★ Spacetime : A manifold of events endowed by a metric.

A set of pts with well understood connection props

When and where something happens

A notion of distance btwn events on a manifold.

★ STR = (GR - Gravity) = The theory of relativity on FLAT SPACTIME

Minkowski Space

★ Natural units : $c = 1$

$\Rightarrow$ [energy] = [mass] = [(length)$^{-1}$] = [(time)$^{-1}$]

★ $\Delta\vec{x} \underset{\mathcal{O}}{\doteq} \Delta x^{\mu}$ $\qquad \Delta x^{\mu} \equiv (\Delta t, \Delta x, \Delta y, \Delta z) \equiv (\Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3)$

Abstract notation

obs

Frame dependent component notn

★ LORENTZ TRANSFORMATION $\boxed{\Delta x^{\mu'} = \Lambda^{\mu'}{}_{\nu}\, \Delta x^{\nu}}$ — flat space convention

relates components in 2 frames

More general form for 4-vector transforms : $\boxed{\Delta x^{\mu'} = \dfrac{\partial x^{\mu'}}{\partial x^{\nu}}\, \Delta x^{\nu}}$

★ Basis Vectors : $\Delta\vec{x} = \Delta x^{\mu}\hat{e}_{\mu} = \Delta x^{\nu'}\hat{e}_{\nu'}$

Has meaning indep of chosen cood

$\cancel{\Delta x^{\mu}}\,\hat{e}_{\mu} = \Lambda^{\nu'}{}_{\mu}\,\cancel{\Delta x^{\mu}}\,\hat{e}_{\nu'} \Rightarrow \Lambda^{\nu'}{}_{\mu}\,\hat{e}_{\nu'} = \hat{e}_{\mu} \Rightarrow \boxed{\hat{e}_{\nu'} = \Lambda^{\mu}{}_{\nu'}\,\hat{e}_{\mu}}$

inverse Lorentz matrix

$\Lambda^{\mu}{}_{\alpha'}\Lambda^{\alpha'}{}_{\nu} = \delta^{\mu}{}_{\nu}$ $\qquad \Rightarrow \boxed{\hat{e}_{\nu'} = \dfrac{\partial x^{\mu}}{\partial x^{\nu'}}\,\hat{e}_{\mu}}$

Kronecker

★ Invariant space-time Interval : $\Delta s^2 \equiv$ Lorentz Scalar

$\Delta s^2 = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = \Delta x^{\mu}\,\eta_{\mu\nu}\,\Delta x^{\nu}$

$\Delta s^2 = \Delta\vec{x}\cdot\Delta\vec{x}$ {Inner prod}

$$\eta \equiv \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

MINKOWSKI METRIC

$$\Delta s^2 \begin{cases} < 0, & \text{timelike} \\ > 0, & \text{spacelike} \\ = 0, & \text{Null} \end{cases}$$

Infinitesimal "Distance"

$$\boxed{ds^2 = dx^{\mu}\,\eta_{\mu\nu}\,dx^{\nu} = dx^{\mu'}\,\eta_{\mu'\nu'}\,dx^{\nu'}}$$

$\eta^{\mu\nu} \equiv$ Inverse of $\eta_{\mu\nu} \Rightarrow \eta^{\alpha\beta}\eta_{\beta\gamma} = \delta^{\alpha}{}_{\gamma}$

★ NOTE : when : $d\vec{x} = dx^{\mu}\,\vec{e}_{\mu}$, we say $\hat{e}_{\mu}$ is a coordinate basis
this isn't always true. Ex. spherical coord. with $\hat{e}_i \equiv \hat{r}, \hat{\theta}, \hat{\phi}$

★ Important 4-vecs : $\vec{U} = \dfrac{d\vec{x}}{d\tau}$ — Time elapsed in rest frame of obj whose velo is measured

4-VELOCITY; $d\tau$: Proper time

$$\boxed{\vec{U} \underset{Q}{\doteq} (\gamma, \gamma \underline{v})}$$

Here $\gamma = \dfrac{1}{\sqrt{1-v^2}}$ and $\underline{v} = \dfrac{d\underline{x}}{dt}$ is the "usual" 3-velo

$\left\{ \begin{array}{l} \vec{V} \equiv \text{4-vector} \\ \underline{V} \equiv \text{3-vector} \end{array} \right\}$

$\vec{U} \underset{\text{Rest frame}}{\doteq} (1, \underline{0})$; $\vec{U}\cdot\vec{U} = U^{\mu}\eta_{\mu\nu}U^{\nu} = -1$

4 - MOMENTUM : $\vec{p} = m\vec{U}$ — m: rest mass — is Lorentz invariant

$= (\gamma m, \gamma m \underline{v}) = (E, \underline{p})$ — $\underline{p}$: Relativistic 3-momentum; E: Energy

$\vec{p}\cdot\vec{p} = -m^2\,\vec{U}\cdot\vec{U} = -m^2 = -E^2 + |\underline{p}|^2 \Rightarrow \boxed{E^2 = m^2 + |\underline{p}|^2}$

★ Consider an obs Q and another obs A.
Let $\vec{p}$ be the 4-mom of A and $\vec{U}$ the 4-velo of Q.
What does Q measure the energy of A to be?

$\vec{p} \underset{Q}{\doteq} (E_Q, \underline{P}_Q)$

$\vec{U} \underset{Q}{\doteq} (1, \underline{0})$ — Q's 4-velo in his own ref

Observe: $\underbrace{\vec{p}\cdot\vec{U}}_{\text{Lorentz invariant}} = -E_Q$ $\Rightarrow$ $\boxed{\vec{p}\cdot\vec{U} = -E}$ — Holds no matter in which frame you are.

★ 4-acceleration : $\vec{a} = \dfrac{d\vec{U}}{d\tau} = \dfrac{d^2\vec{x}}{d\tau^2}$

Note : $\dfrac{d}{d\tau}(\vec{U}\cdot\vec{U}) = \dfrac{d}{d\tau}(-1) = 0 = 2\,\vec{a}\cdot\vec{U} \Rightarrow \boxed{\vec{a}\cdot\vec{U} = 0}$

★ Tensors more generally : a $\binom{0}{N}$ Tensor would be a Mapping from N 4-vectors to a Lorentz invariant Scalar.

$\tilde{\eta}(\vec{A}, \vec{B}) = \vec{A}\cdot\vec{B} = A^{\alpha}\eta_{\alpha\beta}B^{\beta} = a$ — Scalar

The metric $\eta_{\alpha\beta}$ is a $\binom{0}{2}$ tensor

Linearity : $\tilde{\eta}(\lambda\vec{A}, \vec{B}) = \lambda\,\tilde{\eta}(\vec{A}, \vec{B})$

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Linearity : $\tilde{\tilde{\eta}}(\lambda\vec{A}, \vec{B}) = \lambda \tilde{\tilde{\eta}}(\vec{A}, \vec{B})$

$$\eta_{\alpha'\beta'} = \Lambda^\mu{}_{\alpha'} \Lambda^\nu{}_{\beta'} \eta_{\mu\nu} = \frac{\partial x^\mu}{\partial x^{\alpha'}} \cdot \frac{\partial x^\nu}{\partial x^{\beta'}} \eta_{\mu\nu}$$

★ One form : $\binom{0}{1}$ Tensor a.k.a. dual vectors

$\tilde{\tilde{\eta}}(\vec{e}_\alpha, \vec{e}_\beta) = \eta_{\alpha\beta} \longrightarrow$ similarly for a 1-form $\tilde{p}$: $\tilde{p}(\vec{e}_\alpha) = p_\alpha$

$\tilde{p} = p_\alpha \tilde{\omega}^\alpha$ ↪ dual basis

$\underbrace{p_\alpha A^\alpha = \tilde{p}(\vec{A})}_{\text{A Scalar}}$ and $\tilde{\omega}^\beta \vec{e}_\alpha = \delta^\beta{}_\alpha$

★ Contraction, raising & Lowering indices.

$$p^\mu \eta_{\mu\nu} A^\nu = p_\nu A^\nu = p^\mu A_\mu = \vec{p} \cdot \vec{A} = \tilde{p}(\vec{A}) = \tilde{A}(\vec{p})$$

★ 3D space : $\frac{d\phi}{dt} = \frac{dx}{dt} \cdot \frac{\partial \phi}{\partial x} + \frac{dy}{dt} \cdot \frac{d\phi}{dy} + \frac{dz}{dt} \cdot \frac{d\phi}{dz} = \underline{v} \cdot \underline{\nabla} \phi$ ↪ "usual" 3D gradient

4D spacetime : $\frac{d\phi}{d\tau} = \frac{dt}{d\tau} \cdot \frac{\partial \phi}{\partial t} + \frac{dx}{d\tau} \cdot \frac{\partial \phi}{\partial x} + \frac{dy}{d\tau} \cdot \frac{\partial \phi}{\partial y} + \frac{dz}{d\tau} \frac{\partial \phi}{\partial z} = \underbrace{u^\alpha}_{\text{4-velo}} \underbrace{\partial_\alpha \phi}_{\text{Gradient 1-form.}}$

one-form

$\tilde{\partial} \phi \equiv \{\partial_\alpha \phi\}$, NOTATION : $\boxed{\frac{d\phi}{d\tau} = u^\alpha \partial_\alpha \phi = \nabla_{\vec{u}} \phi}$

★ ABSTRACT NOTATION OF THE BASIS :

$\partial_\beta x^\alpha = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta$; $\tilde{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha{}_\beta$

$\boxed{d\tilde{x}^\mu(\vec{\partial}_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu{}_\nu}$

$d\tilde{x}^\alpha = \tilde{\omega}^\alpha$ and $\vec{e}_\beta = \vec{\partial}_\beta$

★ Tensor of type $\binom{M}{N}$ is a multilinear map from M-1forms & N-vectors to a Lorentz scalar.

★ Raising & Lowering indices : $\eta_{\alpha\beta} T^\beta{}_{\gamma\delta} = T_{\alpha\gamma\delta}$ and $\eta^{\alpha\beta} T_{\beta\gamma\delta} = T^\alpha{}_{\gamma\delta}$

★ Basis of Tensors :

$\boxed{\tilde{\tilde{T}} = T^\gamma{}_{\alpha\beta} \, \vec{e}_\gamma \otimes \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta}$

outer product — order matters!

★ Electromagnetizm : Field strength tensor → $F_{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_1 & -B_2 \\ E_2 & -B_1 & 0 & B_3 \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix} = -F_{\nu\mu}$

• 4-vector potential $\vec{A} = (\phi, \underline{A})$ for $c=1$

$$F_{\mu\nu} = \begin{bmatrix} \cdots \\ E_1 & 0 & B_1 & -B_2 \\ E_2 & -B_1 & 0 & B_3 \\ E_3 & B_2 & -B_3 & 0 \end{bmatrix} \quad F_{\nu\mu}$$

- 4-vector potential $\vec{A} = (\phi, \underline{A})$ for $c=1$
- 4-vector current $\vec{J} = (\rho, \underline{J})$

$F^{0i} = E^i$ and $F^{ij} = \bar{\epsilon}^{ijk} B_k$

$\bar{\epsilon}^{ijk} \equiv$ Levi-civita Symbol (Tensor densities soon)

$$\boxed{\begin{aligned} \partial_j F^{ij} - \partial_0 F^{0i} &= J^i \\ \partial_i F^{0i} &= J^0 \end{aligned}} \longrightarrow \text{Maxwell's Eqns}$$

Tensorial form: $\boxed{\partial_\mu F^{\nu\mu} = J^\nu}$ and $\boxed{\partial_{[\mu} F_{\nu\lambda]} = \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0}$

★ ENERGY & MOMENTUM:

Definition of $T^{\mu\nu}$ - (Energy Momentum Tensor) is the flux of 4-Momentum p^μ across a surface of constant x^ν.

$$\left.\begin{array}{l} T^{00} \equiv \text{flux of } p^0 \text{ (energy) in the } x^0 \text{ (time) direction} \Rightarrow \text{ENERGY DENSITY } \rho \\ T^{0i} = T^{i0} \equiv \text{Momentum flux} \\ T^{ij} \equiv \text{stress terms - forces btwn infinitesimal elements of the fluid - Viscosity} \\ T^{ii} \equiv x^{i\text{th}} \text{ component of force being exerted (per unit Ar.)} \equiv \text{Pressure } P_i \end{array}\right\} \text{VVIP}$$

DUST: non-iteracting matter. $N^\mu = n U^\mu$; $\rho = mn$

No.-flux-4-vector (N^μ); n: No. density in rest frame; ρ: rest frame en. density

Rest frame of dust: $N^\mu \equiv (n, 0, 0, 0)$ & $p^\mu \equiv (m, 0, 0, 0) \Rightarrow T^{00} = \rho$, rest all 0.

Guess $\Rightarrow$ $\boxed{T^{\mu\nu}_{dust} = p^\mu N^\nu = \rho U^\mu U^\nu}$

PERFECT FLUID: $T^{\mu\nu} \equiv \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} \Rightarrow \boxed{T^{\mu\nu} = (\rho + P) U^\mu U^\nu + P \eta^{\mu\nu}}$

Equation of state $P = P(\rho)$

$$\left.\begin{array}{l} \text{- for dust : } P = 0 \\ \text{- for isotropic gas of photons : } P = \tfrac{1}{3}\rho \\ \text{- for vacuum energy : } T^{\mu\nu} = -(\rho_{vac})\, \eta^{\mu\nu} \end{array}\right\} \text{- Imp in G.R.}$$

Conservation of $T^{\mu\nu}$: $\boxed{\partial_\mu T^{\mu\nu} = 0}$ (flat space)

★ Projection Tensor : $\boxed{P^\sigma{}_\nu = \delta^\sigma{}_\nu + U^\sigma U_\nu}$ $\rightsquigarrow$ projects a vector, say V^μ orthogonal to U^α