★ EINSTEIN FIELD EQUATION:

Newton: $\vec{a} = -\vec{\nabla}\Phi$ and $\boxed{\nabla^2 \Phi = 4\pi G\rho}$

Recipie for generalizing laws to curved space:

1) Take a law of physics valid in inertial coords in flat spacetime.
2) Write in a coordinate invarient (tensorial) form.
3) Assert that the resulting law remains true in curved space.

Example: $\eta_{\mu\nu} \rightarrow g_{\mu\nu}$ or $\partial_\mu \rightarrow \nabla_\mu$

Free fall: $\frac{d^2 x'^\mu}{d\lambda^2} = \frac{dx^\nu}{d\lambda}\left(\partial_\nu \frac{dx^\mu}{d\lambda}\right) = 0 \rightarrow \frac{dx^\nu}{d\lambda}\nabla_\nu \frac{dx^\mu}{d\lambda} = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$

Energy momentum conservation: $\partial_\mu T^{\mu\nu} = 0 \longrightarrow \nabla_\mu T^{\mu\nu} = 0$

★ Newtonian limit of gravitation:

- Slowly moving particle $v \ll 1$ $(c=1) \rightarrow \boxed{\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau} = \frac{dt}{d\tau}}$
- Static field
- Weak field

$\Downarrow$

Geodesic: eqn $\boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{00}\left(\frac{dt}{d\tau}\right)^2 = 0}$

$\Rightarrow \boxed{\partial_0 g_{\mu\nu} = 0 \Rightarrow \Gamma^\mu{}_{00} = -\frac{1}{2} g^{\mu\lambda}\partial_\lambda g_{00}}$

$\boxed{\begin{array}{l} g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \\ g^{\mu\nu} \simeq \eta^{\mu\nu} - h^{\mu\nu} \end{array}}$ with $|h_{\mu\nu}| \ll 1$

$h^{\mu\nu} = \eta^{\mu\sigma}\eta^{\nu\rho} h_{\sigma\rho}$

$\Gamma^\mu{}_{00} = -\frac{1}{2}\eta^{\mu\nu}\partial_\lambda h_{00} + \mathcal{O}(h^2 \dots)$, $\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\left(\frac{dt}{d\tau}\right)^2$

Using $\partial_0 h_{00} = 0$, the $\mu = 0$ component is: $\frac{d^2t}{d\tau^2} = 0$

$\rightarrow \frac{d^2x^i}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \partial_i h_{00} \Rightarrow \frac{d^2x^i}{d\tau^2} = \frac{1}{2}\partial_i h_{00}$ (and $\vec{a} = -\vec{\nabla}\Phi$)

$\rightarrow \boxed{h_{00} = -2\Phi} \Rightarrow \boxed{g_{00} = -(1+2\Phi)}$

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☆ Now: $\nabla^2 \Phi = 4\pi G \rho$

$\nabla^2 = \delta^{ij}\partial_i \partial_j$

$\longrightarrow [\nabla^2 g]_{\mu\nu} \propto T_{\mu\nu}$

$\nabla^\mu R_{\mu\nu} \neq 0$

$\nabla^\mu T_{\mu\nu} = 0$

Guesses: $\cancel{R_{\mu\nu} = k T_{\mu\nu}}$

or: $\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k T_{\mu\nu}}$

or $\boxed{R_{\mu\nu} = k(T_{\mu\nu} - T g_{\mu\nu})}$

$T_{\mu\nu} = \rho\, U_\mu U_\nu \; ; \; U^\mu = (U^0, 0, 0, 0)$ (dust rest frame)

$g_{\mu\nu} U^\mu U^\nu = -1$ and $g_{00} \simeq -1 + h_{00}$ & $g^{00} \simeq -1 - h^{00}$

$\Rightarrow U^0 \simeq 1 + \frac{1}{2} h_{00} \Rightarrow T_{00} \simeq \rho \Rightarrow T \simeq g^{00} T_{00} \simeq -\rho$

substitute

$R_{00} = -\frac{1}{2} \vec{\nabla}^2 h_{00}$ and $R_{\mu\nu} = \frac{1}{2} k \rho \Rightarrow \boxed{k\rho = \vec{\nabla}^2 h_{00}}$ but $h_{00} = -2\Phi$

$\Rightarrow \boxed{k = 8\pi G}$

FIELD EQUATION

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}}$$

$$\boxed{R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)}$$