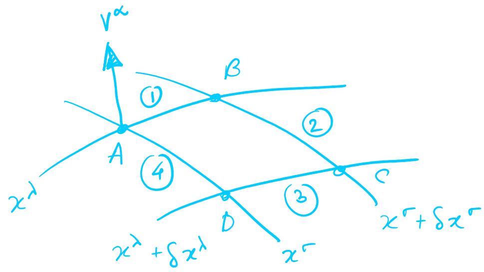
★ Quantifying Curvature : Parallel transport V^α along

$A \to B \to C \to D \to A$



$A \to B$: x^λ is constant ; moving along $\vec{e}_\sigma$

$\nabla_{\vec{e}_\sigma} V^\alpha = 0$ (parallel transport) $\Rightarrow$ $\partial_\sigma V^\alpha + \Gamma^\alpha{}_{\sigma\mu} V^\mu = 0$

$$\Rightarrow \quad \frac{\partial V^\alpha}{\partial x^\sigma} = -\Gamma^\alpha{}_{\sigma\mu} V^\mu \quad \xrightarrow{\text{integrate this}} \quad V^\alpha(B) = V^\alpha_{\text{initial}} - \int_{1} \Gamma^\alpha{}_{\sigma\mu} V^\mu \, dx^\sigma$$

Similarly :
$$V^\alpha(C) = V^\alpha(B) - \int_{2} \Gamma^\alpha{}_{\lambda\mu} V^\mu \, dx^\lambda$$
$$V^\alpha(D) = V^\alpha(C) + \int_{3} \Gamma^\alpha{}_{\sigma\mu} V^\mu \, dx^\sigma$$
$$V^\alpha_{\text{final}}(A) = V^\alpha(D) + \int_{4} \Gamma^\alpha{}_{\lambda\mu} V^\mu \, dx^\lambda$$

Now :
$$\delta V^\alpha = V^\alpha_{\text{final}} - V^\alpha_{\text{initial}} = \overbrace{\int_{4} \Gamma^\alpha{}_{\lambda\mu} V^\mu \, dx^\lambda}^{@\, x^\sigma} - \overbrace{\int_{2} \Gamma^\alpha{}_{\lambda\mu} V^\mu \, dx^\lambda}^{@\, x^\sigma + \delta x^\sigma}$$
$$+ \underbrace{\int_{3} \Gamma^\alpha{}_{\sigma\mu} V^\mu \, dx^\sigma}_{@\, x^\lambda + \delta x^\lambda} - \underbrace{\int_{1} \Gamma^\alpha{}_{\sigma\mu} V^\mu \, dx^\sigma}_{@\, x^\lambda}$$

$\{ f(x_0 + dx) \simeq f(x_0) + f'(x_0)\, dx \}$

Use Taylor expansion:
$$\delta V^\alpha \simeq -\int_{x^\lambda}^{x^\lambda + \delta x^\lambda} \delta x^\sigma \frac{\partial}{\partial x^\sigma}\left(\Gamma^\alpha{}_{\lambda\mu} V^\mu\right) dx^\lambda + \int_{x^\sigma}^{x^\sigma + \delta x^\sigma} \delta x^\lambda \frac{\partial}{\partial x^\lambda}\left(\Gamma^\alpha{}_{\sigma\mu} V^\mu\right) dx^\sigma$$

$$\left\{ \int_\alpha^{\alpha+\varepsilon} f(x)\, dx \simeq \varepsilon \cdot f(\alpha) \quad \text{for } \varepsilon \ll 1 \right\}$$

$$\delta V^\alpha = \delta x^\lambda \delta x^\sigma \left[\partial_\lambda \Gamma^\alpha{}_{\sigma\mu} \cdot V^\mu + \Gamma^\alpha{}_{\sigma\mu} \cdot \partial_\lambda V^\mu - \partial_\sigma \Gamma^\alpha{}_{\lambda\mu} \cdot V^\mu - \Gamma^\alpha{}_{\lambda\mu} \cdot \partial_\sigma V^\mu \right]$$

for $\parallel$ transport : $\partial_\sigma V^\alpha + \Gamma^\alpha{}_{\sigma\mu} V^\mu = 0$

$$\Rightarrow \quad \delta V^\alpha = \delta x^\lambda \delta x^\sigma V^\mu R^\alpha{}_{\mu\lambda\sigma}$$

$$\boxed{R^\alpha{}_{\mu\lambda\sigma} = \partial_\lambda \Gamma^\alpha{}_{\sigma\mu} - \partial_\sigma \Gamma^\alpha{}_{\lambda\mu} + \Gamma^\alpha{}_{\lambda\nu} \Gamma^\nu{}_{\sigma\mu} - \Gamma^\alpha{}_{\sigma\nu} \Gamma^\nu{}_{\lambda\mu}} \to \text{Riemann Curvature Tensor}$$

Equivalently :
$$V^\mu R^\alpha{}_{\mu\lambda\sigma} = [\nabla_\lambda, \nabla_\sigma] V^\alpha$$
$$P_\mu R^\mu{}_{\alpha\lambda\sigma} = [\nabla_\lambda, \nabla_\sigma] P_\alpha$$

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- Symmetries of the Riemann tensor reduce the no. of indep components from n^4 to $\boxed{\frac{n^2(n^2-1)}{12}}$ $\longrightarrow$ for 4D S.T. it is $\frac{4^2 \times 15}{12} = \boxed{20}$

★ Symmetries of the RCT :

① $R^\alpha{}_{\mu\lambda\sigma} = -R^\alpha{}_{\mu\sigma\lambda}$ } antisymmetry in last 2 indices. $\equiv$ Reversal of direction of transport

$R_{\alpha\mu\lambda\sigma} = g_{\alpha\nu} R^\nu{}_{\mu\lambda\sigma}$. Now we go to Locally Lorentz frame. $\Rightarrow$ Γ vanish but not $\partial\Gamma$

$$\left(R_{\alpha\mu\lambda\sigma}\right)_{L.L.F.} = \partial_\lambda \Gamma_{\alpha\sigma\mu} - \partial_\sigma \Gamma_{\alpha\lambda\mu} = \frac{1}{2}\left(\partial_\lambda\partial_\mu g_{\alpha\sigma} - \partial_\lambda\partial_\alpha g_{\sigma\mu} - \partial_\sigma\partial_\mu g_{\alpha\lambda} + \partial_\sigma\partial_\alpha g_{\lambda\mu}\right)$$

We get :

① $R_{\alpha\mu\lambda\sigma} = -R_{\alpha\mu\sigma\lambda}$ (antisymm in last two)

② $R_{\alpha\mu\lambda\sigma} = -R_{\mu\alpha\lambda\sigma}$ (—"— first two)

③ $R_{\lambda\sigma\alpha\mu} = -R_{\alpha\mu\lambda\sigma}$ (first two $\longleftrightarrow$ last two)

④ $R_{\alpha\mu\lambda\sigma} + R_{\alpha\lambda\sigma\mu} + R_{\alpha\sigma\mu\lambda} = 0 \Rightarrow R_{\alpha[\mu\lambda\sigma]} = 0$

★ Ricci Tensor : contract index 1, 3 of RCT :

$$\boxed{R^\alpha{}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\beta\mu\alpha\nu} = R_{\mu\nu}} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \underbrace{\partial_\nu \Gamma^\alpha{}_{\alpha\mu}}_{= \partial_\nu\partial_\mu(\ln|g|^{1/2})} + \Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{\nu\mu} - \Gamma^\alpha{}_{\nu\beta}\Gamma^\beta{}_{\alpha\mu}$$

Symmetric $\binom{0}{2}$ Tensor

★ Curvature Scalar : $\boxed{R = g^{\mu\nu} R_{\mu\nu} = R^\mu{}_\mu}$

★ $0 = g^{\nu\sigma} g^{\mu\lambda} \left(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu}\right) = \nabla^\mu R_{\rho\mu} - \nabla_\rho R + \nabla^\nu R_{\rho\nu}$

$$\boxed{\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R}$$

★ The Einstein tensor: $\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}$ $\rightsquigarrow$ Trace reversed Ricci tensor.

$$\boxed{\nabla^\mu G_{\mu\nu} = 0}$$

★ Geodesic Deviation:

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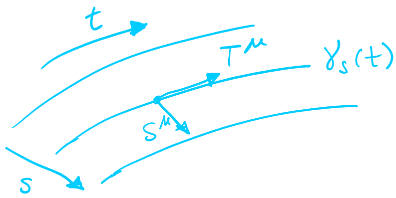
- Tangent vectors to Geodisics: $T^\mu = \frac{\partial x^\mu}{\partial t}$

- Deviation vectors: $S^\mu = \frac{\partial x^\mu}{\partial s}$



- $\boxed{V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu}$ → relative velocity of Geodisics
- $\boxed{A^\mu = (\nabla_T V)^\mu = T^\rho \nabla_\rho V^\mu}$ → relative accl^n of Geodisics

The accl^n of a path away from being a geodisic: $\boxed{a^\mu = T^\sigma \nabla_\sigma T^\mu}$

Since S & T are basis vectors adapted to a cood system, their commutator vanishes:

$[S, T] = 0$; also: $\boxed{S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho T^\mu}$

The acceleration $\boxed{A^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu) = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma}$

$$A^\mu = \frac{D^2}{dt^2} S^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$ ⇝ Geodisic deviation