TISE: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\cdot\psi(x) = E\psi(x)$

Free particle: $V(x) = 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow$ TISE reduces to $\quad -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) = E\psi(x)$

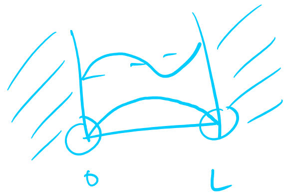
Let $\frac{2mE}{\hbar^2} = k^2 \Rightarrow \boxed{\frac{\partial^2}{\partial x^2}\psi(x) = -k^2\psi(x)}$ —— (i)

$$\psi(x) = Ce^{ikx} + De^{-ikx}$$

$$= C(\cos(kx) + i\sin(kx)) + D(\cos(kx) - i\sin(kx))$$

$$= \underbrace{(C+D)}_{:=B}\cos(kx) + \underbrace{(Ci - Di)}_{:=A}\sin kx = A\sin(kx) + B\cos(kx)$$

PIB: $V(x) = 0$ for $x \in (0, L)$ and $V(x) \to \infty$ if $x \notin (0, L)$



for $x \notin (0, L)$ $\qquad \psi(x) = 0$

for $x \in (0, L)$ $\qquad -\frac{\partial^2}{\partial x^2}\psi(x) = \frac{2mE}{\hbar^2}\psi(x)$

$\psi(x) = A\sin(kx) + B\cos(kx)$ ⤳ Here $k = \sqrt{\frac{2mE}{\hbar^2}}$

But Continuity @ $x = 0$ and $x = L$ demand:

$\psi(0_+) = \psi(0_-) = 0 \Rightarrow A \times 0 + B = 0 \Rightarrow \boxed{B = 0}$

$\psi(L_+) = \psi(L_-) = 0 \Rightarrow A\sin(kL) = 0 \Rightarrow \boxed{kL = n\pi}$ ⤳ Quantization!

$$\psi(x) = A\sin\left(\frac{n\pi x}{L}\right)$$

We need $\int_0^L |\psi(x)|^2 dx = 1 \Rightarrow |A|^2 \int_0^L dx \sin^2\left(\frac{n\pi x}{L}\right) = 1 \Rightarrow |A|^2 \cdot \frac{L}{2} = 1$

$A = \sqrt{2}$ $e^{i\phi}$ … we take $A = \sqrt{2}$ without loss of Generality

$A = \sqrt{\frac{2}{L}}\, e^{i\phi}$ $\rightsquigarrow$ we take $A = \sqrt{\frac{2}{L}}$ without Loss of Generality

doesn't affect the "physics" of the problem

$$\therefore \boxed{\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)} \rightsquigarrow E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = n^2 E_1$$

$$\Psi_n(x,t) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) e^{-iE_n t/\hbar}$$