DISCRETE MATHEMATICS ASSIGNMENT 2

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DIV: CS-D

ROLL NO : 37

1. Suppose S is any set of $n + 1$ natural numbers among $\{1, 2, 3, \ldots, 2n\}$.
   (a) Show that S must have two distinct numbers a, b such that $a|b$
   (b) Show that S must have two distinct numbers a, b such that $gcd(a, b) = 1$

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Solution 2.1.

a.) To show that S must have two distinct numbers a, b such that $a | b$. i.e one is multiple of other.

$\Rightarrow$ For each odd number $\alpha = 2k - 1$ $k = 1 \ldots n$, let C_α be the set of elements x in S such that $x = 2^i \alpha$ for some i. The sets $C_1, C_2, C_3 \ldots \ldots C_{2n-1}$ are a classification of S into n classes. By the pigeon hole principle, given $n+1$ elements of S, at least two of them will be in the same class. But any two elements of the same class C_α verify that one is a multiple of the other one. Hence Proved.

b.) To prove that S must have two distinct numbers a, b such that $\gcd(a, b) = 1$ i.e. they are relatively prime.

$\Rightarrow$ We divide the set into n classes $\{1, 2\}, \{3, 4\} \ldots \{2n-1, 2n\}$. By the pigeonhole principle, given $n+1$ elements at least two of them will be in the same class $\{2k-1, 2k\}$ $(1 \leq k \leq n)$. But $2k-1$ and $2k$ are relatively prime because their difference is 1. Hence proved.

2. State and Prove divisibility tests for primes 13, 41 (By divisibility test we mean an algorithm which is significantly simpler than carrying out actual division, e.g. divisibility test of 9 is: number is divisible by 9 iff sum of digits of n is divisible by 9, note that we plan to apply divisibility test for large numbers). (Hint: $13|1001, 41|99999$).

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Solution 2.2

Divisibility Test for 13.

For a given number, form alternating sums of blocks of three numbers from the right and moving towards the left. Suppose $(n_1, n_2, n_3, n_4, n_5, n_6 \dots)$ is a number N, then if the number formed by the alternative sum of blocks of 3-3 digits from right to left $(n_1, n_2, n_3, - n_4, n_5, n_6, + \dots)$ is divisible by 13, then the number N is additionally divisible by 13

Example : 2, 453, 674.

Sollution : $674 - 453 + 2 = 223$ is not divisible by 13.

Hence 2, 453, 674 is not divisible by 13.

Divisibility Test for 41.

4 times the last digit and subtract it from the rest of the number

If the answer is divisible by 41 then the number is also divisible by 41. Apply this rule again to the answer if necessary.

eg. Check 2993. $2993 \Rightarrow 299 - 3 \times 4$

$\Rightarrow 299 - 12 = 287$.

$287 \Rightarrow 28 - 7 \times 4 = 28 - 28 = 0$.

Hence 2993 is divisible by 41.

3. Consider a circular track on which there are n petrol pumps $P_1, \ldots, P_n$ located at arbitrary positions. Assume that pump P_i has L_i litters of petrol for $i = 1$ to n. The total quantity of fuel at all the pumps together is just enough to complete a traversal round the track with a car with unit fuel efficiency. Prove that there always exist a pump where one can start with a (unit fuel efficiency) car with empty fuel tank and complete the traversal round the track (in clockwise or anticlockwise direction). Note that car collects all the fuel available at any pump which it encounters during the traversal (assume cars fuel tank is large enough to hold all the fuel available at all the pumps).

Solution:

4. Let $a_i, b_i \in \mathbb{R}$ for $i = 1$ to n. Then Cauchy-Schwartz inequality states that

$$\sum_{i=1}^{n} a_i \cdot b_i \leq \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2} \cdot \sqrt{b_1^2 + b_2^2 + \ldots + b_n^2}$$

Prove Cauchy-Schwartz inequality using strong mathematical induction. (Note: Only inductive proof will be given credit)

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Solution 2.4

Proof of cauchy schwarz inequality by mathematical induction.

Beginning the induction at 1, the $n=1$ case is trivial.

Now,

$$(a_1b_1 + a_2b_2)^2 = a_1^2b_1^2 + 2a_1b_1a_2b_2 + a_2^2b_2^2$$
$$\leq a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_2^2 + a_2^2b_1^2$$
$$= (a_1^2 + a_2^2)(b_1^2 + b_2^2),$$

which implies that inequality holds for $n=2$.

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Assume that inequality holds for an arbitrary integer k i.e.

$$\left(\sum_{i=1}^{k} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right).$$

Using the induction hypothesis, we have

$$\sqrt{\sum_{i=1}^{k} a_i^2} \cdot \sqrt{\sum_{i=1}^{k+1} b_i^2}$$

$$= \sqrt{\sum_{i=1}^{k} a_i^2 + a_{k+1}^2} \cdot \sqrt{\sum_{i=1}^{k} b_i^2 + b_{k+1}^2}$$

$$\geq \sqrt{\sum_{i=1}^{k} a_i^2} \cdot \sqrt{\sum_{i=1}^{k} b_i^2} + |a_{k+1} b_{k+1}|$$

$$\geq \sum_{i=1}^{k} |a_i b_i| + |a_{k+1} b_{k+1}| = \sum_{i=1}^{k+1} |a_i b_i|.$$

It means that inequality holds for $n = k+1$, we thus conclude that inequality holds all natural numbers n.