Theory of Relativity

Gravitation and Cosmology

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Greetings reader, this section aims to give you a heads up on what this Report covers and how will we go about studying some of the basic ideas that were laid out by Albert Einstein in the early 20^{th} century.

The first half of this report basically focuses on assembling the arsenal of Mathematical tools required to understand the famous theory of Relativity, and the application of these ideas starts only after that. The report doesn't do enough justice to most topics in SR but this is because the breadth of GR demands much more work to be done, so it is assumed that the reader is comfortable with basic SR.

Contents

1 Theory of Special Relativity and Flat spacetime

1.1 Inertial frames, Lorentz transformations

Getting straight to the point, we first come to the idea of spacetime. Einstein corrected the Newtonian belief of time being an independent parameter which was same everywhere in the universe to it being another intertwined coordinate of the 4-D spacetime we live in. Special Relativity essentially describes the properties of this spacetime in absence of Gravity. Formally, *Spacetime* is a *Manifold* of *Events* endowed by a *Metric*. A Manifold is a set of points with well understood connection properties. A connection is a mathematical object that connects the value of some function at two different points on the manifold. An Event is when and where something happens labelled with coordinates but itself having a meaning independent of the coordinates selected.
A Metric is a notion of distance between events on the manifold. So in our familiar 3-D Euclidean space the metric is a 3x3 Matrix that gives us the notion of distance i.e., the Pythagorean theorem. The metric is also what defines the norm of a vector and scalar products.:

$$dl^2 = dx^2 + dy^2 + dz^2 = \begin{pmatrix} dx & dy & dz \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

For a curved surface or for that matter any curved space the metric isn't as straightforward but its essence remains the same. A Metric can be thought of something that distinguishes a doughnut from your coffee mug, which are topologically same.

Inertial motion is that of an object not subjected to any forces and an Inertial frame of reference (I.F.O.R.) is any coordinate frame attached to such a body (or non rotating F.O.R. whose origin can be thought of as undergoing inertial motion). Special Relativity is the simplest theory of a flat spacetime in the absence of gravity.

It is based on two basic ides, first being the *principle of relativity* stating that the laws of physics must be the same for all inertial observers and second being the constancy of the speed of light in all inertial frames. Naively one can argue that speed of light is a constant because it arises from the fundamental constants μ_0 and ϵ_0 which are frame independent.

Flat 4-D spacetime: In Einstein's theory of relativity time is not an independent dimension but rather one of the intertwined dimensions of a 4-D spacetime. The special Pythagorean theorem for such a Manifold goes as follows:

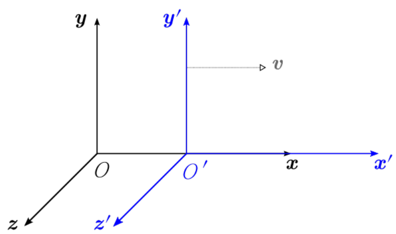
$$\boldsymbol{\Delta s^2 = -c^2(\Delta t)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2} \tag{1}$$

Here Δs^2 is the **spacetime interval** between two events and is analogous differential distance element squared in 3-D. The four dimensions are often represented in Index notation with time component being denoted by $t \equiv x^0$ and the usual spacial dimensions $x \equiv x^1, y \equiv x^2, z \equiv x^3$. Here the major difference from Euclidean norm is the $-c^2$ (c is the speed of light in vacuum) factor for the time component which can be thought as a consequence of the constancy of the speed of light.

Note: Henceforth we choose basic unit of length to be a light second if our unit of time is seconds i.e. in our units c=1. This system of units is called natural units and the dimension of time is that length and dimension of energy becomes that of mass.

Lorentz transformations

Having considered the basic ideas of special relativity now lets consider motion of two inertial observers O and O', O' moving with a relative spatial velocity $\bar{v}$ w.r.t. O (without loss of generality we take the x axes of both their coordinate frames along one another in the direction of $\bar{v}$) and call respective I.F.O.R.s S and S'. Say, they measure the distance (both spatial and temporal) between two spacetime events A and B.



We define the displacement *4- vector* as the separation between the two events in 4-D spacetime.

$\Delta\vec{x} = (\Delta t, \Delta x, \Delta y, \Delta z) = (\Delta t, \Delta x^1, \Delta x^2, \Delta x^3) = (\Delta t, \Delta\overline{x})$ in F.O.R. S

$\Delta\vec{x} = (\Delta t', \Delta x', \Delta y', \Delta z') = (\Delta t', \Delta x^{1'}, \Delta x^{2'}, \Delta x^{3'}) = (\Delta t', \Delta\overline{x}')$ in F.O.R. S'

The postulates of S.R. demand that the spacetime interval be invariant under coordinate transformations. [Notation: overhead arrow → 4-Vector, Bar → 3-vector]

$$\Delta s^2 = -(\Delta t)^2+(\Delta x^1)^2+(\Delta x^2)^2+(\Delta x^3)^2 = -(\Delta t')^2+(\Delta x^{1'})^2+(\Delta x^{2'})^2+(\Delta x^{3'})^2$$

Note: As far as possible an arrowhead will be reserved for 4-Vectors and a bar for 3-vectors.

The in-variance of the spacetime interval leads to a special type of transformation law for coordinate transformation between such inertial frames.
For our example the matrix transformation is easily obtainable by solving the two equations since y and z components are the same. This special transformation called the Lorentz transform is as follows:

$$\begin{pmatrix} \boldsymbol{\Delta x^{0'}} \\ \boldsymbol{\Delta x^{1'}} \\ \boldsymbol{\Delta x^{2'}} \\ \boldsymbol{\Delta x^{3'}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Delta x^{0}} \\ \boldsymbol{\Delta x^{1}} \\ \boldsymbol{\Delta x^{2}} \\ \boldsymbol{\Delta x^{3}} \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma} & \boldsymbol{-\gamma\beta} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{-\gamma\beta} & \boldsymbol{\gamma} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{2}$$

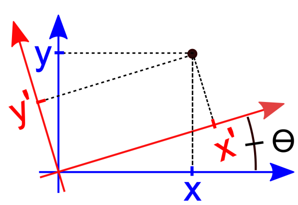
$\gamma = \sqrt{1 - \frac{v^2}{c^2}} = \sqrt{1 - v^2}$, $\beta = v/c = v$, taking c=1.
Here the Lorentz transformation matrix is is denoted symbolically as:

$$\Lambda = [\Lambda^{\mu'}_{\nu}]$$

Henceforth we frequently use the convenient index notation for components for such mathematical objects, more on that later.
Lorentz transformation as a Hyperbolic Rotation:
First let us consider the familiar case of rotation of axes by an angle theta in 2-D. The transformation law goes as:

$$x' = cos(\theta)x + sin(\theta)y$$

$$y' = -sin(\theta)x + cos(\theta)y$$

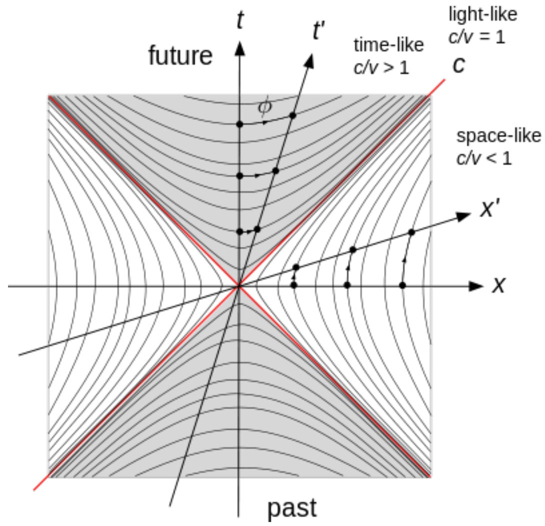


Now consider Hyperbolic rotation by an angle ϕ such that $cosh(\phi) = \gamma$ and $sinh(\phi) = \gamma\beta$. (cosh and sinh are hyperbolic functions with $cosh^2 - sinh^2 = 1$) Then the Lorentz transformation takes the form:

$$t' = cosh(\phi)t - sinh(\phi)x$$

$$x' = -sinh(\phi)t + cosh(\phi)x$$

Hyperbolic rotation is symmetric about line with slope c=1. The function maps the points on the fixed hyperbola analogous to how rotation just maps points on the fixed circle through original coordinates about the origin.



Here ϕ is $tanh^{-1}(\beta)$. In natural units $\beta = v < 1$.
One more thing to note is that though at first sight it looks as though t' and x' are no more orthogonal but that is not the case and this can be mathematically shown using the properties of our space. Notice that in both these coordinates x=t and x'=t' coincide, which shouldn't come as a surprise since c=1 is constant.

The Lorentz transformations can be written in component form as where $\Lambda_{\nu}^{\mu'}$ is $(\mu, \nu)^{th}$ indexed component of the matrix:

$$\boldsymbol{x^{\mu'} = \sum_{\nu=0}^{3} \Lambda^{\mu'}_{\nu} x^{\nu} = \Lambda^{\mu'}_{\nu} \Delta x^{\nu}} \tag{3}$$

We start using the Einstein summation convention where we drop the summation symbol and the sum is implicitly assumed wherever indices are repeated i.e. repeated indices are summed over and these are called dummy indices because the final expression remains the same even if we swap the index with any other index not present in the equation. Remember, in an indexed equation like the one above the free i.e., the non dummy variables must be the same on both sides. In the above example that variable is μ'.

We will use Greek indices for spacetime entities and Latin for while considering only spatial.
In the language of differential geometry:

$$\Lambda^{\mu'}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} = \partial_{\nu} x^{\mu'}$$

This can be understood by going back to the familiar 2-D rotations: The matrix of transformation is:

$$\begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix} = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix}$$

1.2 Transformation of Basis vectors, Dual vectors or 1-forms.

Basis vectors

Any vector in our flat 4-D space can be formed by a set of orthonormal basis ($\vec{e}_\mu$ along x^μ). We have looked at how the components of the vectors transform but now we look at how the basis (orthogonal in this case) transform. Assume that the basis transform as $\vec{e}_{\nu'} = L^{\mu}_{\nu'}\vec{e}_\mu$ where $\mathbf{L}$=[$L^{\mu}_{\nu'}$] is the transformation matrix.

$$\vec{x} \equiv x^\mu \vec{e}_\mu = x^{\nu'}\vec{e}_{\nu'}$$

Now substitute the transformation matrices for both basis (3) and components:

$$\Rightarrow x^\mu \vec{e}_\mu = (x^\mu \Lambda^{\mu}_{\nu'})(L^{\mu}_{\nu'}\vec{e}_\mu) = x^\mu \vec{e}_\mu(\Lambda^{\mu}_{\nu'}L^{\mu}_{\nu'})$$

$$\Rightarrow \Lambda^{\nu'}_{\mu}L^{\mu}_{\nu'} = 1$$

$$\Rightarrow \Lambda^{\nu'}_{\mu}L^{\sigma}_{\nu'} = \delta^{\sigma}_{\mu}$$

Here δ^{σ}_{μ} is the Kronecker delta ($\delta^{\sigma}_{\mu} = 1$ if $\alpha = \mu$ else 0) that is Identity matrix. Thus $L^{\mu}_{\nu'}$ is inverse of $\Lambda^{\mu}_{\nu'}$

$$\boldsymbol{\vec{e}_\mu = \Lambda^{\nu'}_{\mu}\vec{e}_{\nu'}} \tag{4}$$

Dual vectors or 1-Forms

A dual vector can be abstractly thought of as a mathematical object that operates on a vector to produce a scalar. (here, tilde $\equiv$ dual vector)

$$\tilde{p}(\vec{A}) = \tilde{p}(a^\mu \vec{e}_\mu) = p_\mu a^\mu$$

For the output to be a Lorentz invariant scalar the components of the dual vector must transform inversely as that of vector components:

$$\boldsymbol{p_{\mu'} = \Lambda^{\nu}_{\mu'}p_\nu = \frac{\partial x^\nu}{\partial x^{\mu'}}p_\nu} \tag{5}$$

And the dual basis acts on vector basis to give Kronecker delta

$$\tilde{\omega}^\mu(\vec{e}_\nu) = \delta^{\mu}_{\nu}$$

Thus it can be worked out the transformation law for dual basis is the same as that of vector components.

$$\boldsymbol{\tilde{\omega}^{\mu'} = \Lambda^{\nu}_{\mu'}\tilde{\omega}^\nu} \tag{6}$$

$$Grad(\vec{v}) = \partial_\alpha v^\alpha = Scalar$$

Gradient is therefore a 1-form with components: ∂_α which is a shorthand for $\frac{\partial}{\partial x^\alpha}$. Note that a scalar is index free i.e., it is frame independent and hence an invariant quantity under Lorentz transforms.

Tensors

We defined a dual-vector as an object acting on a vector to give a scalar but the idea is really interchangeable i.e. we can think of the dual being acted upon by the vector. A $\binom{m}{n}$ Tensor can be defined as a mathematical object that acts on n vectors and m dual-vectors to give a Lorentz invariant Scalar.

Thus a vector is a $\binom{1}{0}$ tensor.

A dual-vector is a $\binom{0}{1}$ tensor.

Thus our expression for coordinate transforms in flat 4-D space can be easily extended to tensor components.

$$\boldsymbol{T^{\mu_1,\mu_2\ldots}_{\nu_1,\nu_2\ldots} = (\frac{\partial x^{\mu_1}}{\partial x^{\alpha'_1}}\frac{\partial x^{\mu_2}}{\partial x^{\alpha'_2}}\ldots\frac{\partial x^{\beta'_1}}{\partial x^{\nu_1}}\frac{\partial x^{\beta'_2}}{\partial x^{\nu_2}}\ldots)T^{\alpha'_1,\alpha'_2\ldots}_{\beta'_1,\beta'_2\ldots}} \tag{7}$$

where the basis of the tensor are formed by tensor product of vectors and dual-vectors:

$$\mathbf{T} = T^{\mu_1,\mu_2\ldots}_{\nu_1,\nu_2\ldots}\vec{e}^{\mu_1} \otimes \vec{e}^{\mu_2}..\tilde{\omega}_{\nu_1} \otimes \tilde{\omega}_{\nu_2}..$$

The properties of the general metric $g_{\mu\nu}$will be discussed in detail later though the basic notion is still the same: It defines a notion of distances. The metric for flat space $\eta_{\mu\mu}$ which we soon encounter or for that matter the Euclidean norm are just a special cases.

The metric of any manifold lets us convert nature of slots on a tensor. These operations are called contractions on the tensor. Special type of contractions called traces produce scalar functions and will also come in handy at later stages.

$$g_{\alpha\mu}T^{\mu}_{\beta} = T_{\alpha\beta}$$

$$g^{\alpha\mu}T_{\mu\beta} = T^{\alpha}_{\beta}$$

$$g^{\mu\nu}T_{\mu\nu} = T^{\mu}_{\mu} = T$$

Moving forward we must remember that a proper tensorial equation is valid in any frame of reference, for any observer. Often, the goal will be to represent or modify known equations to bring them into a proper tensorial form . Such equations are then called relativistically covariant.

1.3 Minkowski metric and scalar product

Scalar Product

Having learnt how vectors transform we now want a physical interpretation of the length of a vector or equivalently, it's norm. For this we first need to define scalar product. Scalar Product can be motivated by the expression for the Lorentz invariant spacetime interval:

$$\Delta s^2 = -c^2(\Delta t)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = \Delta\vec{x}\cdot\Delta\vec{x} = |\Delta\vec{x}|^2$$

The scalar product of a vector with another vector in flat spacetime is defined as:

$$\vec{a}\cdot\vec{b} = -a^{(0)}b^{(0)} + a^{(1)}b^{(1)} + a^{(2)}b^{(2)} + a^{(3)}b^{(3)}$$

Now in index notation,

$$\vec{a}\cdot\vec{b} = a^\mu b^\nu \eta_{\mu\nu} \tag{8}$$

$\eta_{\mu\nu}$ are components of the matrix

$$\boldsymbol{\eta} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

This is the **Minkowski** metric and it defines the notions of norms (scalar product of vector with itself) and distances on our flat 4-D spacetime.

The metric can be defined in terms of the basis vectors as:

$$\vec{e}_\mu \cdot \vec{e}_\nu = \eta_{\mu\nu}$$

. Clearly the Minkowski metric inputs 2 vectors to give a scalar product as an output$\Rightarrow$ $\eta_{\mu\nu}$ are components of a $\begin{pmatrix}0\\2\end{pmatrix}$ Tensor.

Also observe that $\eta_{\mu\nu}$ is symmetric under exchange of indices.

1.4 Proper Time, 4-Velocity, 4-Momentum

Proper time is The time elapsed in the rest frame of an object, i.e., frame in which spatial displacement of the object is zero. Denoted by symbol τ.

4-Velocity is defined as

$$\vec{u} = d\vec{x}/d\tau \tag{10}$$

Here τ is proper time of the object whose motion is considered and $\vec{x}$ is the 4-displacement in some frame S relative to which object is moving with 3-velocity $\overline{u}$.
Recollect: $\Delta s^2 = -\Delta t^2 + |\Delta \vec{x}|^2 = -\Delta \tau^2$

Thus in a particular F.O.R. (referring back to Lorentz transform matrix: (2)):

$\vec{u}$(in frame S)$\equiv d\vec{x}/d\tau = (dt/d\tau, d\overline{x}/d\tau) = (\gamma, \gamma\overline{u})$
$\vec{u}$(rest frame)$\equiv d\vec{x}/d\tau = (d\tau/d\tau, \overline{0}) = (1, \overline{0})$

Hence we see that irrespective of F.O.R.:(using (8))

$$\boldsymbol{\vec{u} \cdot \vec{u} = -1} \tag{11}$$

Thus, every object moves with the same speed through spacetime, objects moving faster through space have lower temporal speed and visa versa.

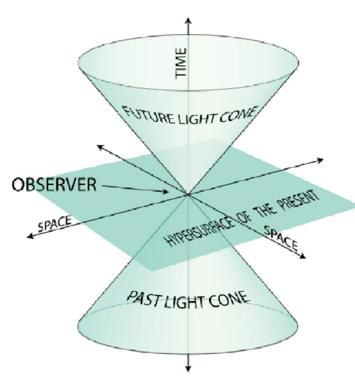
4-Momentum is defined as

$$\boldsymbol{\vec{p} = m\vec{u} = (\gamma m, \gamma m\overline{u}) = (E, \overline{p})} \tag{12}$$

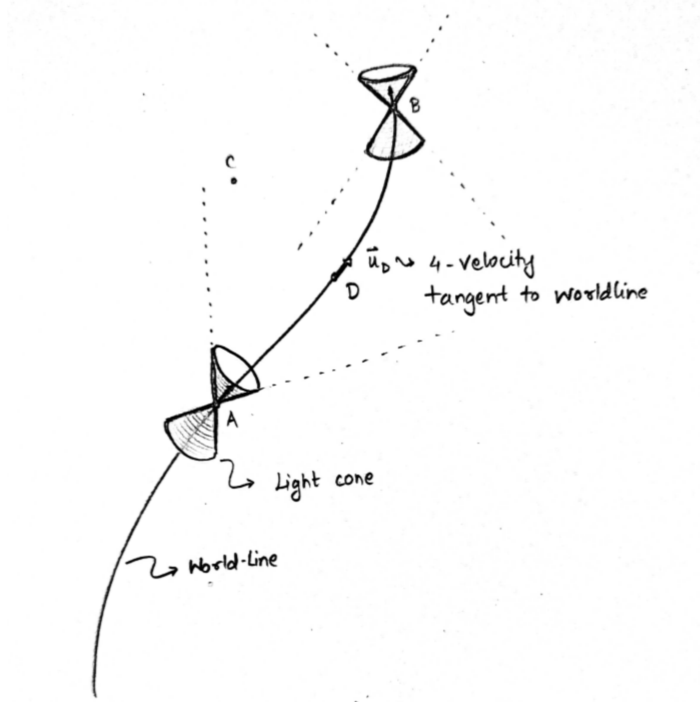
here m is the mass of the object and $\gamma = \sqrt{1-u^2}$, u $=|\overline{u}|$ Taking norm of the above expression gives the famous equation of Mass energy equivalence (taking c=1):

$$E^2 = m^2 + |\overline{p}|^2$$

World line of an object is the trajectory it follows trough spacetime. The finite speed limit restricts the motion of every massive object within a light cone at each instant on its world line. Here we consider time and only two spatial dimensions for easier visualization.



Paths along the light cone ($\Rightarrow$ slope = speed = c = 1)are called null paths because on computation $\Delta s^2 = -\Delta\tau^2 = 0$. Only mass-less photons move along null paths. Paths lying inside the light cone are called Time like since the $\Delta s^2 < 0$ for these paths and all points within the light cone are causally connected (event at one point can be the cause of another event) . Paths lying outside the light cone are called Space like since the $\Delta s^2 > 0$ for these paths and all points outside the light cone are causally disconnected from the object under consideration.



The above image depicts the path taken by an object in spacetime (flat). The direction of motion is from $A \rightarrow D \rightarrow B$. It is clear that both events A and B are causally connected to event D, D being in the future of A and in the past of B. In fact both A and B will be causally connected to every point on the world line. One can see that event C is causally connected to A because it lies within its light cone but that's not the case with B. Another thing to note is that the tangent to the worldline is given by the object's 4-Velocity at every point.

1.5 Electromagnetic field tensor, Stress energy Tensor.

Special relativity was motivated by the goal to make electromagnetism a covariant concept, so this is a glance at some of the key concepts. **Current density 4-Vector** is defined as:($\rho \equiv$ charge density, $\overline{J} \equiv$ usual current density 3-Vector, c=1): $\vec{J} = (\rho, \overline{J}) = (J^0, \overline{J})$.

The continuity equation in 4-D spacetime (charge conservation) simply reduces to: $\partial_\alpha J^\alpha = 0$.

The electromagnetic potential 4-Vector ($\Phi \equiv$ Electric scalar potential, $\bar{A} \equiv$ is the Magnetic potential 3-Vector, c=1) : $\vec{A} = (\Phi, \overline{A}) = (A^0, \overline{A})$.

These 4-Vectors are relativistic covariant ideas and thus encode all the equations of classical electromagnetism.

The Electromagnetic Field tensor is an anti-symmetric 4x4 tensor which encodes all components of the electric and magnetic field, it is constructed by anti-symmetrising derivatives of the EM Potential 4-Vector:

$$[F^{\mu\nu}] = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix}$$

Maxwell's equations simply become:(setting $\mu_0, \epsilon_0 = 1$ in natural units)

$$\partial_\nu F^{\mu\nu} = 4\pi J^\mu$$

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

The above expressions make equations of Electromagnetism relativistically covariant.

Stress Energy Tensor of a Perfect Fluid is another important tensor for modelling cosmological systems.

The components of this tensor encode Energy and Momentum fluxes in the 4 spacetime directions. In absence of lateral stresses the non diagonal terms of the tensor become zero as element $T^{\alpha\beta}$ equals flux of energy-momentum component alpha in direction x^β.

$$[T^{\mu\nu}] = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Here ρ is the energy density and P is isotropic spatial stress i.e., pressure. The components can be written in terms of the metric and the 4- velocity of the fluid in its rest frame.

$$\boldsymbol{T^{\mu\nu} = u^\mu u^\nu (\rho + P) + P\eta^{\mu\nu}} \tag{13}$$

2 Geometry of Curved Spaces

2.1 Basis in curved spaces, The Metric, Christoffel symbols and the covariant Derivative

So far we were working with a orthonormal basis which were constant under translation, but this need not be the case, the basis need not be orthogonal, normalized or even constants under translation. This also means that the elements of the metric themselves can be functions of coordinates on the manifold. The ideas that we have been working with in Minkowski spacetime can be extended to curved spaces keeping in mind that a proper Tensorial equation must be independent of the chosen representation i.e., the laws of physics are the same everywhere.

The notion of a spacetime interval now takes the form:

$$\boldsymbol{ds^2 = g_{\mu\nu} dx^\nu dx^\mu} \tag{14}$$

Where $g_{\mu\nu}$ is the metric endowed upon the manifold.The Metric is symmetric due to the nature of definition of the spacetime interval.

The example of spherical surface parameterized by θ and ϕ with constant radius R: We know that the differential arc-length square:

$$ds^2 = R^2 d\theta^2 + R^2 sin^2(\theta) d\phi^2$$

Thus $[g_{\mu\nu}] = diag(R^2, R^2 sin^2\theta)$ which is clearly not a constant but is symmetric.

So far we have dealt with partials as though they transform tensorially because our basis vectors were constants but that is not the case in on a general curved manifold because Leibniz rule on partial differentiation gives:

$$\frac{\partial \vec{v}}{\partial x^\beta} = \frac{\partial v^\alpha}{\partial x^\beta} \vec{e}_\alpha + v^\alpha \frac{\partial \vec{e}_\alpha}{\partial x^\beta} \tag{15}$$

The second term is non zero and it doesn't transform tensorially. We can write the derivative of any basis vector as a linear combination of the basis vectors:

$$\boldsymbol{\partial_\beta \vec{e}_\mu = \Gamma^\mu{}_{\beta\alpha} \vec{e}_\mu} \tag{16}$$

$\Gamma^\mu{}_{\beta\alpha}$ are called the **Christoffel** symbols. Note that in the second term α and μ are dummy indices and thus we can swap them.Thus we can rewrite (15):

$$\partial_\beta \vec{v} = (\partial_\beta v^\alpha + v^\mu \Gamma^\alpha{}_{\beta\mu}) \vec{e}_\mu$$

The quantity in the bracket are components of a tensor. The reason for it's tonsorial nature is apparent when we look at the nature of transformation of indices on L.H.S. and R.H.S.. Thus we can now define a notion of derivative which is covariant under transformations i.e., it follows transformation law of a tensor.

$$\boldsymbol{\nabla_\beta v^\alpha = \partial_\beta v^\alpha + v^\mu \Gamma^\alpha{}_{\beta\mu}} \tag{17}$$

Henceforth we reserve ∇ for **covariant derivative**.The covariant derivative is linear,follows Leibniz rule but doesn't commute. Another notation:

$$\tilde{\nabla}\vec{v} = (\nabla_\beta v^\alpha)\,\tilde{\omega}^\beta \otimes \vec{e}_\alpha$$

For scalars the partial and covariant derivative are the same. (Simple intuition is that scalars don't need a basis thus no Christoffel symbols are involved in the partial differentials)

$$\boldsymbol{\nabla_\mu \phi = \partial_\mu \phi} \tag{18}$$

For 1-forms the expression for covariant derivative can be derived by covariantly differentiating contraction of a 1-form with a vector. (using (17))

$$\nabla_\beta(p_\alpha v^\alpha) = \partial_\beta(p_\alpha v^\alpha)$$

$$(\nabla_\beta p_\alpha)v^\alpha + p_\alpha(\nabla_\beta v^\alpha) = \partial_\beta(p_\alpha)v^\alpha + p_\alpha(\partial_\beta v^\alpha)$$

Substituting known expressions we get:

$$\boldsymbol{\nabla_\beta p_\alpha = \partial_\beta p_\alpha - p_\mu \Gamma^\mu_{\beta\alpha}} \tag{19}$$

Thus the basis 1-form transform as:

$$\boldsymbol{\partial_\beta \tilde{\omega}^\alpha = -\Gamma^\alpha_{\beta\mu}\tilde{\omega}^\mu} \tag{20}$$

Using above expressions we can easily generalise the covariant derivative for higher rank tensors.

$$\nabla_\beta T^{\mu\nu..}_{\rho\tau..} = \partial_\beta T^{\mu\nu..}_{\rho\tau..} + [\Gamma^\mu_{\ \beta\alpha} T^{\alpha\nu..}_{\rho\tau..} + \Gamma^\nu_{\ \beta\alpha} T^{\mu\alpha..}_{\rho\tau..}..] + [-\Gamma^\alpha_{\ \beta\rho} T^{\mu\nu..}_{\alpha\tau..} - \Gamma^\alpha_{\ \beta\tau} T^{\mu\nu..}_{\rho\alpha..}..]$$

2.2 Cristoffel symbols, the Metric and their symmetries.

We know that the metric is a symmetric tensor. Analysis of double covariant derivative of a scalar function reveals that the Christoffel symbols are also symmetric under exchange of the two lower indices:

$$\Gamma^\mu_{\ \beta\alpha} = \Gamma^\mu_{\ \alpha\beta} = \Gamma^\mu_{\ (\alpha\beta)}$$

Notation: Define Symmetrization as $A^\mu_{\ (\alpha\beta)} = \frac{1}{2}(A_{\ \alpha\beta} + A_{\ \beta\alpha})$
Antisymmetrization as $A^\mu_{\ [\alpha\beta]} = \frac{1}{2}(A_{\ \alpha\beta} - A_{\ \beta\alpha})$

Using the fact that $\nabla_\gamma \eta_{\alpha\beta} = 0$ and equivalence arguments one can conclude that the covariant derivative of the metric must be identically zero:

$$\boldsymbol{\nabla_\gamma g_{\alpha\beta} = 0} \tag{21}$$

$$\Rightarrow \nabla_\gamma g_{\alpha\beta} = \partial_\gamma g_{\alpha\beta} - \Gamma^\mu_{\ \gamma\alpha} g_{\mu\beta} - \Gamma^\mu_{\ \gamma\beta} g_{\alpha\mu} = 0$$

$$\Rightarrow \nabla_\alpha g_{\beta\gamma} = \partial_\alpha g_{\beta\gamma} - \Gamma^\mu{}_{\alpha\beta} g_{\mu\gamma} - \Gamma^\mu{}_{\gamma\alpha} g_{\beta\mu} = 0$$

$$\Rightarrow \nabla_\beta g_{\gamma\alpha} = \partial_\beta g_{\alpha\gamma} - \Gamma^\mu{}_{\beta\gamma} g_{\mu\alpha} - \Gamma^\mu{}_{\beta\alpha} g_{\gamma\mu} = 0$$

Mathematical manipulation of all three permutations of the indices gives a relation between the Christoffel symbols and the derivatives of the Metric,

$$\boldsymbol{\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma}(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta})} \tag{22}$$

For getting comfortable with the idea of Christoffel symbols, continuing with our previous example of a sphere we can now calculate the Christoffel symbols of the manifold:

Recollect: $[g_{\mu\nu}] = diag(R^2, R^2 sin^2)$ Using (22) the derivative of the metric we get:

$$\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{\theta\phi} = \Gamma^\phi_{\theta\theta} = \Gamma^\phi_{\phi\phi} = 0$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = cot(\theta)$$

$$\Gamma^\theta_{\phi\phi} = -sin(\theta)cos(\theta)$$

2.3 Equivalence Principle

Principle of Equivalence: over sufficiently small regions of spacetime the motions of freely falling particles due to gravity cannot be distinguished from uniform acceleration, this is equivalently saying that the inertial mass of an object is equal to its gravitational mass.

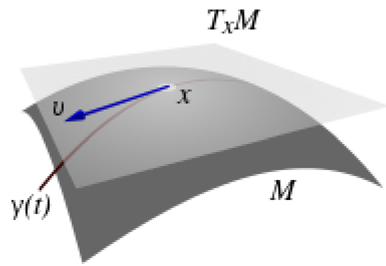
Reformation of the equivalence principle: In sufficiently small regions of spacetime we can find a coordinate system such that laws of physics reduce to those of special relativity.

Using Taylor expansion one can always represent the spacetime metric at a point on a manifold in terms of the Minkowski metric and some correction terms:

$$g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}[(\partial^2 g)(\delta x)^2]$$

The equivalence principle basically states that in sufficiently small regions the correction consisting of the 2^{nd} derivatives of the metric goes to zero. We will soon see that these terms quantify the curvature of our manifold which is intuitively obvious as curvature is exactly what causes the metric to deviate from the Minkowski metric.

On a **curved manifold** initially parallel lines or trajectories do not remain parallel. On a flat manifold all points have the same tangent space but that is not the case for curved manifolds. In mathematics, the tangent space of a manifold facilitates the generalization of vectors to general manifolds. In the case of curved spaces one cannot simply subtract vectors at two points to obtain a vector thus we need a mathematical notion for transporting a vector from one point to another.



2.4 Parallel transport, Lie derivative, Killing vectors

Consider two points on a curve $\gamma(\lambda)$ on a curved manifold say P and Q infinitesimal distance apart. Now consider a vector field A^α with values $A^\alpha(P)$ and $A^\alpha(Q)$ at P and Q respectively.

Now if one wants to define a derivative it must be with respect to a basis. Some thought reveals that this means we cannot directly define a meaningful notion of the derivative for two vectors at different points on a curved manifold because they belong to different tangent spaces all together. So we need a mathematical tool that relates the values of the field at the two points or perhaps the basis of the tangent spaces - The **Connection**.

This means that we need some way to transport the vector from one point to another. Let $A^\alpha(P \to Q)$ denote the transported vector.

We define the transported vector as: $A^\alpha(P \to Q) = A^\alpha(P) - L^\alpha_{\beta\mu} dx^\beta A^\mu$
Here $L^\alpha_{\beta\mu}$ is called the Connection- object that connects values of functions on different points on a manifold. Now we are ready to define a derivative 'D' of a Vector field.

$$D_\beta A^\alpha = \frac{A^\alpha(Q) - A^\alpha(P \to Q)}{dx^\beta}$$

Now, we demand that the connection be such that $D_\beta g_{\mu\nu} = 0$ and D be a relativistic covariant i.e. a tonsorial quantity. Then we find that the the connection in this case is nothing but the Christoffel symbols and the derivative is the covariant derivative
$L^\alpha_{\beta\mu} = \Lambda^\alpha_{\beta\mu}$ and $D_\beta = \nabla_\beta$

Recollect that our initial argument for the need for defining the notion of covariant derivative was to generalise the expression of derivatives to curved spaces such that the notion becomes a relativistic covariant (frame or coordinate independent) idea. The above section gives a way to visualise how we can arrive at the same expression through the geometric idea of transport.

Parallel transport:In geometry, parallel transport is a special way of transporting geometrical data along smooth curves in a manifold. Naively put, we take the vector along the curve preserving its component along the tangents to the curve.

Reconsider the curve $\gamma(\lambda)$ graduated with uniform tick-marks λ, the tangent vector to this curve is:

$$\boldsymbol{u^{\alpha} = \frac{dx^{\alpha}}{d\lambda}} \tag{23}$$

Now, an expression that depicts how the vector changes when we move along the curve specifically in the direction of the tangent to the curve at any point is:

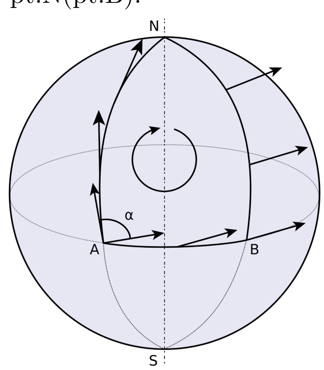
$$u^{\beta}(\nabla_{\beta}A^{\alpha}) = u^{\beta}(\partial_{\beta}A^{\alpha} + \Gamma^{\alpha}_{\beta\mu}A^{\mu}) = \frac{DA^{\alpha}}{d\lambda}$$

The above contraction of the covariant derivative with the tangent vector expresses how the vector changes as is transported along the curve.

By definition of parallel transport:

$$\boldsymbol{u^{\beta}(\nabla_{\beta}A^{\alpha}) = u^{\beta}(\partial_{\beta}A^{\alpha} + \Gamma^{\alpha}_{\beta\mu}A^{\mu}) = \frac{DA^{\alpha}}{d\lambda} = 0} \tag{24}$$

Consider the example where The vector at point A on a sphere is parallel transported along a triangular path formed by three great circles. Along the curve AN the scalar product of the vector with the tangent vector at any point remains constant and equals the initial value at pt.A. Same applies for curve NB(BA) with initial value at pt.N(pt.B).



The transported vector is clearly rotated with respect to original one. In case of flat spaces the vector would've remained unchanged.

Lie derivative Another covariant notion of transport along a vector field is given by the lie derivative also known as the lie commutator. Lie derivative of a vector field along tangent $u^\alpha = \frac{dx^\alpha}{d\lambda}$ field to curve $\gamma(\lambda)$ is:

$$\boldsymbol{\mathcal{L}_{\vec{u}} A^\alpha = u^\beta \nabla_\beta A^\alpha - A^\beta \nabla_\beta u^\alpha} \tag{25}$$

The commutator bracket notation: $\mathcal{L}_{\vec{u}} A^\alpha = [\vec{u}, \vec{A}]$

The lie commutator is tensorial, anti-symmetric and follows Leibniz rule of differentiation.

Lie Derivative of a Scalar: $\mathcal{L}_{\vec{u}}\phi = u^\beta \nabla_\beta \phi - A^\beta \nabla_\beta \phi$

Lie Derivative of a 1-form: $\mathcal{L}_{\vec{u}} p_\alpha = u^\beta \nabla_\beta p_\alpha + p_\beta \nabla_\beta u^\alpha$

Lie Derivative of a $\begin{pmatrix}1\\1\end{pmatrix}$ Tensor: $\mathcal{L}_{\vec{u}} T^\alpha_\beta = u^\mu \nabla_\mu T^\alpha_\beta - T^\mu_\beta \nabla_\mu u^\alpha + T^\alpha_\mu \nabla_\beta u^\mu$

The key significance of this is to study tensors whose lie derivative is zero along a vector field: $\mathcal{L}_{\vec{u}}(Tensor) = 0$. In this case the tensor is said to be lie transported and there exists a special parameterization $\lambda = x^0$.

Suppose there exists a special vector field $\vec{\xi}$ such that the metric is lie transported along it then, $\mathcal{L}_{\vec{\xi}}(g_{\mu\nu}) = 0$ and there exits some coordinates such that:

$$\frac{\partial g_{\mu\nu}}{\partial x^0}$$

If $\mathcal{L}_{\vec{\xi}}(g_{\alpha\beta}) = 0$

$$\xi^\mu \nabla_\mu g_{\alpha\beta} + g_{\alpha\mu} \nabla_\beta \xi^\mu + g_{\mu\beta} \nabla_\alpha \xi^\mu = 0$$

The first term is identically zero and the metric commutes with covariant derivative, thus we have the **Killing equation**:

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$$

The above equation is very useful when studying conservation or invariance of various tensorial quantities along one spacetime dimension.

For example if one were to consider a freely falling particle. Equation of its worldline in terms of 4-velocity being $u^\alpha \nabla_\alpha u^\beta$. Suppose the metric is time independent then there exists a killing vector field such that $u^\alpha \xi_\alpha = E$ is a constant of motion, and time independence means $E \equiv energy$. Similarly if one is working in an equatorial plane in polar coordinates and the metric is rotation independent then there exists a killing field such that $u^\mu \xi'_\mu = L$ is a constant of motion, and angle independence means $L \equiv angular\ momentum$. (here $u^\alpha = dx^\alpha / d\tau$)

2.5 Tensor densities

Tensor densities are quantities that transform almost like tensors. Their transformation is law off by a factor that is the determinant of coordinates transformation matrix.

Two important tensor densities:

- Levi-Civita Symbol
- Determinants

Levi-Civita: (tilde emphasises tensor density)
$\tilde{\epsilon}_{\alpha\beta\gamma\delta} = +1$ for even permutations of 0,1,2,3
$\tilde{\epsilon}_{\alpha\beta\gamma\delta} = -1$ for odd permutations of 0,1,2,3
$\tilde{\epsilon}_{\alpha\beta\gamma\delta} = 0$ otherwise
Theorem: for any 4x4 matrix $[M^{\alpha}_{\mu}]$
$\tilde{\epsilon}_{\alpha\beta\gamma\delta} M^{\alpha}_{\mu} M^{\beta}_{\nu} M^{\gamma}_{\rho} M^{\delta}_{\sigma} = \tilde{\epsilon}|M|$ ($|M|$ denotes determinants)
Suppose $M \to \frac{\partial x^{\mu}}{\partial x^{\mu'}}$

$$\tilde{\epsilon}_{\alpha'\beta'\gamma'\delta'} = \tilde{\epsilon}_{\alpha\beta\gamma\delta} \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \frac{\partial x^{\delta}}{\partial x^{\delta'}} \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right|$$

The extra factor $\left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right|$ is the Jacobian determinant. The Levi-Civita symbol is a tensor density of weight 1.

Determinant of the metric, first step is to look at transformation and then take determinant.

$$g_{\alpha'\beta'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} g_{\alpha\beta}$$

$$g' = \left| \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \right|^{-2} g$$

Determinant of $g_{\mu\nu}$ is a tensor density of weight -2. Using the determinant of the metric we can covert any tensorial density of weight w by multiplying by $|g|^{w/2}$

Example:

$$\epsilon_{\alpha\beta\gamma\delta} \equiv \sqrt{|g|}\tilde{\epsilon}_{\alpha'\beta'\gamma'\delta'}$$

$$\epsilon^{\alpha\beta\gamma\delta} \equiv \frac{1}{\sqrt{|g|}}\tilde{\epsilon}^{\alpha'\beta'\gamma'\delta'}$$

We use the Levi-Civita symbol to define volume elements:

In 4-D: $dV^4 = \sqrt{|g|}\tilde{\epsilon}_{\alpha\beta\gamma\delta} dx^{\alpha}_{(0)} dx^{\beta}_{(1)} dx^{\gamma}_{(2)} dx^{\delta}_{(3)}$

For Orthogonal basis: $dV^4 = \sqrt{|g|} dx^0 dx^1 dx^2 dx^3$

Example: consider the familiar Spherical polar coordinates:
$dV^3 = \sqrt{|g|}drd\theta d\phi = r^2 sin(\theta)drd\theta d\phi$

2.6 The Geodesic Equation

Consider a curve γ parameterized by λ with tangent vector $u^\alpha = \frac{dx^\alpha}{d\lambda}$. Such a curve on a manifold is called a geodesic if its tangent vector itself parallel transports along the curve. Some thought on the fact that the tangent vector undergoes minimum deviation while moving along the curve hints at the fact that this is indeed the "straightest" path between two points on the curve. **Geodesic equation**:

$$u^\alpha \nabla_\alpha u^\beta = 0$$

$$\Rightarrow \frac{du^\beta}{d\lambda} + \Gamma^\beta_{\alpha\mu} u^\alpha u^\mu = 0$$

$$\boldsymbol{\Rightarrow \frac{d^2 x^\beta}{d\lambda^2} + \Gamma^\beta_{\alpha\mu} \frac{dx^\alpha}{d\lambda}\frac{dx^\mu}{d\lambda} = 0} \tag{26}$$

In absence of any external forces a particle follows a straight line path (a geodesic in flat space) as per Newtonian Physics. It doesn't come as a surprise that the trajectory of a particle in a curved manifold (the curvature itself is gravity but more on that later) in absence of any external forces is a geodesic-the straightest possible path.

Now lets consider the curve γ to be the worldline of a freely falling test particle of mass m, parameterized by its proper time τ. The tangent vector is the 4-Velocity: $u^\alpha = \frac{dx^\alpha}{d\tau}$

Now using $p^\alpha = mu^\alpha$ the geodesic equation in terms of momentum is:

$$m\frac{dp^\beta}{d\tau} + \Gamma^\beta_{\mu\nu} p^\mu p^\nu = 0$$

Now we reparameterize the equitation in terms of $d\kappa = \frac{d\tau}{m}$

$$\boldsymbol{\frac{dp^\beta}{d\kappa} + \Gamma^\beta_{\mu\nu} p^\mu p^\nu = 0} \tag{27}$$

The above equation is more general in the sense that it can also be used for mass-less photons. Caution: In practice the right hand side of the geodesic equation becomes non-zero to include effects of real bodies and fields.

2.7 Curvature.

We have been discussing curved manifolds. Given a metric we have discussed several properties of such a "curved" space. But how do we quantify this curvature? Answer is the **Riemann Curvature** tensor: This section defines the Riemann curvature some intuitive reasoning for its form, leaving the rigorous derivation for the curious reader.

Revisit the equivalence principle which mathematically meant (in sufficiently small intervals of spacetime):

$$g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}[(\partial^2 g)(\delta x)^2]$$

. The metric on a curved manifold reduces to the Minkowski metric not only at the point but also in sufficiently small regions where the correction terms having second order derivatives of the metric are negligible. These correction terms arise due to the curvature which we want to quantify.

In terms of Christoffel symbols these correction terms contain derivatives and product of Christoffel symbols.

Consider parallel transport of a vector along a infinitesimal closed loop on a curved manifold. The final transported vector doesn't equal the initial vector because of the curvature of the manifold. The Riemann curvature tensor quantifies this curvature. Intuitively the change in the value of the vector must depend on the area of the loop, the magnitude of the vector itself and some quantification of curvature.

$$\delta v^\mu \sim \Omega v^\nu \Delta S^{\rho\sigma}$$

Naively, one can argue by matching indices for the above equation Ω that should be a tensor of the form $R^\mu_{\nu\rho\sigma}$ and indeed that is the case.

Definition: The Riemann **Curvature tensor** $R^\alpha_{\mu\rho\sigma}$ is a $\begin{pmatrix}1\\3\end{pmatrix}$ tensor given by:

$$\boldsymbol{[\nabla_\rho, \nabla_\sigma] v^\alpha = R^\alpha_{\ \mu\rho\sigma} v^\mu} \tag{28}$$

$$\boldsymbol{[\nabla_\rho, \nabla_\sigma] p_\alpha = -R^\mu_{\ \alpha\rho\sigma} p_\mu} \tag{29}$$

Writing the Lie commutator in terms of the Christoffel (using (17) and (25)) symbols gives:

$$\boldsymbol{R^\alpha_{\ \mu\rho\sigma} = \partial_\rho \Gamma^\alpha_{\ \sigma\mu} - \partial_\sigma \Gamma^\alpha_{\ \rho\mu} + \Gamma^\alpha_{\ \rho\nu}\Gamma^\nu_{\ \sigma\mu} - \Gamma^\alpha_{\ \sigma\nu}\Gamma^\nu_{\ \rho\mu}} \tag{30}$$

In presence of various forces(we will study gravity) $R^\alpha_{\mu\rho\sigma}$ shows how real objects couple with spacetime.

The Riemann tensor itself denotes the curvature of a manifold and thus it in turn encodes how parallel transport changes the vector along a closed curve, deviations of the metric from the Minkowski metric in the neighbourhood of a point, etc.

The Riemann tensor is a 4 index tensor and in 4 dimensions such a tensor in absence of any symmetries has $4^4 = 256$ independent components, but the nature of the Riemann tensors introduces several symmetries that reduce the number of independent components to $\frac{n^2(n^2-1)}{12}$ in n dimensions.
For 1-D: $\frac{n^2(n^2-1)}{12} = 0$ independent components. No Riemann curvature.
For 2-D: $\frac{n^2(n^2-1)}{12} = 1$ independent components. "Radius of curvature" scalar.
For 4-D: $\frac{n^2(n^2-1)}{12} = 20$ independent components.

Symmetries of the Riemann Tensor(derived using (21), (22), (17) and other properties of the metric and the Chirstoffel symbols)
Last two indices are anti-symmetric under exchange. Physically this can be thought of as reversal of direction of transport of the vector.

$$\boldsymbol{R^{\alpha}{}_{\mu\rho\sigma} = -R^{\alpha}{}_{\mu\sigma\rho}} \tag{31}$$

To study further symmetries lower the index by action of the metric and then move to the local Lorentz frame for ease of manipulation: $R_{\alpha\mu\rho\sigma} = g_{\alpha\nu}R^{\nu}{}_{\mu\rho\sigma}$ The tensorial nature of the equation ensures that the symmetries obtained in the local Lorentz frame (LLF) are the true symmetries in any frame. The Christoffels themselves go to zero in the limit but their derivatives do not vanish,

$$(R_{\alpha\mu\rho\sigma})^{LLF} = \partial_\rho\Gamma_{\alpha\sigma\mu} - \partial_\sigma\Gamma_{\alpha\rho\mu}$$

$$\Rightarrow (R_{\alpha\mu\rho\sigma})^{LLF} = \frac{1}{2}(\partial_\rho\partial_\mu g_{\alpha\sigma} - \partial_\rho\partial_\alpha g_{\sigma\mu} - \partial_\sigma\partial_\mu g_{\alpha\rho} + \partial_\sigma\partial_\alpha g_{\rho\mu})$$

Properties of $(R\alpha\mu\rho\sigma$:
1) $R_{\alpha\mu\rho\sigma} = -R_{\alpha\mu\sigma\rho}$ (exchange of last two)
2) $R_{\alpha\mu\rho\sigma} = -R_{\mu\alpha\rho\sigma}$ (exchange of first two)
3) $R_{\alpha\mu\rho\sigma} = R_{\rho\sigma\alpha\mu}$ (exchange of the first pair second pair)
4) $R_{\alpha\mu\rho\sigma} + R_{\alpha\rho\sigma\mu} + R_{\alpha\sigma\mu\rho} = 0$ (cyclic exchange of last three)

Ricci Tensor and Ricci scalar: The trace of the Riemann tensor on indices 1 and 3:

$$\boldsymbol{R^{\alpha}{}_{\mu\alpha\nu} = g^{\alpha\beta}R_{\beta\mu\alpha\nu} \equiv R_{\mu\nu}} \tag{32}$$

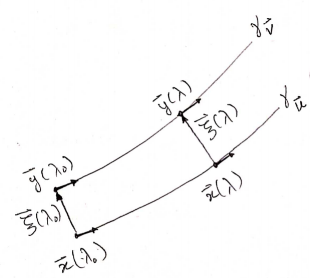
$R_{\mu\nu}$ is the symmetric 4x4 Ricci tensor (thus it has 10 independent components). Symmetry is obtained by mathematical manipulations from:

$$\boldsymbol{R_{\mu\nu} = \partial_\alpha\Gamma^{\alpha}{}_{\mu\nu} - \partial_\nu\Gamma^{\alpha}{}_{\alpha\mu} + \Gamma^{\alpha}{}_{\alpha\beta}\Gamma^{\beta}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\nu\beta}\Gamma^{\beta}{}_{\alpha\mu}} \tag{33}$$

Trace of the Ricci Tensor gives the Scalar curvature:

$$\boldsymbol{R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu} \equiv R} \tag{34}$$

Geodesic Deviation:
As discussed, parallel lines on cease to remain parallel on a curved manifold due to its Curvature. Now consider two initially parallel geodesics $\gamma_{\vec{v}}$ and $\gamma_{\vec{u}}$ parameterized by λ. The spacetime coordinates along these geodesics are $\vec{y}(\lambda)$ and $\vec{x}(\lambda)$ and the tangent vectors $\vec{v}$ and $\vec{u}$ respectively. Let $\vec{\xi}$ denote the separation vector field s.t $\vec{\xi} = \vec{y}(\lambda) - \vec{x}(\lambda)$. We want to find how $\vec{\xi}$ changes as λ changes.



The curvature of the quantifies the geodesic deviation as follows (using the geodesic equation and properties of Riemann tensor, metric):

$$\boldsymbol{\frac{D^2\xi^\alpha}{d\lambda^2} = R^\alpha_{\beta\gamma\delta} u^\beta u^\gamma \xi^\delta} \tag{35}$$

Here, $\frac{D^2\xi^\alpha}{d\lambda^2} = u^\sigma \nabla_\sigma$ and $u^\sigma = \frac{dx^\sigma}{d\lambda}$. All other terms have their usual meaning.

Bianchi Identity and Einstein Tensor
Symmetries of the Riemann Tensor and its Lie derivative reveal an important anti-symmetrization identity of the Riemann Tensor itself called the Bianchi Identity:

$$\nabla_{[\alpha} R_{\beta\gamma]\mu\delta} = 0$$

$$\boldsymbol{\Rightarrow \nabla_\alpha R_{\beta\gamma\mu\delta} + \nabla_\beta R_{\gamma\alpha\mu\delta} + \nabla_\gamma R_{\alpha\beta\mu\delta} = 0 = 0} \tag{36}$$

Mathematically manipulating the above equation and taking contractions with the metric of curved space yields a very important result:

$$\nabla^\mu (R_{\alpha\mu} - \frac{1}{2} g_{\alpha\mu} R) = 0$$

$$\boldsymbol{\nabla^\mu G_{\alpha\mu} = 0} \tag{37}$$

Here the Einstein tensor $G_{\alpha\mu}$ is defined as

$$\boldsymbol{G_{\alpha\mu} = R_{\alpha\mu} - \frac{1}{2} g_{\alpha\mu} R} \tag{38}$$

3 The Einstein Field Equation

Now we are armed with all the mathematical tools required for us to describe curved spaces and to finally lay down Einstein's ideas. This section focuses on giving a brief sketch on the derivation of the Einstein field equation using the weak field static limit i.e., the Newtonian limit. The familiar Newtonian equation relating the Laplacian ∇^2 Scalar potential Φ to the mass(energy) density ρ and the gravitational constant G is:

$$\boldsymbol{\nabla^2\Phi = \delta^{ij}\partial_i\partial_j\Phi = 4\pi G\rho} \tag{39}$$

The equation of motion, rather acceleration of a particle freely falling in this field is given by: $\frac{d^2x^i}{dt^2} = -\delta^{ij}\partial_i\Phi$
Now moving to the model we have developed so far this equation of motion must be the geodesic equation (26) in our spacetime of interest: $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$
The Newtonian limit should be the weak field, static limit of slow motion of the particle i.e., $\frac{dx^0}{d\tau} = \frac{dt}{d\tau} >> \frac{dx^i}{d\tau}$

$$\Rightarrow \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}(\frac{dt}{d\tau})^2 = 0$$

Also, since we are considering the weak field limit the metric of spacetime ($g_{\mu\nu}$) can be approximated by the Minkowski metric ($\eta_{\mu\nu}$) with a relatively small linear order perturbation ($h_{\mu\nu}$). Also note that in the we can ignore the terms second order ($\mathcal{O}(h^2)$) in $h_{\mu\nu}$ as a fair approximation because $|h_{\mu\nu}| << 1$.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

The inverse metric $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2)$. In the geodesic equation substitute:

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\nu}(\partial_0 g_{\nu 0} + \partial_0 g_{0\nu} - \partial_\nu g_{00}) \simeq -\frac{1}{2}g^{\mu\nu}\partial_\nu g_{00} = -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00} + \mathcal{O}(h^2)$$

The weak field limit with $\Gamma^0_{00} = 0$ yields:

$$\frac{d^2x^i}{dt^2} = -\delta^{ij}\partial_i h_{00}$$

This must be identical to the Newtonian limit which gives: $h_{00} = -2\Phi$

$$\boldsymbol{g_{00} = \eta_{00} + h_{00} = -(1+2\Phi)} \tag{40}$$

Now we come to Einstein's key insight that matter and energy cause space to curve and this curvature is what is perceived as gravity. Recollect that the stress energy tensor $T_{\mu\nu}$ models the energy content of the spacetime and local conservation means that it is a divergence-less (covariant derivative) tensor. Also recollect that the Einstein tensor $G_{\mu\nu}$ is a divergence-less tensor which quantifies curvature. A valid guess for the field equation can take the form: $G_{\mu\nu} = \kappa T_{\mu\nu}$. Our next goal is to find κ and this is where the expressions derived for the Newtonian limit come in.

The Einstein Field equation can be written in two equivalent forms using either trace of Ricci or trace of stress energy tensor: [Trace($V_{\mu\nu}$) $\equiv g^{\mu\nu}V_{\mu\nu}$]

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \text{ or } R_{\mu\nu} = \kappa(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T)$$

Einstein's theory must naturally reproduce Newton's gravity for weak static fields. To find κ we use the previously derived conditions for the weak field, but first we must consider a source. For the static Newtonian limit we pick a static Perfect Fluid as a source of gravity (extending (13) to curved space).

$$\Rightarrow \boldsymbol{T_{\mu\nu} = u_\mu u_\nu(\rho + P) + Pg_{\mu\nu}} \tag{41}$$

For static source: $\rho >> P$ and $u^\mu \equiv (u^0, 0, 0, 0)$.

Normalization condition gives: $g_{\mu\nu}u^\mu u^\nu = -1 \Rightarrow g_{00}u^0u^0 = -1$.

But in this limit we know that $g_{00} = -1 + h_{00}$. Thus, $u^0 \simeq 1 + \frac{1}{2}h_{00} + \mathcal{O}(h^2)$

$g_{\mu\nu}u^\nu = u_\mu \Rightarrow u_0 = -u^0$

Thus, $T_{00} = \rho u_0 u_0 = \rho(1 + h_{00})$ and the trace $T = g^{\mu\nu}T_{\mu\nu} = \rho u^\mu u_\mu = -\rho$

Thus, upon computing the right hand side of the field equation we get:

$$T_{00} - \frac{1}{2}g_{00}T = \rho(1 + h_{00}) - \frac{1}{2}(-1 + h_{00})(-\rho) \simeq \frac{1}{2}\rho$$

$$R_{00} = \kappa\frac{1}{2}\rho$$

Now we need $R_{00} = R^\mu_{\ 0\mu0}$ which upon simplification using conditions of the static field limit gives:

$$R_{00} = -\frac{1}{2}\delta^{ij}\partial_i\partial_j h_{00} = -\frac{1}{2}\nabla^2 h_{00}$$

We get:

$\nabla^2 h_{00} = -\kappa\rho$. But, $h_{00} = -2\Phi$ and $\nabla^2\Phi = 4\pi G\rho$,now we can solve for κ.

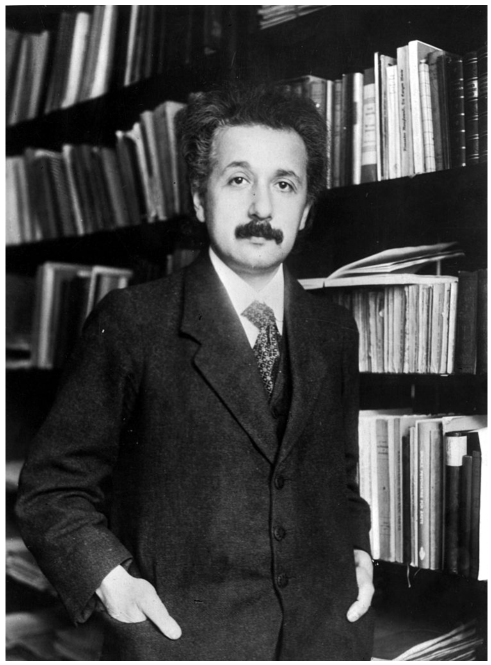
We get $\kappa = 8\pi G$

The Einstein Field equation has been worked out using the fact that both the Einstein tensor and the Stress Energy tensor are divergence-less, thus the consistency of the equation is unaltered upon addition of a multiple of the metric with appropriate coefficient because the metric is a divergence-less tensor associated with the space:

$$\boldsymbol{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}} \tag{42}$$

Λ is the cosmological constant whose effects are negligible even at galactic scales but will become important while discussing effects at cosmological scales.

Man of the century: Albert Einstein
1879-1955



The Field Equation

$$\boldsymbol{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}}$$

(published: 25 November, 1915)

4 The Schwarzschild Solution

4.1 The Schwarzschild metric and Birkhoff's Theorem

Armed with the Field equation we now move to spherically symmetric sources of Gravitation (with Mass M). So the goal is to find the metric of spacetime describing the exterior of a non-rotating spherically symmetric objects (idealised stationary stars). As one might already guess this solution will help model slow rotating stars like our sun to a pretty good degree of accuracy.

Recollect that for Newtonian gravity we solve the Laplace's equation outside such a source (say a star):

$\nabla^2\Phi = 0$ and thus, $\Phi = -\frac{GM}{r}$

In General Relativity we need to find a metric which solves $R_{\mu\nu} = 0$ taking into consideration the symmetries involved. Spherical symmetry and time independence means that the metric must be angle and time independent. Thus a general form for the spacetime interval can be:

$$ds^2 = -e^{2\Pi(r')}dt^2 + e^{2\Sigma'(r')}dr'^2 + e^{2\Upsilon'(r')}d\Omega^2$$

Here $\Omega = d\theta^2 + sin^2(\theta)d\phi^2$ and all coefficients are functions of only the radial coordinate. We can reparametrize this form to a more physically useful form by choosing $e^{\Upsilon'} = r$ and $e^{\Sigma'(r')}dr' = e^{\Sigma(r)}dr$. The equation becomes:

$$ds^2 = -e^{2\Pi(r)}dt^2 + e^{2\Sigma(r)}dr^2 + r^2d\Omega^2$$

Now we find components of the Ricci tensor using equations (22) and (33). We get:

1. $R_{tt} = e^{\Pi-\Sigma}(\partial_r^2\Pi + (\partial_r\Pi)^2 - \partial_r\Pi\partial_r\Sigma + \frac{2}{r}\partial_r\Pi)$
2. $R_{rr} = -\partial_r^2\Pi - (\partial_r\Pi)^2 + \partial_r\Pi\partial_r\Sigma + \frac{2}{r}\partial_r\Sigma$
3. $R_{\theta\theta} = e^{-2\Sigma}(r(\partial_r\Sigma) - \partial_r\Pi) - 1) + 1$
4. $R_{\phi\phi} = sin^2(\theta)R_{\theta\theta}$

Using the above equations with $R_{\mu\nu} = 0$:

$\Pi = -\Sigma$ and $e^{-2\Sigma} = \left(1 - \frac{R_s}{r}\right)$. Here R_s is a constant of integration obtained while solving the differential equations. Thus the spacetime interval becomes:

$$\boldsymbol{ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1}dr^2 + r^2d\Omega^2} \tag{43}$$

From equation (40) and the usual Newtonian potential in 3-D the Newtonian limit for a metric component g_{00} of such spherical static source is:

$$g_{00} = -(1 + 2\Phi) = -\left(1 - \frac{2GM}{r}\right) = -\left(1 - \frac{R_s}{r}\right)$$

The constant $R_s = 2GM$ is called the **Schwarzschild Radius** and the Schwarzschild solution takes the form:

$$\boldsymbol{ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2} \tag{44}$$

We observe:

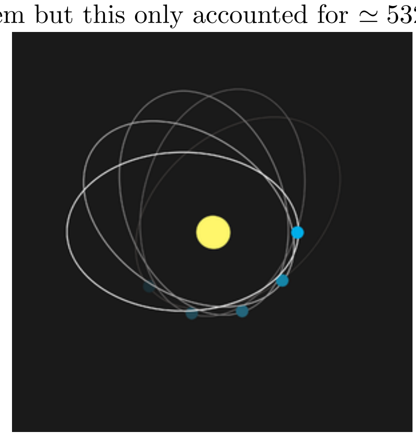
1. for the limit $r \to \infty$ we approach flat space.
2. for the limit $r \to R_s$ something weird happens.
3. for the limit $r \to 0$ there is a Singularity.

For now we are concerned with the region $r > R_s$ and on top of this we consider only exterior of the source. The sun for example has $R_s \simeq 3km$ which is much smaller than the Sun's radius hence we don't need to worry about the singularities for now.

Birkhoff's Theorem: This is a very powerful theorem which states that the exterior vacuum of ANY spherically symmetric body is UNIQUELY described by the Schwarzschild Metric!

4.2 Mercury's Orbit

Now, we arrive at the first test of General Relativity. **Precession of Mercury's orbit** : The plant Mercury's orbit was observed to precess i.e., the perigee of it's elliptical orbit shifts slightly over time. Quantitatively the shift observed was: $\simeq$ 574 arcsec/century, where 1 arcsec $\simeq$ 0.000277778 degrees. Initially it was thought that this was due the gravitational purturbances of fields of other planets in the solar system but this only accounted for $\simeq$ 532.5 arcsec/century.



This mystery was solved by Einstein's General Relativity which predicted the shift to be $\simeq$ 573 arcsec/century which matched the observed shift to a staggering accuracy (only about 0.17% off). Let's have a brief look at how this was done. We want to find the geodesic equation in such a spacetime.

The Schwarzschild Solution for the Field equation for a geodesic at $\theta = \pi/2$ (using (43))

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + d\phi^2)$$

$$\Rightarrow g_{\mu\nu} = diag\left(-\left(1 - \frac{R_s}{r}\right), \left(1 - \frac{R_s}{r}\right)^{-1}, r^2, r^2\right)$$

A massive object has a time-like geodesic with $x^\mu \equiv (t(\tau), r(\tau), \theta(\tau), \phi(\tau))$ where τ is the proper time of the object moving on the geodesic. Now one can either choose to solve for all Christoffels, find the curvature, etc. or exploit the symmetries of the problem and simplify using the killing equation.

The metric has time translation and rotational symmetries thus we can find two killing fields. [the reader can have a quick look at (2.4)]
The killing vector fields and duels (lowering indices using $g_{\mu\nu}$) are:

1. Time(t): $\xi^{(t)\nu} \equiv (1, 0, 0, 0) \Rightarrow \xi^{(t)}_{\mu} \equiv \left(-\left(1 - \frac{R_s}{r}\right), 0, 0, 0\right)$
2. Angle(ϕ): $\xi^{(\phi)\nu} \equiv (0, 0, 0, 1) \Rightarrow \xi^{(\phi)}_{\mu} \equiv (0, 0, 0, r^2)$

The contraction of the 4-velocity ($u^\mu = dx^\mu/d\tau$) with the respective killing fields gives a conserved quantity ($E \equiv Energy\ and\ L \equiv Angular\ momentum$):

1. $u^\mu \xi^{(t)}_{\mu} = \frac{dt}{d\tau}\left(-\left(1 - \frac{R_s}{r}\right)\right) = -E \Rightarrow \frac{dt}{d\tau} = \left(1 - \frac{R_s}{r}\right)^{-1} E$
2. $u^\mu \xi^{(\phi)}_{\mu} = \frac{d\phi}{d\tau} r^2 = L \Rightarrow \frac{d\phi}{d\tau} = \frac{L}{r^2}$

Now using property of norm of 4-velocity: $\vec{u} \cdot \vec{u} = g_{\mu\nu} u^\mu u^\nu = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1$.
Substituting values,

$$-1 = -\left(1 - \frac{R_s}{r}\right)^{-1} E^2 + \left(1 - \frac{R_s}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{r^2}$$

$$\Rightarrow \boldsymbol{\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) = \tilde{E}} \tag{45}$$

$$\boldsymbol{V_{eff} = -\frac{R_s}{2r} + \frac{L^2}{2r^2} - \frac{R_s L^2}{2r^3} = -\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}} \tag{46}$$

Here $\tilde{E} = \frac{1}{2}E^2 - \frac{1}{2}$ is a constant denoting total energy, V_{eff} is the effective energy and the above equation is the net energy content of the system.

Foe Newtonian Gravity the effective potential takes the form as follows:

$$\boldsymbol{V_{eff(N)} = -\frac{R_s}{2r} + \frac{L^2}{2r^2} = -\frac{GM}{r} + \frac{L^2}{2r^2}} \tag{47}$$

The above potential has conic sections as its solutions for eq (45) given L and E.

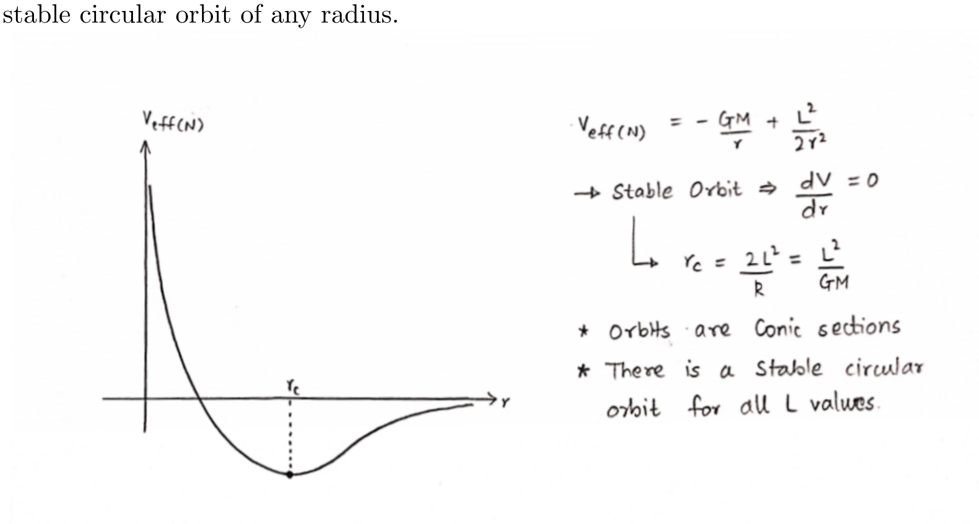
The Newtonian solution geodesics are conic sections. That is not the case for the solution obtained from potential in (46). The correction term $\frac{R_s L^2}{2r^3}$ means that the orbits will not close on themselves! This correction term is minute when $r >> R_s$, this is realised when we go back to S.I. from natural units being used so far $\frac{R_s L^2}{2r^3}(N.U.) \equiv \frac{R_s L^2}{2r^3 c^2}(S.I.)$. This is what causes the Precession of orbits of all planets though as expected the effect is maximum in case of Mercury. The exact calculation of shift per orbit is left for the curious reader but the above section covers the key ideas.

4.3 Analysing geodesics in the Schwarzschild spacetime, Black holes

Having derived the equation for the potential for geodesics for the Schwarzschild metric in the previous section we are now equipped with enough mathematical equations to study the nature of these orbits. But first, Newtonian gravity.

$$V_{eff(N)} = -\frac{R_s}{2r} + \frac{L^2}{2r^2} = -\frac{GM}{r} + \frac{L^2}{2r^2}$$

The nature of the graph is the same for all values of the parameter L. A stable circular orbit exits for all L values. Also there is no restriction on r. That means if an object is given the necessary angular momentum it can trace a stable circular orbit of any radius.

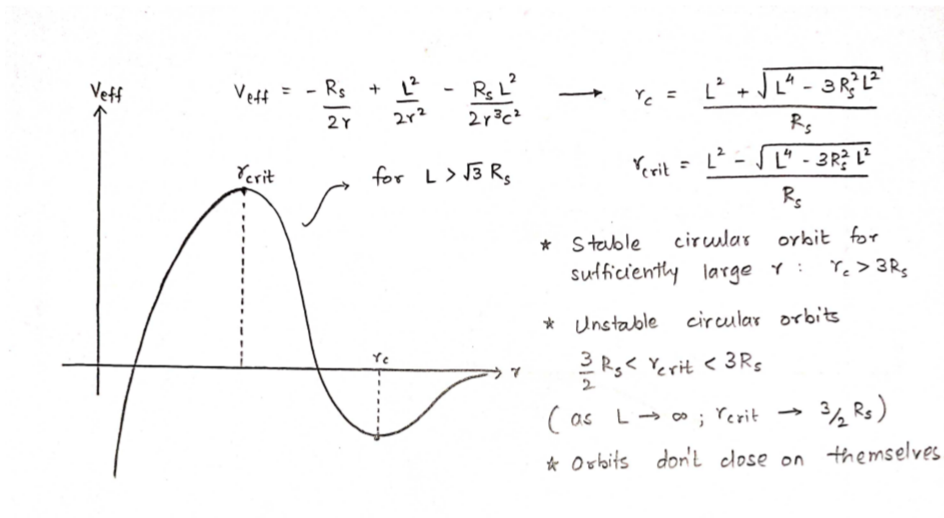


The General Relativistic potential is:

$$V_{eff} = -\frac{R_s}{2r} + \frac{L^2}{2r^2} - \frac{R_s L^2}{2r^3}$$

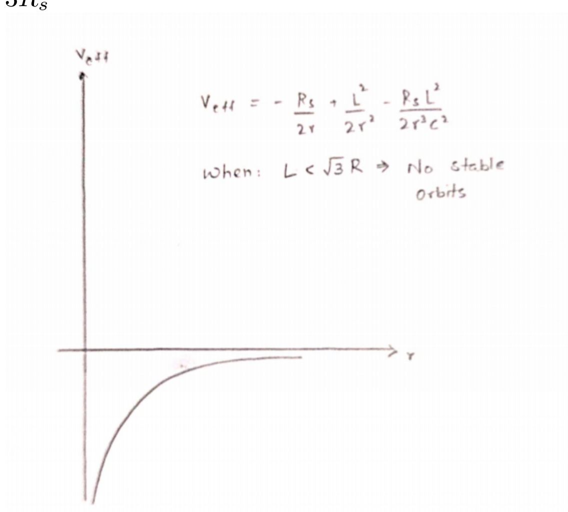
Simply differentiating reveals that the above potential has different number of critical points based on the value of L. Let us examine the nature of these orbits graphically.

Case 1: $L > \sqrt{3}R_s$



The Graph has a Potential peak at critical radius: $r_{crit} = \frac{L^2 - \sqrt{L^4 - 3R^2L^2}}{R_s}$. Stable orbits are possible only at sufficiently large r value beyond r_{crit}. The value of r at which there is a potential valley meaning that we can have a stable orbit is (The orbit will still not close on itself since it isn't a conic section): $r_c = \frac{L^2 + \sqrt{L^4 - 3R^2L^2}}{R_s}$.

Case 2: $L < \sqrt{3}R_s$



The graph has no critical points if the value of L is below a certain minimum. Thus the object simply cannot have a stable orbit around the source if it doesn't have a minimum angular momentum parameter.

So far we haven't talked about the absurd behavior of the time coordinate at the Schwarzschild radius R_s and the singularity at r=0 in the Schwarzschild metric. This was fine so far since we were anyways concerned about the exterior of most ordinary stars (like our own sun) whose radius is much bigger than the Schwarzschild radius $R_* >> R_s$. Note that we'll skip the study of interior of ordinary stars which is modelled using pressure density relations $P = P(\rho)$ for the stress energy tensor. Recollect how a classical point charge in Electrodynamics works, we describe it using a delta function with infinite charge density at single point generating the electric field.

A Black Hole is a remnant of a collapsed star with all its mass concentrated at a single point of infinite density. Space around a black-hole is completely described by the Schwarzschild metric with coordinates centered at the center of a black-hole.

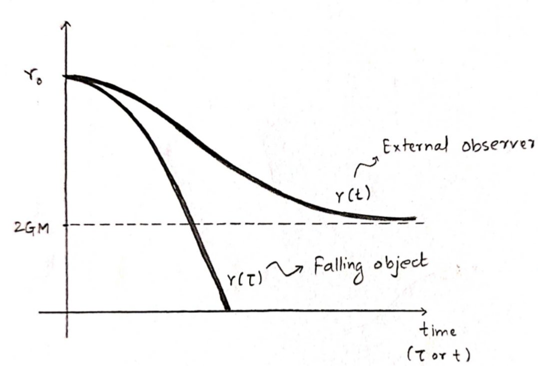
$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

We see the expected singularity at $r = 0$ but something weird happens at $r = R_s = 2GM$.

To understand this consider a thought experiment. Before discussing the experiment we need to understand what the coordinate t denotes. If we take the limit $r \to \infty$ then we arrive at the Minkowski metric, thus the t coordinate must be the time measured by a distant observer. Now let's drop a particle at $r = r_0 > R_s$. First find the Christoffel symbols using the metric (22). Using the geodesic equation (26), we can integrate it to obtain the parameter τ and t in terms of the radius. We get:

$$\frac{\tau}{R_s} = \frac{2}{3}\left[\left(\frac{r_0}{R_s}\right)^{3/2} - \left(\frac{r}{R_s}\right)^{3/2}\right]$$

$$\frac{t}{R_s} = ln\left[\frac{(r/R_s)^{1/2} + 1}{(r/R_s)^{1/2} - 1}\right] + 2\sqrt{\frac{r}{R_s}}\left(1 + \frac{r}{6GM}\right)$$



The striking observation of the above graph is that the external observer never sees the falling object cross the surface at $r = R_s = 2GM$ whilst the falling object in its own frame falls at an ever increasing rate.

Now, consider another thought experiment. Our falling object now emits a pulse of light (EM wave) during the course of its fall at some $r > R_s$. We measure this light pulse at some radial distance $R >> R_s$, what do we observe? Far from the source (nearly flat spacetime), 4-momentum of photon is $p_\alpha = h\nu(-1, 1, 0, 0)$ and for an observer with 4-velocity $\vec{u}$ the observed energy of the photon is $E_{\vec{u}} = -\vec{p} \cdot \vec{u} = -p_\alpha u^\alpha$ For a static observer $\vec{u} = (u^t, 0, 0, 0)$. The condition $g_{\alpha\beta} u^\alpha u^\beta = -1 \Rightarrow u^t = \left(1 - \frac{2GM}{r}\right)^{-1/2}$. Energy of the photon at a distance r' is $E(r') = -\vec{p} \cdot \vec{u} = -p_\alpha u^\alpha = -p_t u^t(r')$ where p_t is a constant.

$$\boldsymbol{\frac{E_{obs}(R)}{E_{emit}(r)} = \sqrt{\frac{1 - \frac{2GM}{r}}{1 - \frac{2GM}{R}}} = \sqrt{\frac{1 - \frac{R_s}{r}}{1 - \frac{R_s}{R}}}} \tag{48}$$

Consider a far away observer. As $r \to R_s$ $\frac{E_{obs}(R)}{E_{emit}(r)} \to 0$. No matter how energetic the light pulse is, it never reaches the observer. In fact for $r \leq 2GM$ **all time-like trajectories point towards r=0**! As radius decreases light-cones seem to tilt toward r=0 at an ever increasing rate. Every direction is inwards which means trying to move anywhere just makes the journey to r=0 faster, the slowest way to reach the center is to do nothing and meet the inevitable end. Thus the name, Black hole: a stellar object from which not even light can escape.

This coordinate singularity at $r = 2GM = R_s$, the point of no return is called the **Event Horizon**. No event inside the Event horizon can affect anything outside and nothing outside can affect anything inside. The event horizon in some sense is the horizon of causal connection.

We have discussed how equatorial geodesics of massive objects behave in this spacetime. What about mass-less objects i.e., photons which travel on null geodesics. For null geodesics $\vec{p} \cdot \vec{p} = 0$. (similar to section (4.2))

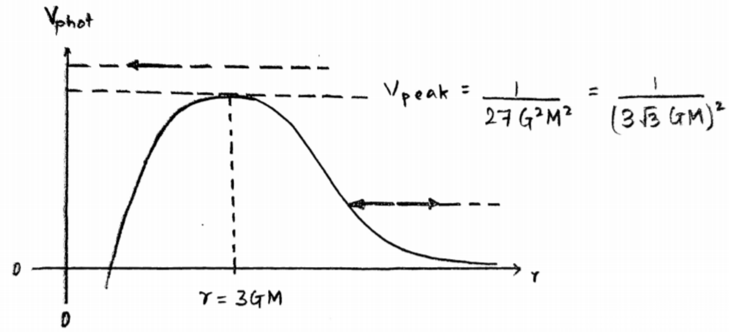
$$\Rightarrow -\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

$$\Rightarrow \left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{L^2}{r^2}\left(1 - \frac{2GM}{r}\right)$$

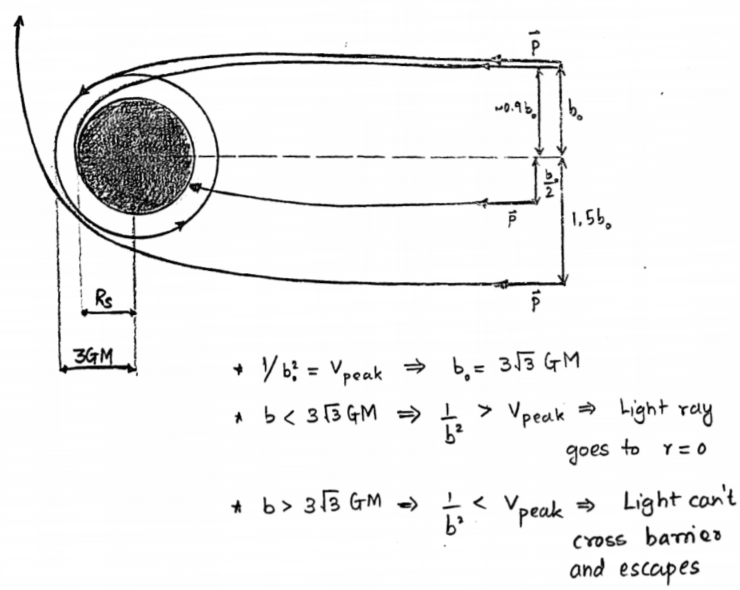
Here, λ is the affine parameter. $E = -p_t = \frac{dt}{d\lambda}\left(1 - \frac{2GM}{r}\right)$ is the apparent energy parameter and $L = r^2 \frac{d\phi}{d\lambda}$ is the apparent angular momentum parameter. Reparameterize $\lambda \to L\lambda$ and define Impact Parameter $b = L/E$ to get:

$$\boldsymbol{\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right) = \frac{1}{b^2} - V_{phot}(r)} \tag{49}$$

Graphically V_{phot} is:



The nature of the orbit depends on the value of the impact parameter b which can be thought of as quantifying angular momentum. A photon can assume an orbit of radius $3GM$ about the black hole only if the impact parameter $b = 3\sqrt{3}GM$. For any values higher the photon escapes the gravitational field and for any lower value the photon falls into the black hole. The region $r = 3GM$ is called the **Photon Sphere**.



The above diagram is a rough sketch to show how photons with different angular momenta (w.r.t black hole's center) behave under the field of a black hole.

5 Cosmology

5.1 Maximally Symmetric Spaces

Cosmology is the study of the large scale structure of the universe. We construct theoretical models which closely resemble the real universe by applying several symmetry restrictions.
On cosmological scales (where even galaxies can be treated as point particles) the universe is spatially homogeneous and isotropic. In simple words, homogeneous means that at these scales any two points have identical neighbourhood. Isotropic means that all directions are identical, there are no preferred directions in such a space. A space with the above properties is a Maximally Symmetric Space (MSS). Note that 4-D spacetime is neither homogeneous nor isotropic on cosmological scales. Thus, our metric must be of the form:

$$ds^2 = -dt^2 + a(t)^2 d\sigma_k^2$$

Here t is the time coordinate, σ_k is the spatial (3-D) part with k taking one of three values. $k = 0$ for flat space. $k = 1$ for space with positive curvature and $k = -1$ for negative curvature. $a(t)$ is the scale factor, more on this later. In the familiar 3-D case a flat plane has no curvature ($\mathbb{R}^2$), positively curved surface is a 2-Sphere ($\mathbb{S}^2$) and a negatively curved surface is a hyperboloid ($\mathbb{H}^2$).
For MSS: (Ricci scalar R computed using (22) (33) (34))

1. Zero curvature Euclidean space (open spatial slices)
   $\mathbb{R}^3 : k = 0 \Rightarrow d\sigma_0^2 = dx^2 + dy^2 + dz^2$
   $R = 0$

2. Positive curvature 3-Sphere (closed spatial slices)
   $\mathbb{S}^3 : k = 1 \Rightarrow d\sigma_1^2 = d\theta^2 + sin^2(\theta)(d\phi^2 + sin^2(\phi)d\psi^2) = d\theta^2 + sin^2(\theta)d\Omega^2$
   $R = 6$

3. Negative curvature Hyperboloid (open spatial slices)
   $\mathbb{H}^3 : k = -1 \Rightarrow d\sigma_2^2 = d\theta^2 + sinh^2(\theta)(d\phi^2 + sin^2(\phi)d\psi^2) = d\theta^2 + sinh^2(\theta)d\Omega^2$
   $R = -6$

Here $d\phi^2 + sin^2(\phi)d\psi^2 = d\Omega^2$.The above three equations can be reduced to one by a convenient change of coordinates where we set coefficient of $d\Omega^2$ equal to r^2. Thus, the spatial part of the line element becomes:

$$\boldsymbol{d\sigma_k^2 = \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2} \tag{50}$$

$k \in \{-1, 0, 1\}$
Spatial Isotropy and Homogeneity of the 3-D spatial parts lead to:
$R^{(3)} = 6k$ and $R_{ij}^{(3)} = 2kg_{ij}^{(3)}$. (sup(3)≡ 3-D)

5.2 FRW metric and Friedmann equations

The Metric of these Maximally Symmetric Spaces become:

$$\boldsymbol{ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right)} \tag{51}$$

This is called the **Friedmann–Robertson–Walker** metric and this model is called the FRW model universe. This describes the nature of our universe on large scales. The matter or energy content of this MSS is modelled using the Stress Energy tensor of a Perfect fluid.

$$T_{\mu\nu} = u_\mu u_\nu (\rho + P) + P g_{\mu\nu}$$

But MSS$\Rightarrow$ $u^\mu = (1, 0, 0, 0)$. Thus, $T_{00} = \rho$. Spatial part $T_{ij} = P g_{ij}$.

Plugging the metric from ds^2 and $T_{\mu\nu}$ into the Einstein field equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$ we get the **Friedmann equations** describing the dependence of scale factor $a(t)$ and its derivatives on energy Density, Pressure and curvature parameter (k):

$$\boldsymbol{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}} \tag{52}$$

$$\boldsymbol{\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3}(\rho + 3P)} \tag{53}$$

Here P and ρ are functions of time. Using local conservation of energy:

$$\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \nabla_\mu T^\mu_0 = 0 = -\dot{\rho} - 3\frac{\dot{a}}{a}(P + \rho) \Rightarrow \frac{\dot{\rho}}{P + \rho} = -3\frac{\dot{a}}{a}$$

If the perfect fluid has $P = \omega\rho$ then:

$$\frac{\dot{\rho}}{\rho}\frac{1}{1 + \omega} = -3\frac{\dot{a}}{a}$$

$$\boldsymbol{\rho = \rho_0 \; a^{-3(1+\omega)}} \tag{54}$$

Here ρ_0 is a constant. The above expression tells us how the energy density evolves with time. Different types of matter/energy evolve in different ways:

1. Dust: Pressure-less matter i.e, $\omega = 0 \Rightarrow \;\rho = \rho_0 \; a^{-3}$.
2. Radiation: has $\omega = 1/3 \Rightarrow \;\rho = \rho_0 \; a^{-4}$.
3. Vacuum Energy: has $\omega = -1 \Rightarrow \;\rho = \rho_0$. The energy density doesn't dilute upon expansion of space!
4. Curvature of spatial slices: can be thought of as an energy density term with $\rho = \rho_0 \; a^{-2}$.

Here the curvature term in eq(52) is expressed as an energy density so that:

$$\rho_k = -\frac{3k}{8\pi G}\frac{1}{a^2}$$

Equation (52) can be equivalently written as:

$$\boldsymbol{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i} \tag{55}$$

Where ρ_i are the contributions due to different types of matter-energy densities.
The **Hubble Parameter** (with dimensions $\sim L^{-1}$) is defined as $H = \left(\frac{\dot{a}}{a}\right)$.

$$H^2 = \frac{8\pi G}{3}\sum_i \rho_i$$

The expansion of the universe is controlled by a combination of the above types of matter-energy densities. For simplicity we now look at how the universe must expand if it is dominated by one kind of matter-energy. Consider a form of matter with $P = \omega\rho \Rightarrow \rho \sim a^{-3(1+\omega)}$. We'll be ignoring constants of proportionality since in the final equation they can be absorbed by appropriate reparameterization of coordinates. Eq(55) gives:

$$\left(\frac{\dot{a}}{a}\right)^2 \sim a^{-3(1+\omega)}$$

$$\dot{a}\; a^{\left(\frac{3\omega}{2}+\frac{1}{2}\right)} = constant$$

Integrating the above relation (for $\omega \neq -1$) we get the time dependence of $a(t)$ as:

$$a \sim (t-t_0)^{\left(\frac{2}{3(\omega+1)}\right)}$$

$$\boldsymbol{ds^2 = -dt^2 + t^{\left(\frac{4}{3(\omega+1)}\right)} d\sigma_k^2} \tag{56}$$

Metric for such a Universe is called FRW metric with polynomial expansion. What about the case when $\omega = -1 \Rightarrow \left(\frac{\dot{a}}{a}\right) = H = constant$? For $\omega = -1$ the mysterious Vacuum Energy (Cosmological constant Λ) dominates the expansion of the universe. The Universe expands Exponentially!

$$\boldsymbol{ds^2 = -dt^2 + e^{Ht}(|d\vec{x}|)^2} \tag{57}$$

5.3 Our Universe

Having developed such models for our universe what is actually the observed state of our universe today? The first observation is that the contribution of Radiation energy is negligible. Another observation is that out of the total matter in the universe only about 15% is visible matter, the rest is Dark Matter- a weakly interacting form of matter which doesn't interact with light. It's presence is known because of its observed gravitational effects. We don't understand composition of dark matter but can observe its gravitational effects. The matter-energy terms for dust (ordinary matter), curvature, cosmological constant are written by incorporating Hubble's Parameter H:

1. $\Omega_m = \frac{8\pi G}{3H^2}\rho_m$. Observed value of $\Omega_m \simeq 0.25$.
2. $\Omega_c = -\frac{k^2}{H^2 a^2}$. Observed value of $\Omega_c \simeq 0$. Currently universe is flat on large scales.
3. $\Omega_\Lambda = \frac{8\pi G}{3H^2}\rho_\Lambda$. Observed value of $\Omega_\Lambda \simeq 0.75$.

(For FRW: $\sum \Omega_i = 1$). It is quite interesting to ponder about why we live in the special point of time in the Universe's long life when the Matter and Cosmological constant terms are of the same order with the Universe being nearly flat on the grandest of scales.

The mysterious Vacuum Energy is also called **Dark energy**. Dark energy appears to be a non-diluting energy density which causes space to expand exponentially. It is some form of energy of Space itself, thus the name Vacuum energy. The only thing opposing the effect of Dark matter and keeping our universe from expanding at this crazy rate is the presence of matter. But in the distant future of our currently expanding universe, Dark energy will only become stronger compared to gravity and will take over.

Observation and Measurement gives the current rate of expansion of the universe in Astrophysically useful units to be:

$$H_{today} \simeq 70\, km\ s^{-1}\ MpC^{-1}$$

$1\ MpC = 3.262 \cdot 10^6\ lightyear$.

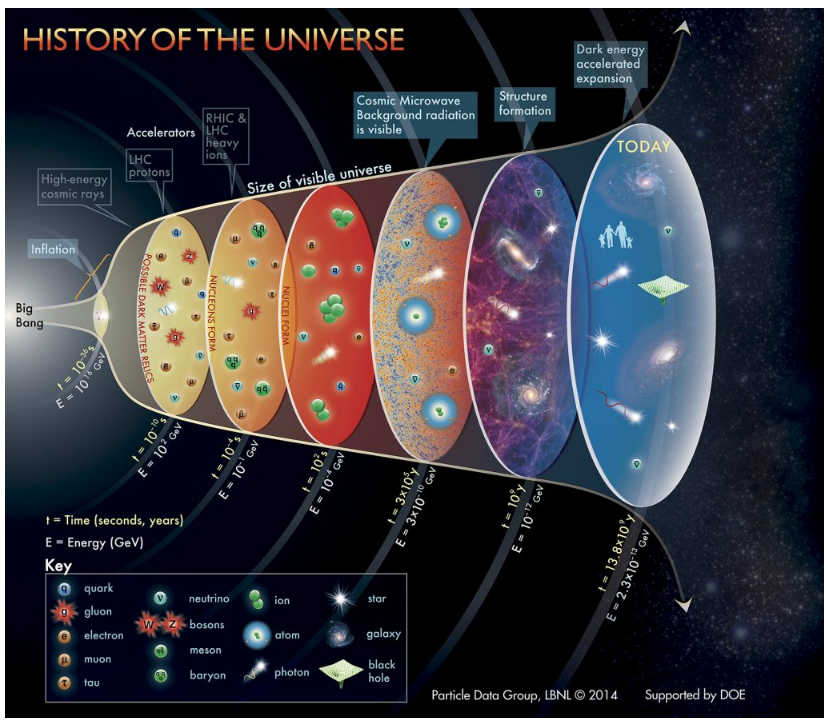
Thus a Galaxy at a distance of 2 MpC from us appears o be receeding away at a speed of 140 km/s

5.4 The Early Universe

Observations of the night sky give us not only the nature of our universe today but billions of years back in time. The light of some of the distinct stars and galaxies that we observe has taken millions if not billions of years to reach us. In some sense the night sky is a time machine which allows us to see the distant past of our universe going back to the beginning of time itself! Our universe is a staggering 13.77 billion years old. Light from various objects has had nearly 14 billion years to reach us which encompasses everything that constitutes our Observable universe. There simply hasn't been enough time for light from objects farther away from the edge of the observable universe to reach us. So, what was the early universe like?

The expansion of the early universe was controlled by very different circumstances compared to what we see now. The universe was a hot and dense plasma of Gluons and Quarks. Over time it cooled to form nucleons-Nuclear matter by the process of confinement. Further cooling and recombination of the Nuclear matter led to formation of the first atoms, predominantly Hydrogen and Helium. With the Cosmic Microwave background and any cosmological observation we can only see back in time up to when recombination started and the universe was no more opaque to light, everything before that is theory. The universe was opaque to light meaning that any photon couldn't have escaped without being reabsorbed by the hot dense plasma. This is in brief a part of the theory of Big Bang Nucleosynthesis.

Inflation: We have already seen that the Cosmic Microwave back-ground along with the observations of the night sky reveal that our universe is homogeneous and isotropic on large spatial scales. But is this even possible? It is as though distant patches or regions of space which were never in causal contact somehow attained thermal equilibrium in the distant past. The deviation from homogeneity is merely $\frac{\delta\rho}{\rho} = 10^{-5}$, which means that these seemingly isolated regions must have been causally connected. These observation are explained by what is called Inflation: the early universe expanded exponentially to grow about e^{60} in volume in a small fraction of its lifetime. Modern physics explains the minute deviations $\frac{\delta\rho}{\rho}$ as being due to quantum fluctuations.



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