$$\boxed{T^{\mu'\nu'}{}_{\rho'\tau'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \cdot \frac{\partial x^{\nu'}}{\partial x^{\nu}} \cdot \frac{\partial x^{\rho}}{\partial x^{\rho'}} \cdot \frac{\partial x^{\tau}}{\partial x^{\tau'}} \cdot T^{\mu\nu}{}_{\rho\tau}}$$

★ The Metric in curved spacetime : Symmetric $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ tensor $g = |g_{\mu\nu}| \neq 0$

$\boxed{g_{\mu\nu}}$ and $\underbrace{g^{\mu\nu} g_{\nu\sigma} = g_{\lambda\sigma} g^{\lambda\mu}}_{\text{Inverse of the metric.}} = \delta^{\mu}_{\sigma}$

$$\left.\begin{aligned} ds^2 &= g_{\mu\nu}\, dx^{\mu} dx^{\nu} \equiv \tilde{\tilde{g}}(d\tilde{x}, d\tilde{x}) \\ g_{\mu\nu} V^{\mu} W^{\nu} &= \tilde{\tilde{g}}(\vec{V}, \vec{W}) \end{aligned}\right\}$$ Metric Tensor defines the notion of Norms / distances

★ In the local lorentz frame $g_{\hat{\mu}\hat{\nu}}(p) = \eta_{\hat{\mu}\hat{\nu}}$ and $\partial_{\hat{\sigma}}\, g_{\hat{\mu}\hat{\nu}}(p) = 0$

@ point p

$$g_{\hat{\mu}\hat{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}} \cdot \frac{\partial x^{\nu}}{\partial x^{\hat{\nu}}} \cdot g_{\mu\nu}$$

– Expand this using Taylor around pt. p, we observe that we have enough DOFs to set ∂g's $= 0$ but not $\partial^2 g$'s. → Curvature.

★ Locally Inertial coods are very useful : solve something in L.L.F. and generalize the TENSORIAL EQN to any frame.

★ An expanding universe : $ds^2 = -dt^2 + a^2(t)\,[dx^2 + dy^2 + dz^2]$

$a(t) = t^{q}$ $\quad q \in (0,1) \Rightarrow t = 0$ is a singularity.

$\{ q = 1/2$: matter dominated flat , $q = 2/3$: radiation dominated flat $\}$

for y & z const : $\quad \boxed{\frac{dx}{dt} = \pm t^{-q} \Rightarrow t = (1-q)^{1/(1-q)} (\pm x - x_0)^{1/(1-q)}}$

leads to causal Horizons.

★ Tensor Densities :

(1) Levi Civita Symbol : $\tilde{\epsilon}_{\mu_1 \mu_2 \cdots \mu_n} = \begin{cases} +1, & \text{even perm of } 0,1,2 \ldots (n-1) \\ -1, & \text{odd} \quad \text{—— " ——} \end{cases}$

A Tensor Densities:

① Levi Civita Symbol: $\bar{\epsilon}_{\mu_1 \mu_2 \cdots \mu_n} = \begin{cases} +1, & \text{even perm of } 0,1,2 \ldots (n-1) \\ -1, & \text{odd} \text{ — " —} \\ 0, & \text{otherwise.} \end{cases}$

for any $n \times n$ matrix: $\bar{\epsilon}_{\mu_1' \mu_2' \cdots \mu_n'} |M| = \bar{\epsilon}_{\mu_1 \mu_2 \cdots \mu_n} M^{\mu_1}{}_{\mu_1'} M^{\mu_2}{}_{\mu_2'} \cdots M^{\mu_n}{}_{\mu_n'}$

set $M^{\mu'}{}_{\mu} = \dfrac{\partial x^{\mu'}}{\partial x^{\mu}}$ ↳ This gives: $\bar{\epsilon}_{\mu_1' \mu_2' \cdots \mu_n'} = \left| \dfrac{\partial x^{\mu'}}{\partial x^{\mu}} \right| \bar{\epsilon}_{\mu_1 \mu_2 \cdots \mu_n} \dfrac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdot \dfrac{\partial x^{\mu_2}}{\partial x^{\mu_2'}} \cdots$

determinant of Jacobian

② Determinant of the metric: $g(x^{\mu'}) = \left| \dfrac{\partial x^{\mu'}}{\partial x^{\mu}} \right|^{-2} g(x^{\mu})$ g is a scalar density of weight -2.

Power of Jacobian ≡ weight (w)

Tensor density $\xrightarrow{\text{multiply by } |g|^{w/2}}$ Tensor.

☆ Levi Civita Tensor:

$$\epsilon_{\mu_1 \mu_2 \cdots \mu_n} = \sqrt{|g|}\, \bar{\epsilon}_{\mu_1 \mu_2 \cdots \mu_n} \qquad \text{wt. } w = +1$$

$$\epsilon^{\mu_1 \mu_2 \cdots \mu_n} = \frac{1}{\sqrt{|g|}}\, \bar{\epsilon}^{\mu_1 \mu_2 \cdots \mu_n}$$

☆ Integral of a scalar function over a n-manifold:

$$\boxed{I = \int \phi(x) \sqrt{|g|}\, d^n x} \qquad dV \equiv \sqrt{|g|}\, d^n x$$

☆ Curved Spaces: One imp thing that we need to take into account is that basis vectors aren't const like in cartesean cood space.

$$\vec{V} = V^{\alpha} \vec{e}_{\alpha} \Rightarrow \frac{\partial \vec{V}}{\partial x^{\beta}} = \left(\frac{\partial V^{\alpha}}{\partial x^{\beta}} \right) \vec{e}_{\alpha} + \left(\frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}} \right) V^{\alpha}$$

$$\boxed{\frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}} = \Gamma^{\mu}{}_{\beta\alpha}\, \vec{e}_{\mu}}$$

↳ Any vec ≡ linear combo of basis

a.k.a. Connection coeffs.

$\Gamma^{\mu}{}_{\beta\alpha}$ are the christoffel Symbols - Show how the basis vectors change from pt to pt.

☆ Defining a Tensorial notion of a derivative of vector components.

$$\partial_{\beta} \vec{V} = \partial_{\beta} (V^{\alpha} \vec{e}_{\alpha}) = (\partial_{\beta} V^{\alpha}) \vec{e}_{\alpha} + (\partial_{\beta} \vec{e}_{\alpha}) V^{\alpha}$$

$$= (\partial_{\beta} V^{\alpha}) \vec{e}_{\alpha} + (\Gamma^{\mu}{}_{\beta\alpha} \vec{e}_{\mu}) V^{\alpha} = \vec{e}_{\alpha} \underbrace{(\partial_{\beta} V^{\alpha} + \Gamma^{\alpha}{}_{\beta\mu} V^{\mu})}$$

μ & α are dummy indices
relabel by $\mu \leftrightarrow \alpha$

$\vec{e}_\alpha \left(\nabla_\beta V^\alpha \right)$

COVARIENT DERIV: $\boxed{\nabla_\beta V^\alpha = \partial_\beta V^\alpha + \Gamma^\alpha{}_{\beta\mu} V^\mu}$

Notation $\left\{ \tilde{\nabla} \vec{V} = \left(\nabla_\beta V^\alpha \right) \tilde{\omega}^\beta \otimes \vec{e}_\alpha \right\}$

★ for Scalars: $\partial_\beta \Phi = \nabla_\beta \Phi \Rightarrow \partial_\beta (p_\alpha A^\alpha) = \nabla_\beta (p_\alpha A^\alpha)$ component of a 1-form

∇_β is a linear differential operator that follows Leibniz rule

$$\Rightarrow \left(\partial_\beta p_\alpha \right) A^\alpha + \cancel{\left(\partial_\beta A^\alpha \right)} p_\alpha = \cancel{\left(\partial_\beta A^\alpha \right)} p_\alpha + \left(\Gamma^\alpha{}_{\beta\mu} A^\mu \right) p_\alpha + \left(\nabla_\beta p_\alpha \right) A^\alpha$$

$$\Rightarrow \boxed{\partial_\beta p_\nu - \Gamma^\alpha{}_{\beta\nu} p_\alpha = \nabla_\beta p_\nu}$$

★ Similarly $\nabla_\beta T^{\mu\nu} = \partial_\beta T^{\mu\nu} + \Gamma^\mu{}_{\beta\alpha} T^{\alpha\nu} + \Gamma^\nu{}_{\beta\alpha} T^{\mu\alpha}$

$$\nabla_\beta T_{\mu\nu} = \partial_\beta T_{\mu\nu} - \Gamma^\delta{}_{\beta\mu} T_{\delta\nu} - \Gamma^\delta{}_{\beta\nu} T_{\mu\delta}$$

★ We will work with Torsion free connections $\Rightarrow \boxed{\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha} = \Gamma^\mu{}_{(\alpha\beta)}}$

Torsion tensor $T^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\beta\alpha}$

★ Metric compatibility: $\boxed{\nabla_\rho g_{\mu\nu} = 0} \Rightarrow g_{\mu\nu} \nabla_\rho V^\nu = \nabla_\rho V_\mu$

this gives

$$\nabla_\rho g_{\mu\nu} = 0 = \partial_\rho g_{\mu\nu} - \Gamma^\lambda{}_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda{}_{\rho\nu} g_{\mu\lambda}$$

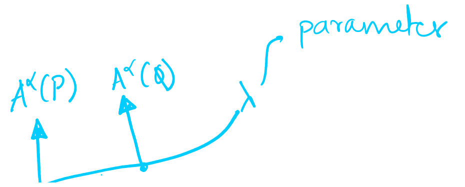
$$\nabla_\mu g_{\nu\rho} = 0 = \partial_\mu g_{\nu\rho} - \Gamma^\lambda{}_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda{}_{\mu\rho} g_{\nu\lambda}$$

$$\nabla_\nu g_{\rho\mu} = 0 = \partial_\nu g_{\rho\mu} - \Gamma^\lambda{}_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda{}_{\nu\mu} g_{\rho\lambda}$$

$$\boxed{\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right)}$$

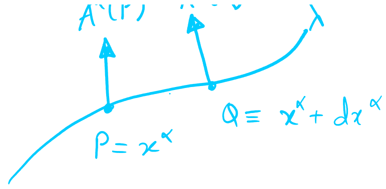
★ other properties: $\Gamma^\mu{}_{\mu\nu} = \frac{1}{\sqrt{|g|}} \partial_\nu \sqrt{|g|} \rightarrow \nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} V^\mu \right)$

★ Why we need the Cov. deriv & Parallel transport.



On a curved manifold both $\vec{A}(P)$ and $\vec{A}(Q)$ belong to different Tangent spaces and the

On a curved manifold both $\vec{A}(P)$ and $\vec{A}(Q)$ belong to different Tangent spaces and the bases change from one tang space to another.



$$\partial_{\beta'} A^{\alpha'} = \frac{\partial A^{\alpha'}}{\partial x^{\beta'}} = \frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial A^{\alpha'}}{\partial x^{\beta}} = \frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial}{\partial x^{\beta}}\left(\frac{\partial x^{\alpha'}}{\partial x^{\alpha}} A^{\alpha}\right)$$

$$= \frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\left(\partial_{\beta} A^{\alpha}\right) + \underbrace{\frac{\partial x^{\beta}}{\partial x^{\beta'}} \cdot \frac{\partial^2 x^{\alpha'}}{\partial x^{\beta}\partial x^{\alpha}} \cdot A^{\alpha}}$$

spoils tensorial nature.

$$\frac{\partial A^{\alpha}}{\partial x^{\beta}} = \lim_{dx^{\beta}\to 0} \frac{A^{\alpha}(Q) - A^{\alpha}(P)}{dx^{\beta}}$$

we need some way to transport $A^{\alpha}(P)$ to Q i.e $A^{\alpha}(P\to Q)$ to take a deriv that is Tensorial

$$\nabla_{\beta} A^{\alpha} = \lim_{dx^{\beta}\to 0} \frac{A^{\alpha}(Q) - A^{\alpha}(P\to Q)}{dx^{\beta}}$$

$$\frac{DA^{\alpha}}{d\lambda} = \frac{dx^{\beta}}{d\lambda} \cdot \nabla_{\beta} A^{\alpha} = u^{\beta} \nabla_{\beta} A^{\alpha}$$

u^{β} ↓ TANGENT VECTOR

If a vector is parallel transported along a path

$$\boxed{\frac{DA^{\alpha}}{d\lambda} = 0}$$

★ Eq^n of parallel transport: $\frac{d}{d\lambda} V^{\mu} + \Gamma^{\mu}{}_{\sigma\rho} \frac{dx^{\sigma}}{d\lambda} V^{\rho} = 0$

- The metric is always parallel transported
- Inner product of two parallel transported vectors is preserved.
- Parallel transport preserves norm.

★ GEODESIC: The "straight line in curved space"

Curve along which the tangent vector is parallel transported

$$\frac{Du^{\alpha}}{d\lambda} = 0 \Rightarrow \frac{dx^{\beta}}{d\lambda} \cdot \nabla_{\beta} x^{\alpha} = 0 \Rightarrow \boxed{\frac{d^2 x^{\mu}}{d\lambda} + \Gamma^{\mu}{}_{\sigma\rho} \frac{dx^{\rho}}{d\lambda} \cdot \frac{dx^{\sigma}}{d\lambda} = 0}$$

If the $\underline{\text{affine parameter } \lambda}$ is chosen to be τ: proper time

↓

$$\frac{d^2 x^{\mu}}{} + \Gamma^{\mu} \quad \frac{dx^{\rho}}{} \cdot \frac{dx^{\sigma}}{} = 0$$

If the affine parameter λ is chosen to be τ: proper time

related to τ as
$\lambda = a\tau + b$

$$\boxed{\frac{d^2x^\mu}{d\tau} + \Gamma^\mu{}_{\sigma\rho}\frac{dx^\rho}{d\tau}\cdot\frac{dx^\sigma}{d\tau} = 0}$$

$$\Rightarrow \boxed{u^\lambda \nabla_\lambda u^\mu = 0} \qquad u^\mu = \frac{dx^\mu}{d\tau} \quad (4\text{-velo})$$

☆ Another notion of transport: (rather, another notion of tensorial derivative)

Lie Derivate:

$$\boxed{\begin{aligned}\mathcal{L}_{\vec{u}} A^\alpha &= u^\beta \partial_\beta A^\alpha - A^\beta \partial_\beta u^\alpha \\ &= u^\beta \nabla_\beta A^\alpha - A^\beta \nabla_\beta u^\alpha\end{aligned}} \to \text{perfectly Tensorial.}$$

$$\mathcal{L}_{\vec{u}} \Phi = u^\alpha \nabla_\alpha \Phi - \Phi \nabla_\alpha u^\alpha$$

$$\mathcal{L}_{\vec{u}} p_\alpha = u^\beta \nabla_\beta p_\alpha + p_\beta \nabla_\alpha u^\beta$$

Tensor: $\mathcal{L}_{\vec{u}} T^\alpha{}_\beta = u^\mu \nabla_\mu T^\alpha{}_\beta + T^\alpha{}_\mu \nabla_\beta u^\mu - T^\mu{}_\beta \nabla_\mu u^\alpha$

★ $\boxed{\text{Lie Transported: } \mathcal{L}_{\vec{u}}(\text{Tensor}) = 0}$ ⇝ Then we say that tensor is Lie transported.

- If a vector is Lie transported then define coordinates centered on the curve for which $\vec{u}$ is tangent : $x^0 = \lambda$; x^1, x^2, x^3 are all constants along the curve.

Then $u^\alpha = \delta^\alpha{}_0 \Rightarrow \partial_\mu u^\alpha = 0$

$\therefore$ $\boxed{\mathcal{L}_{\vec{u}}(\text{Tensor}) = \frac{\partial}{\partial x^0}(\text{Tensor}) = 0}$ → Tensor doesn't vary with this parameter along the curve.

- Suppose the Tensor is $g_{\mu\nu}$ and it is Lie transported along $\vec{\xi}$ then $\boxed{\mathcal{L}_{\vec{\xi}}\, g_{\mu\nu} = 0}$

if $\exists$ such a $\vec{\xi}$ then $\exists$ coordinates s.t. $\boxed{\frac{\partial}{\partial x^0} g_{\alpha\beta} = 0}$ (converse is also true)

Here $\mathcal{L}_{\vec{\xi}}\, g_{\mu\nu} = 0 \Rightarrow \cancel{\xi^\gamma \nabla_\gamma g_{\mu\nu}}^{\,0} + g_{\mu\gamma}\nabla_\nu \xi^\gamma + g_{\gamma\nu}\nabla_\mu \xi^\gamma = 0$

$$\boxed{\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0 \Rightarrow \nabla_{(\alpha}\xi_{\beta)} = 0}$$

If the spacetime has a killing vector $\vec{\xi}$ then $\boxed{c = \overbrace{u^\alpha}^{4\text{-velo}} \xi_\alpha}$ is a const of motion

- Example: if $g_{\mu\nu}$ is time indep, c for the associated killing vector is Energy.
  — " — ϕ — " — " — " — " — Angular momt.