Quantum Effects in Inflation

Supervised Learning Project Report

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1 Introduction

With the demise of the steady state theory in the mid-1960s, the Big Bang theory became the accepted cosmology. The Big Bang theory was widely accepted owing to it's immense success in explaining abundances of elements in the early universe, predicting the number of different types of elementary particles in the universe, predicting a radiation dominated era of the universe at $\sim 3K$, and more. Despite these successes a huge shortcoming of the Big Bang theory is it's inability to explain the initial conditions that have led to the universe we live. This is what the theory of Inflation tries to resolve. In 1981, Dr. Guth put forth the theory of Inflation as an explanation to the initial conditions that led to our extremely homogeneous and "flat" universe.

This report starts with a brief review of the relevant ideas from Inflationary Cosmology and then moves on to the project's main topic, i.e., **Quantum effects in Inflation**. The aim is to be able to compute the power spectrum of density (or metric) perturbations that arise as a consequence of zero point quantum fluctuations in the very early universe. This will form the basic theoretical foundation for making contact with experimental observations and analyzing if the origin of the observed fluctuations, say in the CMB temperature, are of quantum origin.

For the course of the SLP, apart form the "TASI lectures on Inflation"[2] I mainly referred to the books "Modern Cosmology"[3] and "Introduction to Quantum Effects in Gravity"[4]. The Wikipedia ([1]) pages on the horizon problem, the Friedmann equation, and the flatness problem were also useful.

Note: Derivation of most equations is sketched out, and some important derivations are explicitly done. Natural units with $c = 1$,$M_p = 1$, and $\hbar = 1$ are used unless otherwise stated.

2 The Homogeneous and Isotropic Universe

A homogeneous space is one which is invariant under translation, or the same at every point whereas an isotropic space is one which is invariant under rotation, or the same in every direction.
Observations suggest that our universe looks (almost) the same in every direction at the largest of scales. In fact it is "flat" within error bounds, i.e., the universe seems to have zero curvature. The **Friedmann–Robertson–Walker (FRW) metric** is an **exact solution** of Einstein's field equations of general relativity describing such a homogeneous and isotropic universe with uniform curvature. so in fact the FRW metric combined with Einstein's field equations explain the very large scale dynamics of universe extremely well. Even if one wants to account for the inhomogeneities in the universe, just perturbing around the FRW metric to first order does the great job. Thus, the first section of the project is devoted to the analysis of the FRW spacetime.

2.1 The FRW Metric and Friedmann Equations

The differential line element or equivalently the metric for the FRW spacetime is given as:

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}\mathbf{\Sigma}^2 \tag{1}$$

$$\mathrm{d}\mathbf{\Sigma}^2 = \frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\mathbf{\Omega}^2, \text{ where } \mathrm{d}\mathbf{\Omega}^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2. \tag{2}$$

Here, the scale factor $a(t)$ characterizes the relative size of spacelike hypersurfaces Σ at different times. The curvature parameter k is $+1$ for positively curved Σ, 0 for flat Σ, and -1 for negatively curved Σ. More intuitively, $a(t)$ describes the evolution of how physical distances scale ($d_{physical} = a \cdot d_{comoving}$).
To work out the line element for the homogeneous FRW metric with constant positive curvature, consider the embedding of a three-dimensional sphere (a $3D$ hypersurface of constant positive curvature Σ) in a four-dimensional Euclidean space ($x^2+y^2+z^2+w^2 = a^2$). The induced metric (spatial part g_{ij}) on the surface is as given in equation (1) and (2) for $k = +1$. The derivation goes as follows.
Consider the coordinate transform:

$$x = ucos(\phi)sin(\theta),\ y = usin(\phi)sin(\theta),\ z = ucos(\theta)$$

$$\implies \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}u^2 + u^2\mathrm{d}\theta^2 + u^2sin^2(\theta)\mathrm{d}\phi^2$$

$$\therefore w^2 = a^2 - u^2$$

Compute the differential element $\mathrm{d}w$ in terms of u, a:

$$(\mathrm{d}w)^2 = \frac{u^2(\mathrm{d}u)^2}{a^2 - u^2}$$

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2$$

$$\implies \mathrm{d}s^2 = \frac{u^2\mathrm{d}u^2}{a^2 - u^2} + \mathrm{d}u^2 + u^2\mathrm{d}\Omega^2$$

Substitute $r = u/a$ to get the desired form of the line element:

$$\implies \mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = a^2\left(\frac{\mathrm{d}r^2}{1-r^2} + r^2\mathrm{d}\Omega^2\right)$$

The procedure can be followed to derive the metric for a maximally symmetric (homogeneous and isotropic) with a constant curvature (+ve, −ve, or 0) given in equation (2).

An important quantity characterizing the FRW spacetime is the **Hubble parameter**:

$$H \equiv \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\dot{a}}{a} \tag{3}$$

The Hubble parameter is an indicator of the expansion rate of the universe, hence the dynamics of a FRW universe is described most conveniently in terms of H. **Einstein's field equations** govern the evolution of the universe on the largest of scales: ($8\pi G = 1$)

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} = T_{\mu\nu} \tag{4}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{5}$$

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \Gamma^\rho{}_{\mu\nu}\Gamma^\alpha{}_{\rho\alpha} - \Gamma^\rho{}_{\mu\alpha}\Gamma^\alpha{}_{\nu\rho} \tag{6}$$

$$\Gamma^\mu{}_{\alpha\beta} \equiv g^{\mu\nu}[\partial_\beta g_{\alpha\nu} + \partial_\alpha g_{\nu\beta} - \partial_\nu g_{\alpha\beta}] \tag{7}$$

Here. $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci Scalar, and Γs are the Christoffel symbols.
The **Stress Energy Tensor** for a perfect fluid takes the form:

$$T^\mu_\nu = g^{\mu\kappa}T_{\kappa\nu} = (\rho + P)u^\mu u_\nu - P\delta^\mu_\nu \tag{8}$$

Where ρ is the energy density and P is the isotropic pressure. In a frame comoving with the fluid with $u^\mu \equiv (1, 0, 0, 0)$:

$$T^\mu_\nu \equiv \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{bmatrix} \tag{9}$$

In general ρ and P refer to the sum of all components of the energy density and pressure contributed by different species like, baryons, dark matter, dark energy, etc. $w \equiv \frac{P}{\rho}$ is the equation of state parameter. Thus,

$$\rho \equiv \sum_j \rho_j,\ \ P = \sum_j P_j,\ \mathit{and}\ w_j = \frac{P_j}{\rho_j} \tag{10}$$

Now, the procedure to calculate the evolution equations for the homogeneous FRW space-time using the Einstein field equations is:

- Calculate $\Gamma^{\mu}_{\alpha\beta}$ using eqn. (7)
- Calculate components of the Ricci Tensor $R_{\mu\nu}$ using (6)
- Substitute in the Einstein field equations (4)

For the FRW metric, the 00^{th} and ii^{th} components of the Einstein field equations give the **Friedmann equations** for the evolution of the universe:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} - \frac{k}{a^2} \tag{11}$$

and

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{1}{6}\left(\rho + 3P\right) \tag{12}$$

For each species 'j', define the present ratio of the energy density relative to the *critical energy density* $\rho_{crit} \equiv 3H_0^2$ as:

$$\Omega_i \equiv \frac{\rho_0^i}{\rho_{crit}} \tag{13}$$

The curvature parameter is defined as $\Omega_k \equiv -k/a_0^2 H_0^2$.
Here, ρ_{crit} is the energy density for which the universe is "flat" ($k = 0$). In fact, most accurate observations reveal that the observable universe is flat within error bars.
Subscript '0' denotes quantities being evaluated at present time t_0. The scale factor is normalized such that the scale factor today is $a_0 = a(t_0) = 1$.
Friedmann Equations (11) and (12) can be used to obtain the **continuity equation**.
Multiplying equation (11) by a^2 and differentiating:

$$2\dot{a}\ddot{a} = \frac{\dot{\rho}}{3}a^2 + 2a\dot{a}\frac{\rho}{3}$$

$$\therefore\ \dot{\rho} + 2\frac{\dot{a}}{a}\rho - 6\frac{\dot{a}\ddot{a}}{a^2} = 0$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3H(\rho + P) = 0 \tag{14}$$

Expressing pressure in terms of density and equation of state parameter and rearranging:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3H(\rho + P) = 0 \implies \frac{\mathrm{d}\rho}{\rho} + 3\frac{\mathrm{d}a}{a}(1 + P/\rho) = 0$$

$$\frac{\mathrm{d}\,ln\,\rho}{\mathrm{d}\,ln\,a} = -3(1 + w)$$

If $w = \textit{constant}$ over time then integrating the above equation gives:

$$\rho \propto a^{-3(1+w)} \tag{15}$$

Using the Friedmann Equation (11) we can find the time evolution of the scale factor $a(t)$:

$$a(t) \propto \begin{cases} t^{2/3(1+w)} & w \neq -1 \\ e^{Ht} & w = 1 \end{cases} \tag{16}$$

For a universe with it's energy density dominated by Matter (MD), Radiation (RD), or by a Cosmological Constant (Λ) the evolution is summarized in the table (Here τ is the conformal time defined in (22)):

	w	$\rho(a)$	$a(t)$	$a(\tau)$	τ_i
MD	0	a^{-3}	$t^{2/3}$	τ^2	0
RD	1/3	a^{-4}	$t^{1/2}$	τ	0
Λ	-1	a^0	e^{Ht}	$-\tau^{-1}$	$-\infty$

Table 1: Table summarizing $a(t)$ and $a(\tau)$ for different species

Using expressions (15) and (13):

$$\left(\frac{H}{H_0}\right)^2 = \sum_i \Omega_i a^{-3(1+w_i)} + \Omega_k a^{-2} \tag{17}$$

Evaluating the equation above equation implies the consistency relation $\sum_i \Omega_i + \Omega_k = 1$. The second Friedmann Equation (12) evaluated at $t = t_0$ describes the accelerated expansion of the universe:

$$\frac{1}{a_0 H_0^2}\frac{\mathrm{d}^2 a_0}{\mathrm{d}t^2} = -\frac{1}{2}\sum_i \Omega_i(1+3w_i) \tag{18}$$

2.2 Conformal Time and Comoving Distance

Having discussed the basic FRW geometry, we now look at a special coordinate transformation.
The FRW metric in terms of a transformed coordinate X can be written as:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\mathbf{\Omega}^2\right) \tag{19}$$

$$= -\mathrm{d}t^2 + a(t)^2(\mathrm{d}X^2 + \Phi_k(X^2)\mathrm{d}\Omega^2) \tag{20}$$

where

$$r^2 = \Phi_k(X^2) \equiv \begin{cases} sinh^2 X & k=-1 \\ X^2 & k=0 \\ sin^2 X & k=1 \end{cases} \tag{21}$$

The **conformal time** τ is defined as the amount of time it would take a photon to travel from where we are located to the furthest observable distance, provided the universe ceased expanding.

$$\tau = \int_0^t \frac{\mathrm{d}t'}{a(t')} \implies \mathrm{d}\tau = \mathrm{d}t/a \tag{22}$$

In an isotropic background we may consider radial propagation of light as determined by the two dimensional line element:

$$\mathrm{d}s^2 = a(\tau)^2[-\mathrm{d}\tau^2 + \mathrm{d}X^2] \tag{23}$$

For light (null geodesics):

$$\mathrm{d}s^2 = 0 \tag{24}$$

$$\implies X(\tau) = \pm\tau \text{ upto a constant} \tag{25}$$

Thus, the trajectories followed by light form a 45° light cone in the $X - \tau$ plane. (figure (1))

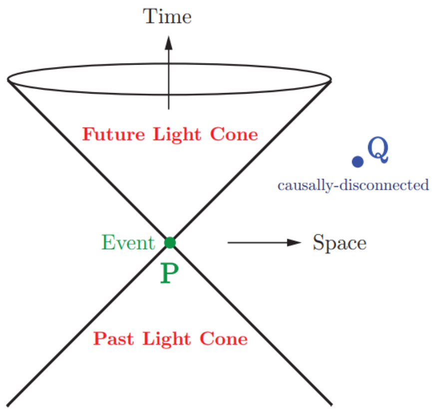

Figure 1: Light trajectories in $\chi - \tau$ plane (Reference [2])

The **comoving distance** χ between a distant light source and us is the coordinate distance traveled by light that began its journey from an object at time t when the scale factor was equal to $a(t)$. It is given by:

$$\chi(t) = \int_{t_0}^{t} \frac{\mathrm{d}t'}{a(t')} = \int_{a(t)}^{1} \frac{\mathrm{d}a'}{a'^2 H(a')} = \int_{0}^{z} \frac{\mathrm{d}z'}{H(z')} \tag{26}$$

Here z is the redshift defined in terms of the wavelength of light and the scale factor as:

$$\frac{1}{1+z} = \frac{\lambda_{obs}}{\lambda_{emit}} = \frac{a_{obs}}{a_{emit}} = \frac{1}{a_{emit}} \tag{27}$$

The **comoving particle horizon** $\chi_p(\tau)$ is the maximum comoving distance (χ) that light can propagate between an initial time t_i and some later time t:

$$\chi_p(\tau) = \tau - \tau_i = \int_{t_i}^{t} \frac{\mathrm{d}t'}{a(t')} \implies \text{physical horizon } d_p(t) = a(t)\chi_p \tag{28}$$

Having discussed the basic terminology and math for the FRW spacetime, we now discuss the drawbacks of the standard Big Band model and motivate the need for a more complete theory.

3 The Big Bang Puzzles

3.1 The Horizon Problem

At an age of 380,000 years (Formation of the Cosmic Microwave Background), the observable universe was extremely uniform, to roughly 1 part in 10^5 (the typical amplitude of the temperature fluctuations in the CMB). One way this high degree of homogeneity can be explained is if different patches of the universe were in thermal contact, then eventually the entire universe will equilibrate at the same, shared temperature. But this solution does not work, because different parts of the universe observed in the map of the CMB were so far apart at the time of recombination that they were not in causal contact with one another, even light could not have traveled from one region to another. Therefore, they were seemingly never able to thermalize. During the formation of the CMB, the particle horizon was small enough that if we could see it today at the spherical surface of last scattering, its diameter will subtend an angle of $\sim 1°$.

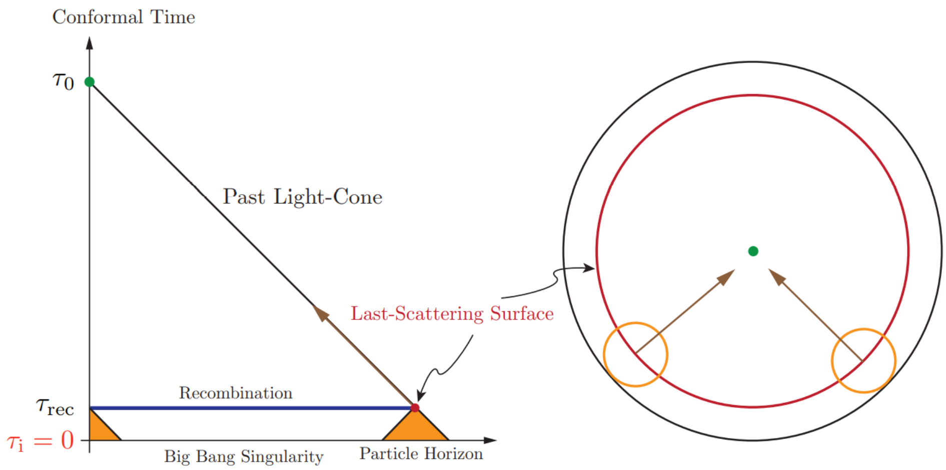

Figure 2: Conformal diagram of Big Bang cosmology. The CMB at last-scattering (recombination) consists of $\sim 10^5$ causally disconnected regions. (Reference [2])

This can be qualitatively understood by analyzing the evolution of the comoving particle horizon (τ) based on the **comoving Hubble radius** $(aH)^{-1}$ for a universe dominated by a fluid with equation of state w.

$$(aH)^{-1} = H_0^{-1} a^{(1+3w)/2} \tag{29}$$

$$\tau = \int_0^a \mathrm{d}\, ln\, a \left(\frac{1}{aH}\right) \tag{30}$$

$$\implies \tau \propto a^{\frac{1}{2}(1+3w)} \tag{31}$$

$$Ex. : \tau \propto \begin{cases} a & Radiation Dominated \\ a^{1/2} & Matter Dominated \end{cases} \tag{32}$$

The nature of the exponent depends on $(1 + 3w)$, thus the qualitative behavior depends on whether $(1 + 3w)$ is positive or negative. During the conventional Big Bang expansion with

($w \geq 0$), $(aH)^{-1}$ grows monotonically and the comoving horizon τ or the fraction of the universe in causal contact increases with time.
This implies that comoving scales entering the horizon today have been far outside the horizon at CMB decoupling but the near-homogeneity of the CMB tells us that the universe was extremely homogeneous at the time of last-scattering on scales encompassing many regions that seem to be causally independent. (figure (2))
In the case of standard Big Bang cosmology the Big Bang Singularity is at $\tau_i = 0$ for $(1 + 3w) > 0$. The Big Bang singularity is the limit $a \to 0$.

3.2 The Flatness Problem

Consider the Friedmann Eqn. (10) and recast it in terms of Ω as in Eqn. (12), writing all the terms as functions of $a(t)$:

$$H^2 = \frac{\rho(a)}{3} - \frac{k}{a^2} \implies 1 - \Omega(a) = -\frac{k}{(aH)^2} \tag{33}$$

$$\implies |1 - \Omega(a)| = \frac{|k|}{(aH)^2} \tag{34}$$

In standard cosmology the comoving Hubble radius, $(aH)^{-1}$, grows monotonically with time which implies that the quantity $|\Omega - 1|$ must diverge with time. The critical value $\Omega = 1$ is an *unstable fixed point*. Therefore, in standard Big Bang cosmology, the near-flatness observed today, $\Omega(a_0) \sim 1$, requires an extreme "artificial" fine-tuning of Ω close to 1 in the early universe.
The flatness and horizon problems are severe shortcomings in the predictive power of the Big Bang model rather than strict inconsistencies. The dramatic flatness of the early universe cannot be predicted, but must instead be assumed in the initial conditions. Likewise, the striking large-scale homogeneity of the universe is not explained or predicted by the model, but instead must simply be assumed. A theory that explains these phenomena would surely be a step froward in understanding out universe.

4 Inflation

Both the Big Bang puzzles arise since in conventional cosmology the comoving Hubble radius is strictly increasing. If this behaviour were to invert for a short duration in the very early universe, i.e. if the comoving Hubble radius decreases sufficiently in the very early universe then we just might solve these problems. **An epoch during which the comoving Hubbel radius $(aH)^{-1} = \dot{a}^{-1}$ decreases corresponds to one of increasing $\dot{a}$, or $\ddot{a} > 0$: the condition for an accelerated expansion of the universe. This postulated epoch is called inflation.**
The main epochs are clearly visible in figure (3): inflation at early times (with $H = H_{inf} \simeq$ *constant* with $a \sim e^{Ht}$), transitioning to radiation domination around $a \sim a_e$, and finally matter domination at $a \sim 10^{-4}$. Dark energy domination is barely visible as a further flattening at $a \sim 0.5$. Scales of cosmological interest (horizontal shaded band) were larger than the Hubble radius when $a \sim 10^{-5}$. They later entered the Hubble radius where we are able to observe them. Very early on during inflation, however, *all scales of interest were smaller than the Hubble radius and therefore within causal contact.*

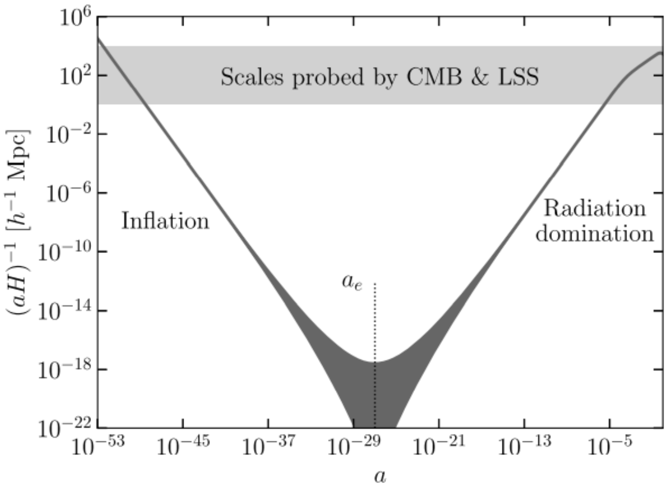

Figure 3: The comoving Hubble radius as a function of scale factor. (Reference [2])

The evolution of the Hubble radius around the end of inflation (at a_e) is uncertain, as indicated by the dark shaded band. However, since all modes of interest were far outside the horizon at that time, they are largely oblivious to the details of that epoch.

4.1 Solving the Big Bang Puzzles

The flatness problem

Equation (33) implies that for a non-flat universe:

$$|1 - \Omega(a)| = \frac{1}{(aH)^2} \tag{35}$$

If the comoving Hubble radius decreases, i.e., $(aH)^{-1}$ decreases, then the solution $\Omega = 1$ is an *attractor (stable fixed point)* during inflation. This solves the flatness problem.

The horizon problem

A decreasing comoving horizon means that large scales entering the present universe were inside the horizon before inflation (figure (4)). Causal physics before inflation therefore established spatial homogeneity, explaining the extreme uniformity of the CMB.

4.2 Mathematical conditions for Inflation

The shrinking of the comoving Hubble radius for Inflation imposes some conditions on the acceleration and the pressure of the universe.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{aH}\right) < 0 \tag{36}$$

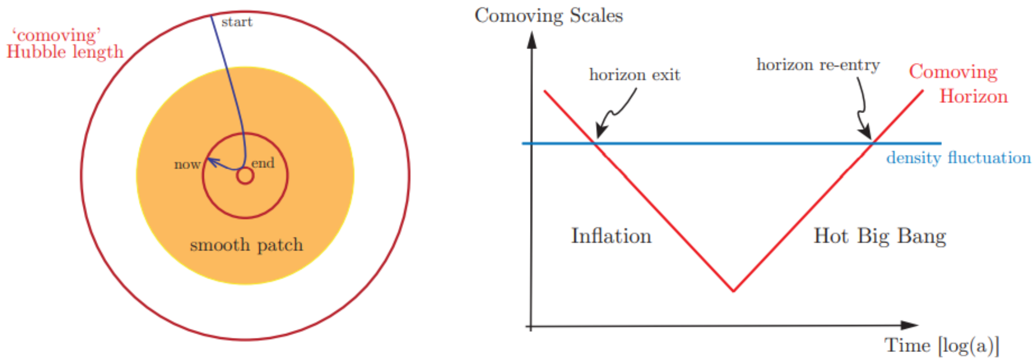

Figure 4: *Left*: Evolution of the comoving Hubble radius, $(aH)^{-1}$, in the inflationary universe. *Right*: Solution of the horizon problem (blue line is a density fluctuation at a particular scale). (Reference [2])

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\dot{a}}\right) = -\frac{\ddot{a}}{\dot{a}^2} < 0 \implies \frac{\mathrm{d}^2 a}{\mathrm{d}t^2} > 0 \tag{37}$$

Thus, Inflation is a period of **accelerated expansion** of the universe.

$$\frac{\ddot{a}}{a} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 = H^2\left(1 + \frac{\dot{H}}{H^2}\right) > 0 \tag{38}$$

Acceleration ($\ddot{a} > 0$) corrosponds to:

$$\implies -\frac{\dot{H}}{H^2} = -\frac{\mathrm{d}\, ln\, H}{\mathrm{d}N} < 1 \tag{39}$$

Here, $\mathrm{d}N = H\mathrm{d}t = \mathrm{d}\, ln\, a$, measures the number of **e-folds N of inflationary expansion**. Thus, the fractional change of the Hubble parameter per e-fold (increase in a) is small. This is precisely why $H \simeq constant$ during inflation.
In equation (12), $\ddot{a} > 0$ requires the presence of **negative pressure** with:

$$P < -\frac{1}{3}\rho \tag{40}$$

4.3 Conformal Diagram of Inflation

The FRW metric in conformal time ($\mathrm{d}\tau = \mathrm{d}t/a(t)$) is given as: $\mathrm{d}s^2 = a^2(\tau)[-\mathrm{d}\tau^2 + \mathrm{d}X^2]$. Results summarized in table (1) along with equation (16) show that if the universe had always been dominated by matter or radiation, then this would imply the existence of the Big Bang singularity for which $a = 0$ at $\tau_i = 0$.

During inflation $\{(H \simeq const)$ from equation (39)$\}$ and the scale factor is:

$$a(t) = e^{Ht} \tag{41}$$

$$\mathrm{d}\tau = \mathrm{d}t/a(t) = e^{-Ht}\mathrm{d}t \tag{42}$$

$$\implies \tau = -\frac{1}{He^{Ht}} = -\frac{1}{Ha} \tag{43}$$

$$\implies a(\tau) = -\frac{1}{H\tau} \tag{44}$$

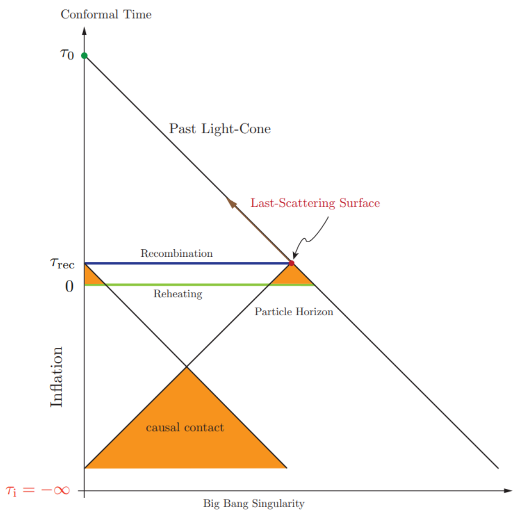

Figure 5: Conformal diagram of inflationary cosmology. Inflation extends conformal time to negative values. (Reference [2])

Thus, the singularity, $a = 0$ is pushed to the infinite past $\tau_i \to -\infty$. It looks like the scale factor becomes infinite as $\tau \to 0$. This 'singularity' at $\tau = 0$ corresponding to $t \to \infty$ is just an artifact of our incorrect assumption that $H \simeq constant$ forever which actually breaks down towards the end of inflation. In reality, there is no singularity at $\tau = 0$ and the light cones at the last scattering intersect at an earlier time if inflation lasts for at least 60 e-folds (figure (5)).

5 Physics of Inflation

The simplest underlying field that can cause the exponential expansion of the universe within a fraction of a second is a single scalar field:The **Inflaton Field** ϕ. We don't specify the physical nature of the field ϕ, but simply use it as an order parameter to parameterize the time-evolution of the inflationary energy density.
The dynamics of such a field **minimally coupled** to gravity is governed by the action:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] = S_{EH} + S_\phi \tag{45}$$

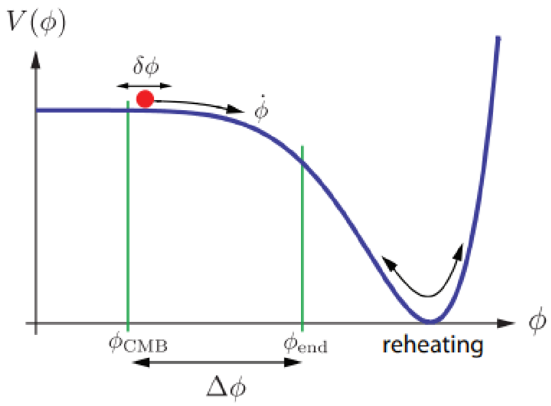

Figure 6: Example of an inflaton potential. Acceleration occurs when the potential energy of the field, $V(\phi)$, dominates over its kinetic energy, $\frac{1}{2}\dot{\phi}^2$. Inflation ends at ϕ_{end} when the kinetic energy has grown to become comparable to the potential energy, $\frac{1}{2}\dot{\phi}^2 \simeq V(\phi)$. CMB fluctuations are created by quantum fluctuations $\delta\phi$ about 60 e-folds before the end of inflation. At reheating, the energy density of the inflaton is converted into radiation. (Reference [2])

The action is the sum of the gravitational Einstein-Hilbert action, S_{EH}, and the action of a scalar field with canonical kinetic term, S_ϕ. The potential $V(\phi)$ describes the self-interactions of the scalar field.
The energy-momentum tensor for the scalar field is:

$$T^{(\phi)}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\phi\partial_\sigma\phi + V(\phi)\right) \tag{46}$$

The equation of motion is:

$$\frac{\delta S_\phi}{\delta\phi} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) + \frac{\mathrm{d}V}{\mathrm{d}\phi} \tag{47}$$

5.1 Slow-Roll Inflation

Assuming the FRW metric (1) for $g_{\mu\nu}$ and restricting to the case of a homogeneous field $\phi(t,x) \equiv \phi(t)$, contracting equation (46) we can show that the scalar Stress Energy tensor takes the form of a perfect fluid (9) with:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{48}$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{49}$$

$$\implies w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{50}$$

Thus, a scalar field can lead to accelerated expansion due to negative pressure (with $w_\phi < -1/3$) if V dominates over the kinetic energy $\frac{1}{2}\dot{\phi}^2$.

Substituting these expressions for the energy density and pressure in the Friedmann equations (11), (12) and the continuity equation (14) for a flat FRW geometry gives:

$$\dot{\rho} + 3H(\rho + P) = \dot{\phi}\ddot{\phi} + \dot{V} + 3H\dot{\phi}^2 = 0$$

$$\Longrightarrow \ddot{\phi} + 3H\dot{\phi} + \mathrm{d}_\phi V = 0 \text{ and } H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) = \frac{\rho}{3} \tag{51}$$

Using equation (38):

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho_\phi + 3P_\phi) = H^2\left(1 + \frac{\dot{H}}{H^2}\right) = H^2(1-\varepsilon) \tag{52}$$

For a universe dominated by a homogeneous inflaton field, first expressing pressure in terms of density and the equation of state parameter in the Friedmann equation gives:

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho_\phi + 3P_\phi) = -\frac{1}{2}\left(\frac{\rho_\phi}{3} + \frac{\rho_\phi}{3}(3w_\phi)\right)$$

$$\Longrightarrow \frac{\ddot{a}}{a} = -\frac{1}{2}H^2(1 + (3w_\phi)) = H^2\left(1 - \frac{3}{2}(1 + w_\phi)\right)$$

$$\therefore \frac{\ddot{a}}{a} = H^2(1-\varepsilon)$$

Here, using the equations derived above we define the first **Hubble slow-roll parameter** ε as:

$$\varepsilon \equiv \frac{3}{2}(1 + w_\phi) = \frac{\dot{\phi}^2}{2H^2} = -\frac{\dot{H}}{H^2} \tag{53}$$

Clearly, accelerated expansion occurs if $\varepsilon < 1$. The limit $P_\phi \to -\rho_\phi$ corresponds to $\varepsilon \to 0$. In this limit the potential energy dominates over the kinetic energy:

$$\dot{\phi}^2 << V(\phi) \tag{54}$$

This is the so called slow-roll limit for which the universe undergoes accelerated expansion in the presence of negative pressure (with $w_\phi \simeq -1$). The discussion in this report is restricted to this special case.
Accelerated expansion persists 'long enough' only if the second time derivative of ϕ in equation (51) is small enough. Thus during inflation (slow-roll):

$$\Longrightarrow |\ddot{\phi}| << |3H\dot{\phi}|, \left|\frac{\mathrm{d}V}{\mathrm{d}\phi}\right| \tag{55}$$

Equation (55) enforces the smallness of the second **Hubble slow-roll parameter** η defined as:

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{56}$$

In the slow-roll limit ($\dot{\phi}^2 << V(\phi)$) the slow-roll conditions, $\varepsilon, |\eta| < 1$ can be equivalently expressed in terms of the shape of the potential $V(\phi)$ as:

$$\varepsilon_V(\phi) \equiv \frac{1}{2}\left(\frac{\mathrm{d}_\phi V}{V}\right)^2, \;\; \eta_V(\phi) \equiv \frac{\mathrm{d}_\phi^2 V}{V} \tag{57}$$

In the slow-roll regime the **potential slow-roll parameters** satisfy $\varepsilon_V, |\eta_V| << 1$. The background evolution is thus given by (using (54), (55), and (51)):

$$H^2 \simeq \frac{1}{3}V(\phi) \simeq constant, \;\; \dot{\phi} \simeq -\frac{\mathrm{d}_\phi V}{3H} \tag{58}$$

The spacetime is approximately de Sitter (maximally symmetric) with $a(t) \sim e^{Ht}$. Using equations (53), (56), (57), and (58) in the slow-roll limit:

$$\varepsilon_V = \frac{1}{2}\left(\frac{\mathrm{d}_\phi V}{V}\right)^2 \simeq \frac{1}{2}\left(\frac{3H\dot{\phi}}{V}\right)^2 \simeq \frac{3}{2}\left(\frac{\dot{\phi}^2}{V^2}\right) \simeq \frac{\dot{\phi}^2}{2H^2} = \varepsilon$$

Differentiate the second term in equation (58):

$$\mathrm{d}_\phi^2 V \simeq -3\dot{H}\dot{\phi} - 3H\ddot{\phi}$$

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} = -\frac{\dot{H}\dot{\phi}}{H^2} + \frac{\mathrm{d}_\phi^2 V}{3H^2} = \frac{\mathrm{d}_\phi^2 V}{V} - \frac{1}{2}\left(\frac{\mathrm{d}_\phi V}{V}\right)^2 = \eta_V - \varepsilon_V$$

As long as the slow-roll approximation holds, the Hubble slow-roll parameters and the potential slow-roll parameters are related as follows:

$$\varepsilon_V \simeq \varepsilon \tag{59}$$

$$\eta \simeq \eta_V - \varepsilon_V \tag{60}$$

Inflation ends when the slow-roll conditions are violated, i.e, $\varepsilon(\phi_{end}) = 1$ and $\varepsilon_V(\phi_{end}) \simeq 1$. To solve the flatness and horizon problem the number of e-folds before inflation ends must be $\gtrsim 60$:

$$N_{tot} \equiv ln\left(\frac{a_{end}}{a_{start}}\right) \gtrsim 60 \tag{61}$$

Using the expressions derived previously (along with equation (39)), this condition can be expressed in terms of the slow roll parameters as:

$$N(\phi) = ln\left(\frac{a_{end}}{a_{start}}\right) = \int_{t_{start}}^{t_{end}} H\mathrm{d}t \tag{62}$$

$$= \int_{\phi_{start}}^{\phi_{end}} \frac{H}{\dot{\phi}}\mathrm{d}\phi \simeq \int_{\phi_{end}}^{\phi_{start}} \frac{V}{\mathrm{d}_\phi V}\mathrm{d}\phi \tag{63}$$

$$\simeq \int_{\phi_{end}}^{\phi_{start}} \frac{\mathrm{d}\phi}{\sqrt{2\varepsilon}} \simeq \int_{\phi_{end}}^{\phi_{start}} \frac{\mathrm{d}\phi}{\sqrt{2\varepsilon_V}} \sim 60 \tag{64}$$

5.2 Quantum Effects during Inflation

So far, the discussion assumed perfect homogeneity. Now, we analyze perturbations around the homogeneous classical background. The key focus will be the fact that these primordial fluctuations in say the filed ϕ were a consequence of quantum mechanics. These fluctuations induce the density fluctuations $\delta\rho(t, \mathbf{x})$ as the universe evolves. Quantum fluctuations during inflation are the source of the primordial power spectra of scalar and tensor fluctuations, $\mathcal{P}_s(k)$ and $\mathcal{P}_t(k)$. The amplitude of quantum fluctuations scales with the expansion parameter H during inflation.

Fluctuations are created on all length scales, i.e. with a spectrum of wavenumbers k (equation (66). Cosmologically relevant fluctuations start their lives inside the horizon (Hubble radius),

$$\frac{1}{k} \sim \lambda < (aH)^{-1}$$
$$\text{Subhorizon:} k > aH \tag{65}$$

The comoving wavenumber is constant but the comoving Hubble radius shrinks during inflation, so eventually all fluctuations exit the horizon,

$$\frac{1}{k} \sim \lambda > (aH)^{-1}$$
$$\text{Superhorizon: } k < aH \tag{66}$$

Observing the large scale (small k) fluctuations that have just entered the horizon can be a crucial in drawing some important conclusions regarding the origin of the structure in the universe, and testing the theory of Inflationary expansion.

5.3 Perturbation around the FRW background

All quantities $\xi(t, \mathbf{x})$ split into a homogeneous spatially invariant background $\bar{\xi}(t)$ and a spatially dependent perturbation $\delta\xi(t, \mathbf{x})$. Since the fluctuations are small $\delta\xi(t, \mathbf{x}) << \bar{\xi}(t)$, Einstein's field equations up to linear order ($\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}$) suffice to solve for the evolution of the universe to a very high accuracy.
The spatially flat, homogeneous and isotropic background spacetime possesses a great deal of symmetry. These symmetries allow a decomposition of the metric and the stress-energy perturbations into independent scalar (S), vector (V) and tensor (T) components. The importance of the SVT decomposition is that the perturbations of each type evolve independently (at the linear order) and can therefore be treated separately.
A key consequence of translation invariance of the linear equations of motion for the perturbations means that the different **Fourier Modes** in the Fourier decomposition of a quantity $\xi_{\mathbf{k}}(t)$ modes do not interact:

$$\xi_{\mathbf{k}}(t) = \int \mathrm{d}^3x \, \xi(t, \mathbf{x}) e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{67}$$

Gauge freedom allows "distribution" of the perturbations in the metric components or the components of the stress energy tensor.

Metric Perturbations

Define perturbations around the homogeneous background solutions for the inflaton $\bar{\phi}(t)$ and the metric $\bar{g}_{\mu\nu}(t)$ as:

$$\phi(t, \mathbf{x}) = \bar{\phi}(t) + \delta\phi(t, \mathbf{x}) \, , \;\; g_{\mu\nu}(t, \mathbf{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \mathbf{x}) \tag{68}$$

Where,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{69}$$
$$= -(1 + 2\Phi)\mathrm{d}t^2 + 2aB_i \mathrm{d}x^i \mathrm{d}t + a^2[(1 - 2\Psi)\delta_{ij} + E_{ij}]\mathrm{d}x^i \mathrm{d}x^j \tag{70}$$

We ignore the vector perturbations since they aren't created by inflation and they decay with the expansion of the universe, hence we only consider the scalar and tensor perturbations which are observed as density fluctuations and gravitational waves in the late universe. SVT decomposition of the metric perturbations is:

$$B_i \equiv \partial_i B \tag{71}$$

$$E_{ij} \equiv 2\partial_{ij} E + h_{ij}\ , \quad \partial^i h_{ij} = 0 \tag{72}$$

Tensor perturbations are gauge invariant, but scalar perturbations change under a change of coordinates. Consider the gauge transformation:

$$t \to \tilde{t} = t + \alpha \tag{73}$$

$$x^i \to \tilde{x}^i = x^i + \delta^{ij}\partial_j \beta \tag{74}$$

Imposing the invariance of the spacetime interval ($g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \tilde{g}_{\mu\nu}\mathrm{d}\tilde{x}^\mu \mathrm{d}\tilde{x}^\nu$) in both these coordinate systems gives us how the metric perturbations transform upto linear order:

$$\Phi \to \tilde{\Phi} = \Phi - \dot{\alpha} \tag{75}$$

$$B \to \tilde{B} = B + a^{-1}\alpha - a\dot{\beta} \tag{76}$$

$$E \to \tilde{E} = E - \beta \tag{77}$$

$$\Psi \to \tilde{\Psi} = \Psi + H\alpha \tag{78}$$

Matter Perturbations

The Einstein field equations govern the coupling between the metric perturbations and the matter perturbations. After inflation, the perturbations to the total stress-energy tensor of the universe are:

$$T^0_{\ 0} = -(\bar{\rho} + \delta\rho) \tag{79}$$

$$T^0_{\ i} = (\bar{\rho} + \bar{P})av_i \tag{80}$$

$$T^i_{\ 0} = -(\bar{\rho} + \bar{P})(v^i - B^i)/a \tag{81}$$

$$T^i_{\ j} = \delta^i_{\ j}(\bar{P} + \delta P) + \Sigma^i_{\ j} \tag{82}$$

The anisotropic stress $\Sigma^i_{\ j}$ is gauge invariant while the density, pressure, and momentum density ($\partial_i \delta q \equiv (\bar{\rho} + \bar{P})v_i$) transform as follows:

$$\delta\rho \to \delta\tilde{\rho} = \delta\rho - \dot{\bar{\rho}}\alpha \tag{83}$$

$$\delta P \to \delta\tilde{P} = \delta P - \dot{\bar{P}}\alpha \tag{84}$$

$$\delta q \to \delta\tilde{q} = \delta q + (\bar{\rho} + \bar{P})\alpha \tag{85}$$

To extract useful information regarding the perturbations that is not merely an artifact of the coordinate choice, we choose **Gauge Invariant** combinations of metric and matter perturbations. The comoving curvature perturbation is a gauge invariant scalar:

$$\mathcal{R} \equiv \Psi - \frac{H}{\bar{\rho} + \bar{P}}\delta q \tag{86}$$

We work in the comoving Gauge ($\delta\phi = 0\,,\ \ \delta q = 0\,, and\ \ E = 0$) for the dynamical fields. Thus, equations (69) and (83) imply:

$$\Longrightarrow \mathcal{R} = \Psi \tag{87}$$

$$\Longrightarrow E_{ij} = h_{ij} \tag{88}$$

$$\Longrightarrow g_{ij} = a^2[(1-2\mathcal{R})\delta_{ij} + h_{ij}] \tag{89}$$

In this Gauge, the inflaton field is unperturbed and all the scalar degrees of freedom are parameterized by the metric perturbation $\mathcal{R}(t,\mathbf{x})$.
The metric perturbations Φ and B are related to $\mathcal{R}$ by Einstein field equations in the comoving gauge.

5.4 Mukhanov Sasaki Action

The next step would be to analyze how the scalar perturbations evolve during inflation. This will in turn allow us to calculate the power spectrum due to the scalar perturbation. We start by analyzing scalar perturbations to the metric. Expanding the action (45) to second order in $\mathcal{R}$ using the ADM formalism gives (won't be derived):

$$S_{(2)} = \frac{1}{2}\int \mathrm{d}^4x\, a^3\frac{\dot{\phi}^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i\mathcal{R})^2\right] \tag{90}$$

Defining the Mukhanov variable:

$$v \equiv z\mathcal{R}\,,\ where\ z^2 \equiv a^2\frac{\dot{\phi}^2}{H^2} = 2a^2\varepsilon \tag{91}$$

We transform to conformal time τ using $\mathrm{d}\tau = \mathrm{d}t/a$ and write the action in terms of the Mukhanov variable. First, we note that z is purely time dependent:

$$\mathcal{R} = \frac{v}{z} \implies \partial_i\mathcal{R} = \frac{\partial_i v}{z}$$

Now, differentiating with respect to t and converting to derivatives with respect to τ ($\dot{\xi} = \xi'/a$):

$$\dot{\mathcal{R}} = \frac{\dot{v}}{z} - \frac{v\dot{z}}{z^2} = \frac{1}{a}\left(\frac{v'}{z} - \frac{vz'}{z^2}\right)$$

Writing the action in terms of Mukhanov variable, integrating by parts, and setting the boundary term to zero:

$$\begin{aligned}
S_{(2)} &= \frac{1}{2}\int \mathrm{d}\tau\mathrm{d}^3x\,\left[(v')^2 - (\partial_i v)^2 - 2\frac{vv'z'}{z} + \frac{v^2z'^2}{z^2}\right]\\
&= \frac{1}{2}\int \mathrm{d}\tau\mathrm{d}^3x\,\left[(v')^2 - (\partial_i v)^2 + v^2\left(\frac{z'}{z}\right)' + \frac{v^2z'^2}{z^2}\right] + \text{B.T.}\\
&= \frac{1}{2}\int \mathrm{d}\tau\mathrm{d}^3x\,\left[(v')^2 - (\partial_i v)^2 + \frac{z''}{z}v^2\right]
\end{aligned}$$

The **Mukhanov Sasaki action**:

$$S_{(2)} = \frac{1}{2}\int \mathrm{d}\tau\mathrm{d}^3x\,\left[(v')^2 - (\partial_i v)^2 + \frac{z''}{z}v^2\right] \tag{92}$$

Here $(..)' = \partial_\tau(..)$ corresponds to derivative w.r.t. conformal time.
The equation of motion for v is calculated using the Euler Lagrange equations:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu v)} \right) = \frac{\partial \mathcal{L}}{\partial v} \tag{93}$$

The Lagrange density of the system is:

$$\mathcal{L} = (v')^2 - (\partial_i v)^2 + \frac{z''}{z} v^2 \tag{94}$$

Thus, the equation of motion is that of a driven, time dependent harmonic oscillator:

$$v'' - \nabla^2 v - \frac{z''}{z} v = 0 \tag{95}$$

Computing the Fourier transform:

$$\begin{aligned}
v(t, \mathbf{x}) &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \, v_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} \\
v''(t, \mathbf{x}) &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \, v_k''(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} \\
\nabla^2 v(t, \mathbf{x}) &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \, (-k^2) v_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}}
\end{aligned}$$

The equation of motion in terms of the fourier modes:

$$\int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left(v_k''(\tau) + k^2 v_k(\tau) - \frac{z''}{z} v_k(\tau) \right) e^{i\mathbf{k}\cdot\mathbf{x}} = 0 \tag{96}$$

Multiplying by $e^{-i\tilde{\mathbf{k}}\cdot\mathbf{x}}$ and integrating w.r.t. $\mathrm{d}^3 x$:

$$\int\int \mathrm{d}^3 x \frac{\mathrm{d}^3 k}{(2\pi)^3} \left(v_k''(\tau) + k^2 v_k(\tau) - \frac{z''}{z} v_k(\tau) \right) e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\tilde{\mathbf{k}}\cdot\mathbf{x}} = 0 \tag{97}$$

$$\int \mathrm{d}^3 k \left(v_k''(\tau) + k^2 v_k(\tau) - \frac{z''}{z} v_k(\tau) \right) \delta(\mathbf{k} - \tilde{\mathbf{k}}) = 0 \tag{98}$$

Thus, each Fourier mode is an uncoupled harmonic oscillator with time dependent frequency:

$$v_k''(\tau) + \left(k^2 - \frac{z''}{z} \right) v_k(\tau) = 0 \tag{99}$$

This is also called the **Mukhanov Sasaki** equation.
Before moving to the full quantum mechanical treatment of the field v, we start by analyzing the simpler $1D$ time dependent quantum harmonic oscillator. The physics in that case can be directly carried over to each of the Fourier modes derived above.

5.5 1D Time Dependent Quantum Harmonic Oscillator

The classical action of a harmonic oscillator with time-dependent frequency is:

$$S = \int dt \left(\frac{\dot{x}^2}{2} - \frac{\omega(t)}{2} x^2 \right) = \int dt L \tag{100}$$

Classical Equation of Motion:

$$\frac{\delta S}{\delta x} = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \tag{101}$$

$$\implies \ddot{x} + \omega(t)^2 x = 0 \tag{102}$$

The conjugate momentum that is promoted to the momentum operator upon quantization is:

$$p = \frac{\partial L}{\partial \dot{x}} = \dot{x} \tag{103}$$

Promote the classical canonical variables to operators satisfying the canonical fixed time commutation relation:

$$[\hat{x}, \hat{p}] = [\hat{x}, \dot{\hat{x}}] = i\hbar \tag{104}$$

Note that $\hat{x}$ is a Heisenberg picture operator. The operator $\hat{x}$ is then expanded in terms of creation and annihilation operators (which are time independent). The time dependence of $\hat{x}$ is contained in the complex mode function $v(t)$:

$$\hat{x} = v(t)\hat{a} + v^*(t)\hat{a}^\dagger \tag{105}$$

$$\dot{\hat{x}} = \dot{v}(t)\hat{a} + \dot{v}^*(t)\hat{a}^\dagger \tag{106}$$

The mode function satisfies the classical equation of motion:

$$\ddot{v} + \omega^2(t) v = 0 \tag{107}$$

Calculating the canonical commutator in terms of $\hat{a}$ and $\hat{a}^\dagger$:

$$\begin{aligned} [\hat{x}, \dot{\hat{x}}] &= [(v(t)\hat{a} + v^*(t)\hat{a}^\dagger), (\dot{v}(t)\hat{a} + \dot{v}^*(t)\hat{a}^\dagger)] \\ &= ((\partial_t v^*)v - v^*(\partial_t v))[\hat{a}, \hat{a}^\dagger] = i\hbar \end{aligned}$$

Without lass of generality, we choose a normalization of v such that $\frac{i}{\hbar}(v^*(\partial_t v) - (\partial_t v^*)v) = \langle v, v \rangle = 1$ to enforce the standard time independent commutation:

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{108}$$

Inverting equations (106) and (105):

$$\hat{a} = \langle v, \hat{x} \rangle \tag{109}$$

$$\hat{a}^\dagger = -\langle v^*, \hat{x} \rangle \tag{110}$$

The vacuum state $|0\rangle$ corresponding to the mode functions v or equivalently, to the operators $\hat{a}, \hat{a}^\dagger$:

$$\hat{a} |0\rangle = 0 \tag{111}$$

We haven't yet determined unique mode functions and hence we haven't fixed the vacuum state. For the simple harmonic oscillator with time-dependent frequency $\omega(t)$ (and for quantum fields in curved spacetime) there is no unique choice for the mode function $v(t)$. Hence, there is no unique decomposition of $\hat{x}$ into annihilation and creation operators and no unique notion of the vacuum. Different choices for the solution v(t) give different vacuum solutions.
For now, we consider the time independent case. We evaluate the Hamiltonian in terms of $v(t)$:

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2 \tag{112}$$
$$= \frac{1}{2}\left[(\dot{v}^2 + \omega^2 v^2)\hat{a}\hat{a} + (\dot{v}^2 + \omega^2 v^2)^*\hat{a}^\dagger\hat{a}^\dagger + (|\dot{v}|^2 + \omega^2|v|^2)(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})\right] \tag{113}$$

The action of the Hamiltonian on the vacuum state:

$$\hat{H}\left|0\right\rangle = \frac{1}{2}(\dot{v}^2 + \omega^2 v^2)^*\hat{a}^\dagger\hat{a}^\dagger\left|0\right\rangle + \frac{1}{2}(|\dot{v}|^2 + \omega^2|v|^2)\left|0\right\rangle \tag{114}$$

The requirement that $|0\rangle$ be an eigenstate of $\hat{H}$ means that the first term must vanish which implies the condition:

$$\dot{v} = \pm i\omega v \tag{115}$$

Imposing the normalization for $v(t)$ defined previously we get:

$$v(t) = \sqrt{\frac{\hbar}{2\omega}}e^{-i\omega t} \tag{116}$$

With this special choice of v the Hamiltonian is:

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right) \tag{117}$$

For the vacuum state $|0\rangle$ with energy $\hbar\omega/2$ we can calculate the zero point fluctuations:

$$\langle|\hat{x}|^2\rangle \equiv \left\langle 0\right|\hat{x}^\dagger\hat{x}\left|0\right\rangle$$
$$= \left\langle 0\right|(v^*\hat{a}^\dagger + v\hat{a})(v^*\hat{a}^\dagger + v\hat{a})\left|0\right\rangle = |v(\omega,t)|^2$$

$$\langle|\hat{x}|^2\rangle = |v(\omega,t)|^2 = \frac{\hbar}{2\omega} \tag{118}$$

We can now move on to the calculation of the power spectrum for scalar fluctuations.

5.6 Mukhanov Sasaki Equation

We have previously derived the evolution equation for the Mukhanov variable and its Fourier decomposition:

$$v_k''(\tau) + \left(k^2 - \frac{z''}{z}\right)v_k(\tau) = 0$$

As in the case of the $1D$ harmonic oscillator we promote the field v and its conjugate momentum v' to quantum operators:

$$v \to \hat{v} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[v_k(\tau)\hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + v_k^*(\tau)\hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} \right] \tag{119}$$

Equivalently, the Fourier components v_k are promoted to operators and expressed via the following decomposition:

$$v_{\mathbf{k}} \to v_k(\tau)\hat{a}_{\mathbf{k}} + v_k^*(\tau)\hat{a}_{-\mathbf{k}}^\dagger \tag{120}$$

where the creation and annihilation operators $\hat{a}_{\mathbf{k}}^\dagger$and $\hat{a}_{\mathbf{k}}$ satisfy the canonical commutation relation:

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \tag{121}$$

The above commutation holds if and only if the mode functions are normalized as follows:

$$\langle v_k, v_k \rangle \equiv \frac{i}{\hbar}(v_k^* v_k' - (v_k^*)' v_k) = 1 \tag{122}$$

The above normalization provides one of the boundary conditions on the solutions of Mukhanov Sasaki equation. The second boundary condition that fixes the mode functions completely comes from vacuum selection.

5.7 Boundary conditions and Bunch-Davies Vacuum

To choose a vacuum state for the fluctuations, $\hat{a}_{\mathbf{k}} |0\rangle = 0$ we need to completely specify the mode functions v_k by specifying an additional boundary condition (besides (122)). The standard choice is the Minkowski vacuum of a comoving observer in the far past (when all comoving scales were far inside the Hubble horizon).
This limit corresponds to $\tau \to -\infty$ or $|k\tau| >> 1$ or $k >> aH$. In this limit, the mode functions satisfy:

$$v_k'' + k^2 v_k = 0 \tag{123}$$

This is the equation of a harmonic oscillator with time-independent frequency. For this case a unique solution exists if we require the vacuum to be the minimum energy state. Hence we impose the initial condition:

$$\lim_{\tau \to -\infty} v_k = \frac{e^{ik\tau}}{\sqrt{2k}} \tag{124}$$

The boundary conditions (122) and (124) completely specify v_k on all scales.

5.8 Solution in de Sitter Space

If we consider the de Sitter limit $H \simeq constant \implies \varepsilon \to 0$ then using equation (91) we can write:

$$\frac{z''}{z} = \frac{a''}{a} \tag{125}$$

Now, for the de Sitter limit for inflationary expansion: (using equation (41))

$$
\begin{aligned}
a &= -\frac{1}{H\tau} \\
\implies a' &= \frac{1}{H\tau^2} \\
\implies a'' &= -\frac{2}{H\tau^3} \\
\implies \frac{a''}{a} &= \frac{2}{\tau^2}
\end{aligned}
$$

Thus, the equation of motion is:

$$v_k'' + \left(k^2 - \frac{2}{\tau^2}\right) v_k = 0 \tag{126}$$

It is easy to verify that this equation admits an exact solution of the form:

$$v_k = \alpha \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) + \beta \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right) \tag{127}$$

The free parameters α and β characterize the non-uniqueness of the mode functions. However, we may fix α and β to unique values by considering the condition (122) together with the subhorizon limit, $|k\tau| >> 1$, equation (124). This fixes $\alpha = 1$, $\beta = 0$ and leads to the unique **Bunch-Davies mode functions**:

$$v_k = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) \tag{128}$$

5.9 Power Spectrum

We are now equipped to compute the power spectrum of the field $\hat{\psi}_{\mathbf{k}} = \hat{v}_{\mathbf{k}}/a$.

$$\langle \hat{\psi}_{\mathbf{k}}(\tau)\hat{\psi}_{\mathbf{p}}(\tau)\rangle = \frac{1}{a^2} \langle 0| \hat{v}_{\mathbf{k}}(\tau)\hat{v}_{\mathbf{p}}(\tau) |0\rangle \tag{129}$$

Expanding $\hat{v}_{\mathbf{k}}$ in terms of creation and annihilation operators:

$$
\begin{aligned}
\frac{1}{a^2} \langle 0| \hat{v}_{\mathbf{k}}(\tau)\hat{v}_{\mathbf{p}}(\tau) |0\rangle &= \frac{1}{a^2} \langle 0| (v_k(\tau)\hat{a}_{\mathbf{k}} + v_k^*(\tau)\hat{a}^\dagger_{-\mathbf{k}})(v_k(\tau)\hat{a}_{\mathbf{p}} + v_k^*(\tau)\hat{a}^\dagger_{-\mathbf{p}}) |0\rangle \\
&= \frac{1}{a^2} v_k(\tau)v_k^*(\tau) \langle 0| \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{p}} |0\rangle \\
&= \frac{|v_k(\tau)|^2}{a^2} \langle 0| \hat{a}^\dagger_{-\mathbf{p}}\hat{a}_{\mathbf{k}} + [\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{-\mathbf{p}}] |0\rangle \\
&= \frac{|v_k(\tau)|^2}{a^2} (2\pi)^3 \delta(\mathbf{k} + \mathbf{p})
\end{aligned}
$$

We can further simplify this expression by using the Bunch-Davies mode functions and the de Sitter limit for the scale factor $a = -1/H\tau$:

$$\begin{aligned}\langle\hat{\psi}_{\mathbf{k}}(\tau)\hat{\psi}_{\mathbf{p}}(\tau)\rangle &= \frac{\left|\frac{e^{-ik\tau}}{\sqrt{2k}}\left(1-\frac{i}{k\tau}\right)\right|^2}{a^2}(2\pi)^3\delta(\mathbf{k}+\mathbf{p}) \\ &= \frac{1}{2k}\left(1+\frac{1}{k^2\tau^2}\right)H^2\tau^2(2\pi)^3\delta(\mathbf{k}+\mathbf{p})\end{aligned}$$

$$\langle\hat{\psi}_{\mathbf{k}}(\tau)\hat{\psi}_{\mathbf{p}}(\tau)\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{p})\frac{H^2}{2k^3}(1+k^2\tau^2) \tag{130}$$

On superhorizon scales, $|k\tau| << 1$ we can write:

$$\langle\hat{\psi}_{\mathbf{k}}(\tau)\hat{\psi}_{\mathbf{p}}(\tau)\rangle \simeq (2\pi)^3\delta(\mathbf{k}+\mathbf{p})\frac{H^2}{2k^3} \tag{131}$$

This allows us to calculate the power spectrum of $\mathcal{R} = \frac{H}{\dot{\phi}}\psi$ at horizon crossing when $a(t_\star)H(t_\star) = k$:

$$\langle\mathcal{R}_{\mathbf{k}}(t)\mathcal{R}_{\mathbf{p}}(t)\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{p})\frac{H_\star^2}{2k^3}\frac{H_\star^2}{\dot{\phi}_\star^2} = (2\pi)^3\delta(\mathbf{k}+\mathbf{p})\mathcal{P}_{\mathcal{R}}(k) \tag{132}$$

Here, $(..)_\star$ indicates that a quantity is to be evaluated at horizon crossing. We then define the dimensionless power spectrum $\Delta^2_{\mathcal{R}}(k)$ as:

$$\Delta^2_{\mathcal{R}}(k) \equiv \frac{k^3}{(2\pi)^2}\mathcal{P}_{\mathcal{R}}(k) = \frac{H_\star^2}{(2\pi)^2}\frac{H_\star^2}{\dot{\phi}_\star^2} \tag{133}$$

The real space variance of $\mathcal{R}$ is $\langle\mathcal{R}\mathcal{R}\rangle = \int_0^\infty \Delta^2_{\mathcal{R}}(k)\mathrm{d}lnk$.
Since $\mathcal{R}$ approaches a constant on super-horizon scales the spectrum at horizon crossing determines the future spectrum until a given fluctuation mode re-enters the horizon. The fact that we computed the power spectrum at a specific instant (horizon crossing, $a_\star H_\star = k$) implicitly extends the result for the pure de Sitter background to a slowly time-evolving **quasi-de Sitter** space. Different modes exit the horizon as slightly different times when $a_\star H_\star$ has a different value. This procedure gives the correct result for the power spectrum during slow-roll inflation. For non-slow-roll inflation the background evolution will have to be tracked more precisely and the Mukhanov Equation typically has to be integrated numerically.
A similar exercise for the tensor perturbation h_{ij} allows us to calculate the power spectrum $\mathcal{P}_t(k)$. I will not be covering it in this report but the behaviour turns out to be exactly the same, with the Mukhanov Sasaki equation governing the evolution in that case too.
Having calculated the power spectrum, I conclude the discussion on quantum effects in Inflation.

6 Conclusion

Starting from FRW cosmology, I studied the origin of Quantum fluctuations during inflationary expansion as a part of my project. The next step is to use this background knowledge to probe the possibility of conclusively pinning the origin of observed fluctuations in say the CMB temperature to Quantum mechanical coherence effects during inflation, over purely classical phenomena. This will involve studying quantum coherence and coherent states in detail.

References

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