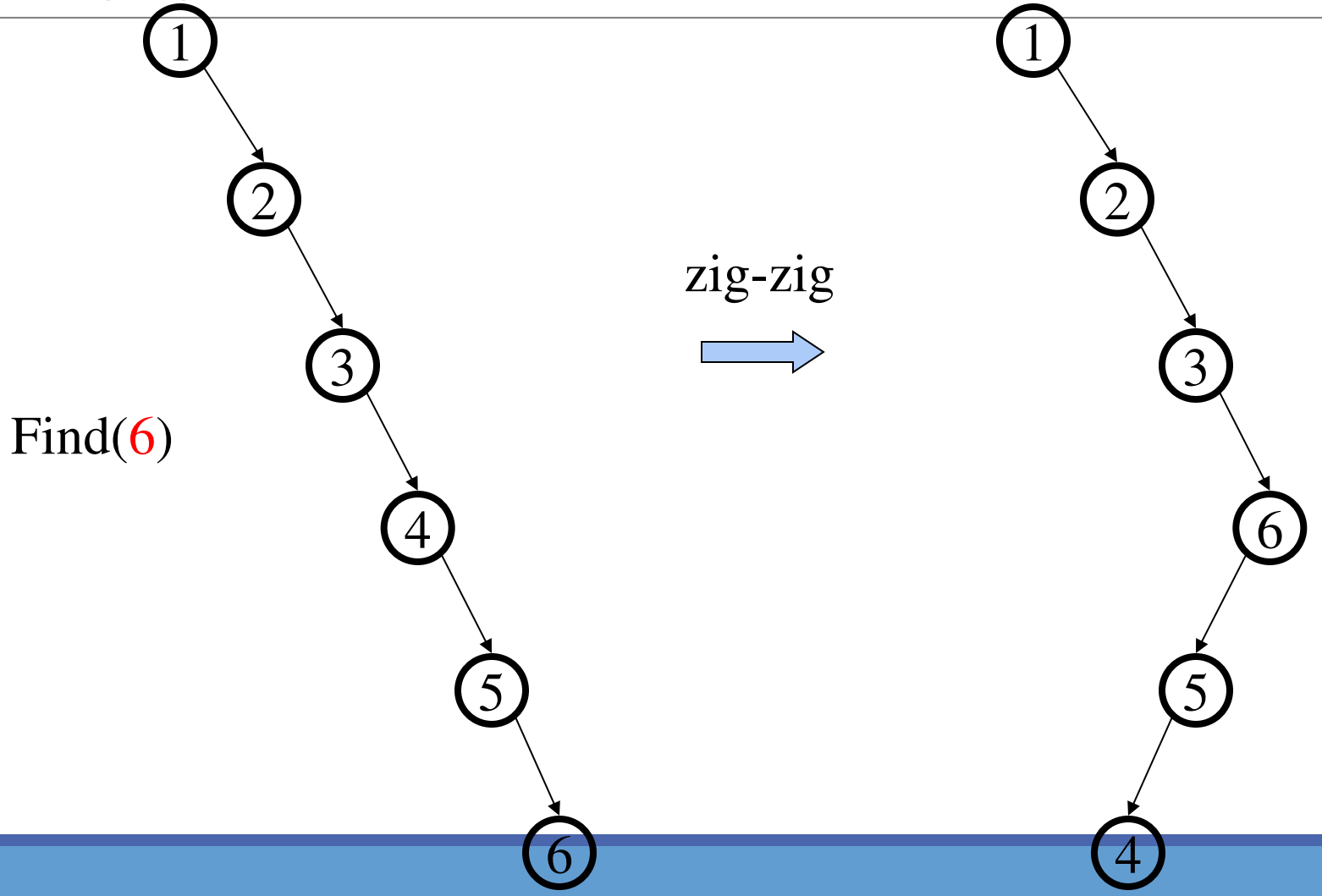
Splay Trees: Example – 40 is accessed



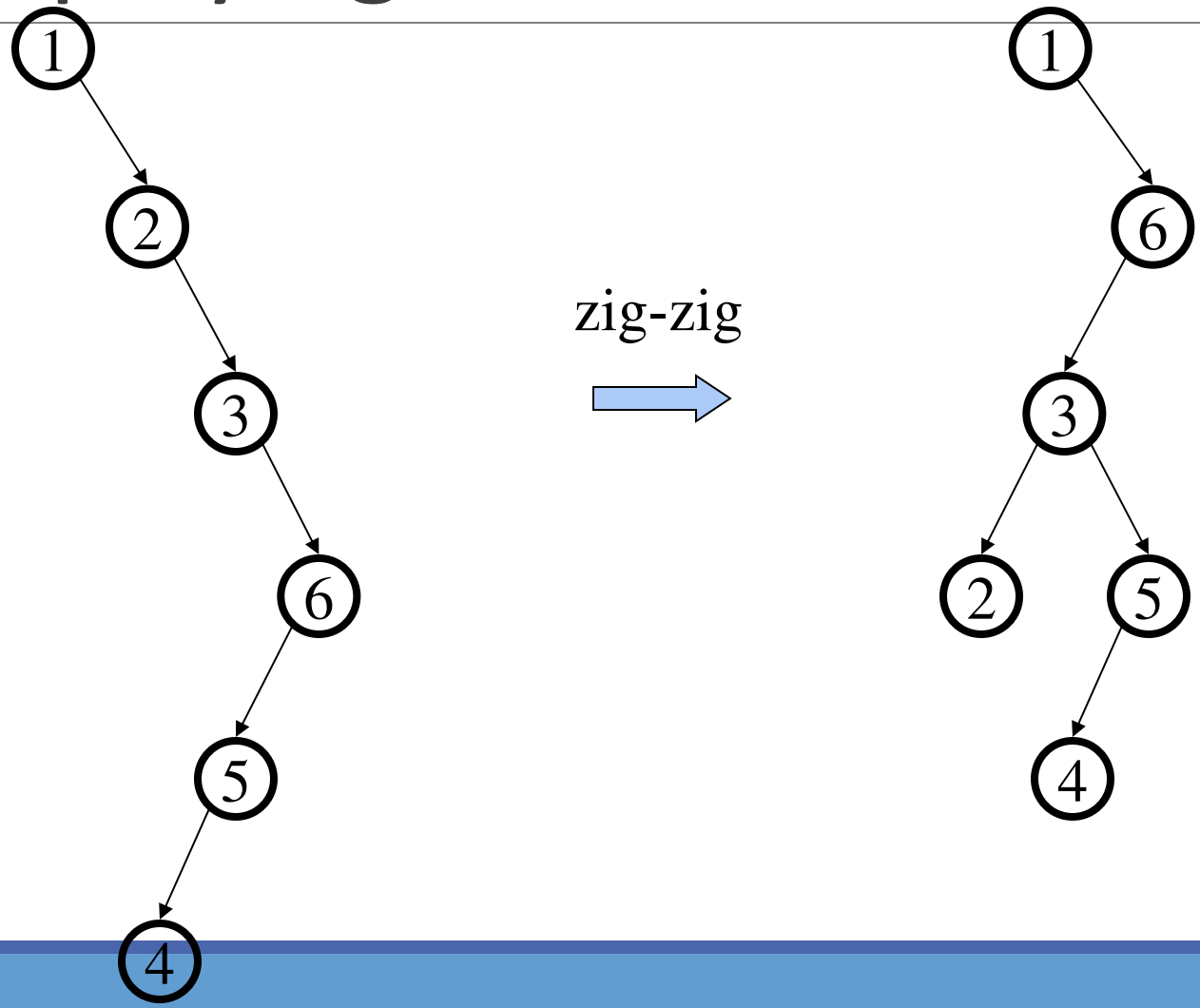
Splay Trees: Example – 60 is accessed



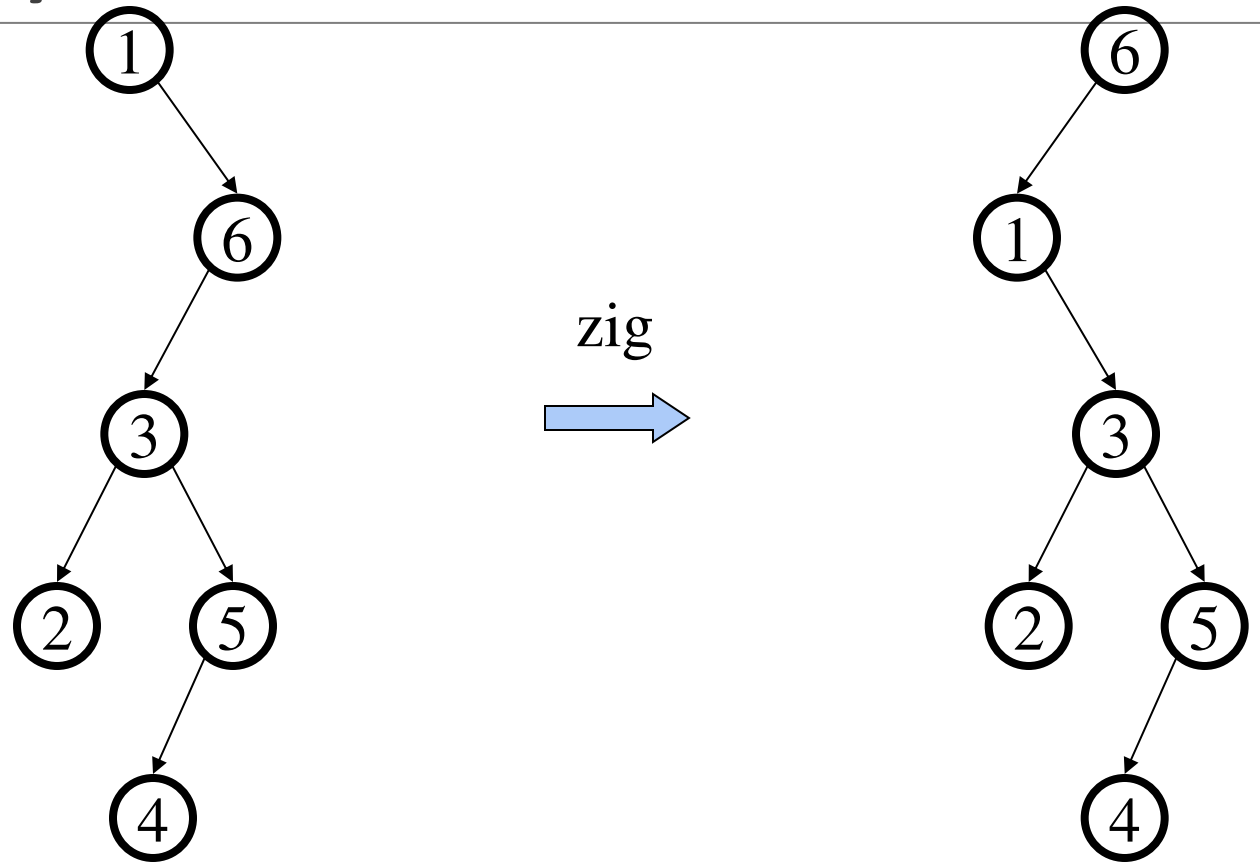
Splaying Example:
Find(6)



… still splaying …

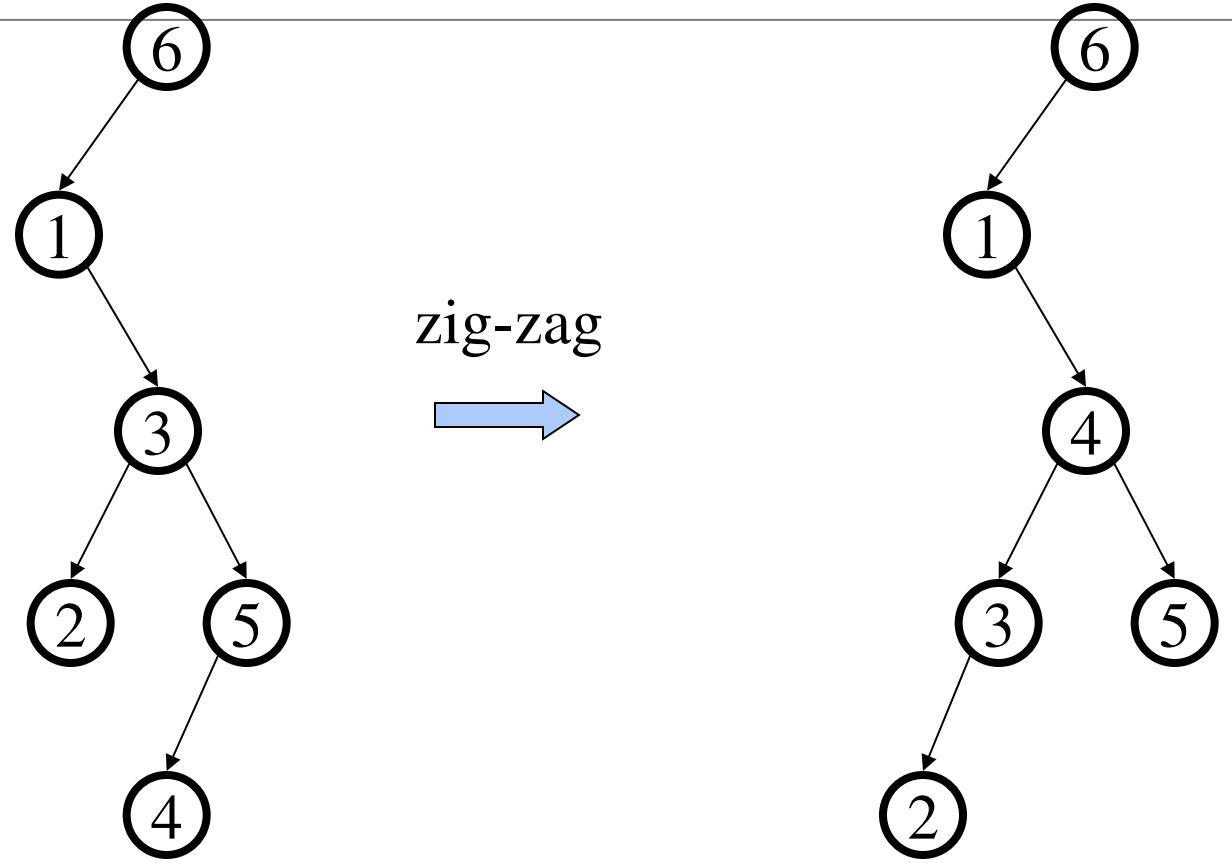


… 6 splayed out!

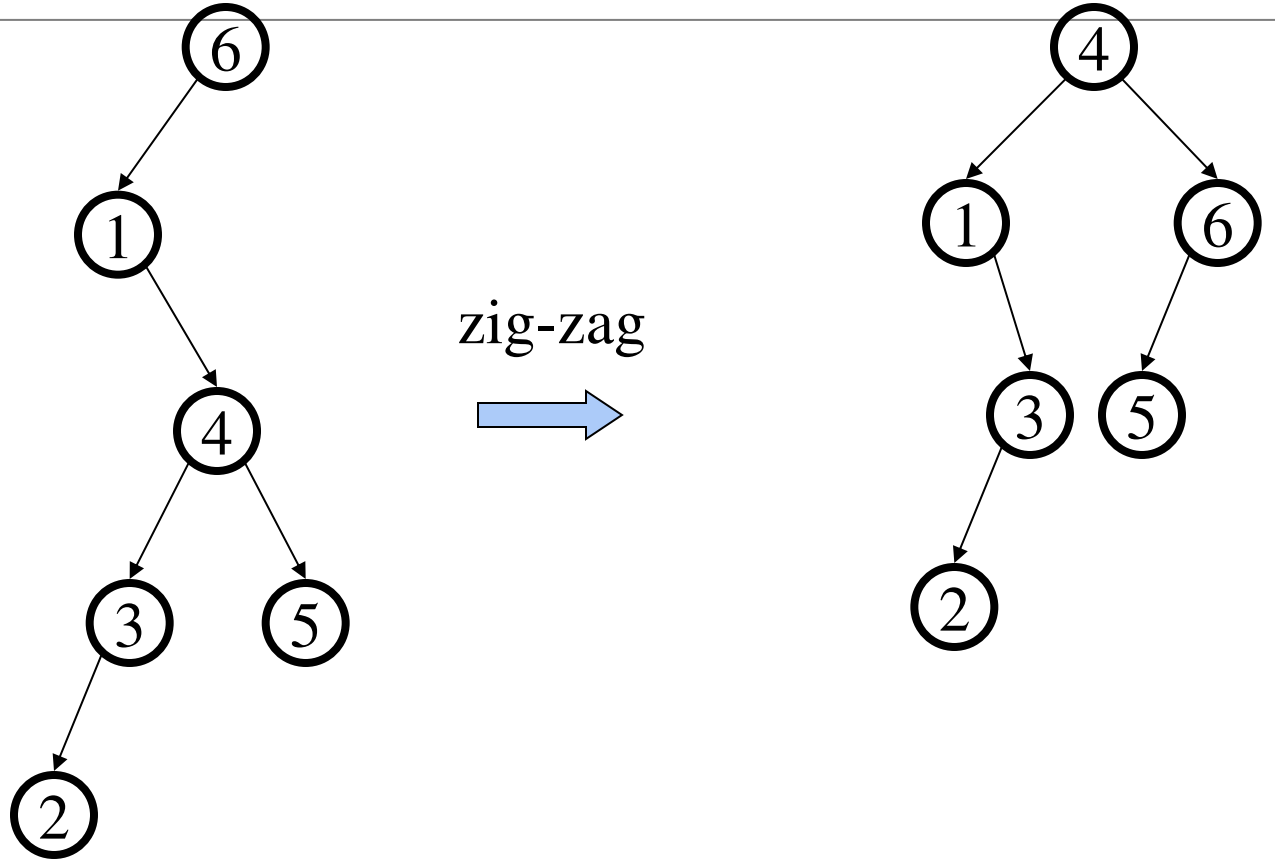


Splay it Again, Sam!
Find (4)

Find(4)



… 4 splayed out!



Insertion Operation in Splay Tree

The insertion operation in Splay tree is performed using following steps...

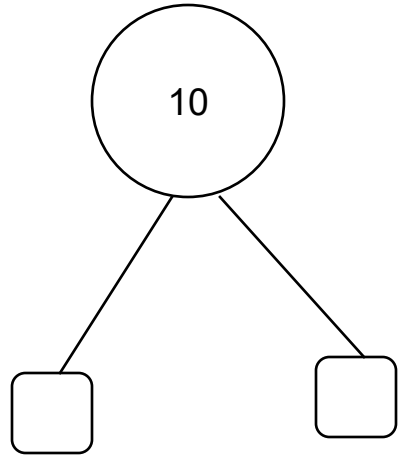
Step 1: Check whether tree is Empty.

Step 2: If tree is Empty then insert the **newNode** as Root node and exit from the operation.

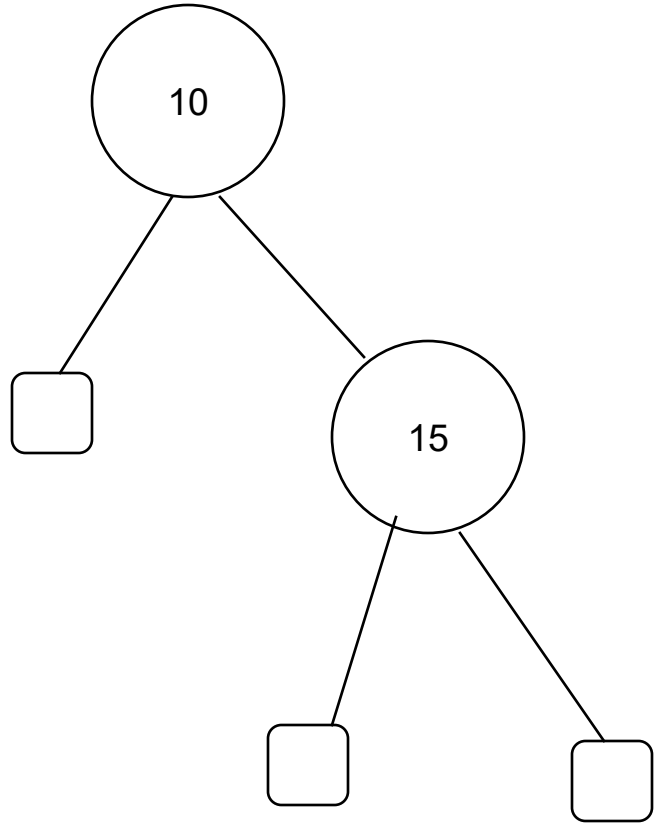
step 3: If tree is not Empty then insert the newNode as a leaf node using Binary Search tree insertion logic.

Step 4: After insertion, **Splay** the **newNode**

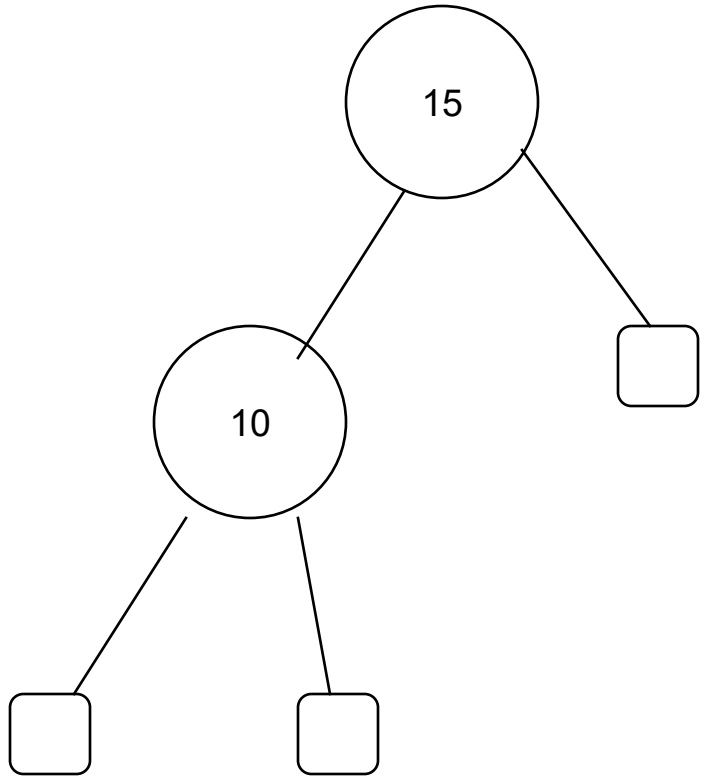
Original tree



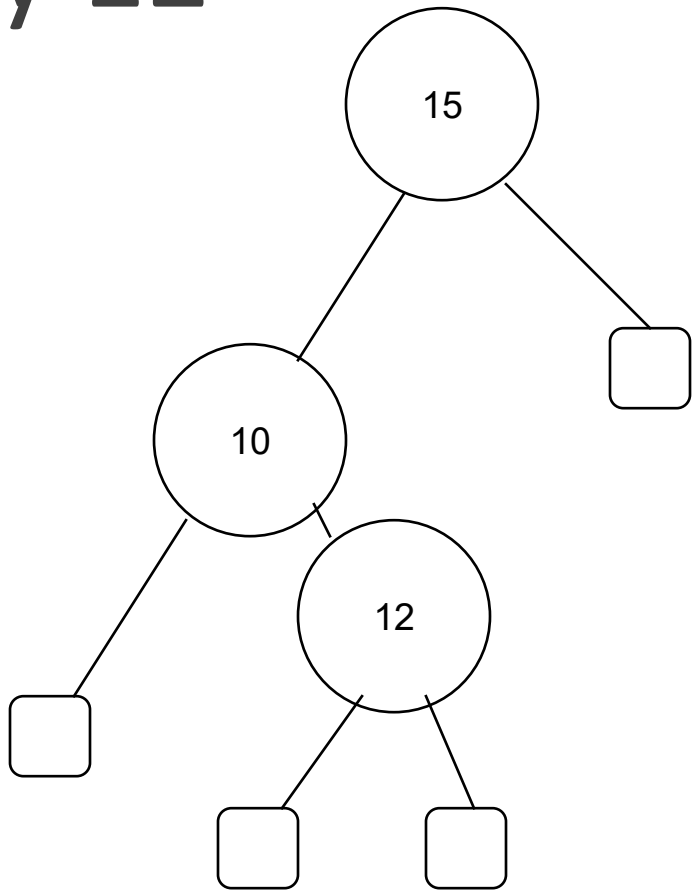
After insert key 15



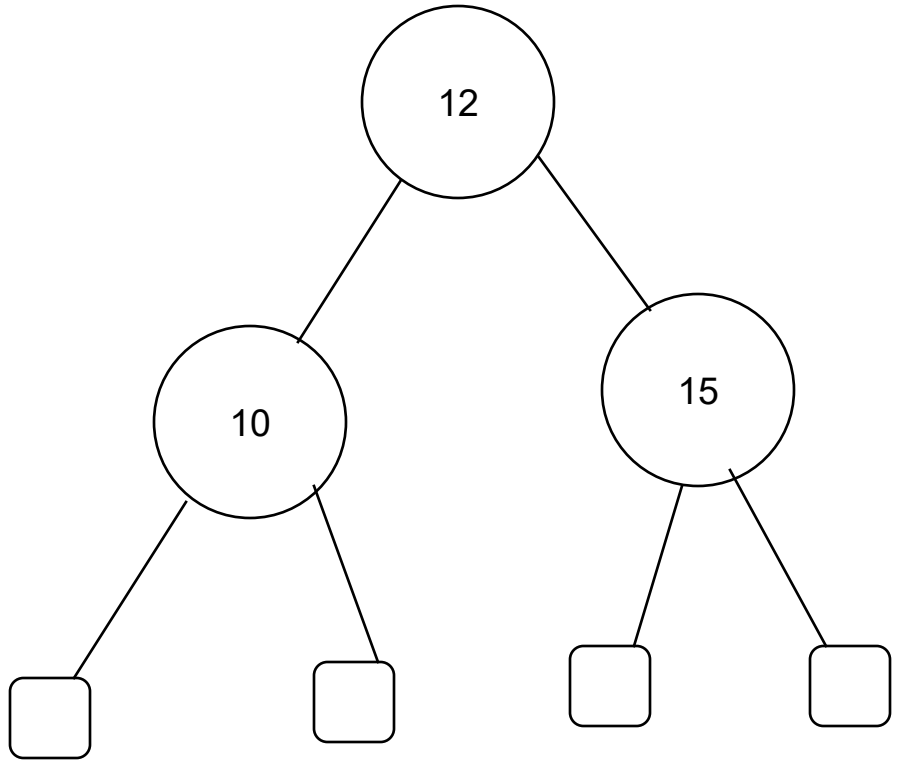
After splaying



After insert key 12



After splaying

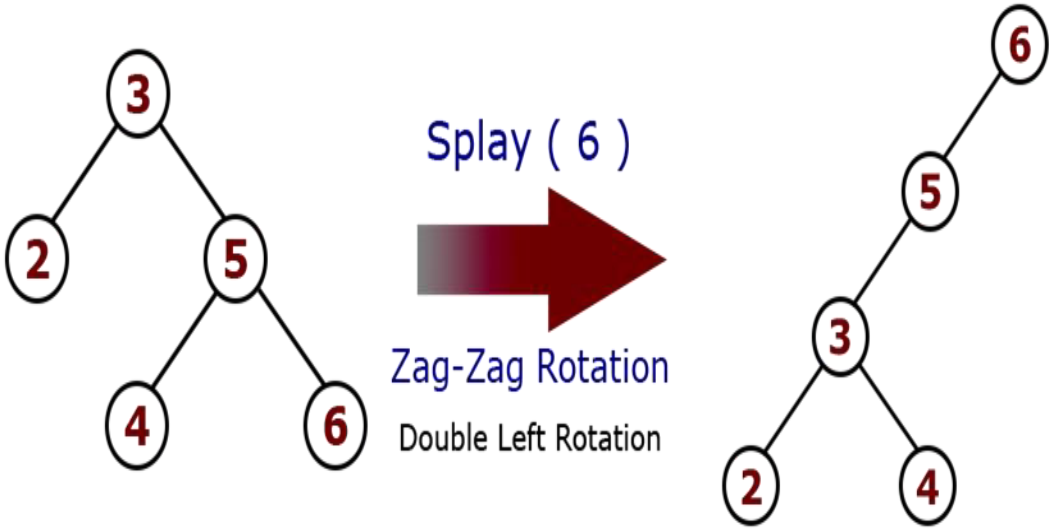


Deletion Operation in Splay Tree

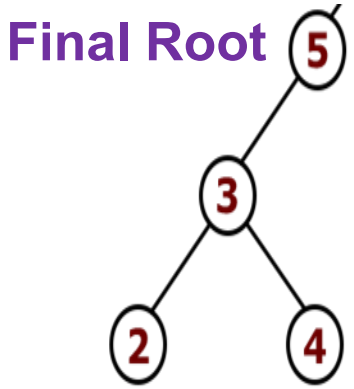
- Before deleting the element first we need to **splay** that node then delete it from the root position then join the remaining tree.
- The tree is split into two trees, the left subtree (Root_left) and the right subtree (Root_right)
- If Root_left is a NULL value, then the Root_right will become the root of the tree. And vice versa.
- But if both Root_left and Root_right are not NULL values, then select the maximum value from the left subtree and make it the new root by connecting the subtrees.

Deletion Operation in Splay Tree

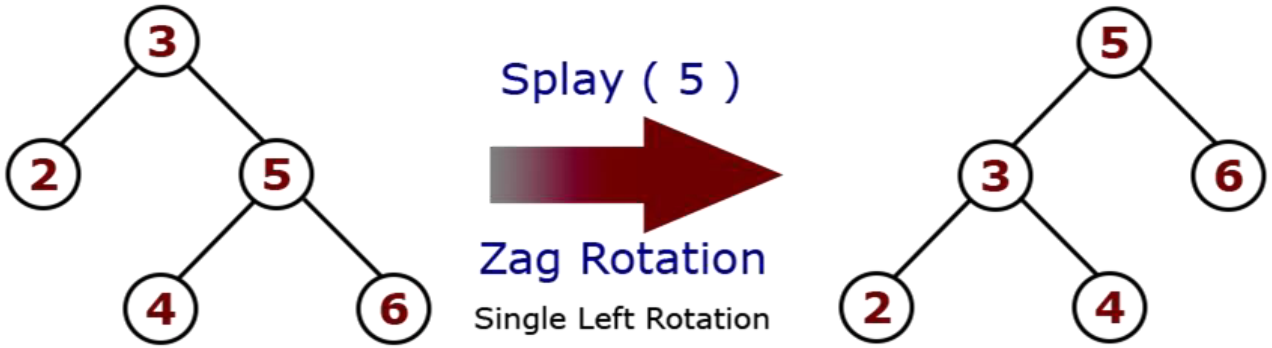
If Root_left is a NULL value, then the Root_right will become the root of the tree. And vice versa.

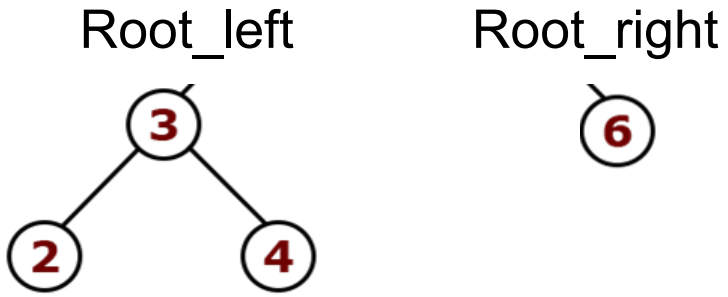


Root_left Root_right = NULL

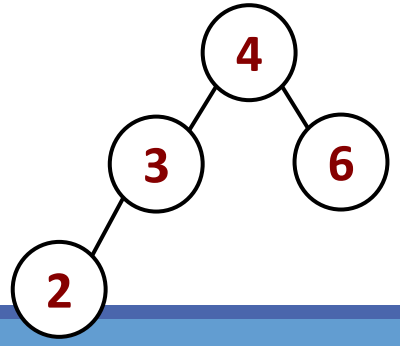


Deletion Operation in Splay Tree





Both Root_left and Root_right are not NULL values, then select the maximum value from the left subtree and make it the new root by connecting the subtrees.



Red-black trees: Overview

- Red-black trees are a variation of binary search trees to ensure that the tree is ***balanced***.
  - Height is $O(\log n)$, where n is the number of nodes.
- Operations take $O(\log n)$ time in the worst case.

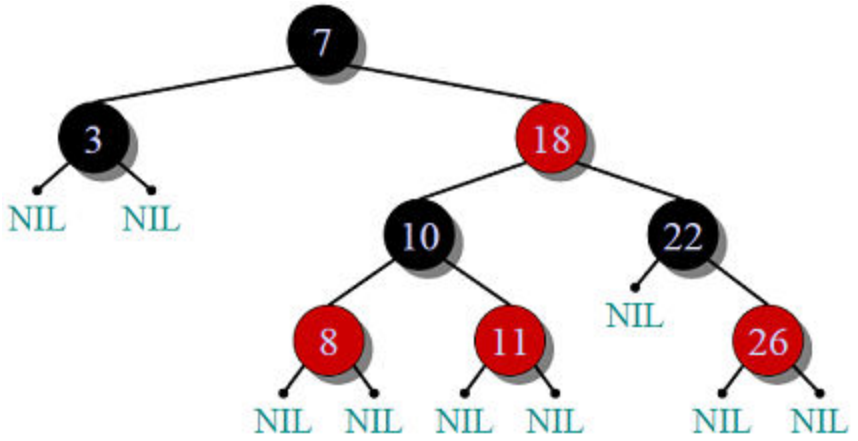
Red-black Tree

- Binary search tree + 1 bit per node: the attribute *color*, which is either **red** or **black**.
- All other attributes of BSTs are inherited:
  - *key*, *left*, *right*, and *p*.

- All empty trees (leaves) are colored black.
  - We use a single sentinel, *nil*, for all the leaves of red-black tree *T*, with *color*[*nil*] = black.
  - The root's parent is also *nil*[*T*].

Red-Black Trees

- Definition: A red-black tree is a binary search tree where:
  - Every node has a color either red or black.
  - Root of tree is always black.
  - There are no two adjacent red nodes (A red node cannot have a red parent or red child).
  - Every path from root to a NULL node has same number of black nodes.



Why Red-Black Trees?

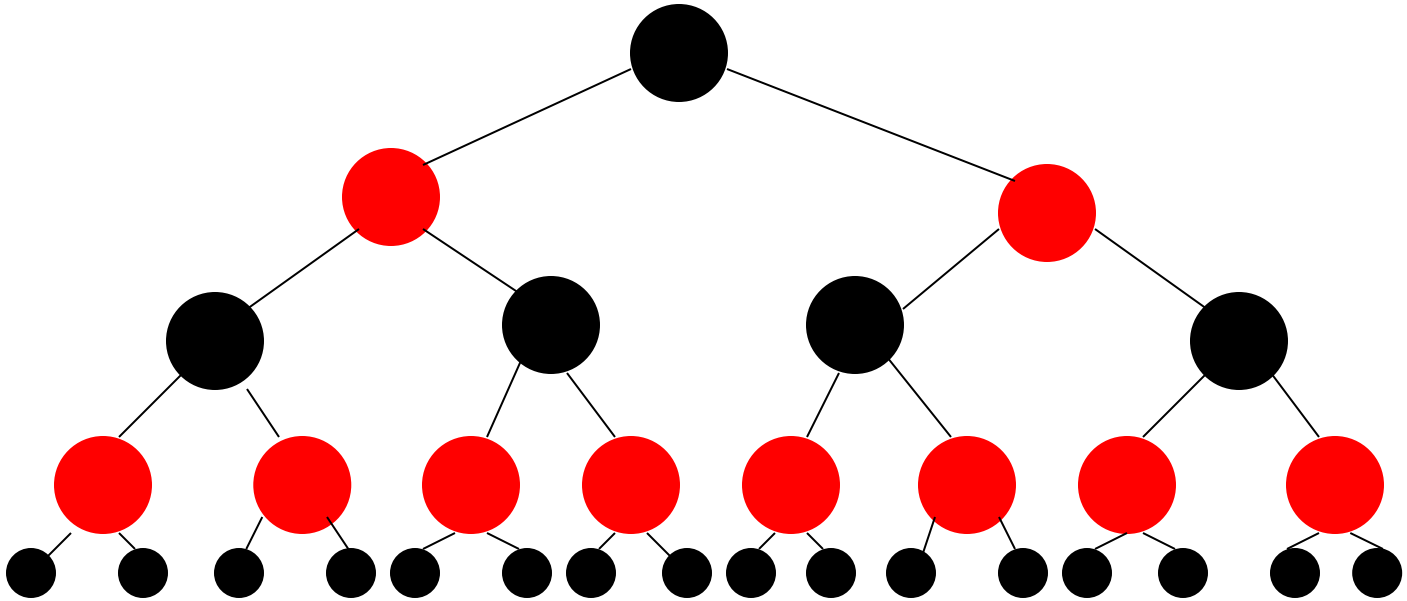
- Most of the BST operations (e.g., search, max, min, insert, delete.. etc) take O(h) time where h is the height of the BST.
- The cost of these operations may become O(n) for a skewed Binary tree.
- If we make sure that height of the tree remains O(logn) after every insertion and deletion, then we can guarantee an upper bound of O(logn) for all these operations.
- The height of a Red Black tree is always O(logn) where n is the number of nodes in the tree.

Comparison with AVL Tree

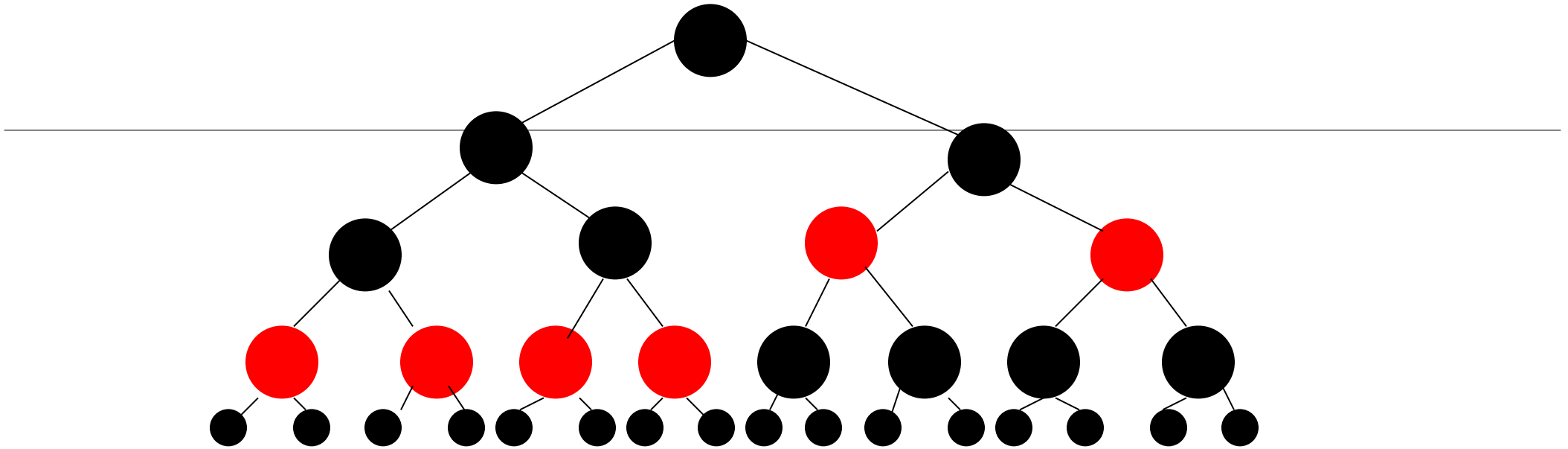
- The AVL trees are more balanced compared to Red Black Trees, but they may cause more rotations during insertion and deletion.
- So if your application involves many frequent insertions and deletions, then Red Black trees should be preferred.
- And if the insertions and deletions are less frequent and search is more frequent operation, then AVL tree should be preferred over Red Black Tree.

The height of red-black tree

Definition: The black-height of a node, x, in a red-black tree is the number of black nodes on any path to a leaf, not counting x.



Black-Height of the root = 2

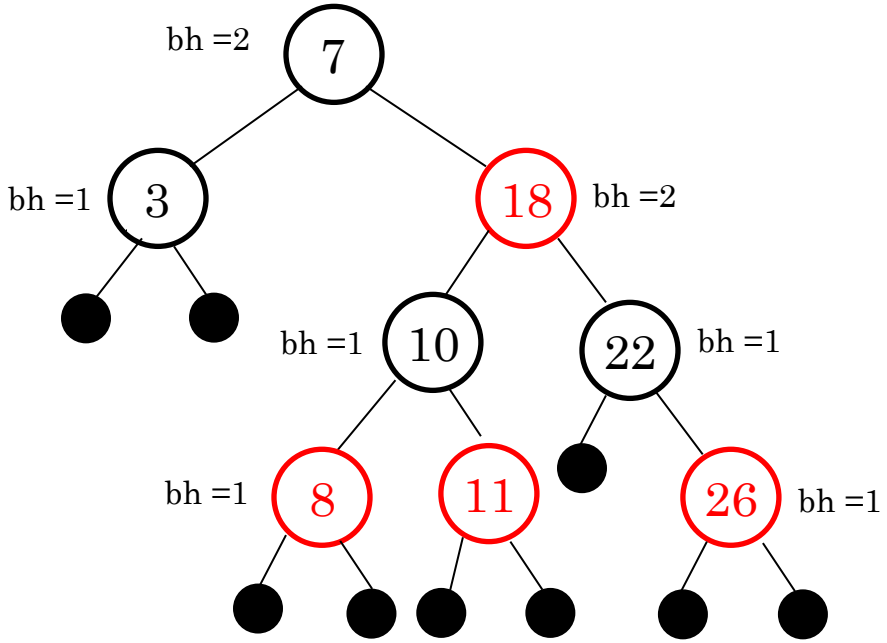


Black-Height of the root = 3

Height-balanced trees

We have to maintain the following balancedness property

- Each path from the root to a leaf contains the same number of black nodes



- In AVL tree insertion, we used rotation as a tool to do balancing after insertion caused imbalance.
- In Red-Black tree, we use two tools to do balancing.

  1) Recoloring
  2) Rotation

- The algorithms has mainly two cases depending upon the color of uncle.

- If uncle is red, we do recoloring.
- If uncle is black, we do rotations and/or recoloring.

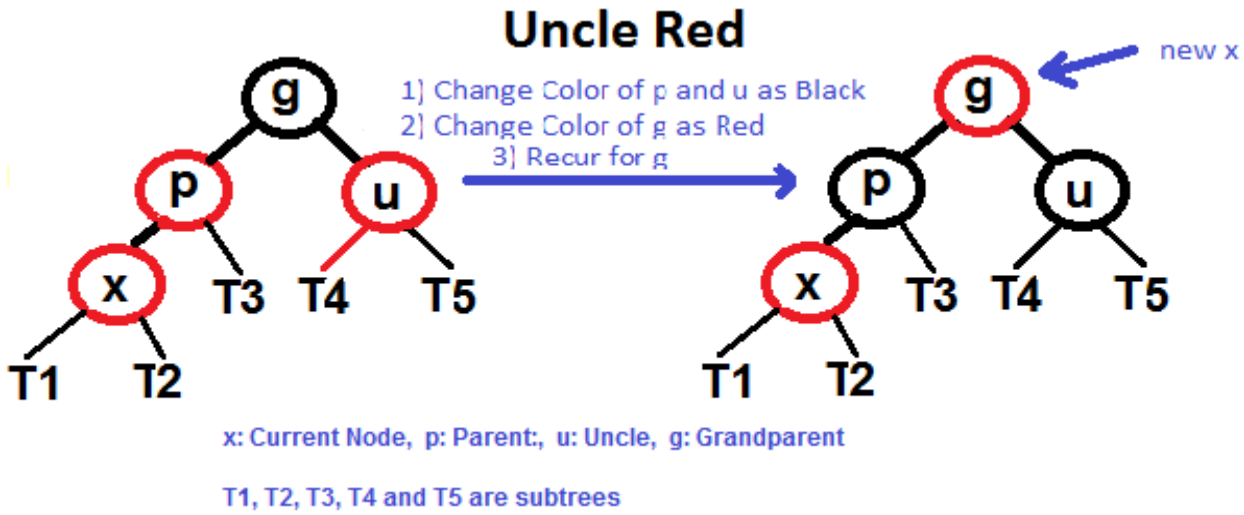
- Color of a NULL node is considered as BLACK.

Let x be the newly inserted node.

1. Perform standard BST insertion and make the color of newly inserted nodes as RED.

2. If x is root, change color of x as BLACK (Black height of complete tree increases by 1).

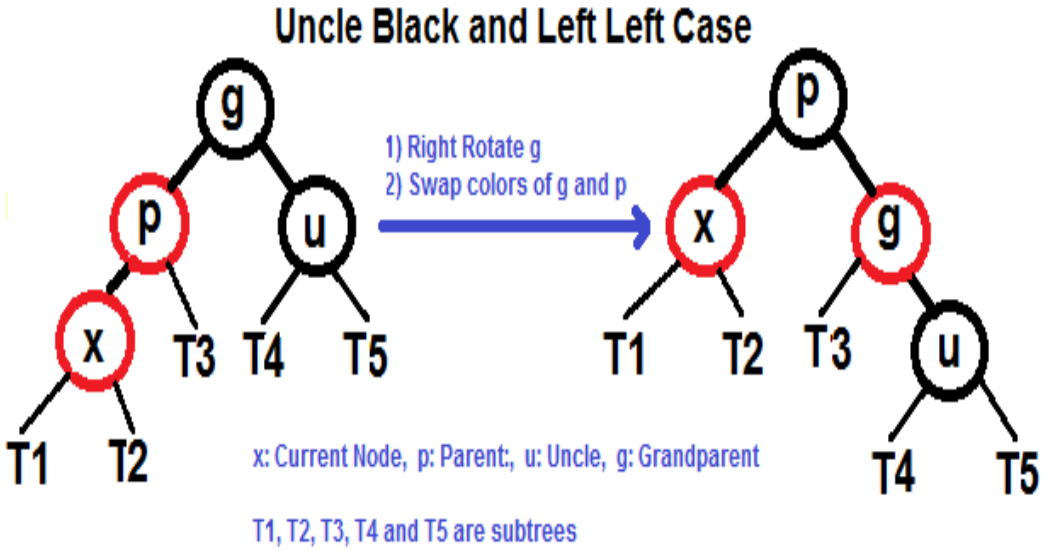
3. Do following if color of x's parent is not BLACK or x is not root.
….**a) If x's uncle is RED** (Grand parent must have been black from property 4)
……..**(i)** Change color of parent and uncle as BLACK.
……..**(ii)** color of grand parent as RED.
……..**(iii)** Change x = x's grandparent, repeat steps 2 and 3 for new x.



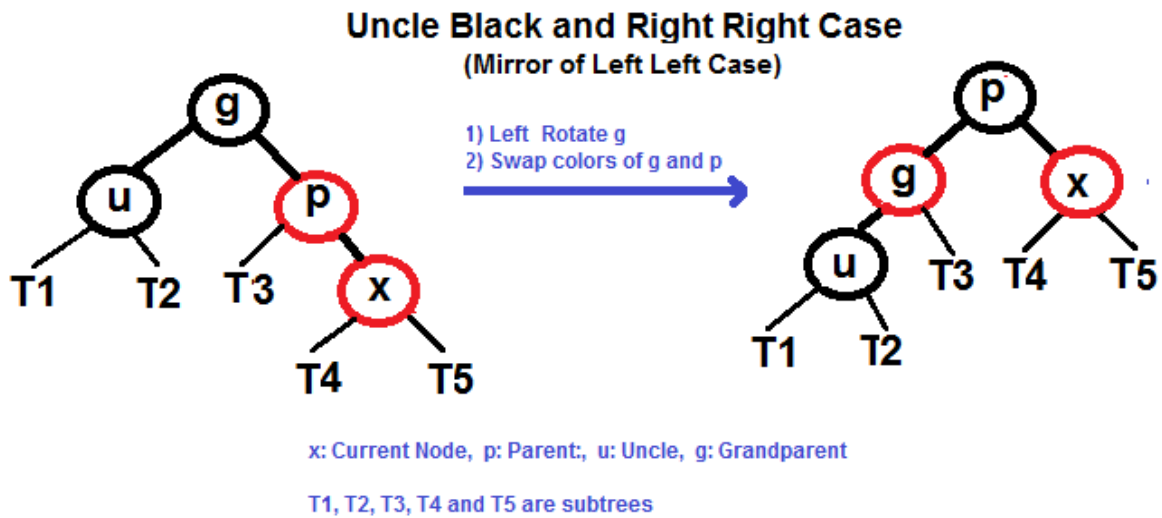
- 3.b) If x's uncle is BLACK, then there can be four configurations for x, x's parent (p) and x's grandparent (g) (This is similar to AVL Tree)
-

  ……..i) Left Left Case (p is left child of g and x is left child of p)
  ……..ii) Left Right Case (p is left child of g and x is right child of p)
  ……..iii) Right Right Case (Mirror of case a)
  ……..iv) Right Left Case (Mirror of case c)

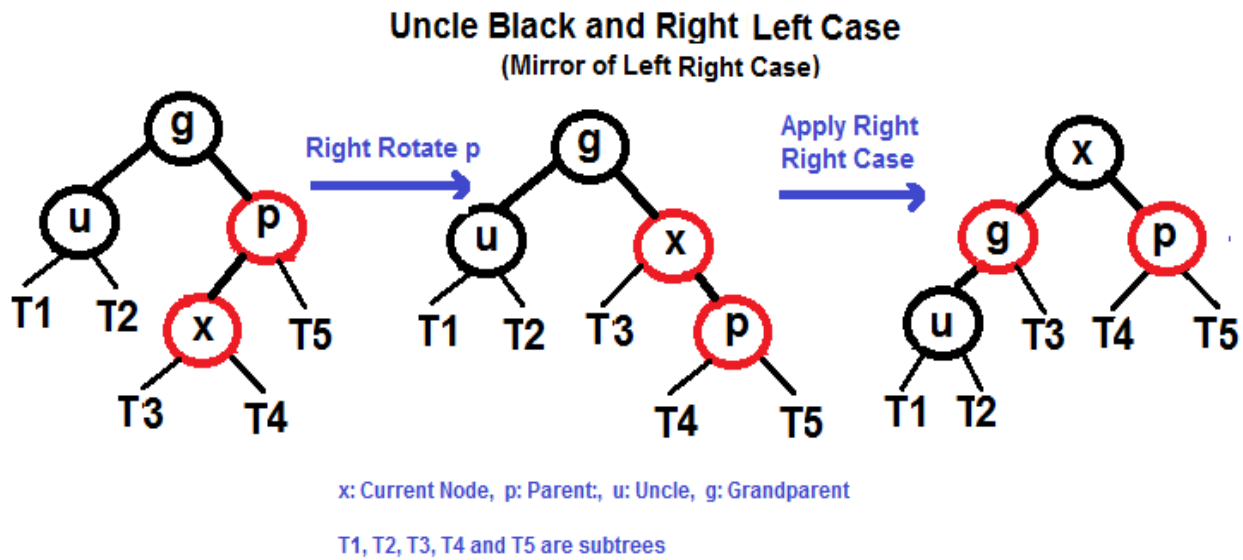
- Following are operations to be performed in four subcases when uncle is BLACK.
- **All four cases when Uncle is BLACK**
- **Left Left Case (See g, p and x)**



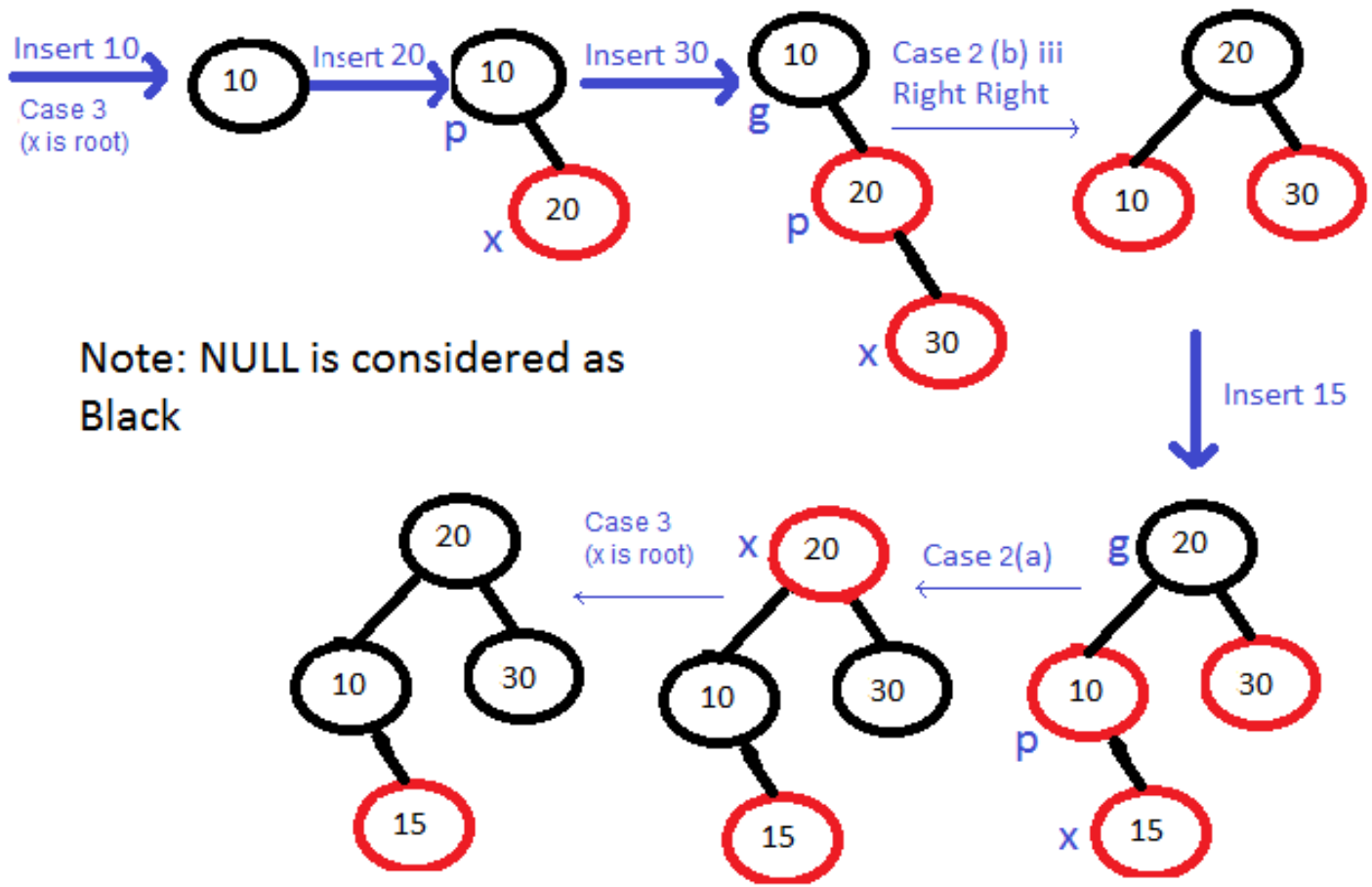
- **Right Right Case (See g, p and x)**



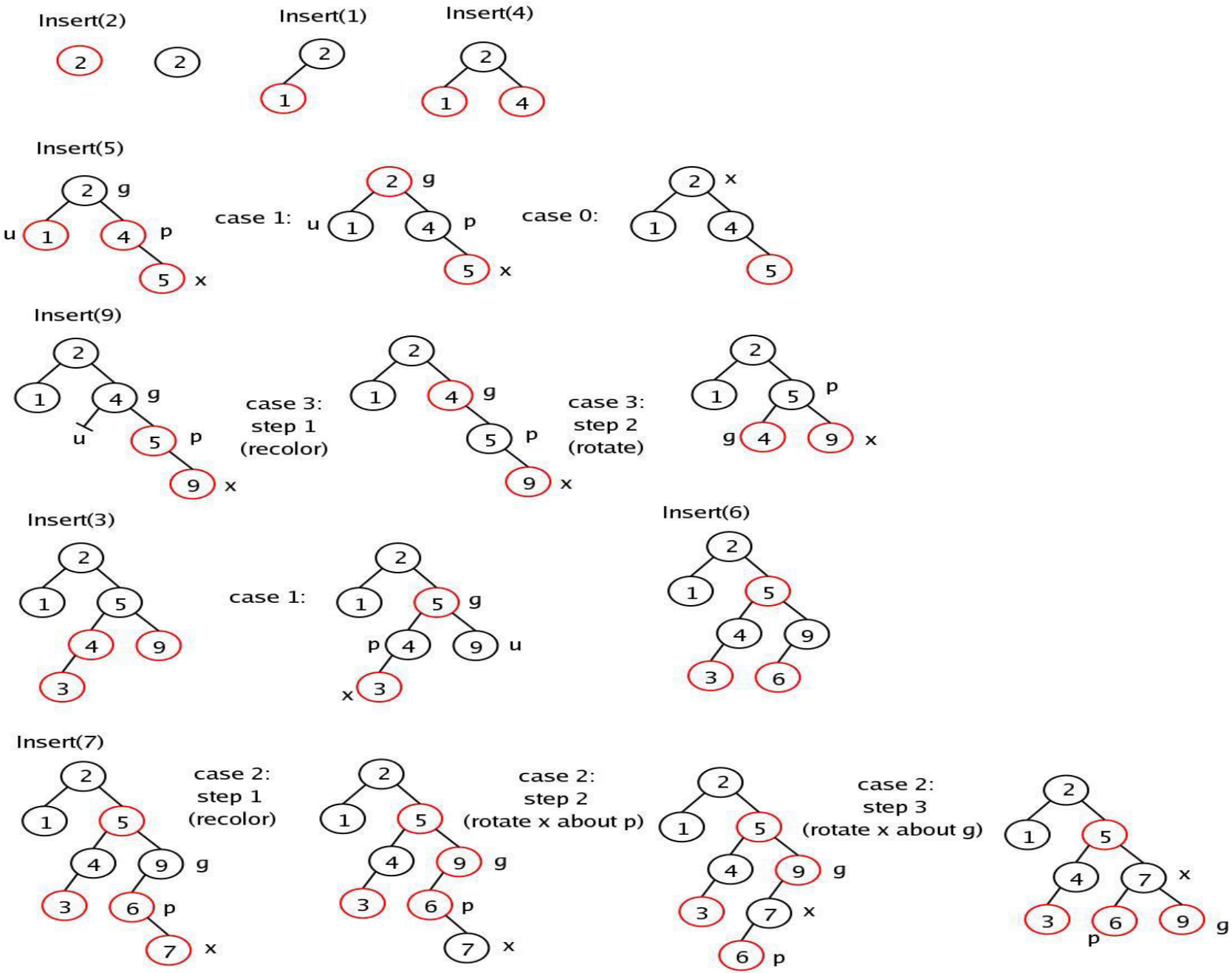
- **Right Left Case (See g, p and x)**



Insert 10, 20, 30 and 15 in an empty tree



Insert 2, 1, 4, 5, 9, 3, 6, 7



Problems

- What red-black tree property is violated in the tree below? How would you restore the red-black tree property in this case?
  - Property violated: if a node is red, both its children are black
  - Fixup: color 7 black, 11 red, then right-rotate around 11