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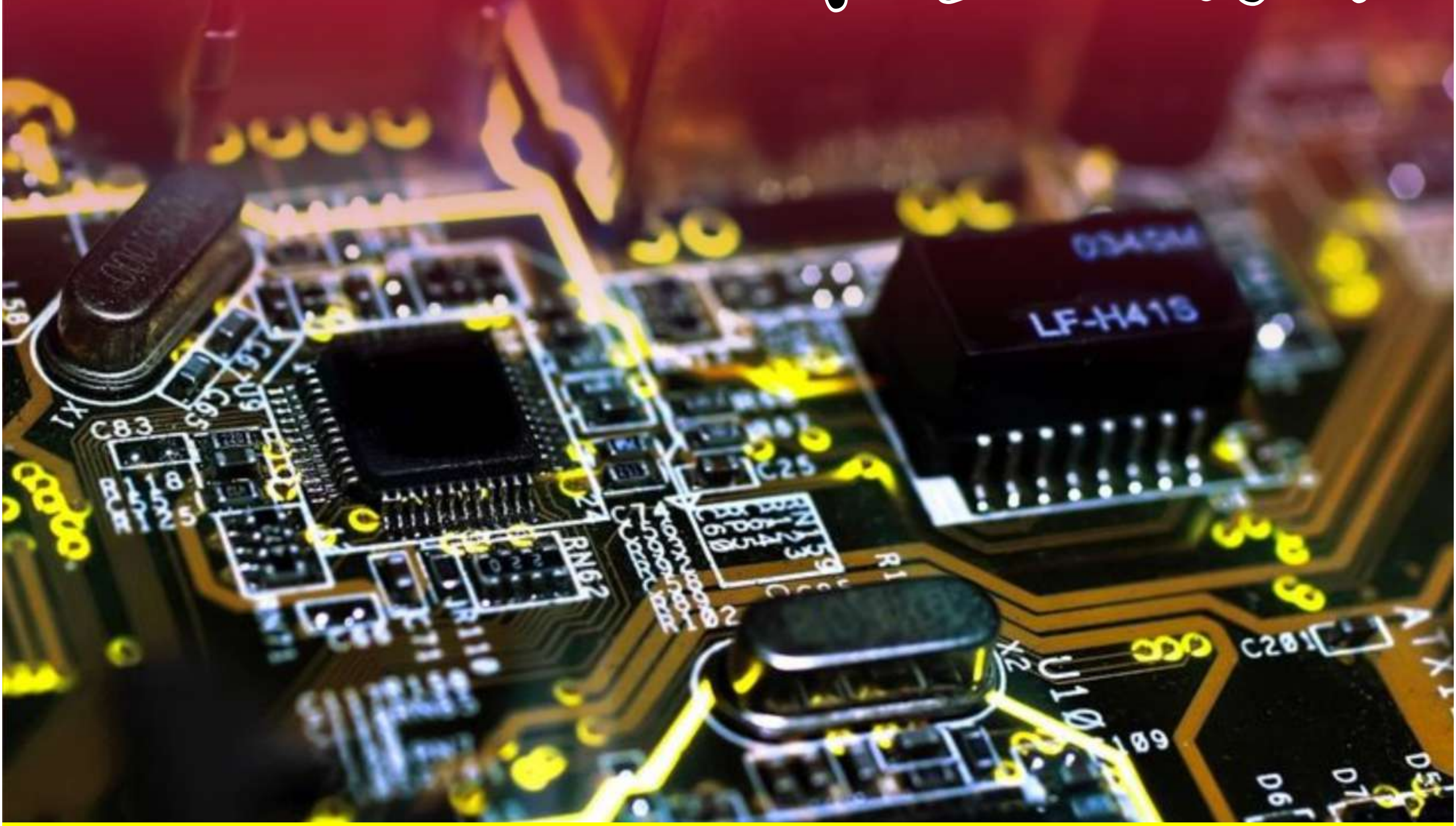

ELECTRONICS & COMMUNICATION ENGINEERING

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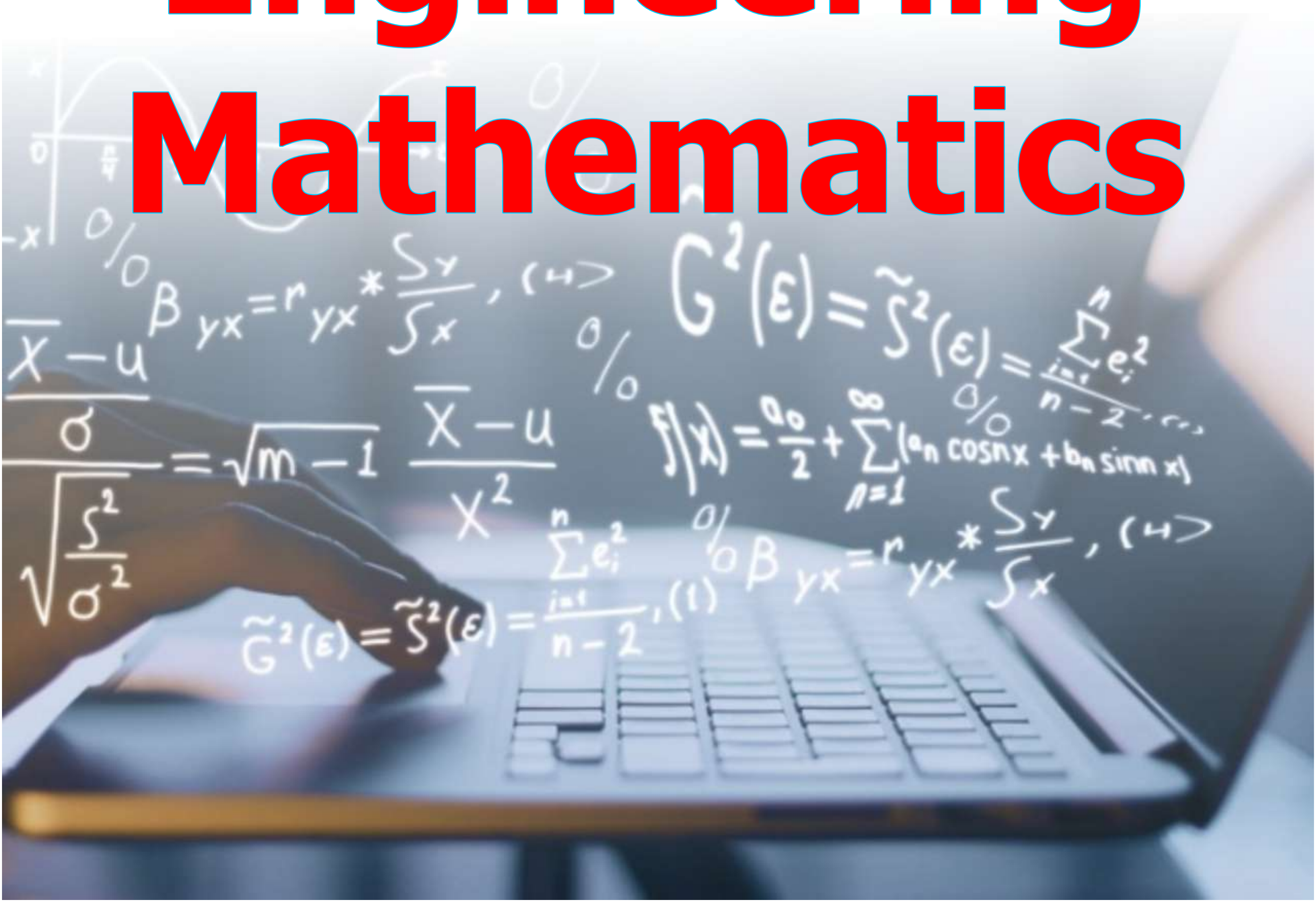
Engineering Mathematics



ENGINEERING MATHEMATICS

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1 BASIC CALCULUS

1.1. Introduction

1.1.1 Limits, Continuity and Differentiability

(a) As x tends to a ($x \to a$) $\Rightarrow$ x is moving towards a

A value l is said to be limit of a function $f(x)$ at $x \to a$ if $f(x) \to l$ as $x \to a$.

It is mathematically defined as

$$\lim_{x \to a} f(x) = l = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

That is, Limit exist at any point, if LHL = RHL

A function f(x) is said to be continuous at x = a, if

$$\lim_{x \to a} f(x) = l = f(a) = f(x)|_{x=a}$$

That is, for a function to be continuous at any point, RHL = LHL = Value of function at point $x = a$.

Note:
- For $\lim_{x \to a} f(x)$ to exist, the function need not be continuous at $x = a$.
- But for $f(x)$ to be continuous at x = a, $\lim_{x \to a} f(x)$ should exist.

- Continuity from Left : $\lim_{x \to a^-} f(x) = f(a)$
- Continuity from Right : If $\lim_{x \to a^+} f(x) = f(a)$

Continuity in an Open Interval

A function 'f' is said to be continuous in open interval (a, b), if it is continuous at each point of open interval.

Continuity in a Closed Interval

Let 'f' be a function defined on the closed interval (a, b) then 'f' is said to be continuous on the closed interval [a, b], if it is :

1. Continuous from the right at a and
2. Continuous from the left at b and
3. Continuous on the open interval (a, b).

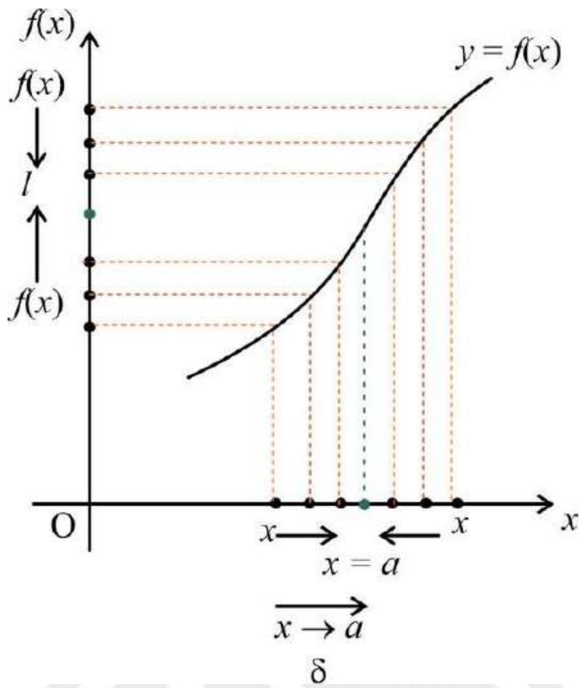

Fig. 1.1

(b) Concept of differentiability

A continuous function $f(x)$ is said to be differentiable at $x = a$, if $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists, that is, RHL and LHL exist at a point under consideration in $f'(x)$.

$f'(x)|_{x=a} = f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

$f'(a) = \tan\theta$, where θ is the angle made by the tangent to the curve at x=a with x – axis.

(c) Some Standard Derivatives

(i) $\frac{d}{dx}(x^n) = n.x^{n-1}$

(ii) $\frac{d}{dx}(\sin x) = \cos x$

(iii) $\frac{d}{dx}(\cos x) = -\sin x$

(iv) $\frac{d}{dx}(\tan x) = \sec^2 x$

(v) $\frac{d}{dx}(\cot x) = -cosec^2 x$

(vi) $\frac{d}{dx}(\sec x) = \sec x . \tan x$

(vii) $\frac{d}{dx}(cosec\, x) = -cosec\, x \cot x$

(viii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}; -1 < x < 1$

(ix) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$

(x) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(xi) $\frac{d}{dx}(cot^{-1} x) = \frac{-1}{1+x^2}$

(xii) $\frac{d}{dx}(sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

(xiii) $\frac{d}{dx}(cosec^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$; $|x| > 1$

(xiv) $\frac{d}{dx}(log_a x) = \frac{1}{x\, log_e a}$

(xv) $\frac{d}{dx}(log_e x) = \frac{1}{x}$

(xvi) $\frac{d}{dx}(a^x) = a^x. log_e a$

(xvii) $\frac{d}{dx}(e^x) = e^x$

(xviii) $\frac{d}{dx}(|x|) = \frac{|x|}{x},\ (x \neq 0)$

(xix) $\frac{d}{dx}(x^x) = x^x(1 + log_e x)$

(xx) $\frac{d}{dx}(sinh\, x) = cos\, h\, x$

(d) Product rule of differentiation

(i) $\frac{d}{dx}\big(f(x). g(x)\big) = f(x). g'(x) + f'(x). g(x)$

(ii) $d(uvw) = uvw' + uv'w + u'vw$

(e) Quotient rule of differentiation

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).f'(x)-f(x).g'(x)}{\left(g(x)\right)^2}, (g(x) \neq 0)$

(f) Logarithmic differentiation:

Taking log might help in differentiation of a function. For example if $y = v^u$ then we can take log both side and differentiable to get $\frac{dy}{dx}$

(g) Differentiation in parametric from :

If we write x and y in term of find variable 't' that is x = f(t), y = ω(t), then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

(h) Greatest Integer function / step function / integer part function

$f(x) = [x] = n, \forall\, n \leq x < n + 1 \text{where}, n \in Z$

$\lim_{x\to a}[x] = \nexists$ if a is an integer (∴ ∄ = do not exist)

L.H.L. = $\lim_{x\to a^-}[x] = a - 1$

R.H.L. = $\lim_{x\to a^+}[x] = a$

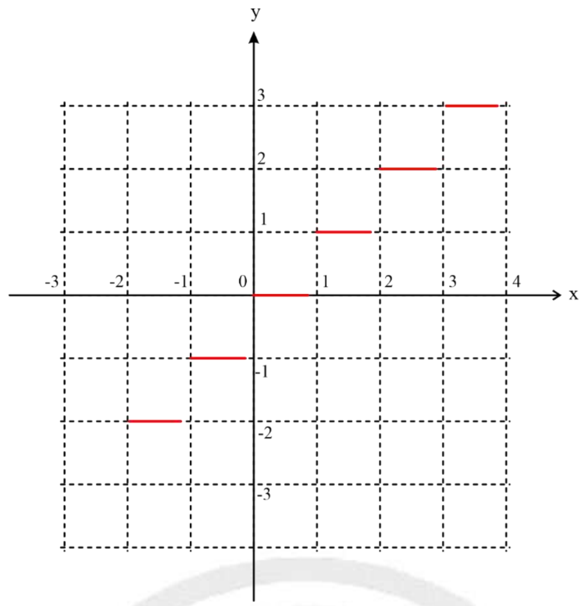

Fig.1.2. Greatest Integer

(i) Properties of Limits

(i) $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

(ii) $\lim_{x\to a}\left(f(x).g(x)\right) = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$

(iii) $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \left(\lim_{x\to a} g(x) \neq 0\right)$

(iv) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x) = \nexists$, then $\lim_{x\to a} f(x).g(x)$ **may exist**

Example: Let $f(x) = \sin x$, $g(x) = \frac{1}{x}$, $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0}\frac{1}{x} = \nexists$

But $\lim_{x\to 0} \sin x.\frac{1}{x} = 1$

(v) Indeterminate form III (0^0, 1^∞, ∞^0)

If $y = \lim_{x\to a}\left[f(x)\right]^{\phi(x)}$

Then, $\log y = \lim_{x\to a} \phi(x)\log\left[f(x)\right]$

Thus 0^0, 1^∞, ∞^0 will convert into $\infty\times 0$ from which can be solved easily.

(vi) If $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ (or) $\frac{\infty}{\infty}$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \neq \left(\frac{0}{0}\right)$

If $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{0}{0}$ (or) $\frac{\infty}{\infty}$, then $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$ and so on

(vii) If $\lim_{x\to a}\left(f(x).g(x)\right) = 0 \times \infty \Rightarrow \lim_{x\to a}\frac{f(x)}{\left(\frac{1}{g(x)}\right)} = \frac{0}{0}$ (Apply L- Hospital Rule again)

(j) Some Standard Limits

(i) $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$

(ii) $\lim\limits_{x\to 0}\frac{\tan x}{x}=1$

(iii) $\lim\limits_{x\to 0}\frac{1-\cos ax}{x^2}=\frac{a^2}{2}$

(iv) $\lim\limits_{x\to\infty}\frac{\sin x}{x}=0$

(v) $\lim\limits_{x\to\infty}\frac{\cos x}{x}=0$

(vi) $\lim\limits_{x\to 0}(1+ax)^{b/x}=e^{ab}$

(vii) $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}=e^{ab}$

(viii) $\lim\limits_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{1/x}=\sqrt{ab}$

(ix) $\lim\limits_{x\to 0}\left(\frac{1^x+2^x+3^x+....+n^x}{n}\right)^{1/x}=\sqrt[n]{n!}$

(x) $\lim\limits_{x\to 0}\frac{a^x-1}{x}=\log_e a\,;\lim\limits_{x\to 0}\frac{e^x-1}{x}=1$

(xi) $\lim\limits_{x\to 0}x.\sin\left(\frac{1}{x}\right)=0$

1.2 Mean Value Theorems

1.2.1 Lagrange's Mean Value Theorem (LMVT):

If $f(x)$ is continuous in [a, b] and it is differentiable in (a, b) then ∃ at least one point 'c' such that c $\in$ (a, b) and

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Here $f'(c)$ slope of tangent to $f(x)$ at $x = c$.

Tangent at x = c is parallel to the line connecting the points A and B

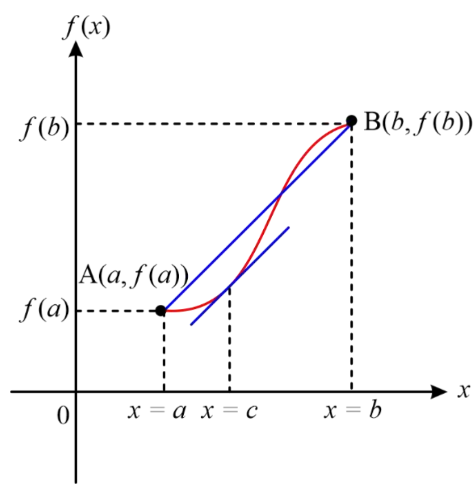

Fig.1.3. LMVT

1.2.2 Rolle's Mean Value Theorem

If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) and $f(a) = f(b)$ then $\exists$ at least one-point c $\in (a, b)$ such that $f'(c) = 0$.

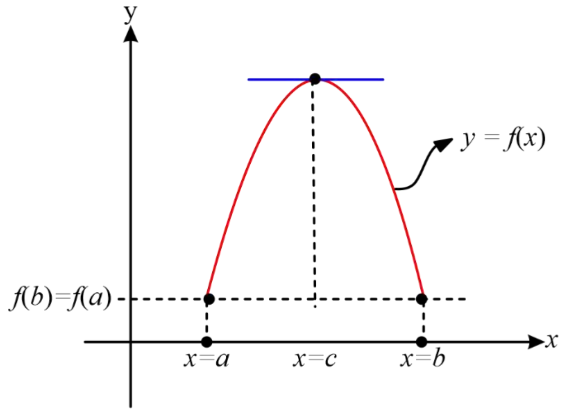

Fig. 1.4. Rolle's mean value

1.2.3 Cauchy's Mean Value Theorem

If $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b) then $\exists$ at least one value of 'c' such that $c \in (a, b)$ and $\frac{g'(C)}{f'(C)} = \frac{g(b)-g(a)}{f(b)-f(a)}$

1.3 Increasing and Decreasing Functions

1.3.1 Increasing Functions

A function $f(x)$ is said to be increasing, if $f(x_1) < f(x_2)\ \forall\ x_1 < x_2$

Or

A function $f(x)$ is said to be increasing, if $f(x)$ increases as x increases.

For a function $f(x)$ to be increasing at the point x=a, $f'(a) > 0$.

Example:

e^x, $\log_e x$ $\rightarrow$ Monotonically increasing functions

$\sin x$ in $(0, \pi/2) \rightarrow$ non-monotonic functions

1.3.2 Decreasing Functions

A function $f(x)$ is said to be a decreasing function, if $f(x_1) > f(x_2) \forall x_1 < x_2$

A function $f(x)$ is said to be decreasing function, if $f(x)$ decreases as x increases.

Example: e^{-x} $\rightarrow$Monotonically decreasing function, $\sin x$ in $\left(\frac{\pi}{2}, \pi\right)$

1.4. Concept of Maxima and Minima

Let $f(x)$ be a differentiable function, then to find the maximum (or) minimum of $f(x)$.

(1) Find $f'(x)$ and equate to zero.

(2) Solve the resulting equation for x. Let its roots be $a_1, a_2, \ldots$ then $f(x)$ is stationary at $x = a_1, a_2, \ldots\ldots$ Thus $x = a_1, a_2, \ldots\ldots$ are the only points at which $f(x)$ can be maximum or a minimum.

(3) Find $f''(x)$ and substitute in it by terms $x = a_1, a_2, \ldots\ldots$ wherever $f''(x)$ is negative, we have a maximum and wherever $f''(x)$ is positive, we have a minimum.

(4) If $f''(a_1) = 0$, find $f'''(x)$ put $x = a_1$ in it. If $f'''(a_1) \neq 0$, there is neither a maximum nor a minimum at $x = a_1$. If $f'''(a_1) = 0$, find $f^{iv}(x)$ and put $x = a_1$ in it. If $f^{iv}(a_1)$ is negative, we have maximum at $x = a_1$, if it is positive there is a minimum at $x = a_1$. If $f^{iv}(a_1)$ is zero, we must find $f^{v}(x)$, and so on. Repeat the above process for each root of the equation $f'(x) = 0$.

Example: $x = 0$ is a critical point of $f(x) = x^3$

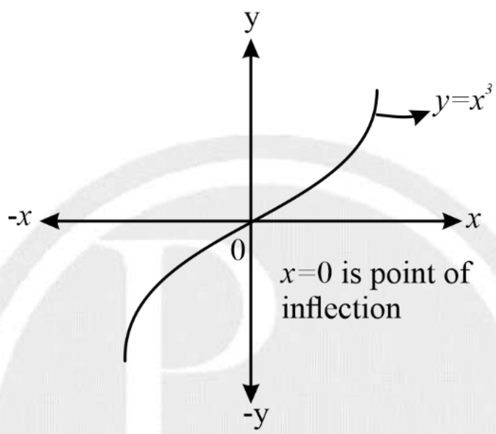

Fig. 1.5. Graph of x^3

$$f(x) = x^3$$

$$\Rightarrow \quad f'(x) = 3x^2 = 0 \quad \Rightarrow \quad x = 0$$

$$f''(x) = 6x \Rightarrow f''(0) = 6(0) = 0$$

- **Global maxima and minima :**
  We first find local maxima and minima and then calculate the value of '*f*' at boundary points of interval given e.g. (a, b) we find f(a) and f(b) and compare it with the values of local maxima and minima. The absolute maxima and minima can be decided then.

1.5. Taylor Series

If $f(x)$ is continuously differentiable ($f'(x), f''(x), f'''(x), \ldots\ldots$ exists) then the Taylor series expansion of $f(x)$ about the point $x = a$ is given by

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 . + \ldots \infty$$

If $a = 0$, then $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots\ldots\infty$ (Remember that Mc-Lauren Series is same as Taylor Series if a = 0)

The coefficient of $(x - a)^n$ in the Taylor series expansion of $f(x)$ is $\frac{f^n(a)}{n!}$.

The general expansion of Taylor series is given by $f(x+h) = f(x) + h.\frac{f'(x)}{1!} + h^2.\frac{f''(x)}{2!} + h^3.\frac{f'''(x)}{3!} + \ldots\ldots\infty$

- Finding the expansion of e^x about $x = 0$

$f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1; f''(0) = f'''(0) = f''''(0) = \ldots\ldots = 1$

$f(x) = e^x = 1 + (x-0)\frac{1}{1!} + (x-0)^2.\frac{1}{2!} + (x-0)^3.\frac{1}{3!} + \ldots\ldots$

$\Rightarrow \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

1.6 Integral Calculus

If $F(x)$ is anti-derivative of $f(x)$. That is, continuous and differentiable in (a, b), then we write $\int_{x=a}^{x=b} f(x)\, dx = F(b) - F(a)$. Here $f(x)$ is integrand

If $f(x) > 0\ \forall a \leq x \leq b, then\ \int_a^b f(x)\, dx$ represents the shaded area in the given figure.

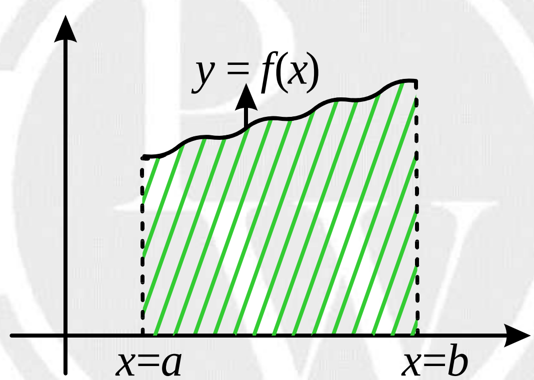

Fig.1. 6. Integration of continuous function

1.6.1 Mean Value Theorem of Integration

If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) then '∃' atleast one-point $c \in (a, b)$ such that

$f(c) = \frac{\int_a^b f(x)dx}{(b-a)}$

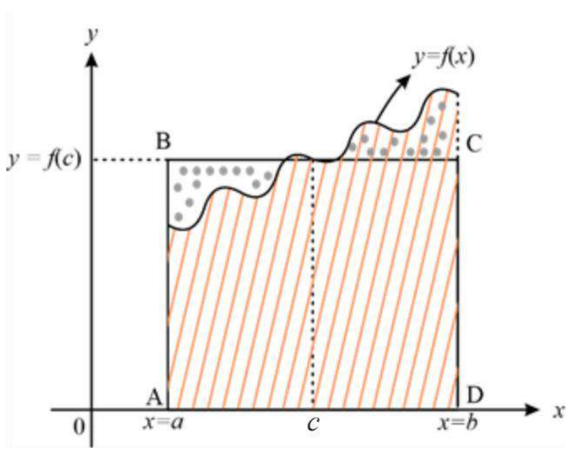

Fig. 1.7. Mean value of integration

1.7. Newton-Leibnitz Rule

If $f(x)$ is continuously differentiable and $\phi(x)$, $\Psi(x)$ are two functions for which the 1st derivative exists, then

$$\frac{d}{dx}\left(\int_{\phi(x)}^{\psi(x)} f(x)dx\right) = f\big(\psi(x)\big).\psi'(x) - f\big(\phi(x)\big).\phi'(x)$$

Example: $\frac{d}{dx}\left(\int_x^{x^2} \sin x\, dx\right) = \sin(x^2)\,.2x - \sin x\,.1 = 2x\sin(x^2) - \sin x$

1.8. Some Standard Integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ (n \neq -1)$

2. $\int \frac{1}{x} dx = \log_e|x| + C$

3. $\int \sin x\, dx = -\cos x + C$

4. $\int \cos x\, dx = \sin x + C$

5. $\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)| + C$

6. $\int \tan x\, dx = -\int -\frac{\sin x}{\cos x} dx = -\log_e|\cos x| + C$

$\Rightarrow \int \tan x\, dx = \log_e|\sec x| + C$

7. $\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \log_e|\sin x| + C = -\log_e|\mathrm{cosec}\, x| + C$

8. $\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx = \log_e|\sec x + \tan x| + C$

9. $\int \mathrm{cosec}\, x dx = \log_e|\mathrm{cosec}\, x - \cot x| + C$

10. $\int a^x dx = \frac{a^x}{\log_e a} + C$

11. $\int \frac{1}{x.\log_e a} dx = \log_a x + C$

12. $\int x^x(1 + \log_e x)\, dx = x^x + C$

13. $\int f(x)\,.f'(x)dx = \frac{1}{2}\big(f(x)\big)^2 + C$

14. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2.\sqrt{f(x)} + C$

15. If $f(x)$, $g(x)$ are two functions. that are differentiable, then

$\int f(x)\, g(x)dx = f(x).\int g(x)\, dx - \int[f'(x)\, g(x)]dx + C$

Before integrating the product, the functions $f(x)$ and $g(x)$ are to be arranged according to the ILATE Principle.

Here, ILATE stands for INVERSE LOGARITHMIC ALGEBRAIC TRIGONOMETRIC EXPONENTIAL.

1.9 Properties of Definite Integrals

1. If $f(x)$ is differentiable in interval (a, b), then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. If $\exists$ a point $c \in (a, b)$ such that $f(x)$ is not differentiable, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

3. If $f(x)$ is continuously differentiable function,

$\int_{-a}^a f(x)\,dx = 2 \times \int_0^a f(x)\,dx$; if $f(-x) = f(x)$, ("$f(x)$ is even function")

$= 0$; if $f(-x) = -f(x)$, ("$f(x)$ is odd function")

4. $\int_0^{2a} f(x)\,dx = 2 \times \int_0^a f(x)\,dx$, if $f(2a - x) = f(x)$

5. $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

6. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} = \left(\frac{b-a}{2}\right)$

Example:

(i) $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x} = \frac{\pi}{4}$

(ii) $\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}dx = \int_0^{\pi/2} \frac{1}{1+\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}}\right)}dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx = \frac{\pi}{4}$

(iii) $\int_2^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}} = \left(\frac{3-2}{2}\right) = \frac{1}{2}$

(iv) $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx = \frac{\pi}{4}$

7. $\int_0^{\pi/2} \sin^m x\,dx = \int_0^{\pi/2} \cos^m x\,\,dx = \frac{(m-1)\times(m-3)\times(m-5)}{m\times(m-2)\times(m-4)} \times \ldots \left(\frac{1}{2}\right)$ (or) $\frac{2}{3} \times K$

Where $K = \pi/2$ if m is even

$= 1$ if m is odd.

8. $\int_0^{\pi} \frac{dx}{a^2\cos^2 x+b^2\sin^2 x} = \frac{\pi}{ab}$

9. $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x+b^2\sin^2 x} = \frac{\pi}{2ab}$

1.10 Length of a Curve

(a) The length of the arc of the curve $y = f(x)$ between the points where x = a and x = b is $s = \int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$

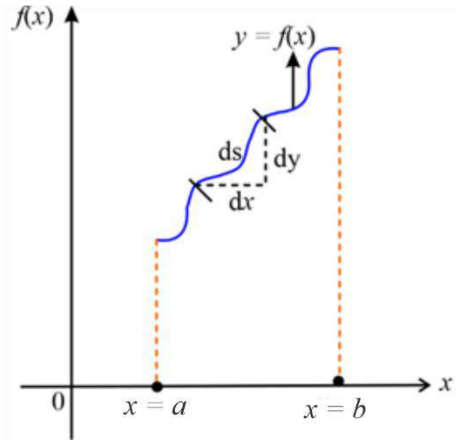

Fig.1.8. Length of the curve

(b) The length of the arc of the curve $x = f(y)$ between the points where y = a and y = b, is $s = \int_a^b \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]}dy$

(c) The length of the arc of the curve $x = f(t), y = f(t)$ between the points where t = a and t = b, is $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

(d) The length of the arc of the curve $r = f(\theta)$, between the points where $\theta = \alpha$ and $\theta = \beta$, is $s = \int_\alpha^\beta \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}d\theta$

1.11 Surface Area of Solid generated by revolving a curve about a fixed axis.

Elemental Surface Area

$dA = 2\pi y \times ds = 2\pi y ds$

$\Rightarrow$ Total surface area = A = $\int_{x=a}^{x=b} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

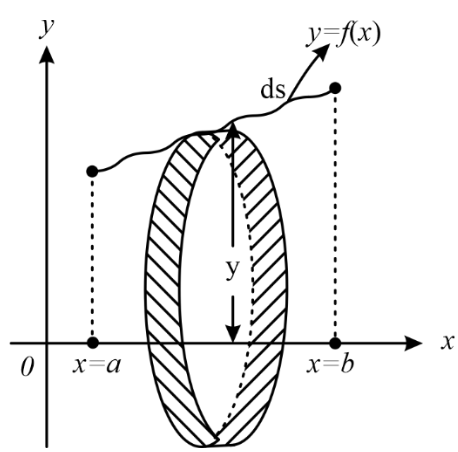

Fig.1.9. Surface area

1.12 Volume of the solid

A. The volume of the solid obtained by revolving the curve y = $f(x)$ between the lines $x = a$ and $x = b$ is given by

$\Rightarrow \quad V \approx \int_{x=a}^{x=b} \pi y^2 dx$

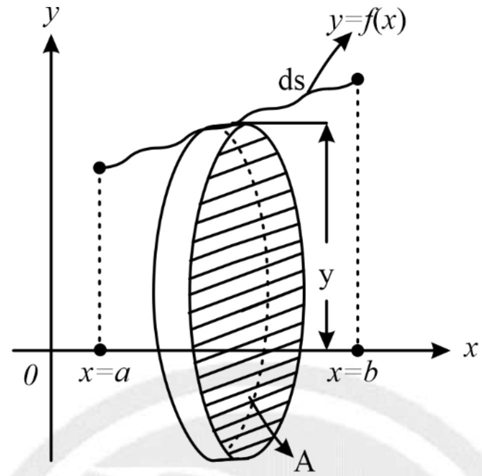

Fig. 1.10. Volume of the solid

B. **Revolution about the y-axis.** Interchanging x and y in the above formula, we see that the volume of the solid generated by the revolution, about y-axis, of the area, bounded by the curve $x = f(y)$, the y-axis and the abscissa y = a, y = b is $\int_a^b \pi x^2 dy$.

1.13 Gamma Function

The integral $\int_0^\infty e^{-x}.x^{n-1}\,dx, (n > 0)$ is called Gamma function of n. It is denoted by $\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$.

Note : $$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left[\frac{m+n+2}{2}\right]}$$

Where Γ(x) is called the gamma function.

1.13.1 Properties of Gamma Function

(i) $\Gamma n = (n-1)!$

(ii) $\Gamma(n+1) = (n)!$

(iii) $\Gamma(n+1) = n\Gamma n$

(iv) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

1.14 Beta Function

The function β $(m, n) = \int_0^1 x^{m-1}.(1-x)^{n-1}\,dx$ $(m, n > 0)$ is called β function of m and n.

1.14.1 Properties of β function

(i) $\beta(m,n) = \frac{\Gamma m.\Gamma n}{\Gamma(m+n)}$

(ii) $\beta(m,n) = \beta(n,m)$

(iii) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$\beta(n,m) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$

(iv) $sin^p\theta.cos^q\,\theta\,dx = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right), (p,q > -1)$

1.15 Area between the curves

If the function f(x) > g(x) for all values of x between x=a and x=b then

$A = \int_a^b f(x)dx - \int_a^b g(x)dx \quad \Rightarrow \quad A = \int_a^b \big(f(x) - g(x)\big)\,dx$

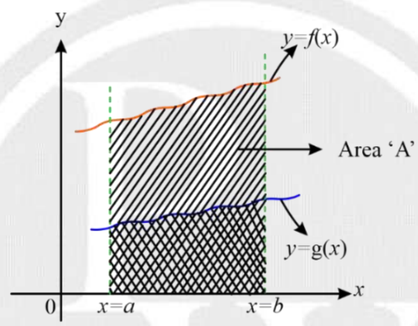

Fig. 1.11. Area under curve

Note : Area bounded by curve $r = f(\theta)$ between $\theta = \alpha$ and β is $\frac{1}{2}\int_\alpha^\beta r^2 d\theta$

1.16. Multi Variable Calculus

(a) Continuity of a function

A function *f*(x, y) is said to be continuous at (*a*, *b*), if $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b)$

(b) Differentiation of a two-variable function

If *f*(*x*, *y*) is a continuous function, then the derivative of *f*(*x*, *y*) with respect to *x* treating *y* as constant is given by

$$p = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

The derivative of *f*(*x*, *y*) with respect to *y* treating *x* as constant is given by

$$q = \frac{\partial f}{\partial y} = \lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k}$$

(c) Homogenous Function

A function f(x, y) is said to be homogenous function of degree 'n' if $f(kx,ky) = k^n.f(x,y)$.

Example: $f(x,y) = x^3 - 3x^2y + 3xy^2 + y^3$

$\Rightarrow$ $f(kx,ky) = (kx)^3 - 3(kx)^2(ky) + 3(kx).(ky)^2 + (ky)^3$

= $k^3(x^3 - 3x^2y + 3xy^2 + y^3)$

= $k^3.f(x,y) \Rightarrow f(x,y)$ is a homogenous function of degree '3'.

(d) Euler's Theorem

If $f(x, y)$ is a homogeneous function of degree 'n' then

(i) $x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = nf$

(ii) $x^2.\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$

If $f(x, y) = g(x, y) + h(x, y) + \phi(x, y)$ where $g(x, y)$, $h(x, y)$ and $\phi(x, y)$ are homogenous functions of degrees m, n and p respectively, then

$x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = m.g(x,y) + n.h(x,y) + p.\phi(x,y)$

$x^2.\frac{\partial^2 f}{\partial x^2} + 2xy.\frac{\partial^2 f}{\partial x \partial y} + y^2.\frac{\partial^2 f}{\partial y^2} = m(m-1).g(x,y) + n(n-1).h(x,y) + p(p-1).\phi(x,y)$

(e) Total derivative:

(i) If u = f(x, y) and if x = ω(t), y = v(t) then $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$

(ii) If u be a function of x and y, where y is a function of x, then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}$

(iii) If u = f(x, y) and $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$, then

$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_1}$ and $\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_2}$

(iv) If x and y are connected by an equation of the form f(x, y) = 0, then $\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}$

(f) Concept of Maxima and Minima in Two Variables

If $f(x, y)$ is a two-variable differentiable function, then to find the maxima (or) minima.

Step-1: Calculate $p = \frac{\partial f}{\partial x}$ and $q = \frac{\partial f}{\partial y}$ and equate $p = 0$, $q = 0$

Let (x_0, y_0) be a stationary point.

Step-2: Calculate r, s, t where $r = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)}$; $s = \left.\frac{\partial^2 f}{\partial x.\partial y}\right|_{(x_0,y_0)}$; $t = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)}$

Case (i): If $rt - s^2 > 0$ and $r > 0$, then the function $f(x, y)$ has minimum at (x_0, y_0) and the minimum value is $f(x_0, y_0)$.

Case (ii): If $rt - s^2 > 0$ and $r < 0$, then the function $f(x, y)$ has maximum at (x_0, y_0) and the maximum value is

$f(x_0, y_0)$.

Case (iii): If $rt - s^2 < 0$; then we cannot comment on the existence of maxima and minima.

Such stationary points where $rt - s^2 = 0$ are called **saddle points**.

(g) Concept of Constraint Maxima and Minima (Method of Lagrange's unidentified multipliers).

If $f(x, y, z)$ is a continuous and differentiable function, such that the variables x, y and z are related/constrained by the equation $\phi(x, y, z) = C$ then to calculate the extreme value of $f(x, y, z)$ using Lagrange's Method of unidentified multipliers.

Step-1: Form the function $F(x, y, z) = f(x, y, z) + \lambda\{\phi(x, y, z) - C\}$, where λ is a multiplier.

Step-2: Calculate $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ and equate them to zero

Step-3: Equate the values of λ from the above 3 equations and obtain the relation between the variables x, y and z.

Step-4: Substitute the relation between x, y and z in $\phi(x, y, z) = C$ and get the values of x, y, z. Let they be (x_0, y_0, z_0).

Step-5: Calculate $f(x_0, y_0, z_0)$

The value $f(x_0, y_0, z_0)$ is the extreme value of $f(x, y, z)$.

(h) Multiple Integrals

If $f(x, y)$ is continuous and differentiable at every point within a region 'R' bounded by some curves is given by
$I = \iint_R f(x, y)\, dxdy$

If there is a double integral,
$I = \int_{x=a}^{x=b} \int_{y=\phi(x)}^{y=\psi(x)} f(x, y) dydx$ [Let $\psi(x) > \phi(x)$]

Then I = area of the region bounded by the curves, $y = \phi(x)$; $y = \Psi(x)$, $x = a$ and $x = b$ if $f(x, y) = 1$

Value of x – co-ordinate of centroid of the region bounded by $y = \phi(x)$; $y = \psi(x)$; $x = a$, $x = b$ if $f(x, y) = x$

(i) Change of Orders of Integration

$$I = \int_{x=a}^{x=b} \int_{y=\phi(x)}^{y=\psi(x)} f(x, y) dydx \rightarrow I = \int_{y=c}^{y=d} \int_{x=g(y)}^{x=h(y)} f(x, y) dxdy$$

In change of order of Integration, the order of the Integrating variables changes and the limits as well.

Note : When limits are constants, the order of integration does not matter,

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)\, dxdy = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y)\, dydx$$

1.17 Jacobian of the Transformation

(i) The Jacobian of the transformation, $x = f_1(u,v), y = f_2(u,v)$ is defined as,

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

(ii) The Jacobian of the transformation,

$x = f_1(u,v,w), y = f_2(u,v,w), z = f_3(u,v,w)$ is defined as

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

1.18 Change of Variables Formula

(i) $\iint\limits_R f(x,y)dx\,dy = \iint\limits_R f(f_1(u,v), f_2(u,v))\,|J|\,du\,dv$

(ii) $\iiint\limits_R f(x,y,z)dxdydz = \iiint\limits_R f(f_1(u,v,w), f_2(u,v,w), f_3(u,v,w))\,|J|\,du\,dv\,dw$

1.19 Change of Variables

(i) Cartesian to polar co-ordinates :

$x = r\cos\theta$

$y = r\sin\theta$

J = r

$dx\,dy = rdrd\theta$

(ii) Cartesian to cylindrical polar co-ordinate :

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$J = r$

$dxdydz = rdr\,d\theta\,dz$

(iii) Cartesian to spherical polar co-ordinates :

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$J = \rho^2\sin\phi$

$dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

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2 Ordinary Differential Equation

2.1 Differential Equation

The equation involving differential coefficients is called a Differential Equation (DE).

1. $x^2.\frac{dy}{dx} + y^2 = 0$

2. $\frac{\partial^2 T}{\partial x^2} = k \cdot \frac{\partial T}{\partial x}$

3. $x^2.\frac{\partial^2 u}{\partial x^2} + y^2.\frac{\partial^2 u}{\partial y^2} = 0$

2.1.1 Ordinary Differential Equations (ODE)

The DEs involving only one independent variable is called ordinary differential equation.

Example:

(1) $x^2 \frac{dy}{dx} + y^2 = 0;$

(2) $e^{-x} \cdot \frac{dy}{dx} + y^2 = e^x$

2.1.2 Partial Differential Equations

The DEs involving two (or) more independent variables are called Partial Differential Equations (PDEs).

Example:

$$\frac{\partial^2 u}{\partial x^2} = C^2 \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{K} \cdot \frac{\partial u}{\partial t}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

2.1.3 Order of a Differential Equation

The order of the highest derivative that occurs in a DE is called order of a DE.

Example:

(1) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - y = 0$ → Order = 2

(2) $\frac{dy}{dx} + 2 \cdot \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} - 3x^2 = e^x$ → Order = 3

(3) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2} \cdot \frac{\partial^2 u}{\partial t^2}$ → Order = 2

(4) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t}$ → Order = 2

2.1.4 Degree of a Differential Equation

The Degree of the highest order derivative that occurs in a DE, when the DE is free from fractional (or) radical powers.

Example:

(1) The Degree of the DE $\left(\frac{d^2y}{dx^2}\right)^1 + 2\left(\frac{dy}{dx}\right)^3 - 3y = 0$ is 1.

(2) The Degree of the DE $\left(\frac{d^2y}{dx^2}\right)^1 + \sqrt{\left(\frac{dy}{dx}\right)^3 + 4y} = 0$ is 2

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = \left(-\sqrt{\left(\frac{dy}{dx}\right)^3 + 4y}\right)^2$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = \left(\frac{dy}{dx}\right)^3 + 4y$$

(3) It is not possible every time that we can find the degree of a given differential equation. The degree of any differential equation can be found, when it is in the form of a polynomial; otherwise, the degree cannot be defined.

Example degree of the DE is not defined

$$\frac{d^2y}{dx^2} + \cos\frac{d^2y}{dx^2} = 5x$$

2.2. Formation of Differential Equations

If a solution $y = f(x)$ contains n arbitrary constants in it, then differentiate y for n times and calculate $y', y'', y''', \ldots . y^{(n)}$ So, from the $(n + 1)$ equations available, try to eliminate the arbitrary constants in $y = f(x)$

- The different equation formed for the solution, $y = C_1 e^{K_1 x} + C_2 e^{K_2 x}$ where C_1, C_2 are arbitrary constants is

  $\frac{d^2y}{dx^2} - (K_1 + K_2)\frac{dy}{dx} + (K_1 . K_2)y$

- If the solution is $y = C_1 e^{K_1 x} + C_2 e^{K_2 x} + C_3 e^{K_3 x}$ where C_1, C_2, C_3 are arbitrary constants, then the DE is $y''' - (K_1 + K_2 + K_3)y'' + (K_1K_2 + K_2K_3 + K_3K_1)y' - (K_1K_2K_3)y = 0$

2.2.1 First Order DE

The general form of a 1st order DE is given by $\frac{dy}{dx} = f(x, y)$

If $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = f(x, y)$

- $N(x, y)dy + M(x, y)dx = 0$
- $Mdx + Ndy = 0$ where M, N are functions of x and y.

2.2.2 Linear ODE:

An ODE is said to be linear, if it do not contains the higher power terms of dependent variable $\left(y^2, y^3, y^4, \ldots.., \left(\frac{dy}{dx}\right)^2, \left(\frac{dy}{dx}\right)^3, \ldots.\right)$ and also the terms containing the product of dependent variable and its differential coefficient $\left(y.\frac{dy}{dx}, y^2\frac{dy}{dx}, y\left(\frac{dy}{dx}\right)^2, \ldots..\right)$

Example:

(1) $x^2.\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 6y = 0$

(2) $\frac{dy}{dx} - 5y = \sin x$

- $\frac{d^2y}{dx^2} - 5.\frac{dy}{dx} + \sin y = 0 \rightarrow$ Non-linear DE
  - Here, $\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} \ldots.$
- $\frac{d^2y}{dx^2} - 5.\frac{dy}{dx} + \sin x = 4y \rightarrow$ Linear DE

2.3 Solving of Differential Equations

2.3.1 Solving of 1st Order DE

(i) Variable-separable form

If the 1st order DE is given by $\frac{dy}{dx} = \phi(x).\psi(y)$

$\Rightarrow \int \frac{1}{\Psi(y)}dy = \int \phi(x)dx$

On integrating we have solution of the given DE

(ii) Homogenous 1st Order

If the 1st order DE is of the form $\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$

Such that both $M(x, y)$ and $N(x, y)$ are homogenous functions of same degree, then we say that the DE is homogeneous.

Example:

(1) $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

(2) $\frac{dy}{dx} = \frac{ax+by}{a'x+b'y}$

(a and b are not zero at the same time)

If the DE $\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$ is a homogeneous DE, then the equation can be converted to Variable Separable form if we substitute $y = Vx$; $\frac{dy}{dx} = V + x\frac{dV}{dx}$

2.3.2 Exact Differential Equations

The DE $Mdx + Ndy = 0$ where M, N are functions of x and y is said to be an Exact Differential Equation, if there exist a function f(x,y) such that $Mdx + Ndy = d(f(x,y))$

Mathematical condition to check the Exactness of a differential equation is

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

(i) Solution of an Exact DE

If $M(x, y)$ dx + $N(x, y)$ dy = 0 is an Exact differential Equation, then the solution of the DE is given by $\int_{y=\text{const}} M(x,y)dx + \int(\text{terms not containing } x \text{ in } N)\, dy = C$

(ii) Integrating Factor

The function which, when multiplied to a non-exact DE converts the DE to exact DE.

Example:

(1) $\frac{1}{y^2}$ is an integrating factor of $ydx - xdy = 0$

(2) $\frac{1}{y}$ is an integrating factor of $x^2dy - xydx = 0$

2.3.3 Methods of Writing the Integrating Factors (I.F.)

(i) If $M(x, y)dx + N(x, y)dy = 0$ is a homogeneous DE, then I.F. $= \frac{1}{Mx+Ny}$, $(Mx + Ny \neq 0)$

(ii) If $Mdx + Ndy = 0$ is of the form $yf(xy)dx + xg(xy)dy = 0$ then I.F. $= \frac{1}{Mx-Ny}$, $(Mx - Ny \neq 0)$

(iii) For a DE, $Mdx + Ndy = 0$, If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$, then $e^{\int f(x)dx}$ is the integrating factor.

(iv) For the DE, $Mdx + Ndy = 0$, if $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = g(y)$ then $e^{\int g(y)dy}$ is the integrating factor.

2.3.4 Leibnitz Linear Equation

The DE of the form $\frac{dy}{dx} + Py = Q$, where P, Q are functions of x alone, is called Leibnitz Linear Equation

Integrating factor of the equation is $e^{\int Pdx}$

Solution of the Differential Equation is : $y.e^{\int Pdx} = \int Q.(e^{\int Pdx})\, dx$+ C, where C is arbitrary constant.

2.3.5. Non-linear Equations Convertible to Leibnitz Linear Form

Bernoullis Equation

$$\frac{dy}{dx} + Py = Q.y^n, (n > 1, n \neq 1)$$

(P, Q are functions of x alone)

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^n}Py = \frac{Qy^n}{y^n}$$

$$\Rightarrow \quad y^{-n} \cdot \frac{dy}{dx} + y^{1-n} P = Q$$

Let $y^{1-n} = z$

$$\Rightarrow \quad \frac{dz}{dx} + (1-n)Pz = (1-n)Q \quad \text{[Leibnitz Linear Equation]}$$

2.3.6 Applications of 1st order DE

Newton's Law of Cooling

The rate of change of temperature of a body placed in an ambience of temperature T_∞ is directional proportional to the temperature difference between the body and the ambience.

$\frac{dT}{dt} \propto -(T - T_\infty)$ where $T_\infty \to$ Ambient Temperature ($T > T_\infty$)

$$\frac{dT}{dt} \propto (T_\infty - T)$$

$$\frac{dT}{dt} = -K(T - T_\infty)$$

Radioactive Growth / Decay

The rate of growth/decay on any radioactive substance at any instant is directly proportional to concentration of the substance that is available at that instant.

- $\frac{dN}{dt} \propto N \to$ For growth

  $\Rightarrow \frac{dN}{dt} = KN$

  $\Rightarrow \int \frac{1}{N} dN = \int K dt$

- $\frac{dN}{dt} \propto -N \to$ For decay

  $\Rightarrow \frac{dN}{dt} = -KN$

  $\Rightarrow \log_e N = Kt + C$

  $\Rightarrow N = e^{Kt} C$

2.4 Higher Order Differential Equations

The general form of Higher order Differential Equations is given by

$$K_1 \frac{d^n y}{dx^n} + K_2 \frac{d^{n-1} y}{dx^{n-1}} + K_3 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + K_n y = X \qquad \ldots(1)$$

- If $K_1, K_2, K_3, K_4, \ldots K_n, X$ are functions of x alone then (1) is called Linear Higher Order Linear DE with variable coefficients.
- If $K_1, K_2, K_3, K_4, \ldots K_n$ are constants and X is a function of 'x' alone, then (1) is called Higher Order Linear DE with constant coefficients.

2.5 Higher Order Linear Differential Equations with Constant Coefficients

The DE $K_1 \frac{d^n y}{dx^n} + K_2 \frac{d^{n-1} y}{dx^{n-1}} + K_3 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + K_n y = X$ …. (1) is said to be a higher order linear DE with constant coefficients if $K_1, K_2, K_3, K_4, \ldots K_n$ are constants and 'X' is a function of x alone.

If $X = 0$, then (1) is called Homogeneous DE

If $X \neq 0$, then (1) is called Non-Homogeneous DE.

2.5.1 Solution of Higher Order Linear Differential Equation

$Y = y_c + y_p$

$y_c \rightarrow$ Complimentary function; $y_p \rightarrow$ Particular Integral

(Solution of homogeneous part; ($X = 0$); (Solution of Non-Homogeneous Part; ($X \neq 0$))

If $K_1 \frac{d^n y}{dx^n} + K_2 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + K_{n-1}\frac{dy}{dx} + K_n y = X$ …. (1) is a linear DE with constant coefficients.

2.5.2 Rules for Writing the Complete Solution of $\left(f(D)\right)y = X$:

(i) Form the auxiliary equation of $\left(f(D)\right)y = X$ i.e. $f(M) = 0$

(ii) Depending on the roots of the auxiliary equation $(f(M) = 0)$, we write the complimentary function.

(iii) Calculate the Particular Integral, $y_P = \frac{1}{(f(D))}X$.

(iv) Write the total solution of the equation $y = y_C + y_P$.

2.5.3 Rules for Writing the Complementary Function

(i) If the roots of $f(M) = 0$ are $M_1, M_2, M_3, \ldots$ ($M_1, M_2, M_3, \ldots \in$ Rational)

Then $y_C = C_1 e^{M_1 x} + C_2 e^{M_2 x} + C_3 e^{M_3 x} + \ldots$ where $C_1, C_2, C_3, \ldots$ Are arbitrary constants)

(ii) If the roots of $f(M) = 0$ are $M_1, M_1, M_3 \ldots \ldots$ ($M_1, M_3 \ldots \ldots \in$ Rational)

Then $y_C = (C_1 x + C_2)e^{M_1 x} + C_3 . e^{M_3 x} + \ldots$ Where $C_1, C_2, C_3, \ldots$ are arbitrary constants).

(iii) If the roots of $f(M) = 0$ are $M_1, M_1, M_1, M_4 \ldots$ (Where $\in M_1, M_4 \ldots \ldots \in$ Rational)

Then $y_C = (C_1 x^2 + C_2 x + C_3)e^{M_1 x} + C_4 e^{M_4 x} + \ldots$ (where $C_1, C_2, C_3, \ldots$ Are arbitrary constants)

(iv) If the roots of $f(M) = 0$ are $\alpha + i\beta, \alpha - i\beta, M_3, M_4 \ldots$ then, $y_C = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + C_3 e^{M_3 x} + C_4 e^{M_4 x} + \ldots$

(v) If the roots of $f(M) = 0$ are $\alpha + i\beta, \alpha - i\beta, \alpha + i\beta, \alpha - i\beta, M_5, M_6, \ldots$ then

$y_c = e^{\alpha x}((C_1 x + C_2)\cos\beta x + (C_3 x + C_4)\sin\beta x) + C_5 e^{M_5 x} + C_6 e^{M_6 x} + \ldots$

(vi) If the roots of $f(M) = 0$ are $\alpha + \sqrt{\beta}$, $\alpha - \sqrt{\beta}$, M_3, M_4, …. then $y_C = e^{\alpha x}\{C_1 \sinh\sqrt{\beta}\, x + C_2 \cosh\sqrt{\beta}\, x\} + C_3 e^{M_3 x} + C_4 e^{M_4 x} + \ldots$

(vii) If the roots of $f(M) = 0$ are $\alpha + \sqrt{\beta}, \alpha - \sqrt{\beta}, \alpha + \sqrt{\beta}, \alpha - \sqrt{\beta}, M_5, M_6, \ldots$ then $y_C = e^{\alpha x}\{(C_1 x + C_2)\sinh\sqrt{\beta}\, x + (C_3 x + C_4)\cosh\sqrt{\beta}\, x\} + C_5 e^{M_5 x} + C_6 . e^{M_6 x} + \ldots$

2.5.4 Rules for writing the Particular Integral

(i) If $X = e^{ax}$,

$y_P = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{aX}$, (if $f(a) \neq 0$)

If $f(a) = 0$, then $y_P = x \frac{1}{f'(a)} e^{ax}$ (if $f'(a) \neq 0$)

If $f'(a) = 0$, then $y_p = x^2 . \frac{1}{f''(a)} e^{ax}$ (if $f''(a) \neq 0$) and so on.

Solve $\frac{d^2y}{dx^2} - 5.\frac{dy}{dx} + 6y = e^{2x}$

Sol. Aux. Eqn $\rightarrow M^2 - 5M + 6 = 0 \Rightarrow M = 2,3$

$y_C = C_1 e^{2x} + C_2 e^{3x}$

$y_P = \frac{1}{D^2 - 5D + 6} e^{2x}$ $since\ f(2) = 0$

$\Rightarrow \quad y_P = x \frac{1}{(2D-5)} e^{2x} = x . \frac{1}{(2(2)-5)} e^{2x}$

$\frac{x}{-1} e^{2x} = -x . e^{2x}$

(ii) If $X = sin(ax + b)\ (or)\ \cos(ax + b)$

$y_P = \frac{1}{f(D)} sin(ax + b)$

Replace D^2 by $-a^2$ in $f(D)$

If the denominator is the form $cD + d$ then rationalize the denominator and replace D^2 by $-a^2$

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = sin(2x + 3)$

$y_p = \frac{1}{D^2 - 5D + 6} \cdot sin(2x + 3)$

$a = 2 \Rightarrow -a^2 = -4$

$\Rightarrow \quad y_p = \frac{1}{-4 - 5D + 6} sin(2x + 3)$

$\Rightarrow \quad y_p = \frac{1}{2 - 5D} \times \frac{2 + 5D}{2 + 5D} . sin(2x + 3)$

$\Rightarrow \quad y_p = \frac{2 + 5D}{4 - 25D^2} sin(2x + 3) = \frac{2 + 5D}{4 - 25(-4)} \sin(2x + 3) = \frac{1}{104} (2. \sin(2x + 3) + 10. \cos(2x + 3))$

(iii) If $X = x^m$

$y_P = \frac{1}{f(D)} x^m$

$\Rightarrow \quad y_P = [f(D)]^{-1} x^m$

Calculate y_P for the DE $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$

$y_P = \frac{1}{D^2 - 5D + 6} x^2$

$= \frac{1}{6\left(1 + \left(\frac{D^2 - 5D}{6}\right)\right)} x^2$

$= \frac{1}{6}\left(1 + \left(\frac{D^2 - 5D}{6}\right)\right)^{-1} x^2$

$= \frac{1}{6} \cdot \left\{1 - \left(\frac{D^2 - 5D}{6}\right) + \left(\frac{D^2 - 5D}{6}\right)^2 - \left(\frac{D^2 - 5D}{6}\right)^3 + \ldots\ldots\right\} x^2$

$= \frac{1}{6}\left\{x^2 - \frac{1}{6}\left(2 - 5(2x)\right) + \frac{1}{36}\{25(2)\}\right\}$

$= \frac{1}{6}x^2 - \frac{1}{18} + \frac{5x}{18} + \frac{25}{108}$

$= \frac{1}{6}x^2 + \frac{5x}{18} + \frac{19}{108}$

(iv) If $X = e^{ax}V$, then

$y_P = \frac{1}{f(D)} \cdot e^{ax} \cdot V = e^{ax}\frac{1}{f(D+a)}V$

2.6 Euler Cauchy Equation (Higher order linear DE with Variable Coefficients)

The DE of the form $x^n \frac{d^n y}{dx^n} + K_1 \cdot x^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}} + K_2 \cdot x^{n-2} \cdot \frac{d^{n-2}y}{dx^{n-2}} + \ldots + K_{n-1} \cdot x \cdot \frac{dy}{dx} + K_n y = X$ Where $K_1, K_2, K_3, \ldots K_n$ are constants is called Euler-Cauchy Equation

2.6.1 Procedure to solve Euler Cauchy Equations

Let $x^n \frac{d^n y}{dx^n} + K_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + K_2\, x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots + K_{n-1}\, x \frac{dy}{dx} + K_n y = X$ ...(1)

$\left(x^n \frac{d^n}{dx^n} + K_1 x^{n-1} \frac{d^{n-1}}{dx^{n-1}} + K_2\, x^{n-2} \frac{d^{n-2}}{dx^{n-2}} + \ldots + K_{n-1}\, x \frac{d}{dx} + K_n\right) y = X$

Let $x = e^z \Rightarrow z = \log_e x$

$x\frac{d}{dx} = \frac{d}{dz} = D$

$x^2 \cdot \frac{d^2}{dx^2} = \frac{d}{dz}\left(\frac{d}{dz} - 1\right) = D(D-1)$

$x^3 \cdot \frac{d^3}{dx^3} = D(D-1)(D-2)$ and so, on

(1) $\Rightarrow \{D(D-1)(D-2) \ldots\ldots (D-(n-1)) + K_1 D(D-1)(D-2) \ldots (D-(n-2)) + \ldots K_{n-1}D + K_n\}y = X$

Where $D = \frac{d}{dz}$

$\Rightarrow \quad (f(D))y = z \rightarrow$ Higher order linear DE with constant

$f(D) \rightarrow$ Polynomial in terms of D with constant coefficients.

(2) If the differential equation is of form

$$(ax+b)^n\left(\frac{d^n y}{dx^n}\right) + k_1(ax+b)^{n-1}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) + \ldots + k_{n-1}(ax+b)\left(\frac{dy}{dx}\right) + k_n y = f(x)$$

i.e., $\left[(ax+b)^n D^n + k_1(ax+b)^{n-1}D^{n-1} + \ldots + k_{n-1}(ax+b)D + k_n\right]y = f(x)$

where, f(x) is a function of 'x'.

It can be reduced to linear differential equations with constant coefficients, by putting $(ax+b)=e^z$

Or z = log (ax + b)

Then, $(ax + b)\,Dy = aD_1y$

$$(ax+b)^2 D^2 y = a^2 D_1 (D_1 - 1) y$$

$$(ax+b)^3 D^3 y = a^3 D_1 (D_1 - 1)(D_1 - 2) y$$

Where, $D = \frac{d}{dx}$ and $D_1 = \frac{d}{dz}$

❑❑❑

3 VECTOR CALCULUS

3.1 Basics of Vectors

1. **Equality of Vectors:** If two vectors have equal length and same direction then they are equal.

2. **Vector between two points P, Q :** Initial point P : (x_1, y_1, z_1) and terminal point $Q:(x_2, y_2, z_2)$ the three numbers,

   (a) $a_1 = x_2 - x_1, a_2 = y_2 - y_1, a_3 = z_2 - z_1$: are called the components of the vector a with respect to that coordinate system, and we write simply $a = [a_1, a_2, a_3]$.

   (b) $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

   (c) **Position vector:** The vector with origin as start point and A (x, y, z) as end point.

3. **Vector addition and multiplication:**

   (a) Addition of vector:

   $\vec{a} + \vec{b}$ of two vector $a = [a_1, a_2, a_3]$ and $b = [b_1, b_2, b_3]$

   $\Rightarrow \quad \vec{a} + \vec{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$

   (b) Vector addition is commutative i.e., $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

   (c) Vector addition is associative i.e., $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

4. **Unit vector :** Unit vector along $\vec{A} = \frac{\vec{A}}{|A|}$, this vector is in direction of $\vec{A}$ but has unit magnitude

Note : **Length and Direction of Vectors**

Any vector $\vec{a}$ may be written as a product of its length and direction as follows :

$$\vec{a} = |\vec{a}| \left(\frac{\vec{a}}{|\vec{a}|} \right)$$

Here $|\dot{a}|$ is the length of vector and $\frac{\vec{a}}{|\vec{a}|}$ is a unit vector in direction of ' $\vec{a}$ '.

3.2 Vector Product / Cross Product

If $\vec{a}$ and $\vec{b}$ are two vectors, then the cross product of the two vectors is denoted by $\vec{a} \times \vec{b}$ and it is given by $\vec{a} \times \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot sin\,\theta \cdot \hat{n}$

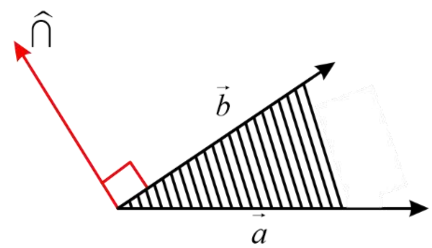


Fig. 3.1. Cross product

$\hat{n} \rightarrow$ unit vector passes through the intersection point of intersection of a and b and lies perpendicular to the plane containing $\vec{a}$ and $\vec{b}$.

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, then, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

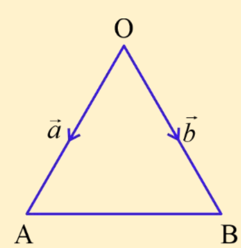
Properties :

(a) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ not commutative rather anti commutative

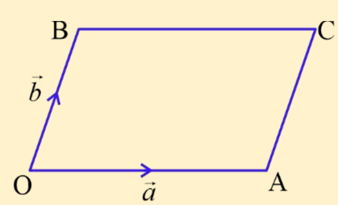
(b) $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

Note :

- Area of triangle $OAB = \frac{1}{2}\left|\vec{OA} \times \vec{OB}\right| = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$



- Area of rectangle or parallelogram $= \left|\vec{a} \times \vec{b}\right|$



- If $\vec{a}, \vec{b}, \vec{c}$ are the vectors then, volume of parallelepiped $= \left|\left[\vec{a}, \vec{b}, \vec{c}\right]\right| = \vec{a} \cdot (\vec{b} \times \vec{c}) = [\vec{a}\vec{b}\vec{c}]$

3.3 Dot / Scalar Product:

(i) If $\vec{a}$ and $\vec{b}$ are two vectors, then the dot /scalar product of the two vectors is denoted by $\vec{a}.\vec{b}$ and it is given by $\vec{a}.\vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot cos\,\theta$ where $\theta$is the angle between the vectors $\vec{a}$ *and* $\vec{b}$.

Note:

$$|(\vec{a}\cdot\vec{b})|^2 + |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2\cdot|\vec{b}|^2 . cos^2\theta + |\vec{a}|^2\cdot|\vec{b}|^2\cdot sin^2\theta = |\vec{a}|^2\cdot|\vec{b}|^2$$

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

$\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$ then $\vec{a}\cdot(\vec{b}\times\vec{c}) = [\vec{a}\vec{b}\vec{c}] \Rightarrow [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(ii) $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

(iii) If $\vec{a}.\vec{b} = 0$, then the two vectors are orthogonal

(iv) Properties of dot product:

(a) $(\vec{a}+\vec{b}).\vec{c} = \vec{a}.\vec{c} + \vec{b}.\vec{c}$ (Distributive)

(b) $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ (Symmetry)

(c) $\vec{a}.\vec{a} \geq 0$ (Positive-definiteness)

(d) $\vec{a}.\vec{a} = 0$ if and only if a = 0

(e) $|\vec{a}.\vec{b}| \leq |\vec{a}||\vec{b}|$ (Schwarz inequality)

(f) $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$ (Triangle inequality)

(g) $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2\left(|\vec{a}|^2 + |\vec{b}|^2\right)$ (Parallelogram equality)

(v) Projection of vector $\vec{A}$ on $\vec{B}$ is $\left\{\dfrac{\vec{A}.\vec{B}}{|B|}.\vec{B}\right\}$

3.4 Scalar/ Vector triple product

Scalar Triple Product

- $\vec{a}.(\vec{b}\times\vec{c}) = [\vec{a},\vec{b},\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
- $\vec{a}.(\vec{b}\times\vec{c}) = \vec{b}.(\vec{c}\times\vec{a}) = \vec{c}(\vec{a}\times\vec{b})$
- $[\vec{a},\vec{b},\vec{c}] = -[\vec{b},\vec{a},\vec{c}] = -[\vec{c},\vec{b},\vec{a}] = -[\vec{a},\vec{c},\vec{b}]$
- $\vec{a}.(\vec{b}\times\vec{c})$ = volume of parallelepiped created by $\vec{a},\vec{b},\vec{c}$.
- If $\vec{a}.(\vec{b}\times\vec{c}) = 0$ then the three vectors are coplanar
- If $\vec{a}.(\vec{b}\times\vec{c}) = 0$ then the three vectors are linearly dependent.

Vector triple product:

$\vec{a}\times\vec{b}\times\vec{c} = \vec{b}.(\vec{a}.\vec{c}) - \vec{c}(\vec{a}.\vec{b})$

3.5 Differentiation of Vector Point functions

If $\vec{R}(t)$ is a vector point function, then the derivative of $\vec{R}(t)$ is given by

$$\frac{d\vec{R}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{R}(t+\Delta t) - \vec{R}(t)}{\Delta t}$$

If $\vec{R}(t) = f(t)\,\hat{\imath} + g(t)\,\hat{\jmath}$ then $\frac{d\vec{R}(t)}{dt} = f'(t)\hat{\imath} + g'(t)\hat{\jmath}$

Example:

If $\vec{R}(t) = \sin t\,\hat{\imath} + \cos t\,\hat{\jmath} \Rightarrow \frac{d\vec{R}(t)}{dt} = \cos t\,\hat{\imath} - \sin t\,\hat{\jmath}$

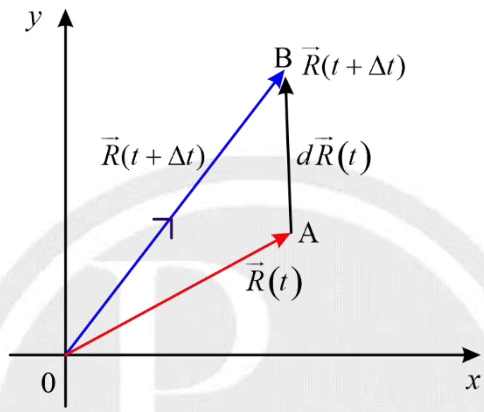


Fig. 3.2

3.5.1. Differentiation of Product of two vectors

$$\frac{d}{dt}\left(\vec{a}(t)\cdot\vec{b}(t)\right) = \vec{a}(t)\cdot\frac{d\vec{b}(t)}{dt} + \frac{d\vec{a}(t)}{dt}.\vec{b}(t)$$

$$\frac{d}{dt}\left(\vec{a}(t)\times\vec{b}(t)\right) = \vec{a}(t)\times\frac{d\vec{b}(t)}{dt} + \frac{d\vec{a}(t)}{dt}\times\vec{b}(t)$$

If $\vec{F}(t)$ is a vector point function with constant magnitude, then $\vec{F}(t)\cdot\frac{d}{dt}\vec{F}(t) = 0$.

If $\vec{F}(t)$ is a vector point function with constant direction, then $\vec{F}(t)\times\frac{d}{dt}\vec{F}(t) = \vec{0}$.

3.6 Dell operator

The Vector operator $\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ is called the differential operator in vector and it is denoted as Del (or) ∇

$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$.

3.6.1 Gradient of a Scalar Point function:

If $\phi(x,y,z)$ is a Scalar Point function, then the gradient (change) of $\phi(x,y,z)$ is denoted by grad $\phi (or) \nabla\phi$ and it is given by

$$\nabla\phi = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}.$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}.$$

Note: If $\vec{F}(x,y,z)$ is an irrotational vector field ($\nabla \times \vec{F} = \vec{0}$), then definitely there exists a scalar point function $\emptyset(x,y,z)$ such that $\vec{F}(x,y,z) = grad\ \emptyset$.

If $\phi(x,y,z) = c$ is a level surface then $\nabla\phi|_{P(x_0,y_0,z_0)}$ gives the gradient of $\phi(x,y,z)$ at Point ' P '.

$$|\nabla\phi|_P| = \sqrt{\left(\frac{\partial\phi}{\partial x}\Big|_P\right)^2 + \left(\frac{\partial\phi}{\partial y}\Big|_P\right)^2 + \left(\frac{\partial\phi}{\partial z}\Big|_P\right)^2}.$$

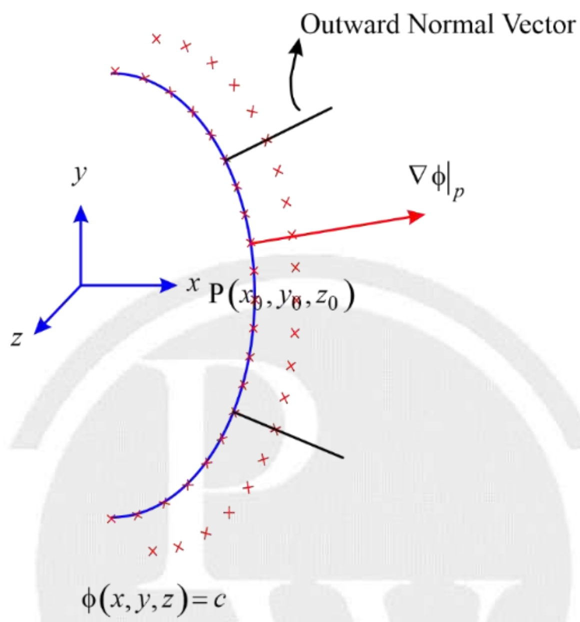


Fig. 3.3

→ $\nabla\phi|_p$ gives the change of $\phi(x,y,z)$ in the direction Normal to the surface $\phi(x,y,z) = c$ at P(x, y, z).

Note:

- Angle between two surfaces:
  Let $f(x,y,z)=C_1$ and $g(x,y,z)=C_2$ and be two surfaces and 'θ' be the angle between them, then $\cos\theta = \frac{\nabla f.\nabla g}{|\nabla f||\nabla g|}$.
- The directional derivative of $f(x,y,z)$ is maximum in the direction of ∇f.
- Maximum value of directional derivative $f(x,y,z) = |\nabla f|$.
- Gradient of a scalar field is normal to the surface given by scalar field.
- If any vector $\vec{A}$= Gradient of scalar field $f(x,y,z)$, then $f(x,y,z)$ is called potential of $\vec{A}$. Vector $\vec{A}$ is called conservative field.

3.7 Directional Derivative:

If $\phi(x,y,z) = c$ is a level surface, then the derivative of $\phi(x,y,z)$ at Point ' P ' in the direction of $\vec{a}$ is called Directional Derivative of $\phi(x,y,z)$ in the direction of $\vec{a}$.
It is given by

Direction Derivative = $\nabla\phi|_p \cdot \hat{a}$

$$= \nabla\phi\big|_p \cdot \frac{\vec{a}}{|\vec{a}|} = \left|\nabla\phi|_p\right| \cdot |\vec{a}| \cdot \frac{\cos\theta}{|\vec{a}|} = \left|\nabla\phi|_p \cdot \cos\theta\right|$$

The directional Derivative of $\phi(x, y, z)$ at P in the direction of $\vec{a}$ is

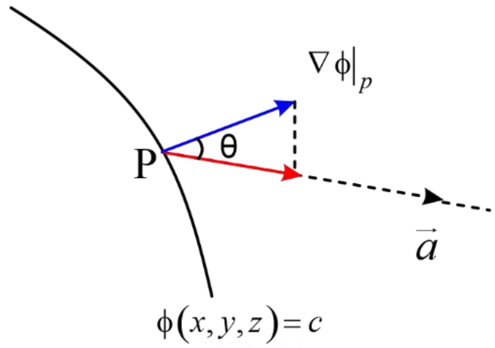


Fig. 3.4

Directional derivative
$DD = |\nabla\phi|_p| \cdot \cos\theta$ where ' θ ' is the angle between $\nabla\phi|_p$ and $\vec{a}$.

For Directional derivative to be maximum $\cos\theta = 1 \Rightarrow \theta = 0°$
$\Rightarrow$ The change of $\phi(x, y, z)$ at Point ' P ' is Maximum in the direction of Normal to $\phi(x, y, z)$

Maximum Change of $\phi(x, y, z)$ at $'p' = |\nabla\phi|_p|$

3.8. Del operated-on Vector Point functions

If Del is a differential operator and $\vec{F}(x, y, z)$ is a vector Point function then the Del operator is operated on $\vec{F}(x, y, z)$ in two Ways.

(i) $\nabla \cdot \vec{F} \rightarrow$ Divergence

(ii) $\nabla \times \vec{F} \rightarrow$ Curl

(i) Divergence of a Vector Point function:

If $\vec{F}(x, y, z)$ is a Vector Point function, then the divergence of $\vec{F}(x, y, z)$ is denoted by div $\vec{F}$(or)$\nabla \cdot \vec{F}$ and for any $\vec{F}(x, y, z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ the divergence is given by

$$\text{div} \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(F_x\vec{i} + F_y\vec{j} + F_z\vec{k}\right) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

If div $\cdot \vec{F} = 0$, then $\vec{F}(x, y, z)$ is called a Solenoidal (or) Incompressible flow Vector.

(ii) Curl of a Vector Point Function:

If $\vec{F}(x, y, z)$ is a Vector Point function, then the curl of $\vec{F}(x, y, z)$ is denoted by $Curl\,\vec{F}(or)\nabla \times \vec{F}$ and for any $\vec{F}(x, y, z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$, the curl of $\vec{F}(x, y, z)$is given by

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \times \left(F_x\vec{i} + F_y\vec{j} + F_z\vec{k}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

If curl $\vec{F} = \vec{0}$; then $\vec{F}$ is called Irrotational Flow Vector.

Note : Geometric interpretation of divergence:

Divergence (div) is the amount of flux per unit volume entering or leaving.

(i) $\nabla.\vec{F} > 0$ at point P, then P is source.

When the divergence is positive it gives the rate at which fluid is flowing away from the point per unit volume.

(ii) $\nabla.\vec{F} < 0$ at point P, then P is sink.

When the divergence is negative it gives the rate at which fluid is flowing towards the point per unit volume.

(iii) $\nabla.\vec{F} = 0$ at point P, then P is neither source or sink.

(iii) Properties of div, Curl & Grad:

If $\phi(x, y, z)$ and $\vec{F}(x, y, z)$ are a scalar point function and a vector point function respectively, then

(a) curl (grad ϕ)=$\vec{0}$

(b) div(curl $\vec{F}$) $= 0$

(c) div(grad ϕ) $= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \nabla^2\phi$

Note : Curl measures the rotatory effects of the vector field.

(i) If curl F > 0 rotation is anticlockwise direction.

(ii) If curl F < 0 rotation is clockwise direction.

(iii) If there is no rotation of fluid anywhere then, $\nabla \times \vec{F} = \vec{0}$. Such a vector field is said to be irrotational or conservative.

(iv) Angular velocity $\omega = \frac{1}{2}\text{curl}\,\vec{v}$

(iv) Angle between two Intersecting Surfaces:

If $\phi_1(x, y, z) = c_1$ & $\phi_2(x, y, z) = c_2$ are two surfaces intersecting at 'P', then the angle of Intersection 'θ' is given by $\cos\theta = \frac{\nabla\phi_1|_p \cdot \nabla\phi_2|_p}{|\nabla\phi_1|_p| \cdot |\nabla\phi_2|_p|}$

3.9 Vector Integration:

3.9.1 Line Integrals

If $\vec{F}(x, y, z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ is a continuous & differentiable Vector Point function at every point along the path C, then the Integral of $\vec{F}(x, y, z)$ from Point 'A' to point 'B' along a path is given by $\int_{A,C}^{B} \vec{F} \cdot d\vec{r}$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\int_{A,C}^{B} \vec{F} \cdot d\vec{r} = \int_{A,C}^{B} (F_x\vec{i} + F_y\vec{j} + F_z\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

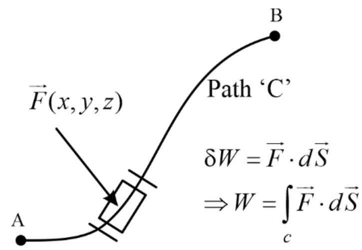


Fig. 3.5. Line Integral

If $\vec{F}(x, y, z)$ is an Irrotational Vector Point Function, (i.e., Curl $\vec{F} = \vec{0}$), then $\int_A^B \vec{F} \cdot d\vec{r}$ is independent of the path between points A and B.

If $\vec{F}(x, y, z)$ is an Irrotational Vector Point Function, then

$\int_A^B \vec{F} \cdot d\vec{r} = \int_A^B \nabla\phi \cdot d\vec{r}$ where $\vec{F} = \nabla\phi$

$= \phi|_B - \phi|_A$

3.9.2 Surface Integral

If $\vec{F}(x, y, z) = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k}$ is a continuous & differentiable Vector Point function at every point on a surface 'S', then the surface integral of $\vec{F}(x, y, z)$ on the surface 'S' is given by $\int_S \vec{F} \cdot d\vec{s}$

Where $d\vec{s} = ds.\hat{n}$ and $\hat{n}$ is the outward unit normal vector to the surface at ds and

$ds = \frac{dx.dy}{|\hat{n}.\hat{k}|} = \frac{dy.dz}{|\hat{n}.\hat{\imath}|} = \frac{dx.dz}{|\hat{n}.\hat{\jmath}|}$:

3.9.3 Volume Integral

If $\vec{F}(x, y, z) = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k}$ is a continuous & differentiable Vector Point function at every point over a volume V, then the volume integral of $\vec{F}(x, y, z)$ on the volume 'V' is given by $\int_V \vec{F} \cdot dv$.

3.9.4 Greens Theorem: (Connects closed line Integral to surface Integral)

If $\vec{F}(x, y) = F_x\vec{\imath} + F_y\vec{\jmath}$ and if the first order derivatives of F_x& $F_y$are continuous at every point with in a region 'R' bounded by a closed path 'C', then

$$\oint_C \vec{F} d\vec{r} = \oint_C F_x dx + F_y dy = \iint_R \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy$$

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

3.9.5 Gauss – Divergence Theorem: (Connects closed surface integral to a Volume Integral)

If 'S' is a closed surface enclosing a volume 'V' and $\vec{F}$ is continuous and differentiable at every point on the closed surface 'S', then the closed surface integral $\oint_S \vec{F}.d\vec{s} = \iiint_V div\, \vec{F}.dV$

3.9.6 Stokes Theorem: (Connect Closed line integral to surface Integral)

If $\vec{F}$ is continuous and differentiable at every point with in a region 'R' (on a surface S) bounded by a closed path 'C', then $\oint_C \vec{F} d\vec{r} = \iint_R curl\, \vec{F} \,. d\vec{S}$

3.10 Properties of Position Vector

Position vector of a point (x, y, z) in space $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$$\text{Length : } r = \sqrt{x^2 + y^2 + z^2}$$

Results :

(a) (i) $\frac{\partial r}{\partial x} = \frac{x}{r}$ (ii) $\frac{\partial r}{\partial y} = \frac{y}{r}$ (iii) $\frac{\partial r}{\partial z} = \frac{z}{r}$

(b) $\text{grad}(r) = \nabla r = \frac{\vec{r}}{r}$

(c) $\text{div}(\vec{r}) = \nabla . \vec{r} = 3$

(d) $\text{curl}(\vec{r}) = \nabla \times \vec{r} = \vec{0}$

(e) $\nabla f(r) = f'(r) \nabla r$

(f) $|\vec{F}| = r^n$ if and only if $\vec{F} = r^{n-1}\vec{r}$

(g) $\nabla . \left(r^n \vec{r}\right) = 0$ if $n = -3$

❑❑❑

4 LINEAR ALGEBRA

4.1. Matrix

An array of elements in horizontal lines (Rows) and Vertical Lines (Columns) is called a Matrix.

Example: $A = \begin{bmatrix} i & n & d & i & a \\ j & a & p & a & n \end{bmatrix}$

4.1.1 Size of Matrix

If a matrix has 'm' rows and 'n' columns, then we say that the size of the matrix is m × n (read as m by n)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13}\dots\dots\dots a_{1n} \\ a_{21} & a_{22} & a_{23}\dots\dots\dots a_{2n} \\ . & . & .. \\ . & . & .. \\ a_{m1} & a_{m2} & a_{m3}\dots\dots\dots a_{mn} \end{bmatrix};\ A = [a_{ij}]_{m\times n} \text{ such that } 1 \le i \le m, 1 \le j \le n \text{ and } a_{ij} = f(i,j)$$

4.1.2 Addition of Matrices:

(i) Two matrices $A = [a_{ij}]_{m\times n}$ & $B = [b_{ij}]_{p\times q}$ can be added only if m = p & n = q.

(ii) Matrix Addition is commutative (A + B = B + A)

(iii) Matrix Addition is Associative. A + (B + C) = (A + B) + C

(iv) Existence of additive identity : If O be m × n matrix each of whose elements are zero. Then, A + O = A = O + A for every m × n matrix A.

(v) Existence of additive inverse : Let $A=\left[a_{ij}\right]_{m\times n}$ then the negative of matrix A is defined as matrix $\left[-a_{ij}\right]_{m\times n}$ and is denoted by –A.

⇒ Matrix –A is additive inverse of A. Because (–A) + A = O = A + (–A). Here O is null matrix of order m × n.

(vi) Cancellation laws holds good in case of addition of matrices, which is X = –A.

⇒ A + X = B + X ⇒ A = B

⇒ X + A = X + B ⇒ A = B

(vii) The equation A + X = 0 has a unique solution in the set of all m × n matrices.

4.1.3 Multiplication of Matrices:

The multiplication of two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$ $(\Rightarrow AB_{m\times q})$ is feasible only if n = P.

$A_{m\times n} \cdot B_{p\times q} = C$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times2} \quad A_{3\times3} \times B_{3\times2}$$

$$\Rightarrow \begin{bmatrix} a_{11}.b_{11} + a_{12}\cdot b_{21} + a_{13}.b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}.b_{11} + a_{22}.b_{21} + a_{23}.b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}.b_{11} + a_{32}\cdot b_{21} + a_{33}\cdot b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}_{3\times2}$$

4.1.4 Properties of Multiplication of Matrices:

(i) Matrix Multiplication Need not be commutative.

(ii) Matrix Multiplication is Associative (A(BC)) = ((AB)C)

(iii) Matrix Multiplication is distributive A(B + C) = (AB + AC)

(iv) The product of two Matrices $A_{m\times n}, B_{n\times q}$ (i.e. $AB_{m\times q}$) can be a zero matrix even if $A \neq O \& B \neq O$.

Example: $A = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- For the multiplication of two matrices $A_{m\times n}$ & $B_{n\times q}$

  (i) The No. of Multiplications required = m n q

  (ii) The number of Additions required = m (n –1) q

4.2 Types of Matrices:

(1) Upper triangular Matrix: A matrix $A = [a_{ij}]; 1 \leq i, j \leq n$ is said to be an upper triangular matrix if

$$a_{ij} = 0 \ \forall \ i > j$$

(2) Lower Triangular Matrix: A matrix $A = [a_{ij}]_{n\times n}; 1 \leq i, j \leq n$ is said to be a lower Triangular Matrix

if $a_{ij} = 0 \ \forall \ i < j$

(3) Diagonal Matrix: A matrix $A = [a_{ij}], \forall \ 1 \leq i, j \leq n$ is said to be a diagonal matrix if $a_{ij} = O \ \forall \ i \neq j$

Example: $A = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$. The diagonal Matrix is also denoted as $A = diag\ [d_1, d_2, d_3]$

(4) Scalar Matrix: A Matrix 'A' = $[a_{ij}]$; $1 \leq i, j \leq n$ is said to be a scalar Matrix if $a_{ij} = \begin{cases} K; i = j \\ 0; 1 \neq j \end{cases}$

If K = 1, then A → Identity Matrix,

If K = 0, then A → Null Matrix.

(5) Idempotent Matrix:

A Matrix '$A_{n\times n}$' is said to be an idempotent matrix if $A^2 = A$.

Example: $A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$

$\Rightarrow A \cdot A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = A$

(6) Nilpotent Matrix: A matrix A is said to be nilpotent of class x or index if $A^x = 0$ and $A^{x-1} \neq 0$ i.e. x is the smallest index which makes $A^x = 0$.

Example: The matrix A = $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent class 3, since A ≠ 0 and $A^2 \neq 0$, but $A^3 = 0$.

(7) Orthogonal Matrix: A matrix A is said to be orthogonal if $\mathrm{A}.\mathrm{A}^{\mathrm{T}} = \mathrm{I}$

Example: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(8) Involutory Matrix: A matrix A is said to be involutory if $\mathrm{A}^2 = \mathrm{I}$

Example: $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$

4.3 Transpose of a Matrix:

For a given matrix $= [a_{ij}]; 1 \leq i \leq m, 1 \leq j \leq n$, we can say that 'B' where $B = [b_{ij}], i \leq i \leq n \;\; i \leq j \leq m$ is the transpose of the Matrix 'A' if $a_{ij} = b_{ji}$

4.3.1 Properties of Transpose of a Matrix:

(i) $(A^T)^T = A$

(ii) $(AB)^T = B^T \cdot A^T$

(iii) $(KA)^T = KA^T$ where 'K' is a scalar.

4.4 Conjugate of a matrix

The matrix obtained by replacing each element of matrix by its complex conjugate.

4.4.1 Properties of conjugate matrix

(a) $(\overline{\overline{A}}) = A$ (b) $(\overline{A+B}) = \overline{A} + \overline{B}$

(c) $(\overline{KA}) = \overline{K}\,\overline{A}$ (d) $(\overline{AB}) = \overline{A}\,\overline{B}$

(e) $\overline{A} = A$ if A is real matrix

$\overline{A} = -A$ if A is purely imaginary matrix

4.5 Transposed Conjugate of a Matrix

The transpose of conjugate of a matrix is called transposed conjugate. It is represented by A^θ.

(a) $(A^\theta)^\theta = A$

(b) $(A+B)^\theta = A^\theta + B^\theta$

(c) $(KA)^\theta = \bar{K}A^\theta$ (K : Complex number)

(d) $(AB)^\theta = B^\theta A^\theta$

4.6 Trace of a Matrix

Trace is simply sum of all diagonal elements of a matrix.

4.6.1 Properties of Trace of a matrix

Let A and B be two square matrices of order η and λ is scalar then

1. $Tr(\lambda A) = \lambda Tr(A)$
2. $Tr(A+B) = Tr(A) + Tr(B)$
3. $Tr(AB) = Tr(BA)$ [If both AB and BA are defined]

4.7. Type of Real Matrix

(a) Symmetric matrix : $(A)^T = A$

(b) Skew symmetric matrix : $(A^T) = -A$

(c) Orthogonal matrix : $(A^T = A^{-1}, AA^T = I)$

Note : (a) If A and B are symmetric, then (A + B) and (A – B) are also symmetric.

(b) For any matrix AA^T is always symmetric.

(c) For any matrix, $\left(\frac{A+A^T}{2}\right)$ is symmetric and $\left(\frac{A-A^T}{2}\right)$ is skew symmetric.

(d) For orthogonal matrices, $|A| = \pm 1$

(e) We can write any matrix A as a sum of symmetric and skew symmetric matrix $A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$

4.8. Type of complex matrix

(a) Hermitian matrix : $(A^\theta = A)$

(b) Skew-Hermitian matrix: $A^\theta = -A$

(c) Unitary matrix : $(A^\theta = A^{-1}, AA^\theta = I)$

Note : (a) $\frac{A+A^\theta}{2}$ is Hermitian and $\frac{A-A^\theta}{2}$ is skew Hermitian matrix.

(b) We can write any matrix as a sum of Hermitian and skew Hermitian matrix $A = \frac{A+A^\theta}{2} + \frac{A-A^\theta}{2}$

4.9. Determinant

The summation of the product of elements of a row(or) column of a matrix with their corresponding Co-factors.

$$A \cdot adj(A) = |A| \cdot I$$

Determinant can be calculated only if matrix is a square matrix.

Suppose, we need to calculate a 3 × 3 determinant,

$$\Delta = \sum_{j=1}^{3} a_{1j} cof(a_{1j}) = \sum_{j=1}^{3} a_{2j} cof(a_{2j}) = \sum_{j=1}^{3} a_{3j} cof(a_{3j})$$

We can calculate determinant along any row or column of the matrix.

4.9.1 Properties of Determinants

(i) If 'A' is a Square Matrix of size ' $n \times n$ ' and 'k' is a Scalar then

$|K \cdot A_{n\times n}| = K^n \cdot |A_{n\times n}|$

(ii) $|adj(A)| = |A|^{(n-1)}$

(iii) $\left|adj\left(adj(A)\right)\right| = (|A|)^{(n-1)^2}$

(iv) $|AB| = |A| \cdot |B|$

(v) $|(AB)^T| = |B^T| \cdot |A^T|$

(vi) If two rows (or) two columns of a determinant are interchanged, then the determinant changes its sign.

(vii) The determinant of an upper triangular Matrix/a lower triangular Matrix/a diagonal Matrix is the product of the principal diagonal elements of the Matrix.

(viii) The determinant of Every Skew-Symmetric Matrix of odd order $(A_{n\times n})('n' \, is\ odd)$ is zero.

(ix) The determinant of an orthogonal Matrix $A_{n\times n}$ is ± 1

(x) The determinant of an Idempotent Matrix is either 0 (or) 1.

(xi) The determinant of an Involuntary Matrix is ± 1

(xii) The determinant of a Nilpotent Matrix is always zero.

(xiii) If the product of two Non-zero Matrices $A_{n\times n} \neq 0; B_{n\times n} \neq 0$ is a zero Matrix $((AB)_{n\times n} = 0)$, then both $|A| = 0$ & $|B| = 0$.

(xiv) If two rows (or) two columns of a Matrix are either equal or Proportional, then the determinant of the Matrix is equal to zero.

(xv) The number of terms in the general expansion of an 'n × n' determinant is $\underline{n!}$

(xvi) Value of the determinant is invariant under row and column interchange i.e., $\left|A^T\right| = |A|$

(xvii) If any row or column is completely zero, then |A| = 0.

(xviii) If any single row or column of the matrix is multiplied by k then the determinant the of new matrix = K|A|

(xix) In a determinant the sum of the product of the element of any row or column with its cofactor gives a determinant of the matrix.

(xx) In determinant the sum of the product of the element of any row or column with a cofactor of another row or column will give zero.

(xxi) |AB| = |A| × |B|

(xxii) Elementary operations don't effect the determinant that is $A \xrightarrow{R_i = R_i + KR_j} B$ then |A| = |B|

$A \xrightarrow{C_i = C_i + KC_j} B$ then |A| = |B|

4.10. Minors, Cofactor and Adjoint of a Matrix

Minor of an element is equal to the determinant of the remaining elements of the matrix, after excluding the row and column containing the particular element. The cofactor of an element can be calculated from the minor of the element. The cofactor of an element is equal to the product of the minor of the element, and -1 to the power of position values of row and column of the element.

$$\text{Cofactor of an Element } = (-1)^{i+j} \times \text{ Minor of an Element}$$

Here i and j are the positional values of the row and column of the element.

Example :

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Minor of element a_{21}: $M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

Co-factor of an element, $a_{ij} = (-1)^{i+j} M_{ij}$

- To design co-factor matrix, we replace each element by its co-factor.
- Adjoint of a matrix = transpose of cofactor matrix
- $A^{-1} = \dfrac{Adj(A)}{|A|}$

4.11 Inverse of a matrix

Inverse of a matrix only exists for square matrices.

$$\left(A^{-1}\right) = \frac{Adj(A)}{|A|} \text{ and } |A| \neq 0$$

Properties:

(a) $AA^{-1} = A^{-1}A = I$

(b) $(AB)^{-1} = B^{-1}A^{-1}$

(c) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(d) $(A^T)^{-1} = (A^{-1})^T$

(e) The inverse of 2 × 2 matrix should be remembered,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(i) Interchange the diagonal elements and put negative sign on the rest.

(ii) Divide by determinant.

4.12. Rank of a Matrix

- The rank of the matrix refers to the number of linearly independent rows or columns in the matrix. ρ(A) is used to denote the rank of matrix A.
- A matrix is said to be of rank zero when all of its elements become zero.
- The rank of the matrix is the dimension of the vector space obtained by its columns.
- The rank of a matrix cannot exceed more than the number of its rows or columns. The rank of the null matrix is zero.
- The nullity of a matrix is defined as the number of vectors present in the null space of a given matrix. In other words, it can be defined as the dimension of the null space of matrix A called the nullity of A. Rank + Nullity is the number of all columns in matrix A.

  A real Number 'r' is said to be the rank of a matrix '$A_{m\times n}$' if

  (1) There is at least one square sub-matrix of A of order r whose determinant is not equal to zero.

  (2) If the matrix A contains any square sub-matrix of order (r + 1) and above, then the determinant of such a matrix should be zero.

  It is mathematically denoted by $\rho(A) = r$

4.12.1 Properties of Rank of a Matrix:

(i) $\rho(A_{m\times n}) \leq (m, n)$

(ii) $\rho(AB) \leq min\{\rho(A), \rho(B)\}$

(iii) Rank of transpose of matrix is equal to rank of matrix

(iv) Elementary operations do-not affect the rank the matrix

(v) $\rho(A + B) \leq \quad \{\rho(A) + \rho(B)\}$

4.12.2 Row Echelon Form

A Matrix $A_{m\times n}$ is said to be in row-echelon form if

(i) Number of zeroes before the 1st Non-zero element in any row is less than the number of such zeroes in its succeeding row.

(ii) Zero rows (if any) should lie at the bottom of the Matrix.

$\rho(A_{m\times n})$ = Number of non-zero rows in the Row-Echelon form of A.

4.13 System of Equations:

The given system of equations

$$a_{11}x_1 + a_{12}x_{12} + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

can be written in Matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \boxed{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

Ax = B

Coefficient Matrix — Variable Matrix — Constants Matrix

The system Ax = B is said to be a homogeneous system if B = 0.

The system of Ax = B is said to be a non-homogeneous system if $B \neq 0$.

4.13.1 Consistency of a non-homogeneous system of Equations:

For the above system of non – homogeneous equations, Ax = B; Augmented Matrix = [A/B] = $\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$

(i) If $\rho(A) = \rho(A/B) =$ Number of unknowns, then the system Ax = B has a unique solution.

(ii) If $\rho(A) = \rho(A/B) <$ Number of unknowns, then the system has infinitely many solutions.

(iii) If $\rho(A) \neq \rho(A/B)$, then the system has no solution.

Number of linearly independent solutions for a system of 'n' equations given by Ax = B is $n - \rho(A)$

4.13.2 Consistency of Homogeneous System of Equations:

$a_{11}x + a_{12}y + a_{13}z = 0$

$a_{21}x + a_{22}y + a_{23}z = 0$

$a_{31}x + a_{32}y + a_{33}z = 0$

$$\text{Ax} = 0 \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; [A/B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{bmatrix}_{3\times 4}$$

If $\rho(A) = \rho(A/B) = n$ $(i.e\ |A| \neq 0)$; the system has a unique solution.

(Trivial solution; x = 0, y = 0, z = 0)

If $\rho(A) = \rho(A/B) < n(|A| = 0)$; the system has infinitely many solutions (Non-trivial solution exists for the system).

4.14 Linear Combination of Vectors:

If $x_1, x_2, x_3, \ldots\ldots, x_n$ are 'n' rows vectors, then the combination $k_1x_1 + k_2x_2 + k_3x_3 + \ldots.. + k_nx_n$ is called a linear combination of $x_1, x_2, \ldots., x_n (k_1, k_2, k_3, \ldots.. k_n$ are scalars)

(1) The linear combination $k_1x_1 + k_2x_2 + k_3x_3 + \ldots.. + k_nx_n$ is said to be linearly dependent if $k_1x_1 + k_2x_2 + k_3x_3 + \ldots.. + k_nx_n = 0$ when $k_1, k_2, k_3, \ldots.., k_n$ (NOT All zeroes).

If $x_1 = [a_1 \quad b_1 \quad c_1]; x_2[a_2 \quad b_2 \quad c_2]; x_3 = [a_3 \quad b_3 \quad c_3]$, then the vectors x_1, x_2, x_3 are said to be linearly dependent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

(2) The combination $k_1x_1 + k_2x_2 + \ldots\ldots + k_nx_n$ is said to be linearly independent if $k_1x_1 + k_2x_2 + \ldots\ldots\ldots + k_nx_n = 0$ when $k_1 = k_2 = k_3 = \ldots\ldots k_n = 0$

4.14.1 Eigen Values and Eigen Vectors

For any square Matrix $A_{n\times n}$, the equation $|A - \lambda I| = 0$ where 'λ' is a scalar is called the characteristic equation.

The roots of the characteristic equation of a Matrix are called Eigen Values.

4.14.2 Properties of Eigen Values

(i) If $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n$ are 'n' Eigen Values of $A_{n\times n}$, then

(a) Sum of Eigen Values of 'A' = $\lambda_1 + \lambda_2 + \lambda_3 + \ldots. + \lambda_n = \sum_{i=1}^{n} \lambda_i = trace(A)$ = Sum of Principal diagonal elements

(b) Product of all the Eigen Values of 'A' = $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ldots\ldots \lambda_n = \prod_{i=1}^{n} \lambda_i = |A|$

(c) Eigen Values of A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m, \ldots\ldots \lambda_n^m$

(d) Eigen Values of adj(A) are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}, \ldots\ldots, \frac{|A|}{\lambda_n}$

(e) Eigen Values of A & A^T are the same.

(f) Eigen Values of $k_1A + k_2I$ (Where k_1 and k_2 are scalar) are

$$k_1\lambda_1 + k_2, k_1\lambda_2 + k_2, k_1\lambda_3 + k_2, k_1\lambda_4 + k_2, \ldots\ldots\ldots k_1\lambda_n + k_2$$

(ii) '0' is always an Eigen Value of an odd-order Skew-Symmetric Matrix.

(iii) Eigen Values of a Real Symmetric Matrix are always real.

(iv) Eigen Values of the Skew-Symmetric Matrix are either zero (or) purely Imaginary.

(v) The Eigen values of an Orthogonal Matrix are of unit modulus.

(vi) If the sum of all the elements in a row (or Column) is constant (= k) for all the rows (or columns) in the matrix respectively, then 'k' is an Eigen Value of the Matrix.

Example: If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ and if $a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = a_3 + b_3 + c_3 = k,$

then 'k' is an Eigen Value of 'A'.

(vii) The Eigen Values of an upper triangular Matrix, a lower triangular Matrix, a diagonal Matrix are the Principal diagonal elements of the Matrix.

4.15. Eigen Vector

A non-zero column vector $X_{n\times 1}$ is said to be an Eigen Vector of $A_{n\times n}$ corresponding to the Eigen Value 'λ', if $AX = \lambda X (X \neq 0)$.

4.15.1 Properties of Eigen Vectors:

(i) Eigen Vectors of A & A^T are not the same.

(ii) Eigen Vectors of A & A^M are same.

(iii) The Eigen Vectors of a Real Symmetric Matrix are always orthogonal.

(iv) The number of linearly independent Eigen Vectors of '$A_{n\times n}$' is equal to the number of distinct Eigen Values of '$A_{n\times n}$'.

4.15.2 Cayley Hamilton Theorem:

Every Matrix satisfies its characteristic equation.

This means that, if $c_0\lambda^n + c_1\lambda^{-1} + ... + c_{n-1}\lambda + c_n = 0$ is the characteristic equation of a square matrix A of order n, then

$c_0A^n + c_1A^{n-1} + ... + c_{n-1}A + c_nI = 0$ …(i)

Note: When λ is replaced by A in the characteristic equation, the constant term c_n should be replaced by c_nI to get the result of the Cayley-Hamilton theorem, where I is the unit matrix of order n.

Also, 0 in the R.H.S. of (i) is a null matrix of order n.

4.16. Subspace (Basis of Dimensions)

4.16.1 Vector

An ordered n-tuple of numbers is called an n-vector.

4.16.2 Linearly Independent and Dependent Vector

Let X_1 and X_2 be the non-zero vectors:

- $\{x_1, x_2, …., x_k\}$ are linearly independent if $r_1x_1 + r_2x_2 + … + r_k x_k = 0$ only for $r_1 = r_2 = … + r_k = 0$.
- The vectors $x_1, r_2, …., x_k$ = are linearly dependent if they are not linearly independent; that is, if there exist scalars $r_1, r_2, … , r_k$ which are not all zero such that

  $r_1x_1 + r_2x_2 + …. + r_k x_k = 0$

Note: Let $X_1, X_2…….X_n$ be 'n' vector of matrix A.

- If rank (A) = number of vectors then vector $X_1, X_2…..X_n$ are linearly independent.
- If rank $(A) \neq$ number of vectors then vector $X_1, X_2 ….. X_n$ are linearly dependent.

4.16.3 Vector Space R^n:

If n positive integer, then an ordered n-tuple is a sequence of n real numbers ($\alpha_1, \alpha_2, \ldots. \alpha_n$). the set of all ordered *n*-tuples is called n-space and is denoted by $\mathbb{R}^n$.

4.16.4 Subspaces of an N-vector space V_n

A non-empty set S, of vectors of $V_n(F)$,is called a subspace of $V_n(F)$, if

- ξ_1, ξ_2 are any two members of S, then $\xi_1 + \xi_2$ is also a member of S; and
- ξ is a member of S, and k is a scalar then $k\xi$ is also a member of S.

  Briefly, we may say that a set S of vectors $V_n(F)$ is a subspace of $V_n(F)$ it closed w.r.t. the compositions of "addition" and "multiplication with scalars".

  Every subspace of V_n contains the zero vector; being the product of any vector with the scalar zero.

4.16.5 Construction of Subspaces

- **A subspace Spanned by a Set of Vectors:** A subspace that arises as a set of all linear combinations of any given set of vectors is said to be spanned by the given set of vectors.
- **Basis of a subspace:** A set of vectors is said to be a basis of a subspace, if
  - ➢ The subspace is spanned by the set, and
  - ➢ The set is linearly independent.

Note: If we have N vectors and they are independent then they span N-dimension space. But if they are dependent then they span only a subspace of N-dimension space.

4.16.6 Orthogonality of Vectors

- Two vectors are orthogonal if each is non-zero and $X_1^T X_2 = 0$
- If n vectors $X_1, X_2 \ldots. X_n$ each of n dimensions is orthogonal then they are surely linearly independent and form the basis for n-dimension space.
- The set of the vector is orthonormal if they are orthogonal and have unit magnitude.

4.17. Similar Matrices

- Two matrix A and B are similar if there exist a non singular matrix P such that $B = P^{-1}AP$
- Similar matrix has same eigen valves
- If A is similar to B then B is also similar to A
- If A is similar to B and B is similar to C then A is similar to C.

4.18. Diagonalization of a matrix:

Finding a matrix D which is a diagonal matrix and which is similar to A is called diagonalization i.e., we wish to find a non-singular matrix M such that $A = M^{-1}DM$ where D is a diagonal matrix.

Condition for a Matrix to be Diagonalizable

1. A necessary and sufficient condition for a matrix $A_{n \times n}$ to be diagonalizable is that the matrix must have n linearly independent eigen vectors.

2. A sufficient (but not necessary) condition for a matrix $A_{n \times n}$ to be diagonalizable is that the matrix must have n linearly independent eigen values.

   This is because if a matrix has n linearly independent eigen values then it surely has n linearly independent eigen vectors (although the converse of this is not true).

❑❑❑

5 PROBABILITY AND STATISTICS

5.1. Random Experiment:

The experiment in which the outcome is uncertain is called a Random Experiment (RE).

Example: Flipping a coin, rolling a pair of dice, Picking a ball from a bag.

5.1.1 Sample Space

The set contains all the possible outcomes of a random experiment. It is denoted by 'S'.

If RE is flipping a coin, S = {Head, Tail}

If RE is rolling a dice, S = {1,2,3,4,5,6}

5.2. Event

Any subset of sample space 'S' is called an Event.

Example: If RE is flipping a coin, then the occurring of a Head is an Event.

If RE is rolling a dice, then getting an odd number is an Event.

5.2.1 Probability of an Event:

If 'A' is any event with in the sample space 'S' of a Random experiment, then the probability of event 'A' is given by

$$P(A)=\frac{\text{No. of outcomes favouring event 'A' to happen}}{\text{Total number of elements in 'S'}}=\frac{n(A)}{n(S)}$$

Probability of getting an Even Number when a dice is rolled.

$$\text{P(Even Number)} = \frac{3}{6} = 0.5$$

S = {1,2,3,4,5,6},

A = {2,4,6}

Note: Probability can also be expressed as odds if favour and odds against an event:

- **Odds is favour of an event:**

  Odds in **favour** of an event = Number of successes : Number of failures = m: (n – m).

- **Odds against an event:**

  Odds against an event = Number of failures : Number of successes = (n – m) : m.

5.2.2 Axioms Probability:

(i) If 'A' is any event with in the sample space 'S' of a RE, then $0 \le P(A) \le 1$

$$\frac{0}{n(S)} \le \boxed{\frac{n(A)}{n(S)}} \le \frac{n(S)}{n(S)}$$

$$\boxed{0 \le P(A) \le 1}$$

(ii) P(S) = 1

When a RE is conducted the experiment yields a possible outcome.

5.2.3 Types of Events:

(i) Mutually Exclusive Events:

If A, B are two events within a sample space 'S', then A & B are said to be mutually exclusive if $A \cap B = \phi$.

Example: If 'A' is the event of getting a prime number when a dice is rolled and 'B' is the event of getting a composite number when a dice is rolled then

$S = \{1,2,3,4,5,6\}$, $A = \{2,3,5\}$, $B = \{4,6\} \Rightarrow A \cap B = \phi \Rightarrow P(A \cap B) = 0$

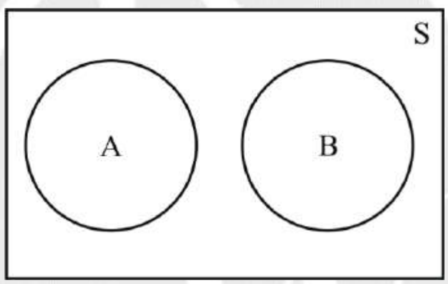


Fig. 5.1. Mutually exclusive event

(ii) Mutually Exhaustive Events:

If 'A', and 'B' are two events within a sample space 'S', then 'A' & 'B' are said to be mutually exhaustive if $A \cup B = S$

Example: If 'A' is the event of getting an odd number when a dice is rolled and 'B' is the event of getting an Even Number, then

$$A \cup B = S$$

$$S = \{1,2,3,4,5,6\}$$

$$A = \{1,3,5\},\ B = \{2,4,6\}$$

$$\Rightarrow \quad A \cup B = S$$

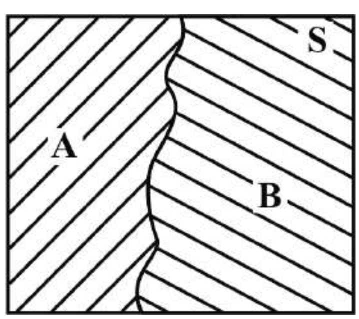


Fig. 5.2. Mutually exhaustive event

(iii) Independent Events:

Two events 'A' & 'B' within the sample space 'S' (or) within two different sample spaces 'S_1' & 'S_2' are said to be independent if $P(A \cap B) = P(A) \cdot P(B)$.

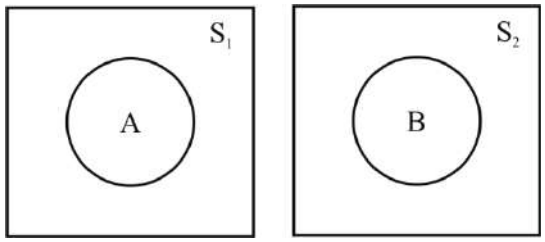


Fig. 5.3. Independent Event

(iv) Impossible Event (ϕ):

The event with zero probability is called an Impossible Event $P(\phi) = 0$.

5.3. Addition Theorem of Probability

If A, and B are two events with a sample space 'S' of a Random Experiment, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

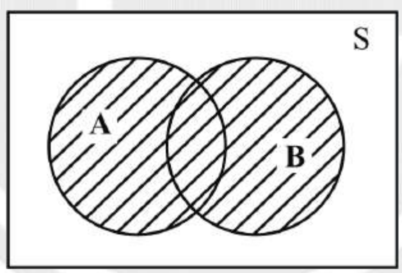


Fig. 5.4. Addition theorem

$\Rightarrow$ $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

When A, and B are mutually exclusive events, $A \cap B = \phi$.

$\Rightarrow$ $$P(A \cap B) = 0$$

$$P(A \cup B) = P(A) + P(B)$$

- If $E_1, E_2, E_3, \ldots\ldots. E_n$ are mutually exclusive events ($E_i \cap E_j = \phi$), then $P(E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n) = \sum_{i=1}^{n} P(E_i)$

  $= P(E_1) + P(E_2) + P(E_3) + \ldots\ldots\ p(E_n)$

5.3.1. De Morgan's Law

- $(A \cup B)^C = A^C \cap B^C$
- $(A \cap B)^C = A^C \cup B^C$

5.3.2. Union and Intersection properties

For any two events A and B:

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(b) $P(A^c \cap B^c) = 1 - P(A \cup B)$

For any three events A, B and C:

(a) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

(b) $P(A^c \cap B^c \cap C^c) = 1 - P(A \cup B \cup C)$

5.3.3 Conditional Probability:

The probability of the happening of event 'A' when it is known that event 'B' has already occurred is given by P(A/B)

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)}$$

5.3.4 Joint Probability:

- $f(x, y)$ is the joint probability of two RV'S x, y.
- If the two RV are Independent then
  $f(x, y) = f(x) \cdot f(y)$
- $P(a \le x \le b, c \le y \le d) = \int_a^b \int_c^d f(x, y) dy dx$
- $f(x) = \int_{-\infty}^{\infty} f(x, y) dy$
- $f(y) = \int_{-\infty}^{\infty} f(x, y) dx$

5.3.5 Multiplication Theorem of Probability:

If A, and B are two events within a sample space 'S', then P(A/B) · P(B) = P(B/A) · P(A)

P(A/B) = $\frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A/B) \cdot P(B) \rightarrow (1)$

P(B/A) = $\frac{P(B \cap A)}{P(A)} \Rightarrow P(B \cap A) = P(B/A) \cdot P(A) \rightarrow (2)$

From (1) & (2)

$P(A/B) \cdot P(B) = P(B/A) \cdot P(A)$

5.3.6 Total Theorem of Probability:

If E_1, E_2, E_3,……E_n are 'n' mutually exclusive $(E_i \cap E_j = \phi; \forall i \neq j)$ and collectively exhaustive event ($E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n = S$) and 'A' is any event with in the sample space 'S', then

$P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + \ldots\ldots + P(E_n) \cdot P(A/E_n)$

$$\boxed{P(A) = \sum_{i=1}^{n} P(E_i) \bullet P(A/E_i)}$$

5.3.7. Baye's Theorem:

If $E_1, E_2, E_3, \ldots\ldots E_n$ are mutually exclusive $(E_i \cap E_j = \phi \forall i \neq j)$ and collectively exhaustive event $(E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n = S)$ and 'A' is any event with in the sample space 'S', then

$$\boxed{P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)}}$$

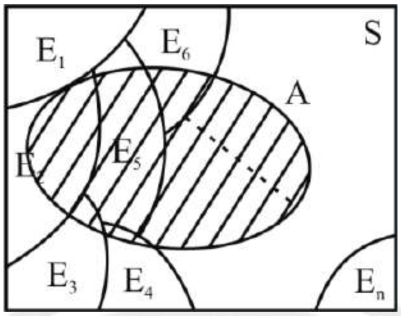


Fig. 5.5. Baye's theorem

5.3.8. Use of permutation and combination

What is combination?

A combination of 'n' objects taken 'r' at a time (r-combination of 'n' objects is an unordered selection of 'r' of the objects).

Number of ways of combining of 'r' object out of 'n' objects without repetition

$$^nC_r = \frac{n!}{(n-r)!r!}$$

What is permutation?

A combination of 'n' objects taken 'r' at a time (r-combination of 'n' objects is an ordered selection of 'r' of the objects).

Number of ways of selection of r object out of n objects without repetition

$$^nP_r = \frac{n!}{(n-r)!}$$

Result :

(i) $^nC_r = {}^nC_{n-r}$

(ii) $^nC_0 + {}^nC_1 + {}^nC_2 + \ldots. + {}^nC_n = 2^n$

(iii) $^nC_0 + {}^nC_2 + {}^nC_4 + \ldots. + = 2^{n-1}$

(iv) $^nC_1 + {}^nC_3 + {}^nC_5 + \ldots. + = 2^{n-1}$

(v) $0.{}^nC_0 + 1.{}^nC_1 + 2.{}^nC_2 + \ldots. + n.{}^nC_n = n.2^{n-1}$

Permutations with Repetition

The number of permutations of n objects, where p objects are of one kind, q objects are of another kind and the rest, if any, are of a different kind is $\frac{{}^{n}P_{r}}{p!q!}$.

Combination with Repetition

Number of combinations of 'n' distinct things taking 'r' at a time when each thing may be repeated any number of times is given by ${}^{n-1+r}C_{r}$.

5.4. STATISTICS

Statistics $\rightarrow$ Collection and Analysis of Data

Data → Ungrouped data / Row data
Data → Grouped Data

5.4.1. Analysis of Ungrouped Data:

If $x_1, x_2, x_3, \ldots, x_n$ are 'n' observations, then

(1) The range of the data = R = $\max\{x_1, x_2, \ldots, x_n\} - \min\{x_1, x_2, x_3, \ldots, x_n\}$

(2) Arithmetic mean **:** Mean of the data is equal to sum of observaions divided by the total number of observations.

$$\bar{x}(or)\mu = \frac{x_1 + x_2 + \ldots\ldots + x_n}{n} = \boxed{\frac{\sum_{i=1}^{n} x_i}{n} = \bar{x} = \mu}$$

- The mean of 1st 'n' natural numbers = $\frac{\left(\frac{n(n+1)}{2}\right)}{n} = \frac{n+1}{2}$
- The mean of 1st 'n' odd numbers = $\frac{n^2}{n} = n$
- The mean of 1st 'n' even numbers = n +1

5.4.2 Median:

The middle most observation of the data $(x_1, x_2, x_3, \ldots\ldots x_n)$ When the data is arranged in either ascending or descending order.

If $x_1, x_2, x_3, x_4, \ldots\ldots\ldots x_n$ are 'n' observations that are arranged in ascending/descending order then

(i) Median of the Data = $\left(\frac{n+1}{2}\right)^{th}$ observation, if 'n' is odd.

(ii) Median of the Data = Mean of $\left(\frac{n}{2}\right)^{th}$ & $\left(\frac{n}{2}+1\right)^{th}$ observations, if 'n' is even.

5.4.3 Mode:

The observation with highest frequency is called mode.

Any Data with two Modes is called $\rightarrow$ Bimodel Data

If $x_1, x_{2,} x_3, \ldots\ldots, x_n$ are 'n' data points, $\bar{x} = \mu = \frac{x_1 + x_2 + \ldots\ldots + x_n}{n}$

Mean Deviation of the observation $(x_i) = d_i = x_i - \bar{x}$

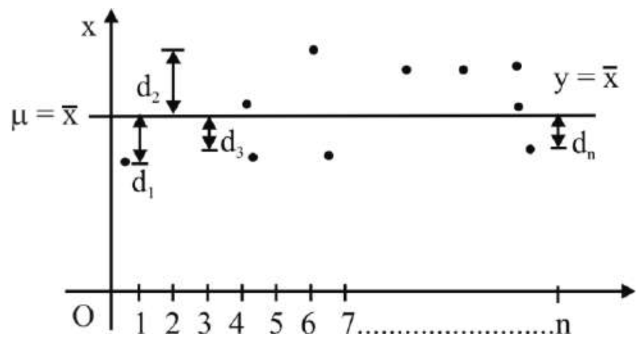


Fig. 5.6. Discrete data

Sum of derivations of all the observations = $\Sigma d_i = (x_1 - \bar{x}) + (x_2 - \bar{x}) + \ldots\ldots 0 + (x_n - \bar{x})$

$$= \Sigma d_i = (x_1 + x_2 + \ldots\ldots + x_n) - n\bar{x}$$

$$\boxed{\Sigma d_i = 0}$$

The sum of mean deviations of all the observations is equal to zero.

5.4.4 Absolute Mean Deviation:

If $x_1, x_2, x_3, \ldots\ldots, x_n$ are 'n' data points with Mean = $\bar{x}$, then the absolute mean deviation of x_i about $\bar{x}$ is given by $|d_i| = |x - \bar{x}|$
The sum of absolute mean derivations of given data is not zero.

$$(\Sigma|d_i| \neq 0) \Rightarrow (|x_1 - \bar{x}| + |x_2 - \bar{x}| + \ldots\ldots + |x_n - \bar{x}| \neq 0)$$

5.4.5 Standard Deviation:

If $x_1, x_2, x_3, \ldots\ldots, x_n$ ('n' is very large), then the standard deviation of the data is given by

Population Standard Deviation $\sigma = \sqrt{\frac{1}{n}\Sigma(x_i - \bar{x})^2}$, n $\to$ size of population

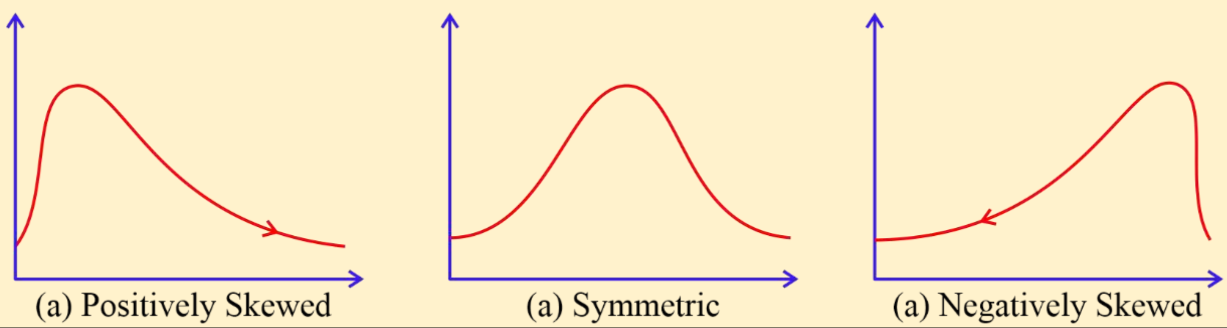
Sample Standard derivation: $\sigma = \sqrt{\frac{1}{(n-1)}\Sigma(x_i - \bar{x})^2}$, n$\to$ size of sample

Generally (n > 29 $\to$ population) (n < 29 $\to$sample)

Note: **Measures of skewness (The degree of asymmetry)**

A lack of symmetry is skewness.

- For symmetric distribution mean (M) = Median (M_d) = Mode (M_e)
- For negatively skewed distribution mean (M) < Median (M_d) < Mode (M_e)
- For positively skewed distribution Mean (M) > Median (M_d) > Mode M_e).



5.5. Random Variables

The variable that connects the outcome of a Random Experiment to a real number.

Example: 'x' is the value of the number that a dice shows when it is rolled.

Discrete RV → The RV whose value is obtained by counting, defined by PMF

Random Variable

Continuous RV → The RV whose value is obtained by Measuring, defined by PDF

- If a data consists of 'f_1' data points with value 'x_1', 'f_2' data points with value 'x_2'......'f_n' data point with value 'x_n', then
  (i) Expectation of 'x' $= E(x) = \sum_{i=1}^{n} x_i P(x = x_i)$
  (ii) Variance of 'x' $= \sigma^2 = E(x^2) - (E(x))^2$ and σ is the standard deviation.

5.5.1 Probability Mass Function (PMF)

The PMF p(x) of a discrete random variable X taking values $x_1, x_2, ...x_n$ is defined such that,

(i) $p(x_i) \geq 0$

(ii) $\sum_{i=1}^{n} p(x_i) = 1$

(iii) $p(x_i) = p(X = x_i)$

5.5.2 Probability Density Function (PDF)

The pdf f(x) of a continuous random variable X is defined such that,

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

(iii) $P(a < X < b) = \int_{a}^{b} f(x)dx$

5.5.3 Expected Value

1. Expected value of a random variable X, E [X], is defined as, $E[X]\begin{cases} \sum xp(x); & \text{X is discrete rv} \\ \int_{-\infty}^{\infty} xf(x)dx; & \text{X is continuous rv} \end{cases}$

2. Expected value of X^2 is,

$$E[X^2] = \begin{cases} \sum x^2 p(x); & \text{X is discrete rv} \\ \int_{-\infty}^{\infty} x^2 f(x)dx; & \text{X is continuous rv} \end{cases}$$

Note: $E[X^n]$ is called nth moment.

5.5.4 Mean of Random Variable 'X'

Mean $= \mu = E[X]$

5.5.5 Variance of a Random Variable 'X'

Var $(X) = E[(X-\mu)^2]$

Or, Var $(X) = E[X^2] - \mu^2$

5.5.6 Properties of Expectation

(i) E[c] = c, c is a constant.

(ii) E[ax] = aE[X]

(iii) E(aX + b) = aE(X) + b

(iv) If X and Y are random variable E[X ± Y] = E(x) ± E(Y).

(v) If X and Y are random variables E(X. Y) = E(X). E(Y / X).

(vi) If X and Y independent random variables E(X. Y) = E(X). E (Y).

5.5.7 Properties of Variance

(i) Var[C] = 0, C is constant.

(ii) Var (aX) = a^2V(X) where X is random variable and 'a' constant.

Var(–X) = $(-1)^2$ Var(X) = Var(X) Variance is always positive.

(iii) Var(ax + b) = a^2 Var(X) + 0

(iv) If X and Y are independent random variables.

Var(X + Y) = Var(X) + Var(Y)

Var(X – Y) = Var(X) + Var(Y)

(v) Var(ax + by) = a^2 v(x) + b^2 v(y) + 2ab Cov (x, y)

(vi) Cov (x, y) = E(x, y) – E(x) E(y)

(vii) For independent random variables Cov(x, y) = 0

5.5.8 Continuous RV

The value of the Random Variable is obtained by Measuring.

5.6. Probability Distribution Function (PDF)

A continuous & differentiable function P(x) is said to be a probability distribution/density function of a continuous random variable 'x' if $P(a \leq x \leq b) = \int_a^b P(x)dx$

5.6.1 Mean (or) Expectation

If P(x) is a probability distribution/density function of a continuous Random Variable 'x' then the Mean of 'x' = E(x) $= \int_{-\infty}^{\infty} x \cdot P(x)dx$

5.6.2 Median

The value of 'x' for which the total probability is exactly divided into two equal halves is called Median.

5.6.3 Mode

The valueof 'x' at which P(x) is maximum is called mode.

5.6.4 Variance

$= \sigma^2 = E(x^2) - (E(x))^2$

$$\boxed{\Rightarrow \sigma^2 = \int_{-\infty}^{\infty} x^2 \cdot P(x)dx - \left(\int_{-\infty}^{\infty} x \cdot P(x)dx\right)^2}$$

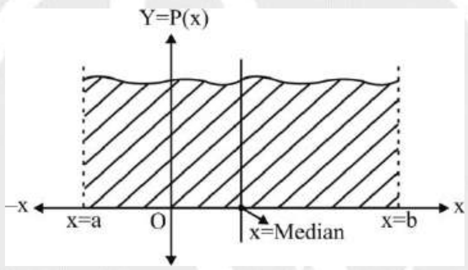


Fig. 5.7. Continuous random variables

5.7. Continous RV distributions

(1) Gaussian/Normal Distributon:

If 'x' is a continuous Random variable with mean 'μ' and standard deviation 'σ', then the probability distribution/density function of normally distributed variable 'x' is given by

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)}$$

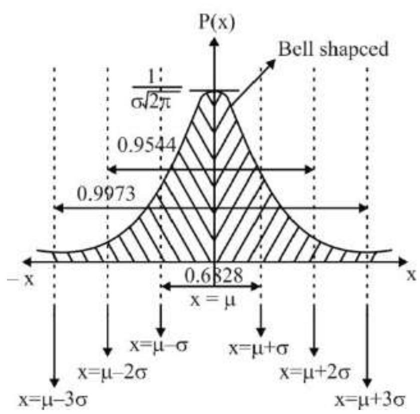


Fig. 5.8. Normal distribution

$$\text{Mean} = \text{Median} = \text{Mode} = \mu$$
$$P(\mu - \sigma \le x \le \mu + \sigma) = 0.6828$$
$$P(\mu - 2\sigma \le x \le \mu + 2\sigma) = 0.9544$$
$$P(\mu - 3\sigma \le x \le \mu + 3\sigma) = 0.9973$$
$$P(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} . e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

(2) Standard Normal Distribution:

Assuming $z = \frac{x-\mu}{\sigma}; \mu = 0; \sigma = 1, P(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-z^2}{2}}$

$$P(-1 \le z \le 1) = 0.6828$$
$$P(-2 \le z \le 2) = 0.9544$$
$$P(-3 \le z \le 3) = 0.9973$$

Note:
1. The normal distribution curve is bell shaped curve
2. The points of infelection of the normal distribution curve are at $x = \mu + \sigma \text{ and } x = \mu - \sigma$.
3. The cumulative function graph is of 'S' Shape.
4. For a given normal distribution, Mean = median = Mode

(3) Uniform Distribution:

If 'x' is a uniformly distrbuted random variable such that $a \le x \le b$ then the Pdf is given by

$$P(x) = \frac{1}{(b-a)}$$

Mean = $\int_a^b x \cdot P(x)dx = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{(b-a)} \int_a^b x \cdot dx$

$$\boxed{\left(\frac{b+a}{2}\right) = \text{Mean}}$$

$\Rightarrow$ $Variance = \sigma^2 = \frac{(b-a)^2}{12}$

Std. deviation = $\sigma = \frac{(b-a)}{\sqrt{12}}$

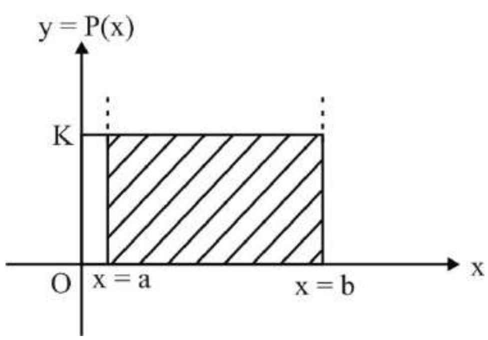


Fig. 5.9. Uniform Distribution

5.7.1 Properties of Mean and Variance:

$E(ax + by) = a \cdot E(x) + b \cdot E(y)$

$$V(ax + by) = a^2 \cdot V(x) + b^2 \cdot V(y) - 2abCOV(x, y)$$

where $COV(x, y) = E(xy) - E(x) \cdot E(y)$

If x,y are indpendent random variables, then $E(xy) = E(x) \cdot E(y) \Rightarrow COV(x, y) = 0$

(1) Exponential Dirtibution:

If 'x' is a continous random variable with mean as $\frac{1}{\lambda}$ then the exponential distribution of 'x' is given by the function

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & ; x \geq 0 \\ 0 & : otherwise \end{cases}$$

Mean = $\frac{1}{\lambda}$

$$\sigma^2 = \frac{1}{\lambda^2}$$

$\boxed{\text{Mean = Standard Deviation} = \frac{1}{\lambda}}$

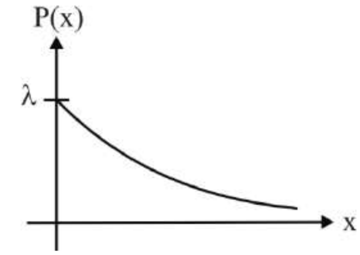


Fig. 5.10. Exponential distribution

5.8. Discrete Random Variable Distributions

If a Random experiment has **only two Possible outcomes**, (one is Success & other is failure) and the Probability of Success doesn't depend on time, then the probability of occuring of exactly 'r-successes' in 'n-trials' is given by

$$P(X = r) = {}^nC_r \cdot P^r \cdot q^{n-r}$$

Where, P $\rightarrow$ Probability of Success,

q $\rightarrow$ Probability of Failure

p + q = 1

Mean = np,Variance = npq = σ^2, standard deviation = $\sigma = \sqrt{npq}$

5.8.1 Poisson Distribution:

If a random experiment has only two possible outcomes, and the average number of successes in a given time 't' is λ, then the probability that exactly 'r' successes occur within the same time 't' given by

$$P(x = r)\frac{e^{-\lambda} \cdot \lambda^r}{r!}$$

Mean = λ.

Mean = Variance = λ

$\Rightarrow \sigma = \sqrt{\lambda}$

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6 COMPLEX CALCULUS

A number of the form $z = x + iy$ where $x, y \in R$ is called a complex number.

x is called real part of z, $x = Re(z)$

y is called imaginary part of z, $y = Im(z)$

6.1. Modulus – Amplitude form of a Complex Number

Every Complex number $z = x + iy$ can be written as $z = r.e^{i\theta}$ where

$r = \sqrt{x^2 + y^2}$ is called the modulus of the complex number and

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is called the amplitude (or) argument of the complex number.

$e^{i\theta} = \cos\theta + i.\sin\theta$ and

$e^{-i\theta} = \cos\theta - i.\sin\theta$

6.2. Arithmetic Operations with Complex Numbers

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers then

(i) $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$

(ii) $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_1y_2 + x_2y_1)$

(iii) $\frac{z_1}{z_2} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$

(iv) $|z_1 + z_2| \le |z_1| + |z_2|$

(v) $|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$

(vi) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

If r_1, θ_1 are the modulus and angle of a complex number z_1 and r_2, θ_2 are the modulus and angle of a complex number z_2 respectively, then

(i) The modulus of $z_1 \cdot z_2$ is $r_1 \cdot r_2$ and the angle of $z_1 \cdot z_2$ is $\theta_1 + \theta_2$

(ii) The modulus of $\frac{z_1}{z_2}$ is $\frac{r_1}{r_2}$ and the angle of $\frac{z_1}{z_2}$ is $\theta_1 - \theta_2$.

If $z = x + iy$ is a complex number, then the conjugate of the complex number is given by z^* (or) $\bar{z} = x - iy$.

$$Re(z) = \frac{z+z^*}{2} \quad \text{and} \quad Img(z) = \frac{z-z^*}{2i}$$

$$z \cdot z^* = |z|^2$$

- **n^th root of unity :**

$Z = (1)^{1/n}$ We write $1 = e^{j2\pi k}$

Thus we get nth root of unity equal to $1, \alpha, \alpha^2, \alpha^3 .... \alpha^{n-1}$. Here $\alpha = e^{i2\pi/n}$

Properties :

1. n^{th} root of unity form a GP
2. Sum of all nth root of unity = 0
3. Product of all n^{th} root of unity is $(-1)^{n-1}$
4. n^{th} root of unity lie on circle of unit radius

$\log_e$ of a complex number

If z = x + iy, to find $\log_e(z)$ we write z in polar form so $z = re^{j\theta}, \log_e(z) = \log_e(re^{i\theta}) = \log_e r + i\theta$.

Since angle $\theta = \theta + 2n\pi$

So $\log(z) = \log_e r + i(\theta + 2n\pi)$

6.3. De-moivre's Theorem

For any complex number x and any integer n,

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

The cube roots of unity when plotted on an argand plane form an equilateral triangle.

6.4. Standard Complex Functions

If $z = x + iy$ is a complex number, then

(i) $\ln z = \frac{1}{2} \cdot \ln(x^2 + y^2) + i \cdot \tan^{-1}\left(\frac{y}{x}\right)$

(ii) $\exp(z) = e^x \cdot (\cos y + i \sin y)$

6.5. Periodic function

A complex function $f(z)$ is a periodic function if there exists a complex number 'k' such that $f(z) = f(z + k)$

Example: The function $f(z) = e^z$ is a periodic function with period $2\pi i$.

6.5.1 Analytic Functions

A function $f(z)$ is said to be analytic at a point $z = z_0$ if the fucntion $f(z)$ is differentiable at the pount $z = z_0$ and also at every point in the neighbourhood of z_0.

The mathematical conditions for a function $f(z) = u(x, y) + i.v(x, y)$ to be analytic at a point $z_0 = x_0 + i\, y_0$ is

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous and differentiable at (x_0, y_0)

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. These set of equations are called Cauchy – Riemann (C-R) Equations .

Note: If the function $f(z) = u(x, y) + i.v(x, y)$ is analytic then

(i) Both $u(x, y)$ and $v(x, y)$ satisfy laplace equation.

i.e. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

(ii) The family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal to each other.

Cauchy – Riemann Equations in polar form for the function $f(z) = u(x, y) + i.v(x, y)$ are given by

$$\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r\cdot\frac{\partial v}{\partial r}$$

6.6. Complex Integration

If $f(z) = u + iv$ is continuous and differentiable at every point along a path 'C' , then the evaluation of $f(z)$ along the pathe 'C' is given by

$$\int_c f(z)dz = \int_c (u+iv)(dx+idy) = \int_c (u\,dx - v\,dy) + i\int_c (u\,dy + v\,dx)$$

Note: If the function $f(z)$ is analytic, then the integral $\int_{z_1}^{z_2} f(z)dz$ is independent of the path connecting the complex numbers z_1 and z_2.

6.6.1 Cauchy Integral Theorem:

If the function $f(z)$ is analytic at everypoint with in a closed path 'C' then $\oint_C f(z)dz = 0$.

Parametric integration of complex function:

Consider a contour C parameterized by z(t) = x(t) + iy(t) for a ≤ t ≤ b. We defined the integral of the complex function along C to be the complex number

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt\,.$$

Note : If F(z) is an analytic function then the integration $\int_a^b F(z)dz$ is independent of path followed to move from point a to b

6.7. Taylor Series and Laurentz Series

(i) **Taylor series:** If the function $f(z)$ is analytic at every point with in a circle with centre at $z = z_0$, then for any point z with in the circle,

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n$$

Where $\quad a_n = \left(\frac{1}{2\pi i}\right) \cdot \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$

(ii) Laurentz Series: If the function $f(z)$ is analytic at every point with in a region bounded by two concetric circles C and C_1 with radii r, r_1 respectively $(r > r_1)$ with centre at $z = z_0$, then for any point z with in the region,

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n + \sum_{n=1}^{\infty} b_n \cdot (z - z_0)^{-n}$$

where $\quad a_n = \left(\frac{1}{2\pi i}\right) \cdot \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$

and $\quad b_n = \left(\frac{1}{2\pi i}\right) \cdot \oint_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$

Note: All the formulaes above for the cyclic integrals are for counter clockwise case by default, if the questions are asked for clockwise case, the answer evaluted using above formulaes should be written with sign change.

6.8. Residue Theorems

6.8.1. Cauchy Residue Theorem:

If f(z) is analytic in closed curve C except at a finite number of singular points within C,

then $\int_C f(z)dz = 2\pi i \times (\text{Sum of the residue at the singular points within } C)$

Note: In above formula the Contour is anti-clockwise. If the Contour is clockwise then we put a –ve sign in RHS of above equation.

Applying Cauchy theorem we have

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + ... + \oint_{C_n} f(z)dz = 2\pi i[\text{Res}[f(a_1)] + \text{Res}[f(a_2)] + ... + \text{Res}[f(a_n)]]$$

6.8.2 Methods of Evaluating Residues

- If f(z) has a simple pole at z = 0 then $\text{Res}[f(a)] = \lim_{z \to a}(z - a) f(z)$
- If $f(z) = \frac{\phi(z)}{\psi(z)}$ where $\Psi(z) = (z – a) f(z)$, $f(a) \neq 0$

$$\text{Res } f(a) = \frac{\phi(a)}{\Psi'(a)}$$

- If f(z) has a pole of order n at z = a, then Res $f(a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{Z=a}$

6.9. Singularities of an Analytic function and It's Type

A point at which the function ceases to be analytic is called singular point of functions.

6.9.1. Type of Singularities

- **Removable singularity:** A removable singularity is a singular point z_0 of a function f(z) for which it is possible to assign a complex number in such a way that f(z) becomes analytic. A more precise way of defining a removable singularity is as a singularity z_0 of a function f(z) about which the function f(z) is bounded. For example, the point $x_0 = 0$ is a removable singularity in the sin c function sin c(x) = sin x/x, since this function satisfies sin c(0) = 1.
- **Isolated singularity:** A point a in the domain D of function f(z) is said to be a point of isolated singularity, if f(z) is analytic at each point in some neighbourhood $|z - a| < R$ of a, except not being analytic at a.
- **Pole:** Pole is another kind of singularity defined as if there exist a positive integer m such that $z \to a(z-a)^m f(z) \neq 0$, then z = a is called a pole of order m.

  For example, let $f(z) = 1/(z-5)^3$, then

  $$z \to 5(z-5)^3 \frac{1}{(z-5)^3} = 1 \neq 0$$

- **Isolated Essential Singularity:** The definition of isolated essential singularity is, if their does not exist a finite value m such that $z \to (z-a)^m f(z) = k$, where k is a non-zero finite constant. The point z = a is called an isolated essential singularity.

  For example, let f(z) = sin[1/(z – a)] where $\sin[1/(z-a)] = 1/(z-a) - 1/(z-a)^3\, 3! + 1/(z-a)^5 5! - \ldots$

  Here the function has infinite terms in negative power of (z – a), so it is not possible to find a finite value of m.

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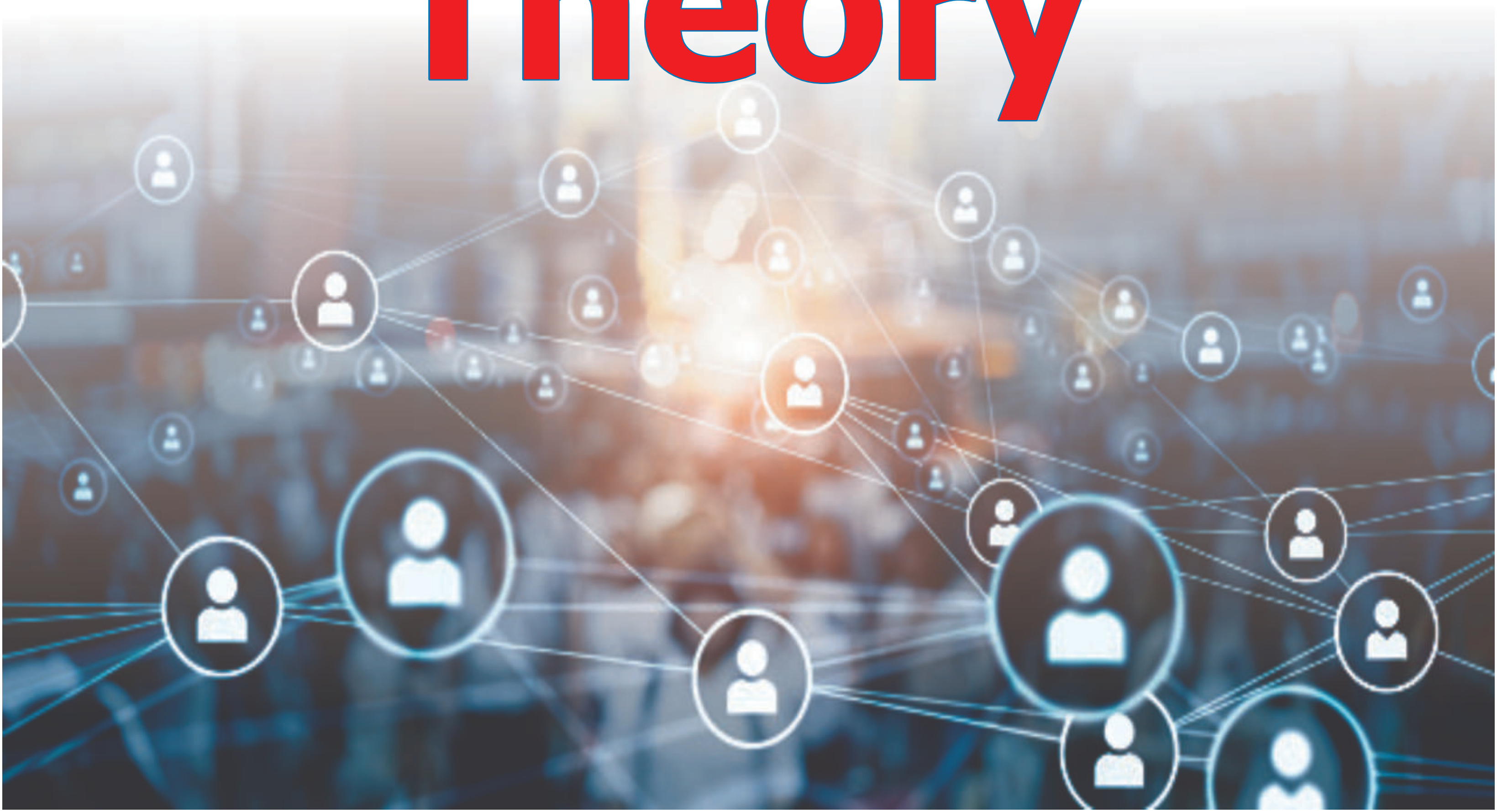
Network Theory



NETWORK THEORY

INDEX

1 BASIC CONCEPTS OF NETWORKS

1.1 Introduction

Network theory is the study of solving the problems of electric circuits or electric networks.

An electric circuit contains a closed path for providing a flow of electrons from a voltage source or current source. The elements present in an electric circuit will be in **series connection, parallel connection,** or in any combination of series and parallel connections and an electric network need not contain a closed path for providing a flow of electrons from a voltage source or current source. Hence, we can conclude that "all electric circuits are electric networks" but the converse need not be true.

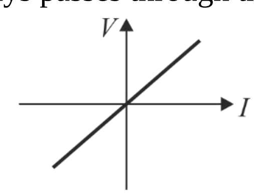
1.1.1. Types of Network Elements

Different types of network elements are

1. Linear Elements and Non-linear Elements.
2. Bilateral Elements and Unilateral Elements.
3. Active Elements and Passive Elements.
4. Time Invariant and Time Variant Elements.
5. Lumped and Distributed Elements.

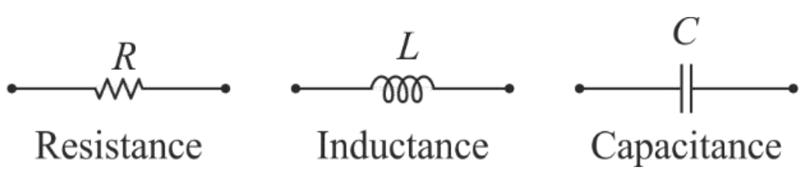
1. Linear Elements

Characteristics of linear elements always passes through the origin in the form of straight line.



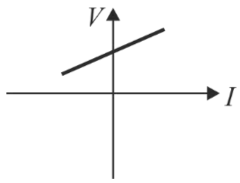
Linear

Example of linear elements:

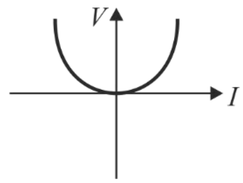


All basic electrical elements are linear (R, L, C).

2. Non-Linear Elements

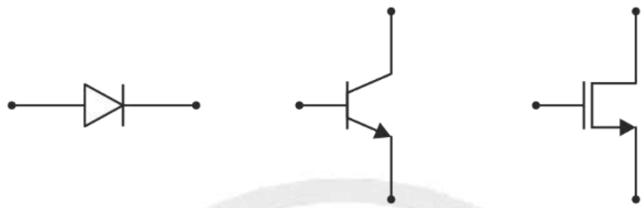


Non linear



Non linear

Example of non-linear elements :



Diode BJT MOSFET

All electronic devices are non-linear (Diode, MOSFET, JFET).

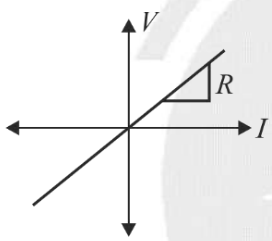


Fig. *V-I* plane

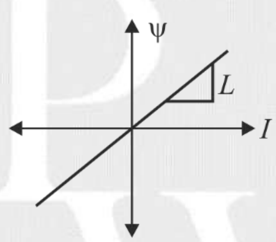


Fig. Ψ-*I* plane

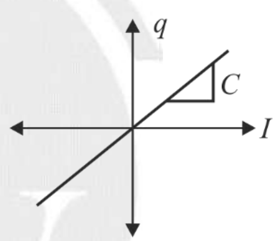
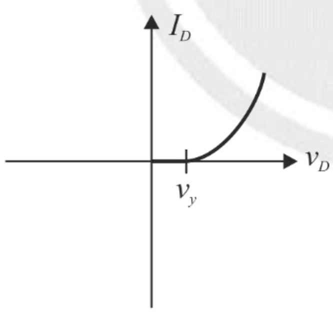
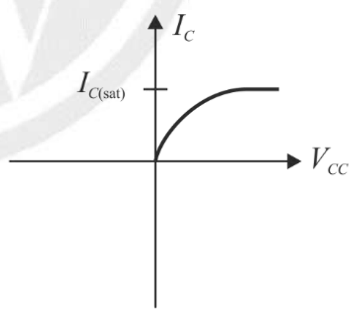


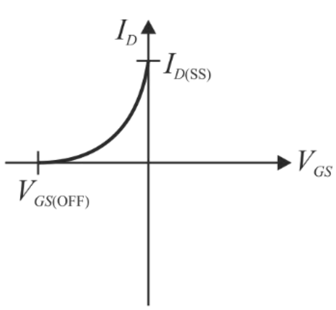
Fig. *q-v* plane



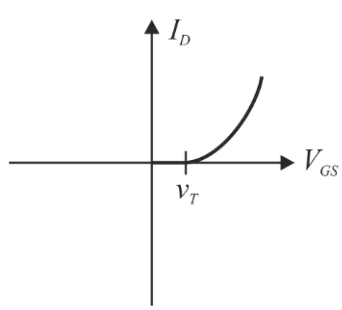
Feedback characteristics of diode



Output characteristics of BJT



Transfer characteristics of JFET



Forward characteristics of MOSFET

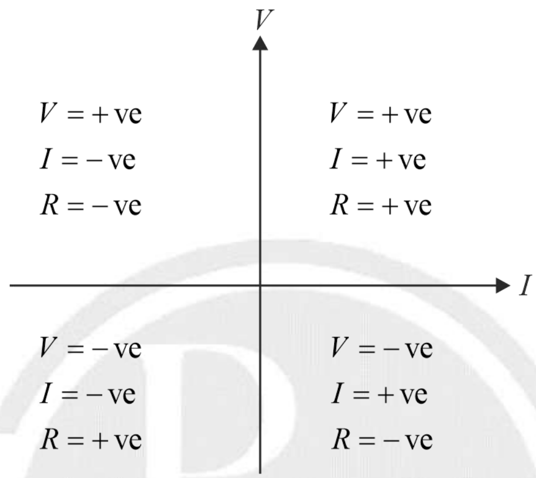
2. Bilateral and Unilateral Elements

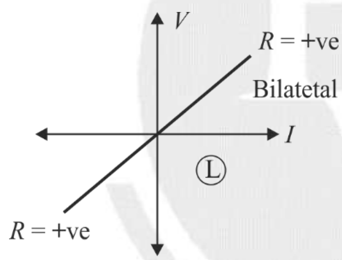
(i) In case of V-I plane, characteristics of **bilateral** elements offer same impedance throughout the characteristics.

(ii) In case of V-I plane, characteristics of **unilateral** elements offer different impedance in different origin.

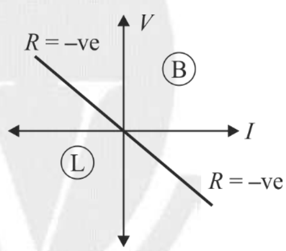
(iii) In case of generalized plane, characteristics of **bilateral** elements is always symmetrical about origin.

Characteristics are same in either I and III quadrant or II and IV quadrant.

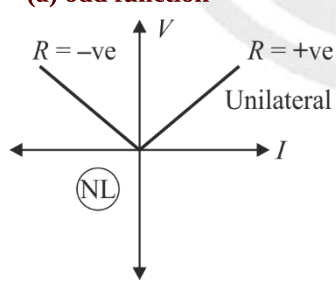




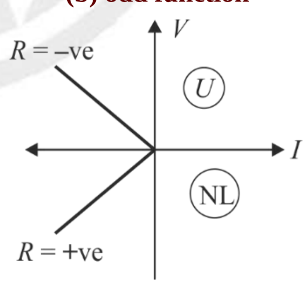
(a) odd function



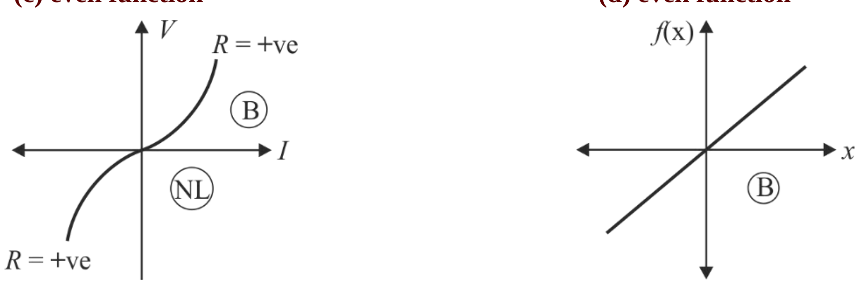
(b) odd function



(c) even function



(d) even function



(e) odd function

Note: L = Linear, NL = Non-linear, B = Bilateral, U = Unilateral

All linear elements are bilateral but reverse is not true.

Network element	Bilateral/Unilateral
R	Bilateral
L	Bilateral
C	Bilateral

Device Element	Bilateral/Unilateral
Diode	Unilateral
BJT	Unilateral
JFET	Unilateral

3. Active and Passive Elements

(i) In case of V-I plane, characteristics of passive element always have positive impedance where as active element offeres negative impedance.

(ii) Passive elements absorb the energy whereas active elements deliver the energy.

(iii) Active element controls, the flow of energy whereas passive elements dissipate or store the energy.

(iv) Elements having capability of delivering the energy is referred as active elements.

Examples of active elements :

(a) Voltage source

(b) Current source

(c) Generator

(d) Biased Transistors

(e) Operational Amplifier

Note: All dependent sources are considers as an active element.

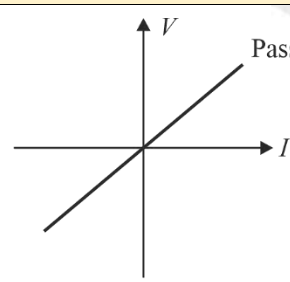


Fig. (a)

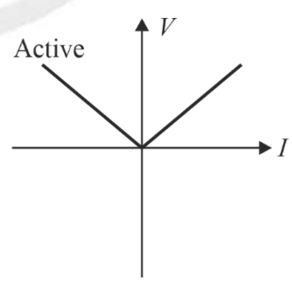


Fig. (b)

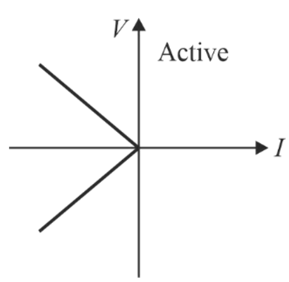


Fig. (c)

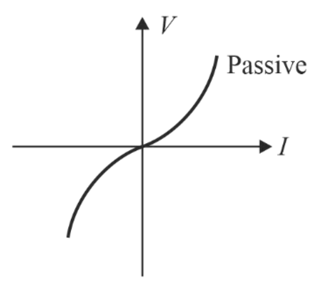
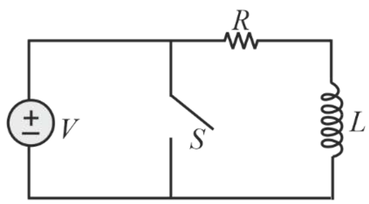


Fig. (d)

Generally, inductors and capacitors are passive elements. As they can not deliver the energy independently for long time? They can give energy only at the time of discharge.



Condition: At $t = 0$, S is closed

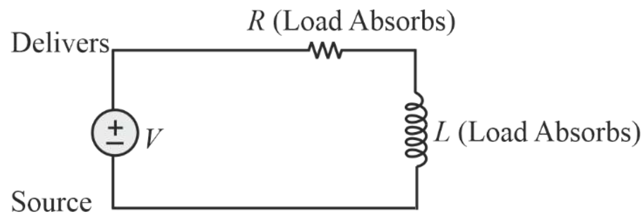


Fig. (i) $t < 0$
Source RL network
(charging RL network)

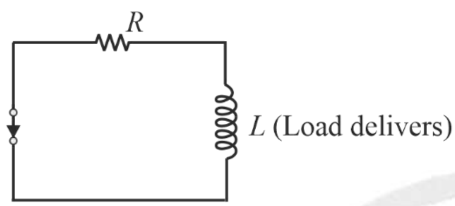


Fig. (iii) $t > 0$
Source RL network
(discharging RL network)

4. Lumped and Distributed Elements

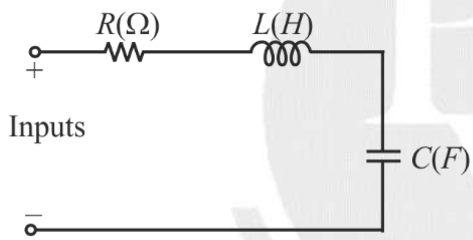


Fig. RLC Network

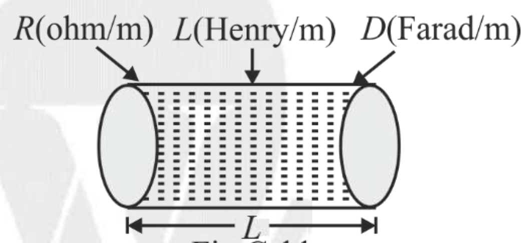


Fig Cable

(i) Physically separated elements in a network is referred as lumped elements.

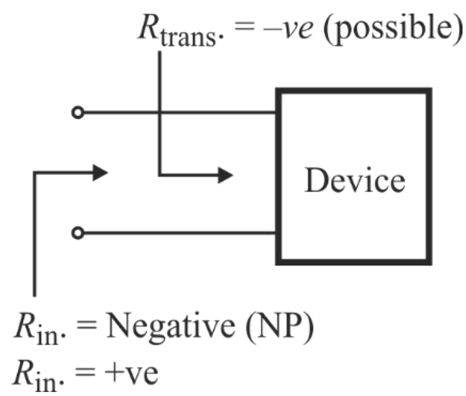
(ii) If elements are distributed along the line, then it is referred as distributed elements.

(iii) Concept of circuit theory is based on lumped elements.

(iv) Concept of field theory is based on distributed elements.

Network Elements	Passive Condition	Active Condition
R	$R \geq 0$	$R < 0$
L	$L \geq 0$	$L < 0$
C	$C \geq 0$	$C < 0$

<table>
<tr><td rowspan="4">$R = 2\ \Omega$ (Passive)
$R = 4\ k\Omega$ (Passive)
$R = -6\ k\Omega$ (NP)
L = 1 mH (Passive)
L = –2 μH (NP)</td><td>Network Theory</td></tr>
<tr><td>$R \geq 0$</td></tr>
<tr><td>$L \geq 0$</td></tr>
<tr><td>$C \geq 0$</td></tr>
</table>



5. Time Invariant and Time Variant Element

(i) If characteristics of elements is varied with time, then it is called **Time Variant.**

(ii) If characteristics of elements is not varied with time, then it is called **Time Invariant.**

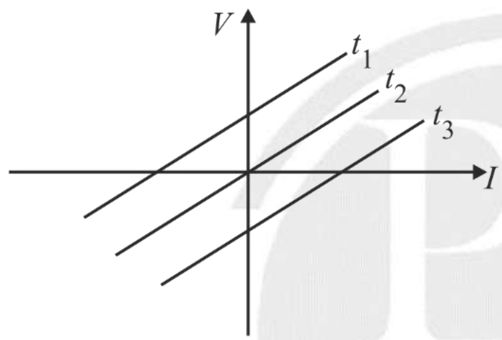


Fig. T-V characteristics

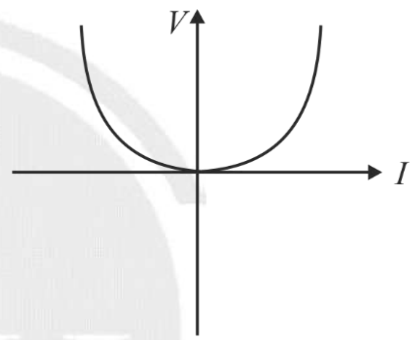


Fig. T-V characteristics

Network Element	TI/TV
R	TI
L	TI
C	TI

1.2. Analysis of Passive Elements

Resistor :

The main functionality of Resistor is either opposes or restricts the flow of electric current.

Relationship between resistance and resistivity is given by,

$$R = \frac{\rho l}{A}$$

$\rho =$ resistivity of the material

$l =$ length of wire

$A =$ area of cross section of the wire

According to Ohm's law, the voltage across resistor is the product of current flowing through it and the resistance of that resistor, provided the temperature is constant.

Mathematically, it can be represented as

$$V = IR$$

$$I = \frac{V}{R}$$

Where, R is the resistance of a resistor and unit of R is ohm (Ω).

From field theory of ohm's law, (Microscopic form of Ohm's Law)

$$J \propto E$$

$$J = \sigma E$$

$J =$ Current density

$E =$ Electric field intensity

$\sigma =$ Conductivity of materials

R can range from zero to infinity, the two extreme possible values. An element with $R = 0$ is called as short circuit, as shown in figure. For a short circuit, $V = IR = 0$.

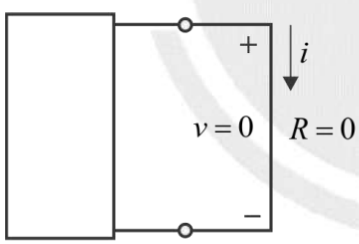


Fig. (A) A short-circuit (*R* = 0)

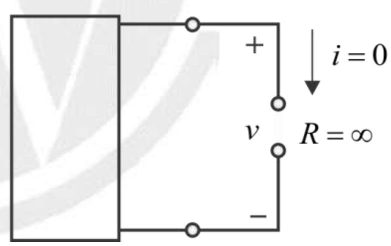


Fig. (b) An open circuit (R = ∞)

Similarly, an element with resistance $R = \infty$ is known as an open circuit, as shown in figure (b).
For an open circuit,

$$I = \frac{V}{R} = 0$$

Network Condition	I	V	R
SC	I_{sc}	0	0
OC	0	V_{sc}	∞

Current :

The current "I" flowing through a conductor is nothing but the time rate of flow of positive charge. Mathematically, it can be written as,

$$i(t) = \frac{dQ}{dt} \qquad \text{[C/sec = Amp]}$$

$$Q = \int i(t)\,dt$$

Q is the charge and its unit is Coulomb.

t is the time and its unit is Second.

I is the current and its unit is Ampere.

Voltage/Potential :

The voltage "V" is nothing but an electromotive force that causes the charge (electrons) to flow.

$$V = \frac{dW}{dQ}$$

$$W = \int V(Q)dQ$$

W is the potential energy and its unit is Joule.

Q is the charge and its unit is Coloumb.

V is the voltage and its unit is Volt.

Power :

The power "P" is nothing but the time rate of flow of electrical energy. Mathematically, it can be written as

$$P = \frac{dW}{dt}$$

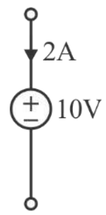
$$P = \frac{dW}{dt} = \frac{dW}{dQ} \times \frac{dQ}{dt}$$

$$P = VI$$

Concept of Absorbed and Delivered Power :

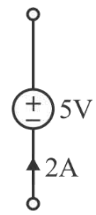
1. If current enters into the positive terminal of voltage source, then it is referred as absorbed power (i.e. power is absorbed by voltage source).
2. If current leaves from positive terminal of voltage source, then it is referred as delivered power (i.e. power is delivered by voltage source).

Examples :

(i) 

$$P_{absorbed} = 2 \times 10 = 20 \text{ watt}$$

$$P_{delivered} = -20 \text{ watt}$$

(ii) 

$$P_{delivered} = 2 \times 5 = 10 \text{ watt}$$

$$P_{absorbed} = -10 \text{ watt}$$

(iii)

$P_{delivered} = 3 \times 4 = 12$ watt

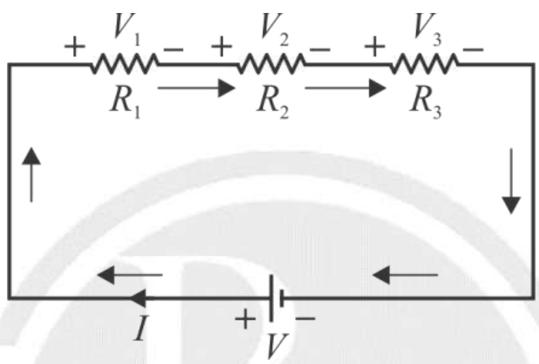
$P_{absorbed} = -12$ watt

(iv)

$P_{absorbed} = 4 \times 2 = 8$ watt

$P_{delivered} = -8$ watt

Series Equivalent Circuit :



$R_{eq} = R_1 + R_2 + R_3$

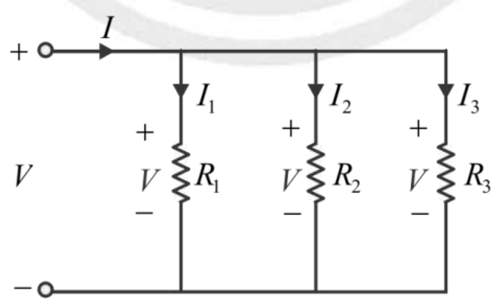
Concept :

(i) $I_1 = I_2 = I_3 = I$

(ii) $R_{eq} = R_1 + R_2 + R_3$

(iii) $R_{eq} > R_{max}$

Parallel Equivalent circuit :



$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Concept :

(i) $V_1 = V_2 = V_3 = V$

(ii) $R_{eq} = \dfrac{R_1 R_2 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3}$

(iii) $R_{eq} < R_{min}$

Inductor :

Inductance is the property of inductor which opposes the change of current.

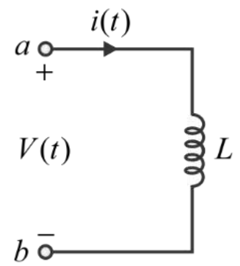
$$I_L(0^+) = I_L(0^-) = I_L(0)$$

$$\psi(t) \propto i(t)$$

$$\psi = Li$$

$$L = \frac{\psi}{i}$$

$$\psi = N\phi$$



$\psi =$ Total magnetic flux (Flux linkage of a coil)

$N =$ Number of turns in coil

$\phi =$ flux per turn

$$L = \frac{N\phi}{i}$$

$$\phi = \frac{MMF}{Reluctance} = \frac{Ni}{l/A\mu}$$

$$L = \frac{N^2 A\mu}{l}$$

$$L \propto N^2$$

$$V(t) = \frac{d\psi}{dt}$$

$\because$ $$\psi = Li$$

$$V(t) = L\frac{di(t)}{dt}$$

$$i(t) = \frac{1}{L}\int V(t)\,dt$$

Energy in inductors :

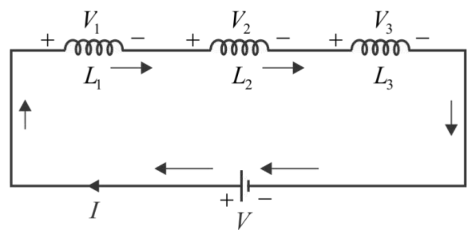
Energy, $$W = \int P(t)\,dt$$

$$W = \frac{1}{2}Li^2 \text{ Joule(J)}$$

Note :

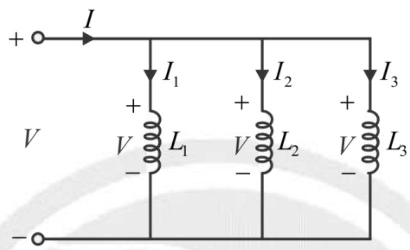
1. Inductor stores the energy in the form of magnetic fields.
2. It is referred as absorbing element like resistor.

Series Equivalent Circuit :



$$L_{eq} = L_1 + L_2 + L_3$$

Parallel Equivalent circuit :



$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

Concept :

(i) $V_1 = V_2 = V_3 = V$

(ii) $L_{eq} = \dfrac{L_1 L_2 L_3}{L_1 L_2 + L_2 L_3 + L_3 L_1}$

(iii) $L_{eq} < L_{\min}$

Capacitor :

Capacitance is the property of capacitor that opposes the change of voltage $V(0^-) = V(0^+)$ of the movement time.

$$q \propto V$$

$$q = CV$$

$$C = \frac{q}{V} \left[\frac{\text{Columb (C)}}{\text{Volt (V)}} \text{ or Farad (F)} \right]$$

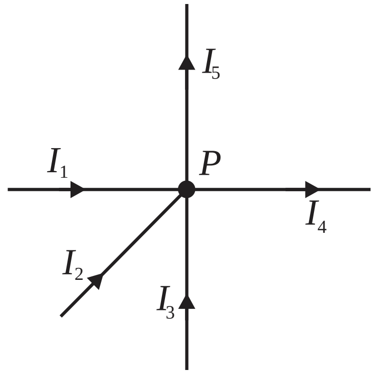
$$i(t) = \frac{dq}{dt}$$

$$i(t) = C\frac{dV(t)}{dt}$$

Kirchoff's Current Law (KCL) :

- In DC circuit KCL states that the algebraic sum of currents entering a node (or a closed boundary) is zero.
- Mathematically, KCL implies that

$$\sum_{n=1}^{N} i_n = 0$$



Applying KCL at node P,

$$-I_1 - I_2 - I_3 + I_4 + I_5 = 0$$

$$I_1 + I_2 + I_3 = I_4 + I_5$$

Sum of incoming currents = Sum of outgoing currents

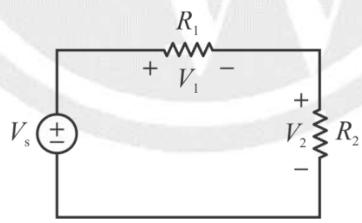
Nodal Analysis :

- It is combination of KCL and Ohm's law (KCL + Ohm's law).
- In Nodal analysis, we will consider the node voltages with respect to Ground. Hence, Nodal analysis is also called as Node-voltage method.

Kirchoff's Voltage Law (KVL) :

- KVL states that the algebraic sum of all voltages around a closed path (or loop) is zero.
- Mathematically, KVL states that

$$\sum_{m=1}^{M} v_m = 0$$



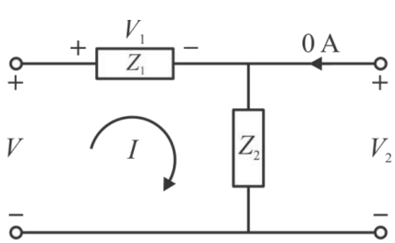
Applying KVL in the above circuit,

$$V_s - V_1 - V_2 = 0$$

$$V_s = V_1 + V_2$$

Sum of voltage drops = Sum of voltage rises

Voltage Division Rule [VDR]



It is used in series equivalent circuit for distribution of voltage.

$$V_1 = \frac{V \times Z_1}{Z_1 + Z_2}$$

$$V_2 = \frac{V \times Z_2}{Z_1 + Z_2}$$

Voltage division rule for Resistor :

$$Z[R_1] = R_1 \qquad Z[R_2] = R_2$$

$$\therefore \quad V_1 = \frac{V \times R_1}{R_1 + R_2} \qquad V_2 = \frac{V \times R_2}{R_1 + R_2}$$

Voltage division rule for Inductor :

$$Z[L_1] = j\omega L_1 \qquad Z[L_2] = j\omega L_2$$

$$\therefore \quad V_1 = \frac{V \times j\omega L_1}{j\omega L_1 + j\omega L_2} \qquad V_2 = \frac{V \times j\omega L_2}{j\omega L_1 + j\omega L_2}$$

$$V_1 = \frac{V \times L_1}{L_1 + L_2} \qquad V_2 = \frac{V \times L_2}{L_1 + L_2}$$

Voltage division rule for Capacitor:

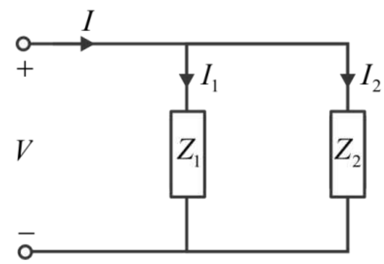
$$Z[C_1] = \frac{1}{j\omega C_1} \qquad Z[C_2] = \frac{1}{j\omega C_2}$$

$$\therefore \quad V_1 = \frac{V \times \frac{1}{j\omega C_1}}{\frac{1}{j\omega C_1} + \frac{1}{j\omega C_2}} \qquad V_2 = \frac{V \times \frac{1}{j\omega C_2}}{\frac{1}{j\omega C_1} + \frac{1}{j\omega C_2}}$$

$$V_1 = \frac{V \times C_2}{C_1 + C_2} \qquad V_2 = \frac{V \times C_1}{C_1 + C_2}$$

Current Division Rule [CDR]

CDR is applicable when passive elements and connected in parallel.



$$I_1 = \frac{I \times Z_2}{Z_1 + Z_2}, \quad I_2 = \frac{I \times Z_1}{Z_1 + Z_2}$$

Current division rule for Resistor :

$$Z[R_1] = R_1 \qquad Z[R_2] = R_2$$

$$\therefore \quad I_1 = \frac{I \times R_2}{R_1 + R_2} \qquad I_2 = \frac{I \times R_1}{R_1 + R_2}$$

Current division rule for Inductor :

$$Z[L_1] = j\omega L_1 \qquad Z[L_2] = j\omega L_2$$

$$\therefore \quad I_1 = \frac{I \times j\omega L_2}{j\omega L_1 + j\omega L_2} \qquad I_2 = \frac{I \times j\omega L_1}{j\omega L_1 + j\omega L_2}$$

$$I_1 = \frac{I \times L_2}{L_1 + L_2} \qquad I_2 = \frac{I \times L_1}{L_1 + L_2}$$

Current division rule for Capacitor :

$$Z[C_1] = \frac{1}{j\omega C_1} \qquad Z[C_2] = \frac{1}{j\omega C_2}$$

$$\therefore \quad I_1 = \frac{I \times \frac{1}{j\omega C_2}}{\frac{1}{j\omega C_1} + \frac{1}{j\omega C_2}} \qquad I_2 = \frac{I \times \frac{1}{j\omega C_1}}{\frac{1}{j\omega C_1} + \frac{1}{j\omega C_2}}$$

$$I_1 = \frac{I \times C_1}{C_1 + C_2} \qquad I_2 = \frac{I \times C_2}{C_1 + C_2}$$

Star to Delta Conversion [Y to Δ] or [T to π] Conversion

Star to Delta conversion [T to π] :

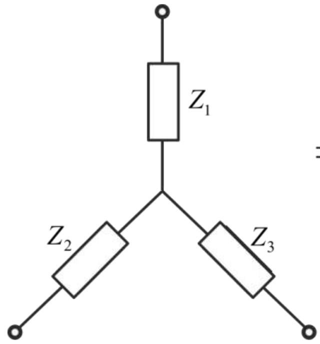


Fig. Star network

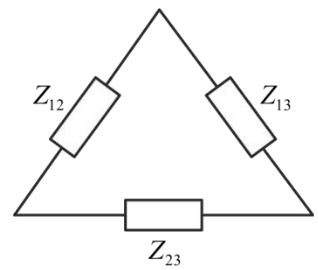


Fig. Delta network

Or

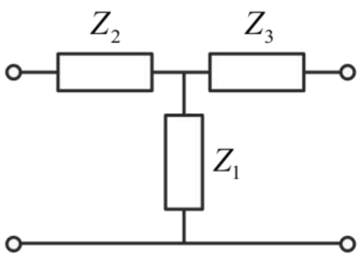


Fig. *T* - network

$\Rightarrow$

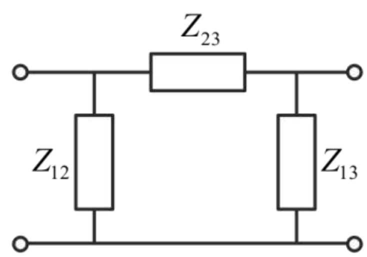


Fig. π- network

$$Z_{12} = \frac{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}{Z_3} \qquad Z_{23} = \frac{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}{Z_1} \qquad Z_{13} = \frac{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}{Z_2}$$

If $Z_1 = Z_2 = Z_3 = Z$ then $Z_{12} = Z_{23} = Z_{13} = 3Z$

$$Z_{eq} = 3Z \qquad R_{eq} = 3R \qquad L_{eq} = 3L \qquad C_{eq} = C/3$$

Delta to Star Conversion [Δ to Y] or [π to T] Conversion

Delta to Star conversion [π to *T*] :

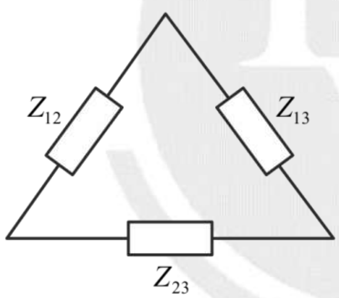


Fig. Delta network

$\Rightarrow$

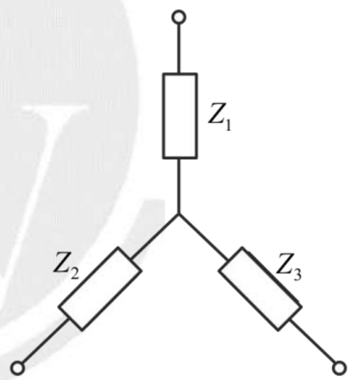


Fig. Star network

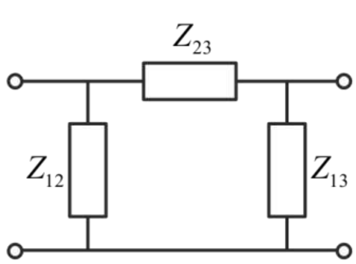


Fig.π- network

$\Rightarrow$

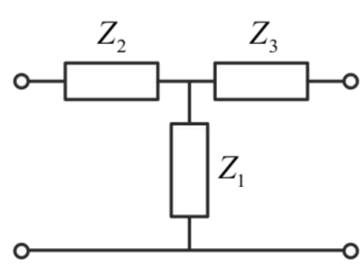


Fig. *T* - network

$$Z_1 = \frac{Z_{12}Z_{13}}{Z_{12} + Z_{23} + Z_{13}} \qquad Z_2 = \frac{Z_{12}Z_{23}}{Z_{12} + Z_{23} + Z_{13}} \qquad Z_3 = \frac{Z_{23}Z_{13}}{Z_{12} + Z_{23} + Z_{13}}$$

If $Z_{12} = Z_{23} = Z_{13} = Z$ then $Z_1 = Z_2 = Z_3 = \dfrac{Z}{3}$

In case of same impedance, delta to star conversion decreases the impedance by factor of 3. (Increases if the element is capacitance by the same factor of 3).

$$Z_{eq} = \frac{Z}{3} \qquad R_{eq} = \frac{R}{3} \qquad L_{eq} = \frac{L}{3} \qquad C_{eq} = 3C$$

Sources

1. Ideal Voltage Source :

An ideal voltage source is a device which has a constant voltage independent of current delivered by it. Ideally, it has zero internal resistance.

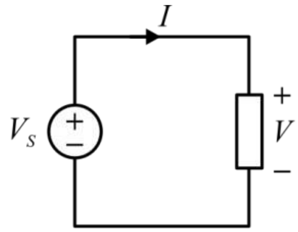


Fig. Ideal voltage source

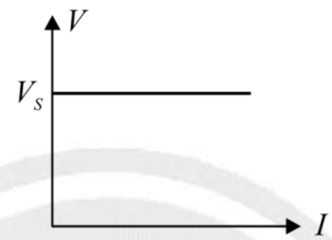


Fig. *V-I* characteristics

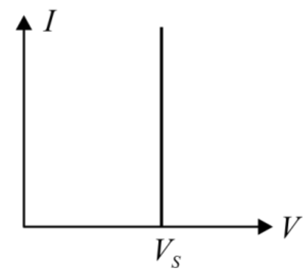


Fig. *I-V* characteristics

2. Practical Voltage Source :

An practical voltage source is a device which has a constant voltage with non-zero internal resistance dependent on the current supplied by the source. Practically its internal resistance should be as small as possible.

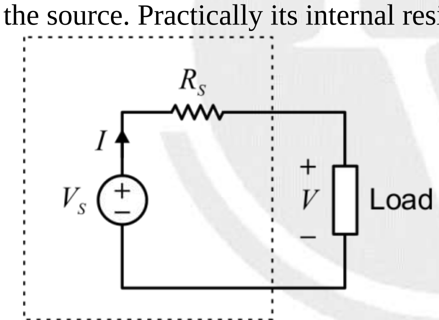


Fig. Practical voltage source

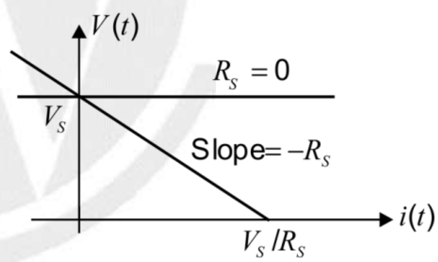


Fig. *V-I* characteristics

3. Ideal Current Source :

An ideal current source is a device which delivers a constant current to any load independent of the voltage across it.

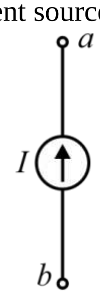


Fig. ideal voltage source

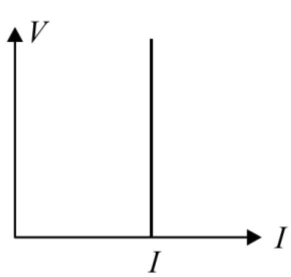


Fig. *V-I* characteristics

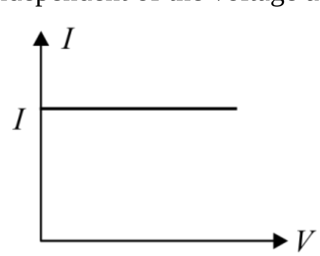


Fig. *I-V* characteristics

4. Practical Current Source :

A practical current source is a device which delivers a constant current to any load independent of the voltage across the source. A practical current source has finite internal resistance.

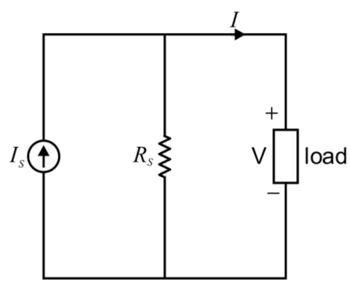


Fig. Practical current source

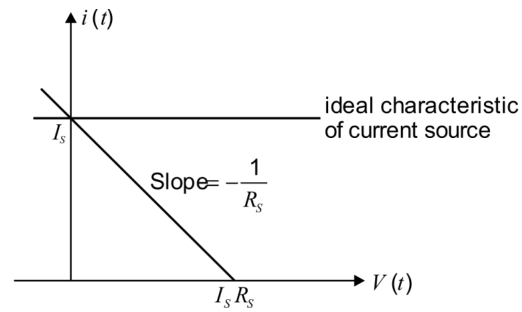
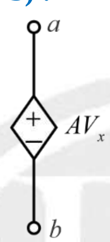


Fig. *I-V* characteristics

Dependent Source :

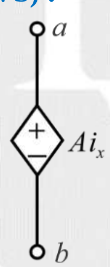
1. A voltage controlled voltage source (VCVS) :



Here, A is unitless

$$V = AV_x$$

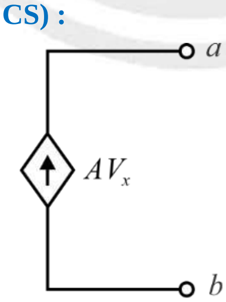
2. A current controlled voltage source (CCVS) :



Unit of A is Ohm.

$$V = Ai_x$$

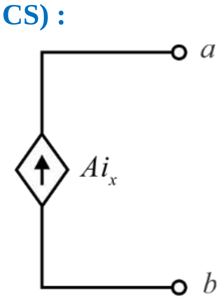
3. A voltage controlled current source (VCCS) :



Here, A is unitless

$$I = AV_x$$

4. A current controlled current source (CCCS) :



Unit of A is Mho.

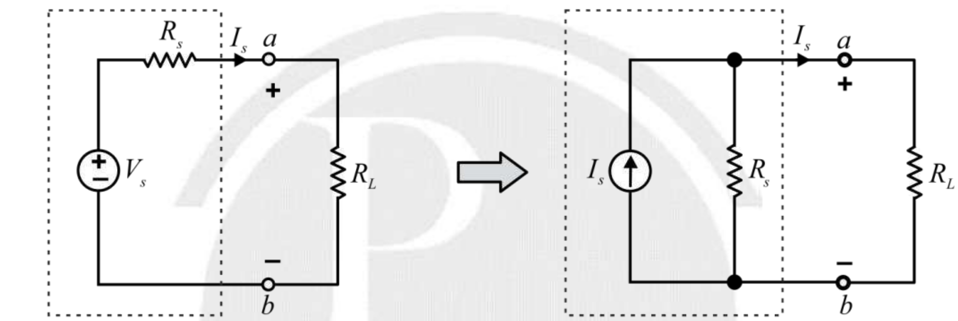
$$I = Ai_x$$

Source/device	Practically Internal resistance	Ideally Internal Resistance
1. Voltage source	$R_s = \text{small}$	$R_s = 0$
2. Current source	$R_s = \text{high}$	$R_s = \infty$
3. Voltmeter	$R_m = \text{high}$	$R_m = \infty$
4. Ammeter	$R_m = \text{small}$	$R_m = 0$

Source Transformation

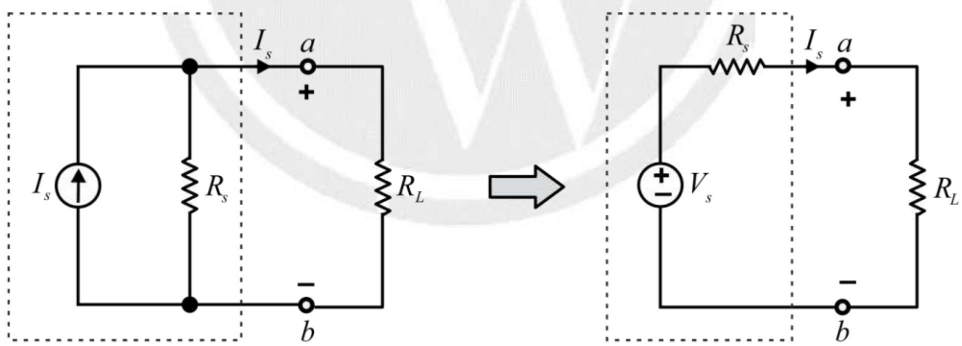
Practical voltage source into a practical current source :

- It states that an independent voltage source V_s in series with a resistance R_s is equivalent to an independent current source, $(I_s = V_s / R_s)$ in parallel with a resistance R_s.



Practical voltage source Practical current source

Practical current source into a practical voltage source :



Practical current source Practical voltage source

Note: Source transformation is not possible in Ideal Source.

Average and RMS Value of Periodic Waveform

Average Value / DC value / Mean value :

$$\text{Average } [x(t)] = \frac{1}{T}\int_0^T x(t)\,dt$$

Average value can be defined with the help of area,

$$\text{Average } [x(t)] = \frac{\text{Area in one period}}{\text{Time period}\,(T)}$$

RMS value / Effective value :

$$\text{RMS }[x(t)] = \text{ Effective }[x(t)] = \sqrt{\frac{1}{T}\int_0^T x^2(t)\,dt}$$

1. Average value can be negative, positive or zero but RMS value is always a positive number.
2. $Avg\begin{bmatrix} A\sin\omega t / A\cos\omega t / A\sin 2\omega t / A\cos 2\omega t / A\sin n\omega t / A\cos n\omega t / \\ A\sin(n\omega t \pm n\phi) / A\cos(n\omega t \pm n\phi) \end{bmatrix} = 0$
3. $RMS\begin{bmatrix} A\sin\omega t / A\cos\omega t / A\sin 2\omega t / A\cos 2\omega t / A\sin n\omega t / A\cos n\omega t / \\ A\sin(n\omega t \pm n\phi) / A\cos(n\omega t \pm n\phi) \end{bmatrix} = \frac{A}{\sqrt{2}}$
4. If $x(t) = DC + AC\sin\omega t$

$$RMS[x(t)] = \sqrt{(DC)^2 + \left(\frac{AC}{\sqrt{2}}\right)^2}$$

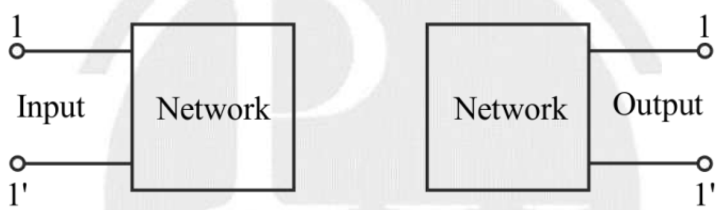
❑❑❑

2 TWO PORT NETWORK

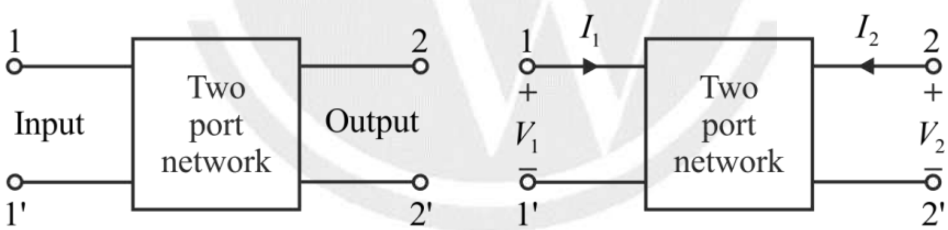
2.1. One Port Network

One port network is a two terminal electrical network in which, current enters through one terminal and leaves through another terminal. i.e. Resistors, inductors and capacitors are the one port network.

Example: Motor, Generator etc.



Two Port Network :



Maximum number of possible parameters for analysis of any port network is given by:

1. Z parameter (impedance parameter)
2. Y parameter (admittance parameter)
3. *h*-parameter (Hybrid parameter)
4. *g*-parameter (Inverse Hybrid parameter)
5. ABCD parameter (Transmission parameter)
6. $[ABCD]^{-1}$ parameter (Inverse transmission parameter)

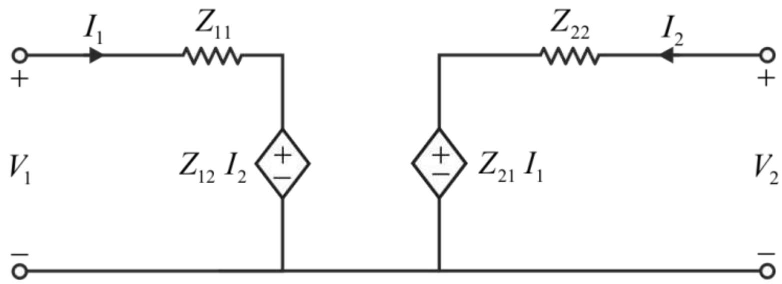
Z Parameter (Impedance Parameter)

In Z-parameter, voltage (V_1, V_2) is the dependent variable and current (I_1, I_2) is the independent variable, the equations of Z – parameter can be written by,

$$V_1 = Z_{11}I_1 + Z_{12}I_2 \quad \text{...(i)}$$

$$V_2 = Z_{21}I_1 + Z_{22}I_2 \quad \text{...(ii)}$$

Circuit diagram of Z parameter,



From equation (i) & (ii),

Case (i) When output port is open circuit, i.e. $I_2 = 0$,

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \text{Driving point input impedance } (\Omega)$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \text{Transfer output impedance } (\Omega)$$

Case (ii) When input port is open circuit, i.e. $I_1 = 0$,

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \text{Transfer input impedance } (\Omega)$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \text{Driving point output impedance } (\Omega)$$

Equation (i) & (ii) can be written in the matrix form,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}_{2\times1} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}_{2\times2} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}_{2\times1}$$

$$[V]_{2\times1} = [Z]_{2\times2}[I]_{2\times1}$$

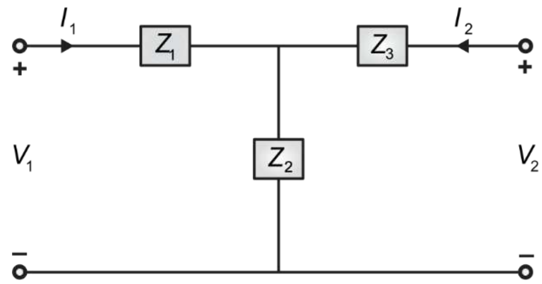
$$\mathbf{[Z]_{2\times2} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}}$$

Condition for Symmetry: $Z_{11} = Z_{22}$

Condition for Reciprocity: $Z_{12} = Z_{21}$

Z-parameter is referred as open circuit impedance parameter.

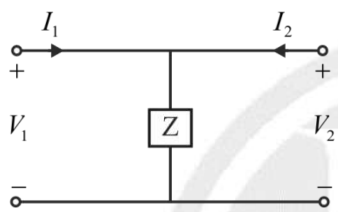
Standard Z Parameter Expression for T Network:



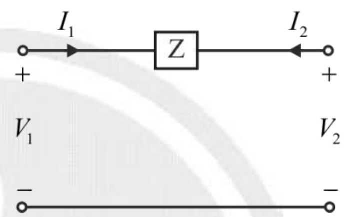
Hence, $$[Z]=\begin{bmatrix} Z_1+Z_2 & Z_2 \\ Z_2 & Z_2+Z_3 \end{bmatrix}$$

Standard Z Parameter Expression for Single Series and Shunt Element:

(i)

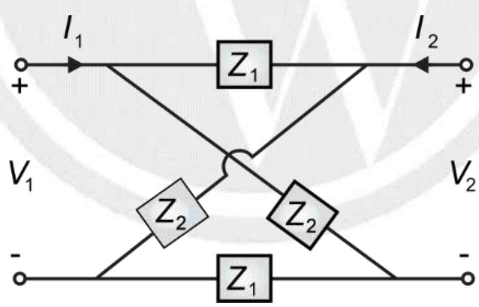


(ii)



$$[Z]=\begin{bmatrix} Z & Z \\ Z & Z \end{bmatrix} \qquad [Z]=\begin{bmatrix} \infty & \infty \\ \infty & \infty \end{bmatrix}$$

Standard Z Parameter Expression for Symmetric Lattice Network :



$$[Z]=\begin{bmatrix} \dfrac{Z_1+Z_2}{2} & \dfrac{Z_2-Z_1}{2} \\ \dfrac{Z_2-Z_1}{2} & \dfrac{Z_1+Z_2}{2} \end{bmatrix}$$

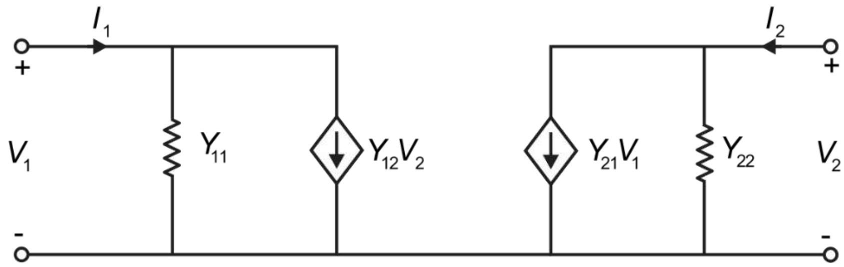
Y Parameter (Admittance Parameter)

In Y-parameter, current (I_1, I_2) is the dependent variable and voltage (V_1, V_2) is the independent variable, the equations of Y – parameter can be written by,

$$I_1 = Y_{11}V_1 + Y_{12}V_2 \qquad \text{...(i)}$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2 \qquad \text{...(ii)}$$

Circuit diagram of Y parameter,



From equation (i) & (ii),

Case (i) When output port is short circuit, i.e. $V_2 = 0$,

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \text{Driving point input admittance } (\mho)$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \text{Transfer output admittance } (\mho)$$

Case (ii) When input port is short circuit, i.e. $V_1 = 0$,

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \text{Transfer input admittance } (\mho)$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \text{Driving point output admittance } (\mho)$$

Equation (i) & (ii) can be written in the matrix form,

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}_{2\times1} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}_{2\times2} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}_{2\times1}$$

$$[I]_{2\times1} = [Y]_{2\times2}[V]_{2\times1}$$

$$[Y]_{2\times2} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

Condition for Symmetry: $Y_{11} = Y_{22}$

Condition for Reciprocity: $Y_{12} = Y_{21}$

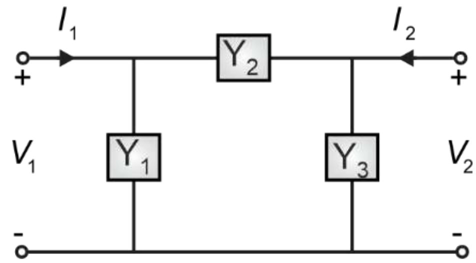
Relationship between Z and Y Parameter:

The relation between Z and Y parameter is given by,

$$[Z] = [Y]^{-1}$$

$$[Y] = [Z]^{-1}$$

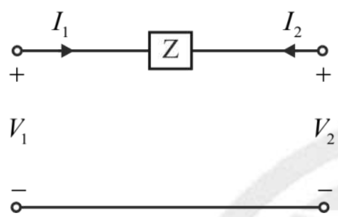
Standard Y Parameter for ∏-Network:



Hence, $$[Y]_{2\times 2} = \begin{bmatrix} Y_1 + Y_2 & -Y_2 \\ -Y_2 & Y_2 + Y_3 \end{bmatrix}$$

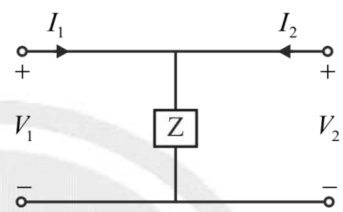
Standard Y Parameter for Single Series and Shunt Element:

(i)



$$[Y]_{2\times 2} = \begin{bmatrix} 1/Z & -1/Z \\ -1/Z & 1/Z \end{bmatrix}$$

(ii)



$$Y = \begin{bmatrix} \infty & \infty \\ \infty & \infty \end{bmatrix}$$

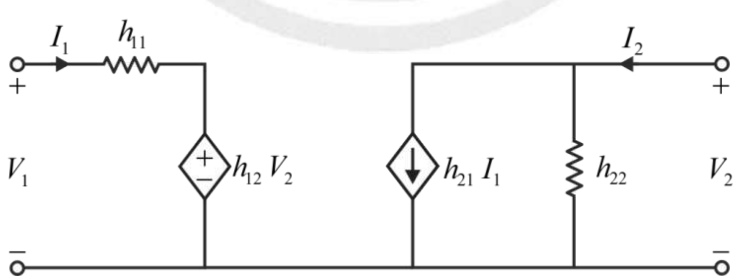
h-parameter (Hybrid Parameter)

In h-parameter, V_1, I_2 are the dependent variables and I_1, V_2 are the independent variables, the equations of h – parameter can be written by,

$$V_1 = h_{11}I_1 + h_{12}V_2 \quad \text{...(i)}$$

$$I_2 = h_{21}I_1 + h_{22}V_2 \quad \text{...(ii)}$$

Circuit diagram of *h*-parameter,



Case (i) When output port is short circuit, i.e. $V_2 = 0$,

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \text{Input impedance } (\Omega)$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \text{Forward current gain}$$

Case (ii) When input port is open circuit, i.e. $I_1 = 0$,

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \text{Reverse voltage gain}$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \text{Output admittance } (\mho)$$

Equation (i) & (ii) can be written in the matrix form,

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix}_{2\times1} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}_{2\times2} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}_{2\times1}$$

Condition of reciprocity and symmetricity in h-parameter is:

Reciprocity: $h_{12} = -h_{21}$

Symmetricity: $\Delta h = 1$

***h*-Parameter in Terms of Z and Y Parameter:**

Sr.	*Z*-Parameter	*Y*-Parameter	*h*-parameter
1.	$Z_{11} = \left.\frac{V_1}{I_1}\right\|_{I_2=0}$	$Y_{11} = \left.\frac{I_1}{V_1}\right\|_{V_2=0}$	$h_{11} = \left.\frac{V_1}{I_1}\right\|_{V_2=0} = \frac{1}{Y_{11}}$
2.	$Z_{12} = \left.\frac{V_1}{I_2}\right\|_{I_1=0}$	$Y_{12} = \left.\frac{I_1}{V_2}\right\|_{V_1=0}$	$h_{12} = \left.\frac{V_1}{V_2}\right\|_{I_1=0} = \frac{Z_{12}}{Z_{22}}$
3.	$Z_{21} = \left.\frac{V_2}{I_1}\right\|_{I_2=0}$	$Y_{21} = \left.\frac{I_2}{V_1}\right\|_{V_2=0}$	$h_{21} = \left.\frac{I_2}{I_1}\right\|_{V_2=0} = \frac{Y_{21}}{Y_{11}}$
4.	$Z_{22} = \left.\frac{V_2}{I_2}\right\|_{I_1=0}$	$Y_{22} = \left.\frac{I_2}{V_2}\right\|_{V_1=0}$	$h_{22} = \left.\frac{I_2}{V_2}\right\|_{I_1=0} = \frac{1}{Z_{22}}$

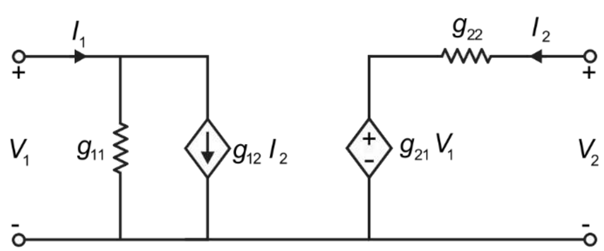
g-Parameter

In g-parameter, I_1, V_2 are the dependent variables and V_1, I_2 are the independent variables, the equations of g – parameter can be written by,

$$I_1 = g_{11}V_1 + g_{12}I_2 \quad \text{...(i)}$$

$$V_2 = g_{21}V_1 + g_{22}I_2 \quad \text{...(ii)}$$

Circuit diagram of g parameter,



Case (i) When output port is open circuit, i.e. $I_2 = 0$,

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0} = \text{Input admittance } (\mho)$$

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0} = \text{Forward voltage gain}$$

Case (ii) When input port is short circuit, i.e. $V_1 = 0$,

$$g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0} = \text{Reverse current gain}$$

$$g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0} = \text{Output impedance } (\Omega)$$

Equation (i) & (ii) can be written in the matrix form,

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix}_{2\times1} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}_{2\times2} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}_{2\times1}$$

Relationship between h and g Parameter:

The relation between h and g parameter is given by,

$$[h] = [g]^{-1}$$

$$[g] = [h]^{-1}$$

Condition for Symmetry: $\Delta g = 1$

Condition for Reciprocity: $g_{12} = -g_{21}$

ABCD Parameter (Transmission Parameter)

In ABCD-parameter, V_1, I_1 are the dependent variable and V_2, $-I_2$ is the independent variable, the equations of ABCD – parameter can be written by,

$$V_1 = AV_2 - BI_2 \qquad \ldots\text{(i)}$$

$$I_1 = CV_2 - DI_2 \qquad \ldots\text{(ii)}$$

Circuit diagram of T parameter cannot be drawn as both dependent variables are at input port.

Case (i) When output port is open circuit, i.e. $I_2 = 0$,

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \text{Reverse voltage gain}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \text{Transfer admittance } (\mho)$$

Case (ii) When output port is short circuit, i.e. $V_2 = 0$,

$$B = \left.\frac{-V_1}{I_2}\right|_{V_2=0} = \text{Transfer impedance } (\Omega)$$

$$D = \left.\frac{-I_1}{I_2}\right|_{V_2=0} = \text{Reverse current gain}$$

Equation (i) & (ii) can be written in the matrix form,

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix}_{2\times1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}_{2\times2} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}_{2\times1}$$

Condition for Symmetry: $A = D$

Condition for Reciprocity: $AD - BC = 1$

Inverse ABCD or Inverse T Parameter ($[ABCD]^{-1}/[T]^{-1}$)

In inverse ABCD-parameter, V_2, I_2 are the dependent variable and $V_1, -I_1$ is the independent variable, the equations of inverse ABCD – parameter can be written by,

$$V_2 = aV_1 - bI_1 \quad \text{...(i)}$$

$$I_2 = cV_1 - dI_1 \quad \text{...(ii)}$$

Case (i) When input port is open circuit, i.e. $I_1 = 0$,

$$a = \left.\frac{V_2}{V_1}\right|_{I_1=0} = \text{Forward voltage gain}$$

$$c = \left.\frac{I_2}{V_1}\right|_{I_1=0} = \text{Transfer admittance } (\mho)$$

Case (ii) When input port is short circuit, i.e. $V_1 = 0$,

$$b = -\left.\frac{V_2}{I_1}\right|_{V_1=0} = \text{Transfer impedance } (\Omega)$$

$$d = -\left.\frac{I_2}{I_1}\right|_{V_1=0} = \text{Forward current gain}$$

Equation (i) & (ii) can be written in the matrix form,

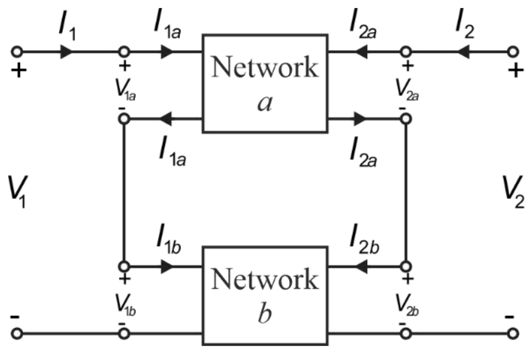
$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix}_{2\times1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times2} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix}_{2\times1}$$

Condition for Symmetry: $a = d$

Condition for Reciprocity: $ad - bc = 1$

Interconnection of Two Port Network

1. Series - series Connection:

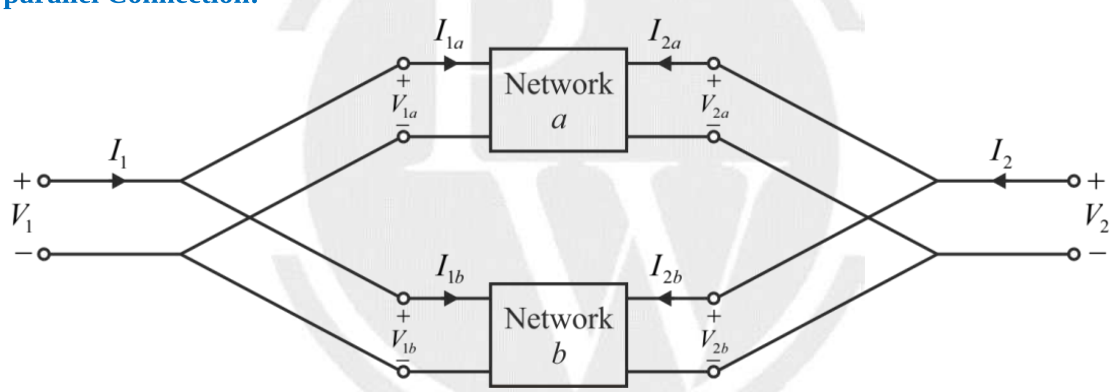


$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_{11a}+Z_{11b} & Z_{12a}+Z_{12b} \\ Z_{21a}+Z_{21b} & Z_{22a}+Z_{22b} \end{bmatrix} = \begin{bmatrix} Z_{11a} & Z_{12a} \\ Z_{21a} & Z_{22a} \end{bmatrix} + \begin{bmatrix} Z_{11b} & Z_{12b} \\ Z_{21b} & Z_{22b} \end{bmatrix}$$

$$[Z] = [Z]_a + [Z]_b$$

In case of series – series connection individual Z parameters are added. Care should be taken for series connection that some current should leave from second terminal of input and output port.

2. Parallel - parallel Connection:

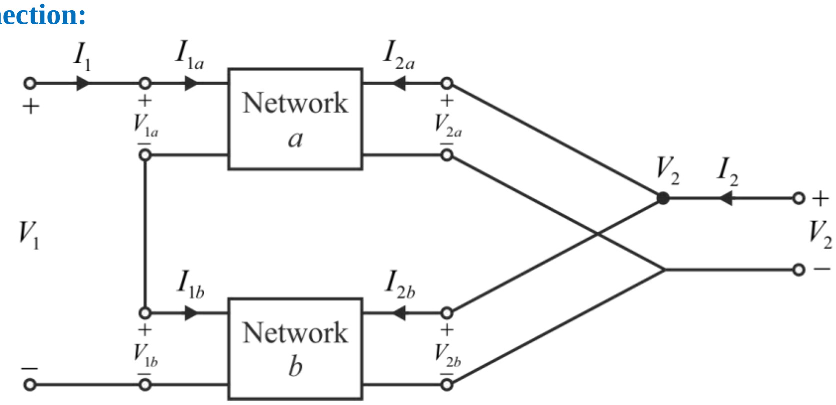


$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Y_{11a}+Y_{11b} & Y_{12a}+Y_{12b} \\ Y_{21a}+Y_{21b} & Y_{22a}+Y_{22b} \end{bmatrix} = \begin{bmatrix} Y_{11a} & Y_{12a} \\ Y_{21a} & Y_{22a} \end{bmatrix} + \begin{bmatrix} Y_{11b} & Y_{12b} \\ Y_{21b} & Y_{22b} \end{bmatrix}$$

$$[Y] = [Y]_a + [Y]_b$$

In case of parallel - parallel connection individual Y parameters are added.

3. Series Parallel Connection:

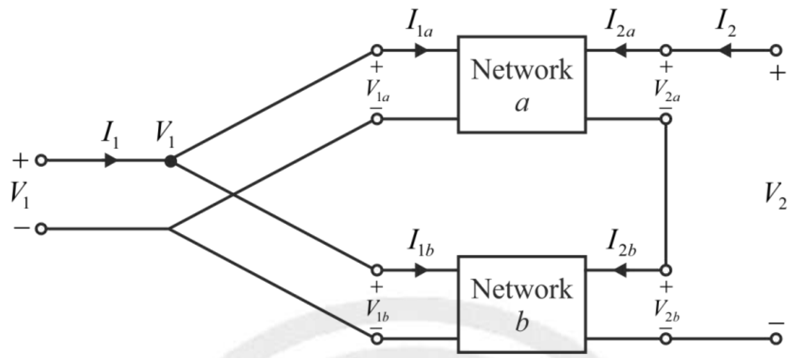


$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} h_{11a}+h_{11b} & h_{12a}+h_{12b} \\ h_{21a}+h_{21b} & h_{22a}+h_{22b} \end{bmatrix} = \begin{bmatrix} h_{11a} & h_{12a} \\ h_{21a} & h_{22a} \end{bmatrix} + \begin{bmatrix} h_{11b} & h_{12b} \\ h_{21b} & h_{22b} \end{bmatrix}$$

$$[h] = [h]_a + [h]_b$$

In case of series - parallel connection individual h parameters are added.

4. Parallel Series Connection :

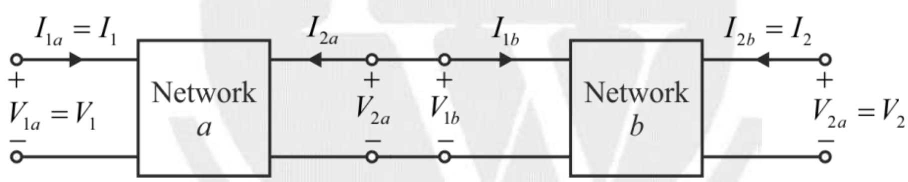


$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} g_{11a}+g_{11b} & g_{12a}+g_{12b} \\ g_{21a}+g_{21b} & g_{22a}+g_{22b} \end{bmatrix} = \begin{bmatrix} g_{11a} & g_{12a} \\ g_{21a} & g_{22a} \end{bmatrix} + \begin{bmatrix} g_{11b} & g_{12b} \\ g_{21b} & g_{22b} \end{bmatrix}$$

$$[g] = [g]_a + [g]_b$$

In case of parallel - series connection individual g parameters are added.

5. Cascade Connection :



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix}$$

$$[ABCD] = [ABCD]_a [ABCD]_b$$

In case of cascade connection individual T parameters matrix are multiplied.

Conclusion :

Interconnection of two port network	Equivalent parameter
Series connection	$[Z] = [Z]_a + [Z]_b$
Parallel connection	$[Y] = [Y]_a + [Y]_b$
Series - parallel connection	$[h] = [h]_a + [h]_b$
Parallel - series connection	$[g] = [g]_a + [g]_b$
Cascade connection	$[T] = [T]_a [T]_b$

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3 NETWORK THEOREM

3.1. Thevenin's Theorem

Thevenin's theorem states that a linear two terminal circuit can be replaced by an equivalent circuit consisting of a voltage source V_{th} (Thevenin's voltage) in series with a resistance R_{th} (Thevenin's resistance).

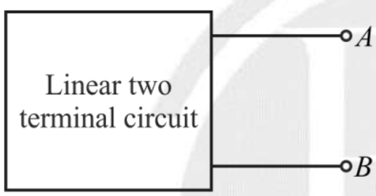


Fig.(a) General network

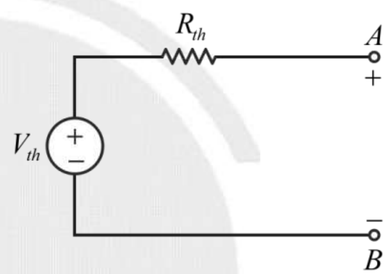
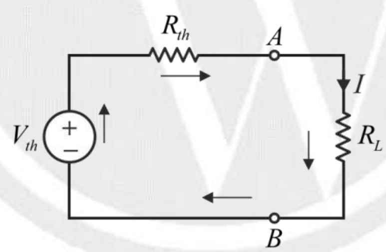


Fig.(b) Thevenin's equivalent



$$I = \frac{V_{th}}{R_{th} + R_L}$$

Case 1 : Circuit having only independent sources

Calculate the thevenin's resistance at load terminals by replacing all independent voltage sources by short circuit or by their internal resistance and all independent current sources by open circuit or by their internal resistances.

$$R_{th} = R_{eq}$$

Case 2 : Circuit having only dependent sources

By Keeping unchanged dependent sources

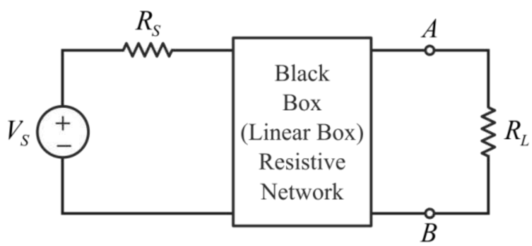
$$R_{th} = \frac{V_{dc}}{I_{dc}}$$

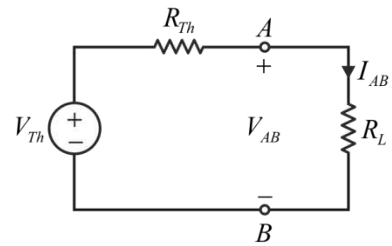
where, V_{dc} = Value of voltage source applied across load terminals

I_{dc} = Direct current supplied by DC voltage source

In this case, put thevenin's voltage,(V_{th})=0. (due to absence of independent source).

Example:





$$I_{AB} = \frac{V_{Th}}{R_{Th} + R_L}$$

$$V_{AB} = I_{AB} R_L = \frac{V_{Th} \times R_L}{R_{Th} + R_L}$$

Case 3 : Circuit having dependent as well as independent sources

Calculate the thevenin's resistance at load terminals by replacing all independent voltage sources by short circuit or by their internal resistance and all independent current sources by open circuit or by their internal resistances.

$$R_{th} = \frac{V_{dc}}{I_{dc}}$$

where, V_{dc} = Value of DC voltage applied across load terminals

I_{dc} = Direct current supplied by DC voltage source

3.2. Norton's Theorem

Norton's theorem states that a linear two terminal circuit can be replaced by an equivalent current source I_N (Norton's current) in parallel with a resistance R_N (Norton's resistance).

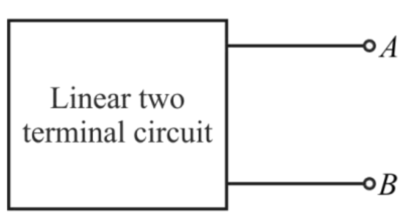


Fig.(a) General network

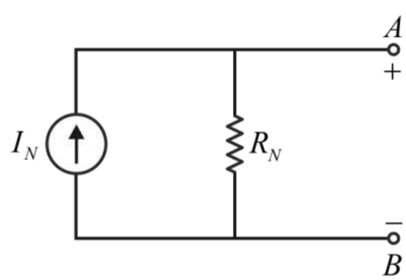
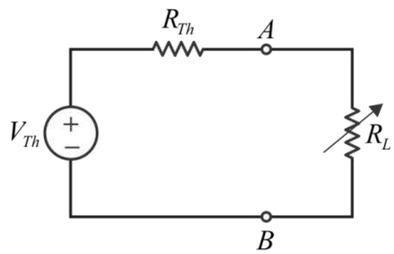


Fig.(b) Norton's equivalent

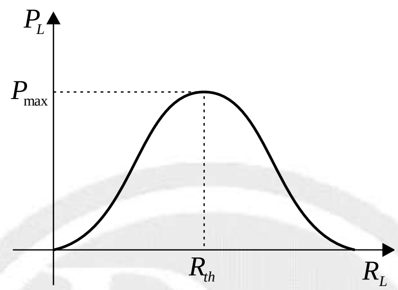
Relation between $V_{OC}\left(\text{or } V_{th}\right), I_{SC}\left(\text{or } I_N\right)$ and Z_{th}

$$I_{SC} = \frac{V_{OC}}{Z_{th}} \quad \Rightarrow \quad Z_{th} = \frac{V_{OC}}{I_{SC}} = \frac{V_{th}}{I_N}$$

3.2.1. Maximum Power Transfer Theorem



Power dissipated to load as a function of R_L is given by,



Maximum power transferred to the load is given by,

$$P_L = \frac{V_{th}^2}{4R_{th}} = \frac{V_{th}^2}{4R_L}$$

$$[R_{th} = R_L]$$

Maximum Power Transfer:

$$DC \div \boxed{P_{max} = \frac{V_{th}^2}{4R_{th}}}$$

- $R_L = R_{th}$, if R_L varying
- $R_{th} = 0$, if R_L constant R_{th} vary

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Case 1 : $\boxed{Z_L = Z_S^*}$, $R_L = R_{th}$, $X_{th} = -X_L$

Case 2 : If only R_L is variable, then $X_L \rightarrow$ constant

$$\boxed{R_L = \sqrt{(R_{th})^2 + (X_L + X_{th})^2}}$$

Then $$\boxed{P_{max} = \frac{V_{th}^2 \cdot R_L}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2}}$$

Case 3 : If R_L is variable $(X_L = 0)$ or $Z_L = R_L$

$$\boxed{R_L = |Z_S| = \sqrt{R_{th}^2 + X_{th}^2}}$$

Then, $$\boxed{P_{max} = \frac{V_{th}^2 \cdot R_L}{(R_L + R_{th})^2 + X_{th}^2}}$$

3.3. Superposition Theorem

Superposition theorem states that in any linear bilateral network containing two or more independent sources, the resultant current or voltage in any branch is the algebraic sum of currents or voltages caused by each independent source acting along with all are other independent sources being replaced by their internal resistances.

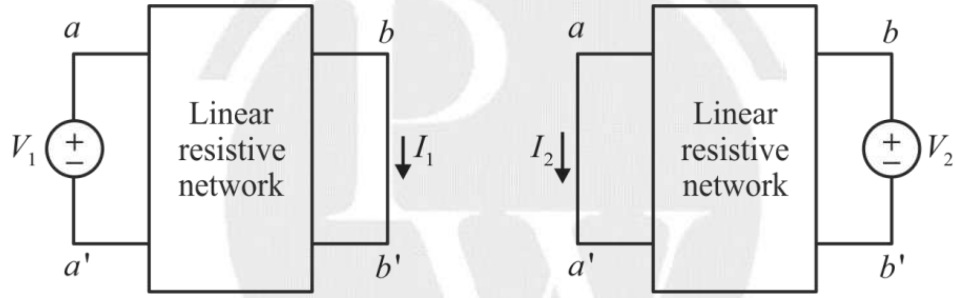
Note:

1. This Theorem is not valid for Non-linear quantities.
2. $P = \left(\sqrt{P_1} + \sqrt{P_2}\right)^2$

The dissipation of total power across any load,when two sources are working simultaneously.

3.4. Reciprocity Theorem

Reciprocity theorem states that in any linear bilateral network, if a source produces a certain current in any other branch, then the same source acting on the second branch produces the same current in the first branch.



$$\frac{V_1}{I_1} = \frac{V_2}{I_2}$$

3.5. Tellegen's Theorem

Tellegen's theorem states that an instantaneous power in an electrical network is zero.

Mathematically it is given by,

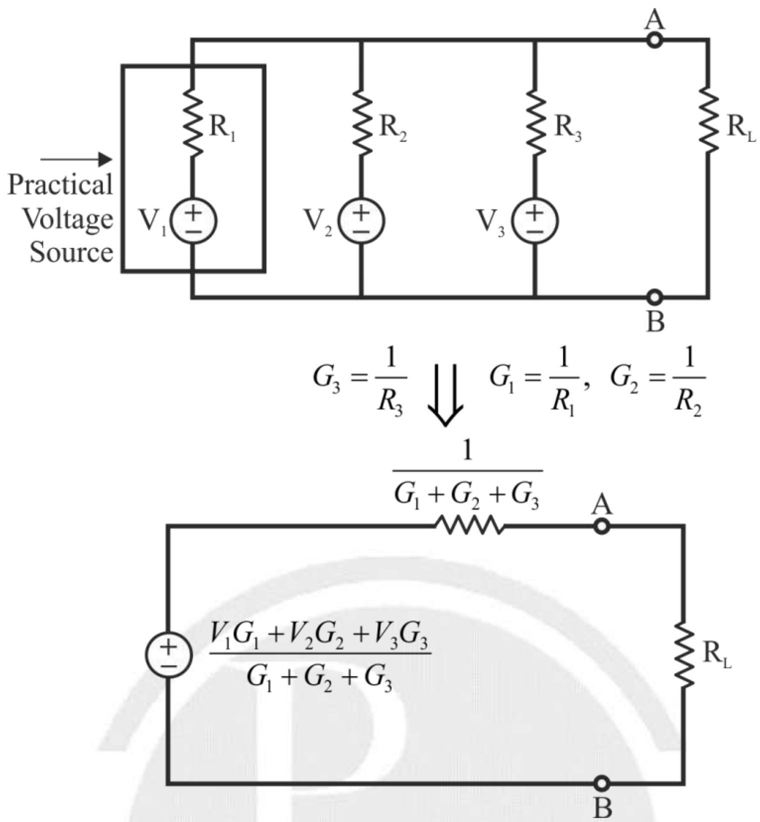
$$\sum_{K=1}^{n} P_K = 0$$

$$\sum_{K=1}^{n} V_K I_K = 0$$

We point out it as Power Conservation theorem.

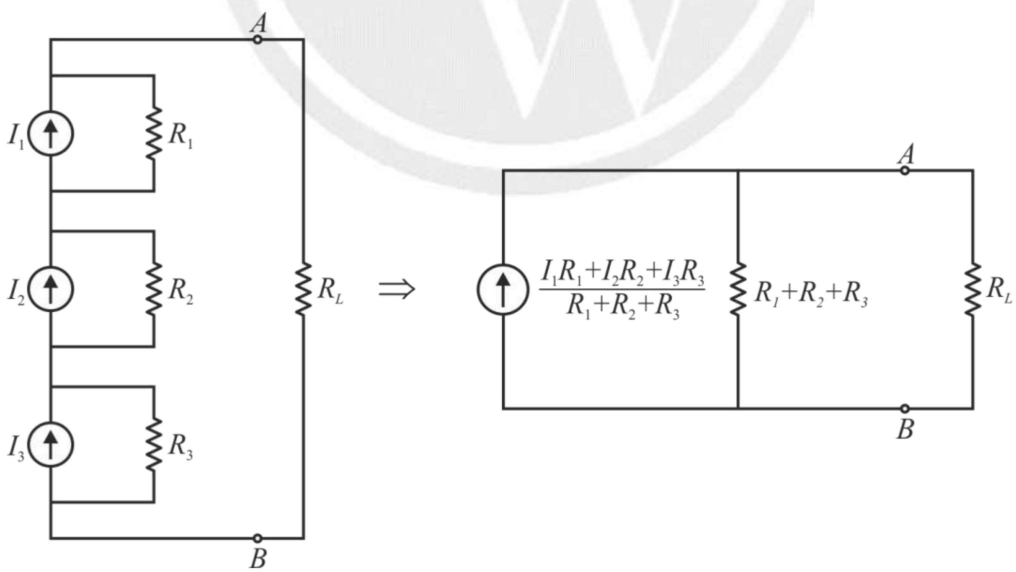
3.6. Millman's Theorem

Millman's theorem states that a number of voltage sources with internal resistance connected in parallel can be replaced by a single equivalent voltage source *V* in series with equivalent resistance *R*.



3.6.1. Dual of Millman's Theorem

Dual of Millman's theorem state that a number of current sources with internal resistance connected in series can be replaced by a single equivalent current source in parallel with resistance.



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4 TRANSIENT ANALYSIS

4.1. Introduction

- Transient analysis is the analysis of the circuits during the time it changes from one steady-state condition to another steady-state condition
- The transient analysis will reveal how the currents and voltages are changing during the transient period.
- Transient will occur when at least one energy storage element is present in circuit.

4.2. Common Aspects of RC and RL Circuits

While doing transient analysis on simple RC and RL circuits, we need remember two facts:

1. The voltage across a capacitor as well as the current in an inductor cannot have a discontinuity i.e. the capacitor voltage must be continuous at time t = 0 and hence $v_{C(0^+)} = v_{C(0^-)}$ and the inductor current must be continuous at time t = 0 and hence $i_{L(0^+)} = i_{L(0^-)}$.
2. With DC excitation, at a steady state, the capacitor will act as an open circuit and the inductor will act as a short circuit.

4.2.1 Time Constant

It is the time required for the response to delay by a factor 1/e or 36.8% of its initial value. It is represented by τ.

- For a RC circuit, $\tau = RC$
- For a RL circuit, $\tau = \dfrac{L}{R}$

R is the Thevenin resistance across inductor or capacitor terminals.

4.2.2 General Method of Analysis

$$x(t) = x(\infty) + \left[x(t_0^+) - x(\infty)\right] e^{-(t-t_0)/\tau},\ t > 0$$

If switching is done at $t = t_0$,

$x(t_0^+)$ = Initial value of $x(t)$ at $t = t_0$

$x(\infty)$ = Final value of $x(t)$ at $t = \infty$

Follow these steps to solve numerical:

Step 1: Choose any voltage and current in the circuit which has to be determined.

Step 2: Assume circuit has reached steady state before switch was thrown at $t = t_0$.

Step 3: Draw the circuit at $t = t_0$ with capacitor replaced by open circuit and inductor replaced by short circuit. Solve for $V_c\left(t_0^-\right)$ and $i_L\left(t_0^-\right)$.

Step 4: Voltage across capacitor and inductor current cannot change instantaneously,

$$V_c\left(t_0^+\right) = V_c\left(t_0^-\right) = V_c\left(t_0\right)$$

$$i_L\left(t_0^+\right) = i_L\left(t_0^-\right) = i_L\left(t_0\right)$$

Step 5: Draw the circuit for $t = t_0^+$ with switches in new position.

Step 6: Replace a capacitor with a voltage source of value $\text{v}_c\left(t_0^-\right) = \text{v}_c\left(t_0^+\right)$ and inductor with a current source of value $i_L\left(t_0^+\right) = i_L\left(t_0^-\right)$. Solve for initial value of variable $x\left(t_0^+\right)$.

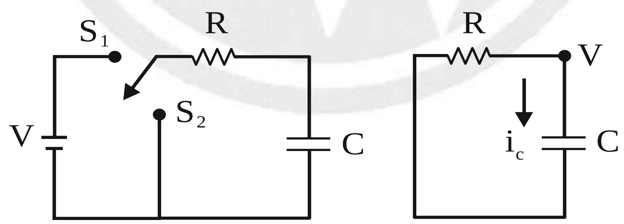
Step 7: Draw the circuit for $t = \infty$, in a similar manner as step- 2 and calculate $x(\infty)$. Calculate time constant of circuit

Step 8: $\tau = R_{\text{th}}C$ or $\tau = L/R_{\text{th}}$

Step 9: Substitute all value to calculate $x(t)$.

4.3. Source Free RC Circuit

A source free RC circuit occurs when its DC source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.



(a) (b)

State variable = V

Let, $V(0) = V_0$

Apply KCL in Fig (b)

$$\left(\frac{V}{R}\right) + C\frac{dV_c}{dt} = 0$$

$$C\frac{dV_c}{dt} = -\left(\frac{V}{R}\right)$$

$$\int \frac{dV_c}{dt} = -\int \frac{dt}{RC}$$

$$\ln(V) = -\frac{t}{RC} + \ln(A)$$

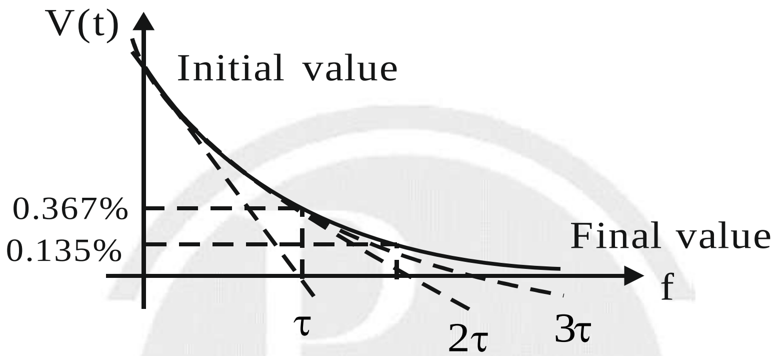
$$\ln\left(\frac{V_c}{A}\right) = -\frac{t}{RC}$$

$$V_c(t) = Ae^{-t/RC}$$

$$V = V_0 \quad \Rightarrow A = V_0$$

But at $t = 0$, $\quad V_c(t) = V_0 e^{-t/RC}$

for $t < 5\tau$, circuit will be in transient state and for $t \geq 5\tau$, circuit will be in steady state. Sometimes 4τ is also considered as settling time.

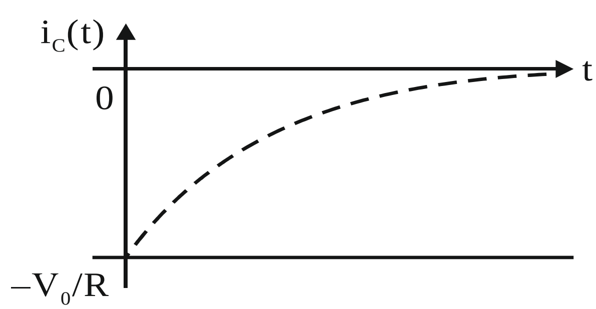


Expression for current through capacitor,

$$i_c = C\frac{dV_C}{dt} = C\frac{d}{dt}\left(V_0 e^{-t/RC}\right)$$

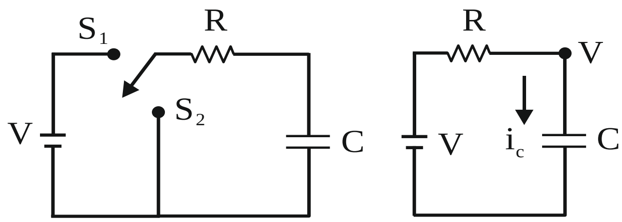
$$i_c(t) = -\left(\frac{V_0}{R}\right)e^{-t/\tau} \text{A}$$

and $\quad V_R(t) = V_0 e^{-t/\tau} V$



4.3.1 RC Circuit with Source:

Again, consider the circuit shown in below Figure. Let us say that the switch was in position S_2 long enough so that $v_c(t) = 0$ and $i_c(t) = 0$ i.e. all the energy in the capacitor is dissipated and the circuit is at rest. Now, the switch is moved to position S_1.



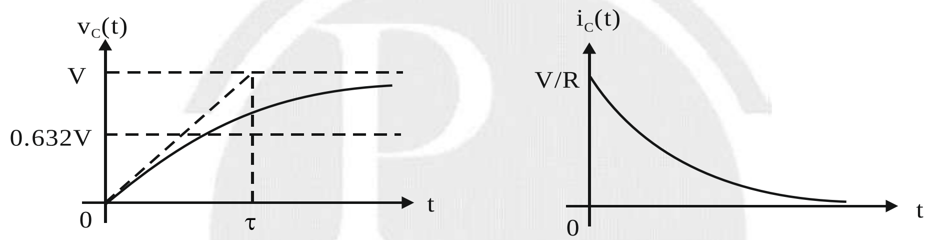
$$v_c(0^+) = v_c(0^-) = 0$$

The voltage through the capacitor is calculated as:

$$v_C(t) = V\left(1 - e^{-\frac{1}{RC}t}\right)$$

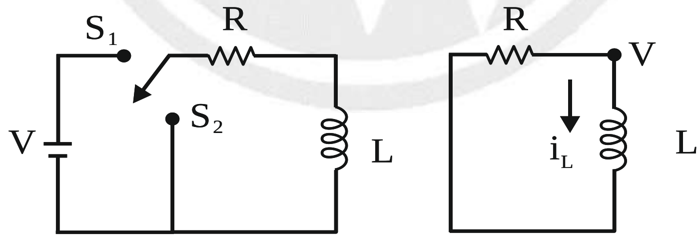
The current through the capacitor is calculated as:

$$i_C(t) = C\frac{dv_C}{dt} = \frac{V}{R}e^{-\frac{1}{RC}t}$$



4.4 Source Free RL Circuit

A source free RL circuit occurs when its dc source is suddenly disconnected. The energy already stored in the inductor is released to the resistors.



(a) (b)

Let, $i(0) = i_0$

Apply KVL, in Fig (b)

$$iR + L\frac{di}{dt} = 0$$

$$L\frac{di}{dt} = -iR$$

$$\int\frac{di}{i} = -\left(\frac{R}{L}\right)\int dt$$

$$\ln\left(\frac{i}{A}\right) = -\left(\frac{R}{t}\right)t$$

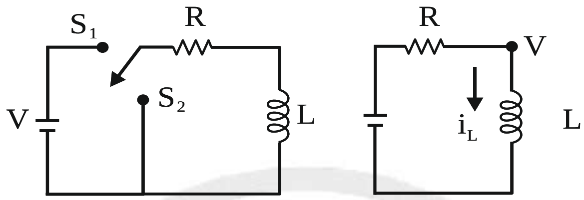
But at $t = 0$, $\quad i = I_0 \Rightarrow A = I_0$

$$i(t) = I_0 e^{-t/(L/R)} A$$

$$\tau = \frac{L}{R} \text{ (Time constant of R-L Circuit)}$$

4.4.1 With Source

Consider the circuit shown in Fig After the circuit has attained the steady state with the switch in position S_2, the switch is moved to position S_1 at time $t = 0$. We like to find the inductor current for time $t > 0$.



The current through the capacitor is calculated as:

$$i_L(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

The voltage through the capacitor is calculated as:

$$v_L(t) = L\frac{di}{dt} = Ve^{-\frac{R}{L}t}$$

4.5. Comparison Table

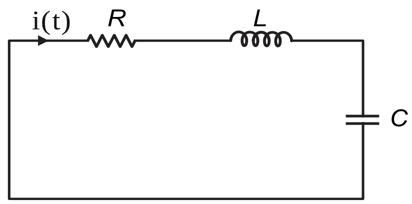
RC Circuit	RL Circuit
1. $\tau = RC$ (Time constant)	1. $\tau = \frac{L}{R}$ (Time constant)
2. With DC, at steady state capacitor acts as open circuit	2. With DC, at steady state inductor acts as short circuit
3. If $v_C(0) \neq 0$; $v_C(\infty) = 0$ Then $v_C(t) = v_C(0)e^{-\frac{1}{RC}t}$	3. If $i_L(0) \neq 0$; $i_L(\infty) = 0$ Then $i_L(t) = i_L(0)e^{-\frac{R}{L}t}$
4. If $v_C(0) = 0$; $v_C(\infty) \neq 0$ Then $v_C(t) = v_C(\infty)e^{-\frac{1}{RC}t}$	4. If $i_L(0) = 0$; $i_L(\infty) \neq 0$ Then $i_L(t) = i_L(\infty)e^{-\frac{R}{L}t}$
5. If $v_C(0) \neq 0$; $v_C(\infty) \neq 0$ Then $v_C(t) = v_C(\infty) + \left[v_C(0) - v_C(\infty)e^{-\frac{1}{RC}t}\right]$	5. If $i_L(0) \neq 0$; $i_L(\infty) \neq 0$ Then $i_C(t) = i_C(\infty) + \left[i_C(0) - i_C(\infty)e^{-\frac{R}{L}t}\right]$

4.6. Series RLC Circuit

Without source:

The voltage across Capacitor in the circuit shown below can be given as;

$$V(0) = \frac{1}{C}\int_{-\infty}^{0} i(t)dt = V_0$$



Characteristic equation of series RLC circuit is given as:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$S_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$S_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$S_1, S_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2};$$

$$\alpha = \frac{R}{2L};\ \omega_0 = \frac{1}{\sqrt{LC}}$$

1. If $\alpha > \omega_0$ roots are real and unequal (over-damped),

   $$i(t) = A e^{s_1 t} + Be^{s_2 t}$$

2. If $\alpha = \omega_0$ roots are real and equal (critically damped),

   $$i(t) = (A + Bt)e^{-\alpha t}$$

3. If $\alpha < \omega_0$, roots are complex conjugate (under damped),

   $$i(t) = e^{-\alpha t}(A\cos\omega_d t + B\sin\omega_d t)$$

   $$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

With a source:

$$v(t) = V_s + (A e^{s_1 t} + Be^{s_2 t}) \Rightarrow \text{(Over-damped)}$$

$$v(t) = V_s + (A + Bt)e^{-\alpha t} \Rightarrow \text{(Critically damped)}$$

$$v(t) = V_s + e^{-\alpha t}(A\cos\omega_d t + B\sin\omega_d t) \Rightarrow \text{(Under damped)}$$

4.7 Parallel RLC Circuit

Without source:

$$i(0)=\frac{1}{L}\int_{-\infty}^{0} v(t)dt$$

Characteristic equation of parallel RLC circuit is given as:

$$s^2+\frac{1}{RC}s+\frac{1}{LC}=0$$

$$S_1, S_2=-\alpha\pm\sqrt{\alpha^2-\omega_0^2};$$

$$\alpha=\frac{1}{2RC};\ \omega_0=\frac{1}{\sqrt{LC}}$$

1. If $\alpha>\omega_0$ roots are real and unequal (over-damped),

$$v(t)=A\,e^{s_1t}+Be^{s_2t}$$

2. If $\alpha=\omega_0$ roots are real and equal (critically damped),

$$v(t)=(A+Bt)e^{-\alpha t}$$

3. If $\alpha<\omega_0$, roots are complex conjugate (under damped),

$$v(t)=e^{-\alpha t}(A\cos\omega_d t+B\sin\omega_d t)$$

$$\omega_d=\sqrt{\omega_0^2-\alpha^2}$$

With step input:

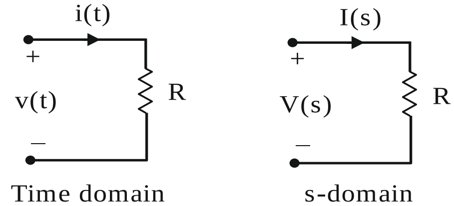
$$i(t)=I_s+\left(A\,e^{s_1t}+Be^{s_2t}\right)\ \Rightarrow\ \text{(Over-damped)}$$

$$i(t)=I_s+(A+Bt)e^{-\alpha t}\ \Rightarrow\ \text{(Critically damped)}$$

$$i(t)=I_s+e^{-\alpha t}(A\cos\omega_d t+B\sin\omega_d t)\ \Rightarrow\ \text{(Under damped)}$$

4.8 Representation of Circuit Elements in s-domain

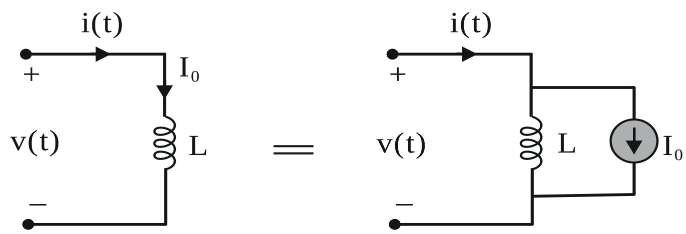
Resistor:



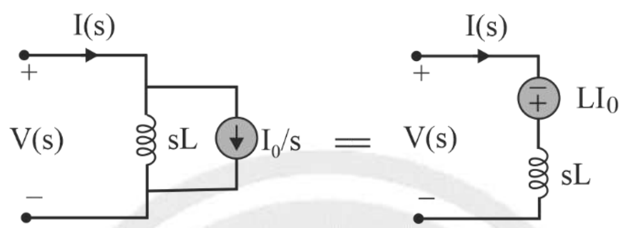
1. In Time domain V(t) = R.i(t)
2. In s-domain V(s) = R.I(s)

Inductor 'L' with initial current 'I_0'

In time domain:



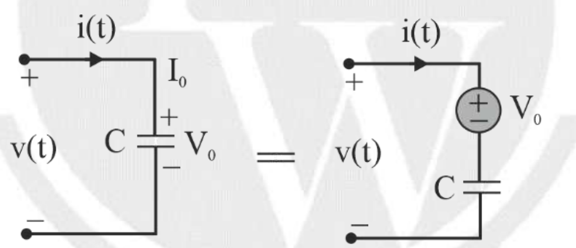
In frequency domain:



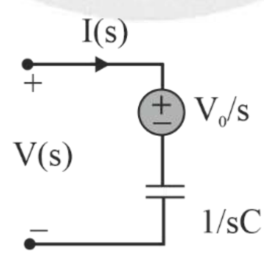
$$V(s) = sLI(s) - LI_0 \quad \text{or} \quad I(s) = \frac{V(s)}{sL} + \frac{I_0}{s}$$

Capacitor 'C' with initial Voltage 'V_0'

In time domain:



In frequency domain:



$$I(s) = sCV(s) - CV_0 \quad \text{or} \quad V(s) = \frac{I(s)}{sC} + \frac{V_0}{s}$$

4.9 Transient Analysis using Laplace Transform

Laplace Transform Method:

Step 1: Choose any voltage and current in the circuit which has to be determined.

Step 2: Assume circuit has reached steady state before switch was thrown at $t = t_0$.

Step 3: Draw the circuit at $t = t_0$ with capacitor replaced by open circuit and inductor replaced by short circuit. Solve for $V_c\left(t_0^-\right)$ and $i_L\left(t_0^-\right)$.

Step 4: Laplace-transform a circuit, including components with non-zero initial conditions.

Step 5: Analyze a circuit in the s-domain and Check the s-domain answers using the initial value theorem (IVT) and final value theorem (FVT).

Step 6: Inverse Laplace-transform the result to get the time domain solutions.

Following equations can be used to find response of RL and RC circuits after switching:

1. If the switching occurs at $t = 0$.

$$i(t) = i(\infty) + \left[i(0) - i(\infty)\right]e^{-t/\tau} \qquad v(t) = v(\infty) + \left[v(0) - v(\infty)\right]e^{-t/\tau}$$

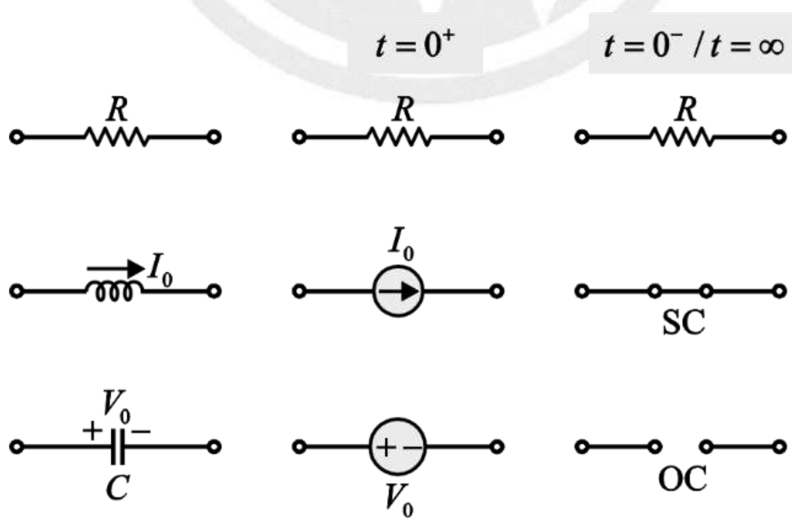
2. If the switching occurs at $t = t_0$.

$$i(t) = i(\infty) + \left[i(t_0) - i(\infty)\right]e^{-(t-t_0)/\tau} \qquad v(t) = v(\infty) + \left[v(t_0) - v(\infty)\right]e^{-(t-t_0)/\tau}$$

4.10. Limitations of Transient Equation

1. It is not applicable in circuits having two energized capacitors in series.
2. It is not applicable in circuits having two energized inductors in parallel.
3. If time dependent source is present in the circuit then transient equation is not applicable.
4. For inductor or capacitor if excitation is impulse signal then transient equation is not applicable.
5. If the source is sinusoidal in nature, transform domain approach is not valid.

Behavior of *R*, *L* and *C* at $t = 0^-$, $t = 0^+$ & $t = \infty$:



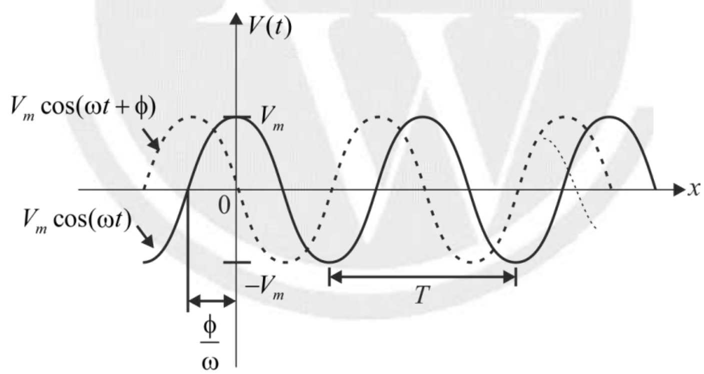
5 SINUSOIDAL STEADY STATE ANALYSIS

5.1. Introduction

- When we apply sinusoidal across any reactive network or complex network, then all responses either current or voltage are referred as sinusoidal steady state response or steady state response or sinusoidal response.
- These is no concept of open circuit and short circuit for sinusoidal input.
- In case of dc source, at steady state, capacitor is replaced by open circuit and inductor is replaced by short circuit.
- The sinusoidal varying voltage can be written as,

$$V(t) = V_m \cos(\omega t + \phi)$$

- To aid discussion of the parameters of the sinusoidal voltage equation, below is the figure.

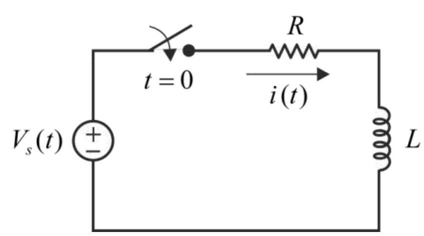


where, V_m = Amplitude

ϕ = Phase angle

ω = Angular frequency which is related to time period T as $\omega = \frac{2\pi}{T}$. The argument ωt changes 2π radians (360^0) in one period.

Example : Series RL circuit,



$$i_{tr}(t) = -\frac{V_m \cos\left(\phi - \tan^{-1}\frac{\omega L}{R}\right)}{\sqrt{R^2 + \omega^2 L^2}} e^{-\frac{R}{L}t} \quad \text{[Transient response, dies out as } t \to \infty]$$

$$i_{ss}(t) = -\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right) \text{[Steady state response, lasts even } t \to \infty]$$

5.2. Phasor

The phasor is a constant complex number that carries the amplitude and phase angle information of a sinusoidal function.

A sinusoidal function can be represented by the real part of a phasor times the "Complex carrier".

$$V_m \cos(\omega t + \phi) = V_m \operatorname{Re}\left\{e^{j(\omega t + \phi)}\right\} = \operatorname{Re}\left\{(V_m e^{j\phi}) e^{j\omega t}\right\}$$

$$= \operatorname{Re}\left\{ \underset{\substack{\uparrow \\ Phasor}}{V} \; \underset{\substack{\uparrow \\ Carrier}}{e^{j\omega t}} \right\}$$

(i) Polar form

$$V = V_m e^{j\phi} = V_m \angle\phi$$

(ii) Rectangular form

$$V = V_m \cos\phi + jV_m \sin\phi$$

$$H^2 = \text{real}^2 + \text{Img}^2$$

$$H^2 = V_m^2 \cos^2\phi + V_m^2 \sin^2\phi$$

$$H^2 = V_m^2 (\cos^2\phi + \sin^2\phi)$$

$$H = V_m$$

Impedances of the Passive Circuit Element

1. Generalize resistance to impedance.
2. Impedance of R, L, C
3. In phase and quadrature.

(i) In a resistor, the ratio of voltage $V(t)$ to the current $i(t)$ is a real constant R.

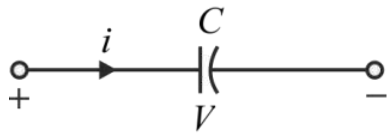
$$R = \frac{V(t)}{i(t)} \quad \ldots\text{(Resistance)}$$

(ii) For two terminals of a linear circuit driven by sinusoidal source, the ratio of voltage phasor V to the current phasor I is a complex constant Z.

$$Z = \frac{V}{I} \quad \ldots\text{(Impedance)}$$

Relation between *i*, *v* and Impedance of a Capacitor :

Let us take, $$V(t) = V_m \cos(\omega t + \theta_v)$$



The time t_0 is referred as switching time or transient free time

$V_{in}(t)$	s	Condition of transient free
$V_m \sin(\omega t + \phi)$	$t = 0$	$\phi - \tan^{-1}\dfrac{\omega L}{R} = 0 \quad \left[\tau = \dfrac{L}{R}\right]$ $\tau = RC$ for RC circuit $\phi - \tan^{-1}\omega RC = 0$
$V_m \sin(\omega t + \phi)$	$t = t_0$	$\omega t_0 + \phi - \tan^{-1}\dfrac{\omega L}{R} = 0$
$V_m \cos(\omega t + \phi)$	$t = 0$	$\phi - \tan^{-1}\dfrac{\omega L}{R} = \dfrac{\pi}{2}$
$V_m \cos(\omega t + \phi)$	$t = t_0$	$\omega t_0 + \phi - \tan^{-1}\dfrac{\omega L}{R} = \dfrac{\pi}{2}$

$$t_0 = \frac{\tan^{-1}\frac{\omega L}{R} - \phi_{rad}}{\omega(rad/sec)} \quad \text{(sin excitation)}$$

$$t_0 = \frac{\frac{\pi}{2} + \tan^{-1}\frac{\omega L}{R} - \phi}{\omega} \quad \text{(cosine excitation)}$$

1. Transient free response is only possible for ac excitation. It is not possible for dc excitation.
2. Transient free response is not applicable for series RLC as well as parallel RLC network (2nd order network)
3. Consider 2nd order system with complex conjugate roots $s = -\alpha \pm j\beta$

$$(t) = \underbrace{e^{-\alpha t}\left[K_1 \cos\beta t + K_2 \sin\beta t\right]}_{\text{Transient}} + \underbrace{\frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left[\omega t + \phi - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right]}_{\text{Steady state}}$$

For Transient part to be 0, $K_1 \cos\beta t + K_2 \sin\beta t$ should be 0, which is not possible at same time because of sin and cos.

Transient free response is possible only for series and parallel first order RL and RC circuits, with ac excitation.

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6 RESONANCE

6.1 Resonance

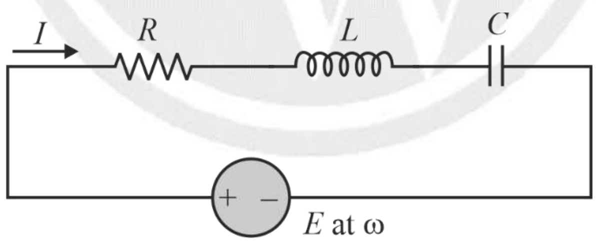
A.C Circuits made up of resistors, inductors and capacitors are said to be resonant circuits when the current drawn from the supply is in phase with the applied sinusoidal voltage. Then

1. The resultant Reactance or Susceptance is zero.
2. The circuit behaves as a resistive circuit.
3. The power factor is unity.

6.2. Series Resonance

Figure represents a series resonant circuit. Resonance can be achieved by

1. Varying frequency ω
2. Varying the inductance L
3. Varying the capacitance C



The current in the circuit is

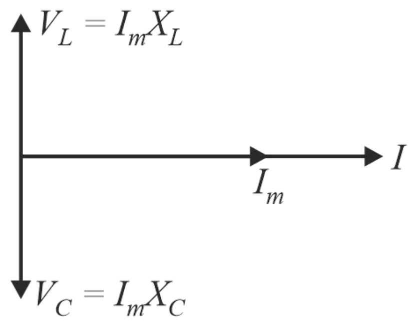
$$I = \frac{E}{R + j(X_L - X_C)} = \frac{E}{R + jX}$$

At resonance. X is zero. If ω_0 is the frequency at which resonance occurs, then $\omega_0 L = \frac{1}{\omega_0 C}$ or $\omega_0 = \frac{1}{\sqrt{LC}}$ = resonant frequency.

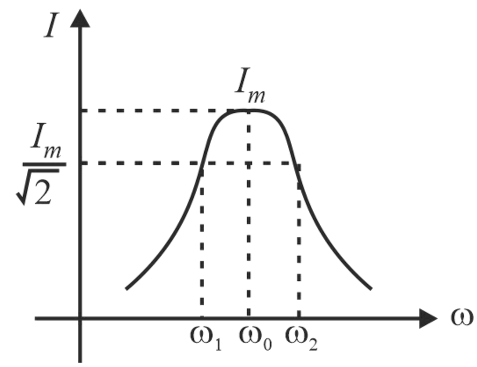
The current at resonance is $I_m = \frac{V}{R}$ = maximum current.

The phasor diagram for this condition is shown in Fig. (a).

The variation of current with frequency is shown in Fig. (b).



(a)



(b)

Points to be remembered:

At Series Resonance,

- The voltage across inductor and capacitor are equal in magnitude i.e. $|V_L| = |V_C|$ and 180° out of phase.
- Imaginary part of input impedance is equal to zero.
- The net impedance is Minimum i.e. $Z_{in}|_{\omega=\omega_0} = \text{R}(\text{Minimum})$
- The net current flow in the circuit is Maximum i.e. $I|_{\omega=\omega_0} = \frac{V_S}{R}(\text{Maximum})$

Impedance of a Series Resonant Circuit:

The impedance of a series RLC circuit is given by,

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right) \qquad |Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

Variation of impedance with change in frequency is shown in figure below,

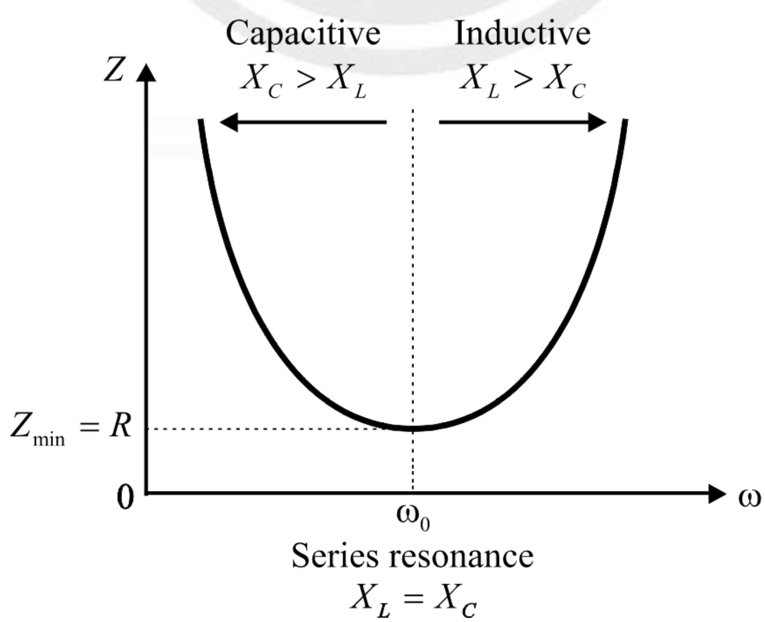


Fig. Impedance vs frequency curve for series RLC circuit

6.1.1 3dB Frequency

The range of frequencies at which the current drawn by network becomes 0.707 or $\frac{1}{\sqrt{2}}$ times of its maximum value.

Higher and lower 3dB cut off frequencies of series RLC circuit are given by,

$$\omega_H = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \text{ rad/sec} \quad \text{...(i)}$$

$$\omega_L = \frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \text{ rad/sec} \quad \text{...(ii)}$$

Quality factor (Q-factor):

- The "sharpness" of the resonance in a resonant circuit is measured quantitatively by the quality factor Q
- The Q factor is also known as the voltage amplification factor OR current amplification factor and it's value must be high for any tuned circuit.
- Q-factor describes the energy storage capability of inductor and capacitor in RLC network.

$$Q[L] = 2\pi \times \frac{\text{Energy stored by inductor}}{\text{Energy dissipated by resistance per cycle}}$$

$$Q[L] = \frac{2\pi \times \frac{1}{2} L I_0^2}{\frac{I_0^2 R}{2} \cdot T_0}$$

$$Q[L] = \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Similarly, $$Q[C] = 2\pi \times \frac{\text{Energy stored by capacitor}}{\text{Energy dissipated by resistance per cycle}} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

The relationship between bandwidth, frequency of resonance and quality factor is given by,

$$Q = \frac{\omega_0}{\text{BW}}$$

Note:

For any tuned network the Q-factor must be high and the bandwidth should be small and therefore the resistance used in the network should be small as $Q \propto \frac{1}{BW}$ & $BW \propto R$

6.1.2 Parallel RLC Resonance Circuit

The Dual of a series resonant circuit is often considered as a parallel resonant circuit and it is as shown in figure.

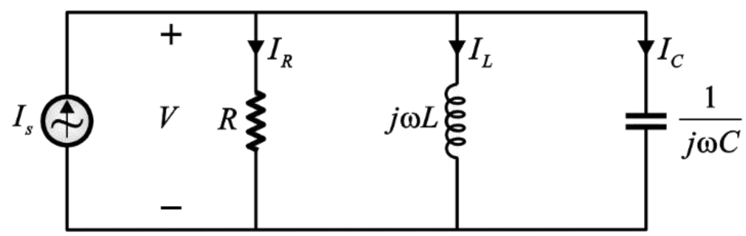


Fig. A parallel resonance circuit

The voltage across the parallel combination of RLC is given by,

$$V = \frac{I_s}{\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)} \qquad |V| = \frac{|I_s|}{\sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}}$$

The value of ω that satisfies the condition of parallel resonance is called resonant frequency.

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/sec}$$

At resonance, $\omega L = 1/\omega C$, the admittance will be minimum and voltage will be maximum as given below,

$$|V_R| = |I_s| R$$

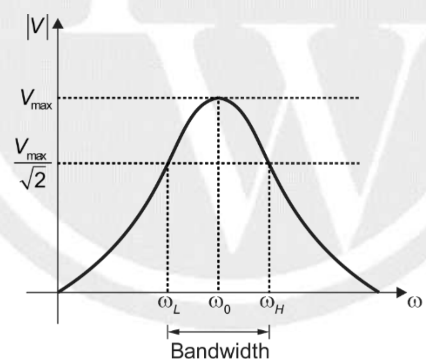


Fig. Voltage variation in a parallel resonant circuit

Admittance of the parallel RLC circuit given by,

$$Y = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

From above equation, admittance is dominated by inductive term at lower frequencies and by capacitive term at higher frequencies. The plot of Y against frequency is shown below.

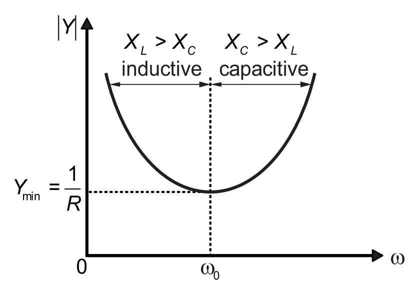


Fig. Admittance vs frequency curve in parallel RLC circuit

3dB Bandwidth

3 dB Bandwidth is given by,

$$\text{BW} = \omega_H - \omega_L \qquad \text{BW} = \frac{1}{RC}\ \text{rad/sec} \qquad \text{BW} = \frac{1}{2\pi RC}\ \text{Hz}$$

To obtain the relationship between bandwidth and frequency of resonance, multiply equation (i) and (ii),

$$\therefore \qquad \omega_H . \omega_L = \frac{1}{LC} \qquad \omega_H . \omega_L = \omega_0^2$$

$$\therefore \qquad \omega_0 = \sqrt{\omega_H . \omega_L}\ \text{rad/sec}$$

3dB Frequencies

Higher 3dB cut frequencies of parallel RLC circuit are given by,

$$\omega_H = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}\ \text{rad/sec} \qquad \ldots\text{(i)}$$

Lower 3dB cut frequencies of parallel RLC circuit are given by,

$$\omega_L = \frac{-1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}\ \text{rad/sec} \qquad \ldots\text{(ii)}$$

3 dB Frequency in terms of bandwidth is given by

$$\omega_H = \frac{\text{BW}}{2} + \sqrt{\left(\frac{\text{BW}}{2}\right)^2 + \omega_0^2} \qquad \omega_L = -\frac{\text{BW}}{2} + \sqrt{\left(\frac{\text{BW}}{2}\right)^2 + \omega_0^2}$$

Quality factor for parallel RLC circuit:

- The "sharpness" of the resonance in a resonant circuit is measured quantitatively by the quality factor Q
- *Q*-factor describes the energy storage capability of inductor and capacitor in RLC network.

$$Q[L] = \frac{I_L}{I_R} = Q[C] = \frac{I_C}{I_R} = R\sqrt{\frac{C}{L}}$$

The relationship between bandwidth and quality factor is given by,

Quality factor, $\qquad Q = \dfrac{\omega_0}{\text{BW}}$

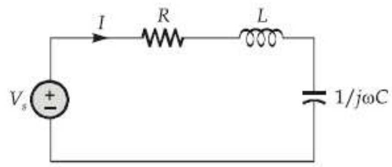
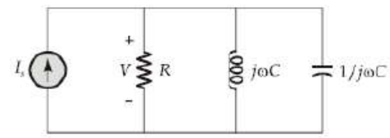
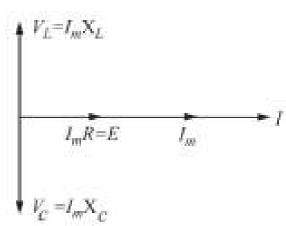
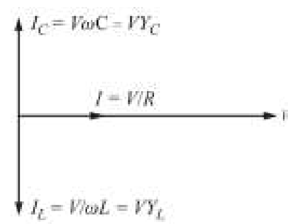
6.1.3 Resonance in Series RLC vs Parallel RLC

Parallel resonant circuit is the **DUAL** of the series resonant circuit and hence the results of parallel resonant circuit can be obtained just by following replacement in the results of series resonant circuit,

$$R \rightarrow \frac{1}{R}$$

$$L \rightarrow C,\ C \rightarrow L$$

The key differences between series resonance and parallel resonance are given in the following table −

Parameters	Series Resonance	Parallel Resonance
Circuit Diagram		
Impedance	The impedance of a series RLC circuit becomes minimum at series resonance. The imaginary part of the impedance is 0 i.e. $X_L - X_C = 0$ $X_L = X_C$	The impedance of a parallel RLC circuit becomes maximum at parallel resonance.
Admittance	Series resonance has maximum admittance.	Parallel resonance has minimum admittance. The imaginary part of the impedance is 0 i.e. $Y_L - Y_C = 0$ $Y_L = Y_C$
Phasor Diagrams	Figure 6.2	
Upper cut-off frequency	$\omega_H = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$ rad/sec	$\omega_H = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$ rad/sec
Lower cut-off frequency	$\omega_L = -\left[\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right]$ rad/sec	$\omega_H = -\left[\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}\right]$ rad/sec
Bandwidth	$BW = \omega_H - \omega_L = \frac{R}{L}$ rad/sec	$BW = \omega_H - \omega_L = \frac{1}{RC}$ rad/sec
Behave of the circuit	It acts as a voltage amplifier circuit or acceptor circuit.	It acts as current amplifier circuit or rejector circuit.
Current	The series resonance results in the maximum current through the circuit.	The current in circuit at parallel resonance is minimum.
Filter Characteristics	It works as a Band Pass Filter	It works as a Band Stop Filter and Band Reject Filter.

Parameters	Series Resonance	Parallel Resonance
Magnification Feature	The series resonance magnifies the voltage in the circuit.	The parallel resonance magnifies the current in the circuit.
Equation of effective impedance	The effective impedance is given by, Z = R	The effective impedance is given by, Z = L/CR
Quality factor (Q-factor)	$Q[L]=\frac{\omega_0 L}{R}=\frac{1}{R}\sqrt{\frac{L}{C}}$	$Q[L]=\frac{R}{\omega_0 L}=R\sqrt{\frac{C}{L}}$
Applications	The series resonance is widely used in tuning, oscillator circuits, voltage amplifiers, high-frequency filters, etc.	The parallel resonance is used in current amplifiers, induction heating, filters, radio-frequency amplifiers, etc.

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7 COMPLEX POWER

7.1. Introduction

Complex power is "the complex sum of real and reactive powers", it is also termed as apparent power, measured in terms of volt-amps (or), in kilo volt-amps (KVA).

The rectangular form of complex power is given below,

$$S = P + jQ$$

$$S = \sqrt{P^2 + Q^2} \angle \tan^{-1}\left(\frac{Q}{P}\right)$$

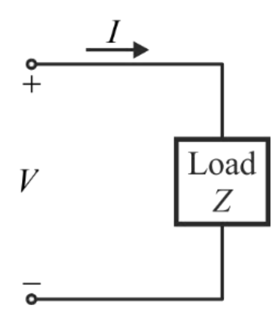
$$S = VI^*$$

where, I^* is the complex conjugate of current.

In case of AC circuit, complex power is given by, $S = V_{rms} I_{rms}^*$ VA.

7.2. Analysis of Complex Power

Consider the AC load figure in below.



$$S = V_{rms} I_{rms}^*$$

$$V_{rms} = V_{rms} \angle \theta_v = \frac{V_m}{\sqrt{2}} \angle \theta_v$$

The expression of RMS current in phasor form is given by;

$$I_{rms} = I_{rms}\angle\theta_i = \frac{I_m}{\sqrt{2}}\angle\theta_i$$

$$I^*_{rms} = I_{rms}\angle-\theta_i = \frac{I_m}{\sqrt{2}}\angle-\theta_i$$

$$S = V_{rms}I_{rms}\angle\theta_v - \theta_i \qquad \text{[in polar form]}$$

$$S = V_{rms}I_{rms}\,e^{j(\theta_v-\theta_i)} \qquad \text{[in exponential form]}$$

$$S = V_{rms}I_{rms}\cos(\theta_v-\theta_i) + jV_{rms}I_{rms}\sin(\theta_v-\theta_i)$$

$$S = \frac{V_mI_m}{2}\cos\phi + j\frac{V_mI_m}{2}\sin\phi$$

$$\phi = \angle\theta_v - \theta_i$$

7.3. Real Power or Reactive Power

The complex power may be expressed in terms of the load impedance *Z*.

The load impedance *Z* may be written as,

$$Z = \frac{V}{I} = \frac{V_{rms}}{I_{rms}} = \frac{V_{rms}}{I_{rms}}\angle\theta_v - \theta_i$$

$$V_{rms} = Z \times I_{rms}$$

$$S = I^2_{rms}Z = \frac{V^2_{rms}}{Z^*} = V_{rms}I^*_{rms}$$

Since, $Z = R + jX$

$$S = I^2_{rms}(R+jX) = P + jQ$$

$$S = I^2_{rms}(R+jX)$$

$$S = P + jQ = I^2_{rms}(R+jX)$$

where *P* and *Q* are the real and imaginary parts of the complex power, that is

$$P = \text{Re}(S) = I^2_{rms}R$$

$$Q = \text{Im}(S) = I^2_{rms}X$$

- *P* is the average or real power and if depend *S* on the loads resistance *R*.

- Q depend on the loads reactance X and is called the reactive (or quadrature) power.

Comparing equations (i) and (ii), we notice that,

$$P = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$$

$$Q = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$$

Points to be Remembered:

- The real power P is the average power in watts delivered to a load, it is the only useful power, it is actual power dissipated by the load.
- The reactive power Q is a measure of the energy exchange between the source and the reactive part of the load.

The nature of reactive power for different types of loads:

Reactive Power	Power factor	Load Types
$Q = 0$	Unity	Resistive
$Q < 0$	Leading	Capacitive
$Q > 0$	Lagging	Inductive

7.4. Power Triangle

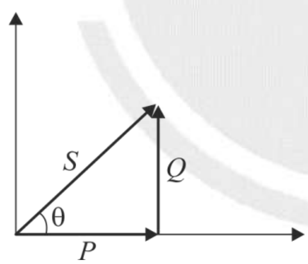


Fig.(a)

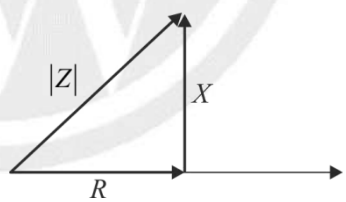


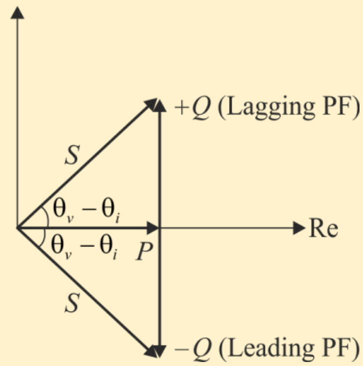
Fig.(b)

$$|S| = \sqrt{P^2 + Q^2}$$

$$\cos\phi = \frac{P}{S} = \text{ Power factor.}$$

$$\sin\phi = \frac{Q}{S} = \text{ Reactive factor}$$

Note:



1. $P = VI$ (In DC circuit)
2. $P = VI\cos\phi$ (In single phase AC circuit)
3. $P = \sqrt{3}V_L I_L \cos\phi$ (In three phase AC circuit)
4. $P = \sqrt{(S^2 - Q^2)}$
5. $P = \sqrt{(\text{VA})^2 - (\text{VAR})^2}$
6. $Q = VI\sin\phi$
7. Reactive power $= \sqrt{(\text{Apparent power})^2 - (\text{True power})^2}$
8. $Q = \sqrt{S^2 - P^2}$
9. $\text{KVAR} = \sqrt{(\text{KVA})^2 - (\text{KW})^2}$

Where: $P =$ Power in watt
$V =$ Voltage in volt
$I =$ Current in amperes
$\cos\phi =$ Power factor (phase angle difference)
$V_L =$ Line voltage
$I_L =$ Line current
$S =$ Apparent power in VA (volt ampere)
$Q =$ Reactive power in VAR (volt ampere reactive).

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8 Magnetic Coupling

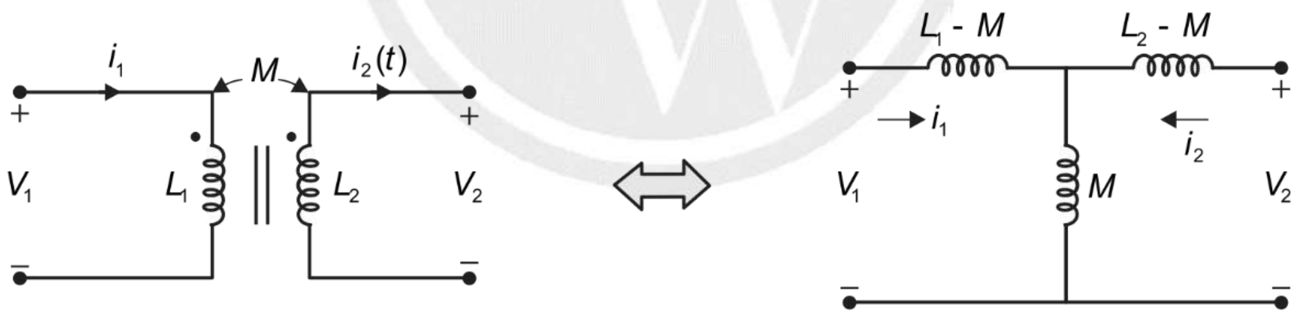
8.1 Introduction

When two loops with or without contacts between them affect each other through the magnetic field generated by one of them, they are said to be magnetically coupled. Whenever a current flows through a conductor, a magnetic field is generated (magnetic flux). When time varying magnetic field generated by one loop penetrates a second loop, a voltage induced between the ends of the second wire.

8.1.1 Important Points to Remember

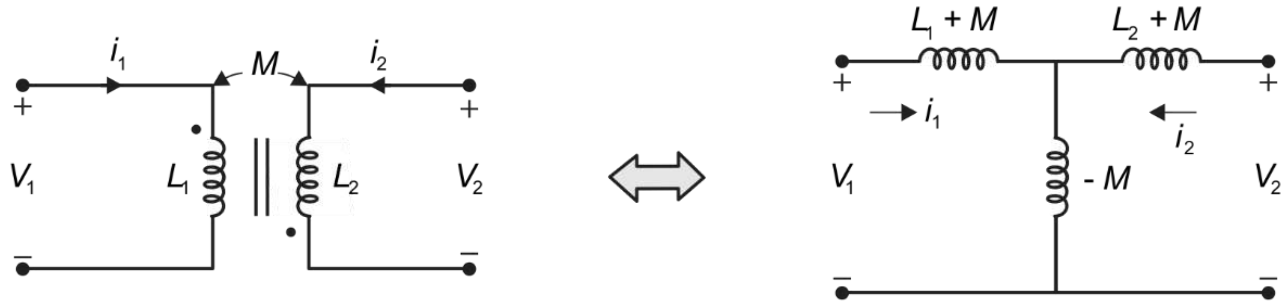
Magnetic Aiding:

If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal of the second coil.



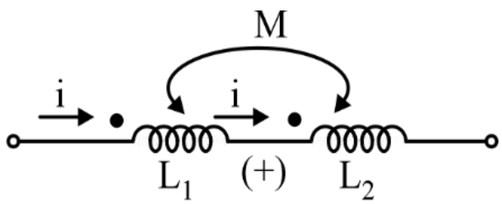
Magnetic Opposition:

If a current leaves the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is negative at the dotted terminal of the second coil.



Coupled inductors in series:

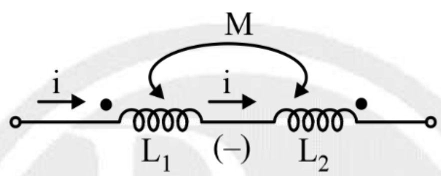
(i) Aiding



(a)

$$L = L_1 + L_2 + 2M \quad \text{(Series-aiding connection)}$$

(ii) Opposing



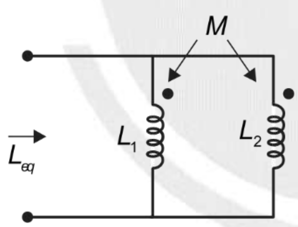
(b)

$$L = L_1 + L_2 - 2M \quad \text{(Series-opposing connections)}$$

Coupled inductors in parallel:

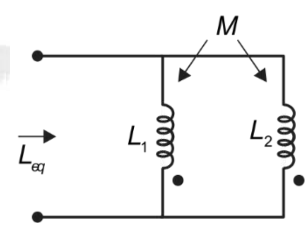
(i) Aiding

$$L_{eq} = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$$



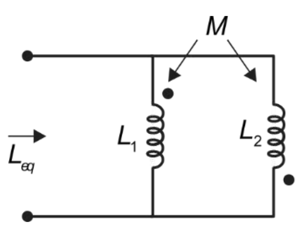
OR

$$L_{eq} = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$$



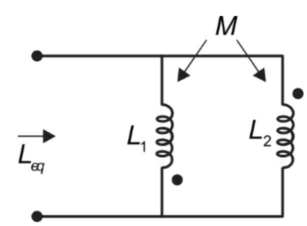
(ii) Opposing

$$L_{eq} = \frac{L_1L_2 - M^2}{L_1 + L_2 + 2M}$$



OR

$$L_{eq} = \frac{L_1L_2 - M^2}{L_1 + L_2 + 2M}$$



8.2. Coupling Coefficient, *k*

- A measure of the amount of magnetic coupling between two inductors

$$0 \le k \le 1$$

- k = 0: completely un-coupled inductors
- k = 1: perfect coupling – all magnetic flux generated by one inductor penetrates the coil of the other inductor.

- Relationship to mutual inductance, M:

$$M = k\sqrt{L_1 L_2}$$

8.1.2 Energy of Coupled Coil

Case (i): Magnetic Opposing Circuit

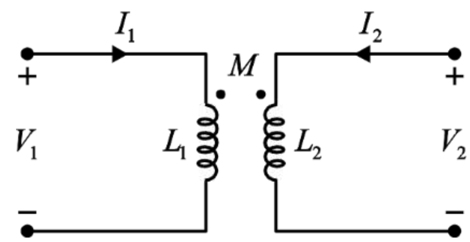


Fig. (a)

$$W = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2$$

Case (ii): Magnetic Adding Circuit

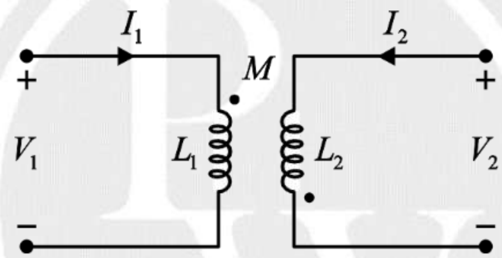


Fig. (b)

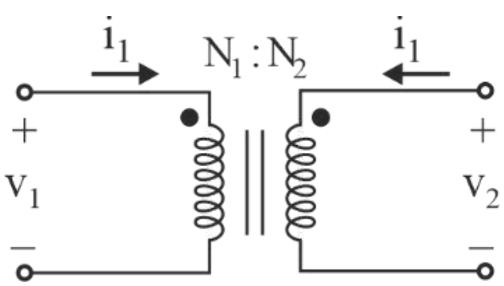
$$W = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 - M I_1 I_2$$

8.1.3 Dot Convention in Transformer

Transformation ratio or turn ratio is given by,

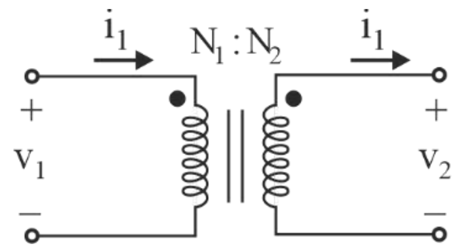
$$n = \frac{N_1}{N_2}$$

Case 1:



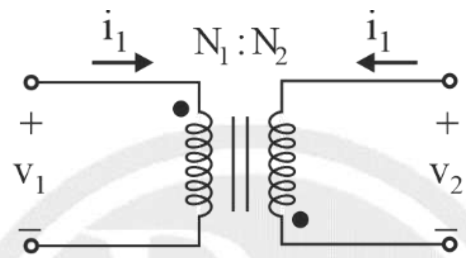
$$\frac{v_1}{v_2} = \frac{N_1}{N_2} \quad \frac{i_1}{i_2} = -\frac{N_2}{N_1}$$

Case 2:



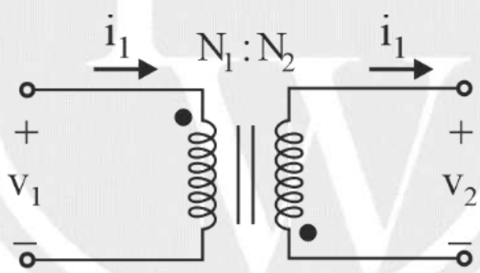
$$\frac{v_1}{v_2} = \frac{N_1}{N_2} \quad \frac{i_1}{i_2} = \frac{N_2}{N_1}$$

Case 3:



$$\frac{v_1}{v_2} = -\frac{N_1}{N_2} \quad \frac{i_1}{i_2} = \frac{N_2}{N_1}$$

Case 4:



$$\frac{v_1}{v_2} = -\frac{N_1}{N_2} \quad \frac{i_1}{i_2} = -\frac{N_2}{N_1}$$

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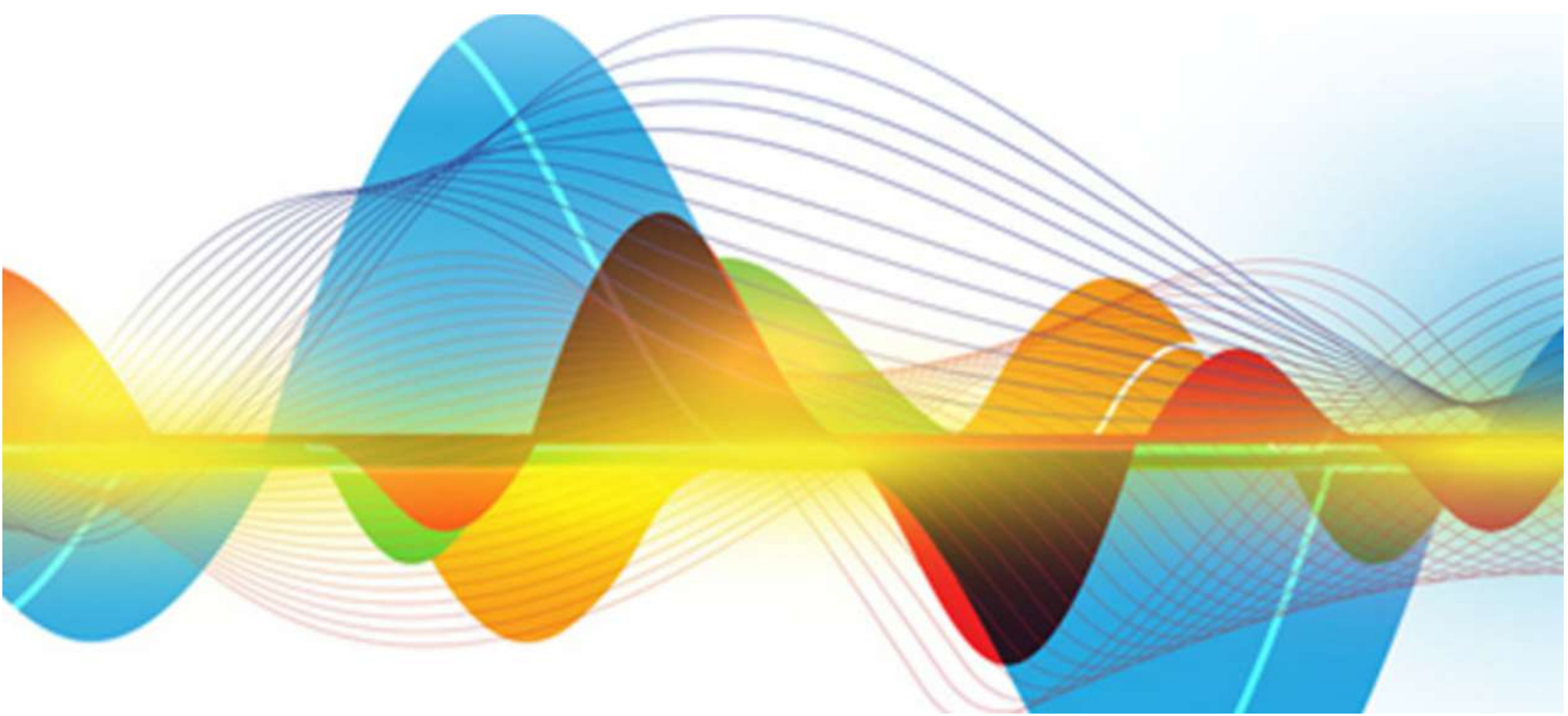
Signals and Systems



SIGNAL AND SYSTEMS

INDEX

1 BASIC SIGNALS AND SYSTEMS

1.1. Introduction

1.1.1. Continuous Time Signal

When independent variable is it continuous in time

Discrete Time Signal:

- Obtained from CTS by uniform sampling given a

$$t = nT_S$$

$$x(nT_S) = f(nT_S) \qquad a \le nT_S \le b$$

$$x(n) = f(nT_S) \qquad \frac{a}{T_S} \le n \le \frac{b}{T_S}$$

Continuous Time Signal $x(t)\ v\ \&t$

operations

an independent variable | an independent variable

On D.V.

(1) Amplitude: Given x(t) vs t, plot Ax(t) vs t every vertical axis parameter is multiplied by A

(2) Amplitude Reversal: Given x(t) vs t, plot $-x(t)\ vs\ t$ Take mirror image w. r. to horizontal axis

(3) Modulus - |x(t)| vs t

- ➢ Retain graph above horizontal axis.
- ➢ Take the mirror image of graph below horizontal axis.

(1) Addition or subtraction of dc value

Plot $x(t) \pm A\ vs\, t$

$x(t) + A \rightarrow$ Shift up

$x(t) - A \rightarrow$ Shift down

Operation on independent variable: Let x(t) is given

Every operation on t only $(t_0 > 0)$

(1) Time Shifting - Plot $x(t-t_0)$ or $x(t+t_0)$

$x(t-t_0)$ v & t $\rightarrow$ Shift x(t) vs to unit rightward

(Delay)

$x(t+t_0)$ v & t $\rightarrow$ Shift x(t) vs t to unit leftward

(Advance)

(2) Time scaling Plot x(at) vs t a > 0

Divide time axis by a

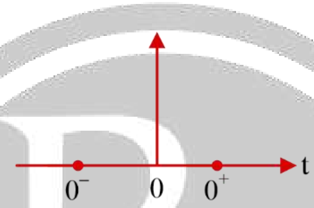
(3) Time Reversal Plot x(-t) v & t

Mirror image w.r.t. to vertical axis

Natural : Time shifting $\rightarrow$ Time scaling $\rightarrow$ Time Reversal

1.1.1. Standard Signals:

(1) Unit impulse



$$\delta(t) = 0 \quad t \neq 0$$
$$\delta(t) = \infty \quad t = 0$$
$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$
$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \neq 0 & t = 0 \rightarrow \infty \\ \int_{0^-}^{0^+} \delta(t)dt = 1 & \end{cases}$$

Properties

(1) $\delta(t) = \delta(-t)$: Even signal

(2) $\delta(t \pm t_o) \Rightarrow$ Not even signal

(3) $\delta(bt) = \frac{1}{|b|}\delta(t)$

(4) $\delta(-bt) = \frac{1}{|-b|}\delta(t)$

(5) $\delta(-bt+c) = \frac{1}{|-b|}\delta\left(t - \frac{c}{b}\right)$

(6) $\delta(-bt-c) = \frac{1}{|-b|}\delta\left(t + \frac{c}{b}\right)$

(7) $\delta(bt-c) = \frac{1}{|b|}\delta\left(t - \frac{c}{b}\right)$

(8) $\delta(bt+c) = \frac{1}{|b|}\delta\left(t+\frac{c}{b}\right)$

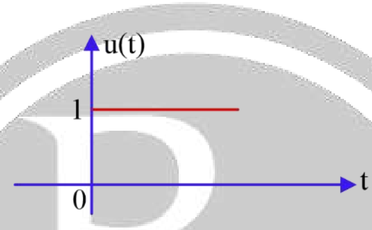
(9) $\delta[g(t)] = \sum_i \frac{\delta(t-t_i)}{|g(t_i)|}$ where t_i is root of $g(t)=0$

(10) $x(t)\delta(t) = x(0)\delta(t)$
$\downarrow$
$t=0$

(11) $\int_a^b x(t)\delta(t)dt = x(0)\int_a^b \delta(t)dt$

Unit step signal: $u(t) = \begin{cases} 1 : t \geq 0 \\ 0 : t < 0 \end{cases}$

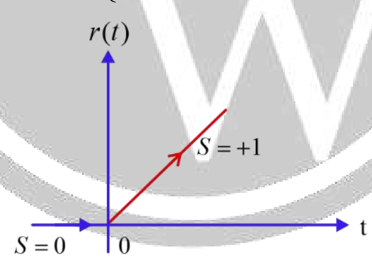
Property:



(1) $u(at) = u(t)$

(2) $2u(at) - 1 = Sgn(at)$

Unit Ramp signal : $r(t) = tu(t) = t\begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$



$r(at) = ar(t)$

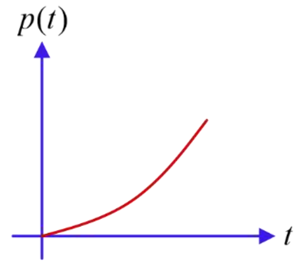
$r(at+b) = ar\left(t+\frac{b}{a}\right)$

$r(-at+b) = ar\left(-t+\frac{b}{a}\right)$

(1)	Impulse divide by a $\downarrow$ Horizontal axis	Ramp Divide by a
(2)	Divide by a (Area) $\downarrow$ Vertical axis	multiplied by a (Slope)

Unit Parabola Signals:

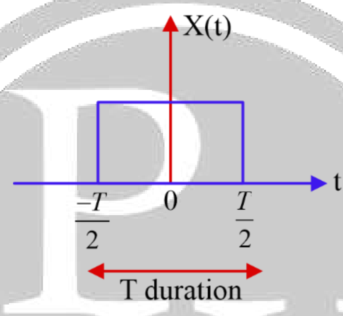
$$p(t)=\frac{t^2}{2}u(t)$$



$$p(t)\xrightarrow{d/dt}r(t)\xrightarrow{d/dt}u(t)\xrightarrow{d/dt}\delta(t)$$

$$\delta(t)\xrightarrow{\int_{-\infty}^{t}dt}u(t)\xrightarrow{\int_{-\infty}^{t}dt}r(t)\xrightarrow{\int_{-\infty}^{t}dt}p(t)$$

Gate pulse or Rectangular Pulse :



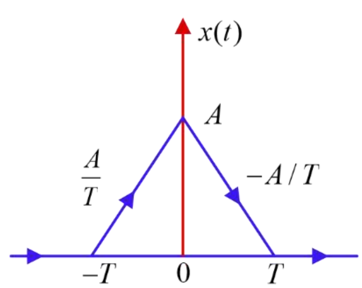
(i) $x(t)=\begin{cases}A & |t|\le T/2\\ 0 & \text{else}\end{cases}$

(ii) $x(t)=Au\left(t+\frac{T}{2}\right)-Au\left(t-\frac{T}{2}\right)$

(iii) $x(t)=A\text{rect}\left(t/T\right)$

$\downarrow$ amplitude (A), $\downarrow$ duration (T)

Triangular Pulse :



(i) $x(t)=\begin{cases}A\left(1-\frac{|t|}{T}\right) & : \ |t|\le T\\ 0 & : \ \text{else}\end{cases}$

$$x(t) = A\ tri\left(\frac{t}{T}\right)$$

(ii) $\underset{\text{peak}}{\downarrow} \quad \underset{\text{duration / 2}}{\downarrow}$

(iii) $x(t) = \begin{cases} A(1+t/T) & -T \le t < 0 \\ A & t = 0 \\ A(1-t/T) & 0 < t \le T \end{cases}$

(iv) $x(t) = \frac{A}{T} r(t+T) - \frac{2A}{T} r(t) + \frac{A}{T} r(t-T)$

SINC Function $\boxed{\sin ct = \frac{\sin \pi t}{\pi t}}$ $\boxed{\sin c(Kt) = \frac{\sin(K\pi t)}{K\pi t}}$

$\frac{\sin at}{bt} = \frac{a}{b} \sin c\left(\frac{at}{\pi}\right)$ # $\frac{\sin t}{t} = \sin c\left(\frac{t}{\pi}\right)$

Properties of $\sin c(t)$ -

(1) $\lim_{t\to 0} \sin c(t) = \lim \frac{\sin \pi t}{\pi t} = 1 = \sin c(0)$

(2) $\lim_{t\to\pm\infty} \sin c(t) = \lim_{t\to+\infty} \frac{\sin \pi t}{\pi t} = 0$

(3) $\sin c(-t) = \sin c(t)$ Even graph

$$\frac{\sin \pi(-t)}{\pi(-t)} = \frac{\sin \pi t}{\pi t}$$

(4) $t = n \quad n \in I,\ n = \pm 1$

$n \ne 0 \quad n = \pm 2$

(5) $\int_{-\infty}^{\infty} \sin c(t) = 1 \quad \Rightarrow \quad 2\int_{-\infty}^{\infty} \sin c(t)dt$

(6) $\int_{-\infty}^{\infty} \sin c(Kt)dt = 1/K$

(7) $\int_{-\infty}^{\infty} \sin c^2(t)dt = 1$

(8) $\int_{-\infty}^{\infty} \sin c^2(Kt)dt = \frac{1}{K}$

Sampling Function:

$$Sa(t) = \frac{\sin t}{t}, Sa(Kt) = \frac{\sin Kt}{Kt}, \frac{\sin at}{bt} = \frac{a}{b} Sa[at]$$

$$Sa(t) = \frac{\sin t}{t} = \sin c\left(\frac{t}{\pi}\right)$$

Properties:

(1) $\lim\limits_{t \to 0} Sa(t) = 1$

(2) $\lim\limits_{t \to \pm\infty} Sa(t) = 0$

(3) $Sa(-t) = Sa(t)$

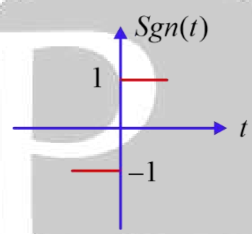
(4) Zero crossover - $t = n\pi, n \in I \quad n \neq 0$

(5) $\int\limits_{-\infty}^{+\infty} Sa(t)dt = \pi$

(6) $\int\limits_{-\infty}^{\infty} Sa^2(t)dt = \pi$

Signum Function:

$$Sgn(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$



$Sgn\ (Sgn\ (Sgn\ t)) = Sgn(t)$

$Sgn(t) = 2u(t) - 1 = \dfrac{t}{|t|}$

Discrete Time Signal:

$$(DTS) \rightarrow \begin{matrix} \nearrow \text{ Natural DTS} \\ \searrow \text{ Converted DTS} \end{matrix}$$

Important Points:

(1) $x(n) = \{1, \underset{\uparrow_{n=0}}{2}, 3\}$ Finite duration

(2) $x(n) = \{1, \underset{\uparrow}{2}, 3 - - - - - \}$ Infinite duration + Right sided

(3) $x(n) = \{- - - - 3, \underset{\underset{n=0}{\uparrow}}{2}, 1\}$ Infinite duration + left sided

(4) $x(n) = \{- - - 3, 3, 2, 1, 4, 4 - - -\} \rightarrow$ Duration infinite

\# $x(n - n_0) \rightarrow$ Left
$x(n + n_0) \rightarrow$ Right

\# $x(-n) VS\ \ n \rightarrow$ Mirror image about vertical axis.

Time Scaling: plot $x(an)$ VS n

Case 1. $a > 1$ $x(n) = \left\{1, 2, 3, \underset{\underset{n-0}{\uparrow}}{4}, 5, 6, 7, 9\right\}$ Decimation ,

$x(2n) = \left\{\ 2, \underset{\uparrow}{4}, 6, 8\right\}$

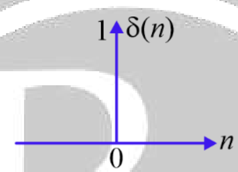
Case 2. $a < 1$ $x(n) = \left\{1, 2, \underset{\uparrow}{3}, 4\right\}$

$x\left(\frac{n}{2}\right) = \{1\ \ 0\ \ 2\ \ 0\ \ 3\ \ 0\ \ 4\}$

- Interpolation of zero

Unit Impulse Signal :

$$\delta = \begin{cases} 1 & : \quad n = 0 \\ 0 & : \quad n \neq 0 \end{cases}$$



Properties:

(1) $\delta[-n] = \delta[n]$: Even

(2) $\delta[an] = \delta[n]$

(3) $\delta[-an+b] = \delta[-a(n-b/a)] = \delta\left[n - \frac{b}{a}\right]$

(4) $x(n)\delta(n) = x(0)\delta(n)$
$n = 0$

(5) $x(n)\delta(n-n_0) = x(n_0)\delta(n-n_0)$

(6) $x(n)\delta(-an+b) = x\left(\frac{b}{a}\right)\delta\left[n - \frac{b}{a}\right]$

(7) $\delta(n) \times \delta(n) ----- = \delta(n)$

(8) $\delta[n] + \delta[-n] = 2\delta[n]$

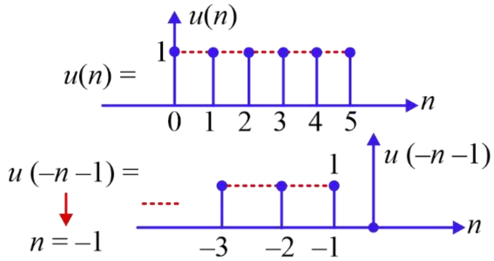
(9) $\delta[n] - \delta[-n] = 0$

(10) $\sum_{K=-\infty}^{\infty} \delta(K) = 1$

(11) $\sum_{K=n_1}^{n_2} \delta(K)$ $\nearrow$ if $\delta[K]$ lies between $n_1 \leq K \leq n_2$ $\searrow$ 0 else where

(12) $\sum_{n=n_1}^{n_2} x(n)\delta(-an+b) = x\left(\frac{b}{a}\right)\sum_{n-n_1}^{n_2}\delta\left(n - \frac{b}{a}\right)$ $\nearrow x(b/a)$ $\searrow 0$

Unit Step Signal:



(1) $u(n)+u(-n-1)=(1)^n$

$u(-t)\xleftrightarrow[\text{Analogy}]{}u(-n-1)$

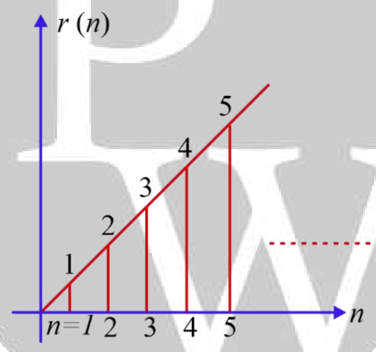
(2) $u(n)u(-n-1)=0$

(3) $u[n]+u[-n]=\begin{cases}2 & : \quad n=0\\ 1 & : \quad n\neq 0\end{cases}$

(4) $u(n)\times u(-n)=\delta(n)$

(5) $u(n)+u(-n-1)=1$

$\boxed{u(n)=\sum_{K=0}^{\infty}\delta[n-K]}$ $\boxed{u(n)=\sum_{K=-\infty}^{n}\delta[K]}$ $\boxed{\delta[n]=u[n]-u[n-1]}$



Unit Ramp Sequence:

$$r(n)=\sum_{K=0}^{\infty}u[n-K-1]$$

$$r(n)=\sum_{K=-\infty}^{n-1}u[K]$$

Even /odd | N.E.N.O:

(1) Even - $x(-t)=x(t)$

$x(-n)=x(n)$

graph , must be symmetrical about the vertical axis.

$\int_{-\infty}^{\infty}x(t)dt=2\int_{-\infty}^{0}x(t)dt \begin{matrix}\nearrow =0\\ \searrow \neq 0\end{matrix}$ $\quad Eg-\delta(t),\delta(n),\sin c(t),|t|,$

$\cos t,\ |\sin t|$

(2) Odd Signal, $x(-t)=-x(t)$ Graph Must be Symmetrical about origin.

$x(-n)=-x(n)$

Eg- $\sin t,\ \text{sgn}(t),\ t,\ 1/t,\ n,\ \sin n$

$$\int_{-\infty}^{\infty} x(t)dt = 0, \qquad \sum_{n=-\infty}^{\infty} x(n) = 0$$

(3) Neither Even nor odd –

Eg- $u(t), r(t), u(n), \delta(t-2), \delta(n-2)$

$$x_e(t) = \frac{x(t)+x(-t)}{2}, \qquad x_e(n) = \frac{x(n)+x(-n)}{2}$$

$$x_0(t) = \frac{x(t)-x(-t)}{2}, \qquad x_0(n) = \frac{x(n)-x(-n)}{2}$$

$x_1(t) \mid x_1(n)$	$x_2(t) \mid x_2(n)$	$x_1 \cdot x_2$	$x_1 \mid x_2$
E	E	E	E
E	0	0	0
0	E	0	0
0	0	E	E

Conjugate Symmetry :

(1) Even Conjugate

(2) $x(-t) = x^*(t)$ $\qquad x(t)$: Even Conjugate $\Rightarrow \text{Re}\,[x(t)] = \text{Even}$

$x(-n) = x^*(n)$ $\qquad x(n)$: $\qquad \text{Im}[x(t)] = \text{odd}$

$x(t) \mid x(n) \rightarrow \text{complex}$

Conjugate Anti Symmetry :

(1) odd conjugate

(2) $\left.\begin{array}{l} x(-t) = -x^*(t) \\ x(-n) = -x^*(n) \end{array}\right] x(t) \mid x(n)$ complex

Periodic & Non periodic Signal :

For continuous time signal –

(1) Graph must repeat itself from $-\infty$ to $+\infty$:- $-\infty < t < \infty$

(2) $x(t+T_0) = x(t_0 - T_0) = x(t)$

To = Smallest duration = fundamental Time period

To = +ve and constant , integer or non integer , rational or Irrational

Complex Exponential

$$x(t) = Ae^{j(\omega_0 t+\phi)}, T_0 = \frac{2\pi}{\omega_0}$$

$$A\cos(\omega_0 t+\phi) \quad T_0 = \frac{2\pi}{\omega_0}$$

$x_1(t)$	$x_2(t)$	$x(t)=x_1(t)+x_2(t)$	$x(t)=x_1x_2$
P	P	?	?
N	NP	NP	NP
NP	P	NP	NP
NP	NP	NP	NP

Continuous time sinusoids or complex exponential are always individually periodic (irrespective of ω_0)
The linear combination of above may or may not be period

Periodicity of Liner combination of C.T sinusoidal –

$$x(t) = A + B\cos(\underset{\substack{\downarrow \\ T_1=\frac{2\pi}{\omega_1}}}{\omega_1 t} + \phi_1) + C\sin(\underset{\substack{\downarrow \\ T_2=\frac{2\pi}{\omega_2}}}{\omega_2 t} + \phi_2) - D\cos(\underset{\substack{\downarrow \\ T_3=\frac{2\pi}{\omega_3}}}{\omega_3 t} + \phi_3)$$

S-1 T_1, T_2, T_3

S-2 $\frac{T_1}{T_2}:R,\ \frac{T_1}{T_3}:R$ $\quad x(t)$ is periodic.

S-3 $T_0 = LCM(T_1,T_2,T_3) = \frac{\text{LCM of Numeratar}}{\text{HCF of Denominator}}$

$$\omega_0 = \frac{2\pi}{\omega_0} = HCF(\omega_1,\omega_2,\omega_3) = \frac{\text{HCF of } N^r}{\text{LCM of } D^r}$$

$\omega_1 = K_1\omega_0 \ K_1th$ Harmonic

$\omega_2 = K_2\omega_0 \ K_{2th}$

$\omega_3 = K_3\omega_0$

Discrete Tie Periodic signal :

Fundamental Time period – Minimum no of samples Which repeats itself

$$\boxed{x(n+n_0)=x(n)}$$

- $N_0 \neq 0,\ N_0 \neq \infty,\ N = +ve,\ N_0 =$ Integer N_0 cannot be negative
- Discrete time sinusoids and complex exponential are not individually periodic always

  Steps – $x(n) = A\cos(\omega_0 n + \phi)$

  S-1 $N = \frac{2\pi}{\omega_0}$ $\nearrow$ R:periodic $\searrow$ IR:Non periodic

  S-2 $FTP = N_0 = N \times r$ (r is smallest integer which makes N_0 integer)

Periodicity of under combination of discrete time signal –

x_1	x_2	$\pm x_1 \pm x_2$
P	P	P
P	NP	NP
NP	P	NP
NP	NP	NP

$$x(n) = A(1)^n + B\cos(\underset{\downarrow N_{0_1}}{\omega_1} n + \phi_1) + C\cos(\underset{\downarrow N_{0_2}}{\omega_2} n + \phi_2) + D\sin(\underset{\downarrow N_{0_3}}{\omega_3} n + \phi_3)$$

$$\boxed{N_0 = LCM(N_{0_1}, N_{0_2}, N_{0_3})}$$

Note:

C.T.S	D.T.S
$x(t) \to T_0$	$x(n) = T_0$
$x(-at+b) = \dfrac{T_0}{\lvert a \rvert}$	$x(-an+b) \to T_0 =$ P check
$P \times NP = NP$	$P \times NP = NP$
NP should not be constant	NP should not be constant

➤ $x(n) = A\cos[\omega_0 T_S]n$

$$N = \frac{2\pi}{\Omega_0} \to \text{Rational}, \quad \frac{2\pi}{\omega_0 T_s} \to \text{Rational}, \quad \frac{T_0}{T_s} \to \text{Rational}$$

Orthogonal – If inner product of two Signal is zero

$$\int_{-\infty}^{\infty} x_1(t).x_2^*(t)dt = 0, \quad \int_{<T_0>} x_1(t)x_2^*(t)dt, \quad \sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = 0$$

$$\sum_{n=<N_0>} x_1(n)x_2^*(n) = 0$$

Energy , Power, NENP:

(1) N.E.N.P $\to$ $x(t)/x(n)$ $\to \pm\infty$ at any signal value of t/n

(2) Energy signal – Must have finite energy for infinite possible duration .

$$\underset{\text{watt}}{\underset{\downarrow}{P}} = \frac{E}{T}\ \frac{\text{(Joules)}}{\text{sec}} \begin{array}{l} \nearrow \text{finite} \\ \searrow \text{Infinite} \end{array} = 0$$

Signals → Periodic → Can never be energy signal

Signals → Non Periodic → Amplitude finite

Amplitude finite → Duration finite

Amplitude finite → Duration = infinite → Amplitude at t or n

- Formula - $E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$, $E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$
- $|x(t)|^2 = x^2(t)$ for real value of $x(t)$.
- If $x(t) = x_1(t) + x_2(t)$

$$\boxed{E_x = E_{x_1} + E_{x_2} + \int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt + \int_{-\infty}^{\infty} x_1^*(t)x_2(t)dt}$$

$$\downarrow$$

If x_1 and x_2 are orthogonal

$E_x = E_{x_1} + E_{x_2}$

Note :

Signal		**Energy**
$x(t)$	$\longrightarrow$	E_x
$x(t-t_0)$	$\longrightarrow$	E_x
$x(-t)$	$\longrightarrow$	E_x
$x(at)$	$\longrightarrow$	$E_x / \vert a \vert$
$x(-at+b)$	$\longrightarrow$	$E_x / \vert a \vert$
$-Kx(-at+b)$	$\longrightarrow$	$\vert K \vert^2 \dfrac{E_x}{\vert a \vert}$

Discrete time Energy Signal:

$$\boxed{E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2} \qquad |x(n)|^2 = x^2(n) \text{ for } x(n) \text{ real}$$

$$E_x = E_{x_1} + E_{x_2} + \sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) + \sum_{n=-\infty}^{\infty} x_1^*(n)x_2(n)$$

Average Value

$$x(t)/x(n) \text{ is periodic } T_0/N_0 \rightarrow \bar{x}(t)\frac{1}{T_0}\int_{T_0} x(t)dt, \bar{x}(n) = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x(n)$$

$$x(t)/x(n) \text{ is non periodic } \Rightarrow \bar{x}(n) = \lim_{N\to\infty}\left[\sum_{n=-N/2}^{n=N/2} x(n)/N\right], \bar{x}(t) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x(t)dt$$

Power Signal

Signals → Periodic → Power signal when it is not NENP

Signals → Non Periodic → Amplitude finite

Duration infinite

$t \to \infty \Rightarrow 0$, $t \to -\infty \neq 0$

$t \to \infty \Rightarrow \neq 0$, $t \to -\infty \Rightarrow 0$

$t \to -\infty \neq 0$, $t \to +\infty \neq 0$

Periodic (T_0 / N_0)	Non Periodic
$P_x = \frac{1}{T_0}\int_{<T_0>} \lvert x(t)\rvert^2\, dt = MSV \quad \underset{\text{Avrage value of } \lvert x(t)\rvert^2}{\underset{\downarrow}{[x/(t)]}}$	$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} \lvert x(t)\rvert^2\, dt = \overline{\lvert x(t)\rvert^2}$
$P_x = \frac{E_{xT_0}}{T_0} = \frac{\text{Energy of } 1\,T_0 \text{ of } x(t)}{T_0}$	

$$P_x = P_{x_1} + P_{x_2} + \lim_{T\to\infty}\int_{\frac{-T}{2}}^{T/2} x_1(t)x_2^*(t)dt + \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x_1^*(t)x_2(t)dt \quad \text{for non periodic}$$

If $x_1(t)$ and $x_2(t)$ are orthogonal $\to P_x = P_{x_1} + P_{x_2}$

Properties for Periodic Signal:

(1) Power signal has finite Energy.

$$\underset{\text{finite}}{\underset{\swarrow}{P}} = \frac{E \to \infty}{T \to \infty}$$

(2) $-Kx(-at+b) = \lvert -K\rvert^2 P$

(3) $Px = \frac{1}{T_0}\int_{T_0} \lvert x(t)\rvert^2\, dt = \frac{ET_0}{T_0}$

Discrete Time Power Signal:

$x(n)$ is power signal

$$x(n) \text{ is non periodic signal - } P_x = \lim_{N\to\infty}\frac{1}{(2N+1)}\sum_{n=-N}^{N}\lvert x(n)\rvert^2$$

$$P_x = \frac{E_{N_0}}{N_0}$$

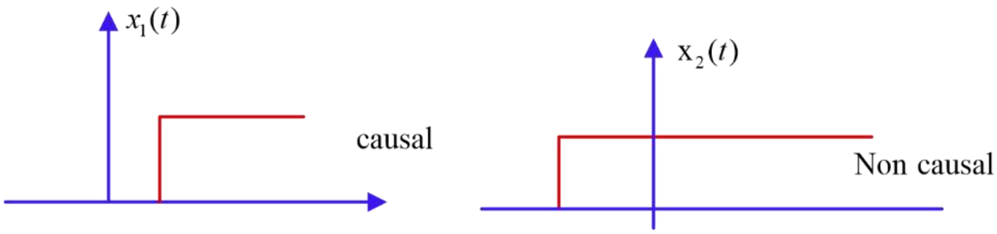
Causal non causal ant Causal:

(a) Causal signal $x(t) = 0$ for $t < 0$

$x[n] = 0$ for $n < 0$, $n \leq -1$

Part of graph for –ve value of time = 0

(b) Non causal – Which is not causal



(c) Anti causal $\rightarrow$ $x(t) = 0 \quad t \geq 0$, $x(n) = 0 \quad n \geq 0$ Graph should be zero for +ve value of time including 0

$u[n]$ – causal   Anti causal $\rightarrow$ Non causal

$u[-n-1]$ - Anti causal

$u[-n]$ - Non causal

➢ $x(t)$ $\nearrow \int_{-\infty}^{\infty} x(t)dt \rightarrow$ finite $\rightarrow$ Integrable

$\searrow \int_{-\infty}^{\infty} |x(t)|dt \rightarrow$ finite $\rightarrow$ Absolutely integrable

➢ $\sum_{n=-\infty}^{\infty} x(n) =$ finite $\rightarrow$ summable

$\sum_{n=-\infty}^{\infty} |x(n)| =$ finite $\rightarrow$ Absolutely summable.

Bounded Signal – x(t) is Bounded

$|x(t)| \leq M < \infty \qquad -\infty < t < \infty$

(finite)

$|x(t)| \leq M < \infty \qquad -\infty < t < \infty$

(finite)

Ex– $\cos t / \sin t,\ \mathrm{sgn}(t), u(t), dc, e^{-a|t|}\ a > 0,\ \delta[n]$

Static and Dynamic System :

Static – output should depends only on present value of input

$Ex - \quad y(t) = \sin[x(t)], y(t) = |x^2(t)|$

Dynamic – Not static

$Ex - \quad y(t) = \text{Even } [x(t)], y(t) = \frac{d}{dt} x(t), y(t) = \int_{-\infty}^{t} x(\tau) d\tau$

Causal and Non causal :

- Causal – output at any instant of time depends on either input at same instant of time or input at past instant of time.

  (OR)

- Output depends on past or present values of input.
- Non causal – which is not causal.
- Anti causal – output depends on future value of input value

Linear – Non liner:

Linear equation : $y = mx + c$

Non linear : $y^2 = x, \sin x, \cos x$

linear system : Additivity + Homogeneity

S.1 $\begin{matrix} x(t) \xrightarrow{S} y(t) \searrow \\ x_1(t) \xrightarrow{S} y_1(t) \nearrow \end{matrix} \oplus \rightarrow y_1(t) + y_2(t)$ ...(i)

S.2 $x_2(t) \xrightarrow{S} y_2(t) \Rightarrow x_1(t) + x_2(t) \longrightarrow y_3(t)$ ...(ii)

Equation (i) = equation (ii)

S.3 $A\,x(t) \xrightarrow{S} y_4(t)$ ...(iii)

$A\,y(t) = ?$ ...(iv)

equation (iii) = equation (iv) $\rightarrow$ Homogeneity is satisfied

Time variant and Invariant :

$$x(t - t_0) \longrightarrow \boxed{S/S} \longrightarrow y(t - t_0)$$

Identity definition of system.

$$x(t) \xrightarrow{S} y(t)$$
$$x_1(t) \longrightarrow y_1(t)$$
$$x_1(t) = x(t - t_0)$$
$$y_1(t) = ? \;\_\_\_\_\_\_\text{(i)}$$

S-3 Mathematical exp. $y[t - t_0]$ ----(iii)

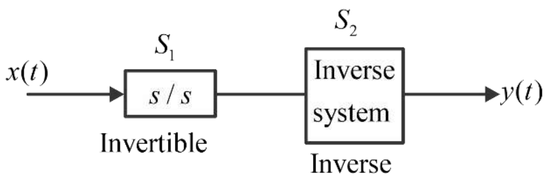
equation (i) = equation (ii) Time Invariant

Invertible and Non Invertible:

Invertible – There must be a one to one mapping between the input and output .

S-1 Replace x and y .

S-2 Obtain y completely in terms of x

S-3 

- Inverse System may or may not be Invertible .

Stable and Unstable :

Stable S/S – Bounded input – Bounded output.

$x(t)/x(n)$ is Bounded –

$$|x(t)| \le M \underset{\to \text{finite}}{<} \infty; -\infty < t < \infty$$

$$|x(n)| \le M < \infty; -\infty < t < \infty$$

$$\text{Ex-} \quad \begin{array}{lccc} & x(t) & x(n) & \\ \to & dc & dc & \\ \to & u(t) & u(n) & \delta(n) \\ \to & \text{sinusoidal} & \text{sinusoidal} & \end{array}$$

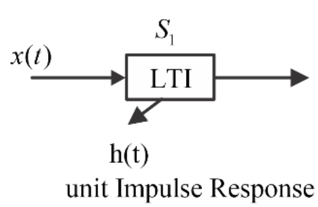
Then y(t) must be bounded

$$\left.\begin{array}{l} y(t) \le N < \infty \\ |y(n)| \le N < \infty \end{array}\right. \quad \text{finite}$$

- Finite $\to$ time duration

  Bounded $\to$ Amplitude / Magnitude

1.2. Continuous Time LTI System



$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t) = x(t) * h(t)$$

$\to$ Convolution operator

Convolution Integral :

$$x(t)*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

Properties Convolution :

(1) $A*B = B*A$

(2) Cumulative: $x(t)*h(t) = h(t)*x(t)$

(3) Distributive: $x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t) + x(t) * h_2(t)]$

(4) Associative: $x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t) * h_2(t)]$

(5) $y(t) = x(t) * h(t) \Rightarrow A = A_1 \times A_2$

(6) $x(t-a) * h(t-b) = y[t-a-b]$

(7) $x(-t) * h(-t) = y(-t)$

(8) $x(at) * h(at) = \dfrac{1}{|a|} y(at)$

(9) $Ax(t) * Bh(t) = ABy(t)$

(10) $\left(\dfrac{d^n x(t)}{dt^n}\right) \times \left(\dfrac{d^m h(t)}{dt^m}\right) \Rightarrow \dfrac{d^{m+n} y(t)}{dt^{m+n}}$

Standard Result :

(1) $x(t) * \delta(t) = x(t)$

(2) $x(t-a) * \delta(t-b) = x(t-a-b)$

(3) $\delta(t) * \delta(t) = \delta(t)$

(4) $\delta(t) * \delta(t) - - - - - - - = \delta(t)$

(5) $\delta(t-a) * \delta(t-b) = \delta(t-a-b)$

(6) $x(t) * u(t) = \int_{-\infty}^{t} x(\tau) d\tau$

(7) $\delta(t) * u(t) = u(t)$

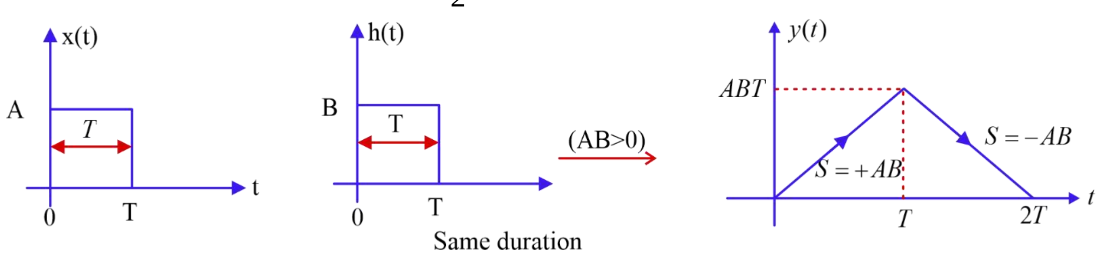
(8) $u(t) * u(t) = r(t)$

(9) $u(t-a) * u(t-b) = r(t-a-b)$

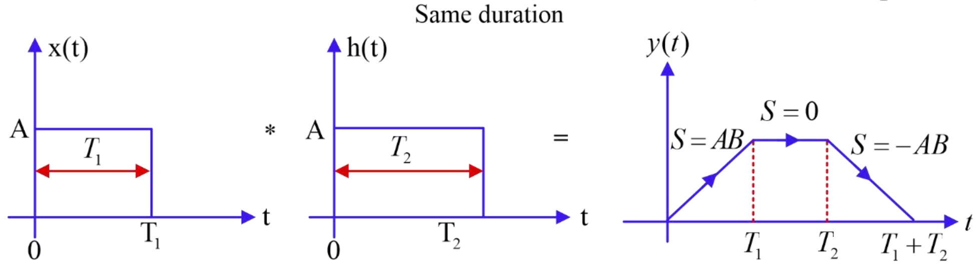
(10) $u(t) \times r(t) = p(t)$

(11) $u(t-a) * r(t-b) = p(t-a-b) = \dfrac{(t-a-b)^2}{2} u(t-a-b)$

(12)



(13)



Differential of a Signal :

Sinusoidal

Non Sinusoidal
- Upto slope variation (Ramp variation)
- Beyond slope variation

$x(t)$	Slope	$Dx(t)dt \rightarrow$ Slope
$S=0$	$S=0 \longrightarrow$	Part of time axis
$S=+m$	$S=+m \longrightarrow$	m
A_1	$S=+\infty \longrightarrow$	Upward Impulse $=A_1$
A_2	$S=-\infty \longrightarrow$	Downward Impulse $=-A_2$

Integration: $x(t), y(t)$

$$y(t)=\int_{-\infty}^{t}=x(t)dt$$

- RunningIntegratio
- Area of $x(t)$ from $-\infty$ upto t

Convolution Method :

Method (1) $x(t)*u(t)=\int_{-\infty}^{t}x(\lambda)d\lambda$

Method (2) Rectangular pulse $\rightarrow$
- Same duaration (Triangle)
- Different duration (Trapezoidal)

Method (3) $y(t)=\int_{-\infty}^{t}[x(t+1)+x(t-1)]dt$

Method (4) Timeline Method

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$$

S -1 Given : $x(t)$ and $h(t)$

S – 2 $x(\tau)$ and $h(t-\tau)$

S – 3 Make time line of $x(\tau)$ vs τ and $h(t-\tau)$ vs τ

S – 4 Vary t and determine the integration

Method (5) **Graphical Method**

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$$

S -1 Given : $x(t)$ and $vs\ t$ and $h(t)\ vs\ t$

S – 2 $x(\tau)\ vs\ t$ and $h(\tau)\ vs\ \tau$

S – 3 $h(t-\tau)\ vs\ \tau$

$$h(\tau) vs\ \tau \xrightarrow{\text{fold}} h(\tau)\ vs\ \tau \xrightarrow[\text{by t}]{\text{Right Shift}} h(t-\tau) vs\ \tau$$

S – 4 Vary t and calculate integration

Note: Before solving the problem of convolution decide the range of convolution

1.2.1. Discrete Time L.T.I. System

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

$$\boxed{y(n) = x(n) * h(n) = \sum_{K=-\infty}^{\infty} x(K)h(n-K)}$$

$h(n)$: unit impulse response of D. T LTI system

Or

Mathematical representation of D. T LTI system

Or

D.T LTI system parameter

$$x(n) * \delta(n) = \sum_{K=-\infty}^{\infty} x(K)\delta(n-K) = x(n)$$

$$x(n) * u(n) = \sum_{K=-\infty}^{\infty} x(K)u(n-K) = \sum_{K=-\infty}^{n} x(K)$$

Standard Result :

(1) $\delta(n-n_1) * \delta(n-n_2) = \delta(n-n_1-n_2)$

(2) $x(n-n_1) * \delta(n-n_2) = x(n-n_1-n_2)$

(3) $u(n) * u(n) = (n+1)u(n)$

(4) $u(n+\alpha) * u(n+\beta) = r(n+\alpha+\beta+1) = (n+\alpha+\beta+1)u(n+\alpha+\beta+1)$

Method Of Discrete Time Convolution:

Either $x(n)$ or h (n) or both are having infinite duration

$$y(n) = \sum_{K=-\infty}^{\infty} x(K)h(n-K)$$

Both $x(n)$ and h (n) are of finite duration
Tabular Method

Basic Methods :

(1) By using standard Method

(2) Time line Method : $y(n)\sum_{K=-\infty}^{\infty} x(K)h(n-K)$

S.1 $x(K), h(n-K)$

S. 2 vary n and calculate summation .

(3) Graphical Method: $y(n)=\sum_{K=-\infty}^{\infty} x(K)h(n-K)$

S.1 x (1) VS K

S.2 $h(K)\ VS\ K \xrightarrow{\text{fold}} h(-K)\ VS\ K \xrightarrow[\text{shift by n}]{\text{Right}} h(n-K)\ VS\ K$

S.3 very n and calculation summation .

$x(n)=l,\ \ h(n)=m,\ y(n)=l+m-1$

Tabulation

	$x(0)$	$x(1)$	$x(2)$
$h(0)$	$x(0)h(0)$	$x(1)h(0)$	$x(2)h(0)$
$h(1)$	$x(0)h(1)$	$x(1)h(1)$	$x(2)h(1)$
$h(2)$	$x(0)h(2)$	$x(1)h(2)$	$x(2)h(2)$

$$y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$\nearrow t<\tau$(above system behaves as non causal system)

$\searrow t\geq\tau$(above system behaves as casual system).

For an LTI system to be causal system:

$h(t-\tau)=0 \qquad t<\tau$

$h(t-\tau)=0 \qquad t-\tau<0\, t-\tau=p \qquad h(t)=0, \text{for } t<0$

$h(p)=0 \qquad p<0$

$h(t)=0 \qquad t<0$

$h(n-K)=0 \ ;\ n<K \ ;\ n\leq K-1$

$h(n-K)=0 \ ;\ n-K<0 \ ;\ n-K\leq -1 \qquad h(n)=0 \text{ for } n<0$

$h(p)=0 \ ;\ p<0 \ ;\ p\leq -1 \qquad n\leq -1$

$h(n)=0 \ ;\ n<0 \ ;\ n\leq -1$

Stability of LTI System :

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Let $|x(t)| \leq M < \infty$

$$|x(t-\tau)| \leq M < \infty$$

$$\boxed{|y(t)| \leq \int_{-\infty}^{\infty} M|h(\tau)|d\tau} \quad N$$

$$N = \int_{-\infty}^{\infty} \underset{\text{finite}}{\underset{\downarrow}{M}}\, |h(\tau)|d\tau \begin{cases} \nearrow N : \text{finite} \\ \searrow \int_{-\infty}^{\infty} |h(\tau)|d\tau \rightarrow \text{finite} \end{cases}$$

For discrete :

$$\boxed{|y(n)| \leq \sum M|h(K)|} \rightarrow N \qquad |x(n-K)| \leq M$$

$$N = M\sum_{-\infty}^{\infty}|h(K)| \rightarrow \text{finite}, \qquad \sum_{-\infty}^{\infty}|h(K)| \rightarrow \text{finite}$$

Note : $h(t) : e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$: stable system when $a > 0$

$h(n) : a^{|n|} = a^n u(n) + a^{-n}u[-n-1]$ stable system when $|a| < 1$

1.3. Static and Dynamic System

For an L.T.I system to be static the unit impulse response $h(t) / h(n)$ must be an impulse signal.

Invertible and Non Invertible system–

$$x(t) \longrightarrow \boxed{h(t)} \xrightarrow[y_1(t)]{} \boxed{h_I(t)} \longrightarrow y(t) = x(t)$$

$$y_1(t) = [x(t) * h(t)]$$

$$y(t) = y_1(t) * h_I(t) = x(t) * [\underbrace{h(t) * h_I(t)}_{S(t)}]$$

$$y(t) = x(t)$$

$$h(t) * h_I(t) = S(t) \Rightarrow \boxed{H_I(S) = \frac{1}{H(S)}}$$

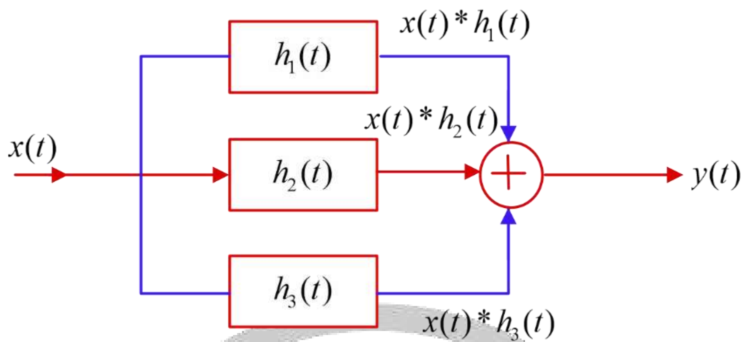
- For discrete $H_I(z) = \dfrac{1}{H(z)}$
- Unit step Response : $s(t) \Rightarrow \dfrac{ds(t)}{dt} = h(t)$ unit impulse Response
- Unit impulse Response : $h(t) \Rightarrow \int_{-\infty}^{t} h(\tau)d\tau = s(\tau)$ unit step response

For discrete :

Unit step - $s(n)$, $s(n)-s(n-1)=h(n)$: unit impulse response

Unit Impulse - $h(n)$, $\sum_{K=-\infty}^{n} h(K)=s[n]$ unit step response

LTI System in Cascaded :



$$x(t) \longrightarrow \boxed{ht} \longrightarrow yt$$

$$ht = h_1(t)*h_2(t)*h_3(t)$$

LTI System in Cascaded:

$$x(t) \longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow y(t)$$

$$x(t) \longrightarrow \boxed{h_1(t)*h_2(t)} \longrightarrow y(t)$$

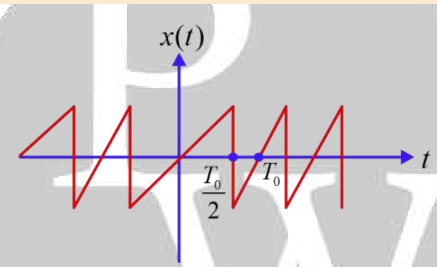
2 CONTINUOUS TIME FOURIER SERIES

2.1. Introduction1

$$x(t) = A\sin\omega_0 t \begin{array}{l} \nearrow \text{Sinusoidal} \\ \searrow \text{Periodic } C \to \omega_0 \end{array}$$

Fourier series is the representation of time domain non sinusoidal periodic signal as the weighted sum of harmonically related , mutually orthogonal sinusoids .

2.1.1. Trigonometric Fourier Series:



$$x_{FS}(t) = a_0 + \sum_{n=1}^{\infty}[(a_n \cos n\,\omega_0 t) + (b_n \sin n\omega_0 t)]$$

$$T_0 = \frac{2\pi}{\omega_0}$$

$a_0, a_n, b_n \to$ Trigonometric Fourier series coefficient

$$\boxed{a_0 = \frac{\int_{T_0} x(t)dt}{T_0}} \quad \frac{\text{area } of \;\; x(t) \text{ in } T_0}{T_0} \Rightarrow a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$$

a_0 D.C value or avg value or mean value of x(t)

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt = f(n\omega_0) : n \geq 1$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t\,dt = g(n\omega_0) : n \geq 1$$

x(t)	a_0	a_n	b_0
Real	Real	Real	Real
Purely Imaginary	P.I	P.I	P.I
Complex	Complex	Complex	Complex

$$\boxed{a_n = a_{-n}} \qquad n \ge 1$$

$$\boxed{b_{-n} = -b_n}$$

$$\boxed{x(t) = r_0 + \sum_{n=1}^{\infty} r_n \cos[n\omega_0 t - \phi_n]}$$

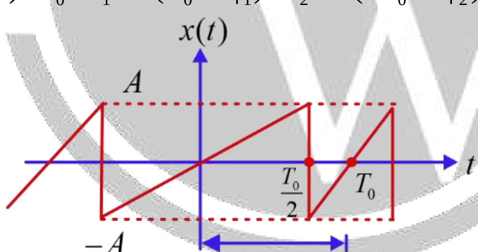
Polar from of T.F.S.

- $r_0 \rightarrow$ dc component of time domain nonsinusoidal periodic signal x(t).
- frequency of dc component = 0Hz
- Amplitude = r_0
- Power = r_0^2
- rms value = r_0

$r_K \cos(K\omega_0 t - \phi_K) \rightarrow K^{th}$ Harmonic of time domain nonsinusoidal periodic signal .

- Frequency of K^{th} harmonic = $K\omega_0$ rad/sec , Kf_0 Hz
- Amplitude of K^{th} harmonic $= r_K = \sqrt{a_K^2 + b_K^2}$
- rms value of K^{th} harmonic $= r_k / \sqrt{2}$
- MSV value of or power of K^{th} harmonic = $\frac{r_K^2}{2}$

$$X_{FS}(t) = r_0 + r_1\cos(\omega_0 t - \phi_1) + r_2\cos(2\omega_0 t - \phi_2) + r_3\cos(3\omega_0 t - \phi_3) + \_\_\_\_\_$$



$$T_0 = \frac{2\pi}{\omega_0}$$

$$\boxed{x_{FS}^2(t) = r_0^2 + \frac{r_1^2}{2} + \frac{r_2^2}{2} + \frac{r_3^2}{2} + \_\_\_\_} = \frac{A^2}{3} \quad \text{Parseval Theorem}$$

How to calculate absent harmonic in Time domain nonsinusoidal periodic signal:

$S-1 \quad \omega_0, T_0$

$S-2 \quad a_0, a_n, b_n$

$S-3 \quad r_0 = a_0,\ r_n = \sqrt{a_n^2 + b_n^2}\ n \ge 1$

S – 4 find value of n for which $r_n = 0$

$r_K = 0 \quad K^{th}$ harmonic is absent .

Complex or Exponential Fourier series – x(t) is real .

$$x(t)=\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_0=\frac{1}{T_0}\int_{T_0} x(t)dt = a_0 = r_0 a$$

$$C_n=\frac{1}{T_0}\int_{T_0} x(t)e^{-j_n\omega_0 t} \quad -\infty<n<\infty$$

(1) $\boxed{C_n=\frac{a_n}{2}-j\frac{b_n}{2}}:n\ge 1$

(2) $\boxed{C_{-n}=\frac{a_n}{2}+j\frac{b_n}{2}}:n\ge 1$

(3) $\boxed{C_0=a_0}$

(4) $\boxed{C_n=C_{-n}^*}$

(5) $|C_n|=\frac{r_n}{2}:n\ge 1$

$\angle C_n=-\tan^{-1}\left(\frac{b_n}{a_n}\right):n\ge 1$

(6) $|C_{-n}|=\frac{r_n}{2} \quad : \quad n\ge 1$

$\angle C_{-n}=\tan^{-1}\left(\frac{b_n}{a_n}\right):n\ge 1$

(7) $|C_n|=|C_{-n}|\rightarrow$ Magnitude spectrum : Even

(8) $\angle C_n=-\angle C_{-n}\rightarrow$ Phase spectrum : Odd

$$\boxed{C_1=-|C_1|}$$

Note: **As Long as x(t) is real .**

$\rightarrow |C_n|$ *vs* $n\omega_0 \rightarrow$Even

$\angle C_n$ *vs* $n\omega_0\rightarrow$ Odd $\rightarrow$ It may looklike even when $\angle C_n$ is multiple of π.

(6) absent frequency – If $|C_n|=0, C_n=0$

$\rightarrow$ n^{th} harmonic will be absent.

(7) Amplitude of K^{th} harmonic : $r_K=\sqrt{a_K^2+b_K^2}=2|C_K|$

rms value of K^{th} harmonic : $\frac{r_K}{\sqrt{2}}=\sqrt{2}|C_K|$

Power of K^{th} harmonic : $\frac{r_K^2}{2}=2|C_K|^2$

Numerical :

Type 1 – validity of Trigonometric Fourier series and calculation of harmonies –

- **Procedure** Check the periodicity of given signal

If $x(t)$ is P | If $x(t)$ is NP

- Given exp is valid F.S
- Given exp is not valid F.S
- Calculate harmonics

Type 2 – Calculation of complex F.S.C of sinusoid or combination of sinusoidal :

S.1 Calculate $\omega_0 \begin{cases} \nearrow 2\pi / T_0 \\ \searrow \omega_0 = HCF(\omega_1, \omega_2, \_\_\_\_\_\_) \end{cases}$

S.2 Write x(t) in exponential from .

S.3 $x(t) = \sum C_n e^{jn\omega_0 t}$ replace ω_0

S.4 Compare S.2 and S.3

➢ Calculation of T.F.S coefficient when sinusoids are mentioned-

S-1 Calculate ω_o

S-2 Calculate the harmonics $\begin{matrix} \omega_1 = K_1\omega_o \\ \omega_2 = K_2\omega_o \end{matrix}$

S-3 Final values of a_n, b_n

Type 3 – Questions based on properties of Fourier series w.r.t complex F.S.C

1. Linearity- $g(t) = A\underset{\substack{\downarrow \\ C_n}}{x_1(t)} + B\underset{\substack{\downarrow \\ d_n}}{x_2(t)} \longleftrightarrow g_n = AC_n + Bd_n$

2. Time shifting property - $g(t) = x(t + t_0) \xleftrightarrow{FSC} d_n = e^{jn\omega_0 t_0} C_n$

$$C_n \longrightarrow x(t)$$

$$|g_n| = |C_n|\ ,\ \angle g_n = \underset{\substack{\downarrow \\ x(t)}}{\angle C_n} - n\omega_0 t_0$$

3. Time Reversal - $x(t) \xleftrightarrow{FSC} C_n \Rightarrow C_n\ vs\ n\omega_0$

$$g(t) = x(-t) \longrightarrow \underset{\substack{\downarrow \\ g(n\omega_0) = f(-n\omega_0)}}{g_n} = C_{-n} \Rightarrow g_n\ vs\ n\omega_0$$

$x(t)$	C_n
E	E
0	0
NENO	NENO

4. Time Scaling – $T_0, \omega_o \quad x(t) \xleftrightarrow{FSC} C_n \Rightarrow C_n \ \text{vs} \ n\omega_o$

$$\left(\frac{T_0}{a}\right), (a\omega_0) \ g(t) = x(at) \xleftrightarrow{FSC} C_n \Rightarrow C_n \ \text{vs} \ n(a\omega_0)$$

Time domain		**Frequency Domain**
Compression	$\longleftrightarrow$	Expansion
Expansion	$\longleftrightarrow$	Compression

5. Complex conjugate –

$$\omega_o, T_o \quad x(t) \xleftrightarrow{FSC} C_n \ \text{vs} \quad n\omega_o$$

$$\omega_o, T_o \ : g(t) = x^*(t) \xleftrightarrow{FSC} g_n = C_{-n}^* \Rightarrow g_n \ \text{vs} \ n\omega_o$$

$x(t)$	C_n	
Real $\longrightarrow$	Conjugate symmetry	$\Rightarrow C_n = C_{-n}^* \Rightarrow \lvert C_n \rvert = \lvert C_{-n} \rvert, \angle C_n = -\angle C_{-n}$
Imaginary $\longrightarrow$	Conjugate Summitry	$\Rightarrow C_n = -C_{-n}^* \Rightarrow \lvert C_n \rvert = \lvert C_{-n} \rvert, \angle C_n = -\angle C_{-n} \pm 180^0$
Conjugate Symmetry $\longrightarrow$	Real	
Conjugate anti Symmetry $\longrightarrow$	Imaginary	

***x*(*t*)**	***C**$_n$*
R E	R E
R O	I O
I E	I E
I O	R O

(6) Multiplication by complex exponential function.

$$T_o, \omega_o \quad x(t) \xleftrightarrow{FSC} C_n \longrightarrow C_n \ \text{vs} \ n\omega_o$$

$$g(t) = e^{jm\omega_o t} \ x(t) \longleftrightarrow g_n = C_{n-m} \Rightarrow g_n \ \text{vs} \ n\omega_o$$

$$g(t) = e^{-jm\omega_o t} \ x(t) \longleftrightarrow g_n = C_{n+m} \Rightarrow g_n \ \text{vs} \ n\omega_o$$

(7) Differentiation : $T_o, \omega_0 : x(t) \xleftrightarrow{FSC} C_n \Rightarrow C_n \text{ vs } n\omega_o$

$$T_o, \omega_o : \frac{d^3 x(t)}{dt^3} \longleftrightarrow (jn\omega_o)^3 C_n$$

(8) Integration Property : $T_o, \omega_o : x(t) \xleftrightarrow{FSC} C_n \Rightarrow C_n \text{ vs } n\omega_o$

$$T_0, \omega_0 : g(t) = \int_{-\infty}^{t} x(\tau) d\tau \xleftrightarrow{FSC} \frac{C_n}{jn\omega_0} = g_n = g_n \text{ vs } n\omega_0$$

(9) Periodic convolution - $x_1(t)$ and $x_2(t)$ are both periodic with some time period To.

$$x_1(t) * x_2(t) = \int_{T_o} x_1(\tau) x_2(t-\tau) d\tau$$

Multiplication in time domain :

$$T_0, \omega_0 \quad x_1(t) \rightarrow C_n$$

$$T_0, \omega_0 \quad x_2(t) \rightarrow d_n$$

$$g(t) = x_1(t) \cdot x_2(t) \longleftrightarrow g_n = C_n * d_n$$

$\longrightarrow$ Tabular Method

Type 4 – Symmetry :

(a) Even :- Even in $\left(-\frac{T_0}{2}, \frac{T_0}{2}\right) or \left(-\frac{T_0^+}{2}, \frac{T_0^+}{2}\right) or \left(-\frac{T_0^-}{2}, \frac{T_0^-}{2}\right)$

(b) Odd :- odd in $\left(\frac{-T_0}{2}, \frac{T_0}{2}\right)$

(c) Half wave symmetry .

(a) Odd HWS - $x\left(t \pm \frac{T_0}{2}\right) = -x(t)$

(b) Even HWS - $x\left(t \pm \frac{T_0}{2}\right) = x(t)$

Effect of symmetry on T.F.S Coefficients .

Case 1: x(t) is even

$a_0 \begin{matrix} \nearrow = 0 \\ \searrow \neq 0 \end{matrix}$ but $b_n = 0$ always , a_n : will not be zero for all value of n.

- dc value may or may not be present.
- Harmonic of cosine decided by a_n

- All sine harmonics are absent.
- Frequency – 0HZ $\rightarrow$decide by a_n
- Other frequency $\rightarrow$decide by a_n

Case 2: $x(t)$ is odd

$a_n = 0$, $a_0 = 0$, $b_n \rightarrow$ will not be zero always.

- dc is absent, all cosine harmonics absent, sine harmonic decided by a_n
- $0\text{HZ} \rightarrow \text{absent}$

Other frequency $\rightarrow$decided by a_n

Case 3: $x(t)$ is HWS-

$$a_0 = 0$$
$$a_n = 0 \quad \text{for } n \text{ even}$$
$$= \frac{4}{T_0}\int_{-T_o/2}^{T_o/2} x(t)\cos n\omega_o t\ dt \quad n:\text{odd}$$

$$b_n = 0 \ n:\text{even}$$
$$b_n = \frac{4}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin + \omega_0 t\, dt$$

- dc is absent
- all even harmonic of sine / cosine are absent.
- all odd harmonic of sine /cosine are present .
- 0HZ: absent and $f_0, 3f_0, 5f_0$_______ will be present .

Case 4: $x(t)$ is Even + HWS (odd)

$$a_0 = 0, \quad a_n = 0 \ \ n:\text{even} \qquad b_n = 0 \quad \forall_n$$
$$a_n \neq 0 \ \ n:\text{odd}$$

- dc absent
- all harmonic of sine and even harmonic of cosine are absent.
- all odd harmonic of cosine are present.
- OHZ $\rightarrow$ absent , $f_0, 3f_0, 5f_0$ $---$prestent

Case 5: $x(t)$ is odd +HWS

$a_0 = 0$ $\qquad$ b_n =0 n:even

$a_n = 0 \ \forall_n$ $\qquad$ $b_n \neq 0 \quad n:\text{odd}$

- dc absent
- all harmonic of cosine and even harmonic of sine $\rightarrow$ absent .
- odd harmonic of sine will be present.
- 0HZ $\rightarrow$ absent , $f_0, 3f_0, 5f_0 \rightarrow$ presens

Fourier Transform:

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt,\; X_{To}(\omega) = \int_{-To/2}^{To/2} x(t)e^{-j\omega t}dt$$

$$X_{To}(n\omega_0) = \int_{-T_0/2}^{T_o/2} x(t)e^{-jn\omega_o t}dt$$

$$x(t) \xleftrightarrow{BLT} X(S) \qquad \underset{L.T}{X(S)} \xrightarrow{S=j\omega} x(\omega)F.T$$

$$X(S) = \int_{-\infty}^{\infty} x(t)e^{st}dt \longrightarrow \text{ROC} \qquad s = j\omega \text{ is part of ROC.}$$

Property $= x(t) \longrightarrow x(\omega)$ then

(1) $x(t - t_o) = e^{-j\omega to}X(\omega)$

(2) $x(t + to) = e^{j\omega to}X(\omega)$

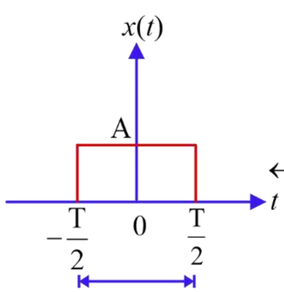
$x(t) \longleftrightarrow X(s)$

$x(t - t_o) \longleftrightarrow e^{-St_o}X(s)$

$x(t + to) \longleftrightarrow e^{St_o}X(s)$

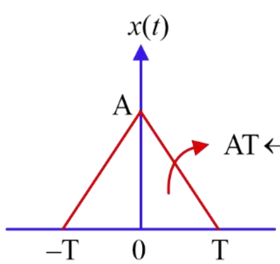
(1) $\delta(t) \xleftrightarrow{F.T} 1$

(2)



$\longleftrightarrow X(\omega) = \underset{\downarrow\ \text{Area}}{AT}\ \ \text{Sa}\left[\frac{\omega T}{2}\right]$ ($\frac{\omega T}{2}$ $\downarrow$ Half of duration T)

(3)



$AT \longleftrightarrow \underset{\text{Area}}{\underbrace{AT}}\ \text{Sa}^2\left[\frac{\omega T}{2}\right] = X(\omega)$

$\frac{T}{2} = \frac{\text{duration}}{4}$

(4) $u(t) \xleftrightarrow{L.I} \frac{1}{s}$

(5) $tu(t) \longleftrightarrow \frac{1}{s^2}$

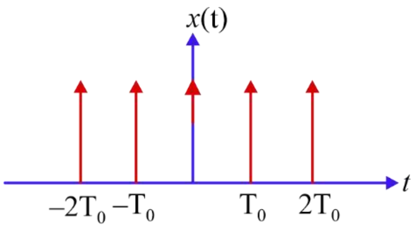
(6) $t^n u(t) \longleftrightarrow \dfrac{n!}{s^{n+1}}$

(7) $\sin\omega_0 t \; u(t) \longleftrightarrow \dfrac{\omega_0}{s^2+\omega_0^2}$

(8) $\cos\omega_0 t \; u(t) \longleftrightarrow \dfrac{s}{s^2+\omega_0^2}$

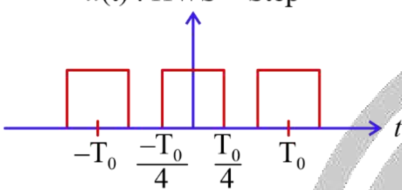
Important Observation :

(1)



$\xleftrightarrow{\text{FSC}} C_n = \dfrac{1}{T_0}$

(2) $x(t)$: HWS + Step



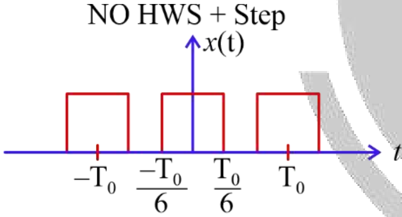
$\longleftrightarrow C_n = C_0 \sin C\left(\dfrac{n}{2}\right)$

$\rightarrow |C_n| \alpha \dfrac{1}{n}$

n : even → $C_n = 0$

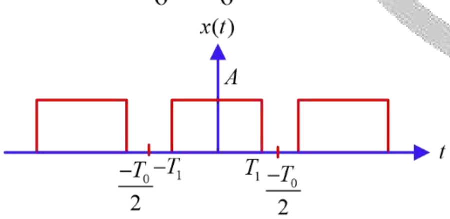
n : odd → $|C_n| = \left(\dfrac{2C_0}{\pi}\right) \times \dfrac{1}{n}$

(3) NO HWS + Step



$\longleftrightarrow C_n = C_0 \sin C\left(\dfrac{n}{3}\right)$

$|C_n| \alpha \dfrac{1}{n}$



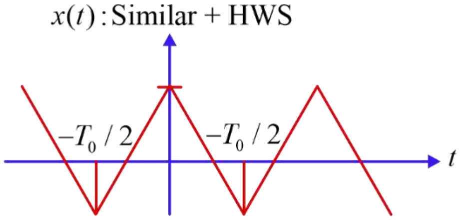
$\longleftrightarrow C_n = C_0 \sin C\left[\dfrac{n(2T_0)}{T_0}\right]$

duration

F.T.P

$|C_n| \alpha \dfrac{1}{n}$

$x(t)$: Similar + HWS

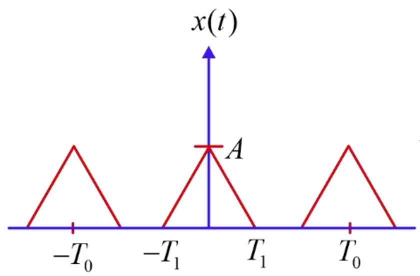


$\longleftrightarrow C_n = \dfrac{2A}{n^2\pi^2}\left[1-(-1)^n\right]$

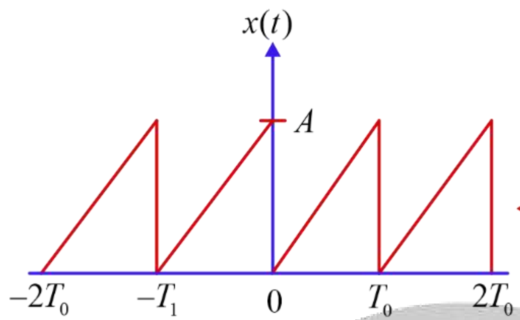
$|C_n| \alpha \dfrac{1}{n^2}$

$C_n = 0 \quad n : \text{even}$

$C_n = \dfrac{4A}{n^2\pi^2} \quad n : \text{odd}$



$\longleftrightarrow \quad C_n = C_0\left[\text{Sin}\,C\left(\frac{nT_1}{T_0}\right)\right]^2$



$\longleftrightarrow C_n = \frac{jA}{n\pi}$

Type 6: Parseval Theorem

$x(t)$: Power signal, which is periodic with F.T.P T_0 absolute or Exact power $x(t)$:

$$P_x = \frac{1}{T_0}\int_{T_0} x^2(t)dt \qquad \text{(If } x(t) \text{ is real)}$$

$$= \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt$$

$$P_x = a_0^2 + \sum_{n=1}^{\infty}\left[\frac{a_n^2}{2} + \frac{b_n^2}{2}\right]$$

Note: $x(t) \xleftrightarrow{FSC} C_n$

(1) $P_x = \sum_{n=-\infty}^{\infty} |C_n|^2$

(2) $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \Rightarrow x(0)\sum_{n=-\infty}^{\infty} |C_n| e^{j\angle C_n}$

Drichlet's condition - Only sufficient condition not necessary

x (t) is periodic with T_0

x (t) follows D.C → fourier series will exist

x (t) does not follows → fourier may or may not be exist

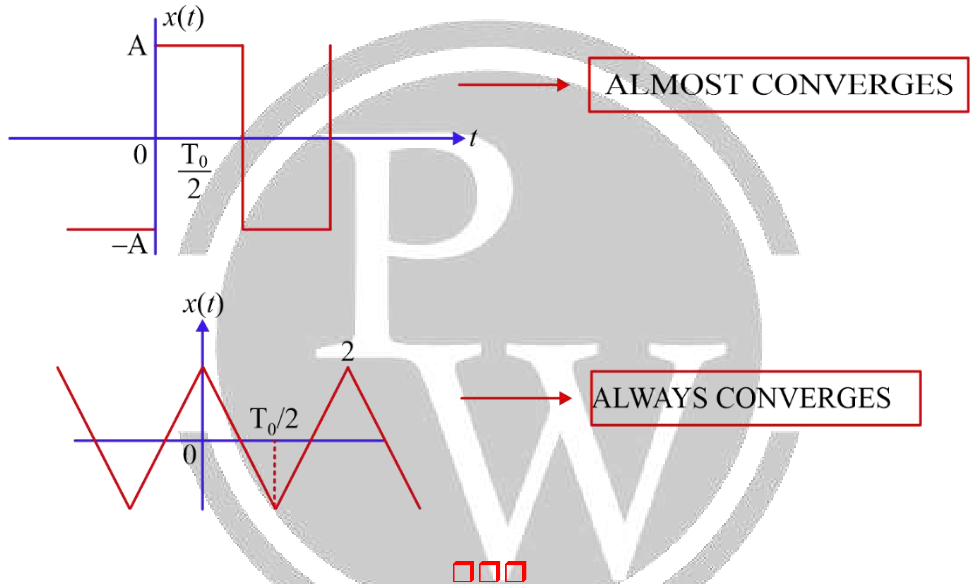
Statement :

(1) Any nonsinusoidal time domain periodic signal can always be Exactly written as weighted sum of Harmonically related naturally orthogonal sinusoids is not completely true.

(2) Fourier series a nonsinusoidal time domain periodic signal converges at all points on the nonsinusoidal time domain periodic signal is not Exactly True.

(3) The Fourier series representation of T.D. non sinusoidal periodic signal converge at ALMOST all the points on time domain non sinusoids periodic signal, except at the point of discontinuity

***x*(t) : N.S. + P**	**Fourier Series**
Continuous in Amplitude $\longrightarrow$	Fourier Series converges at all points
Discontinuous in Amplitude $\longrightarrow$	Fourier Series converges at almost all the point except the point of discontinue



3 FOURIER TRANSFORM

3.1. Continuous Time Fourier Transform

- $x(t)$ is non periodic signal
- $x(t) \xleftrightarrow{F.T} X(\omega)$
- $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ or $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$
- $X(\omega) = \delta(\omega)$
- $X(f) = \delta(2\pi f) = \frac{1}{2\pi}\delta(f)$

Note : For applying F.T formula $x(t)$ should be N.P and absolutely integrable.

x(t)	Formula of F.T	F.T Exist
Energy	Applicable	Yes (always)
Power	Not Applicable	Always Exist $\longrightarrow$ Limited sense
NENP except $\delta(t)$	Not applicable	No
$\delta(t)$	Applicable	Always Exist

$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$

$x(t) \xleftrightarrow{F.T} X(\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X(\omega)}_{\text{volt/(rad/sec)}} e^{j\omega t} d\omega \rightarrow \text{rad/sec}$$

$$\underset{\substack{\downarrow \\ volt}}{x(t)} = \int_{-\infty}^{\infty} \underbrace{X(f)}_{\text{volt/Hz}} e^{j2\pi ft} df$$

$$X(\omega) = |X(\omega)| e^{j\angle x(\omega)}$$

$$X(f) = |X(f)| e^{j\angle x(f)}$$

Properties

(1) Linearity

$$x_1(t) \longleftrightarrow X_1(\omega)$$
$$x_2(f) \longleftrightarrow X_2(\omega)$$

$$g(t) = Ax_1(t) + Bx_2(f) \longleftrightarrow G(\omega) = AX_1(\omega) + BX_2(\omega)$$

Time shift - $x(t) \longleftrightarrow X(\omega)$

$$x(t - t_0) \longleftrightarrow e^{-j\omega t_o} X(\omega) = e^{-j2\pi f t_o} X(f)$$

$$x(t + t_0) \longleftrightarrow e^{j\omega t_o} X(\omega) = e^{j2\pi f t_o} X(f)$$

- Does not affect the magnitude .
- $\dfrac{x(t-a) + x(t+a)}{2} \longleftrightarrow X(\omega)\cos a\omega, \dfrac{x(t+a) - x(t-a)}{2j} \longleftrightarrow X(\omega)\sin a\omega$

$$\frac{x(t-a) + x(t-a)}{2} \longleftrightarrow X(f)\cos(2\pi a) f, \frac{x(t+a) - x(t-a)}{2j} \longleftrightarrow X(f)\sin(2\pi a) f$$

Frequency Shifting $x(t) \longleftrightarrow X(\omega)$

$$e^{j\omega_0 t} x(t) \longleftrightarrow X(\omega - \omega_0)$$

$$e^{-j\omega_0 t} x(t) \longleftrightarrow X(\omega + \omega_0)$$

$$\cos \omega_0 t\, x(t) \longleftrightarrow \frac{X(\omega - \omega_0) + X(\omega + \omega_0)}{2}$$

$$\sin \omega_0 t\, x(t) \longleftrightarrow \frac{X(\omega - \omega_0) - X(\omega + \omega_0)}{2j}$$

$$\cos 2\pi f_0 t x(t) \longleftrightarrow \frac{X(f - f_0) + x(f + f_0)}{2}$$

$$\sin 2\pi f_0 t x(t) \longleftrightarrow \frac{X(f - f_0) - X(f + f_0)}{2j}$$

Modulation Property $x(t) \longrightarrow X(f)$

$$\cos 2\pi f_0 t\, x(t) \longleftrightarrow \frac{X(f - f_0) + X(f + f_0)}{2}$$

$$\sin 2\pi f_0 t\, x(t) \longleftrightarrow \frac{X(f - f_0) - x(f + f_0)}{2j}$$

Time Reversal

$$x(t) \longleftrightarrow X(\omega) \quad \Big| \quad x(t) \longleftrightarrow X(f)$$

$$x(-t) \longleftrightarrow X(-\omega) \quad \Big| \quad x(-t) \longleftrightarrow X(-f)$$

Time Scaling

$$x(t) \longleftrightarrow X(\omega) \quad \Big| \quad x(t) \longleftrightarrow X(f)$$

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right) \quad \Big| \quad x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Differentiation Property:

$$x(t) \longleftrightarrow X(\omega)$$

$$\frac{dx(t)}{dt} \longleftrightarrow j\omega X(\omega) \longrightarrow \text{valid only when } \overline{x(t)} = 0$$

$$\frac{dx(t)}{dt} \longleftrightarrow (2j\pi f) X(f)$$

If $\bar{x}(t) \neq 0, \bar{x}(t) = K$ then $X(\omega) = \dfrac{G(\omega)}{j\omega} + \text{F.T of } [K]$

(1) $\delta(t) \xleftrightarrow{F.T} 1$

(2) $\dfrac{\delta(t-a) + \delta(t+a)}{2} = \cos(a\omega)$

(3) $\dfrac{\delta(t+a) - \delta(t-a)}{2j} = \sin(a\omega)$

(3) One sided exponential , $x(t) = e^{-at}u(t), a > 0$

$$X(\omega) = \frac{1}{(a + j\omega)}$$

$$x(t) = e^{at}u(-t) \longleftrightarrow \frac{1}{(a - j\omega)}$$

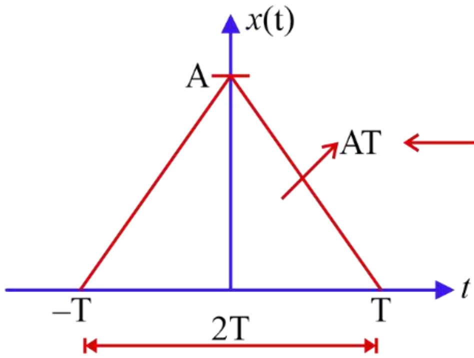
(4) Two sided exponential = $x(t) = e^{-a|t|} \longleftrightarrow X(\omega) = \dfrac{2a}{a^2 + \omega^2}$

(5) $x(t) = e^{-a|t|}\, sgn(t) \qquad a > 0, \longleftrightarrow X(\omega) = \dfrac{-2j\omega}{a^2 + \omega^2}$

(6) Multiplication - $tx(t) = +j\dfrac{dx(\omega)}{d\omega}$

$$t^n[e^{-at}u(t)] = \frac{n!}{(a + j\omega)^{n+1}}$$

(7) Even Triangular pulse:–



$$AT \longleftrightarrow X(\omega) = \underbrace{AT}_{\text{Area}}\ Sa^2\underbrace{\left[\frac{\omega T}{2}\right]}_{\frac{\text{Width}}{4}}$$

Fourier Transform of power signal (Type II)

or

Periodic + Non periodic

- Formula not applicable , properties applicable .
- Limitedly defined F.T so can . not be calculated by L.T.
- Obtained by limiting Type 1 signal.

(1) $1 \xleftrightarrow{F.T} 2\pi\delta(\omega)$

$1 \xleftrightarrow{F.T} \delta(f)$

(2) $\dfrac{dx(t)}{dt} \longleftrightarrow j\omega[X(\omega) - F.T(\overline{x}(t))]$

(3) $\cos\omega_0 t \longleftrightarrow \pi\delta(\omega-\omega_0)+\pi\delta(\omega-\omega_0)$

or

$\dfrac{1}{2}\delta(f-f_0)+\dfrac{1}{2}\delta(f+f_0)$

(4) $\sin\omega_0 t \longleftrightarrow \dfrac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$

or

$\dfrac{1}{2j}[\delta(f-f_0)-\delta(f+f_0)]$

Duality $x(t) \xleftrightarrow{F.T} X(\omega)$ $\quad x(t) \xleftrightarrow{F.T} X(f)$

$X(t) \xleftrightarrow{F.T} 2\pi x(-\omega)$ $\quad X(t) \xleftrightarrow{F.T} x(-f)$

Steps :

(1) Identify the $x(t)$ and try to obtain $X(\omega)$ from $x(t)$

(2) If step 1 fails then

$$x(t) \xrightarrow{t=\omega} G(\omega)$$

or

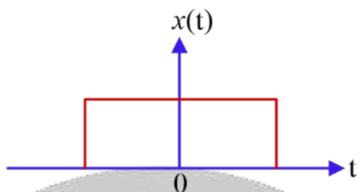
$$x(t)\big|_{t=\omega} = G(\omega)$$

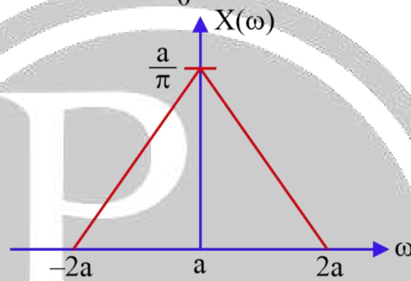
(3) $g(t) \xleftrightarrow{F.T} G(\omega)$

$$G(t) \xleftrightarrow{F.T} 2\pi g(-\omega)$$

Important Result:

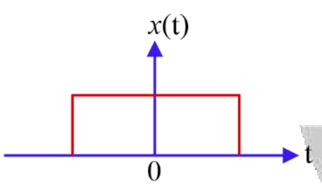
$$\left(\frac{\sin at}{\pi t}\right) \xleftrightarrow{F.T}$$





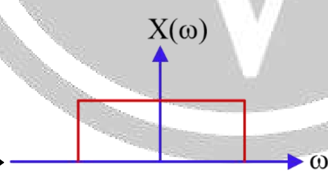
$$\left(\frac{\sin at}{\pi t}\right)^2 \xrightarrow{F.T}$$

(1)

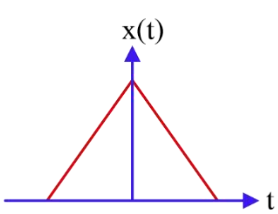


$$\longleftrightarrow Sa(\omega) = \frac{\sin\omega}{\omega} \text{ or } \sin c(\omega) \text{ or } sa(\omega)$$

(2) $\dfrac{\sin at}{\pi t}$ or $sa(t)$ or $\sin c(t) \longleftrightarrow$

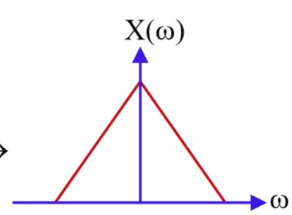


(3)



$$\longleftrightarrow \left(\frac{\sin\omega}{\omega}\right)^2 \text{ or } \sin c^2(\omega) \text{ or } sa^2(\omega)$$

(4) $\sin\, c^2(t)$ or $sa^2(t)$ or $\left(\dfrac{\sin at}{\pi t}\right)^2 \longleftrightarrow$



Area Property:

(1) $X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

$$X(\omega)\big|_{\omega=0} = \int_{-\infty}^{\infty} x(t)dt$$

Area of $x(t) \Rightarrow \int_{-\infty}^{\infty} x(t) \xrightarrow{} F.T \quad X(\omega)\big|_{\omega=0}$

(2) (i) Area of $X(\omega)=\int_{-\infty}^{\infty} X(\omega)d\omega = 2\pi x(0)$

(ii) $x(t)\big|_{t=0} = \int_{-\infty}^{\infty} X(f)df$

Convolution $x_1(t)\longleftrightarrow X_1(\omega)$

$x_2(t)\longleftrightarrow X_2(\omega)$

$x_1(t)*x_2(t)\xleftrightarrow{F.T} X_1(\omega)X_2(\omega)$

Note: A sin c $(\alpha t)*B\sin c(\beta t)=AB\left[\frac{1}{m}\sin c(kt)\right]$ $\quad m=\max(\alpha,\beta)$

$K=\min(\alpha,\beta)$

Multiplication in time domain

$$x_1(t).x_2(t)\longleftrightarrow \frac{1}{2\pi}\left[X_1(\omega)*X_2(\omega)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)X_2(\omega-\lambda)d\lambda$$

$$x_1(t)\,.\,x_2(t)\longleftrightarrow X_1(f)*X_2(f)=\int_{-\infty}^{\infty} X_1(\lambda)X_2(f-\alpha)d\lambda$$

Integration Property –

$$\int_{-\infty}^{t} x(\tau)d\tau = x(t)*u(t)\longleftrightarrow X(\omega)\left[\frac{1}{j\omega}+\pi\delta(\omega)\right]=\frac{X(\omega)}{j\omega}+\pi X(0)\delta(\omega)$$

Complex conjugate - $x^*(t)\longleftrightarrow X^*(\omega)$ or $X^*(-f)$

Important table:

$x(t)$	$X(\omega)$
Even	Even
Odd	Odd
NENO	NENO

$x(t)$	$X(\omega)$
Real	Conjugate symmetry
Imaginary	Conjugate anti symmetry
Conjugate Symmetry	Real
Conjugate anti symmetry	Imaginary

$x(t)$	$X(\omega)$
RE	RE
RO	IO
IE	IE
IO	RO

Parseval's Energy Theorem –

(1) $\int_{-\infty}^{\infty} x(t)h(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)H(-\omega)d\omega = \int_{-\infty}^{\infty} X(f)H(-f)df$

(2) $\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)X(-\omega)d\omega = \int_{-\infty}^{\infty} X(f)X(-f)df$

(3) $\int_{-\infty}^{\infty} x(t)h^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)H^*(\omega)d\omega = \int_{-\infty}^{\infty} X(f)H^*(f)df$

(4) $\int_{-\infty}^{\infty} x(t)x^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)X^*(\omega)d\omega = \int_{-\infty}^{\infty} X(f)X^*(f)df$

F.T of Gaussian Pulse

$$\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi} \qquad e^{-at^2} \longleftrightarrow \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}$$

$$e^{-\pi t^2} \longleftrightarrow e^{-\pi f^2}$$

LTI System

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$X(\omega) \qquad H(\omega) \qquad Y(\omega)$$

$$|Y(\omega)| = |H(\omega)|\,|X(\omega)|$$

$$\angle Y(\omega) = \angle X(\omega) + \angle H(\omega)$$

➢ $E_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} |y(\omega)|^2\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2\,|X(\omega)|^2\, d\omega$

$$= \int_{-\infty}^{\infty} |Y(f)|^2 df = \int_{-\infty}^{\infty} |H(f)|^2 |X(f)|^2 df$$

Eigen values and eigen function –

Eigen function of LTI S/S $\xrightarrow{x(t)} \underset{LTI}{\boxed{ht}} \longrightarrow y(t) = Kx(t)$

↗ eigen value of LTI System

↘ Real or complex or 1

$$x(t) = e^{S_0 t} \longrightarrow \boxed{H(S)} \longrightarrow y(t) = e^{S_0 t} H(S_0)$$

$$x(t) = e^{j\omega_0 t} \longrightarrow \boxed{H(\omega)} \longrightarrow y(t) = e^{j\omega_0 t} H(\omega_0)$$

$$A\cos\omega_0 t \longrightarrow \underset{even}{\boxed{h(t) \to H(\omega)}} \longrightarrow y(t) = A\cos\omega_0 t \underset{\text{eigen value}}{\boxed{H(\omega_0)}}$$

$$A\sin\omega_0 t \longrightarrow \boxed{h(t) \leftrightarrow H(\omega)} \longrightarrow y(t) = A\sin\omega_0 t \boxed{H(\omega_0)}$$

$h(t)$	$H(\omega)$
R E	R E
R O	I O
I E	I E
I O	R O

$$A\cos(\omega_0 t + \theta) \longrightarrow \boxed{h(t) \leftrightarrow H(\omega)} \longrightarrow A\cos(\omega_0 t + \theta) H(\omega_0)$$

$$A\sin(\omega_0 t + \theta) \longrightarrow \boxed{h(t) \leftrightarrow H(\omega)} \longrightarrow A\sin(\omega_0 t + \theta) H(\omega_0)$$

$$\begin{matrix} A\cos(\omega_0 t + \theta) \\ A\sin(\omega_0 t + \theta) \end{matrix} \longrightarrow \overset{\text{Real}}{\boxed{h(t) \leftrightarrow H(\omega)}} \longrightarrow \begin{matrix} A|H(\omega_0)|\cos(\omega_0 t + \theta + \angle H(\omega_0)) \\ A|H(\omega_0)|\sin(\omega_0 t + \theta + \angle H(\omega_0)) \end{matrix}$$

↓ not an eigen function

Case 1 $h(t)$ is even / $H(\omega)$ is even

➢ Both A sin $(\omega_o t + \theta)$, A cos $(\omega_o t + \theta)$ will be eigen function with same eigen value $H(\omega_o)$ not necessarily real .

Case 2 $h(t)$ is real and even

➢ A $\sin(\omega_0 t + \theta)$ and A $\cos(\omega_0 t + \theta)$ are eigen function with same real eigen value $H(\omega_0)$

Case 3 $h(t)$ is real .

➢ A $\cos(\omega_0 t + \theta)$ and A $\sin(\omega_0 t + \theta)$ is not an eigen function.

❒❒❒

4 LAPLACE TRANSFORM

4.1. Introduction

Bilateral T.F $X(S)=\int_{-\infty}^{\infty} x(t)e^{-st}dt = F.T[x(t)e^{-\sigma t}]$

Unilateral T.F $X(S)=\int_{0^-}^{\infty} x(t)e^{-st}dt$

Note: for $X(S)$ to be finite or for $X(S)$ to converge

$S-1$ $x(t)e^{-\sigma t}$ must be absolutely integrable .

$$\int_{-\infty}^{\infty}\left|x(t)e^{-\sigma t}\right|dt \to \text{finite}$$

S – 2 $X(S)=\int_{-\infty}^{\infty} x(t)e^{-st}dt$

$$x(t)=e^{-s_0 t}u(t)\xleftrightarrow{B.L.T}\begin{cases} X(s)=\dfrac{1}{S+S_0} & \text{When } \operatorname{Re}\{S\}>-\sigma_0 \\ X(s)=\infty & \text{When } \operatorname{Re}\{S\}\le-\sigma_0 \end{cases}$$

- $e^{-St_0}u(t)\longleftrightarrow X(S)=\dfrac{1}{S+S_0}$ ROC: $\operatorname{Re}\{S\}=-\operatorname{Re}\{S_0\}$

  Pole : $S=-S_0$ $\operatorname{Re}\{S\}>-\operatorname{Re}\{S_0\}$ RHP

  $e^{St_0}u(t)\longleftrightarrow X(S)=\dfrac{1}{S-S_0}$

  Pole : $S=S_0$ $\operatorname{Re}\{S\}=\operatorname{Re}\{S_0\}$

  RHP $\operatorname{Re}\{S\}>\operatorname{Re}\{S_0\}$

  $-e^{Sto}u(-t)\longleftrightarrow\dfrac{1}{S-S_0}\Rightarrow ROC:$ $\operatorname{Re}\{S\}<\operatorname{Re}\{S_0\}$

Properties

(1) Linearity - $x_1(t) \longleftrightarrow X_1(S)$ $\quad ROC : R_1$

$x_2(t) \longleftrightarrow X_2(S)$ $\quad ROC : R_2$

Case 1 $\quad g(t) = Ax_1(t) + Bx_2(t) \longleftrightarrow G(S) = AX_1(S) + BX_2(S) : ROC : R_1 \cap R_2$

$\rightarrow R.S.R$
$\rightarrow L.S.S$
$\rightarrow$ Double sided

Case 2: $\quad g(t) = Ax_1(t) + Bx_2(t) \longleftrightarrow G(S) = AX_1(S) + BX_2(S)$

Finite duration + absolutely $\quad$ ROC – entire S plane

Inferable too

(2) Time Shifting - $x(t) \xleftrightarrow{BLT} X(S)$ $\quad ROC : R_1$

$x(t - t_o) \longleftrightarrow e^{-st_o} X(S)$ $\quad ROC : R_1$

$x(t - t_o) \longleftrightarrow e^{st_o} X(S)$ $\quad ROC : R_1$

(3) Multiplication with complex exponential

$x(t) \longleftrightarrow X(S)$ $\quad ROC : \text{Re}[S]$

$e^{S_o t} x(t) \longleftrightarrow X(S - S_0)$ $\quad ROC : \text{Re}\{S - S_0\}$

$e^{-S_o t} x(t) \longleftrightarrow X(S + S_0)\, ROC : \text{Re}\{S + S_0\}$

➢ B.L.T always have associated ROC with them .

Properties of R.O.C

(1) R.O.C may or may not include zeros of x(s).

(2) R.O.C can not includes poles of x(s)

be cause $X(S = S_p) \longrightarrow \infty$ ROC is either
(1) Right ward of pole
(2) Left ward of pole
(3) Bounded between poles

(3) If $x(t)$ is absolutely integrable then ROC of X(s) must include $j\omega$ axis.

(4) $x(t) \rightarrow$ finite duration + absolutely integrable . ROC of X(s) will be entire s plane

$(-\infty < \sigma < +\infty)$

(i) Impulse signal

(ii) finite duration + finite amplitude

(5) $x(t)$ is R.S.S
- $\nearrow X(S)$ does not exist even for single value of σ
- $\searrow$ If $X(S)$ exist then ROC will be right of right most pole

(6) $x(t)$ is L.S.S
- $\nearrow X(S)$ does not exist even for single value of σ
- $\searrow$ If $X(S)$ exist then ROC is left of the left most pole.

(7) $x(t)$ is B.S.S
- $\nearrow X(S)$ does not exist even for single value of σ
- $\searrow$ If $X(S)$ exist then ROC will be in strip form bounded between poles.

Some Important Results:

(1) $\delta(t) \longrightarrow 1$ ROC:entire S plane

(2) $u(t) \longrightarrow \dfrac{1}{S}$ $\quad \text{Re}\{S\} > 0$

(3) $-u(-t) \longrightarrow \dfrac{1}{S}$ $\quad \text{Re}\{S\} < 0$

(4) $e^{-at}u(t) \longrightarrow \dfrac{1}{S+a}$ $\quad \text{Re}\{S\} > -a$

(5) $e^{at}u(t) \longrightarrow \dfrac{1}{S-a}$ $\quad \text{Re}\{S\} > a$

(6) $-e^{-at}u(-t) \longrightarrow \dfrac{1}{S+a}$ $\quad \text{Re}\{S\} < -a$

(7) $e^{-a|t|} \longrightarrow \dfrac{2a}{a^2 - S^2}$ $\quad -a < \text{Re}\{S\} < a$

(8) $e^{-j\omega_0 t}u(t) \longrightarrow \dfrac{1}{S + j\omega_0}$ $\quad : \text{Re}\{S\} > 0$

(9) $\cos \omega_o t\, u(t) \longrightarrow \dfrac{S}{S^2 + \omega_0^2}$ $\quad \text{ROC} : \text{Re}\{S\} > 0$

(10) $\sin \omega_0 t\; u(t) \longrightarrow \dfrac{\omega_0}{S^2 + \omega_0^2}$ $\quad \text{ROC} : \text{Re}\{S\} > 0$

$$e^{-at}\cos \omega_0 t u(t) \xleftrightarrow{B.L.T} \frac{(S+a)}{(S+a)^2 + \omega_0^2} \qquad \text{Re}\{S+a\} > 0$$

$$e^{-at}\sin \omega_0 t u(t) \xleftrightarrow{B.L.T} \frac{\omega_0}{(S+a)^2 + \omega_0^2} \qquad \text{Re}\{S+a\} > 0$$

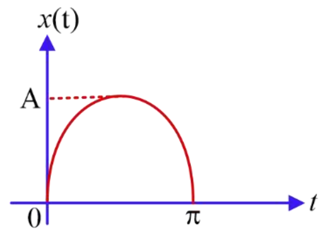
Time Reversal - $x(t) \longleftrightarrow X(S)$ $\qquad ROC : \text{Re}\{S\}$

$x(-t) \longleftrightarrow X(-S)$ $\qquad ROC : \text{Re}\{-S\}$

Multiplication by t $x(t) \longleftrightarrow X(S)$

$$t^n u(t) \longleftrightarrow \frac{n!}{S^{n+1}} \qquad ROC: \text{Re}\{S\} > 0$$

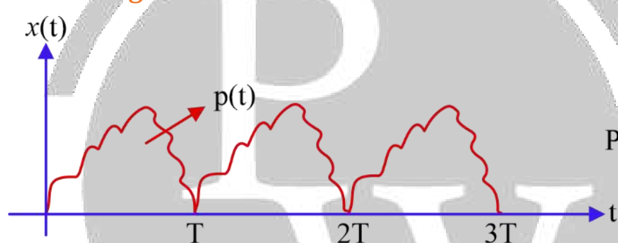
$$tx(t) \longrightarrow -\frac{d}{ds}X(S)$$



$$x(t) = A\sin t[u(t) - u(t-\pi)]$$

$$X(S) = \frac{A(1+e^{-\pi S})}{1+S^2}$$ ROC : entire S plane .

Laplace Transform of Causal Periodic Signal :



P↔P(S)

$$x(t) = p(t) + p(t-T) + p(t-2T) - - -$$

$$\boxed{X(S) = \frac{P(S)}{1-e^{-ST}}}$$ only when $\sigma > 0$

Time Scaling $x(t) \xleftrightarrow{BLT} X(S)$ $ROC: \text{Re}[S]$

$$x(at) \xleftrightarrow{BLT} \frac{1}{|a|}X\left(\frac{S}{a}\right) \qquad ROC: \text{Re}\left\{\frac{S}{a}\right\}$$

Divide by T property $x(t) \longleftrightarrow X(S)$ $\text{ROC}: \text{Re}(S)$

$$\frac{x(t)}{t} \longleftrightarrow \int_{-\infty}^{\infty} X(S)ds \qquad \text{ROC}: \text{Re}[S]$$

Inverse Laplace Transform:

(1) $\frac{1}{(S+a)}$ $\nearrow$ $e^{-at}u(.t)$ When $\text{Re}\{S\} > -a$

$\searrow$ $-e^{-at}u(t)$ When $\text{Re}\{S\} < -a$

(2) $\frac{1}{(S+a)^2}$ $\nearrow$ $te^{-at}u(t)$ $\text{Re}\{S\} > -a$

$\searrow$ $-te^{-at}u(-t)$ $\text{Re}\{S\} < -a$

(3) $\frac{1}{S} \begin{matrix} \nearrow & u(t) & \text{Re}\{S\} > 0 \\ \searrow & -u(-t) & \text{Re}\{S\} < 0 \end{matrix}$

(4) $\frac{\omega_0}{S^2 + \omega_0^2} \begin{matrix} \nearrow & \sin \omega_0 t\, u(t) & \text{Re}\{S\} > 0 \\ \searrow & -\sin \omega_0 t\, u(-t) & \text{Re}\{S\} < 0 \end{matrix}$

(5) $\frac{S}{S^2 + \omega_0^2} \begin{matrix} \nearrow & \cos \omega_0 t\, u(t) & \text{Re}\{S\} > 0 \\ \searrow & -\cos \omega_0 t\, u(-t) & \text{Re}\{S\} < 0 \end{matrix}$

Important Tables:

(1) Table 1 : X(S) : Rational/ Irrational

ROC is known and x(t) to be calculated

ROC	x(t)
R. H. P ⟶	R. S. S
L. H. P ⟶	L. S.S
STRIP ⟶	B.S.S

(2) Table 2 : X(S): Rational/ Irrational

Nature of x(t) is known and ROC to be decided.

x(t)	**ROC**
R. S. S	R. H. P
L. S.S	L. H. P
B.S.S	STRIP

(3) Table 3 : X(S): Rational

ROC is known and x(t) to be calculated

ROC	**x(t)**
R. H. P ⟶	Causal
L. H. P ⟶	Anti causal
STRIP ⟶	Non causal (causal + Anti causal)

(4) Table 4 X(S): Rational

Nature of x(t)is known and ROC is to be decided

x(t)	**ROC**
Causal	R. H. P
Anti causal	L. H. P
Non causal	STRIP

Note : (1) If ROC is entire s plane then x(t) will be finite duration finite amplitude

(2) If X(S) is irritation then always calculate x(t) to check causal, anti-causal non causal nature.

$$\text{No. of R.O.C} = \text{No of I.L.T} = \frac{\begin{pmatrix}\text{no.of non repeated}\\ \text{complex conjugate}\\ \text{poles}\end{pmatrix}}{2} + (\text{no of non Repeated Realpoles}) + 1$$

LTI System

$$X(S):R_1 \longrightarrow \boxed{H(S):R_2} \longrightarrow Y(S)$$

$\nearrow D.N.E$ ROC $\rightarrow R_1 \cap R_2 = \{\phi\}$

$\searrow$ Exist

$Y(S) = X(S)H(S)$ ROC : $R_1 \cap R_2$

Differentiation in time domain .

$$x(t) \xrightarrow{B.L.T} X(S) \qquad ROC : R_1$$

$$\frac{dx(t)}{dt} \xrightarrow{B.L.T} SX(S) \qquad ROC : \text{at least } R_1$$

Integration in time domain .

$$\int_{-\infty}^{t} x(\tau)d\tau \longrightarrow$$

$x(t) * u(t) \longrightarrow$ $\nearrow D.N.E$

R_1 $\quad \text{Re}\{S\} > 0$ $\searrow \frac{X(S)}{S} ROC : R_1 \cap [\text{Re}\{S\} > 0]$

Stability of an LTI system – for an LTI system to be stable

(1) $h(t)$ must be absolutely integrable

(2) For $h(t)$ must be absolutely integrable, $H(S)$ must include $j\omega$ axis.

Causality of an LTI system-

(1) $h(t)$ must be causal signal .

(2) For an LTI system having rational H(S) : ROC of H(S) must be right of right most pole.

Anti causal of an LTI system - $h(t) \longrightarrow$ anti causal

ROC of rational $H(S) \longrightarrow$ Left of left most pole

Non causality of an LIT system - $h(t) \longrightarrow$ Non casual

For rational $H(S)$:ROC must be in strip form.

Causal and stable - $H(S)$ rational $\rightarrow$All the poles of $H(S)$ must be in left hand side S plane

$H(S)$ Irrational $\rightarrow$ (ROC include $j\omega$ axis) $\cap h(t)$ is causal .

Anti causal and Stable $H(S)$ rational : All poles of $H(S)$ must be strictly on right half side of S – plane.

$H(S)$Irrational $\Rightarrow$(ROC include $j\omega$ axis)$\cap$($h(t)$is anti causal)

Non casual and stable - $H(S)$ rational : Poles of $H(S)$ must be located on either side of $j\omega$ axis

$H(S)$ Irrational : (ROC includes $j\omega$ axis) $\cap$ ($h(t)$ is non causal)

Important Table

(1) H(S) : Rational

ROC	LTI System
R.H.P	Causal
L.H.P	Anti causal
STRIP	Non causal

(2)

LTI System	ROC
Causal	RHP
Anti causal	LHP
Non causal	STRIP

Unilateral L.T $X(S)=\int_{0^-}^{\infty} x(t)e^{-St}dt$ No ROC exist

$$\boxed{ULT\{x(t)\}=BLT\{x(t)u(t)\}}$$

Properties of ULT

(1) Differentiation property

$$\frac{dx(t)}{dt}\xleftrightarrow{ULT} SX(S)-x(0^-)$$

$$\frac{d^2x(t)}{dt^2}\xleftrightarrow{ULT} S^2X(S)-Sx(0^-)-\frac{dx(0^-)}{dt}$$

(2) Integration Property –

$$\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{X(S)}{S}+\frac{\int_{0^-}^{\infty} x(\tau)d\tau}{S}$$

(3) Time Shift –

$$\underset{\text{Causal}}{\underset{\downarrow}{x(t-t_0)}} \xleftrightarrow{ULT} e^{-st_0}X(s)$$

(4) Convolution: $x(t)=u(t)*u(t+1)=r(t+1)$

$$X(S)=\frac{1}{S}\times\frac{1}{S}=\frac{1}{S^2}$$

Linear constant coefficient differential equation –

A D.E will represent a liner system if and only if

(i) No higher power of x(t) and its derivative and y(t) and its derivative are allowed.

(ii) No product term of x(t) and y(t) and their derivatives are allowed.

(iii) No addition of constant term

Transfer function by ULT

$$X(S)\longrightarrow\boxed{H(S)}\longrightarrow Y(S)$$

$$H(S)=\frac{Y(S)}{X(S)}$$

If initial conditions are zero:

(1) T.F can be calculated

(2) $y(t)$ can be calculated from T.F

(3) If initial condition not zero – T.F can be calculated but $y(t)$ can not be calculated from T.F.

Types of Responses :

Case 1.

$$x(t)=0\longrightarrow\underset{\text{I.C }\neq 0}{\underset{\downarrow}{\boxed{h(t)\to H(s)}}}\longrightarrow \overset{\text{Transient Response}}{\overset{\uparrow}{y(t)}}=\underset{\text{Zero input Response}}{y_{ZIR}(s)}$$

$$x(t)\neq 0\longrightarrow\underset{\text{I.C = 0}}{\underset{\downarrow}{\boxed{h(t)\leftrightarrow H(S)}}}\longrightarrow \overset{\text{Steady state Response}}{\overset{\uparrow}{y(t)}}=\underset{\text{Zero State Response}}{Y_{ZSR}(S)}$$

$x(t)\longrightarrow\boxed{h(t)\leftrightarrow H(S)}\longrightarrow y_1(t)$: Poles of input forced Response,

$x(t)\longrightarrow\boxed{h(t)\leftrightarrow H(S)}\longrightarrow y(t)$: Poles of system Natural Response,

Initial value Theorem on ULT –

(1) Applicable only when x(t) is causal.

(2) Helps in calculation of initial value $x(0^+)$ not initial condition $x(0^-)$

$$X(s)=\frac{N(s)}{D(s)}$$

Note: while applying I.V.T common factors in N(S) and D(S) must be cancelled out .

$$\boxed{\lim_{t\to o^+} x(t)=\lim_{S\to\infty} SX(S)} \quad \begin{array}{l} x(t) \quad \text{is casual} \\ x(s)\to D^r > N^r \end{array}$$

4.2. Final value Theorem

(1) Applicable only when x(t) is casual .

(2) While applying F.V.T common factor must cancelled out.

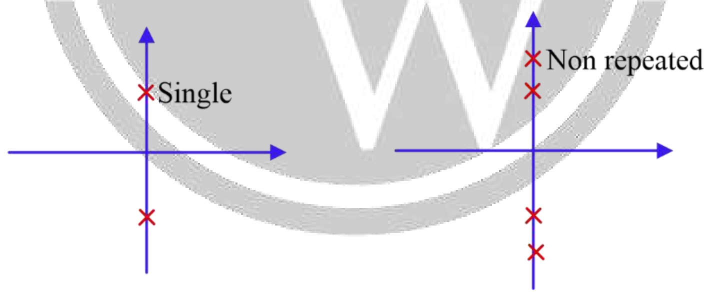
$$\boxed{\lim_{t\to\infty} x(t)=\lim_{S\to 0} SX(S)}$$

Case : 1. If all poles of $SX(S)$ lies strictly in LHP .

(i) Final value is finite

(ii) FVT applicable

Case : 2. If poles location of is $SX(S)$ as shown below .



(i) Final value is indeterminate.

(ii) FVT is not applicable.

Case : 3. In all other cases

(i) Final value is ∞

(ii) F.V.T is not applicable

5 Z TRANSFORM

5.1. Introduction

- Z domain signal
- $X(Z) = \dfrac{N(Z)}{X(Z)}$

- Rotational X(Z)
  - Proper X(Z)
    $D^r > N^r$
  - Improper X(Z)
    $N^r > D^r$
- Irrational X(Z)
  $X(Z) = \dfrac{e^Z}{(Z-1)}$

Laplace Tx	Z.T
$S = \sigma + j\omega$	$Z = re^{j\omega}$
$S = a + jb$:Point	$Z = r_o e^{j\omega_o}$: Point
$\text{Re}[s] = a$: Line parallel to $j\omega$ axis	$\lvert Z\rvert = r_o$:Circle concentric to unity circle $\lvert Z\rvert = 1$
$\text{Re}\{S\} > a$:Region parallel to $j\omega$ axis	$\lvert Z\rvert > r_o$ Region concentric to unity circle.

Relation between Z.T and L.T

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \boxed{Z = e^{st_s}}$$

$$|Z| = e^{\sigma T_s}$$

$$\angle Z = \omega T_s$$

Mapping

$\sigma > 0$	Left Half of s plane	$0 \le \lvert Z\rvert < 1$	Family of circles having radius less then 1.
$\sigma > 0$	Right half of s plane	$1 < \lvert Z\rvert \le \infty$	Family of circles having radius greater then 1.
$\sigma = 0$	$j\omega$ axis	$\lvert Z\rvert = 1$	Unity circle

(1) Vertical line in s plane $\to$ A circle in A.C.W in Z-plane

(2) Left half side of s plane $\to$ Inside unity circle in Z-plane

(3) Left side nature $\rightarrow$ In ward nature in Z – plane

(4) Right hand side of s plane – outside of unity circle in z – plane

(5) Right side nature $\rightarrow$ outside nature in z- plane.

(6) $j\omega$ axis mapped onto unity circle.

(7) origin in s plane is mapped $z = e^{ST} = 1$

Important Analogy

C.T signal	D.T Signal		
u(t)	u(n)		
u(-t)	u(-n-1)	$e^{-at}u(t)$	$a^n u(n)$
$-e^{-at}u(t)$	$-a^n u(-n-1)$		

Z. Transform

B.Z.T
$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

Unilateral Z.T
$$X(Z) = \sum_{n=0}^{\infty} x(n)Z^{-n}$$

B.Z.T

$$(z_0)^n u(n) \longrightarrow \frac{Z}{Z - Z_0} \qquad ROC : |Z| > |Z_0|$$

$$-(z_0)^n u(-n-1) \longleftrightarrow \frac{Z}{Z - Z_0} \qquad ROC : |Z| < |Z_0|$$

(1) $$a^n u(n) \longleftrightarrow \frac{Z}{Z - a} \qquad ROC : |Z| > |a|$$

(2) $$a^{-n} u(n) \longleftrightarrow \frac{Z}{Z - \left(\frac{1}{a}\right)} \qquad ROC : |Z| > \frac{1}{|a|}$$

(3) $$(-a)^n u(n) \longleftrightarrow \frac{Z}{Z - (-a)} \qquad ROC : |Z| > |-a|$$

(4) $$(-a)^{-n} u(n) \longleftrightarrow \frac{Z}{Z - \left(\frac{-1}{a}\right)} \qquad ROC : |Z| > \frac{1}{|-a|}$$

(5) $$-a^n u(-n-1) \longleftrightarrow \frac{Z}{(Z - a)} \qquad ROC : |Z| < |a|$$

(6) $-(a)^{-n}u(-n-1) \longleftrightarrow \dfrac{Z}{Z-\left(\dfrac{1}{a}\right)}$ $ROC: |Z| < \dfrac{1}{|a|}$

(7) $-(-a)^{n}u(-n-1) \longleftrightarrow \dfrac{Z}{Z-(-a)}$ $ROC: |Z| < |-a|$

(8) $-(-a)^{n}u(-n-1) \longleftrightarrow \dfrac{Z}{Z-\left(\dfrac{-1}{a}\right)}$ $ROC: |Z| < \left|\dfrac{1}{-a}\right|$

(9) $u(n) \longleftrightarrow \dfrac{Z}{(Z-1)}$ $ROC: |Z| > 1$

(10) $-u(-n-1) \longleftrightarrow \dfrac{Z}{(Z-1)}$ $ROC: |Z| < 1$

Properties

(1) Linearity: $x_1(n) \longleftrightarrow X_1(z)$ $ROC: R_1$

$x_2(n) \longleftrightarrow X_2(z)$ $ROC: R_2$

Case :1 $g(n) = Ax_1(n) + Bx_2(n)$ $(R_1 \cap R_2) = \{\theta\}$ $Z.T$ $D.N.E$

L.S.S $\neq \{\theta\}$

R.S.S $X(Z)\text{exist} \Rightarrow AX_1(z) + BX_2(z)$

B.S.S

Case:2 $\underbrace{g(n) = Ax_1(n) + Bx_2(n)}_{F.D+Abs\sum} \longrightarrow G(z) = AX_1(z) + BX_2(z)$

ROC:entire z plane except

(2) Time Shifting: $x(n) \longleftrightarrow X(Z)$ $ROC: R_1$

$x(n+1) \longleftrightarrow ZX(Z)$ $ROC: R_1$, except possibly

$|Z| = 0$ or $|Z| = \infty$

inclusion/declusion.

(3) Multiplication by complex exponential:

$x(n) \longleftrightarrow X(Z)$ $ROC: |Z|$

$Z_0^n x(n) \longleftrightarrow X\left(\dfrac{Z}{Z_0}\right)$ $ROC: \left|\dfrac{Z}{Z_0}\right|$

$u(n) = \dfrac{Z}{Z-1}$, $|Z| > 1$

$$\left(e^{j\omega_o}\right)^n u(n) \longleftrightarrow \frac{Z}{Z-e^{j\omega_0}} \qquad |Z|>1$$

$$\cos\,\omega_o n\, u(n) \longleftrightarrow \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \qquad ROC: |Z|>1$$

$$\sin\omega_0 n\, u(n) \longleftrightarrow \frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \qquad |Z|>1$$

$$a^n \cos\,\omega_0 n\, u(n) \longleftrightarrow \frac{Z^2 - az\cos\omega_0}{Z^2 - 2az\cos\omega_0 + a^2} |Z|>|a|$$

$$a^n \sin\omega_0 n\, u(n) \longleftrightarrow \frac{az\sin\omega_o}{Z^2 - 2az\,cas\,\omega_0 + a^2} |Z|>|a|$$

Properties of ROC

(1) ROC may or may not include zeros of x(z).

(2) Will not include poles of x(z).

(3) If x(n) absolutely summable $\rightarrow$ROC of x(z) includes unity circle.

(4) $x(n)$ F.D + Abs Σ $\longrightarrow$ ROC of $X(Z)$, will be entire Z plane except possibly $|Z| = 0$ AND / OR $|Z| = \infty$

(5) $x(n)$ is $L.S.S$
- $\nearrow$ X(z) may not exist, even for signal Value of $|Z|$
- $\searrow$ If X(z):exist, ROC will inside the innermost circle having radius formed by magnitude of innermost non-zero pole

(6) $x(n)$ is L.S.S
- $\nearrow$ $X(Z)$ may not exist, even for signal value of $|Z|$
- $\searrow$ If $X(Z)$: exist, ROC will inside the innermost circle having radius formed by magnitude of innermost non - zero pole

(7) $x(n)$ is B.S.S
- $\nearrow$ $X(Z)$ may not exist, even for signal value of $|Z|/r$
- $\searrow$ If $X(Z)$: exist, ROC will be in form of ring bounded by magnitude of finite/non zero poles

Time Scaling

$$x(n) \longleftrightarrow X(Z) \qquad ROC: |Z|$$

$$x\left(\frac{n}{K}\right) \longleftrightarrow X(Z^K) \qquad ROC: |Z^K|$$

Area or Summation property-

$$X(S)=\int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$\nearrow \int_{-\infty}^{\infty} x(t)dt \text{ when } S=0 \text{ is part of ROC}$$

$$X(S=0)$$

$$\searrow \infty, \text{ when S = 0 is not part of ROC}$$

$$\int_{-\infty}^{\infty} x(t)dt \xrightarrow{\text{F.T}} X(\omega)|_{\omega=0}$$

S = 0 Part of soc $\longrightarrow$ $X(S)|_{s=0} \longrightarrow \infty$

S = 0 is not part of soc $\longrightarrow$ $X(S)|_{s=0} \longrightarrow$ Area can not be calculated from S domain

- $a^n u(n) \rightarrow \dfrac{Z}{Z-a}$ $\quad |Z|>|a|$

  $na^{n-1}u(n) \rightarrow \dfrac{Z}{(Z-a)^2}$ $\quad |Z|>|a|$

Multiplication by *n*

$$nx(n) \longleftrightarrow -z\frac{dx(z)}{dz} : \text{ROC-Remains Same}$$

- $na^n u(n) = \dfrac{az}{(z-a)^2}$ $\quad |z|>|a|$

  Or

  $na^n u(n-1)$

- $(n+1)a^{n+1}u(n+1) \longleftrightarrow \dfrac{az^2}{(z-a)^2}$ $\quad |z|>|a|$

  Or

  $(n+1)a^{n+1}u(n)$

- $a^n u(n) \longleftrightarrow \dfrac{z}{(z-a)}$ $\quad |z|>|a|$

  $\dfrac{na^{n-1}u(n)}{1!} \longleftrightarrow \dfrac{z}{(z-a)^2} : |z|>|a|$

  $\dfrac{n(n-1)(n-2)a^{n-3}u(n)}{3!} \longleftrightarrow \dfrac{z}{(z-a)^4} : |z|>|a|$

Analogy between L.T and Z. T

$S \leftrightarrow (1 - z^{-1})$ analogy

$z = e^{ST}$ equivalent

Inverse Z.T

Table 1 X(Z) : Rational , ROC Known and x(n) to be Calculated

ROC	$x(n)$
Outside outmost finite pole	R.S.S
Inside Innermost nonzero pole	L.S.S
Ring from, bounded by non zero and finite poles	B.S.S

Table 2 X(Z) : Rational x(n) is given and ROC is to be decided .

$x(n)$	R.O.C
R.S.S	Outside outermost finite pole
L.S.S	Inside Innermost nonzero pole
B.S.S	Ring from bounded by finite non zero pole

Table 3 : X(Z) : Rational nature of ROC known and x(n) to be calculated .

ROC	$x(n)$
Outside outermost finite pole, including $\|Z\| = \infty$	Causal
Inside Innermost non zero pole, including $\|Z\| = 0$	Anti casual
Ring form bounded by non zero and finite pole	Non causality

Table 4 : X(Z) : Rational

$x(n)$	R.O.C
Casual	Outside outermost finite pole including $\|Z\| = \infty$
Anti causal	Inside innermost non – zero pole including $\|Z\| = 0$
Non causal	Ring from bounded by finite and non zero pole .

Methods to calculate I.Z.T

$X(Z) = (D)/D(Z)$

(1) By Long division

(i) $D(Z) \geq N(Z)$

$\nearrow$ casual : $N(Z),\ D(Z) \rightarrow$ decreasing power of Z.

(ii) $x(n)$

$\searrow$ Anticausal: $N(Z),\ D(Z) \rightarrow$ Increasing power of Z.

(2) Partial fraction

(i) $X(Z)$:pole – zero cancellation .

(ii) Plot Pole diagram and obtain all possible ROC.

(iii) Perform partial fraction of $\left\{\frac{X(Z)}{Z}\right\}$ if needed and calculate I.Z.T for each ROC.

Convolution Property:

$x(n) \leftrightarrow X(Z)\ R_1$

$h(n) \leftrightarrow H(Z)\ R_2$

y(n)=x(n)*h(n) $\longrightarrow R_1 \cap R_2 = \{\phi\} Y(Z) D.N.E$

$R_1 \cap R_2 \neq \{\phi\}\ y(z) = X(z)H(Z)$

ROC : $R_1 \cap R_2$

Accumulation

$$x(n) \longleftrightarrow X(Z) : ROC - R$$

Case 1. $x(n) * u(n)$

$$\sum_{K=-\infty}^{n} x[K] \longleftrightarrow \frac{x(z)}{(1-Z^{-1})} \quad ROC : R \cap (|z| > 1)$$

Case 2. $x(n) = 0,$ $n<0$ or $n \leq -1$

$$x(n) * u(n)$$

$$\sum_{K=-\infty}^{n} x[K] = \sum_{K=0}^{n} x[K] \longleftrightarrow \frac{X(z)}{(1-Z^{-1})}$$

$n \leq -1$

Generalized eigen function for D.T LTI s/s-

D.T LTI system : exponential $\left(Z_0^n\right)$

$$x(n) = \underset{\text{eigen function}}{\underset{\downarrow}{Z_0^n}} \longrightarrow \boxed{h(n)} \longrightarrow y(n) = z_0^n \underset{\text{scalar eigen value}}{\underset{\downarrow}{H(z_o)}} \begin{array}{l} \nearrow \text{Real} \\ \searrow \text{Complex} \end{array}$$

$$\boxed{y(n) = z_0^n \sum_{K=-\infty}^{\infty} \underset{H(z_o)}{\underset{\downarrow}{h[K] Z_0^{-K}}}}$$

Important Table:

$x(n)$	ROC
R.S.S + causal	Outside outermost finite pole including $\lvert Z\rvert=\infty$
Finite duration + causal	Entire Z plane including $\lvert Z\rvert=\infty$ and possibly including $\lvert Z\rvert=0$
L.S.S + Anti causal	Inside Innermost +Non zero pole including $\lvert Z\rvert=0$
Finite duration + Anti causal	Entire Z plane including $\lvert Z\rvert=0$
R.S.S + Non causal	Outside outmost finite pole , including $\lvert Z\rvert=\infty$
L.S.S + Non Causal	Inside innermost non zero pole not including $\lvert Z\rvert=0$
B.S.S + Non causal	Ring from bounded by finite & Non zero pole.
Finite duration + Non causal	Entire Z plane not including $\lvert Z\rvert=0\,\&\,\lvert Z\rvert=\infty$

Stability of an LTI S/S.

$h(n)\to$ must be absolutely summable

ROC $\to$ will include unity circle.

Causality:

$h(n)\to$ Must be causal signal

ROC $\to$Either outside of outmost pole including $\lvert Z\rvert=\infty$ or entire Z plane including $\lvert Z\rvert=\infty$

Anti Causality :

$h(n)\to$ Anti causal

ROC $\to$ Either inside the innermost pole or entire z plane including $\lvert Z\rvert=0$

Non Causality:

$h(n)\to$ non causal

$H(Z)\to$ Has finite and non zero poles $\nearrow RSS+NC$, $\to LSS+NC$, $\searrow BSS+NC$

$H(Z)\to$ Does not have any finite – non zero pole. ROC entire Z plane not including $\lvert Z\rvert=0\ \&\ \lvert Z\rvert=\infty$

Causal + Stable – All poles must be strictly inside unity circle H(Z) has finite and non zero pole, if not then decide based on common portion of ROC [causal $\cap$ stable]

Anti causal + Stable

H(Z) finite and non zero pole $\longrightarrow$ All the poles must be strictly outside unity circle.

H(Z) does not have finite and non zero pole $\longrightarrow$ (ROC of Stable) $\cap$ (ROC of anti causal)

Unilateral Z. T

$$x(n) \longleftrightarrow X(Z)$$

$$X[Z] = \sum_{n=0}^{\infty} x(n) Z^{-n}$$

$$UZT\{x(n)\} = BZT\{x(n)u(n)\}$$

(1) $1 \longrightarrow \dfrac{Z}{Z-1}$

(2) $2^n \longrightarrow \dfrac{Z}{Z-2}$

(3) $\cos\omega_0 n \xrightarrow{UZT} \dfrac{Z^2 - Z\cos\omega_0 n}{Z^2 + 2Z\cos\omega_0 n + 1}$

Properties of UZT

(1) Time Shifting

$$x(n-1) \longleftrightarrow Z^{-1}X(Z) + x(-1)$$

$$x(n-2) \longleftrightarrow Z^{-2}X(Z) + Z^{-1}x(-1) + x(-2)$$

Types of Response

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) \quad \text{ZIR}$$

$I.C \neq 0$

$$\underset{\neq 0}{x(n)} \longrightarrow \boxed{h(n)} \longrightarrow y(n) \quad \text{ZSR}$$

I.C = 0

If y(n) is only due to input $\Rightarrow$ Forced Response y(n) is only due to system pole $\Rightarrow$ Natural response

Transfer function

If I.C = 0

$$H(z) = \frac{Y(Z)}{X(Z)}$$

Note :

(1) I.C = 0

(a) H(z) can be calculated.

(b) Y(n) can be calculated from T.F

(2) $I.C \neq 0$

(a) H(z) can be calculated

(b) Y(n) can not be calculated from T.F

Initial Value Theorem	Final Value Theorem
$\lim_{n\to 0} x(n) = \lim_{Z\to\infty} X(z)$ Valid only when (1) $x(n)$ is casual $D^r \geq N^r$ (2) $X(z) = N(Z)/D(Z)$	$\lim_{n\to\infty} x(n) = \lim_{Z\to 1}(1-Z^{-1})X(Z)$ $\boxed{\lim_{n\to\infty} x(n) = \lim_{Z\to 1}(Z-1)X(Z)}$ Valid it (a) x(n) is causal (b) all the poles of $(1-z^{-1})X(z)$ or $(z-1)X(z)$ Should strictly be inside unity circle

Note: Before using this theorem , common factors must be cancelled out in $X(Z)$.

Multiplication by *n*

$$nx(n) \longleftrightarrow -z\frac{dX(z)}{dz}$$

6 DTFT

6.1. Introduction

Important Table:

Time domain	Frequency domain
Continuous	Non Periodic
Discrete	Periodic
Periodic	Discrete
Non Periodic	Continuous

Transform	Time domain	Frequency domain
C.T.F.S	C + P	Discrete + Np
C.T.F.T	C + Np	C + Np
DTFS	D + p	D + p
DTFT	D + Np	C + p

$X(e^{j\omega}) = \sum_{n-\infty}^{\infty} x(n)e^{-j\omega n}$ well defined DTFT, calculates from B.Z.T at unity circle

- For well defined DTFT to converge $x(n)$ must be absolutely summable.

For well defined DTFT

(a) Includes all energy signal .

(b) Formula of DTFT applicable

(c) Properties of DTFT applicable .

(d) $X(e^{j\omega})$ will be defined for each and every value of ω.

Limitedly defined DTFT

(a) Includes all power signal

(b) Formula not applicable .

(c) properties applicable.

(d) $X(e^{j\omega})$ will be $\longrightarrow \infty$ for any one value of ω.

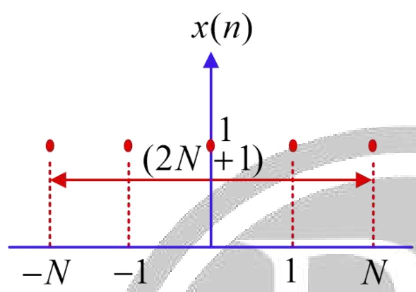
Note: $X(e^{j\omega})$ is periodic with $-\pi \le \omega \le \pi$,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$a^n u(n) \longleftrightarrow \frac{1}{1-ae^{-j\omega}} \quad \text{When } |a|<1$$

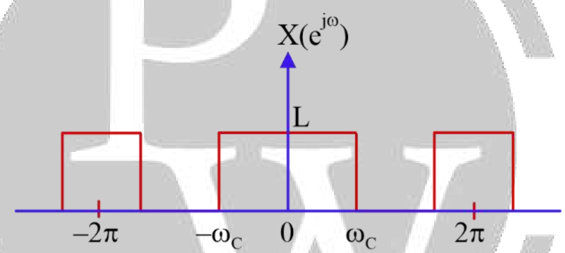
$$\boxed{X(e^{j\omega}) \longleftrightarrow \frac{1}{1-ae^{-j\omega}}} \quad \text{periodic with } 2\pi$$

DTFT of signals



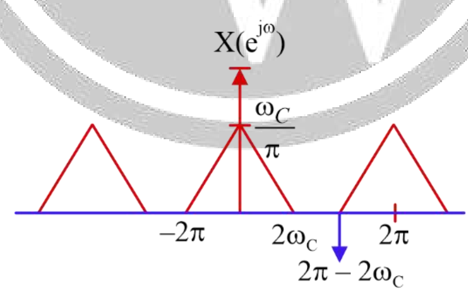
$$\longleftrightarrow X(e^{j\omega}) = \frac{\sin\left\{\frac{\omega}{2}(2N+1)\right\}}{\sin\left(\frac{\omega}{2}\right)}$$

➢ $x(n) = \dfrac{\sin\omega_c n}{\pi n} \xleftrightarrow{DTFT}$



Only valid when $\omega_c < \pi$

➢ $x(n) = \left(\dfrac{\sin\omega_c n}{\pi n}\right)^2$



$\omega_c < \dfrac{\pi}{2}$

Properties of DTFT :

(1) Linearity - $Ax_1(n) + Bx_2(n) \longleftrightarrow Ax_1(e^{j\omega}) + BX_2(e^{j\omega})$

(2) Time shifting $x(n-n_0) \longleftrightarrow e^{-j\omega n_o}X(e^{j\omega})$

$x(n+n_0) \longleftrightarrow e^{j\omega n_o}X(e^{j\omega})$

(3) Frequency shifting

$$e^{j\omega_o n}x(n) \longleftrightarrow X(e^{j(\omega-\omega_o)})$$

$$e^{-j\omega_o n}x(n) \longleftrightarrow X(e^{j(\omega+\omega_o)})$$

$$\cos\omega_o n \longleftrightarrow \pi[\delta(\omega-\omega_o)+\pi\delta(\omega+\omega_o)] - \pi \le \omega \le \pi$$

$$\sin\omega_o n \longleftrightarrow \frac{\pi}{j}[\delta(\omega-\omega_o)-\delta(\omega+\omega_o)] - \pi \le \omega \le \pi$$

$$(-1)^n x(n) = e^{j\pi n}x(n) \longleftrightarrow x(e^{j(\omega-\pi)}) \longleftrightarrow X(-e^{j\omega})$$

(4) Time Reversal - $x(-n) \longleftrightarrow x(e^{-j\omega}) = X\left((e^{j\omega})*\right)$

(5) Complex conjugate - $x*(n) \longleftrightarrow X^*\left((e^{j\omega})*\right) = X^*(e^{-j\omega})$

$x(n)$	$X(e^{j\omega})$
E O NENO	E O NENO

$x(n)$	$X(e^{j\omega})$
R+E	R+E
R+O	I+O
I+E	I+E
I+O	R+O

$x(n)$	$X(e^{j\omega})$
Real	C.S
I	C.A.S
C.S	Real
C.A.S	I

(1) Time Expansion - $x\left[\frac{n}{K}\right] \longleftrightarrow X(e^{j\omega K})$

1st difference or successive difference –

$$x(n)-x(n-1) \longleftrightarrow (1-e^{-j\omega})X(e^{j\omega})$$

$$u(n) \xleftrightarrow{DTFT} \pi\delta(\omega)+\frac{1}{(1-e^{-j\omega})} - \pi \le \omega \le \pi$$

or

$$\sum_{K=-\infty}^{\infty}\pi\delta(\omega-2\pi K)+\frac{1}{(1-e^{-j\omega})}$$

Multiplication with n - $nx(n) \longleftrightarrow +j\dfrac{d}{d\omega}X(e^{j\omega})$

Convolution - $y(n) = x(n) \times h(n) \longleftrightarrow y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

6.1.1. Parseval Energy Theorem

(1) $\displaystyle\sum_{n=-\infty}^{\infty} x(n)h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H(e^{-j\omega})d\omega$

(2) $\displaystyle\sum_{n=-\infty}^{\infty} x(n)h^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H^*(e^{j\omega})d\omega$

(3) $\displaystyle\sum_{n=-\infty}^{\infty} x(n)x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})X(e^{-j\omega})d\omega$

(4) $\displaystyle\sum_{n=-\infty}^{\infty} x(n)x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega$

7 SAMPLING

7.1. Introduction

Instantaneous sampling in time domain:

$$m_s(t) = m(t)c(t)$$

$$m_s(t) = m(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \qquad T_S = \frac{1}{f_s}$$

T_s : sampling interval

f_s : Sampling frequency

Instantaneous sampling in frequency domain

$$m(t) \longleftrightarrow M(\omega)$$

$$m(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_S) \longleftrightarrow f_s \sum_{n=-\infty}^{\infty} M(\omega - n\omega_s)$$

$$m(t) \longleftrightarrow M(f)$$

$$m(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_S) \longleftrightarrow f_S \sum_{n=-\infty}^{\infty} M(f - nf_S)$$

Spectral analysis of Instantaneous Frequency

$$\sum \delta(t - nT_S) \longleftrightarrow \frac{2\pi}{T_S} \sum \delta(\omega - n\omega_S)$$

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_S) \longleftrightarrow f_S \sum_{n=-\infty}^{\infty} \delta(f - nf_S)$$

If $f_S > 2f_m$:– oversampling

Tx : No aliasing PBG = T_S

Rx: practical LPF , Ideal LPF with $f_m \le f_c \le f_s - f_m$

Recovery - $(f_S > 2f_m) \cap (f_m \le f_C \le f_S - f_m)$

If $f_S = 2f_m$: critical sampling

Tx : Aliasing on verge (No aliasing)

Rx : Ideal LPF with $(f_s = f_m)$ & PBG $= T_S$

Recovery - $(f_S = 2f_m) \cap (f_C = f_m)$

Case 3: $f_S < 2f_m$ under sampling

Tx : Aliasing

Rx : Recovery not possible.

Low Pass Sampling Theorem–

A lowpass signal bandlimited to $f_m\ Hz$ can be sampled and reconstructed form its samples if and only

If $\boxed{f_S \geq 2f_m} \cap \boxed{f_m \leq f_c \leq (f_S - f_m)}$

Sampling rate. $\boxed{f_s \geq 2f_m}$

Nyquist rate = minimum sampling rate

$$(f_S)_{\min} = 2f_m$$

$$\text{Nyquist interval } T_s = \frac{1}{(f_s)_{\min}} = \frac{1}{2f_m}$$

$m(t)$	f_{NY}
$\sin c\,(t)$	$1Hz$
$\sin c(at)$	$a\,Hz$
$\sin c^K(at)$	$Ka\ Hz$
$\sin c(at) + \sin c(bt)$	$Max(aHz, bHz)$
$\sin c(at) \times \sin c(bt)$	$(a+b)Hz$
$\sin c(at) * \sin c(bt)$	$\min(aHz, bHz)$
$\frac{d}{dt}\sin c(t)$	1Hz
$\int_{-\infty}^{t} \sin c(\tau)d\tau$	1Hz

Sampling using general carrier pulse train–

$$m(t) \longleftrightarrow M(f)$$

$$c(t) \longleftrightarrow C(f) = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_s)$$

$$M_s(f)=\sum_{n-\infty}^{\infty}C_nM(f-nf_s)$$

If $(f_s > 2f_m)\cap(f_m \le f_c \le f_s - f_m)$

L.P.F(P.B.G)	y(t)
1	$c_0 m(t)$
$1/C_o$	$m(t)$
L	$L\ C_0 m(t)$

When c(t) is rectangular pulse train –

$$C_n=\frac{2A}{a}\sin c\left[n\left(\frac{2}{a}\right)\right]$$

$$M_S(f)=\sum_{n=-\infty}^{\infty}\left(\frac{2A}{a}\right)\sin c\left(\frac{2n}{a}\right)\delta(f-nf_s)$$

Sampling of Sinusoidal Signal:

Note: $f_s < 2f_m$ Recovery is possible through BPF

$f_s < 2f_m$ Recovery not possible through BPF

Calculation of Frequency:

(i) $m(t)=A_m\cos 2\pi f_m t$

$C(t)$: Impulse train with period $T_S \rightarrow 0, f_s, 2f_s, 3f_s, ---$

$m_s(t)=m(t)c(t)\longrightarrow \quad 0\pm fm \begin{cases}\nearrow 0+f_m \searrow \\ \searrow |0-f_m| \nearrow\end{cases}$ same

$f_s \pm f_m \begin{cases}\nearrow f_s+f_m \\ \searrow |f_s-f_m|\end{cases}$

$2f_s \pm f_m \begin{cases}\nearrow 2f_s+f_m \\ \searrow |2f_s-f_m|\end{cases}$

(ii) $m(t)=A_1\cos 2\pi f_1 t + A_2\cos 2\pi f_2 t \longrightarrow f_1, f_2$

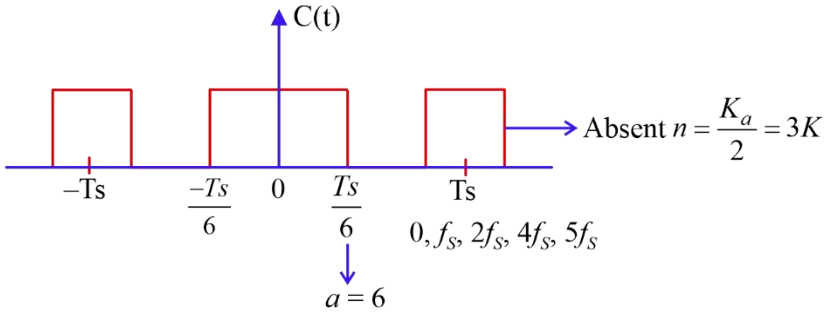
$C(t)=$ Impulse train , $0, f_s 2f_s, 3f_s$

$0\pm f_1 \quad 0\pm f_2$

$f_s \pm f_1 \quad f_s \pm f_2$

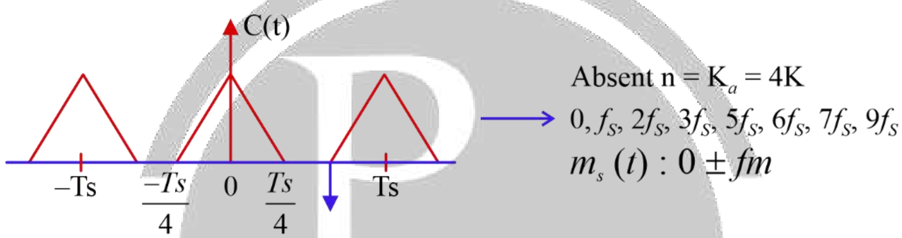
$2fs \pm f_1 \quad 2fs \pm f_2$

(iii) $m(t) = A_m \cos 2\pi f_m t \longrightarrow f_m$



$m_S(t) = 0 \pm f_m, f_s \pm f_m, 2f_s \pm f_m, 3f_s \pm f_m$

(iv) $m(t) = Am \cos 2\pi f_m t$



$f_s \pm f_m$

$2f_s \pm f_m$

$3f_s \pm f_m$

Band pass sampling

m(t) is lowpass signal

S(t) = m (t) cos $2\pi f_c$t

Lowpass signal (Low pass S.T) | Bandpass signal (Bandpass S.T)

$\boxed{f_S \geq \frac{2f_H}{K}}$ $\boxed{K = \left[\frac{f_H}{f_H - f_L}\right]}$ $[\cdot] \to GIF$

$\boxed{\text{Nyauist rate} = 2f_H}$

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8 MISCELLANEOUS

8.1. DFT (Disrete Fourier Transform)

$$\underset{\text{discrete in time}}{\underset{\downarrow}{x(n)}} \xleftrightarrow{DTFT} \underset{\text{Continuous in frequency}}{\underset{\downarrow}{X(e^{j\omega})}}$$

DFT:

Discrete in time + discrete in frequency .

$$x(n) \xleftrightarrow{DFT} X(K)$$

(i) x(n) periodic with length n .

(ii) x(K) periodic with length K

(iii) Information of one period of either $x(n)$ or $X(K)$ will be given .

N point $x(n)$ is given calculate n point X(K)

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\left(\frac{2\pi}{N}\right)Kn} \qquad K = 0,1,2,---N-1$$

$$x(n) \xleftrightarrow{DFT} X(K)$$

$$\boxed{x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K) e^{j\left(\frac{2\pi}{N}\right)Kn}} \quad n = 0,1,2,---N-1$$

$$x(K) \xleftrightarrow{IDFT} x(n)$$

Twiddle factor: $\boxed{W_N = e^{-j\frac{2\pi}{N}}}$

$W_N^0 = 1$	$W_n^{N+1} = W_N$	$W_N^{(n+lN)} = W_N^n$	$W_N = e^{-j\frac{2A}{N}}$
$W_N^N = 1$	$W_N^{n+\frac{N}{2}} = -W_N^n$	$W_N^{lN} = W_N^N = 1$	$W_N^{-1} = W_N^*$
$W_N^{N/2} = -1$	$W_N^{n+N} = W_N^n$	$W_N^{(2l+1)\frac{N}{2}} = -1$	

Matrix Method :

- DFT: $[X(K)]=[W_N^n][x(n)]$

- IDFT: $[x(n)]=\frac{1}{N}[W_N^n]^{-1}[X(K)]=\frac{1}{N}[W_N^n]^*[X(K)]$

2 point DFT / IDFT (N=2)

$$\begin{bmatrix} X(0) \\ X(1) \end{bmatrix}=\begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix}\begin{bmatrix} x(0) \\ x(1) \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ x(1) \end{bmatrix}=\frac{1}{2}\begin{bmatrix} W_N^{-0} & W_N^{-0} \\ W_N^{-0} & W_N^{-1} \end{bmatrix}\begin{bmatrix} X(0) \\ X(1) \end{bmatrix}$$

3 point DFT / IDFT N=3

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^1 & W_3^2 \\ 1 & W_3^2 & W_3^1 \end{bmatrix}\begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-4} \end{bmatrix}\begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix}$$

4 point DFT / IDFT N=4

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}=\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}$$

➢ If $x(n)=x(-n)\rightarrow$ circle

$$DFT\left[DFT\{x(n)\}\right]\longrightarrow\left(\sqrt{N}\right)\left(\sqrt{N}\right)\{x(n)\}$$

$$DFT\left[DFT\left[DFT\left[DFT\{x(n)\}\right]\right]\right]=\left(\sqrt{N}\right)^4 x(n)$$

➢ If $X(-K)=X(K)$

$$IDFT[IDFT[IDFT[IDFT[x(K)]]]]=\left(\frac{1}{\sqrt{N}}\right)^4[X(K)]$$

- $$X(K) = \frac{1}{N^2}\sum_{K=0}^{N-1} x(n)W_N^{-Kn}$$

  If $x(n) = x(-n)$

  $$DFT[DFT(x(n))] = \left(\sqrt{N}\right)^2\left(\frac{x(n)}{N^4}\right) = \frac{x(n)}{N^3}$$

Properties of DFT:

(1) Linearity: $Ax_1(n) + Bx_2(n) \longleftrightarrow AX_1(K) + BX_2(K)$

(2) Periodicity: $x(n+N) = x(n)$

$X(K+N) = X(K)$

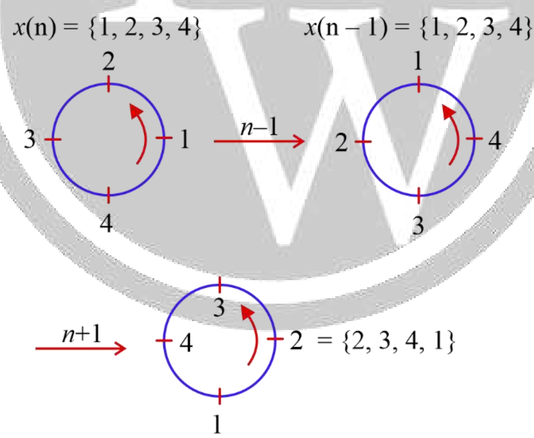
(3) Time Reversal: $[x(n)]_N \longleftrightarrow [X(K)]_N$

$(x(-n))_N \longleftrightarrow (X(-K))_N$

$x(N-n) \longleftrightarrow X(N-K)$

(4) Circular frequency shift $x(n) \longleftrightarrow X(K)$

$e^{j\left(\frac{2\pi}{N}\right)(l)n} x(n) \longleftrightarrow (x(K-l))_{MODN}$



Complex conjugate property $x(n) \longleftrightarrow X(K)$

$x^*(n) \longleftrightarrow X^*(-K)$

$(x^*(n))_{MODN} \longleftrightarrow (X^2(-K))_{MODN} = X^*(N-K)$

$x(n)$	$X(K)$
R+E	R+E
R+O	I+O
I+E	I+E
I+O	R+O

$x(n)$	$X(K)$
Real	C.S
Image	CAS
C.S	Real
C.A.S	Img.

Circular convolution

Case 1: Column Method

$$x_1(n)=\{a,b,c,d\}$$

$$x_2(n)=\{p,q,r,s\}$$

$$x(n)=x_1(n)*x_2(n)=\{\alpha,\beta,\gamma,\delta\}$$

$$\begin{bmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{bmatrix}=\begin{bmatrix}a & d & c & b\\ b & a & d & c\\ c & b & a & d\\ d & c & b & a\end{bmatrix}\begin{bmatrix}p\\ q\\ r\\ s\end{bmatrix}$$

Case 2: Row Method

$$[\alpha,\beta,\gamma,\delta]=[p\ q\ r\ \&]\begin{bmatrix}a & b & c & d\\ d & a & b & c\\ c & d & a & b\\ b & c & d & a\end{bmatrix}$$

- $x_1(n)\otimes x_2(n)\xleftrightarrow{DFT}X_1(K)X_2(K)$

  $x(n)\otimes x(n)\xleftrightarrow{DFT}X^2(K)$

Multiplication in time domain:

$$x_1(n).x_2(n)\xleftrightarrow{DFT}\frac{1}{N}[X_1(K)\otimes X_2(K)]$$

$$x^2(n)\xleftrightarrow{DFT}\frac{1}{N}[X(K)\otimes X(K)]$$

Parseval's Theorem

(1) $\sum_{n=0}^{N-1}x_1(n)x_2(n)=\frac{1}{N}\sum_{K=0}^{N-1}X_1(K)X_2(K)$

(2) $\sum_{n=0}^{N-1}x_1(n)x_2^*(n)=\frac{1}{N}\sum_{K=0}^{N-1}X_1(K)X_2^*(K)$

(3) $\sum_{n=0}^{N-1}x^2(n)=\frac{1}{N}\sum_{K=0}^{N-1}X(K)X(-K)$ $\boxed{\sum_{n=0}^{N-1}|x(n)|^2=\frac{1}{N}\sum_{K=0}^{N-1}|X(K)|}$

(4) $\sum_{n=0}^{N-1}x(n)x^*(n)=\frac{1}{N}\sum_{K=0}^{N-1}X(K)X^*(K)$

Time Expansion

- N point $\quad x(n) \xleftrightarrow{D.F.T} \{X(K)\}^{N\ \text{point}}$

  2N point $\quad x\left(\frac{n}{2}\right) \xleftrightarrow{D.F.T} \{X(K), X(K)\}^{2N\ \text{point}}$

- N point: $\quad X(K) \xleftrightarrow{IDFT} \{x(n)\}$

  2N point: $\quad X\left(\frac{K}{2}\right) \xleftrightarrow{IDFT} \frac{1}{2}[x(n), x(n)]$

Discrete Time Fourier Series

$$\underset{\text{Periodic N}}{\underset{\downarrow}{x(n)}} = \sum_{K=0}^{N-1} C_K e^{jn\left(\frac{2\pi}{N}\right)K}$$

$$C_K = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-jn\left(\frac{2\pi}{N}\right)K}$$

$$\boxed{C_K = \frac{X(K)}{N}} \qquad \boxed{C_{K+N} = C_K}$$

$$N \quad x(n) \xleftrightarrow{DFT} X(K) = N(C_K)$$

$$2N \quad [x(n), x(n)] \longleftrightarrow 2X\left(\frac{K}{2}\right) = 2\left[2N\ \frac{C_K}{2}\right]$$

FAST-FOURIER TRANSFORM : (F.F.T)

↙ Decimation in Time (D.I.T) ↘ Decimation in frequence (D.I.F)

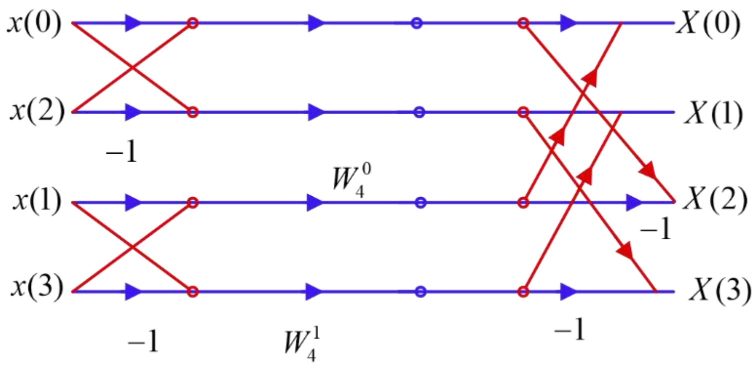
Drawback of DFT Calculation :

$$\boxed{X(K) = \sum_{n=0}^{N-1} x(n) W_n^{Kn}}$$

N Point DFT
- ↗ N^2 Complex multiplication $\longrightarrow 4N^2$ Real Multiplication
- ↘ $N(N-1)$ Complex addition $\rightarrow N(4N-2)$ Real additions

DIT algorithm in FFT :

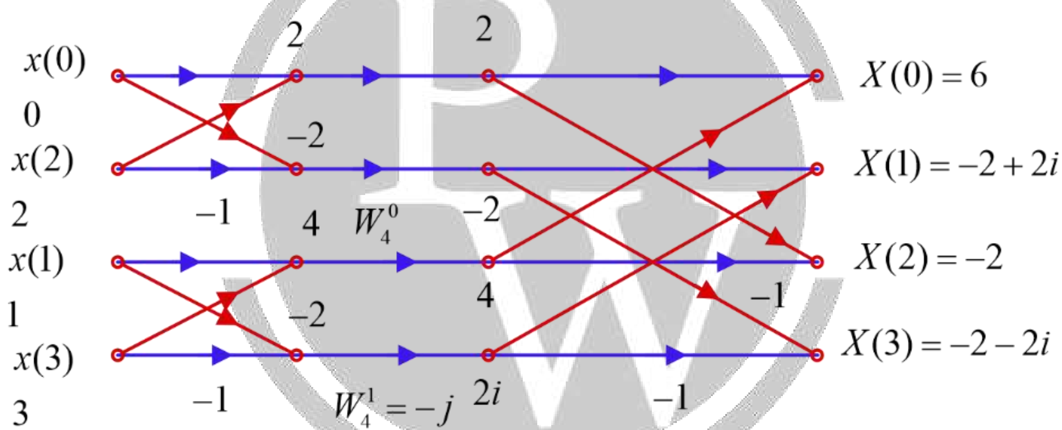
4 point DFT : $x(n) = \{x(o), x(1), x(2), x(3)\}$

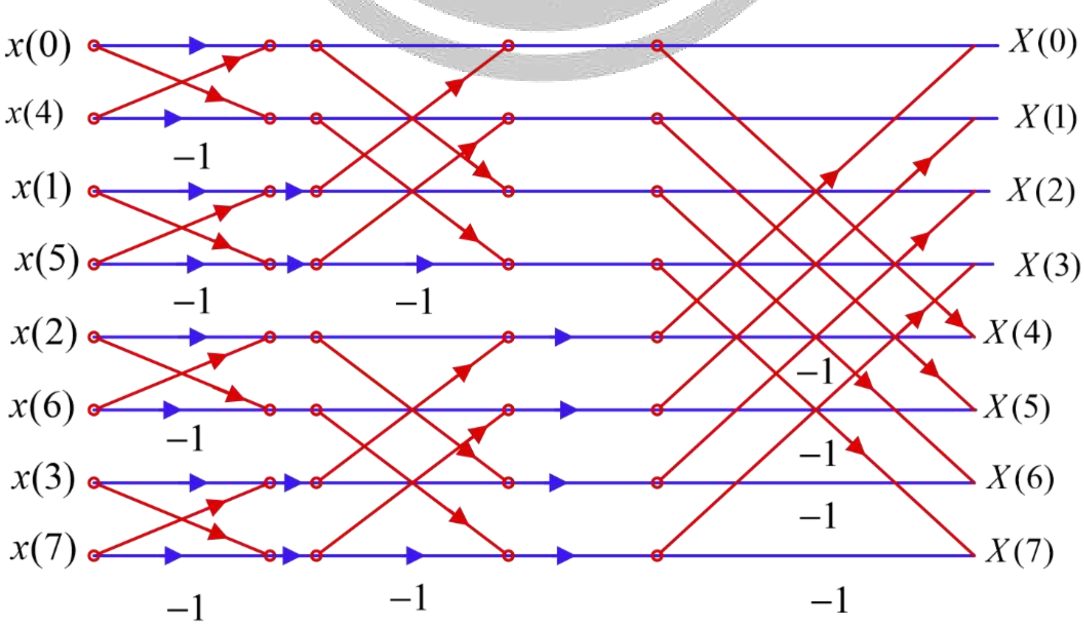


- $X(K)=\sum_{n=0}^{3}x(n)W_N^{Kn}$

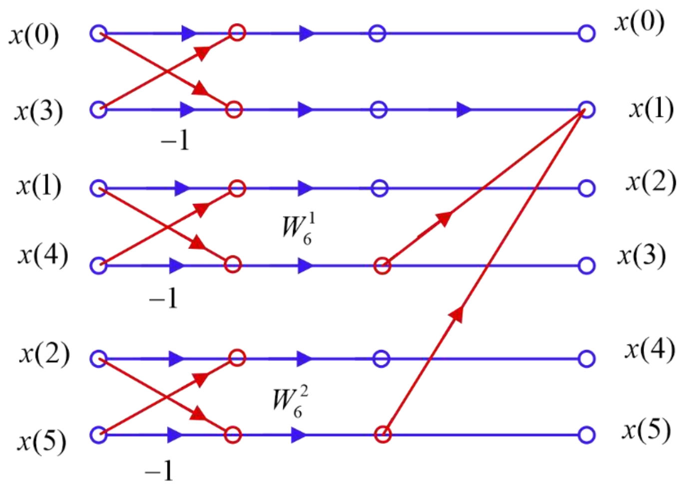
$$X(1)=\sum_{n=0}^{3}x(n)W_4^n=[x(0)-x(2)]+W_4^1[x(1)-x(3)]$$

$x(n)=\{0,1,2,3\}$





6 point DFT : $x(n)=\{x(0),x(1),x(2),x(3),x(4),x(5)\}$



$\boxed{DFT\ w.r.t\ X(1)}$

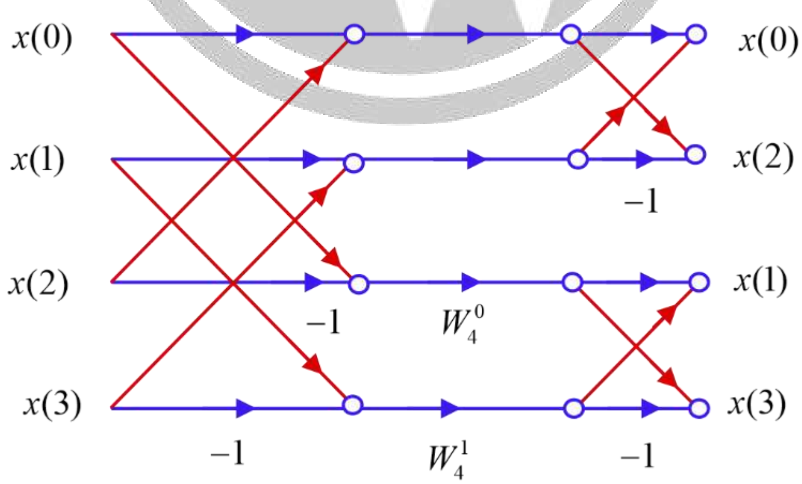
$$X(K)=\sum_{n=0}^{5}x(n)W_6^{Kn}$$

$$X(1)=\sum_{n=0}^{5}x(n)W_6^{n}=[x(0)-x(3)]+(x(1)-x(4))W_6^1+(x(2)-x(5))W_6^2$$

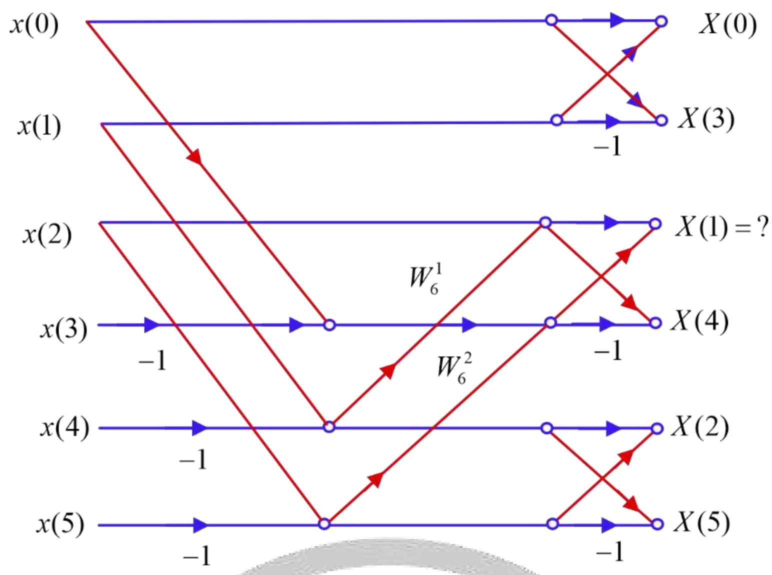
Summary:

Radix 2	Radix Non – 2
↓	↓
Symm Butlerfly	use formula to generate Butterfly

DIF algorithm



6 point DIF :



For Radix N Butterfly for calculation of N point DFT

- No of stages = $\log_2^N$
- No of Butterfly in each stage $= N/2$
- Total no. of Butterflies $= \frac{N}{2}\log_2^N$
- Total no of complex multiplication $= \frac{N}{2}\log_2^N$
- Total number of complex addition $= N\log_2 N$

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Control Systems

CONTROL SYSTEMS

INDEX

1 BLOCK DIAGRAM REPRESENTATION AND SIGNAL FLOW GRAPH

1.1. Introduction

Input Signal Excitation → System → Output Signal Response

1. $x(t) = \delta(t)$ Unit Impulse → System → $y(t)$ = Unit Impulse Response

$x(t) = A\delta(t)$ Impulse → System → $y(t)$ = Impulse Response

2. $x(t) = u(t)$ → System → $y(t)$ = Unit step Response

Similarly if $x(t) = Au(t)$ then $y(t) =$ step Response

$x(t) = r(t)$ then $y(t) =$ unit ramp Response

$x(t) = Ar(t)$ then $y(t) =$ Ramp Response

Linear System:

$Ax_1(t) + Bx_2(t)$ → Linear System → $Ay_1(t) + By_2(t)$

Time Invariant:

$x(t - T)$ → Time Invariant → $y(t - T)$

1.1.1. Linear Time Invariant System

$h(t) \longrightarrow$ unit Impulse Response of L.T.I

$h(t) \xrightarrow{F.T} H(\omega)$ – frequency System

$h(t) \xrightarrow{F.T} H(S)$ – Transfer function of LTI Convolution

$Y(t) = X(t) * h(t)$

$Y(S) = X(S)\, H(S)$

Standard LTI system

$$H(S)=\frac{1/RC}{S+1/RC}=\frac{L.T\ of\ y(t)}{L.T\ of\ x(t)}=\frac{Y(S)}{X(S)}$$

Case 1: T.F to differential equation

$$H(S)=\frac{Y(S)}{X(S)}=\frac{1/RC}{S+1/RC}$$

$$\frac{dy(t)}{dt}+\frac{1}{RC}y(t)=\frac{1}{RC}x(t)$$

Case 2: Differential eq. to T.F

$$\frac{dy(t)}{dt}+\frac{1}{RC}y(t)=\frac{1}{RC}x(t)$$

$$y(s)=\frac{y(0)}{s+\frac{1}{RC}}+\frac{(1/RC)\times(S)}{S+1/RC}$$

Case A Initial condition = 0
$Y(S)=X(S).H(S)$

(a) T.F can be calculated
(b) output can be calculated by using T.F

Case B Initial condition to $\neq 0$

$$Y(S)\neq X(S)H(S)$$

(a) T.F can be calculated by putting initial condition = 0
(b) Output can not be calculated .
- Regenerate initial condition to calculate output

Block Diagram Representation

- Used to represent a system .
- T.F can be calculated by forcing I.C = 0

$$X(S)\longrightarrow\boxed{H(S)}\longrightarrow Y(S)=H(S)\,X(S)$$

$r(t)\longrightarrow R(S)$ input

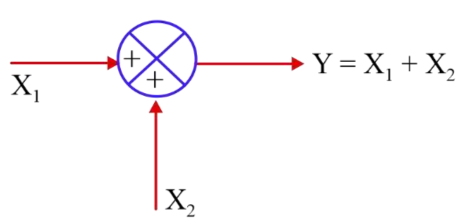
$c(t)\longrightarrow c(S)$ output

$g(t)\longrightarrow C(S)$ Transfet function

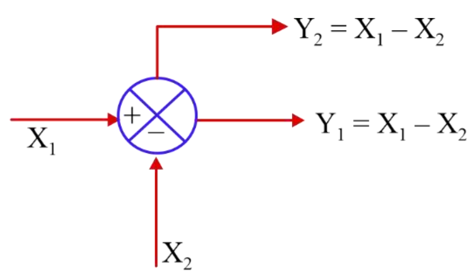
Important Concepts

(1) Summer – It should have 2 or more then 2 inputs.

Symbol

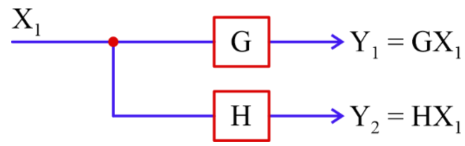


Multi Input Single Output



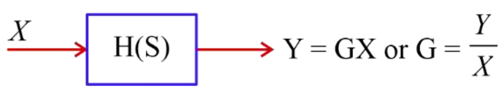
Multi Input Multi Output

(2) Take of points - single input and Multi output



Used for input distributions

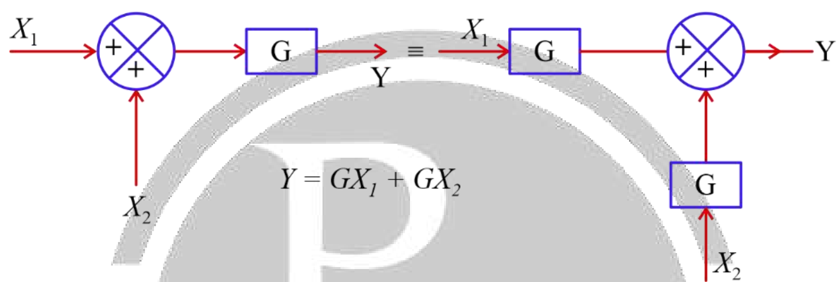
(3) Forward Gain - Direction always from input to output



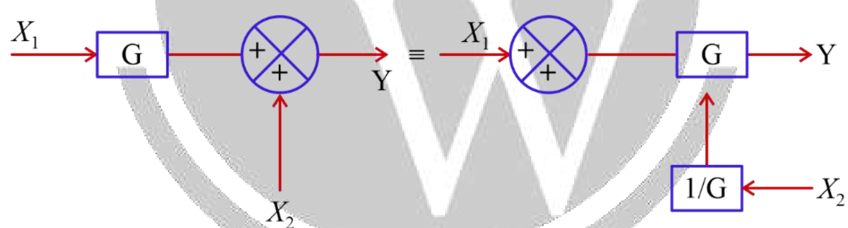
Rules

Case 1 : Summer and forward Gain

Case A

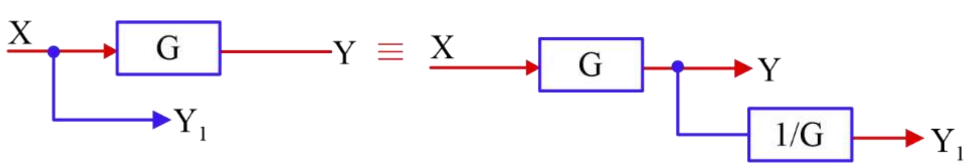


Case B

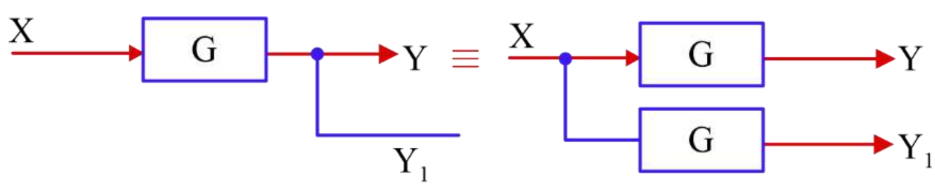


Case 2 : Take off points and forward gain

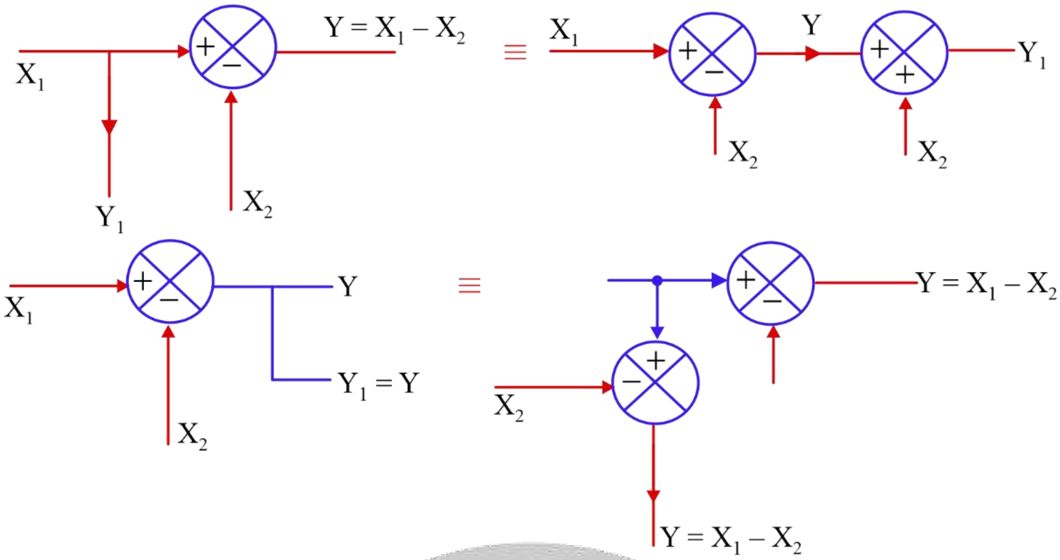
Case A



Case B



Case 3 : Take off point and summer

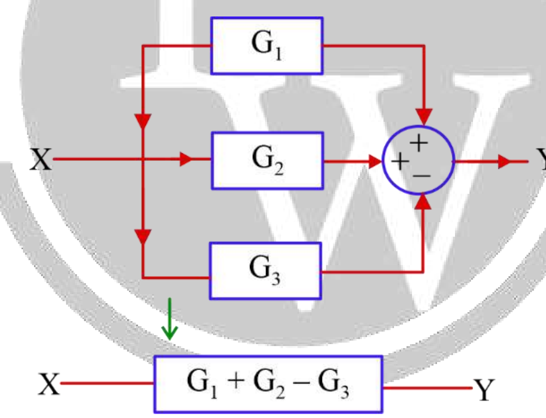


Gain Connected in Parallel

(1) Direction of flow should be from input to output from all the gains .

(2) Summing block should be common.

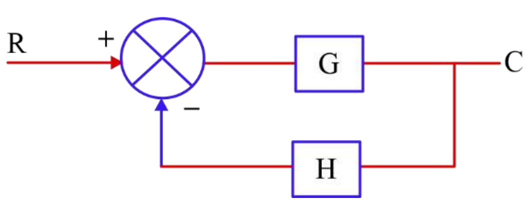
(3) Input should be common.



Gain Connected in Cascade

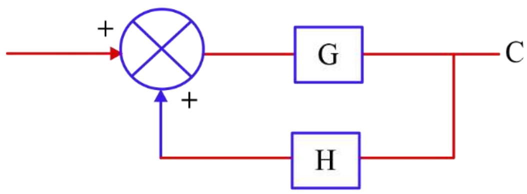
$$\frac{C}{R} = G_1 G_2$$

Feedback –



$$\boxed{\frac{C}{R} = \frac{G}{1+GH}}$$

Negative Feedback



$$\boxed{\frac{C}{R} = \frac{G}{1-GH}}$$

Problem solving Techniques -

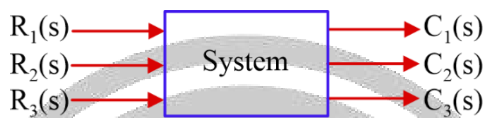
(1) Try to eliminate common node by using parallel paths .

(2) Convert 3 input summer to Two , 2 input Summer

(3) Try to bring two summers side by side by changing their inputs if required.

1.2. MIMO

(1) T.F can not be calculated



(2) $\left.\dfrac{C_1(S)}{R_1(S)}\right|_{\substack{R_2(S)=0\\R_3(S)=0}}$ Ratio parameter

(3) $\left.\dfrac{C_2(S)}{R_3(S)}\right|_{\substack{R_1(S)=0\\R_2(S)=0}}$ Ratio parameter

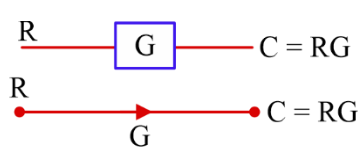
(iii) Output can be calculated by super position .

$$C_1(S) = ? \quad \underbrace{\left.\frac{C_1(S)}{R_1(S)}\right|_{\substack{R_2=0\\R_3=0}}}_{\downarrow\ =H_1(S)} \quad \underbrace{\left.\frac{C_1(S)}{R_2(S)}\right|_{\substack{R_1=0\\R_3=0}}}_{\downarrow\ =H_2(S)} \quad \left.\frac{C_1(S)}{R_3(S)}\right|_{\substack{R_1=0\\R_2=0}} = H_3(S)$$

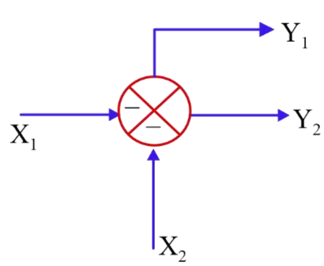
$$\boxed{C_1(S) = R_1(S)H_1(S) + R_2(S)H_2(S) + R_3(S)H_3(S)}$$

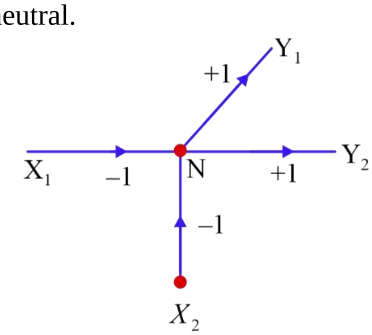
Signal flow Graph - Alternative Representation of a system .

(1) Gain Block Representation



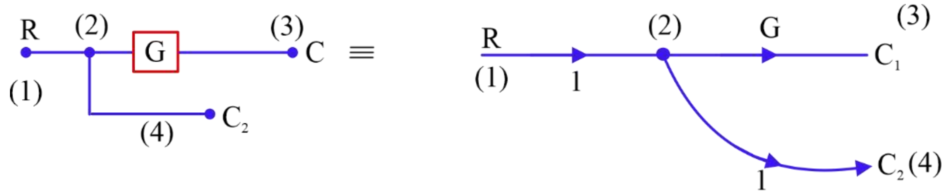
(2) Summer Block - Represented by node , after Making it neutral.





$\boxed{N = -X_1 - X_2}$ $\boxed{Y_1 = N}$ $\boxed{Y_2 = N}$

(3) Take off – Represented by a node.



Note : If take off point comes other summer both of them represented by same node, but if it comes before summer then two nodes .
Input Node - Only outgoing branches
Initial Node - may have incoming and outgoing branches

1.

R: input node

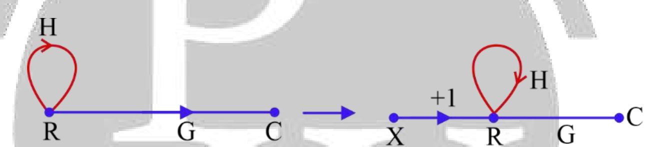
$\frac{C}{R} = G(M.G)$

$R = X$

R: Initial Node

X: Input Node

2.



$\frac{C}{R} \rightarrow$ MG not allowed R=HR+X

R: initial node

$\frac{C}{X} \rightarrow$ MG allowed X: input node

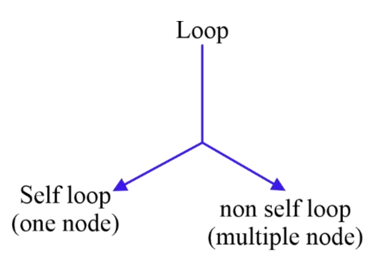
$\frac{R}{X} \rightarrow$ MG allowed

Output node – Having only incoming branches, when this condition not present then forward drugging Fill above condition.

Forward Path - Path connecting input to output.

- Direction from input to output.
- Any node should not traversed twice.
- Not in loop from.

Loop – A path which originate and terminate at same node



Macon Gain formula – used to calculate for

Limitations

(1) $\dfrac{\text{Output Node}}{\text{Input Node}}$ (2) $\dfrac{\text{Intermidiate Node}}{\text{Input Node}}$

$$\boxed{\frac{Y}{X}=\frac{\sum_{k=1}^{n} p_k\Delta_k}{\Delta}}$$

n = no of forward path from X to Y

Δ = 1-(sum of all loop gains) + (sum of product of two non touching loop gain) (sum of product of 3 non touching loop gains)

Δ_K = It is dependent on forward path

Δ_K = 1- [sum of all loop gains not touching K^{th} forward path] + (sum of product of 2 non touching loop gains not touching K^{th} Forword path) + --------------

Steps :

(1) $\dfrac{C}{R}\longrightarrow M.G$ Applicable

(2) Calculate total no of forward paths

(3) Calculate all types of loops .

(4) Calculate Δ and Δ_k

Note :

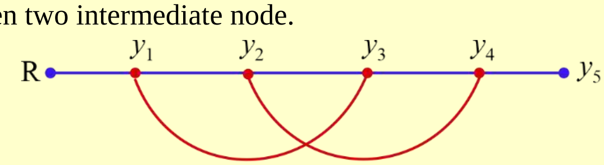
(1) Self loop at initial (first) node of unity /non unity can always be ignored .

(2) Self loop at intermediate nodes having unity gain –

(i) Result in inconsistent nodal equation $\dfrac{C}{R}\rightarrow\infty$,

(ii) To obtain finite $\dfrac{C}{R}$ such loops can be ignored .

(3) MGF can not be applied between two intermediate node.



$$\frac{y_5}{y_2}=?\quad=\left(\frac{\frac{y_5}{R}}{y_2/R}\right)\longrightarrow\text{calculate by MGF}$$

Mapping of Block diagram to SFG

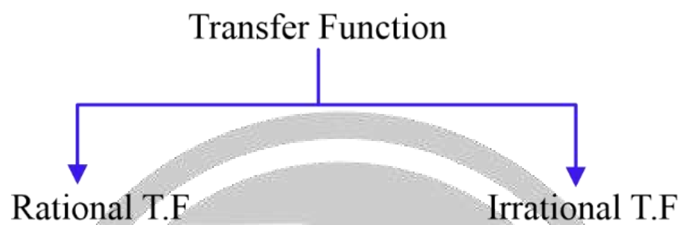
Summer $\longrightarrow$ Node

Take off $\longrightarrow$ Node

Gain $\longrightarrow$ Line

2 TIME RESPONSE ANALYSIS

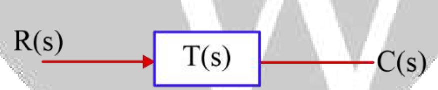
2.1. Introduction



$$T(S)=\frac{N(S)}{D(S)}=\frac{S+1}{S+2} \qquad T(S)=\frac{N(S)}{D(S)}=\frac{e^{S}}{(S+1)}$$

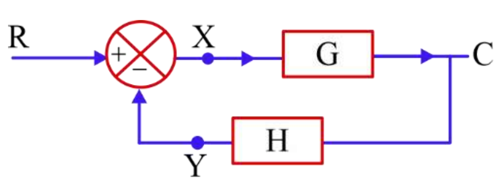
$$\text{Pole – Zero format } =\frac{k(S-z_1)(S+z_2)}{S^N(S-p_1)(S+p_2)}\rightarrow \text{Root locus Nyquist}$$

$$\text{Time constant from } T(S)=\frac{K(1+ST_1)(1+ST_2)}{S^N(1+ST_a)(1+ST_b)}\rightarrow \text{Bode , Nyquist}$$



Open loop system

Open loop Transfer function $T(S)=\dfrac{C(S)}{R(S)}$



Close loop system

$$\text{C.L.T.F} = T(S)=\frac{C(S)}{R(S)}=\frac{G}{1+GH}$$

C.L.S May hare C.L.T.F or O.L.T.F

$$\text{O.L.T.F of C.L.S}=\boxed{\frac{Y}{X}=GH}$$

2.1. Degree of T.F

$\Rightarrow$ Highest order of D(S) after pole zero cancellation.

$$T(S)=\frac{(S+1)}{S^4(S+2)(S+1)(S+3)^3}\text{ , degree = 8}$$

Type of a system

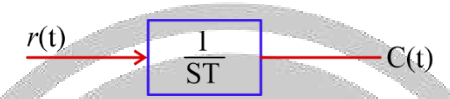
(1) Defined for C.L.S only .

(2) To calculate Type of C.L.S , O.L.T.F or G(S) H(S) of C.L.S is used.

(3) Pole at origin in O.LT.F of C.L.S – Type

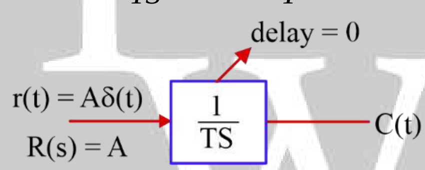
First order system (O.L.S)

$$\frac{C(S)}{R(S)}=\frac{1}{ST}$$



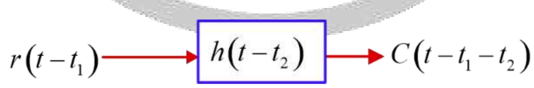
If $\quad r(t)=A\delta(t),\quad C(S)=\frac{A}{TS},\ c(t)=\frac{A}{T}u(t)$

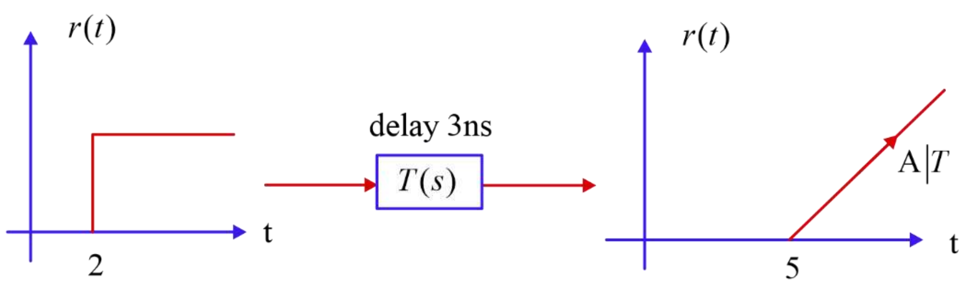
$$r(t)=A\delta(t-t_o)\ \ C(S)=\frac{Ae^{-St_0}}{TS}=c(t)=\frac{A}{T}u(t-t_o)$$



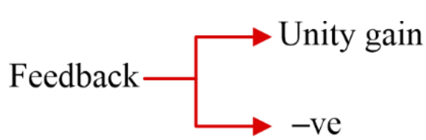
If system delayed by to $h(t-t_o)=H(S)=\frac{e^{-St_o}}{TS}$

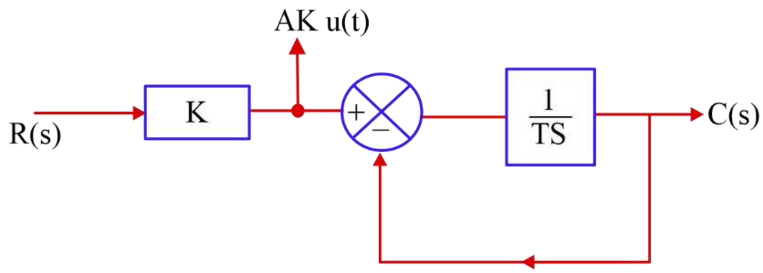
$$C(S)=\frac{Ae^{-st_o}}{TS}\quad ,\ C(t)=\frac{A}{T}u(t-t_o)$$



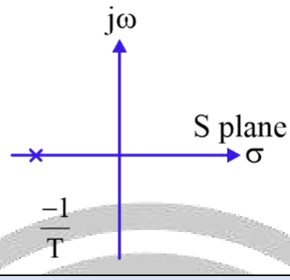


First order Close Loop System

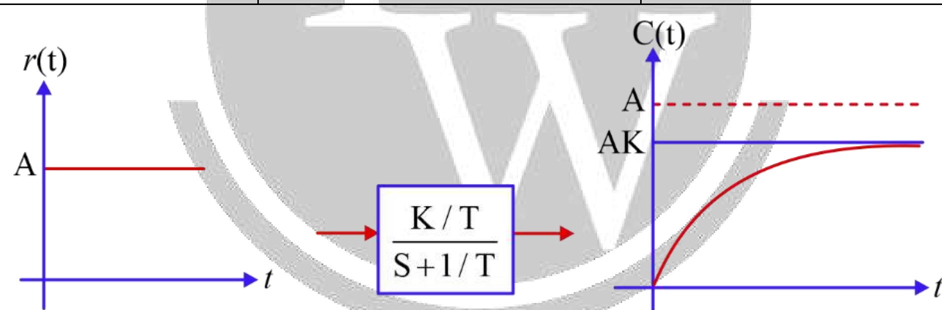




$$\boxed{T(S)=\frac{C(S)}{R(S)}=\frac{K/T}{S+1/T}}\xrightarrow{s=0}K(\text{d.c gain})$$



I/P	C(S)	C(t)
$r(t)=A\delta(t)$	$C(S)=\dfrac{AK/T}{S+1/T}$	$C(t)=\dfrac{AK}{T}e^{-t/T}u(t)$
$r(t)=AU(t)$	$C(S)=AK\left[\dfrac{1}{S}-\dfrac{1}{s+1}\right]$	$C(t)=A\left(1-e^{-\frac{t}{T}}\right)u(t)$



1. Initial slope at t = 0 can be calculate from graph and can be equated to $\left.\dfrac{dc(t)}{dt}\right|_{t=0}=\dfrac{AK}{T}$

2. Net input at summer = output approaches or settled .

Transient and Steady state Response –

$$C(t)=(1-e^{-t/T})u(t)=1-e^{-t/T}$$

$$C_1(t)=1\rightarrow\text{Steady state Response (constant)}$$

$$C_2(t)=-e^{-t/T}\rightarrow\text{Transient Response [exponential]}$$

$$\lim_{t\to\infty} c_1(t)=1,\qquad \lim_{t\to\infty} c_2(t)=0$$

Transient Response

It is part of total step Response which tend to 0, When large time frame is Considered 0.

- Tends to 0 as $t \to \infty$
- To calculate transient response from D.E $IC \neq 0,\ i/p = 0$
- Zero input response

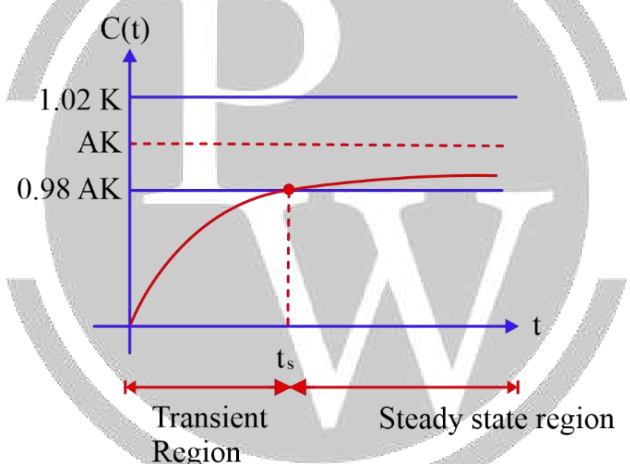
Steady State Response

Part of total step response which remains after transient dies out .

- To calculate the steady state response from D.E = I.C = 0 and input $\neq 0$
- zero state response

$$c(t) = c_{tr}(t) + c_{ss}(t)$$
$$\lim_{t \to \infty} c(t) = c_{ss}(t)$$

Setting Time



(1) t_s = Time required to settle in a predefined "Error Band".

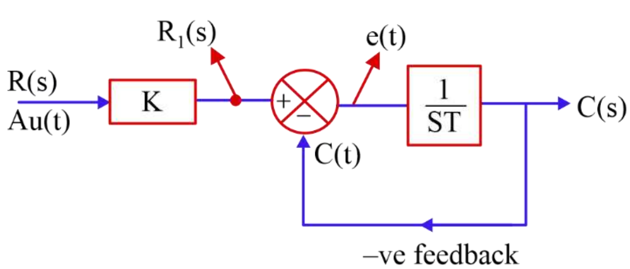
(2) Error Band $= \pm$m% of input + d.c gain

(3) t_s = f (error band)

(4) $t_s \downarrow$ as error band $\uparrow$

(5) $t_s \to \infty$ for 0% error band.

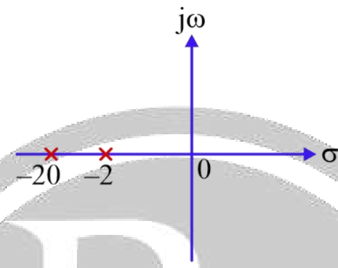
Error Signal



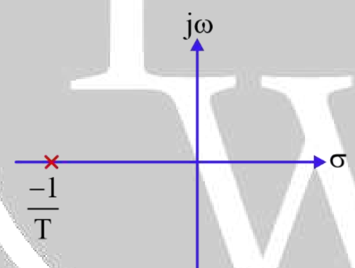
$$e(t) = Kr(t) - c(t)$$
$$e(t) = r_1(t) - c(t)$$
$$e(t) = \text{Error signal (unity f/b)}$$
$$e(t) = Ake^{-t/T}u(t)$$

- Steady state error = final value of error signal = $\lim_{t \to \infty} e(t) = 0$
- Time constant = $\dfrac{1}{|\text{Real part of dominant pole}|}$

$$T = \frac{1}{2}\sec$$



- 3 dB band width of RC circuit $= \dfrac{1}{RC}$ rad/sec $= \dfrac{1}{\text{Time constant}}$



$$T(s) = \frac{K/T}{S + \frac{1}{T}} \longrightarrow$$

Time constant =T

3dB BW=$1/T$ rad/sec

Rise Time

Time taken by step response of first order CLS to reach from 10% to 90% of its final value.

$t_s = 4T$ 2% error band

$t_r = 2.2T$

$t_d = 0.693T$

Second order system :

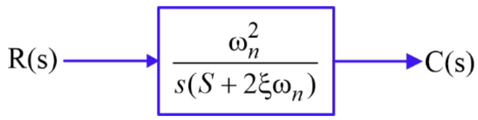
- For first order system = one parameter = Time constant
- For 2nd order $\rightarrow$ ξ : damping ratio

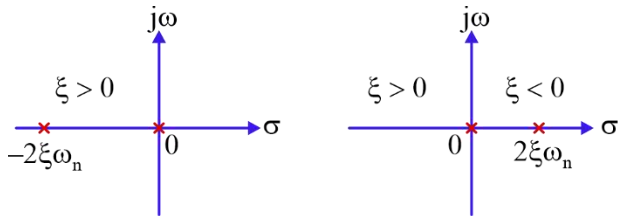
  ω_n : undamp natural frequency

$\xi\omega_n$ = damping tactor $\omega_n\sqrt{1-\xi^2}$ = Damped Natural frequency.

ω_d

2nd order O.L.S :

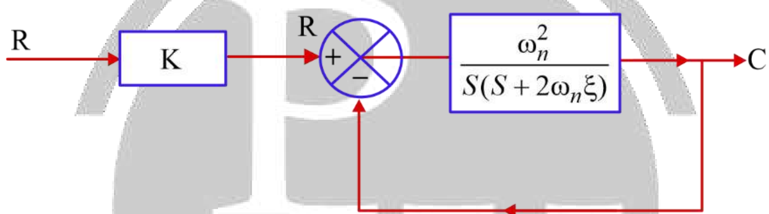




$$\frac{C(S)}{R(S)}=\frac{\omega_n^2}{S(S+2\xi\omega_n)} \quad 0<\omega_n<\infty \qquad -\infty<\xi<\infty$$

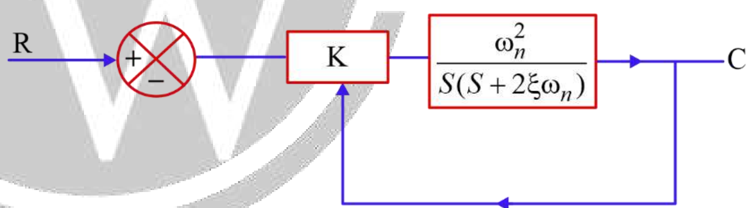
2nd Order C.L.S

(1) $\boxed{\frac{C(S)}{R(S)}=\frac{\omega_n^2}{S^2+2\xi\omega_n s+\omega_n^2}}$ $S=0\to 1$ d.c gain



(2) $\frac{C(S)}{R(S)}=\frac{K\omega_n^2}{S^2+2\xi\omega_n s+\omega_n^2}$

d.c gain = 0



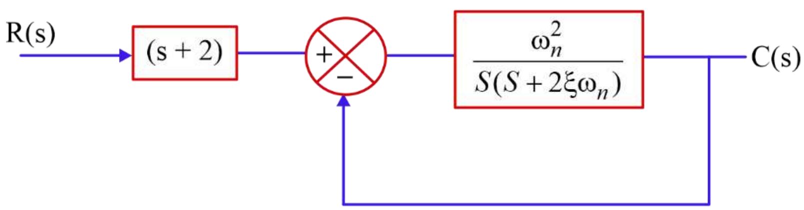
(3) $\frac{C(S)}{R(S)}=\frac{K\omega_n^2}{S^2+2\xi\omega_n S+K\omega_n^2}$

d.c gain = 1

$$T(S)=\frac{\omega_n'^2}{S^2+2\xi'\omega'_n S+\omega_n'^2} \quad \omega'_n=\sqrt{k}\ \omega_n$$

$$\xi'=\frac{\xi}{\sqrt{K}}$$

Non Standard second order s/s



$\boxed{T(S)=\frac{C(S)}{R(S)}=\frac{(S+2)\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}}$ Non Standard T.F

For std. 2nd order C.L.T.F $\Rightarrow$ $\boxed{C.L.T.F = \dfrac{OLTF}{1+OLTF}}$

Important Points :

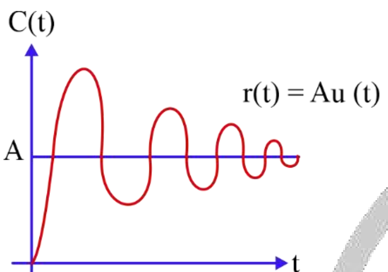
(1) Damping ratio (ξ), dimensionless, represent decay of oscillation in output response .

(2) Undamped natural frequency $(\omega_n) \rightarrow$ rad/sec, frequency of oscillation of output response, in absence of damping force, $\omega_n > 0$.

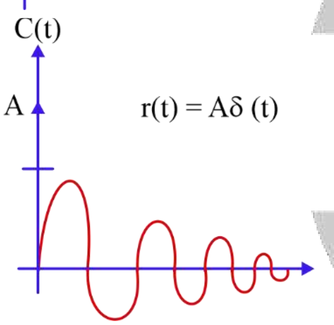
(3) Damped natural frequency $\omega_d = \omega_n\sqrt{1-\xi^2}$ rad/sec frequency of output response oscillation, when Damping force is present, $\omega_d > 0$.

Case 1: $0 < \xi < 1$ (under damp) $c(t)$

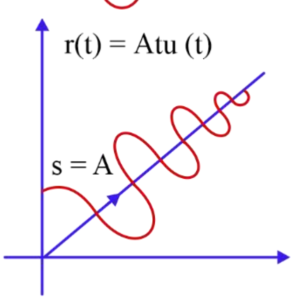
(1)



(2)

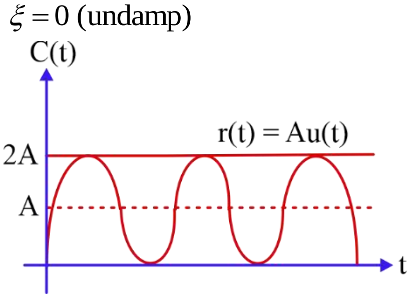


(3)

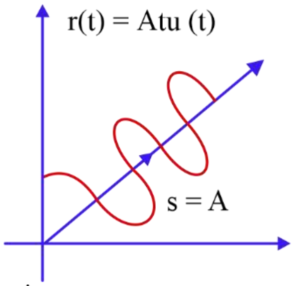


Case 2 : $\xi = 0$ (undamp)

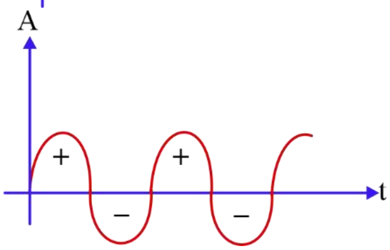
(1)



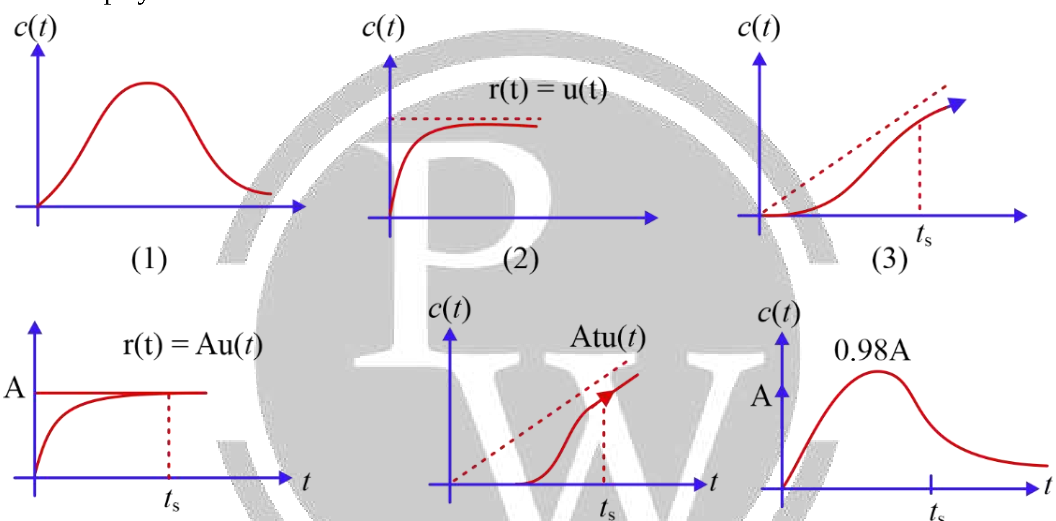
(2)



(3)



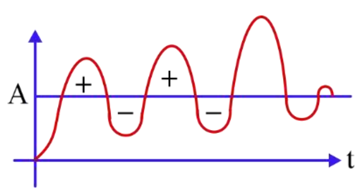
Case 3 : $\xi = 1$ critical damp system



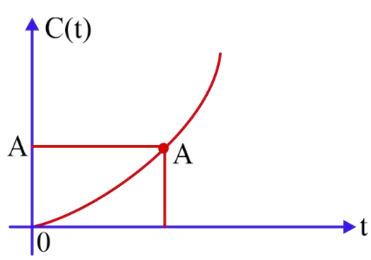
Case 4 : $1 < \xi < 0$ overdamp

- Output response does not oscillate and approaches constant parameter of input not in shortest possible time.

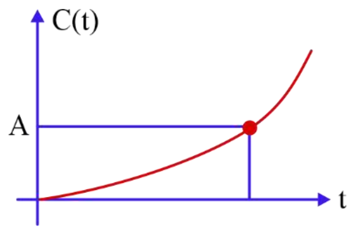
Case 5 : $-1 < \xi < 0$



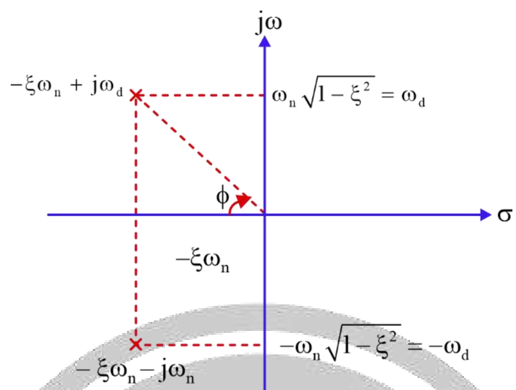
Case 6 : $\xi = -1$



Case 7 : $\xi < -1,\qquad -\infty < \xi - 1$



Under Damped System

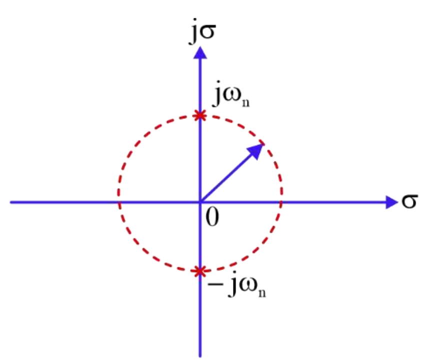


- $0<\xi<1$ $\qquad T(S)=\dfrac{\omega_n^2}{S^2+2\xi\omega_n^2 S+\omega_n^2}=\dfrac{G(s)}{1+G(s)H(s)}=\dfrac{N(s)}{D(s)}$
- Characteristic equation $=\boxed{1+G(S)H(S)=0}$

  $S^2+2\xi\omega_n S+\omega_n^2=0$
- Poles $S=-\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}=-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$
- Complex poles, left half of S-plane.
- Time constant $=T=\dfrac{1}{\xi\omega_n}$
- $t_s=4T$ 2% Error Band

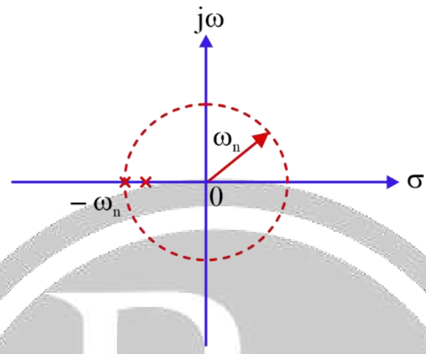
  $=3T$ 5% Error Band
- $\cos\phi=\xi$
- Locus of poles = semi circle of radius ω_n
- Always stable.

Undamp System :



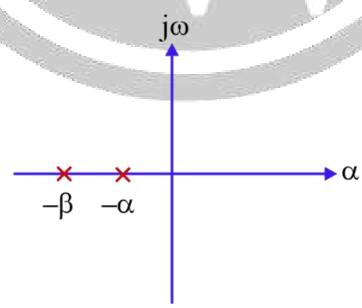
- $\xi = 0$
- $D(s) = 1 + G(S)H(S) = S^2 + \omega_n^2 = 0$
- Poles , $S = \pm j\omega_n$ (purely imaginary)
- $T = \infty$
- $t_s = \infty$
- marginally stable (Non repeated poles on imaginary axis)
- Locus $\rightarrow$ circle of radius ω_n

Critical Damp System :



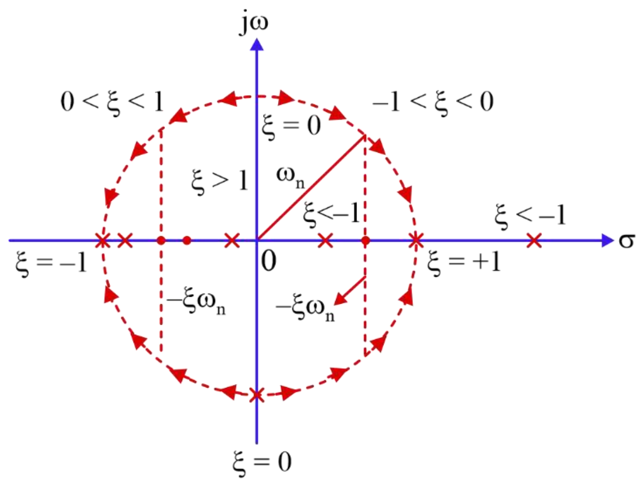
- $\xi = 1$
- $D(S) = S^2 + 2\omega_n S + \omega_n^2 = 0 \qquad (S + \omega_n)^2 = 0$
- Poles $S = -\omega_n, -\omega_n$
- $T = \dfrac{1}{\omega_n}$
- $t_s = 6T$ for 2%
- Always Stable
- Poles lies on circle of radius ω_n

Over damp System :

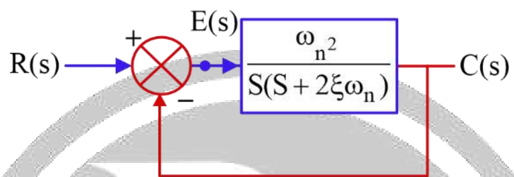


- $1 < \xi < \infty$
- $D(S) = S^2 + 2\xi\omega_n S + \omega_n^2 = 0 = (S + \alpha)(S + \beta)$
- Poles = $-\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$ real poles
- $T = \dfrac{1}{\alpha}$

  $t_s = 4T$ 2%
- Always stable



Step Response of under damp system



$r(t) = Au(t)$

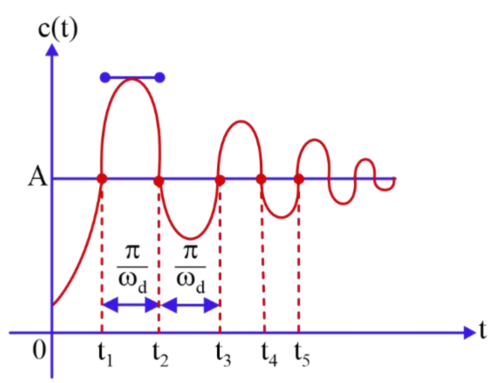
- Error signal E(S) = R(S) - C(S)

- $\boxed{C(S) = \dfrac{A\omega_n^2}{S(S^2 + 2\xi\omega_n S + \omega_n^2)}}$ $\qquad C(t) = A\left\{1 - \dfrac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta\right\} \to u(t)$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\cos\theta = \xi, \quad \tan\phi = \frac{\sqrt{1-\xi^2}}{\xi}$$

- $e(t) = A\left[\dfrac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)\right]u(t)$



(1) t_k [Time Constant When $C(t) = A$] $\qquad \boxed{t = \dfrac{n\pi - \phi}{\omega_d}}$

(2) $n=1,\ \boxed{t_1=\frac{\pi-\phi}{\omega_d}}$ Rise time $\boxed{\Delta t=\frac{\pi}{\omega_d}}$

$n=2$ $\boxed{t_2=\frac{2\pi-\phi}{\varepsilon_d}}$ Time instant of A

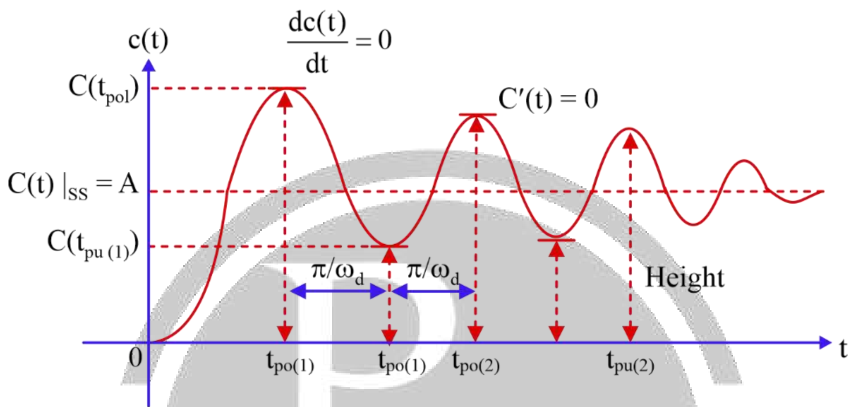
$\Rightarrow$ 2nd order rise time $\Rightarrow$ Time taken to reach from 0 to 100% of final value.

$$t_r=\frac{\pi-\cos^{-1}\xi}{\omega_n\sqrt{1-\xi^2}}$$

$\pi = 3.14$

ϕ = radians

(ii) Peak Overshoot and Peak undershoot time :



Peak Time $\boxed{t=\frac{n\pi}{\omega_d}}$ $n=1,2,3$

(1) $t_1=\frac{\pi}{\omega_d}=t_{po(1)}\rightarrow$ First peak overshoot time

(2) $t_2=\frac{2\pi}{\omega_d}=t_{pu(1)}\rightarrow$ first peak undershoot time

(3) $t_3=\frac{3\pi}{\omega_d}=t_{po(2)}\rightarrow 2^{nd}$ peak overshoot time

(4) $\boxed{t_{po(K)}=\frac{(2K-1)\pi}{\omega_d}}$, K = 1,2

K^{th} peak overshoot time

(5) $\boxed{t_{pu(K)}=\frac{2K\pi}{\omega_d}}$ K = 1, 2

K^{th} peak undershoot time

(6) At peak overshoot time $\frac{dc(t)}{dt}=0,$ same for peak undershoot also

(7) At 1st peak overshoot time value, output is maximum $c\left(t=t_{po(1)}\right)=[c(t)]_{\max}$

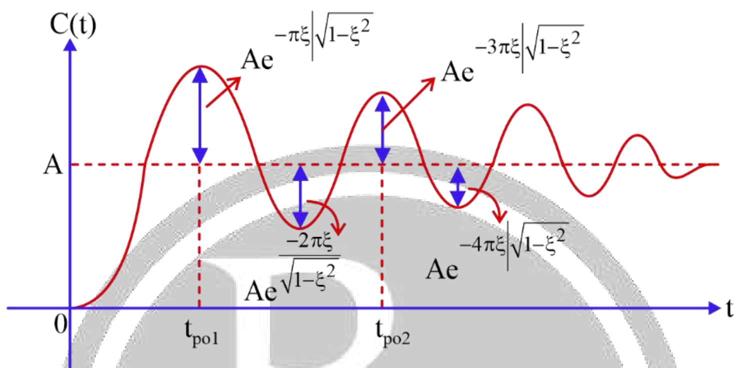
(8) Time gap b/w two successive $\rightarrow p.o|p.u$ is $2\pi/\omega d$

(9) First peak undershoot time = time period of damped oscillation $t_{pu(1)} = T_D = \dfrac{2\pi}{\omega_d}$

(10) $C(t)\big|_{SS} = C(t)\big|_{t=\infty} = A$

(11) Output maxima $\Rightarrow \boxed{C\left(t_{po(K)}\right) = A\left(1+e^{\frac{-\pi\xi(2K-1)}{\sqrt{1-\xi^2}}}\right)} K = 1, 2$

(12) Output Minima $\Rightarrow \boxed{C\left(t_{pu(K)}\right) = A\left(1-e^{\frac{-\pi\xi(2K)}{\sqrt{1-\xi^2}}}\right)} K = 1, 2$



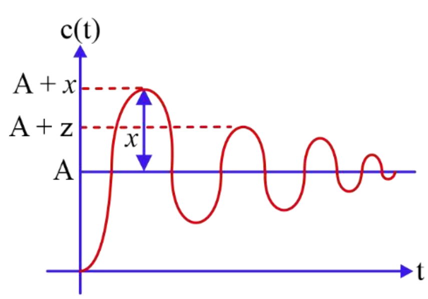
(1) Maxima of $C(t) \Rightarrow C(t)\Big|_{\max} = C(t)\Big|_{\max} = A\left(1+e^{-\pi\xi/\sqrt{1-\xi^2}}\right)$

(2) Peak overshoot = height of overshoot $\Rightarrow C\left(t_{po(K)}\right) - C(t)\big|_{SS} = Ae^{\frac{\pi\xi(2K-1)}{\sqrt{1-\xi^2}}}$

(3) Max. peak overshoot $= C(t)\big|_{\max} - C(t)\big|_{SS} = Ae^{-\pi\xi/\sqrt{1-\xi^2}}$

(4) Max. peak percentage overshoot - $\%M_{po} = \dfrac{C(t)\big|_{\max} - C(t)\big|_{ss}}{C(t)\big|_{ss}} \times 100\% = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100\%$

(5) Graphical Relation



(i) $x = Ae^{-\pi\xi/\sqrt{1-\xi^2}}$

$z = Ae^{-3\pi\xi/\sqrt{1-\xi^2}}$

(ii) $\dfrac{x}{A} \times 100\% = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100\%$

(iii) $\frac{z}{A}\times 100\% = e^{-3\pi\xi/\sqrt{1-\xi^2}}\times 100\%$

(6) Minima of $C(t) \to C(t)\big|_{\min} = C\left(t_{pu_1}\right) = A\left(1-e^{-2\pi\xi/\sqrt{1-\xi^2}}\right)$

(7) Peak undershoot = height of undershoot $C(t)\big|_{ss} - C\left(t_{pu(K)}\right) = Ae^{-\pi\xi(2K)/\sqrt{1-\xi^2}}$

(8) Maxima peak overshoot $C(t)\big|_{ss} - C\left(t_{pu1}\right) = C(t)\big|_{ss} - C(t)\big|_{\min} = Ae^{-2\pi\xi/\sqrt{1-\xi^2}}$

(9) Maximum peak percentage undershoot

$$\%M_{pu} = \frac{C(t)\big|_{ss} - C(t)\big|_{\min}}{C(t)\big|_{ss}}\times 100\% = e^{-2\pi\xi/\sqrt{1-\xi^2}}\times 100\%$$

(10) Decay ratio $= \frac{M_{po(2)}}{M_{po(1)}} = \frac{M_{Pu(2)}}{M_{pu(1)}} = e^{-2\pi\xi/\sqrt{1-\xi^2}}$

(11) $\%M_{p0} = m\% = \%M_{po(1)}$

$$s.1 \quad p = \frac{m}{100}$$

$$s.2 \quad \xi = \sqrt{\frac{(\ln p)^2}{\pi^2 + (\ln p)^2}}$$

(12) $r(t) = Au(t)$, effective input $=$ A

(13) System dependent parameters

$$t_K = \frac{K\pi - \phi}{\omega_d}, t_r = \frac{\pi - \phi}{\omega_d}, t_{po(K)} = \frac{(2K-1)\pi}{\omega_d}, t_{pu(k)} = \frac{2K\pi}{\omega_d}$$

$$t_s = 4T = \frac{4}{\xi\omega_n}, T = \frac{1}{\xi\omega_n}, \omega_d = \omega_n\sqrt{1-\xi^2}, T_D = \frac{2\pi}{\xi_d}$$

Total no. of cycle before oscillation dies out $A = \frac{t_s}{T_D}$

$\%M_{p0(K)}$, $\%M_{pu(K)}$

(14) Input Independent Parameters

(i) $C(t)$

(ii) $C\left(t_{poK}\right)$

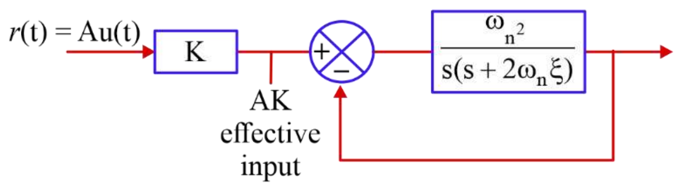
(iii) $C(t)\big|_{\max}$

(iv) Peak overshoot

(v) $C\left(t_{puK}\right)$

(vi) $C(t)_{\min}$

(vii) Peak undershoot



$$\frac{C(s)}{R(s)}=\frac{k\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}$$

$$C(t)\big|_{ss}=AK$$

- System dependent parameters $\rightarrow$ No charge
- Input dependent parameters $\boxed{\rightarrow A \Rightarrow AK}$

Note : If $r(t)\rightarrow Au(t-t_0)$ then all time formulas will be replaced by $\left(t-t_0\right)$

Step Response of undamped System

(1) $C(s)=\left[\dfrac{AK\omega_n^2}{s^2+\omega_n^2}\right]\dfrac{1}{s}\Rightarrow c(t)=Ak\left[1-\cos\omega_n\right]u(t)$

- From c(t) of underdamp
  - critically damp c(t)
  - undamp c(t)
  - over damp c(t) X not posible
- $\cos\phi=0$, $t_s=\dfrac{4}{\xi\omega_n}=\infty$
- $\omega_d=\omega_n$, $\%M_p=100\%$
- $t_r=\dfrac{\pi}{2\omega_n}$,
- $t_{po(K)}=\dfrac{\pi}{\omega_n}(2K-1)$
- $t_{pu}(k)=\dfrac{2k\pi}{\omega_n}$

Step Response of critically damped s/s

$$C(S)=\frac{AK\omega_n^2}{S(S+\omega_n)^2},\quad C(t)=AK[1-e^{-\omega_n t}(1+\omega_n t)]u(t)$$

- Output reaches to steady state without oscillation in short time.

Step response of overdamp s/s-

- $C(S)=\dfrac{AK\omega_n^2}{S(S^2+2\xi\omega_n S+\omega_n^2)}\xrightarrow{\xi>1}\dfrac{AK\omega_n^2}{S(S+p_1)(S+p_2)}\longrightarrow C(\infty)=AK$

$$c(t)=[a_0+a_1e^{-p_1t}+a_2e^{-p_2t}]u(t)$$

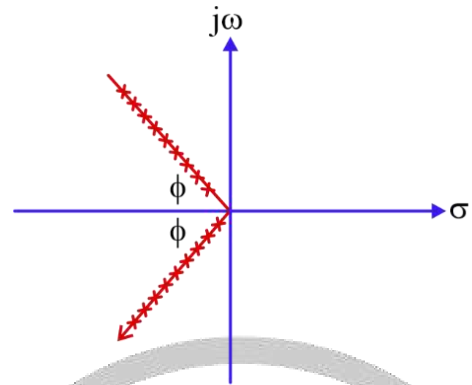
Rise time:

(1) 0 to 100% of s. s value $\rightarrow$ underdamp and undamp system

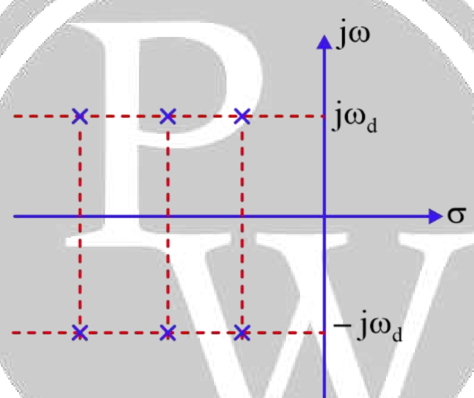
(2) 10 to 90% of s .s value $\rightarrow$ critical and overdamp system

Locus

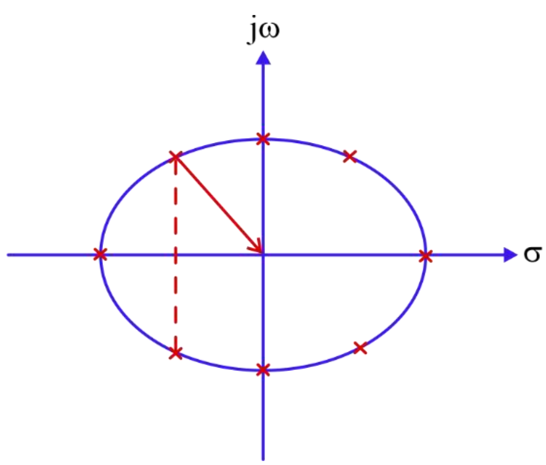
(1) Constant ξ



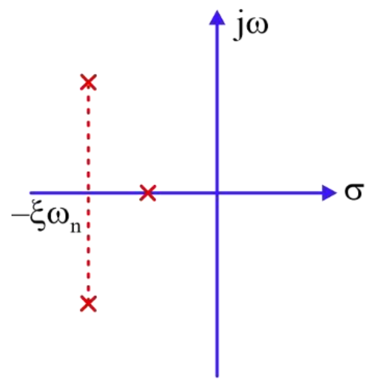
(2) Constant ω_d



(3) Constant ω_n



(4) Constant time constant (setting time) $T = \dfrac{1}{\xi\omega_n}$



Impulse Response of 2nd order s/s

(1) Under damped system $0<\xi<1$

$$C(t)=\frac{AK\omega_n}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin(\omega_d t)u(t)$$

$$t_r=\frac{\pi-\phi}{\omega_d}, t_p(\text{peak time})=\frac{n\pi}{\omega_d}, t_s=\frac{4}{\xi\omega_n}, \%M_p e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}\times 100}$$

(2) Undamp system $\xi=0$

$$C(t)=AK\omega_n\sin\omega_n t$$

(3) Critically damped System $\xi=1$

$$C(t)=AK\omega_n^2 te^{-\omega_n t}u(t)$$

(4) Overdamped System $1<\xi<\infty$

$$C(t)=a_0e^{-p_1t}+a_1e^{-p_2t}$$

Dominant Pole Concept

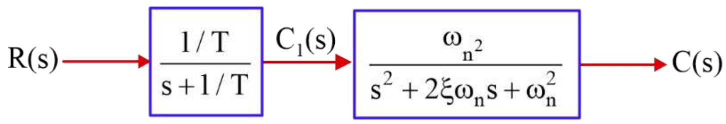
$$\frac{C(S)}{R(S)}=\frac{M}{(S+p_1)(S+p_2)(S+p_3)(S+p_4)}$$

- $p_1, p_2, p_3 p_4 \rightarrow$ Magnitude of real part of pole
- Smallest of $p_{1,} p_2, p_3, p_4 = p_1$
- (i) $\frac{p_2}{p_1}\geq 5 \; S+p_2 \xrightarrow{S=0} p_2$

  (ii) $\frac{p_3}{p_1}\geq 5, \; (S+p_3)\xrightarrow{S=0} p_3$

  (iii) $\frac{p_4}{p_1}\geq 5 \; (S+p_4)\xrightarrow{S=0} p_4$

Cascading of 2nd order under damped System



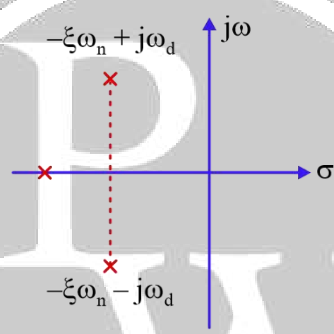
$$\boxed{\frac{C_1(s)}{R(s)}=\frac{1/T}{s+1/T}} \qquad \boxed{\frac{C(s)}{C_1(s)}=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}}_{\text{IInd order}}$$

S – IIst order S – II

$$\boxed{\frac{C(s)}{R(s)}=\frac{\frac{1}{T}\omega_n^2}{\left(s+\frac{1}{T}\right)\left(s^2+2\xi\omega_n s+\omega_n^2\right)}}\ \text{3rd order}$$

Case 1:

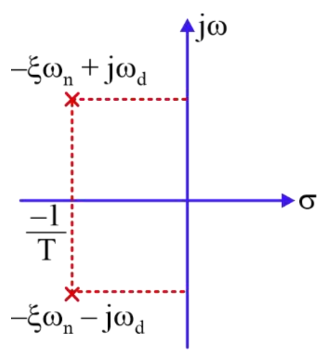
$$\frac{C(s)}{R(s)}=\frac{N(s)}{s^2+2\xi\omega_n s+\omega_n^2}$$



$N(s) = K\omega_n^2$ → $\xi\omega_n$ comparable
→ Formula Applicable

$N(s) = \text{Polynomial}$ → Comparision of ω, ξ allowed
→ Formula Allowed

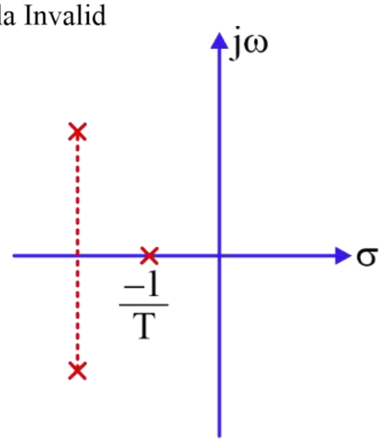
Case 2 :



$N(s)=k\omega_n^2 \rightarrow$ comparision of ξ,ω_n allowed

$$\frac{C(s)}{R(s)}=\frac{N(s)}{s^3+\alpha s^2+\beta s+T}$$

Case 3: N(s) = Polynomial
- Comparison allowed
- Formula Invalid



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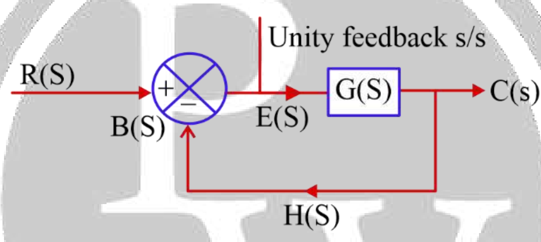
3 Steady State Error and Routh Stability

3.1. Error

- Error Signal
  - System Error
  - Amplitude Error
- Error
  - Unity f/b system
  - Non unity f/b system



$e(t) = r(t) - b(t)$ amplitude error

System Error

(1) Must tends to 0 as $t \to \infty$

(2) $e_{sys}(t) = r(t) - c(t)$ for $r(t) = Au(t)$

(3) $e_{sys}(t) = \int_{-\infty}^{t} r(t) - \int_{-\infty}^{t} c(t)dt$ for $r(t) = A\delta(t)$

(4) $e_{sys}(t) = \dfrac{dr(t)}{dt} - \dfrac{dc(t)}{dt}$ for $r(t) = At\ u(t)$

Amplitude Error

$$e(t) = r(t) - c(t) \to \text{output}$$

Effective input at summer

$r(t) = Au(t)$ $e(t) = Au - c(t)$

r(t) = Atu(t) e(t)= Atu(t) – c(t)

r(t) = $\dfrac{At^2u(+2)}{2}$ $e(t) = \dfrac{At^2}{2}u(t) - c(t)$

Calculation of amplitude error

(1) $e(t)$ or $E(s)$ at the output of summer.

(2) Steady state amplitude error $\lim\limits_{t\to\infty} e(t) = \lim\limits_{S\to 0} SE(S) \to$ will be finite only when all poles of SE(S) will be strictly on LHP.

$$e_{SS} = \lim_{S\to 0} \frac{SR(S)}{1+G(S)}$$

Steps :

(1) Identify feedback (unity)

(2) $ess = \lim\limits_{S\to 0} \dfrac{SR(S)}{1+G(S)}$

(3) Pole location of $\dfrac{SR(S)}{1+G(S)}$ is strictly on L.H.P then perform calculation.

Steady state Error for different inputs.

Input	Check	e_{ss}
Au(t)	Poles of $\dfrac{A}{1+G(S)}$	$e_{SS} = \dfrac{A}{1+K_p}$
Atu(t)	Poles of $\dfrac{A}{S(1+G(S))}$	$e_{SS} = \dfrac{A}{K_V}$
$\dfrac{At^2}{2}u(t)$	Poles of $\dfrac{A}{S^2(1+G(s))}$	$ess = \dfrac{A}{K_a}$

$Kp = \lim\limits_{S\to 0} G(s) =$ Positional Error constant /coefficient

$Kv = \lim\limits_{S\to 0} SG(S) =$ velocity Error constant / coefficient

$Ka = \lim\limits_{S\to 0} S^2 G(S)$ Acceleration Error constant / coefficient

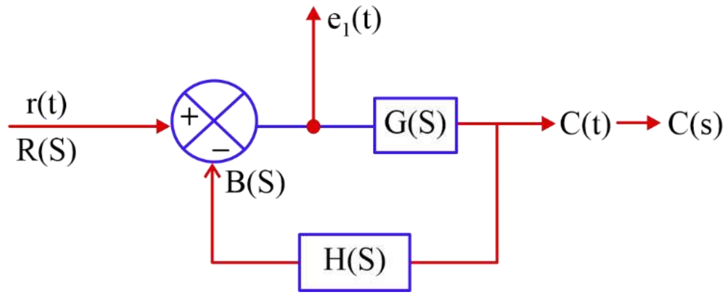
Effect of Type

Input	e_{SS}		
	T-0	T-1	T-2
Au(t)	Finite	0	0
Atu(t)	∞	Finite	0
$\dfrac{At^2}{2}u(t)$	∞	∞	Finite

(1) Finite steady error is independent of shift in input signal.

(2) Steady state error can be undefined or ∞ even if CLS is stable.

Error Analysis for Non unity f/b –

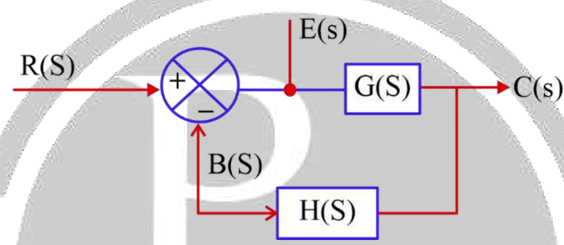


(1) $e_1(t) = r(t) - b(t) \rightarrow$ Can be shown in diagram

(2) $e_2(t) = r(t) - c(t) \rightarrow$ Can not be shown in diagram

(3) $e_3(t) =$ ref signal - $c(t)$

Ref signal = Value of c(t) due to which output of summer is 0.



Case : 1

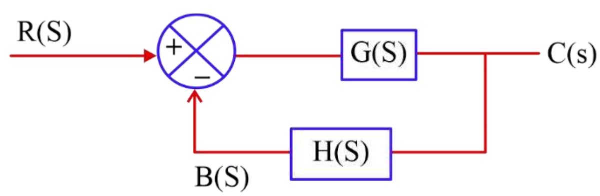
Error Signal $= e(t) = r(t) - b(t)$

$$\boxed{e_{ss} = \lim_{s\to 0} SE(S) = \lim_{s\to 0} \frac{SR(S)}{1+G(S)H(S)}}$$

Steps: (1) feedback – Non unity $\rightarrow$ error was shown in frequency .

(2) $e_{ss} = \lim_{S\to 0}\left[\frac{SR(S)}{1+G(S)H(S)}\right]$ all poles must be in L.H.P

Case: 2 $e(t) = r(t) - c(t)$



$$e_{ss} = \lim_{S\to 0} \frac{SR(S)[1+G(S)H(S) - G(S)]}{1+G(S)H(S)}$$

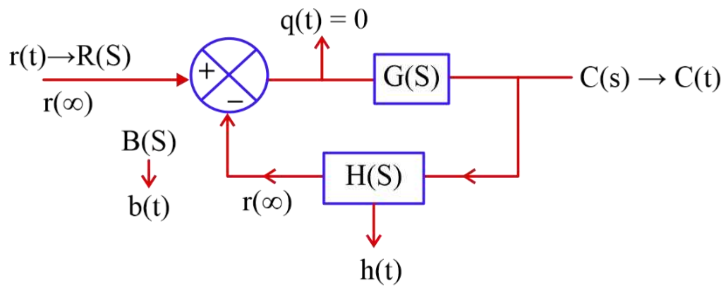
Steps :

(1) f/b $\rightarrow$Non unity $e(t) = r(t) - c(t)$

(2) $ess = \lim_{S\to 0} SE(S) = \lim_{S\to 0} S[R(S) - C(S)]$

(3) All poles of SE(S) must be in L.H.P

Case: 3



$$r(\infty)=H(0)C(\infty)$$

$$H(0)=K_H$$

$$\boxed{e_{ss}=r_f(\infty)-c(\infty)}$$

$r_f(t)=$ Value of C(t) due to which $q(t)=0$

$r_f(\infty)=$ Value of $C(\infty)$ due to which $q(\phi)=0$

$$e_{ss}=\lim_{S\to 0}\frac{SR(S)}{K_H}[1-K_H\ T(S)] \qquad K_H=H(S)|_{S=0}$$

Steps: (1) Non unity f/b, $e(t)=\text{ref signal}-c(t)$

(2) $e_{ss}=\lim_{s\to 0}\left\{\underbrace{\frac{SR(S)}{K_H}[1-K_H T(S)]}\right\}=\frac{r(\infty)}{K_H}-c(\infty)$

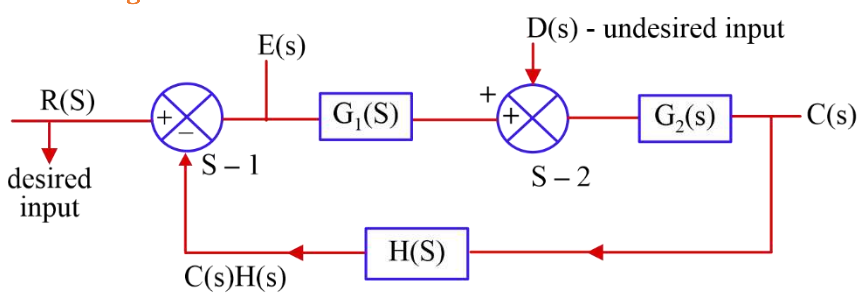
all poles must be in L.H.P

Note : $H(S)=S^N F(S)$

$$K_H=\lim_{S\to 0}F(S)$$

$$e_{ss}=\lim_{s\to 0}\frac{SR(S)}{K_H S^N}(1-K_H T(S)S^N)$$

Concept of Disturbance or Noise signal :



$$\boxed{C=\left[\frac{G_1G_2}{1+G_1G_2H}\right]R+\left[\frac{G_2}{1+G_1G_2H}\right]D}$$

Error signal at output of S_1, $E = R - CH$ S_1, $E = R - CH = \left[\dfrac{R}{1+G_1G_2H}\right] + \left[\dfrac{-D}{1+G_1G_2H}\right]G_2H$

Stead state error, $e_{ss} = \lim\limits_{s\to 0} SE(S)$

for addition at s-2 $\overset{(+)}{}$

$$e_{ss} = \lim_{s\to 0}\left\{\frac{SR(S)}{1+G_1(S)G_2(S)H(S)}\right\} + \lim_{s\to 0}\left\{\frac{-SD(S)}{1+G_1(S)G_2(S)H(S)}\right\}G_2(s)H(s)$$

due to input r(t)[desired input] at s-1

due to undesired input d(t) or Noise or diturbance at S-2

➢ We can reduce e_{ss} if G_1 should be as high as passible and G_2 as low as possible

Sensitivity of a parameter :

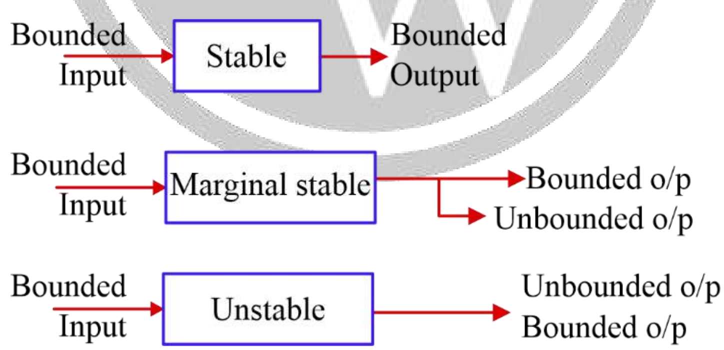
S_T^K = sensitivity of K w.r.t T

$$S_T^K = \frac{(\partial K / K)}{(\partial T / T)}$$

Stability of an LTI System :

(1) For an LTI S/S to be stable must follow BIBO criteria .

(2) Bounded (in amplitude) signal ; $|X(t)| \le M < \infty, M$:finite No. X(t) is bounded signal



(3) LTI system $\rightarrow h(t)$

Unit impulse response must be absolutely integrable

$\int_{-\infty}^{\infty} |h(t)|\, dt \rightarrow h(t)$

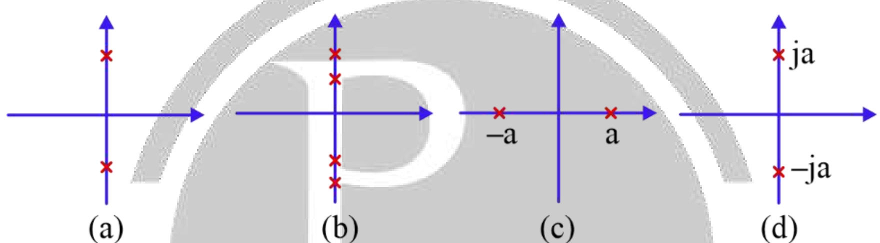
(4) For an LTI s/s to be stable ROC of H(s) must include $j\omega$ axis [ROC never include poles]

Conclusion :

Causal LTI system can be stable, it all poles of H(s) must be strictly on L. H. P

Location of poles in H(s)	Stability
(1) Repeated or Non – repeated poles on L.H.P	Stable
(2) Single pole at origin	Marginal stable
(3) Non repeated poles on $j\omega$ axis	Marginal stable
(4) Multiple Non repeated poles on $j\omega$ axis	Marginal stable
(5) Multiple poles on $j\omega$ axis	Unstable
(6) Repeated poles on $j\omega$ axis	Unstable
(7) Poles on R.H.P	Unstable
(8) No poles	Stable

Imaginary vs image Location :



Imaginary location (a), (b) image location (c), (d)

1st Order Polynomial

If all coefficient are same then roots will be in L.H. P

D(S) = S +2 , D(S) $= -S - 3$

L. H. P L. H. P

2nd Order

All coefficient having same sign and No coefficient is zero, then all roots will be in LHP

D(S) = $S^2 + 2S + 2$ D(S) $= -S^2 - S - 1$

L.H.P L. H. P

3rd Order

No missing power and all coefficient has same sign then –

(A) All real roots in L. H. P

(B) No comment on complex roots

R – H Table

$$T(s) = H(S) = \frac{N(s)}{D(s)}$$ Root of D(s) = Poles of T(s)

$$D(s) = a_0 s^6 + a_1 a^5 + a_2 s^4 + a_3 s^3 + a_4 s^2 + a_5 s + a_6$$

s^6	a_0	a_2	a_4	a_6	$b_2 = \dfrac{a_1a_4 - a_0a_5}{a_1}$
s^5	a_1	a_3	a_5	0	
s^4	$b_1 = \dfrac{a_1a_2 - a_0a_3}{a_1}$	b_2	0	0	$c_1 = \dfrac{b_1a_3 - b_2a_1}{b_1}$
s^3	c_1	c_2	0	0	$c_1 = \dfrac{b_1a_3 + b_2a_1}{b_1}$
s^2	d_1	a_6	0	0	$c_2 = b_1a_5 - a_6a_1 \vert c_1$
s^1	c_1	0	0	0	$d_1 = c_1b_2 - b_1c_2 \vert c_1$
s^0	a_6	0	0	0	$e_1 = \dfrac{d_1c_2 - c_1a_6}{d_1}$

Key Point:

(1) If any now of RH table multiply or divided by + constant result remains same

(2) 1st column elements have same sign $\rightarrow$ No roots in

(3) Any row becomes zero then roots will be at image location

(4) If all elements in 1st column $\rightarrow$ same sign + No row becomes zero

Then all roots in LHP

(5) The no of sign changes in 1st column = No of roots lying in RHP

(6) If any power of s is missing then

(i) 1 or more than 1 root may exist in RHP

$D(s) = s^2 - 1$

Roots $= s = \pm 1$

(ii) Non repeated roots on $j\omega$ axis may exist

$$D(s) = s^2 + 1 \Rightarrow s = \pm j$$

(iii) Repeated roots may exist on $j\omega$ axis

(iv) There may be complex roots

(v) If only even power of s exist then root will be at image location.

(vi) If only odd power of s exist then few roots will be at origin and remaining roots at image location

(vii) In RH table $\rightarrow$ odd ROZ $\rightarrow$ some roots will be at image location

$\downarrow$

Sign of elements in 1st $\rightarrow$ Image location will be imaginary axis

Column is same

(viii) In RH table $\rightarrow$ No odd ROZ $\rightarrow$ No root at image location

$\downarrow$

No root at imaginary location

3.2. Root Calculation when Odd Row is never zero

For nth order polynomial

(1) Form RH Table

(2) Observe the first column

(3) 1st column elements
- → Same sign → all roots in LHP
- → K sign change → K roots in RHP (n - K) roots in LHP

Note :

$$a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

I. P

E. P

I.P – Inner product

E. P – External Product

All coefficient should have same sign

(1) IP > EP → all roots in LHP

(2) IP = EP
- → 1 root LHP
- → 2 root on image location

(3) $IP = EP$
- ↗ 2 root RHP
- ↘ 1 root LHP

Special Cases

(1) 1st element of row = 0 other elements non zero

Ex. $3s^4 + s^3 + 3s^2 + s + 2$

$$\begin{array}{llll} s^4 & 3 & 3 & 2 \\ s^3 & 1 & 1 & 0 \\ s^2 & 0 \to d & 2 & 0 \\ s^1 & \dfrac{d-2}{d} & 0 & 0 \\ s^0 & 0 & & \end{array}$$

$d = +ve$ quantity

$$\frac{d-2}{d} = 1 - \frac{2}{d} = 1 - \frac{2}{0}$$
$$= -ve$$

No odd row is zero, two sign changes

$2 \to R.H.P$

$2 \to L.H.P$

(2) If all element of odd row are zero

(i) Few roots at image location

(ii) Form auxiliary characteristic equation has been formed from the row just above the odd row

$A(s) = 0$

(3) Then

$$\frac{d}{ds} A(s) = B(s)$$

(4) Replace odd row of zeros with B(s) coefficients

(5) Roots of A(s) = Roots of D(s), A(s) roots will be at image location A(s) is always a factor of D(s)

$$\frac{D(s)}{A(s)} = P(s) \rightarrow \text{Remaining roots}$$

(6) Both location and exact value of roots can be calculated

If odd row becomes zero once:

(1) Roots of A(s) will be to image location and non-repeated in nature

(2) If A(s) is of 2nd order and roots of A(s) is on $j\omega$ axis then roots will represent, undamp natural frequency of 2nd order system

(3) $\frac{D(s)}{A(s)} = P(s) = 0 \rightarrow$ Remaining roots

3.2.1. Odd Row becomes Zero Twice

(1) 2 Auxillary equation

A_1 (s) : Higher order AE

A_2 (s) : Lower order AE

(2) Roots of $A_2(s)$ will automatically be covered by $A_1(s)$

(3) Roots of $A_1(s)$ will be image location

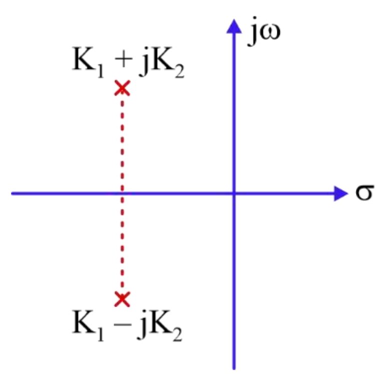
(4) $\frac{D(s)}{A_1(s)} = P(s) \rightarrow \text{Remaining roots}$

$$\begin{matrix}\text{No. of roots of } D(s) \text{ on} \\ \text{Imaginary axis}\end{matrix} = \begin{pmatrix}\text{Highest} \\ \text{order AE}\end{pmatrix} - 2 \times \begin{pmatrix}\text{No. of sign changes} \\ \text{below highest order AE}\end{pmatrix} - C$$

No. of roots of D(s) is RHP = No of sign charge in 1st column ..... (2)

No. of roots of D(s) in LHP = order of D(s) – (i) – (ii)

Conditionally Stable :



$K_1 \rightarrow \text{variable}, K_2 \rightarrow \text{Constent}$

- Stability depends on K_1 or conditionally stable
- Use wavy curve

Marginally Stable :

S – 1 form RH table

S – 2 All sign should be same

S – 3 Odd row become zero once. Non repeated roots on imaginary axis and system becomes marginally stables

Oscillating system with undamp natural frequency

S – 1 form RH table

S -2 All elements should be +ve

S – 3 Odd row zero once Auxiliary C.E. is of 2nd order

Note :

- If polynomial has only even power the roots are symmetrical about origin or image location.
- Random power of *s* missing then at least 1 root in RHP

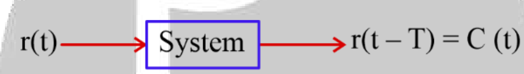
Limitation of Routh:

(1) Applicable to finite order polynomial only

(2) $D(s)=e^{s}, \tan s, \cos s \rightarrow$ RH invalid

(3) Coefficient of polynomial showed be constant

3.3. Transportation Lag System



T : delay time or log time

Common Mistake

$$T(s)=e^{-sT}$$

$$e^{-x}=1-x \text{(When } x \text{ is very small)}$$

$$D(s)=1+Ke^{-ST}=0$$

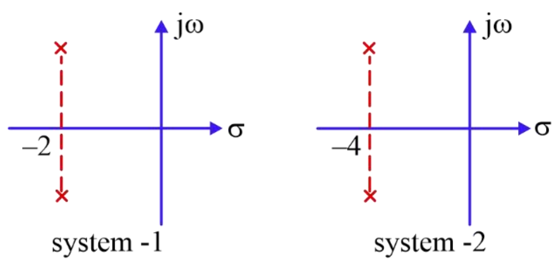
Routh invalid, use basic root calculation approach

- Both polynomial and exponential present thin R.H. applicable

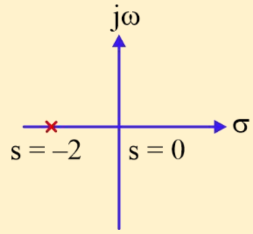
$$s^2+s+Ke^{-sT}=0$$
$$s^2+s+K(1-sT)=0$$

Shifting Of origin

- System 2 is more stable



Note : D(s) = s + 2



shifting the origin S = 0 to S = - 1

$$S \begin{array}{l} \nearrow Z+1 \xrightarrow[Z=0]{} s=1 \\ \searrow Z-1 \xrightarrow[Z=0]{} s=-1 \end{array}$$

Put S = Z – 1

D(s) = Z + 1

Note : $D(s) = s^2 + s + 1$

(1) How many roots are more negative than $\sigma = 0$ $\Rightarrow$ Roots in LHP R – H criteria in D(s)

(2) How many roots are more +ve than $\sigma = 0 \Rightarrow$ Roots in RHP R – H criteria in D(s)

(3) How many roots have $\sigma = 0 \Rightarrow$ Roots on $j\omega$ axis R – H criteria on D(s)

(4) How many roots are more negative than $\sigma = -1$

Put S = Z – 1

D(z) $\rightarrow$ R. H criteria

No of roots in RHP in z plane $\Rightarrow$ No of roots having $\sigma > -1$ in s – plane

No of roots in LHP in z plane $\Rightarrow$No of roots having $\sigma < -1$ in s – plane

❑❑❑

4 Root Locus

4.1. Root Locus

Locus of roots of characteristic equation or Locus of zeros of characteristic equation or Locus of poles of closed loop system .

D.R.L $\Rightarrow$ Direct root locus

C.R.L $\Rightarrow$ Complementary root Locus

4.1.1. Angle and Magnitude Criteria

Case 1: For D.R.L , C.E , KF(S) = 1 , K= +ve constant

$$|KF(S)|=1$$

$$< KF(S)=(2n+1)\pi$$

Case 2: For C.R.L KF(S) =1

$$|KF(S)|=1$$

$$\angle KF(S)=2n\pi$$

G(S)H(S)	K	Feedback	Locus
KF(S)	$0<K<\infty$	-Ve	D.R.L
-KF(S)	$0<K<\infty$	-Ve	C.R.L
KF(S)	$0<K<\infty$	+Ve	C.R.L
-KF(S)	$0<K<\infty$	+Ve	D.R.L

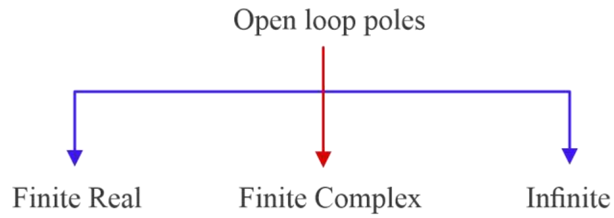
Rules to plot D.R.L

Rule 1 : To plot D.R.L all the coefficient of S should be + ve.

Rule 2 : Origination of D.R.L

(1) DRL originate from open loop poles

(2)



G (S) H (S) = $\dfrac{K}{S+2}$ G (S) H (S) = $\dfrac{K}{S^2+S+1}$ G (S) H (S) = KS^2

(3) Open loop pole : finite Real – DRL branch will originate from the open loop pole and move on the real axis in the section D.R.L present .

(4) Open loop pole : finite complex – D.R.L branch will originate form open loop pole in the directions of angle of departure

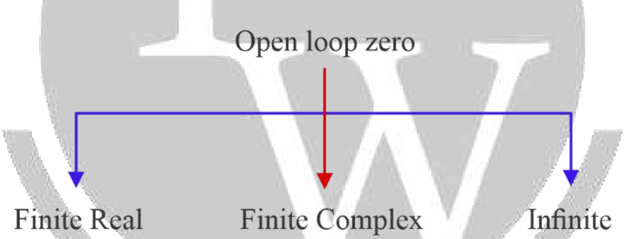
(5) open loop pole present at ∞ - D.R.L branch will originate from ∞ by gazing the asymptotic lines given by angle of asymptotes.

(6) at open loop pole value of K = 0

Rule 3 : Termination of D.R.L

(1) Terminate at open loop zeros .

(2)



(3) open loop zero : finite Real – Terminate at open loop zero by moving on the real axis in the section D.R.L exist .

(4) open loop zero finite complex – DRL branch will terminate at open loop zero in the direction given by angle of arrival.

(5) open loop zero at ∞ - DRL branch will terminate at open loop zero by gazing angle of asymptotes.

(6) Value of K at open loop zero is

$0<K<\infty \rightarrow K=\infty$

$-\infty<K<0 \rightarrow K=-\infty$

Rule 4 : Existence of D.R.L on Real axis.

- Segment of real axis where D.R.L exist
- Segment where DRL exist must hare " ODD no of open loop poles zeros towards its night.

Rule 5 : Identification of a point $S=S_o$ is

(i) Part of root locus

(ii) Poles of C.L.S

(iii) Roots of C.E

(iv) zeros of C.E

Case 1: $S = S_o$ is real

Method 1 check if $S = S_o$ is part of D.R.L

Method 2 Angle sub started by all open loop poles and zeros towards desired point must be odd Multiple of π.

Method 3 (i) put $S = S_0$ in CE , calculate K then if K is real and $+Ve \Rightarrow S = S_0$ is part of D.R.L

K is real and +ve $\Rightarrow S = S_0$ is part of D.R.L

K otherwise $\Rightarrow S = S_0$ is not part of D.R.L.

Method 4 verify magnitude and angle criteria at $S = S_0$

D.R.L $\angle KF(S) = (2n+1)\pi$

$|KF(S)| = 1$

Case 2 : $S = S_0$ Complex.

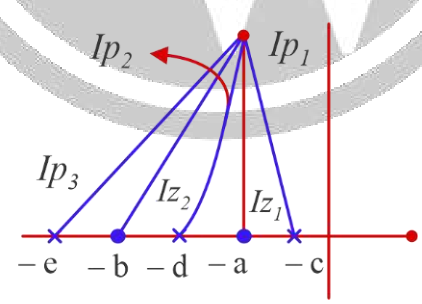
Method 1 fails

Method 2,3,4 are applicable

Rule 6 : Calculation of K at $S = S_0$ (if $S = S_0$ is part of RL)

$$G(S)H(S) = \frac{K(S+a)(S+b)}{(S+c)(S+d)(S+e)} \qquad \begin{matrix} 0 < K < \infty \\ -ve\ f/b \end{matrix}$$

$$\boxed{K = \frac{lp_1 lp_2 lp_3}{lz_1, lz_2}}$$



Rule 7: Angle of asymptotes

Only for those branches which ether originate or terminate at ∞.

Formula :

(1) $P - Z \neq 0,\ n = 0,1,2,3,--P-Z-1$ P = finite open loop poles

(2)

If $P > Z$	$Z > P$
$\theta_n = \dfrac{(2n+1)180^0}{(p-z)}$	$\theta_n = \dfrac{(2n+1)180^0}{(z-p)}$

Z = finite open loop zeros

Rules 8 : Centroid

(i) Calculated When $P \pm Z$

(ii) Needed only when A.O.A are calculated.

(iii) The A.O.A drown from a point on real axis known as centroid, (originating point of asymptotes)

(iv) All the asymptotic lines meets at common point on real axis know as centroid.

(v) Always present on real axis .

(vi) May or may not be part of R.C

(vii) Value of centroid $\rightarrow \sigma = 0, +ve, -ve$

(viii) $\boxed{\text{formula} \quad \sigma = \frac{\sum p - \sum z}{p - z}}$ σ :Real numbers

Rule 9: Break point

(1) Where 2 or more then 2 poles of C.L.S coincides simultaneously . If is part of R.L

Types :

(1) Break away point

(1) 2 or more then 2 poles of C.L.S coincides .

(2) After B.A.P R.L Breaks into some parts and it can not remains on real axis. It moves into different parts in complex conjugate location

(3) BAP means shifting of R.L from real axis into complex conjugate Location.

(4) At BAP k achieves max value for which root remains on real axis if $K \uparrow$ then R.L moves on complex conjugate location

$0 \le K \le K_{BAP}$: Root locus is on real axis .

$K > K_{BAP}$ Root locus on complex conjugate location.

(2) Break In point

(1) 2 or more then 2 poles of C.L.S coincides .

(2) After the BIP R.L Breaks into some parts and if can not remain on complex conjugate location . If move on into different parts on real axis .

(3) BIP means shifting of root locus from complex conjugate location to real axis .

(4) At BIP K achieves min value for which root locus is on real axis . If $K \uparrow$ if remains on real axis $0 < K \le K_{BIP} \rightarrow$ complex conjugate location

$K \ge K_{BIP} \rightarrow$ Real axis

S.1 from the CE $1 \pm G(S)H(S) = 0$

$1 \pm KF(S) = 0$

$$K = \pm \frac{1}{F(S)} = Q(S)$$

$$\frac{dk}{ds} = 0, \text{ Possible Break point}$$

S.2 By $\frac{dk}{ds} = 0,$

$S = S_0$ → S_0 is part of DRC → Invalid Break Point

$S = S_0$ → S_0 is not part of DRC → Invalid

S.3 If so is valid Break point .

S_0 : Real $\left(\frac{d^2K}{ds^2}\right)_{S=S_0} < 0$

$K = Q(S) \rightarrow \text{maxima at } S = S_0 \rightarrow B.A.P$

$\left(\frac{d^2k}{ds^2}\right)_{S=S_0} > 0$

K = Q(S) $\rightarrow$Minima at $S = S_0 \rightarrow B.I.N$

4.2. Properties of Break points

(1) At Break point ,RL branches from an angle of $\pm\frac{180^0}{n}$ with real axis where n is number of closed loop poles arriving or departing from signal breakpoint on the real axis .

(2) If 2 adjacent open loop poles on real axis and segment between them is part of DRL then there will be at least one BAP between then .

(3) For 2 adjacent open zeros of OLS $\rightarrow$At least 1 BIP exist

(4) If 2 or more then 2 poles of OLS coincide at K = 0 this itself represent BAP .

for K = 0 , OLP = CLP

(5) If 2 or more then 2 zeros coincide at $K = \infty$ then this itself becomes Break point for

$K = \infty, OLZ = CLP$

Rule 10 : Angle of departure

- For complex OLP , given originating direction to Branch of DRL

$\theta_d = 180^0 - [\phi_p - \phi_z]$

$\phi_p =$ Angle sustained by remaining OLP towards desired pole

$\phi_z =$ Angle sustained by remaining all OLZ towards desired pole

Rule 11 : Angle of arrival

- For complex OLZ , gives terminating direction .

$\theta_a = 180^0 - [\phi_z - \phi_p]$

$\phi_z =$ Angle sustained by remaining OLZ toward desired OLZ

$d_p =$ Angle sustained by remaining all OLP toward desired OLZ .

Rule 12 : Intersection with jω axis

- Identification of CLP on $j\omega$ axis .

S-1 From C.E and R.H table .

S-2 Odd Row →
- Not Zero
- Zero but K = -ve
- Zero and K = + ve But roots of AE not on $j\omega$ axis

} Root Locus does not intersect $j\omega$ axis

Odd Row $\rightarrow$ Zero and K = +ve and Root of AE on $j\omega$ axis → RL intersect $j\omega$ axis

Rules to PLOT a C.R.L

- Rule 1 – Same as DRL
- Rule 2 – Same
- Rule 3 – Same
- Rule 4 – Replace odd with Even
- Rule 5 – Identification of $S = S_0$ on CRL

Case 1 : $S = S_0$ is real

- Method-1 So is part of CRL
- Method -2 Angle by all OLP and zero are even multiple of $\pi = 2n\pi$
- Method -3 $1 \pm G(S)H(S) = 0 \xrightarrow{S=S_o} K = \text{Real and} + ve$
- Method -4 $|KF(S)|_{S=S_0} = 1$

  $\angle KF(S)|_{S=S_0} = 2\pi$

Case 2 : $S = S_0$ Complex

M-1 Fall

M-2 Angle by all OLP and OLZ should be $2n\pi$

M-3 $C \cdot E \xrightarrow{S=S_0} K$ real and $+ve$

M-4 $|KF(S)| = 1,\ \angle KF(S)|_{S=S_0} = 2n\pi$

Rule 6 : **Angle of asymptotes**

$$P > Z \qquad\qquad P < Z$$

$$\theta_n = \frac{2n\pi}{P-Z} \qquad \theta_n = \frac{2n\pi}{P-Z}$$

Rule 7 : Same

Rule 8 : Calculate K at $S = S_0$, if $S = S_0$, is part of C.R.L

Rule 9 : Breakpoint

S.1 – Same as DRL

S.2 – Validate S = S_0 by following C.R.L criteria

Rule 10: Angle of departure

D.R.L	C.R.L
$\theta_d = 180^0 - (\phi_p - \phi_z)$	$\theta_d = 0^0 - (\phi_p - \phi_z)$

Rule 11: Angle of arrival.

$\theta_a = 0^0 - (\phi_z - \phi_p)$

Rule 12: Same

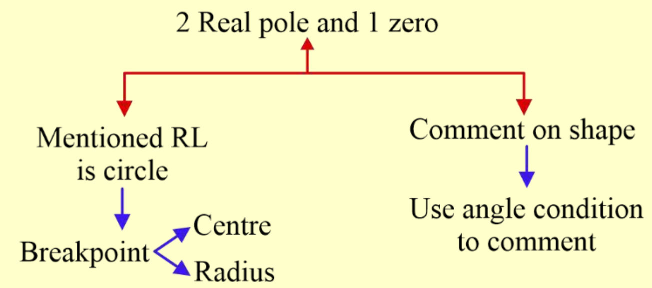
Note – RL always symmetrical about real axis

Few Important Result

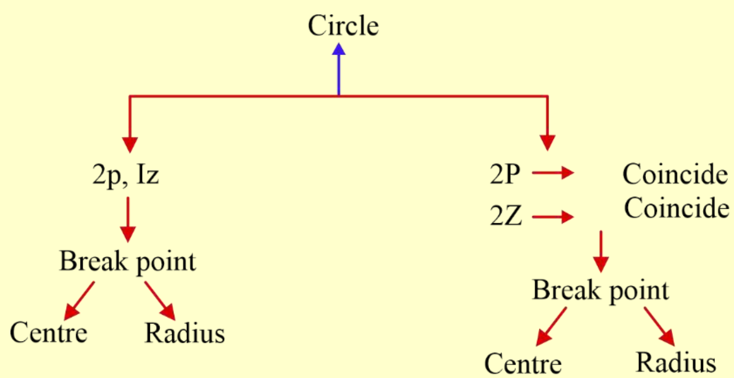
(1) $G(S)H(S) = \frac{K(s+b)}{(s+a)}$ Breakpoint $= -b \pm \sqrt{b^2 - ab}$ Radius of circle = $\sqrt{b^2 - ab}$ Centre = (-b, 0)	(2) $G(S)H(S) = \frac{-K(s-b)}{S(s+a)}$ Breakpoint $\Rightarrow s = b \pm \sqrt{b^2 + ab}$ Centre = (b, 0) Radius $= \sqrt{b^2 + ab}$
(3) $G(S)H(S) = \frac{KS}{(S-a)(S-b)}$ Breakpoint $\Rightarrow s = \pm\sqrt{ab}$ Centre = (0 , 0) Radius $= \sqrt{ab}$	(4) (4) $G(S)H(S) = \frac{-KS}{(S+a)(S+b)}$ Breakpoint $\Rightarrow s = \pm\sqrt{ab}$ Centre = (0, 0) Radius = $\sqrt{ab}$
(5) $G(S)H(S) = \frac{K(S+a^2)}{(S+b)^2}$ Centre $= \left[-\left(\frac{a+b}{2}\right), 0\right]$ Radius $= \frac{1}{2}\lvert a-b \rvert$ Breakpoint , $s = -a, -b$	

Note :

(1)



(2)



Min phase System : All polls and zeros must be in L.H.P

Non Minimum phase system : Which are not minimum phase

5 FREQUENCY RESPONSE ANALYSIS

5.1. Introduction

Test inputs : Sinusoidal input

$$x(t) = A\cos\omega_o t \rightarrow \text{ const frequency} = \omega_o \qquad S = j\omega$$

$$A\cos\omega_o t \longrightarrow \boxed{H(S)} \longrightarrow y(t) = A|H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_o))$$

$$A\sin\,\omega_o t \longrightarrow \boxed{H(S)} \longrightarrow y(t) = A|H(j\omega_o)|\sin(\omega_o t + \angle H(j\omega_o))$$

- Only $j\omega$ axis of S domain needed .
- Steady state output when a sinusoidal signal is applied –

  $\{y(t)\}_{SS} = \lim\limits_{t\to\infty}[a_0 e^{-j\omega_0 t} + a_1 e^{j\omega_o t}]$

Few observations

$$A\cos(\omega_o t + \phi) \longrightarrow \boxed{H(S)} \longrightarrow y(t)$$

$$\boxed{y(t) = A|H(j\omega_o)|\cos(\omega_0 t + \phi + \angle H(j\omega_o))} \rightarrow [y(t)]_{SS}$$

$$\downarrow$$

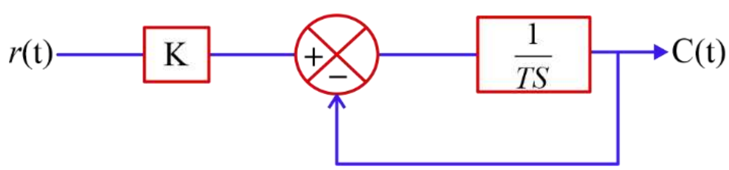
Replace With sin for sin i/p .

$H(j\omega) \rightarrow$ Frequency response of an LTI S/S

$|H(j\omega)| \rightarrow$ Magnitude

$\angle H(j\omega) \rightarrow$ Phase Response

5.1.1. Frequency domain analysis of 1st order



$$T(S) = \frac{K/T}{S + 1/T}$$

$$T(j\omega) = \frac{K/T}{j\omega + 1/T}$$

$$|T(j\omega)| = \frac{K/T}{\sqrt{\omega^2 + 1/T^2}}, \angle T(j\omega) = -\tan^{-1}\omega T$$

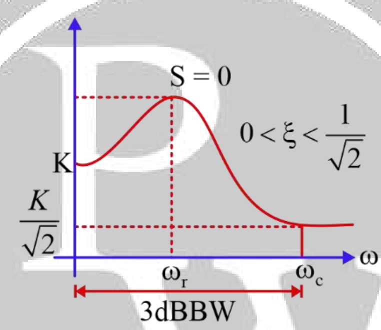
$$3.dB\ \ BW - \omega_c = \frac{|T(j_0)|}{\sqrt{2}} \text{rad/sec}$$

5.1.2. Frequency domain analysis of 2nd Order System

$$T(S) = \frac{K\omega_n^2}{s2 + 2\xi\omega_n s + \omega_n^2} \quad 0 \le \xi \le 1 \text{ stability can be decided.}$$

$$|T(j\omega)| = \frac{K}{\sqrt{\left(i - \left(\frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2\frac{\omega^2}{\omega_n^2}\right)}}, \angle T(j\omega) = -\tan^{-1}\left\{\frac{2\xi\omega/\omega_n}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)}\right\}$$

Plot 1



For Resonant frequency

$$\frac{d}{d\omega}|T(j\omega)| = 0$$

$\omega_r = \omega_n\sqrt{1 - 2\varepsilon^2}$

$\omega_n \to$ undamped Natural frequency

$\omega_r \to$ Resonant frequency

$\omega_d \to$ Damped frequency

Resonant Peak

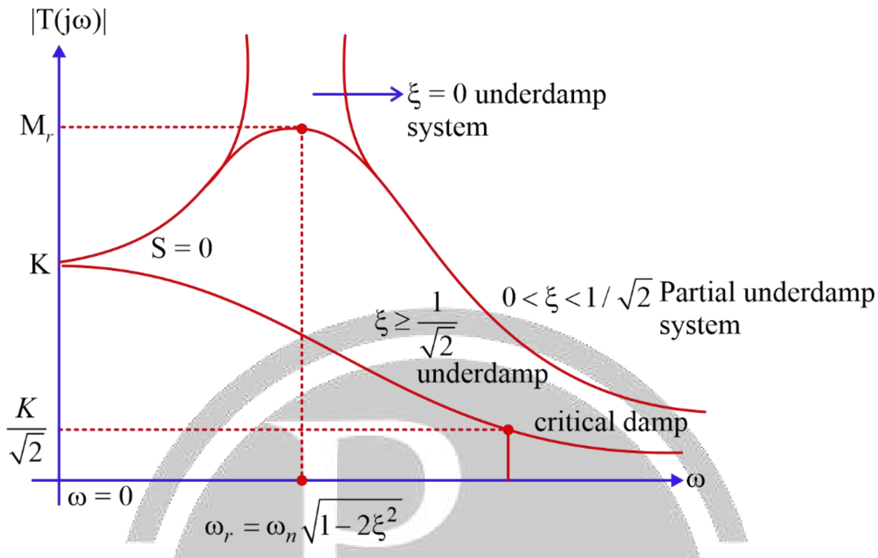
$$\boxed{M_r = \frac{K}{2\xi\sqrt{1-\xi^2}}}$$

ω_r is real only when $\xi < 1/\sqrt{2}$

for $\xi \ge \frac{1}{2}, \omega_r$ does not exist

$$\omega_r = \begin{cases} \omega_n : \xi = 0 \\ \omega_n\sqrt{1 - 2\xi^2} : 0 < \xi < 1/\sqrt{2} \\ 0 \quad\quad : \xi \ge 1/\sqrt{2} \end{cases}$$

$$M_r = \begin{cases} \infty & : \quad \xi = 0 \\ \dfrac{K}{2\xi\sqrt{1-\xi^2}} & : \quad 0 < \xi < 1/\sqrt{2} \\ K & : \quad \xi \geq \dfrac{1}{\sqrt{2}} \end{cases}$$



$\rightarrow \omega_r \downarrow$ as $\xi \uparrow$ if ω_n constant

$\rightarrow \omega_r \alpha\, \omega_n$ if ξ is constant

$\rightarrow M_r \downarrow as\ \xi \uparrow (0 < \xi < 1/\sqrt{2})$

$$\omega_c = \omega_n \sqrt{(1-2\xi^2) + \sqrt{(1-2\xi^2)^2 + 1}}$$

$$\boxed{\omega_c = \omega_n \sqrt{(1-2\xi^2) + \sqrt{4\xi^4 - 4\xi^2 + 2}}}$$

$\rightarrow \omega_c \downarrow$ as $\xi \uparrow$, ω_n constant

$\rightarrow \omega_c\ \alpha\ \omega_n$ When ξ *is* constant

Bode Plot :

$$\frac{\text{Exact frequency Analysis of a system}}{T(S)} :$$

S-1 Put $S = j\omega \quad 0^+ < \omega < +\infty$

S-2 $T(j\omega) = |T(j\omega)| e^{j\angle T(j\omega)}$

S-3 Plot $|T(j\omega)|\ VS\ \omega \rightarrow$ Exact Magnitude plot

$\angle T(j\omega)\ VS\ \omega \rightarrow$ Exact phase plot

➢ Exact plots are non linear in shape ; drawn on normal graphs .
Stability of T(S) $\rightarrow$ can be defined by plotting Bode plot of OLTF G(S)H(S).

Bode Plot:

Let OLTF is G(S)H(S)

1. $S = j\omega \;\; G(j\omega)H(j\omega) = T(j\omega)$

2. $|G(j\omega)H(j\omega)| = |T(j\omega)|$

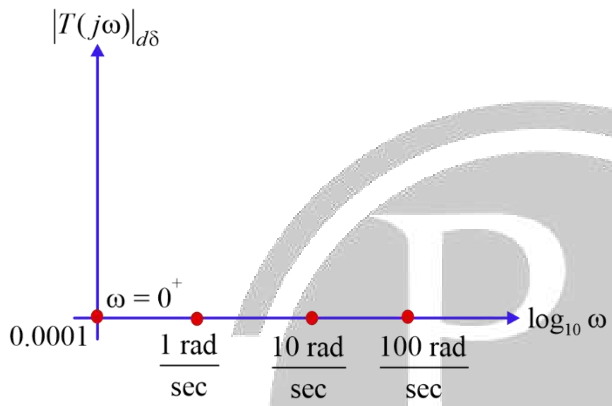
   $20\log_{10}|G(j\omega)H(j\omega)| = 20\log_{10}|T(j\omega)| = |G(j\omega)H(j\omega|_{dB}$

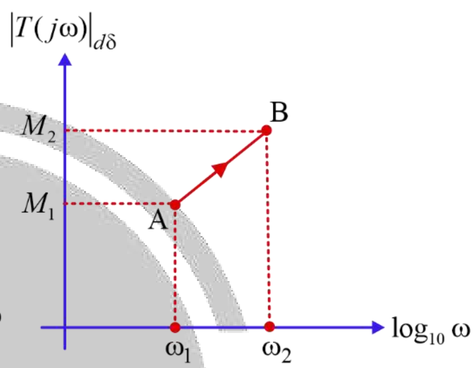
   Plot $|G(j\omega)H(j\omega)|$ vs $\log_{10}\omega \rightarrow$ Should be linear

3. $\angle G(j\omega)H(j\omega) = \angle T(j\omega)^0$

   Plot : $\angle G(j\omega)H(j\omega)$ vs $\log_{10}\omega \rightarrow$ This need not to be linear .

4.

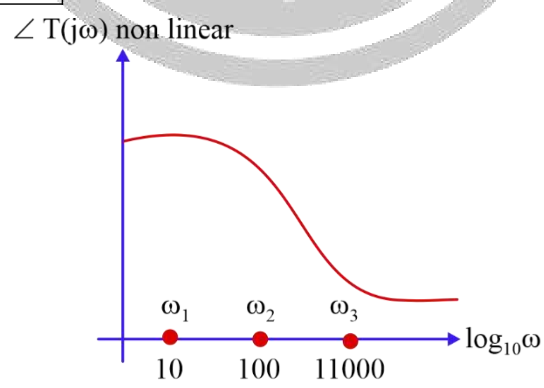




$A(\log_{10}\omega_1, M_1 dB)$ $\qquad$ $B(\log_{10}\omega_2, M_2 dB)$

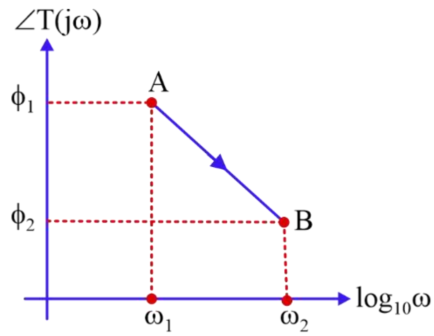
$|T(j\omega)|\,dB$ VS $\log_{10}\omega \rightarrow$ Make sure it is linear.

$$\boxed{\text{Slope} = \frac{(M_2 - M_1)}{\left(\log_{10}\dfrac{\omega_2}{\omega_1}\right)} = \frac{dB}{\text{decade}} = \frac{dB}{\text{Octave}}}$$



$\angle T(j\omega)$ ↗ linear ↘ Non linear

(7)



$$S = \frac{\phi_2 - \phi_1}{\log_{10}\left(\frac{\omega_2}{\omega_1}\right)} \ \frac{\text{degree}}{\text{decode}} \ or \ \frac{\text{degree}}{\text{octave}}$$

$$+\frac{20dB}{\text{decode}} = \frac{+6dB}{\text{octave}}$$

Exact Plot -

(1) Low frequency Range $\omega << \omega_c : |T(j\omega)|_{dB} = 20\log_{10} K$

Summary Table for $T(S) = KS^p$

$G(S)H(S)$	Initial slope	0 dB axis Int.	Slope at $\omega \to \infty$	PHASE
K	0 dB / decode	$\times$	0 dB/decode	0^0
Ks	+20 dB/dec	$\omega = \frac{1}{K}$	+20 dB/dec	$+90^0$
Ks^2	+40 dB/dec	$\omega = \frac{1}{(K)^{1/2}}$	+40 dB/dec	$+180^0$
:	;	;	;	:
KS^p	+20p dB/dec	$\omega = \frac{1}{(K)^{1/p}}$	+20 dB/dec	$+90^0\ p$

Summary table for $G(S)H(S) = K/S^p$

G(S)H(S)	Initials lobe	0dB axis Int .	Final slope $\omega \to \infty$	Phase
$\frac{K}{S}$	-20dB / dec	ω=K	-20dB / dec	-90^0
$\frac{K}{S^2}$	-40dB / dec	$\omega = (K)^{\frac{1}{2}}$	-40dB / dec	-180^0
$\frac{K}{S^3}$	-60dB/dec	$\omega = (K)^{\frac{1}{3}}$	-60dB/dec	-270^0
:	;	:	:	;
$\frac{K}{S^p}$	-20pd B/dec	$\omega = (K)^{\frac{1}{p}}$	-20pdB/dec	$-90\,p^0$

Important Observation :

Slop of Initial line	Initial Phase	Type
+20 p dB /dec	$+90P^0$	"0"
+0dB/decode	0^0	"0"
-20dB / decade	-90^0	"1"
-40dB/decade	-180^0	"2"
-20p dB/decade	$-90^0\,p$	"p"

Steady state error from Bode plot –

Initial Line slope	Information	e_{ss}
0dB / decade	$Amp = 20\log_{10} K_p$	$\dfrac{A}{1+K_p}$
-20dB/decade	0dB axis Intersection $= K_v$	A/K_v
-40dB/decade	0 dB axis Intersection $= \sqrt{K_a}$	A/K_a

Exact Plot

(1) Low frequency Range $\omega << \omega_c : |T(j\omega)|_{dB} = 20\log_{10} K$

(2) Mid frequency Range $\omega = \omega_c : |T(j\omega)|_{dB} = 20\log_{10} K + 3dB$

(3) High frequency Range $\omega >> \omega_c : |T(j\omega)|_{dB} = 20\log_{10} \dfrac{\omega}{\omega_c} + 20\log_{10} K$

G(S)H(S)	Initial Line with slope	Change in slope at $\omega = \omega_c$	error in mag. at $\omega = \omega_c$	Slop at $\omega \to \infty$
$K\left(\dfrac{S}{\omega_c}+1\right)$	0dB /decade with mag $20\text{Log}_{10}K$	+20 dB/dec	+3dB	+20dB / dec
$K\left(\dfrac{S}{\omega_c}+1\right)^2$	0dB /decade with mag $20\text{Log}_{10}K$	+40dB/dec	+6dB	+40dB/dec
⋮				
$K\left(\dfrac{S}{\omega_c}+1\right)^p$	0dB /decade with mag $20\log_{10} K$	+20pdB/dec	+3pdB	+20pdBdecade

G(S)H(S)	Initial Line With slope	Change in slope at $\omega = \omega_c$	Error in Mag at $\omega = \omega_c$	Slope at $\omega = \infty$
$\dfrac{K}{\left(\dfrac{S}{\omega_c}+1\right)}$	0dB/decade with mag $20\log_{10} K$	–20dB/dec	–3dB	–20dB/dec
$\dfrac{K}{\left(\dfrac{S}{\omega_c}+1\right)^2}$	0dB/decade with mag $20\log_{10} K$	–40dB/dec	–6dB	–40dB/dec
$\dfrac{K}{\left(\dfrac{S}{\omega_c}+1\right)^P}$	0dB/decade with mag $20\log_{10} K$	–20pdB/dec	–3pdB	–20pdB/dec

- If magnitude plot given , then recovered T.F is not unique .
- If Bode magnitude and phase plot is given , then reordered T.F is unique

G(S)H(S)

→ Min phase system $\left(\dfrac{K}{1+\dfrac{S}{\omega_c}}\right), \dfrac{K}{\left(\dfrac{-S}{\omega_c}-1\right)}$

→ Non min phase system $\dfrac{K}{\left(\dfrac{S}{\omega_c}-1\right)}, \dfrac{Ke^{-S}}{\left(\dfrac{S}{\omega_c}+1\right)} \Rightarrow \dfrac{K(1-S)}{\left(\dfrac{S}{\omega_c}+1\right)}$

Time constant from

$$T(S) = \frac{1}{\left(1+\dfrac{S}{T}\right)}$$

$\downarrow$

Corner frequency

Approximation of T.F.

$$T(S) = \frac{5(S+20)(S+50)}{(S+10)(S+100)} = \frac{5\left[1+\dfrac{5}{20}\right]\left[1+\dfrac{5}{50}\right]}{\left[1+\dfrac{5}{10}\right]\left[1+\dfrac{5}{100}\right]}$$

(i) $0<\omega<10$ $\quad T(S) = \dfrac{5(1)(1)}{(1)(1)} = 5$

(ii) $\omega = 10$ $\quad T(S) = \dfrac{5(1)(1)}{(1)(1)} = 5$

(iii) $10<\omega<20$ $\quad T(S) = \dfrac{5(1)(1)}{\left(\dfrac{5}{10}\right)(1)} = \dfrac{50}{5}$

(iv) $\omega = 20$ $\quad T(S) = \dfrac{5(1)(1)}{\frac{S}{10}.1} = \dfrac{50}{5}$

(v) $20 < \omega < 50$ $\quad T(S) = \dfrac{5\left(\frac{S}{20}\right)1}{\left(\frac{S}{10}\right).1} = \dfrac{5}{2}$

(vi) $\omega = 50$ $\quad T(S) = \dfrac{5}{2}$

Similany

- How to calculate T.F from Bode plot – 0 dB / decode $\rightarrow$ K

Step - 1 Identify initial slope $20\text{pdB/decode} \rightarrow KS^p$

$-20\text{pdB/decode} \rightarrow K/S^p$

Step - 2 Identify corner frequency (where slope changes change in slop = (final – Initial) slope

$$\Delta S = +20p\left(1+\frac{s}{\omega_c}\right)^p, \Delta S = -20p\frac{1}{\left(1+\frac{S}{\omega_c}\right)^p}$$

Step -3 For calculation of K

M-1 Approximation

M-2 (i) If initial line is 0dB/de code = $20\log_{10} K = M$

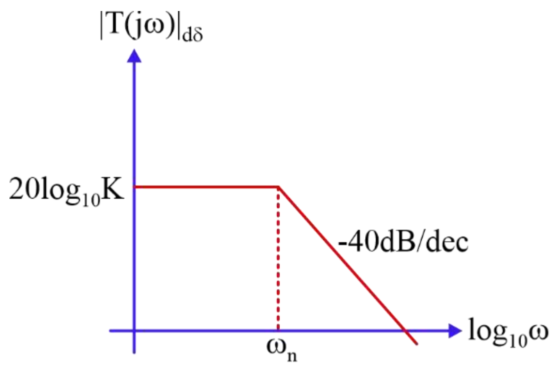
(ii) If slope is there

Magnitude at $\omega = 1 \longrightarrow N$ (magnitude of initial line) $20\log_{10} K = N$

Bode plot of $G(S)H(S) = \dfrac{K\omega_n^2}{S^2 + 2\xi\omega_n S + \omega_n^2} = ?$

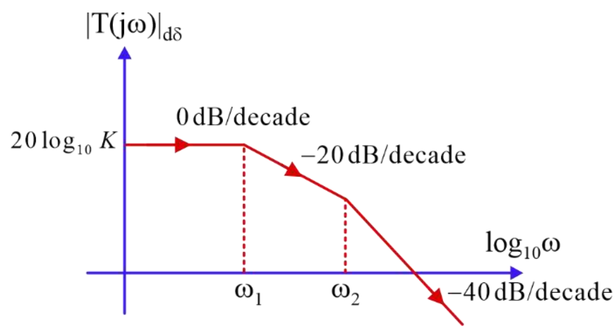
Case - 1 Critical damping $(\xi = 1)$

$$G(S)H(S) = \frac{K}{\left(\frac{S}{\omega_n}+1\right)^2}$$



Case – 2 Overdamped $(\xi > 1)$

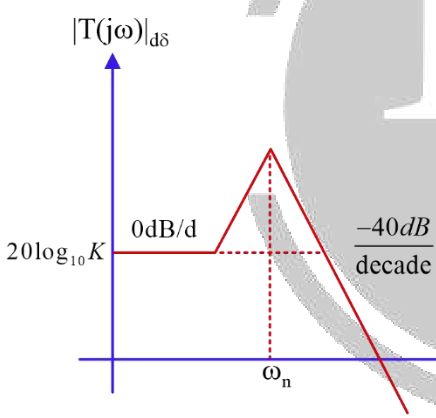
$$G(S)H(S)=\frac{K}{\left(\frac{S}{\omega_1}+1\right)\left(\frac{S}{\omega_2}+1\right)}$$

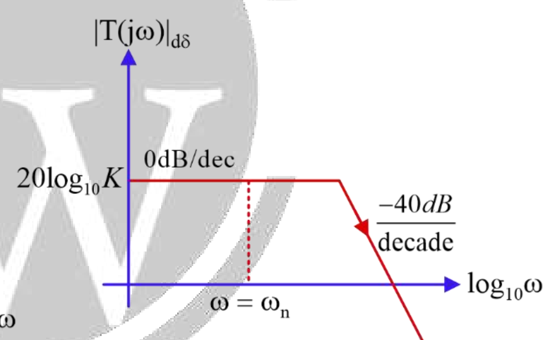


Case -3 Underdamp $(0<\varepsilon<1)$

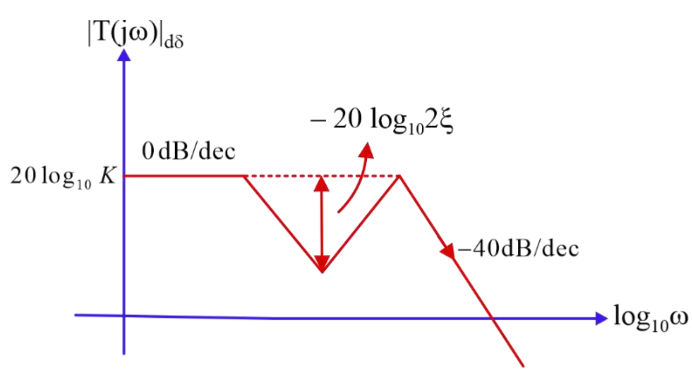
(1) $0<\xi<\frac{1}{2}$

(2) $\xi=\frac{1}{2}$

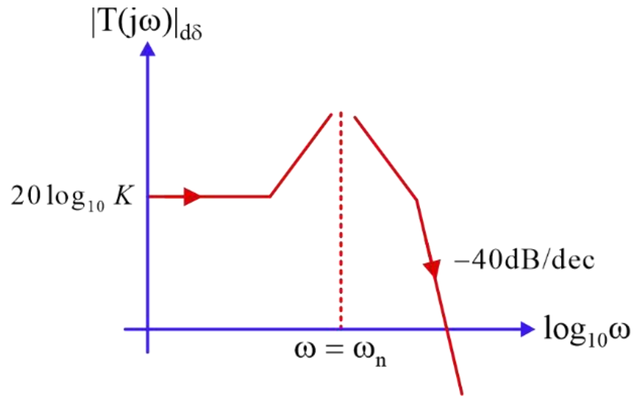




(3) $\frac{1}{2}<\xi<1$



Case-4: $\xi = 0$



Important Points :

(1) Calculation of unknown frequency

change in magnitude between 2 frequencies = ΔM

$\frac{\Delta M}{20p} = r$

factor affecting frequency = 10^r

$\frac{\Delta M}{6p} = r$

factor affecting frequency = 2^r

5.2. Nyquist Stability and Plot

- Contour (closed curve in S – plane) or (specified region in s plane) encircles encircles (contains) m poles of Q(S) strictly inside if .

  ↓

  Q(S) plot in Q(S) plane encircles origin (0,0) m times.
- Contour in splane passes through one or more pole of $\theta(S)$

  Q(S) plot in Q(S) plane remain open curve. Hence poles do not contribute in encirclement of origin.
- Contour in s-plane encircle m zeros of Q(S) strictly inside it

  ↓Same duurection

  Q(S) plot in Q(S) plane encircles the origin m times.
- Contour in s plane has m zero on the boundary of the contour

  ↓ same sense

  (i) Q(S) plot in Q(S) plane is closed.

  (ii) Q(S) plot crosses origin m times.

  (iii) Such zeros do not contribute in encirclement of origin

5.2.1. Rules of mapping from S plane to Q(S) Plane

$$Q(S)=\frac{N(S)}{D(S)},\quad \text{C contour in S plane}$$

P_C = No of poles of Q(S) present strictly inside C,

Z_C = No of Zeros.

N = No . of encirclement of origin by Q(S) plot.

$N = P_C - Z_C$ → N = +ve C and Q(C) are in opposite direction

$N = P_C - Z_C$ → N = –ve C and Q(C) are in same direction

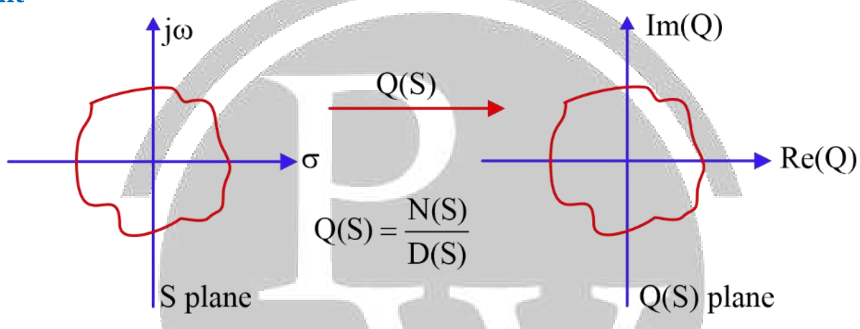
Limitation

(a) If pole of Q(S) lies on boundary of C

(b) If zero of Q(S) lies on boundary of C

Important Points :

(1) Principle of argument



- If P_C = Pole inside Contour, Z_C = zero inside contour , Q(S) plot in Q(S) plane

(i) $P_C > Z_C \rightarrow Q(S)$ plot in Q(S) plane encircles the origin (0,0) $(P_C - P_Z)$ time in direction opposite to the contour in S plane.

(ii) $P_c < Z_c \rightarrow$ Encircle origin $(Z_C - P_C)$ time in same direction .

(iii) $P_C = Z_C \rightarrow$ Does not encircle origin .

(2) Rules of Mapping

valid for T.F $Q(S)=\frac{N(S)}{D(S)}$

$$A+BQ(S)=A+\frac{N(S)}{D(S)}B=\frac{AD(S)+BH(S)}{D(S)}$$

(i) Let T.F $= Q(S)=\frac{N(S)}{D(S)}$

P_C = No of poles of Q(S) inside contour .

Z_C = No of Zeros of Q(S) inside contour

N = No of encirclement of (0,0) by Q(S) plot.

N = +ve , if C and Q(S) has opposite direction

N = - ve , if C and Q(S) has same direction

(ii) Let T.F is A + BQ(S)

N = $P_C - P_Z$

$P_C \rightarrow$ No . of poles of T(S) inside contour

$Z_C \rightarrow$ No. of zero .

$N \rightarrow$ No. of encirclement of (0,0) by A+BQ(S) plot .

(iii) If A+BQ(S) plot encircle origin then Q(S) plot will encircle $\left(-\frac{A}{B},0\right)$.

Or

If Q(S) plot encircles (-A/B,0) then A+BQ(S) plot will encircle (0,0)

Nyquist

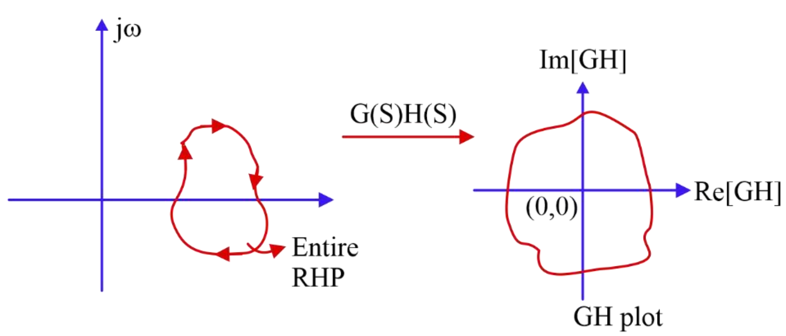
If $G(S)H(S) = \frac{N(S)}{D(S)}$

$$1 + G(S)H(S) = \frac{N(S)+D(S)}{D(S)}$$

$$A + BG(S)H(S) = \frac{AD(S)+BH(S)}{D(S)}$$

- Poles of T.F $A + BG(S)H(S)$ will be same as T.F G(S)H(S).
- Zeros of $1+G(S)H(S) =$ Root of $[1+G(S)H(S)]$ = Poles of CLS
- Poles of $1+G(S)H(S) =$ Poles of G(S)H(S)

Case 1



Fixed : Clockwise

Rule of Mapping : let TF is $G(S)H(S) \Rightarrow N = P_C - Z_C$

N = No of encirclement of (0,0) by G(S)H(S) plot in G(S)H(S) plane.

N → $+ve$ $G(S)H(S)$ plot in ACW direction

N → $-ve$ $G(S)H(S)$ plot in CW direction

P_C = No of poles of G(S)H(S) lying inside contour in S plane .

OR

No of poles of G(S)H(S) lying in right side plane $P_C = P_t$

Z_C = No of zero of G(S)H(S) lying inside contour in S plane .

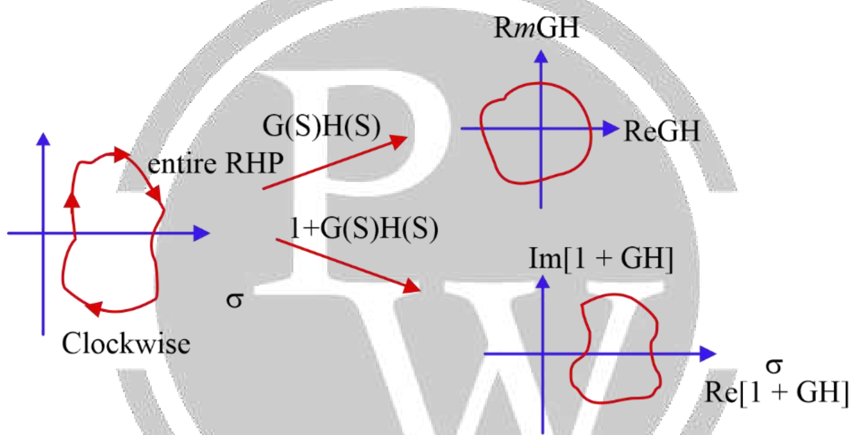
OR

No of zeros of G(S)H(S) lying in RHP $Z_C = Z_t$

$$\boxed{N = P_+ - Z_+}$$

For OLS to be stable $P_C = 0$

Case 2



Rule of Mapping : Let T.F is $1+G(S)H(S) = \dfrac{N(S)+D(S)}{D(S)}$

$$N = P_C - Z_C$$

N = No of encirclement of (0,0) by 1+GH plot in 1+GH plane .

OR

No of encirclement of (-1,0) by GH plot in GH plane .

P_C = no of poles of $[1+G(S)H(S)]$ lying in R.H.P

OR

No of poles of G(S)H(S) lying in $P_C = P_t$

Z_C = no of zeros of [1+G(S)H(S)] lying in RHP.

OR

No of poles of closed loop system lying in R.H.P. $Z_C \rightarrow Z_t$

Note: After Nyquist modified the mapping Rule .

(i) S plane contour : Entire R.H.P.

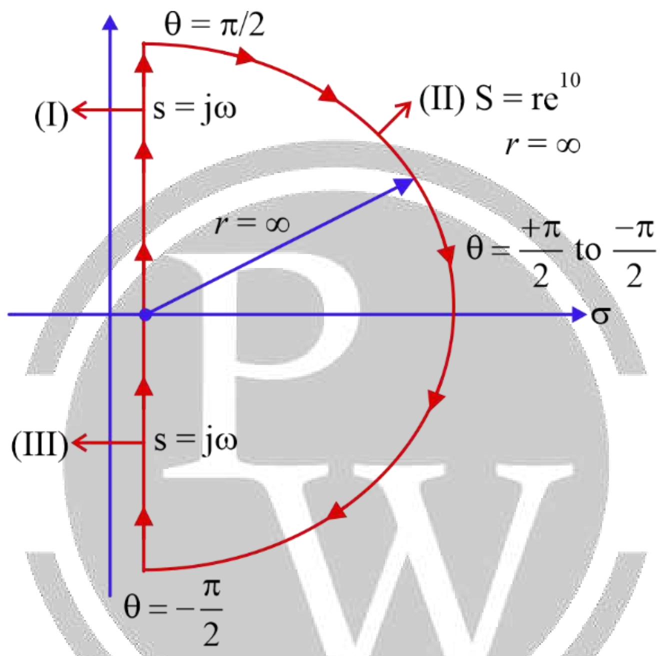
(ii) Plot of G(S)H(S) is needed only

- Stability of all the T.F. of type A+BG(S)H(S) can be determined.
- Also stability of C.L.S can be determined by applying N.S.C. in $1 \pm G(S)H(S)$.

  S plane contour $\rightarrow$ Entire R.H.P. : "NYQUIST CONTOUR "

Nyquist Contour : "Contour containing entire R.H.P."

5.2.2. "Types Of Nyquist Contour"



$$G(S)H(S) = \frac{N(S)}{D(S)}$$

$\downarrow$

Should not have any pole or zero on $j\omega$ axis

Nyquist Contour

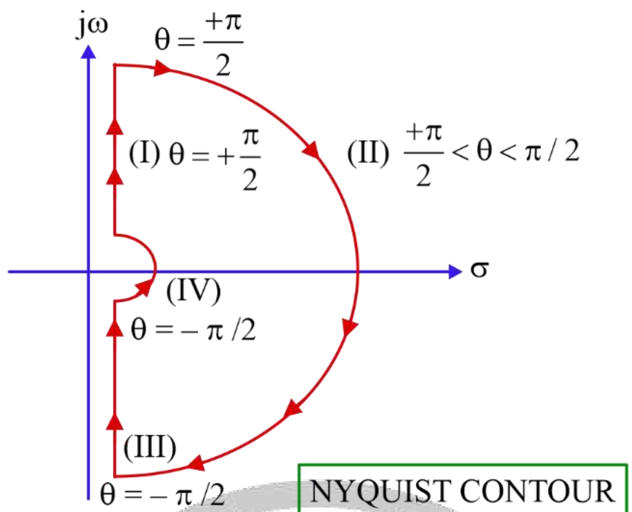
[I]: $S = j\omega \quad 0 < \omega < \infty$

[II]: $S = re^{j\theta} \quad r = \infty$

$$\frac{-\pi}{2} < \theta < \frac{-\pi}{2}$$

[III] $\begin{cases} S = j\omega & -\infty < \omega < 0 \\ S = -j\omega & 0 < \omega < \infty \end{cases}$

2. $G(S)H(S)=\dfrac{N(S)}{D(S)} \rightarrow$ It has poles or zeros at origin



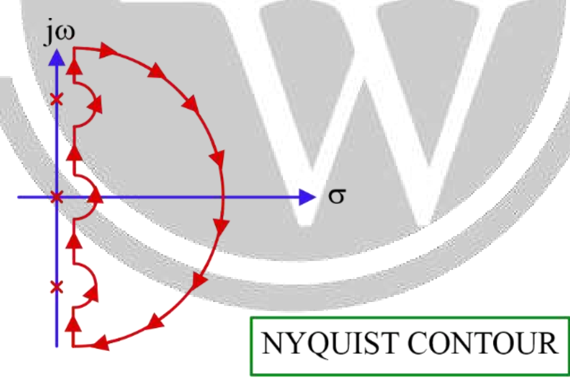
(I) $S=j\omega$ $\quad 0<\omega<\infty$

(II) $s=re^{10}$ $\quad r=\infty$

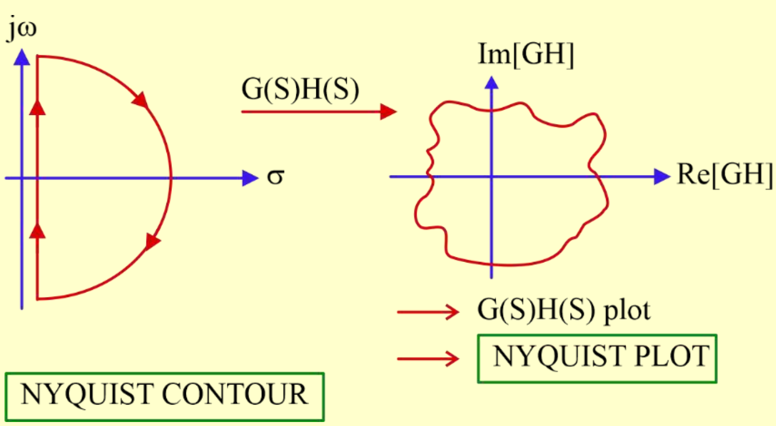
(III) $S=-j\omega$ $\quad o<\omega<\infty$

(IV) $S=re^{10}$ $\quad r\rightarrow 0 \quad -\pi/2<\theta<+\pi/2$

3. $G(S)H(S)=\dfrac{N(S)}{D(S)} \longrightarrow$ It has poles and zeros on $j\omega$ axis.



Note:



N.S.C. for various T.F. Let Nyquist contour is clockwise:

Case 1 : N.S.C. for T.F. G(S)H(S)

$N = P_+ - Z_+$

$N =$ number of encirclement of (0,0) by GH plot in GH plane

N → +ve if GH is ACW
→ –ve if GH is C.W.

P_+ = Number of poles of GH lying in RHP

Z_+ = Number of zeros of GH lying in RHP

For TF GH to be stable : $P_+ = 0$

Case 2 : N. S. C. for T.F. 1 + G(s)H(s)

$N = P_+ - Z_+$

N = Number of encirclement of (0, 0) by 1 + GH plot in 1 + GH plane

Or

No. of encirclement of (–1, 0) by GH plot in GH plane

P_+ = Number of poles of 1 + GH lying in R.H.P.

Or

Number of poles of GH lying in R.H.P

Z_+ = Number of zeros of 1 + GH lying in RHP

Or

Number of poles of $\frac{G}{1+GH}$ (C.L.S) lying in RHP

(i) For TF 1 + GH to be stable $\longrightarrow P_+ = 0$

(ii) For closed loop system $\left(\frac{G}{1+GH}\right)$ to be stable $\longrightarrow Z_+ = 0$

Case 3: N.S.C. for T.F. 1 – G(s)H(s)

$N = P_+ - Z_+$

N = Number of encirclement of (0,0) by 1 - GH plot in 1- GH plane

Or

Number of encirclement of (1,0) by GH plot in GH plane

P_+ = Number of poles of 1- GH lying in RHP

Or

Number of poles of GH lying in RHP

Z_+ = Number of zeros of 1 – GH lying in RHP

Or

Number of poles of $\frac{G}{1+GH}$ (C.L.S.) lying in R.H.P.

(i) For TF 1 - GH to be stable $\longrightarrow P_+ = 0$

(ii) For C.L.S. $\left(\frac{G}{1-GH}\right)$ to be stable $\longrightarrow Z_+ = 0$

Case 4 : N.S.C. for T.F. 4 + 3 G(s)H(s)

$N = P_+ - Z_+$

N = Number of encirclement of (0,0) by 4 + 3GH plot in 4 + 3 GH plane

Or

Number of encirclement of $\left(\frac{-4}{3}, 0\right)$ by GH plot in GH plane

P_+ = Number of poles of $4+3GH$ lying in RHP

Or

No. of poles of GH lying in RHP

Z_+ = Number of zeros of $4+3GH$ lying in RHP

Or

Number of poles of 0 system $\frac{G}{4+3GH}$ lying in R.H.P.

Case 5 : N.S.C. for T.F. $A+BG(s)H(s)$

$N = P_+ - Z_+$

N = Number of encirclement of (0,0) by $A+BGH$ plot in $A+BGH$ plane

Or

Number of encirclement of $\left(\frac{-A}{B}, 0\right)$ by GH plot in GH plane

P_+ = Number of poles of $A+BGH$ lying in RHP

Or

Number of poles of GH lying in RHP

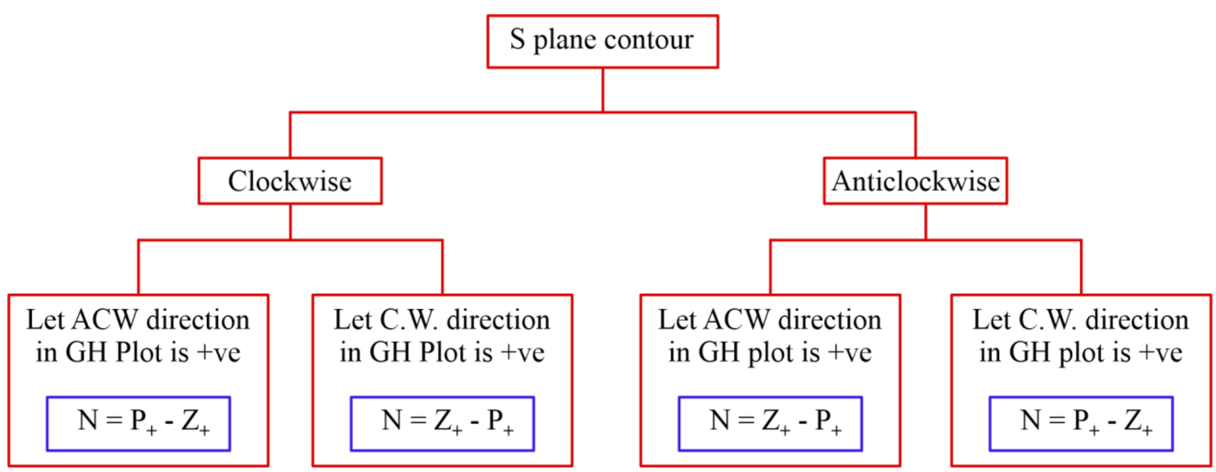
Z_+ = Number of zeros of $A+BGH$ lying in RHP

Or

Number of poles of 0 system $\frac{G}{A+BGH}$ lying in R.H.P.

5.3. Problem Solving Approach

1. Flow Chart



Plotting Nyquist Plot

Assumptions

(1) Nyquist contour is clockwise.

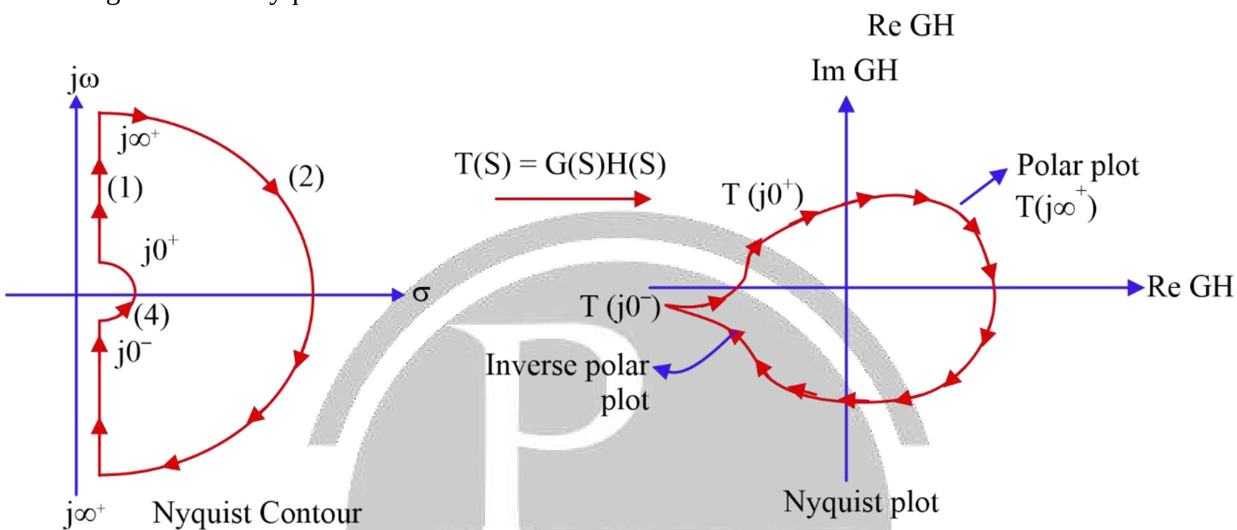
(2) Does not contain pole or zero on jω axis.

(3) Mapping on G(s) H(s) plane

Prerequisite

(1) Rang of $\omega \rightarrow -\infty < \omega < +\infty$

(2) Consider generalized Nyquist contour



Closing of Nyquist Plot

(i) $s = j\omega \rightarrow 0^+ < \omega < \infty^+ \longrightarrow T(s) : T\left(j0^+\right)$ to $T\left(j\infty^+\right)$

(ii) $s = \text{Re}\, j\theta \rightarrow \infty^+ < \omega < \infty^- \longrightarrow T(s) : T\left(j\infty^+\right)$ to $T\left(j\infty^-\right)$

(iii) $s = j\omega \rightarrow \infty^- < \omega < 0^- \longrightarrow T(s) : T\left(j\infty^-\right)$ to $T\left(j0^-\right)$

(iv) $s = \text{Re}\, j\theta \rightarrow 0^- < \omega < 0^+ \longrightarrow T(s) : T\left(j0^-\right)$ to $T\left(j0^+\right)$

Nyquist Contour Nyquist plat

(3)

(i) $s = j\omega \;\; 0^+ < \omega < \infty^+$

(ii) $s = \text{Re}^{j\theta} \;\; R \rightarrow \infty \;\; \theta \rightarrow +\frac{\pi}{2}$ to $\frac{-\pi}{2}$ (C W)

(iii) $s = j\omega \;\; \infty^+ < \omega < 0^-$

(iv) $s = \text{Re}^{j\theta} \;\; R \rightarrow 0 \;\; \theta \rightarrow -\frac{\pi}{2}$ to $\frac{\pi}{2}$ A.C.W

(4)

$\theta \rightarrow \frac{+\pi}{2}$ to $\frac{-\pi}{2}$:CW

$-\theta \rightarrow \frac{-\pi}{2}$ to $\frac{+\pi}{2}$:ACW

$$T(s)=\frac{1}{1+s}$$

(5) segment $-1 s=j\omega$

$$T(j\omega)=\frac{1}{1+j\omega}$$

Case 1: $T(j\omega)=\frac{1}{1+j\omega}$

$$|T(j\omega)|=\frac{1}{\sqrt{1+\omega^2}},\angle T(j\omega)=-\tan^{-1}\omega$$

$\omega=0^+$ $\quad\left|T\left(j0^+\right)\right|=1$ $\quad\omega=\infty^+$ $\quad\left|T\left(j\omega\right)\right|=0$

$\angle T\left(j0^+\right)=0^0$ $\quad\angle T\left(j\omega\right)=-90^0$

Case 2: $T\left(j\omega\right)=\frac{1}{1+j\omega}$

$$\omega=0^+ \quad T\left(j0^+\right)=\frac{1}{1+j0^+}=1=1\angle 0^0$$

$$\left|T\left(j0^+\right)\right|=1,\ \angle T\left(j0^+\right)=0^0$$

$$\omega=\infty \quad T\left(j\infty^+\right)=\frac{1}{1+j\infty^+}=\frac{1}{j\infty^+}=0\angle-90^0$$

- Mapping of segment I on $G(s)H(s)$ plane is polar plot.
- Mapping of segment III on $G(s)H(s)$ Inverse polar plot

  Inverse polar plot = error image of polar plot w.r.t. horizontal axis keeping the same flow direction

Step to Draw Polar Plot

(1) Put $S=j\omega$ $T(j\omega)$ $0^+<\omega<\infty t$

(2) $T(j0^+)=M_1\angle\theta_1$

$T(j\infty^+)=M_2\angle\theta_2$

(3) Rationalise $T(j\omega)=\text{Re}\{T\}+j\,\text{Im}\{T\}$

Nyquist Plot

S-1 Draw pole – zero diagram on S plane and select proper contour.

S-2 Map segment I of Contour and draw polar plot .

S-3 Map segment II of contour and draw the respective mapping (generally circle).

S-4 Map segment III of contour and draw inverse polar plot .

S-5 Map segment IV of contour and draw respective mapping (generally circle) .

All Pass System/filter

Poles and zeros are at mirror image w.r. to $j\omega$ axis . $T(S)=\frac{(1-S)}{1+S}K$

- Nyquist plot of all pass filter is always circle, with radius K and center (0,0).

5.3.1. Closing of Nyquist Plot from Polar Plot

Case 1 If OLTF contains n no. of poles at origin .

- $G(\infty+)$ *and* $G(\infty-)$ will be connected by O^{+} radius circle (short circle)
- $G(O^{+})$ and $G(O^{-}) \rightarrow O.C$
- To close this $n\pi$ clockwise encirclement is performed from $G(O^{-})$ to $G(O^{+})$

Case 2 If Type of OLS = 0, and order of zero is greater then of pole .

- $G(0^{-})$ and $G(0^{+}) \div$ short circuit
- $G(\infty^{+})$ and $G(\infty^{-}) \div O.C$
- $(m-n)\pi$ clockwise encirclement from $G(\infty^{+})$ *to* $G(\infty^{-})$.

Case 3 Type of OLS = 0, order of zero $\leq$ order of pole

- $G(0^{-})$ and $G(0^{+}) \div S.C$
- $G(\infty^{+})$ and $G(\infty^{-}) \div S.C$

Gain Margin – Phase Margin

Minimum Phase System : All poles and zero must be on L.H.P

- Poles and zeros at origin or $j\omega$ axis are allowed

Non Minimum Phase System : Which are not minimum .

- All poles in L.H.P , few zeros in RHP $\rightarrow$ Type A
- All zero in LHP few poles are in RHP $\rightarrow$ Type B

Gain Margin

(1) Can determine stability

(2) Amount of gain K_1 that is needed to multiplied in OLTF such that corresponding C.L.S becomes stable .

(3) Amount of gain (in dB) that need to be added in OLTF such that corresponding C.L.S becomes marginally stable .

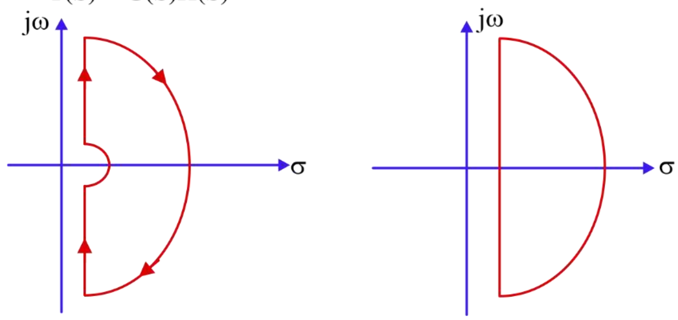
Phase Margin

(1) Can determine stability

(2) Amount of phase angle that is needed to added in of G(S)H(S) such that C.L.S become Marginally stable

5.3.2. Mathematical calculation of G.M

T(S) = G(S)H(S)



S-2 Put $S = j\omega$ and determine range of ω

Contour 1: $0<\omega<\infty \quad T(S)\big|_{S=j\omega} = T(j\omega)$

Contour 2: $0\le\omega<\infty$

S-3 $\angle T(j\omega) = -180^0$

Solve and calculate ω, possible phase crossover frequency .

S-4 Validation of ω

(i) ω is real and +ve (i) n (ii) $\omega = \omega_{pc}$

(ii) $\angle T(j\omega) = -180^0$

S-5 At $\omega = \omega_{PC}$ $|T(j\omega)_{PC}| = M$

Gain Margin = $\frac{1}{M}$

Note: (1) If No valid ω_{PC} then G.M will be either $+\infty$ dB or $-\infty$ dB, depending on nature of OLS and absolute stability of CLS .

(2) $GM = +\infty$ dB or $-\infty$ dB represent absolute stable / unstable nature .

Method 2

S-1 Put $S = j\omega$ and find range of ω

S-2 $T(s = j\omega) = TR(j\omega) + jT_I(j\omega)$: Rationalize

S-3 $T_I(j\omega) = 0$ Possible ω_{pc}

S-4 validity $\omega_{pc} = (\text{Real and +ve}) \cap (TR(j\omega_{pc}) = -ve)$

S-5 $|T(j\omega_{PC})| = M$, $G.M = \frac{1}{M}$

Note : If ω_{pc} is invalid $\rightarrow$ same procedure as Method 1.

Note: G.M cannot be 0 or ∞ in ratio

C.L.T.F	O.L.T.F	GM(dB)	PM in degree
Stable unstable	Min phase system	+ve (dB) -ve(dB)	+ve in degree -ve in degree
Stable unstable	Non Minimum Type –A	+ve(dB) -ve(dB)	+ve in dgree -ve in degree
Stable unstable	Non Minimum Type – B	-ve (dB) +ve(dB)	-ve in degree +ve in degree

Mathematical calculation of phase Margin–

Given T(S)

S-1 $S = j\omega$ and range of ω

S - 2 $|T(j\omega)| = 1$ passible : gain crossover frequency $(\omega_g c)$

S-3 Validity : $\omega \rightarrow$ Real and +ve

S-4 P.M = $\angle T(j\omega_{gc}) + 180^0$

Note :

(1) Changes in ω_{pc}

(2) Change gain margin

Introduction of Transportation log

$\downarrow$

(1) ω_{gc} remain same

(2) PM changes

(3) PM $\downarrow$, so stability $\downarrow$

Shortcut for G.M

$T(S) = G(S)H(S)$

- Let K Multiplied in $G(S)H(S) \to K, G(S)H(S),$ So that roots of $1 + KG(S)H(S)$ represents Marginal Stability
- S-1 from Routh table

S-2

ODD Row		LAST Row		G.M
Invalid	$\to$	Invalid	$\to$	$+\infty / -\infty$
Invalid	$\to$	valid	$\to$	finite
Valid	$\to$	Invalid	$\to$	finite
Valid	$\to$	valid	$\to$	Absured case

S-3 **Valid Odd Rows :**

Odd row = 0 , K= +ve , A.E roots are non repeated on $j\omega$ axis then $K_1 = GM$ in ratio and roots of $AE \to \omega_{PC}$

S-4 **Valid last Row :**

Last Row = 0

$K = +ve \to 0 < K < \infty$

G.M and P.M from Nyquist

G.M (1) $\omega_{pc} = ?$ $\angle T(j\omega) = -180^0$ [Nyq plot must cross –ve real axis $\to \omega_{pc}$ exist]

(2) $G.M = \dfrac{1}{|T(j\omega_{PC})|} = \dfrac{1}{|\text{length on negative real axis tiee } \omega_{PC}|}$

Phase Margin –

(i) $\omega_{gc} \to |T(j\omega_{gc})| = 1$

Nyquist plot intersects unity radius circles then ω_{gc} exist .

(ii) $PM = \angle T(j\omega_{ge}) + 180^0$

$\angle T(j\omega_{gc}) \to$ from+ve real axis

$\to CW = -ve$

$\to ACW = +ve$

For OLS : Min phase

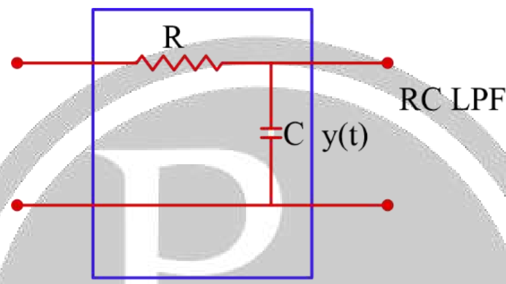
		G.M (in dB)	PM in degree	C.L.S
$K = K_1$	$\omega_{pc} > \omega_{gc}$	+ve (dB)	$+ve^0$	Stable
$K = K_2$	$\omega_{pc} > \omega_{gc}$	0 (dB)	0^0	M.S
$K = K_3$	$\omega_{pc} > \omega_{gc}$	- ve (dB)	$-ve^0$	Unstable

❑❑❑

6 STATE SPACE ANALYSIS

6.1. Introduction

Single I/P single output Input $\rightarrow u(t)$



$$H(s)=\frac{1/RC}{s+1/RC}, I.C=0$$

$$\frac{dy(t)}{dt}+\frac{1}{RC}y(t)=\frac{1}{RC}u(t) \rightarrow \text{ initial condition not zero}$$

$$y(t)=x_1(t) \rightarrow \text{ stable variable (A parameter across memory element)}$$

$$\frac{dy(t)}{dt}=\dot{x}_1(t)$$

$$\boxed{\begin{array}{l}\dot{x}_1(t)=\frac{-1}{RC}x_1(t)+\frac{1}{RC}u(t)\\ y(t)=x_1(t)\end{array}} \rightarrow \text{State equation}$$

$\downarrow$

Output equation

$$\boxed{\begin{array}{l}[\dot{x}_1(t)]_{|X|}=\left[\frac{-1}{RC}\right]_{|X|}[x(t)]_{|X|}+\left[\frac{1}{RC}\right]_{|X|}u(t)\\ [y(t)]_{|X|}=[1][x_1(t)]_{|X|}+[0]_{|X|}[u(t)]\end{array}}$$

State Model of above system

$$\left.\begin{array}{l}[x(t)]=[A][x(t)]+[B][u(t)]\\ [y(t)]=[C][x(t)]+[D][u(t)]\end{array}\right] \text{Mathematical Representating of a physical system}$$

For MIMO System

State Equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}_{n\times 1} = \begin{bmatrix} \\ A \\ \\ \end{bmatrix}_{n\times n} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}_{n\times 1} + \begin{bmatrix} \\ B \\ \\ \end{bmatrix}_{n\times l} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_l(t) \end{bmatrix}_{l\times 1}$$

Output Equation

$$\begin{bmatrix} y_1(t) \\ y_0(t) \\ \vdots \\ y_n(t) \end{bmatrix}_{m\times 1} = \begin{bmatrix} \\ C \\ \\ \end{bmatrix}_{n\times n} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}_{n\times 1} + \begin{bmatrix} \\ D \\ \\ \end{bmatrix}_{m\times l} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_l(t) \end{bmatrix}_{l\times 1}$$

$$\boxed{\begin{array}{l} [\dot{x}(t)]_{n\times 1} = [A]_{n\times n}[x(t)]_{n\times 1} + [B]_{n\times l}[u(t)]_{l\times 1} \\ [y(t)]_{m\times 1} = [C]_{m\times n}[x(t)]_{n\times 1} + [D]_{m\times l}[u(t)]_{l\times 1} \end{array}}$$

State Model Representation from DE

$$\frac{d^3y(t)}{dt^3} + \frac{3d^2y(t)}{dt^2} + \frac{6dy(t)}{dt} + 7y(t) = 6u(t)$$

$y \rightarrow$ output, $\mu \rightarrow$ input

Let $y(t) = x_1(t)$, $\quad dy(t) = x_2(t) = xi(t), \dfrac{d^2y(t)}{dt^2} = x_3(t) = \dot{x}_2(t)$

$$\frac{d^3y(t)}{dt^3} = \dot{x}_3(t)$$

Then solve question

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = [A]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [B]u(t) \qquad \text{and } y(t) = [C]\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + Du(t)$$

From Transfer function

Let T.F is T(S) $= \dfrac{b(c_3d^3 + c_2s^2 + c_1s + c_0)}{s^4 + a_3s^3 + a_2s^2 + a_1s + a_0}$

Case 1 : Controllable canonical from

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}(u)$$

$$[y] = [C_0\ C_1\ C_2\ C_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + [0][u]$$

Note :

1. No of state variable = Highest order of D^r
2. Coefficient of highest order of D^r should be 1.

Case 2 : Observable canonical from

X = AX + BU

Y = CX + DU

$$[A]_{OCF} = [A]^T_{CCF}, [B]_{OCF} = [C]^T_{CCF}, [C]_{OCF} = [B]^T_{CCF}$$

Case 3 : Diagonal canonical form

$$T(S) = \frac{b_1}{(s+p_1)} + \frac{b_2}{(s+p_2)} + \frac{b_3}{(s+p_3)} + \frac{b_4}{(s+p_4)}$$

$$A = \begin{bmatrix} -p_1 & 0 & 0 & 0 \\ 0 & -p_2 & 0 & 0 \\ 0 & 0 & -p_3 & 0 \\ 0 & 0 & 0 & -p_4 \end{bmatrix} [B] \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix}$$

$$[C] = [m_1, m_2, m_3, m_4]$$

$$b_1 = K_1 m_1$$
$$b_2 = K_2 m_2$$
$$b_3 = K_3 m_3$$
$$b_4 = K_4 m_4$$

Case 4 : Jordan Canonical form

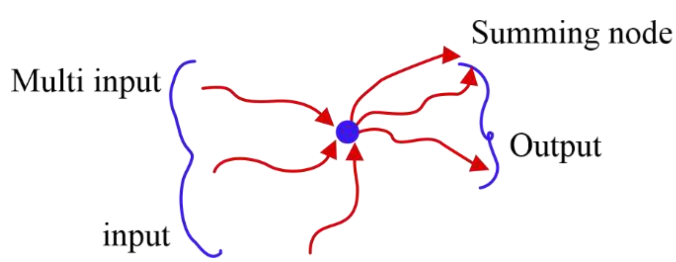
Extension of D.C.F when poles are repeated.

$$T(S) = \frac{b_1}{(s+p_1)} + \frac{b_2}{(s+p_1)^2} + \frac{b_3}{(s+p_1)^3} + \frac{b_4}{(s+p_2)^4}$$

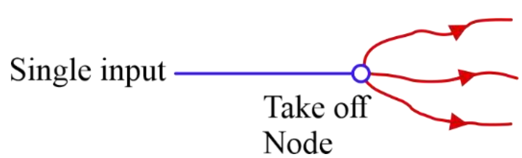
$$[A] = \begin{bmatrix} -p_1 & 0 & 0 & 0 \\ 0 & -p_1 & 0 & 0 \\ 0 & 0 & -p_1 & 0 \\ 0 & 0 & 0 & p_2 \end{bmatrix} \text{ Jordan Block}$$

From S.F.G

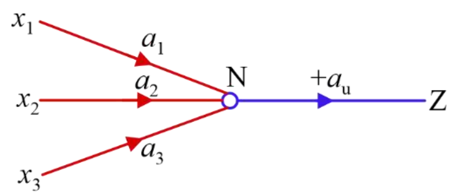
1. Summing Node



2. Take off Node

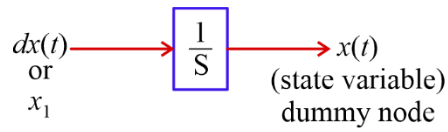


3. Potential of a node

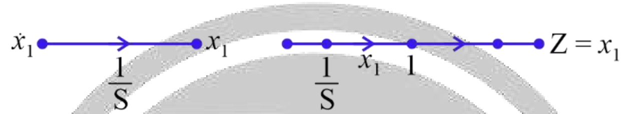


$$N = a_1x_1 + a_2x_2 + a_3x_3$$
$$Z = a_4N$$

4. Integrator Block



5. Integrator SFG



No of integrator = No of state variable

Important :

$$[X(S)] = \underbrace{[SI - A]^{-1}[X(0^-)]}_{\text{Solution of state variable due to non zero I.C}} + \underbrace{[SI - A]^{-1}[B][U(S)]}_{\text{Solution of state Variable due to input.}}$$

State Transition Matrix – (S.T.M)

(a) S.T.M in S-domain = $[SI - A]^{-1}$ $\quad n \times n$

(b) S.T.M in time domain = $[\phi(t)]$ or $[e^{AT}]_{nxn}$

$$[\phi(t)]_{n\times n} \xleftrightarrow{\text{L.T}} [ST - A]^{-1}$$

6.1.2. Properties of STM

$$[e^{At}] = [\phi(t)]$$

(1) $[e^{Ao}] = [\phi(0)] = [I]_{nxn}$

(2) $\left[\left(\dfrac{de^{At}}{dt}\right)_{t=0}\right] = (A)_{nxn}$

(3) $[\phi(-t)] = \left[\phi^{-1}(t)\right]$

(4) $[\phi(t_1 + t_2)] = [\phi_1(t)\phi_2(t)]$

(5) $\left[\phi^K(t)\right] = [\phi(Kt)]$

(a) Homogeneous State equation $\left[\dot{X}(t)\right] = [A][x(t)]$

Sol. $[X(s)] = [SI - A]^{-1}\left[X(0^-)\right]$

$$x(t) = \left[e^{AT}\right]\left[x(0^{-1})\right] \text{ or } [x(t)] = [\phi(t)][x(0^-)]$$

$$\phi(t) = ILT\left\{[SI - A]^{-1}\right\}$$

(b) Non Homogeneous state equation $\left[\dot{X}(t)\right] = [A][x(t)] + [B][u(t)]$

$\dot{X} = AX + BU$

Sol. $X(S) = [SI - A]^{-1}\left[x(0^-)\right] + [\phi(t)] * [B][u(t)]$

Solution of y(t)

$[y(t)] = [C][X(t)] + [D][u(t)]$

$[Y(S)] = [C][X(S)] + [D][U(S)]$

$[Y(S)] = [C]\left\{[SI - A]^{-1}[x(0^-)] + [SI - A]^{-1}[B][U(S)]\right\} + [D][U(S)]$

$$[Y(S)] = \underbrace{[C][SI - A]^{-1}[x(0^-)]}_{\text{Zero Input Response}} + \underbrace{\left\{[C][SI - A]^{-1}[B] + [D]\right\}[U(S)]}_{\text{Zero State Response}}$$

Total Response

For 2 Input 2 Output

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \text{ and } \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

For 1 Input and 1 Output

$$\frac{Y(S)}{U(S)} = \left[[C][SI - A]^{-1}[B] + [D]\right]$$

$$\left[x(0^-)\right] = 0$$

$$[SI - A] = \text{Poles of the system} = \text{eigen values of matrix A} = D(S)$$

$$\boxed{\frac{Y(S)}{U(S)} = \frac{[C]adj[SI - A][B] + [D]|SI - A|}{|SI - A|}}$$

Controllability and observability

$$\dot{X} = AX + BU$$
$$Y_2 = CX + DU$$

$A \rightarrow$ Square matrix

$\rightarrow |A|$

$\rightarrow$ Rank of matrix $A = \rho(A)$

$\rightarrow n \times n$

Method 1

Kalman Test

- **Controllability**

(1) $[Q_C]=\left[B:AB:A^2B:-----A^{n-1}B\right]$ $\nearrow$ Square $\searrow$ Rectangular

(2) Q_C : Rectangular, $\rho(Q_C)=\rho(A)\rightarrow$ Controllable

$\rho(Q_C)<\rho(A)\rightarrow$ Uncontrollable

- **Observability**

(1) $[Q_0]=\left[C^T:A^TC^T:(AT)^2C^T\text{..........}(A^T)^{n-1}C^T\right]$

(2) Q_0 : Square

$|Q_0|=0$ Non observable

$|Q_0|\neq 0$ observable

(3) Q_C : Rectangular, $\rho(Q_0)=\rho(A)\rightarrow$ Observable

$\rho(Q_C)<\rho(A)\rightarrow$ Not observable

Method 2

If A is diagonal Matrix with distinct diagonal

$$\dot{X}=\begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}_{n\times n} X+\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ : & : \\ b_{n1} & b_{n2} \end{bmatrix}\longrightarrow \text{They should not be all zero (Controllable)}$$

$$Y=\begin{bmatrix} C_{11} & C_{12} & ---- & C_{1n} \\ C_{21} & C_{22} & ---- & C_{2n} \end{bmatrix}X+[\ ]U$$

They should not be all zero (observable)

Method 3

Gilbert Test

Upper Triangular Matrix

- UTM having Jordan block
- Jordan block is used when E.V are repeated

$$\begin{bmatrix} d_1 & a_1 & a_2 \\ 0 & d_2 & a_3 \\ 0 & 0 & d_3 \end{bmatrix}\longrightarrow \text{They all Should not be zero}$$

Lower Triangular Matrix

$$\begin{bmatrix} d_1 & 0 & 0 \\ a_1 & d_2 & 0 \\ a_1 & a_3 & d_3 \end{bmatrix}\longrightarrow \text{Should not be all zero}$$

Method 4

- Controllable Canonical form

$$A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}=B$$

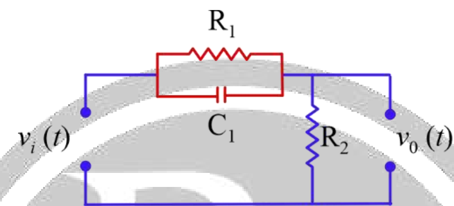
- O.C.F

$$[A]=\begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}[C]=[C, \quad 0, \quad 0]$$

7 CONTROLLER AND COMPENSATOR

7.1. Introduction

(1) Phase lead compensator



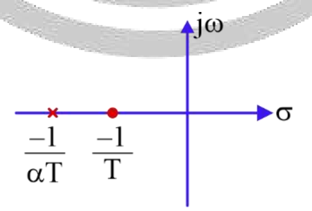
$$T = R_1C_1$$

$$T(S) = \frac{R_2(1+SR_1C_1)}{R_1+R_2\left\{1+\frac{SR_1C_1}{R_1+R_2}\right\}}$$

$$\boxed{T(S) = \frac{\alpha(1+ST)}{1+S\alpha T}} \qquad \frac{1}{\alpha} > 1 \qquad \boxed{\alpha = \frac{R_2}{R_1+R_2} < 1}$$

Pole, $S = -\frac{1}{\alpha T}$

Zero, $S = -\frac{1}{T}$



7.1.1. Zero Dominant Compensator

Phase $\phi = \tan^{-1}(\omega T) - \tan^{-1}(\alpha\omega T)$

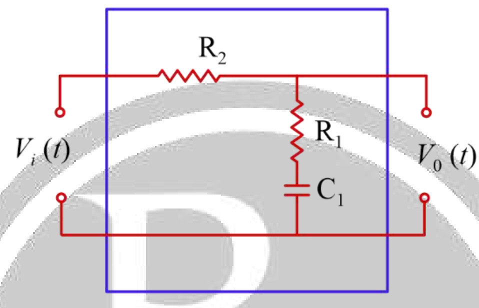
Max. value of phase, $\frac{d}{d\omega}\phi(\omega) = 0$

Max $\leftarrow$ $\boxed{\omega = \frac{1}{T\sqrt{\alpha}}}$ $\qquad$ $\boxed{\tan\phi_{max} = \frac{1-\alpha}{2\sqrt{\alpha}}}$

$\boxed{\sin\phi_{max} = \frac{1-\alpha}{1+\alpha}}$

- Behave as HPF
- Decrease gain of system
- Increase steady state error
- Increase $\omega_{gc}, BW \uparrow$
- Increase P.M, improve relative stability
- Increase $\xi \nearrow$ Mr decreases, $\searrow$ % Mp decreases
- ω_n increases, t_s decreases
- Improve or reduces the transient region
- Increase the speed of system

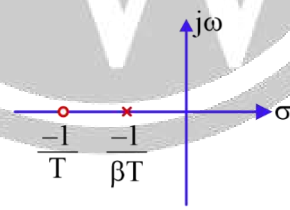
Phase log Compensator



$$T(S)=\frac{1+SR_1C_1}{1+SC_1(R_1+R_2)}$$

$$T(S)=\left(\frac{1+ST}{1+\beta ST}\right)$$

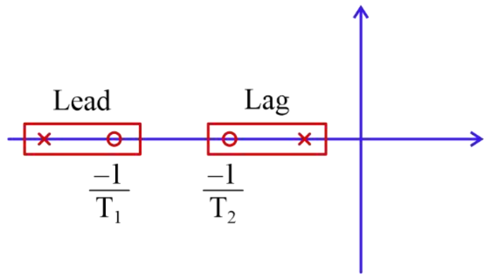
$$S=\frac{-1}{\beta T}, \frac{-1}{T}, \beta=\frac{R_1+R_2}{R_1}>1, T=R_1C_1$$



- Pole dominant
- $\phi=\tan^{-1}\omega T-\tan^{-1}\beta\omega T$
- $\omega_m=\frac{1}{T\sqrt{\beta}}$
- $\sin\phi_m=\left(\frac{1-\beta}{1+\beta}\right)\beta>1$
- LPF
- Gain remains constant
- β → $= 1$ e_{ss} same; → ≥ 1 e_{ss} reduces
- Reduces $\omega_{gc} \rightarrow$ B.W reduced

- Reduces P.M → Relative stability decrease
- $\xi \downarrow \rightarrow M_p \uparrow$ and $M_r \uparrow$
- $\xi \downarrow, \omega_n \downarrow \rightarrow t_s \uparrow$
- Increases transient region, speed of operation decreases.

Lead – Log Compensator



$$T(S) = \frac{\alpha(1+ST_1)}{(1+\alpha ST_2)} \cdot \frac{(1+ST_1)}{(1+\beta ST_2)}$$

$$T_1 = R_1C_2 \qquad T_2 = R_1'C_1'$$

$$\alpha = \frac{R_2}{R_1+R_2} \qquad \beta = \frac{R_1'+R_2'}{R_1'}$$

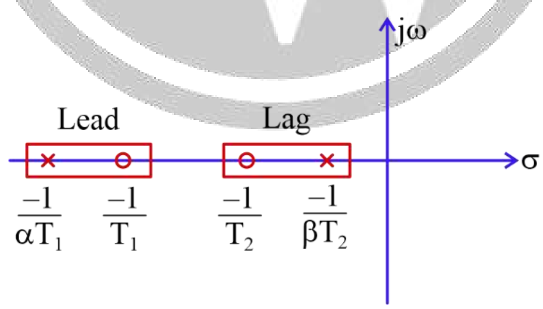
$\boxed{T_1 > T_2}$

➢ B.P.F

LAG – LEAD Compensator

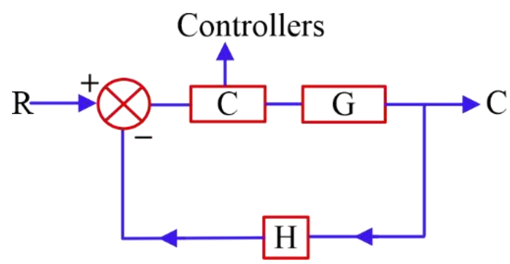
$$T(S) = \left\{\frac{(1+ST)}{(1+\beta ST)}\right\} \times \left\{\frac{\alpha(1+ST)}{(1+\alpha ST)}\right\} \qquad \frac{1}{T_1} > \frac{1}{T_2}$$

$\boxed{T_2 > T_1}_{\text{log}}$

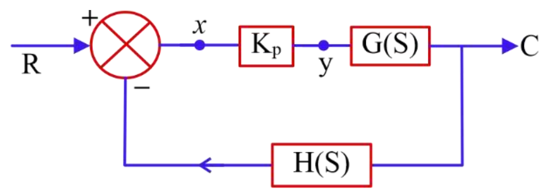


- Band Reject filter

Controllers



7.2. Proportional Controller



T.F of controller: $\frac{Y(S)}{X(S)} = K_P$

$$H(S) = 1$$

$$G(S) = \frac{\omega_n^2}{S(S + 2\xi\omega_n)}$$

$$\omega_n' = \omega_n\sqrt{K_p}, \xi' = \xi/\sqrt{K_p}$$

$$e_{ss} = \frac{A}{K_p}\left(\frac{2\xi}{\omega_n}\right)$$

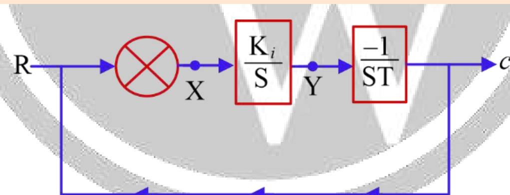
Effects

(1) e_{ss} reduces if $K_P > 1$

(2) $\xi\omega_n =$ constont, $t_s =$ constant, stability same

(3) $\xi\downarrow, \%M_P\uparrow, \omega_d\uparrow, t_r\downarrow$

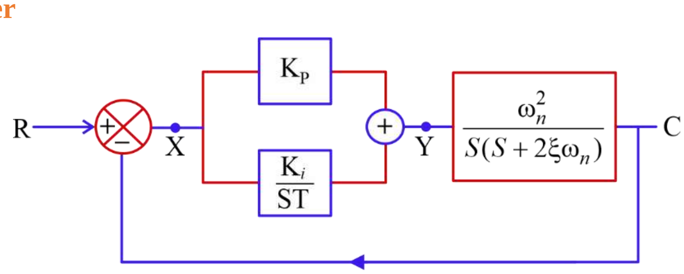
7.2.1. Integral Controller 1st order



$$T.F = \frac{Y(S)}{X(S)} = \frac{K_i}{S}$$

- Increases type of system by 1
- e_{ss} for same input becomes 0.
- It makes $\rightarrow$ $\nearrow$ 1^{st} order CLS to M.S; $\searrow$ 2^{nd} order CLS to unstable

Proportional Integral Controller



$$\frac{Y(S)}{X(S)} = K_P + \frac{K_i}{S}$$

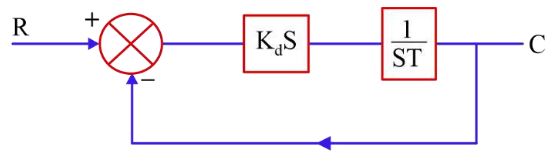
If $K_P = 1$ $\qquad \dfrac{Y(S)}{X(S)} = 1 + \dfrac{K_i}{S}$

$\rightarrow \xi\omega_n \downarrow \rightarrow t_s \uparrow \rightarrow$ No. of oscillations $\uparrow$

$\downarrow$

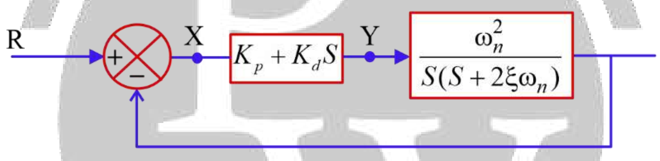
Sluggish $\leftarrow$ Transiend region becomes pronounced

Derivative Controller



- Derivative Controller reduces type of system by 1
- $e_{ss} \uparrow$ for same input
- Transient region reduced

Proportional Derivative Controller (P –D)



$$\frac{Y}{X} = K_P + K_dS \xrightarrow{K_P=1} 1 + K_dS$$

- It reduces $\%M_P, t_s, t_r$
- It improves: relative stability and transient region

PID Controller

$$\text{Transfer function } = K_P + \frac{K_i}{S} + SK_D$$

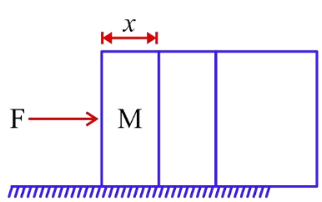
- Improves stability and decreases e_{ss}
- Increases type and decreases e_{ss}

Mathematical Modelling

Mechanical system $\longleftrightarrow$ Electrical system

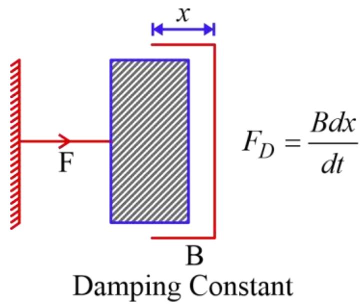
Translational System (Mass Damper System)

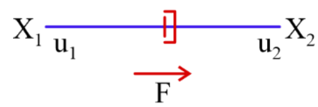
(1) Mass



$$F_M = M\frac{d^2x}{dt^2}$$

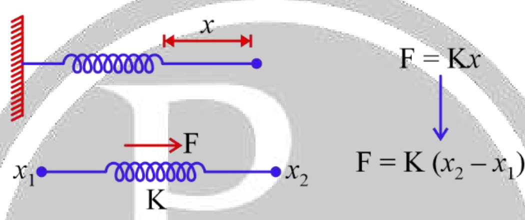
(2) Damper





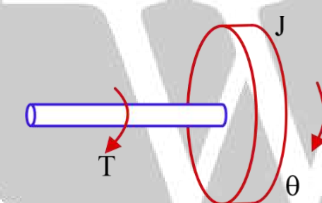
$$F = B\frac{d}{dt}(x_2 - x_1) = B(v_2 - v_1)$$

(3) Spring



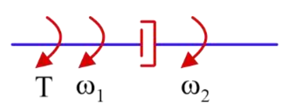
Rotational System

(1) Inertia



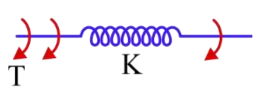
$\boxed{T = J\frac{d^2\theta}{dt^2}}$ J : Moment of Inertia

(2) Damper



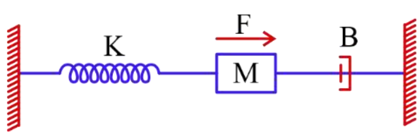
$$\boxed{T = B(\omega_2 - \omega_1) = B\left(\frac{d\theta_2}{dt} - \frac{d\theta_1}{dt}\right)}$$

(3) Spring

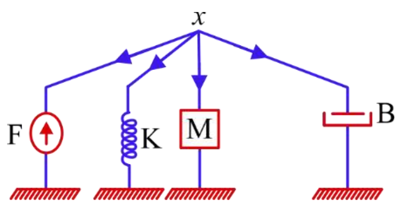


Force voltage – force Current

Given



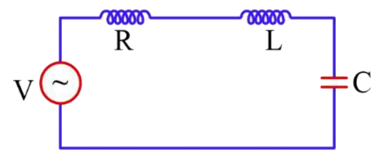
Network



$$\boxed{F = Kx + M\frac{d^2x}{dt^2} + B\frac{dx}{dt}}$$

Force voltage Analogy

$$I = \frac{dQ}{dt}$$

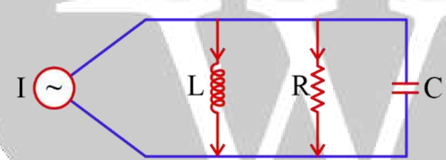


$$V = \frac{Ld^2Q}{dt^2} + \frac{RdQ}{dt} + \frac{1}{C}Q$$

$$F = \frac{Md^2x}{dt^2} + \frac{Bdx}{dt} + K_x$$

$$\left|\begin{array}{l} F \to V \\ M \to L \\ B \to R \end{array}\right| \quad \begin{array}{l} x \to Q \\ K \to \dfrac{1}{C} \end{array} \quad v \longleftrightarrow I$$

Force current Analogy



$$V = \frac{dQ}{dt}$$

$$\boxed{I = C\frac{d^2Q}{dt^2} + \frac{1}{R}\frac{dQ}{st} + \frac{1}{L}Q}$$

$F \to I$

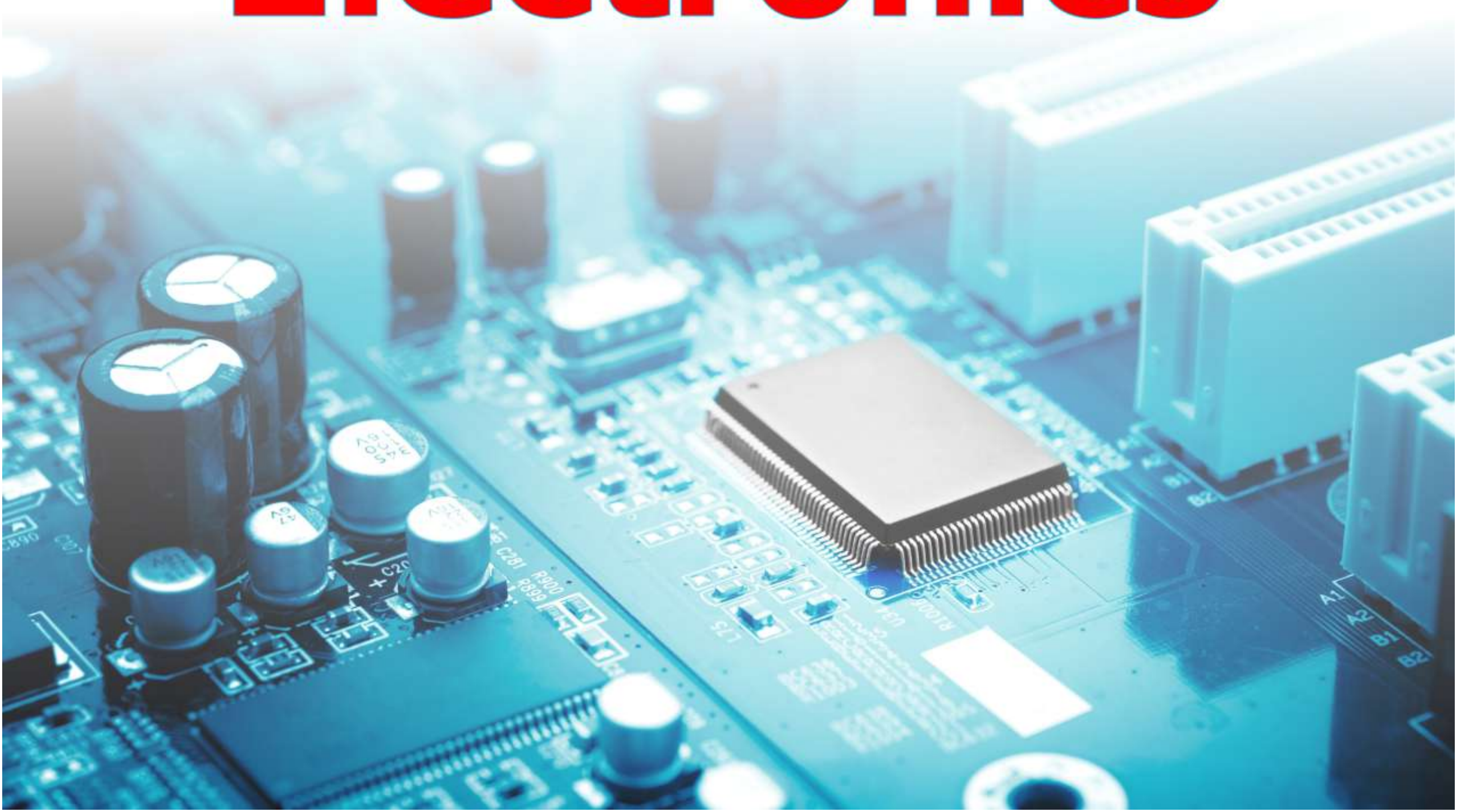
$M \to C$

$B \to 1/R$ Voltage $\to$ Velocity

$K \to 1/L$

$x \to Q$

Digital Electronics



DIGITAL ELECTRONICS

INDEX

1 LOGIC GATE

1.1. Logic Operations

In Boolean algebra, all the algebraic functions performed is logical. The AND, OR and NOT are the basic operations that are performed in Boolean algebra. There are some derived operations such as NAND, NOR, EX-OR, EX-NOR that are also performed in Boolean algebra.

1.1. NOT Operation

Symbol:

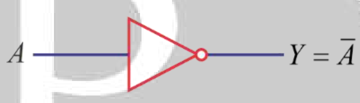


Fig. 1.1.

$A \xrightarrow{\text{NOT}} \bar{A}$ or A' (Complementation law)

and $\bar{\bar{A}} = A \Rightarrow$ Double complementation law

Truth table for NOT operation

Input *A*	Output $Y = \bar{A}$
0	1
1	0

A NOT gate can be represented using switch whose circuit representation is shown in figure below.

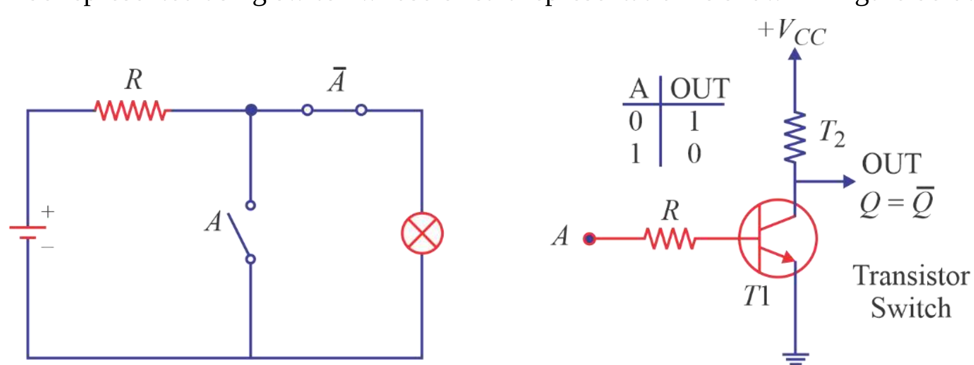
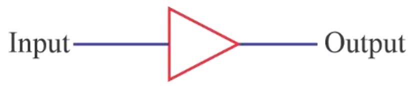


Fig. 1.2.

A buffer is a basic logic gate that passes its input, unchanged, to its output. Its behaviour is the opposite of a NOT gate.

The main purpose of a buffer is to regenerate the input, usually using a strong high and a strong low. Buffers are also used to increase the propagation delay of circuits by driving the large capacitive loads.

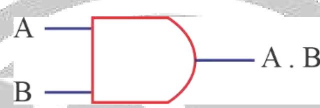
"Buffer" gate



Input	Output
0	0
1	1

1.1.2. AND Operation

Symbol:



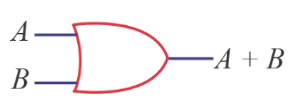
$A.A = A, A.0 = 0, A.1 = A, A\bar{A} = 0$

Truth table for AND operation:

Input		Output
A	**B**	**Y = AB**
0	0	0
0	1	0
1	0	0
1	1	1

1.1.3. OR Operation

Symbol:



$A + A = A, A + 0 = A, A + 1 = 1, \ A + \bar{A} = 1$

Truth table for OR operation:

Input		Output
A	**B**	**Y = A+B**
0	0	0
0	1	1
1	0	1
1	1	1

Example: Reduce the combinational logic circuit shown figure such that the desired output can be obtained using only one gate.

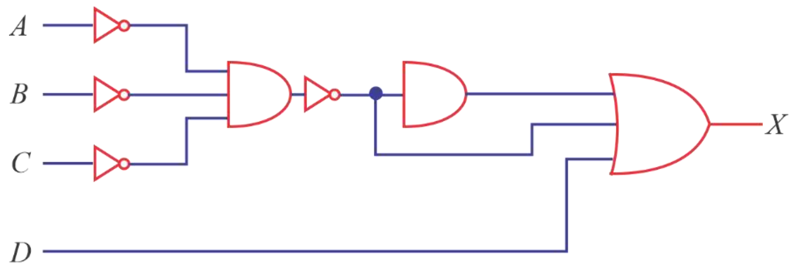


Fig. 1.3.

Solution:

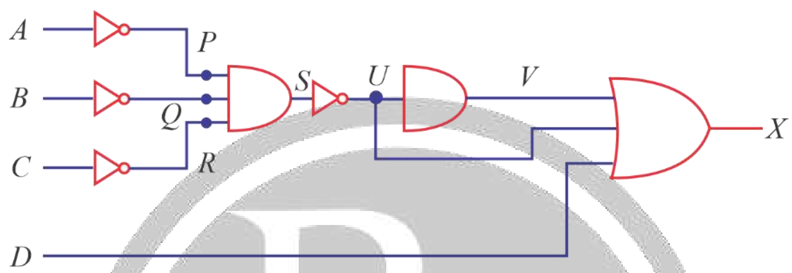


Fig. 1.4.

$$P = \bar{A},\ Q = \bar{B},\ R = \bar{C}$$

$$S = \bar{A} \cdot \bar{B} \cdot \bar{C}$$

$$V = U = \overline{\bar{A}\bar{B}\bar{C}}$$

$$\begin{aligned} X &= U + V + D \\ &= \overline{\bar{A}\bar{B}\bar{C}} + \overline{\bar{A}\bar{B}\bar{C}} + D \\ &= A + B + C + D \end{aligned}$$

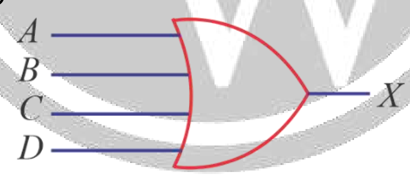


Fig. 1.5.

Enable or Disable Input:

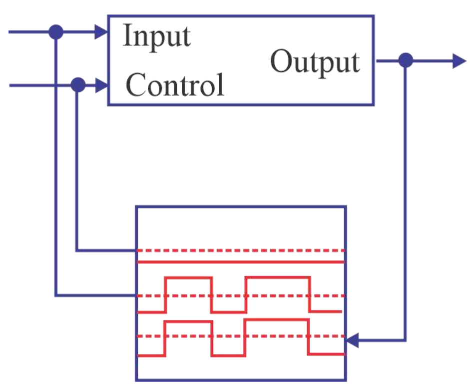


Fig. 1.6.

Enable:

- Allow a signal to pass when the control signal is HIGH.
- Prevent a signal from passing when the control signal is LOW.

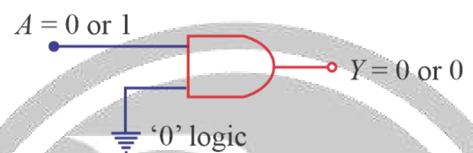
Disable:

- Prevent a signal from passing when the control signal is HIGH.
- Allow a signal to pass when the control signal is LOW.
- Enable and Disable Functions:
- AND and OR gates can both be used to enable or disable a transmitted waveform.

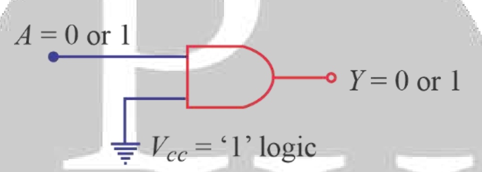
For a two input AND gate:

For a two input OR gate:

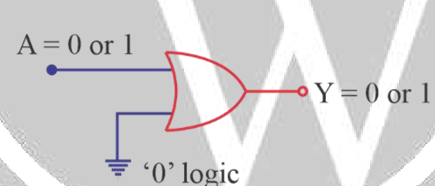
- **Control '0' disable**



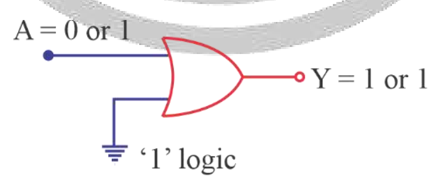
- **Control '1' enable (Buffer)**

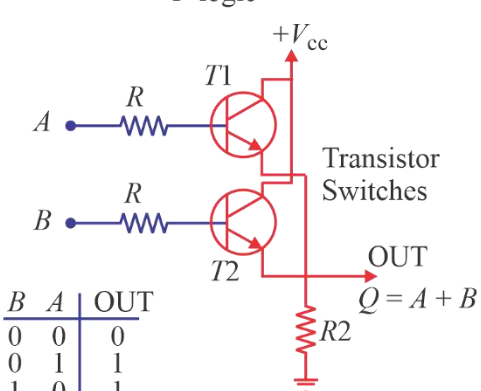


- **Control '0' enable (Buffer)**



- **Control '1' Always enable**





B	A	OUT
0	0	0
0	1	1
1	0	1
1	1	1

1.1.4. Switch Diagram for AND/OR gate

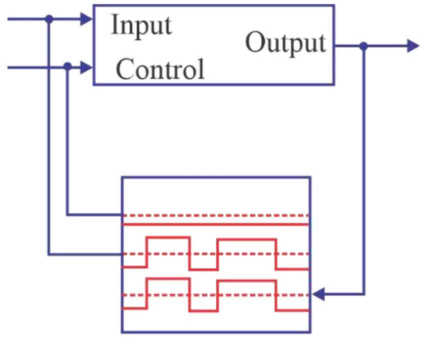


Fig. 1.7.

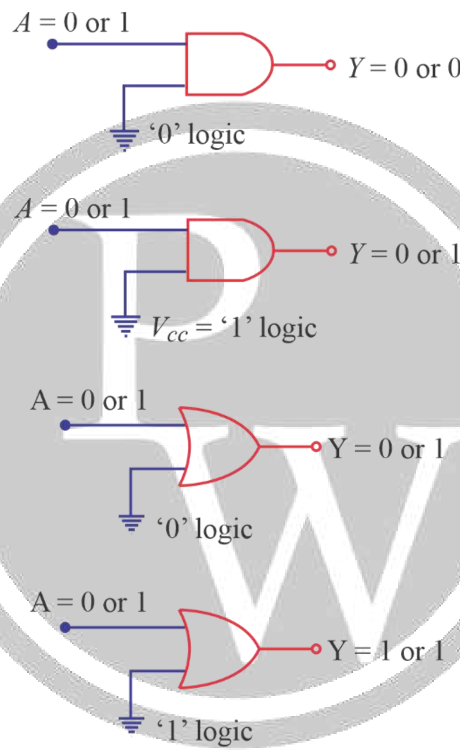


Fig. 1.8.

1.1.5. Basic law applications in AND/OR gate.

(a) Commutative Law:

The commutative law allows change in position of AND or OR variables. There are two commutative laws.

$$A + B = B + A$$

$$A \times B = B \times A$$

(b) Associative Law:

$$(A + B) + C = A + (B + C)$$

$$(A \times B) \times C = A \times (B \times C)$$

1.1.6. Circuit Diagram for AND/OR gate.

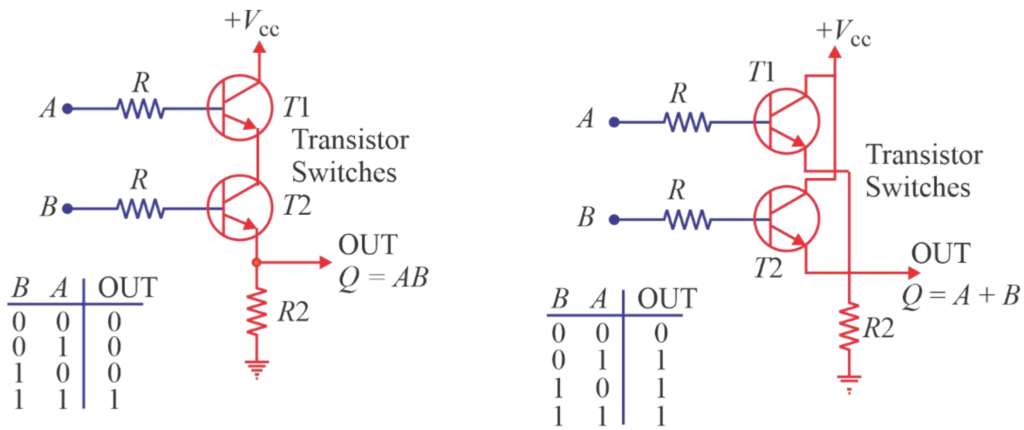


Fig. 1.9.

1.1.7. Switch Diagram for AND/OR Gate

The circuit shown below shows the switch representation of AND gate which is basically the series connection of switches A and B.

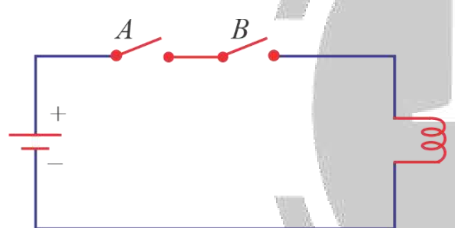


Fig. 1.10.

The circuit shown below shows the switch representation of OR gate which is basically the parallel connection of switches A and B.

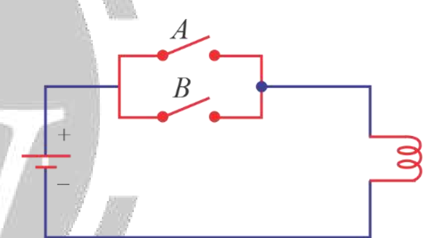
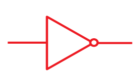
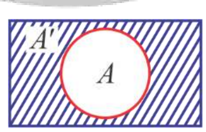
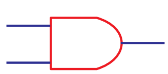
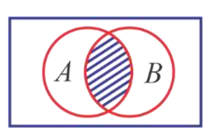
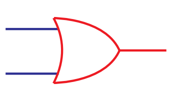
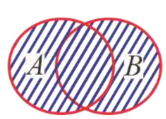


Fig. 1.11.

1.1.8. Venn Diagram:

NOT			$\bar{A}$	A 0 1		Output 1 0
AND			A.B	A 0 0 1 1	B 0 1 0 1	Output 0 0 0 1
OR			A + B	A 0 0 1 1	B 0 1 0 1	Output 0 1 1 1

1.2. Logic Gates

Logic gates are the fundamental building blocks of digital systems.

Types of logic gates: There are three basic logic gates, namely

- OR gate
- AND gate
- NOT gate

And other logic gates that are derived from these basic gates are:

- NAND gate
- NOR gate
- Exclusive OR gate
- Exclusive NOR gate

1.2.1. NAND gate:

The term NAND gate equivalent to AND gate followed by a NOT gate, implies NOT-AND

Symbol:

Fig. 1.12.

Truth table of 2-input NAND gate.

Input		Output
A	B	$Y = \overline{AB}$
0	0	1
0	1	1
1	0	1
1	1	0

Switching and Circuit Diagram for NAND gate.

B	A	OUT
0	0	1
0	1	1
1	0	1
1	1	0

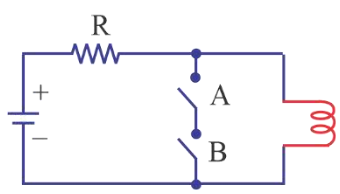


Fig. 1.13.

NAND gate acts as Universal Gate

Logic Gates using only NAND Gates

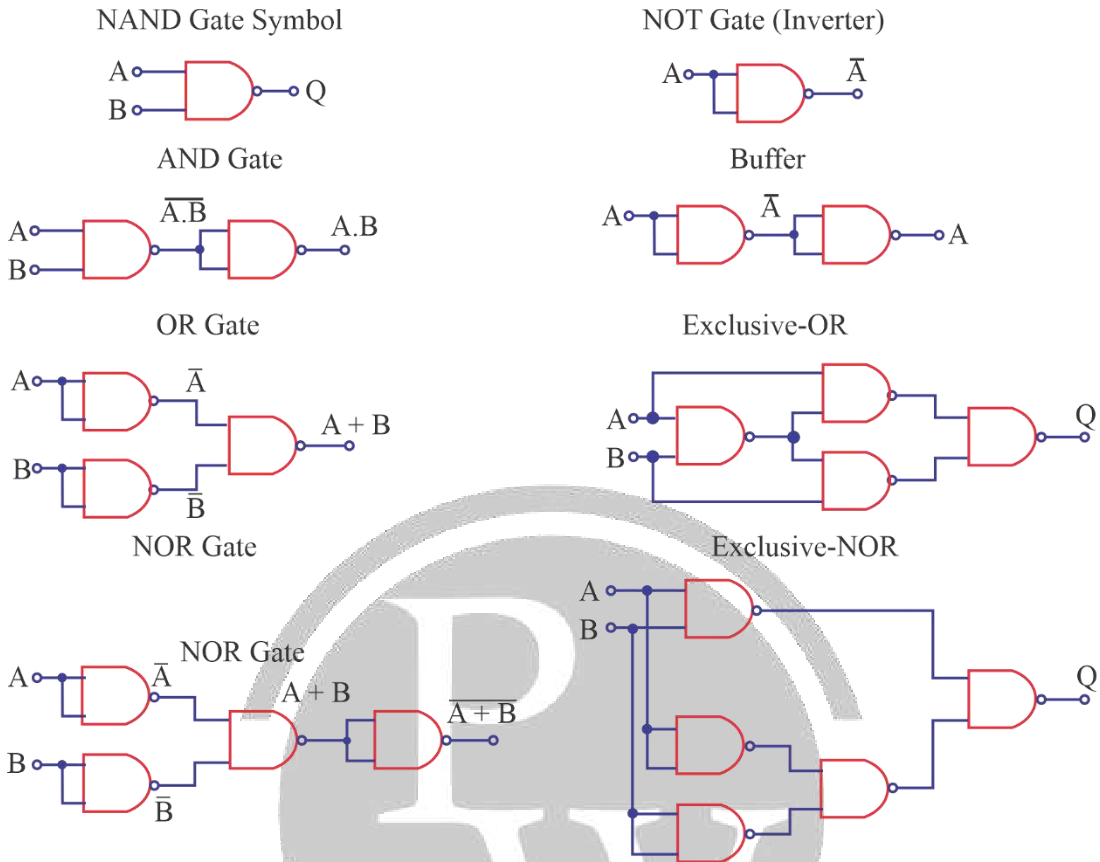


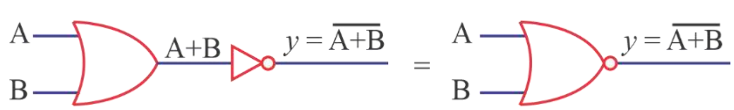
Fig. 1.14.

All the logic gate functions can be created using only NAND gates. Therefore, it is also known as a Universal logic gate.

1.2.2. NOR Gate

A NOR gate is equivalent to OR gate followed by a NOT gate.

Symbol:

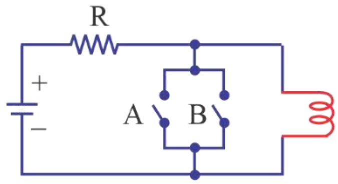


Truth Table for 2-input NOR gate

Input		Output
A	B	$Y = \overline{A + B}$
0	0	1
0	1	0
1	0	0
1	1	0

Switching and Circuit Diagram for NOR gate

B	A	OUT
0	0	1
0	1	0
1	0	0
1	1	0



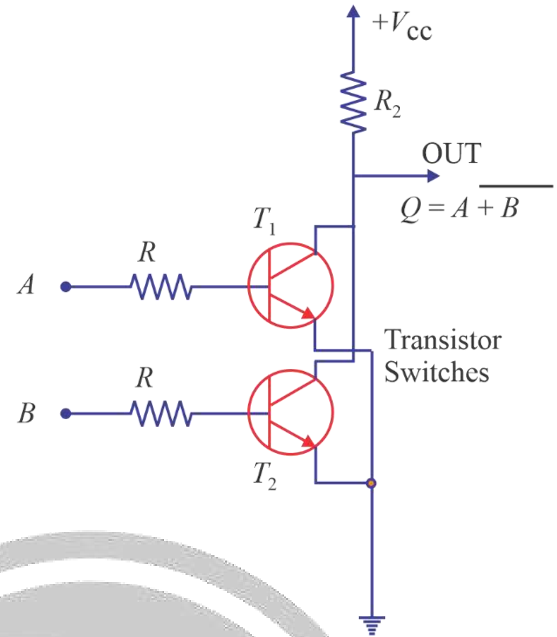


Fig. 1.15.

NOR gate acts as Universal Gate.

Logic Gates using only NOR Gates

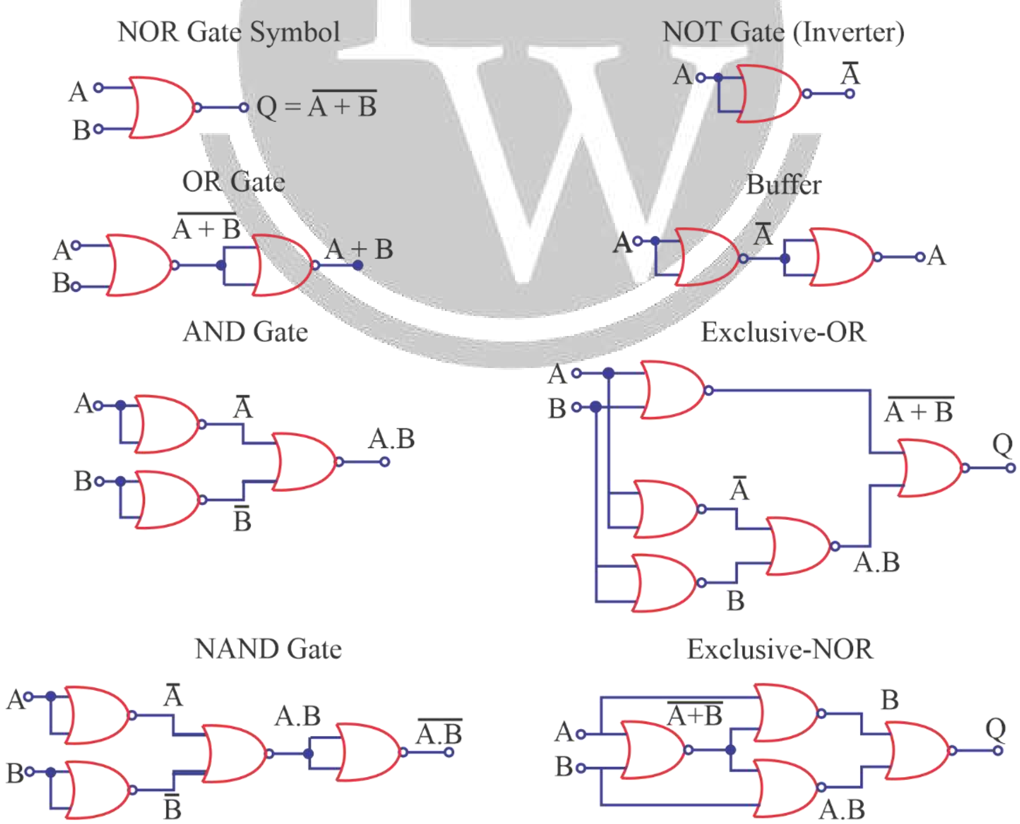
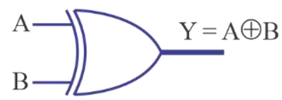


Fig. 1.16.

All the logic gate functions can be created using only NOR gates. Therefore, it is also known as a Universal logic gate.

1.2.3. XOR Gate

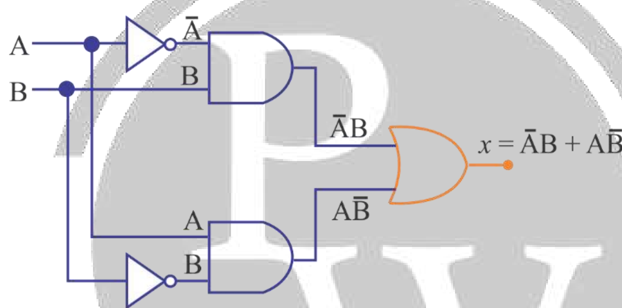
Symbol of two input XOR gate



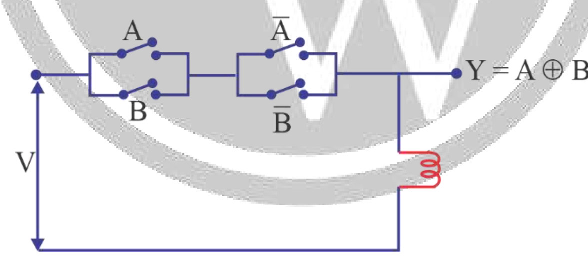
Truth table for 2-input XOR gate

Input		Output
A	**B**	**Y = A ⊕ B**
0	0	0
0	1	1
1	0	1
1	1	0

XOR gate using AND, OR and NOT gate



Switching diagram of XOR gate



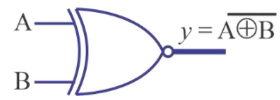
Truth Table:

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	0

$$Y = (A + B)(\bar{A} + \bar{B})$$
$$= \bar{A}B + A\bar{B}$$
$$= A \oplus B$$

1.2.4. X-NOR gate:

Symbol for two input X-NOR gate



Truth table for 2-input X-NOR gate

Input		Output
A	B	Y = A ⊙ B
0	0	1
0	1	0
1	0	0
1	1	1

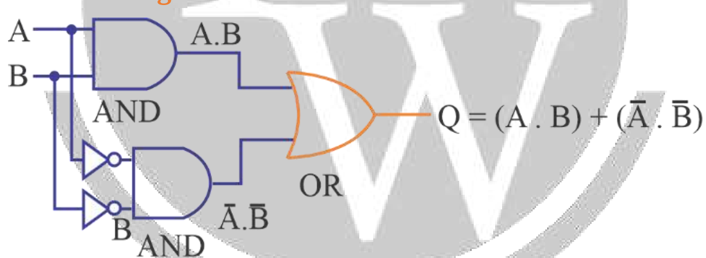
Boolean expression for EX-NOR gate is $Y = \overline{A \oplus B}$

Apply De-Morgan's theorem:

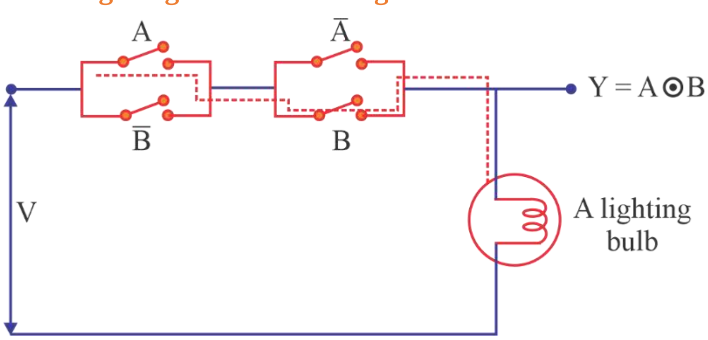
$$\overline{A \oplus B} = \overline{\overline{A}B + A\overline{B}} = \overline{\overline{A}B} \cdot \overline{A\overline{B}} = \left(A + \overline{B}\right)\left(\overline{A} + B\right) = AB + \overline{A}\,\overline{B}$$

The output of a two input EX-NOR gate is logic '1' when the inputs are same and a logic '0' when they are different.

X-NOR gate using AND OR and NOT gate



Switching Diagram of X-NOR gate

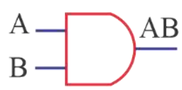
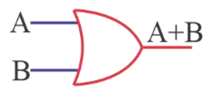
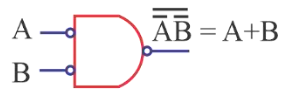
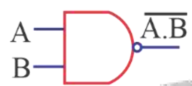
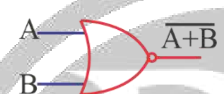


Truth Table:

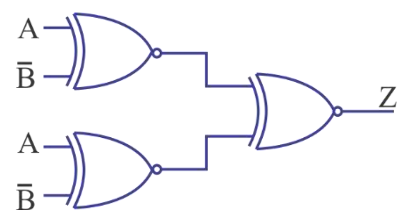
A	B	Y
0	0	1
0	1	0
1	0	0
1	1	1

$$Y = (A + \overline{B})(\overline{A} + \overline{B})$$
$$= AB + \overline{A}\overline{B}$$
$$= A \oplus B$$

1.3. Alternate Logic Gate Representation

Logic	Normal symbol	Alternate symbol
NOT	 A, $\bar{A}$	 A, $\bar{A}$
AND	 A, B, AB	 A, B, $\overline{\bar{A}+\bar{B}}=AB$
OR	 A, B, A+B	 A, B, $\overline{\bar{A}\bar{B}} = A+B$
NAND	 A, B, $\overline{A.B}$	 A, B, $\bar{A}+\bar{B} = \overline{A.B}$
NOR	 A, B, $\overline{A+B}$	 A, B, $\bar{A}.\bar{B} = \overline{A+B}$

Example: In the following circuit, the find the output Z?

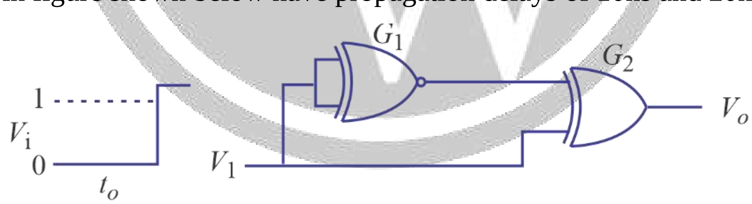


Solution: From the given circuit, we can observe that input to last XNOR is same, so, the XNOR output is given by (let input is X)

$$Z = X.X + \bar{X}.\bar{X} = X + \bar{X} = 1$$

i.e. the output will be high [logic 1] irrespective of the inputs A and B.

Example: The gate G_1 and G_2 in figure shown below have propagation delays of 10ns and 20ns respectively.

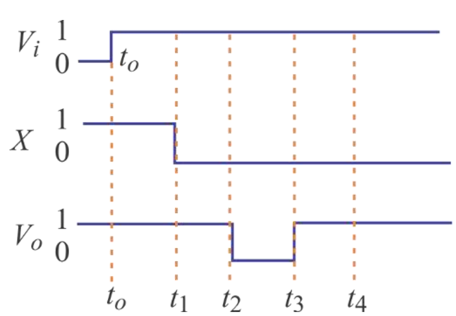


If input V_i makes an abrupt change from logic 0 to 1 at t = t_0, then find the output waveform V_0?

Here, $t_1 = t_0 + 10$ ns, $t_2 = t_1 + 10$ns, $t_3 = t_2 + 10$ ns.

Solution: Let the output of $G_1 = X$

The output waveform will be as shown in figure below.



2 Minimization of Boolean Function

2.1. Boolean Algebra

Boolean algebra is a system of mathematical logic. It is an algebraic system consisting on the set of elements (0, 1) two binary operators called OR, AND and one unary operator NOT. It is the basic mathematical tools in the analysis and the synthesis of switching circuits. It is a way to express logic functions algebraically.

Note: Any functional relation in Boolean algebra can be provided by the method of perfect induction perfect inductions the method of proof, where by a function relation is verified for every possible combination of values that the value may assume.

Axioms of Boolean Algebra:

Axioms of Boolean algebra are a set of logical expressions that we accept without proof and upon which we can build a set of useful theorem.

	AND operator	OR operator	NOT operators
Axioms 1:	0.0 = 0	0 + 0 = 0	$\bar{1} = 0$
Axioms 2:	0.1 = 0	0 + 1 = 1	$\bar{0} = 1$
Axioms 3:	1.0 = 0	1 + 0 = 1	
Axioms 4:	1.1 = 1	1 + 1 = 1	

Logic operations: In Boolean Algebra all the algebraic function performed is logical. These actually represents logic operations. The AND, OR and NOT are the basic operations that are performed in Boolean Algebra. In addition to these operations, there are some derived operations such as NAND, NOR, EX-OR and EX-NOR that are also performed in Boolean Algebra.

1.1.1. NOT Operation

The NOT operation in Boolean algebra is similar to inversion in ordinary algebra

1 : $\bar{0} = 1$

2 : $\bar{1} = 0$

3 : if A = 0 then $\bar{A} = 1$

4 : $\bar{\bar{A}}$ = A (Double inversion)

1.1.2. AND Operation

It is a logical operation that are performed by AND gate. The AND operation in Boolean Algebra is similar to multiplication in ordinary algebra.

1: $A \cdot 0 = 0$ (Null Law)

2: $A \cdot 1 = A$ (Identity law)
3: $A \cdot A = A$
4: $A \cdot \bar{A} = 0$

1.1.3. OR Operation

It is the logical operation that are performed by OR gate. The OR operation in Boolean Algebra is similar to addition in ordinary algebra.

1: A + 0 = A (Null law)
2: A + 1 = 1 (Identity law)
3: A + A = A
4: $A + \bar{A} = 1$

1.1.4. NAND Operation:

The NAND operation in Boolean Algebra is performed by AND operation followed by NOT operation i.e., the negation of AND operation is performed by NAND gate.

1.1.5. NOR Operation:

The NOR operation in Boolean Algebra is performed by OR operation followed by NOT operation i.e., the negation of OR operation is performed by NOR gate.

2.2. Laws of Boolean Algebra

2.2.1. Commutative Law

1. A + B = B + A

   A + B + C = B + C + A = C + A + B = B + A + C

2. AB = BA

   $A \cdot BC = B \cdot CA = C \cdot AB = B \cdot AC$

Violation: Inhibition (1) for Example x/y (x but not y) is not commutative law it means $x/y \neq y/x$

2.2.2. Associative law:

This law arrows grouping of variables

1. (A + B) + C = A + (B + C)

   A + (B + C + D) = (A + B + C) + D

   = (A + B) + (C + D)

2. $(A \cdot B)C = A \cdot (B \cdot C)$

   $A\,(BCD) = (ABC) \cdot D$

   $A\,(BCD) = AB \cdot CD$

   Note:- NAND and NOR gates are not Associative

2.2.3. Distributive Law

1: A(B + C) = AB + AC

A + BC = (A + B) (A + C)

2. 2.4. Redundant Literal Rule

1. $A + \bar{A}B = A + B$
2. $A(\bar{A} + B) = AB$

2.2.5. Idempotent Law

1. $A \cdot A = A$
2. $A + A = A$

2.2.6. Absorption Law

1. $A + AB = A$
2. $A(A + B) = A$

2.2.7. Involutionary Law

The law that for any variable A.

$\bar{\bar{A}} = (A')' = A$

2.2.8. Consensus theorem:

There are two consensus theorems

$$AB + \bar{A}C + BC = AB + \bar{A}C$$

$$(A + B)(\bar{A} + C)(B + C) = (A + B)(\bar{A} + C)$$

2.3. De-Morgan's theorem:

De-Morgan's theorem represents two of the most important rules of Boolean algebra.

I. $\overline{A.B} = \bar{A} + \bar{B}$

II. $\overline{A+B} = \bar{A}.\bar{B}$

The above two laws can be extended for 'n' variables as,

$\overline{A_1.A_2.A_3... + A_n} = \bar{A}_1 + \bar{A}_2 + ...\bar{A}_n$ and $\overline{A_1 + A_2 + ... + A_n} = \bar{A}_1.\bar{A}_2...\bar{A}_n$

2.3.1 Duality theorem:

Duality Theorem states that,

(a) Change each OR sign by an AND sign and vice versa.

(b) Compliment any '0' or '1' appearing in expression

(c) Keep literals as it is.

Note: With n variables, maximum possible distinct logic function = 2^{2^n}

Example : If a function is given as f = $AB + \bar{A}\bar{B}$ then find its complement.

Solution : Given $f = \left(AB + \bar{A}\bar{B}\right)$

$$\text{Complement of } \bar{f} = \overline{AB + \bar{A}\bar{B}} = \overline{AB} \cdot \overline{\bar{A}\bar{B}}$$

$$= \left(\bar{A} + \bar{B}\right)\left(A + B\right)$$

$$= A\bar{A} + A\bar{B} + B\bar{A} + B\bar{B} = A\bar{B} + \bar{A}B$$

Example : Show that

$$AB + B\bar{C} + AC = AC + B\bar{C}$$

Solution : LHS = $AB + B\bar{C} + AC$

$$= AB\left(C + \bar{C}\right) + B\bar{C}\left(A + \bar{A}\right) + A\left(B + \bar{B}\right)C$$

$$= ABC + AB\bar{C} + AB\bar{C} + \bar{A}B\bar{C} + ABC + A\bar{B}C$$

$$= ABC + AB\bar{C} + \bar{A}B\bar{C} + A\bar{B}C$$

$$= AC\left(B + \bar{B}\right) + B\bar{C}\left(A + \bar{A}\right) = AC + B\bar{C}$$

= RHS

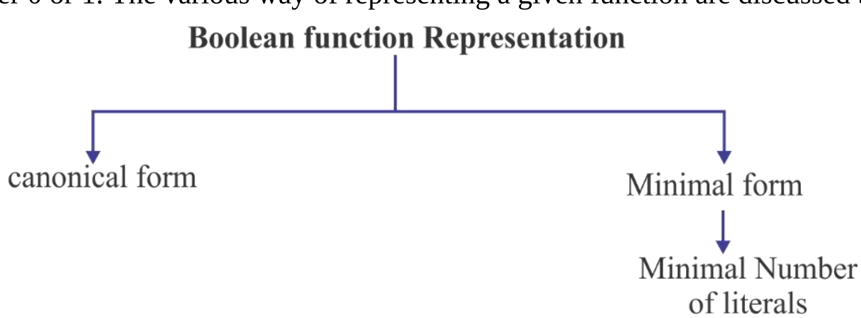
2.4. Minimization of Boolean function:

Every Boolean function expression must be reduced to as simple form as possible before realization because every logic operation in the expression represents a corresponding elements of hardware. Realization of digital circuit with minimal expression has several advantages as:

1. The number of logic gates will reduced.
2. The speed of operation will increase
3. power dissipation will decrease
4. The FAN IN may reduced
5. The complexing of the circuit reduces

The simple method of minimization of Boolean function using certain Algebraic rules which results in the reduction of number of term and/or number of arithmetic operations the various theorem and rules that are already discussed are very useful for the simplification of Boolean expression.

A function of n Boolean variables denoted by f(A_1, A_2.... A_n) is another variable of Algebra and takes one of the two possible values either 0 or 1. The various way of representing a given function are discussed below.



All the terms contain all the variable either in complementary or in uncomplimentary form

The literal means the Binary variable either in complementary or in uncomplimentary form.

2.4.1. Minimization of Boolean function using k-map

- **Using K-map:** The Boolean function can be simplified Algebraically but being not a symmetric method, we can never be sure that whether the minimal expression obtained is the real minimal or not.
- **Karnaugh Map (k-map):** A k-map is a graphical representation of Boolean expression, A two variable k-map will have four cell or squares 3-variable k-map will have 8-cells, n-variable k-map will have 2^n cells.

Note: Adjacent cells differ by 1 bit to maintain adjacently property gray code sequence is used in k-maps (Any two adjacent cells will differ by only one bit)

Min terms & Max terms:

1. n-binary variable have 2^n possible combinations.
2. Min term is a product term, it contains all the variables either complementary or un complementary form for that combination the function output must be ‘1’.

- Max term is a sum term, it contains all the variables either complementary or uncomplimentary form for that combination the function output must be ‘0’.
  For two variable

x	y	Min terms	Max terms
0	0	$m_0 = \bar{x}\,\bar{y}$	$M_0 = x + y$
0	1	$m_1 = \bar{x}\,y$	$M_1 = x + \bar{y}$
1	0	$m_2 = x\,\bar{y}$	$M_2 = \bar{x} + y$
1	1	$m_3 = x\,y$	$M_3 = \bar{x} + \bar{y}$

- In min terms we assigns ‘1’ to each uncomplemented variables and ‘0’ to each complemented variable.
- In Max terms we assign ‘0’ to each uncomplemented variable and ‘1’ to each complemented variables.

2.5. Representation of Boolean Functions

Any Boolean expression can be expressed in two forms

- Sum of Product form (SOP)
- Product of Sum form (POS)

2.5.1. SOP Form

The SOP expression usually takes the forms of two or more variables OR together.

$$Y = \bar{A}BC + A\bar{B} + AC$$

$$Y = A\bar{B} + B\bar{C}$$

SOP forms are used to write logical expression for the output becoming logic '1'.

Example:

Input (3-variables)			Output (Y)
A	B	C	Y
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

∴ Notation of SOP expression is:

$$f(A, B, C) = \Sigma m(3, 5, 6, 7)$$

$$Y = m_3 + m_5 + m_6 + m_7$$

Also, $Y = \overline{A}BC + A\overline{B}C + AB\overline{C} + ABC$

2.5.2. POS Form

The POS expression usually takes the form of two or more OR variables within parentheses, ANDed with two or more such terms.

Example: $Y = \left(A+\overline{B}+C\right)\left(B\overline{C}+D\right)$

Each individual term in standard POS form is called maxterm.

POS forms are used to write logical expression for output be coming logic '0'.

we get f(A, B, C) = $\pi M(0, 1, 2, 4)$

$$Y = M_0.M_1.M_2.M_4$$

$$Y = \left(A + B + C\right)\left(A + B + \overline{C}\right)\left(A + \overline{B} + C\right)\left(\overline{A} + B + C\right)$$

∴ We can also conclude from Table 2 and from above equations:

if $Y = \Sigma m(3, 5, 6, 7)$ or $Y = \pi M(0, 1, 2, 4)$

2.6. Karnaugh Map (K-MAP)

The K-map is a graphical method which provides a systematic method for simplifying the Boolean expressions. In n variable K-map, there are 2^n cells.

2.6.1. Two variable K-map

A \ B	$\bar{B}$ ← 0	B ← 1
$\bar{A}$ ← 0	m_0	m_1
A ← 1	m_2	m_3

For SOP

A \ B	B ← 0	$\bar{B}$ ← 1
A ← 0	M_0	M_1
$\bar{A}$ ← 1	M_2	M_3

For POS

2.6.2. Three variable K-map

A \ BC	($\bar{B}\bar{C}$) 00	($\bar{B}C$) 01	(BC) 11	($B\bar{C}$) 10
$\bar{A}$ ← 0	M_0	M_1	M_3	M_2
A ← 1	M_4	M_5	M_7	M_6

For SOP

A \ BC	(B + C) 00	($B+\bar{C}$) 01	($\bar{B}+\bar{C}$) 11	($\bar{B}+C$) 10
A ← 0				
$\bar{A}$ ← 1				

For POS

2.6.3. Four Variable K-map

AB \ CD	($\bar{C}\bar{D}$) 00	($\bar{C}D$) 01	(CD) 11	($C\bar{D}$) 10
($\bar{A}\bar{B}$) ← 00	m_0	m_1	m_3	m_2
($\bar{A}B$) ← 01	m_4	m_5	m_7	m_6
(AB) ← 11	m_{12}	m_{13}	m_{15}	m_{14}
($A\bar{B}$) ← 10	m_8	m_9	m_{11}	m_{10}

For SOP

AB \ AB	(C+D) 00	($C+\bar{D}$) 01	($\bar{C}+\bar{D}$) 11	($\bar{C}+D$) 10
(A+B) ← 00	M_0	M_1	M_3	M_2
($A+\bar{B}$) ← 01	M_4	M_5	M_7	M_6
($\bar{A}+\bar{B}$) ← 11	M_{12}	M_{13}	M_{15}	M_{14}
($\bar{A}$+B) ← 10	M_8	M_9	M_{11}	M_{10}

For POS

2.6.4. Five variable K-map

- 32 cells
- 32 Minterms (Maxterms)

Here, we have f(A, B, C, D, E)

For $\boxed{A = 0}$

BC \ DE	$\bar{D}\bar{E}$	$\bar{D}E$	DE	$D\bar{E}$
$\bar{B}\bar{C}$	m_0	m_1	m_3	m_2
$\bar{B}C$	m_4	m_5	m_7	m_6
BC	m_{12}	m_{13}	m_{15}	m_{14}
$B\bar{C}$	m_8	m_9	m_{11}	m_{10}

(From 0 - 16)

For $\boxed{A = 1}$

BC \ DE	$\bar{D}\bar{E}$	$\bar{D}E$	DE	$D\bar{E}$
$\bar{B}\bar{C}$	M_0	M_1	M_3	M_2
$\bar{B}C$	M_4	M_5	M_7	M_6
BC	M_{12}	M_{13}	M_{15}	M_{14}
$B\bar{C}$	M_8	M_9	M_{11}	M_{10}

(From 16 - 31)

(For Sop)

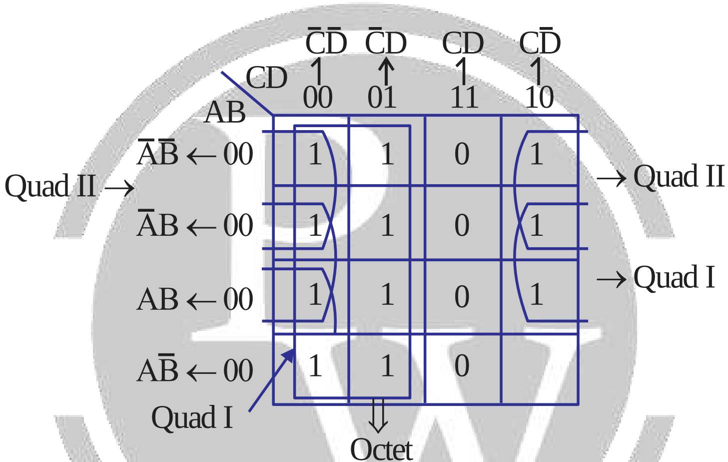
2.7. Simplification Rules

1. Construct the K-map and place 1's in those cells corresponding to the 1's in the truth table. Place the 0's in the other cells.
2. Examine the map for adjacent 1's and loop those 1's which are not adjacent to any other 1's. These are called isolated 1's.
3. Next, look for those 1's which are adjacent to only one other 1. Loop any pair containing such a 1.
4. Loop any octet even if it contains some 1's that have already been looped.
5. Loop any quad that contains one or more 1's which have not already been looped, making sure to use the minimum number of loops.
6. Form the OR sum of all the terms generated by each loop.

Example: Simply a four variable logic function using K-map

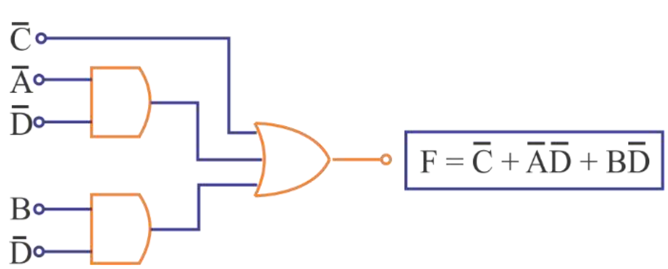
$f(A, B, C, D) = \Sigma m(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$ also implement the simplified expression with AND-OR logic.

Solution:



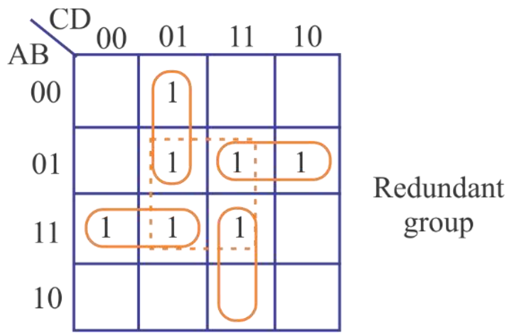
$$\therefore \quad f = \underset{\text{octet}}{\underset{\Downarrow}{\bar{C}}} + \underset{\text{quad - II}}{\underset{\Downarrow}{\bar{A}\bar{D}}} + \underset{\text{quad - I}}{\underset{\Downarrow}{B\bar{D}}}$$

⇒ Gate implementation:



2.8. Redundant Group

If all the 1's in a group are already involved in some other groups, then that group is caused as a redundant group. A redundant group has to be eliminated, because it increases the no of gates required.



2.9. Don't Care Condition

The combinations for which the values of the expression are not specified are called don't care conditions.

Example: Simply the given equation in part (i) and (ii)

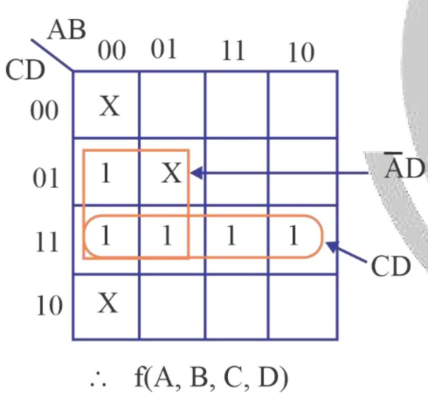
(i) In terms of SOP. and don't care conditions

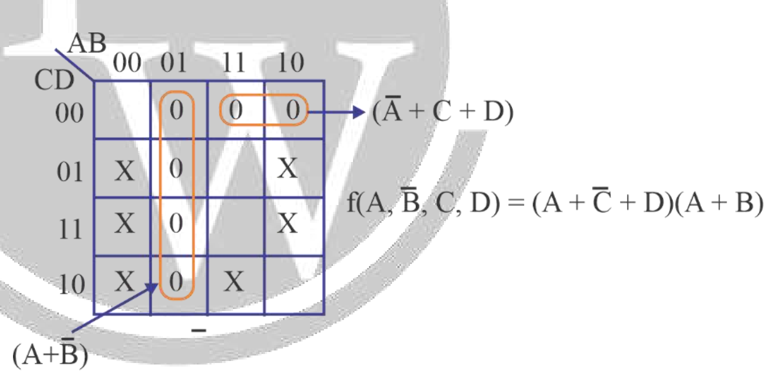
$f(A, B, C, D) = \Sigma m(1, 3, 7, 11, 15) + d(0, 2, 5)$

(ii) In terms of POS and don't care conditions.

$f(A, B, C, D) = \pi M(4, 5, 6, 7, 8, 12) + d(1, 2, 3, 9, 11, 14)$

Solution:





2.10. Implicants, Prime Implicants and Essential Prime Implicants

2.10.1. Implicants

Implicants is a product term on the given function for that combination the function output must be 1.

2.10.2. Prime Implicant (PI)

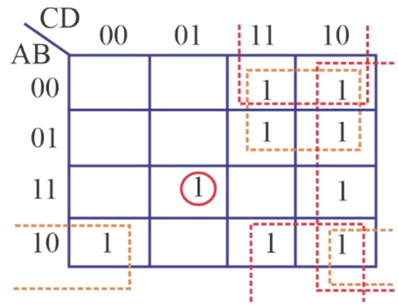
Prime implicant is a smallest possible product term of the given function,

2.10.3. Essential Prime Implicants (EPI)

EPI is a prime implicant it must cover at least one minterms, which is not covered by other PI.

Example: Reduce the expression using mapping $F = \Sigma m(2, 3, 6, 7, 8, 10, 11, 13, 14)$

Solution :

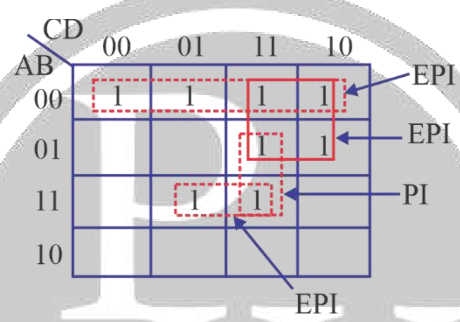


$$F = AB\bar{C}D + A\bar{B}\bar{D} + \bar{A}C + \bar{B}C + C\bar{D}$$

Example : Reduce the following expression using k-map and identify PI's and EPI

$F = \Sigma m(0, 1, 2, 3, 6, 7, 13, 15)$

Solution :



EPI = $\bar{A}\bar{B}$, $\bar{A}C$, ABD

PI = BCD

Minimal $F = \bar{A}\bar{B} + \bar{A}C + ABD$

Example: Given the following Karnaugh map, which one of the following represents the minimal sum of products of the map.

yz \ ωx	00	01	11	10
00	0	x	0	x
01	x	x	x	1
11	0	x	1	0
10	0	1	x	0

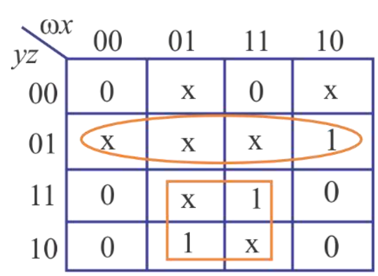
A. $xy + \bar{y}z$

B. $\omega \bar{x}\bar{y} + xy + xz$

C. $\bar{\omega}x + \bar{y}z + xy$

D. $xy + y$

Solution :



$= xy + \bar{y}z$

3 COMBINATIONAL CIRCUITS

3.1. Combinational Circuits

The combinational circuit has 'n' input variables and 'm' output variables. Since, the number of input variables is n, there are 2^n possible combinations of bits at the input. Each output can be expressed in terms of input variables by a Boolean expression.

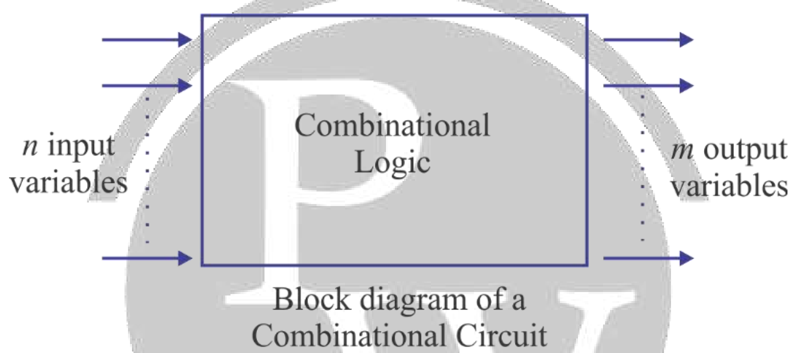


Fig. 3.1. Block diagram of a Combinational Circuit

3.2. Adders

The most basic arithmetic operation is the addition of two binary digits. A combinational circuit that performs the addition of two 1-bit numbers is called as half adder, and the logic circuit that adds three 1-bit numbers is called as full adder.

3.2.1. Half Adder

The logic circuit that performs the addition of two 1-bit numbers is called as half adder. It is the basic building block for addition of two single bit numbers. This circuit has two outputs namely carry (C) and sum (S).

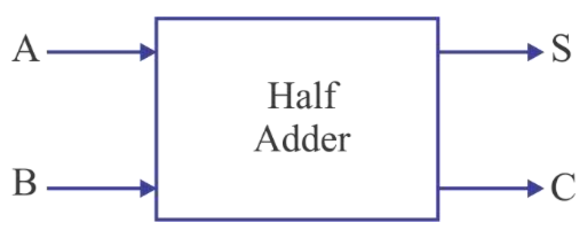


Fig. 3.2. Block Diagram of a 2-bit Half Adder

The truth table of half adder: where A and B are the inputs and sum and carry

Table 1: Truth Table of Half Adder

Inputs		Outputs	
A	**B**	**Sum (S)**	**Carry (C)**
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

K–map simplification for Carry and Sum: Boolean expressions for the sum (S) and carry (C) outputs from K – maps:

K-map for sum output K-map for carry output

Sum, $S = \bar{A}B + A\bar{B} = (A \oplus B)$

Carry, $C = AB$

Logic Diagram:

Fig. 3.3. Logic Diagram of Half Adder

3.2.2. Full Adder

A full adder circuit is an arithmetic circuit block that can be used to add three bits to produce a sum and a carry output. Let us consider A and B as two 1-bit inputs & C_{in} is a carry generated from the previous order bit additions. Let S (sum) and C_{out} (carry) are the outputs of the full adder.

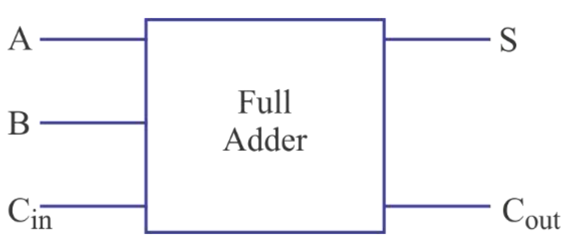


Fig. 3.4. Block Diagram of a Full Adder

The Truth Table for Full Adder is given as:

Table: Truth Table for Full Adder

Inputs			Outputs	
A	**B**	**C_{in}**	**Sum (S)**	**Carry C_{out}**
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

K – map Simplification for Carry and Sum:

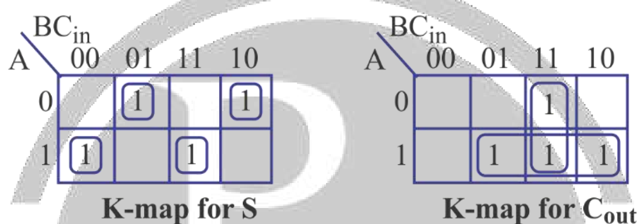


Fig. 3.5.

Simplified Boolean expressions for the sum (S) and carry (C_{out}) output from K-maps is

Sum, $S = \bar{A}\bar{B}C_{in} + \bar{A}B\bar{C}_{in} + ABC_{in} + A\bar{B}\bar{C}_{in} = C_{in}\left(\bar{A}\bar{B} + AB\right) + \bar{C}_{in}\left(\bar{A}B + A\bar{B}\right)$

$= C_{in}\left(A \odot B\right) + C_{in}\left(A \oplus B\right)$

$= C_{in}\left(\overline{A \oplus B}\right) + C_{in}\left(A \oplus B\right)$

Sum, $S = C_{in} \oplus A \oplus B$

Carry, $C_{out} = AB + AC_{in} + BC_{in}$

Logic Diagram:

We can realize logic diagram of a full adder using gates as shown in below figure:

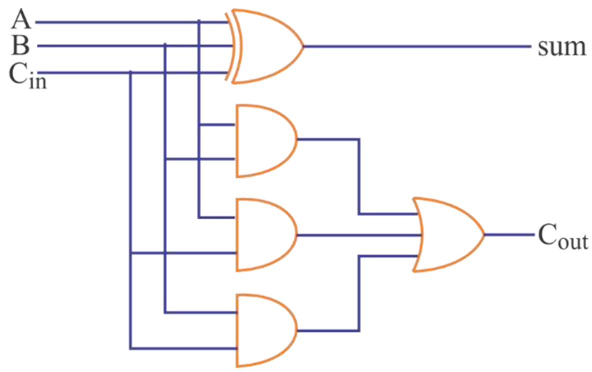


Fig. 3.6. Logic Diagram of Full Adder

Example: A full adder is implemented using two input OR gate and two half adders. Half adder is implemented using two input XOR and two input AND gate. The propagation delays of XOR gate, AND gate and OR gate respectively are 2ns, 1.5ns. and 1ns. The propagation delay of full adder is ….. ns.

Solution:

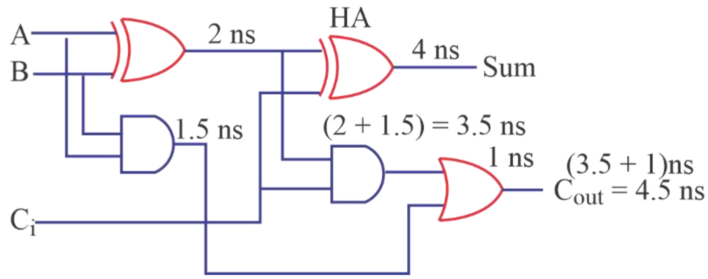


Fig. 3.7.

3.3. Subtractors

3.3.1. Half subtractor

A half subtractor is a combinational logic circuit, which performs the subtraction of two 1-bit numbers. It subtracts one binary digit from another to produce a DIFFERENCE output and a BORROW output.

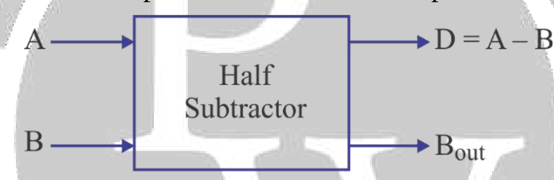


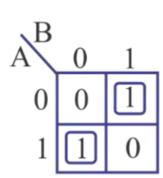
Fig. 3.8. Block diagram of a half subtractor

The truth table of half – subtractor, where A, B are the inputs, and difference (D) and borrow (B) are the outputs.

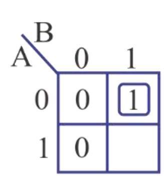
Table: Truth table of Half – Subtractor

Inputs		**Outputs**	
A	**B**	**D**	**B_{out}**
0	0	0	0
0	1	1	1
1	0	1	0
1	1	0	0

K – map Simplification for Difference and Borrow:



K-map for diffrence output



K-map for borrow output

Difference, $D = \bar{A}B + A\bar{B} = A \oplus B$

Borrow, $B_{out} = \bar{A}B$

Logic Diagram:

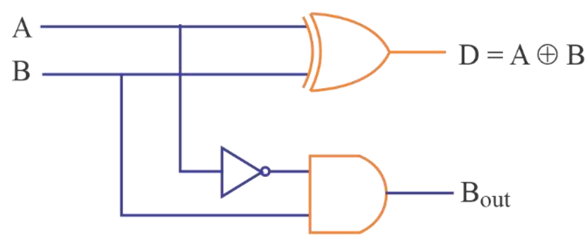


Fig. 3.9. Logic Diagram of a Half subtractor

3.3.2. Full Subtractor

A full subtractor performs subtraction operation on two bits, a minuend and a subtrahend.

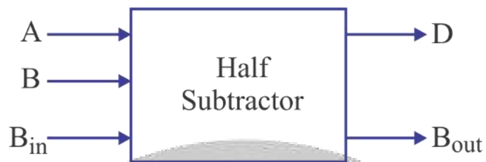


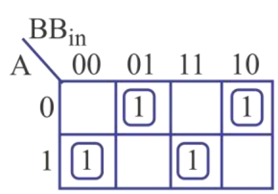
Fig. 3.10. Block Diagram of a Full subtractor

Truth table of Full subtractor:

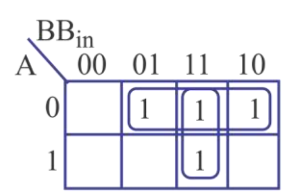
ble: Truth table of Full subtractor

Inputs			Outputs	
A	**B**	**B_{in}**	**D**	**B_{out}**
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	0	1
1	0	0	1	0
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

K – map simplification for Difference and Borrow:



K-map for difference output



K-map for borrow output

Difference, $D = \bar{A}\bar{B}B_{in} + \bar{A}B\bar{B}_{in} + A\bar{B}\bar{B}_{in} + ABB_{in}$

$$= B_{in}\left(AB + \bar{A}\bar{B}\right) + \bar{B}_{in}\left(A\bar{B} + \bar{A}B\right)$$

$$= B_{in}\left(\overline{A \oplus B}\right) + B_{in}\left(A \oplus B\right)$$

$$D = A \oplus B \oplus B_{in}$$

Borrow, $B_{in} = \bar{A}B + \bar{A}B_{in} + BB_{in}$

Logic Diagram

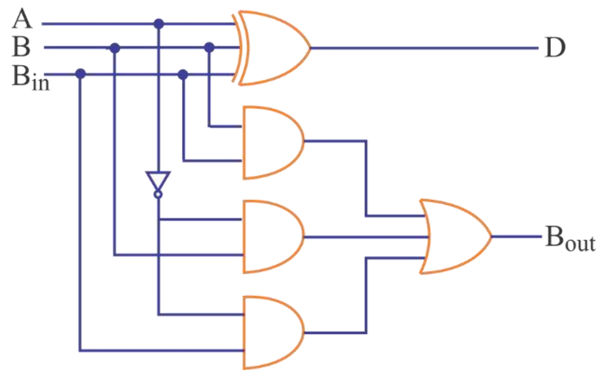


Fig. 3.11. Logic Diagram of a Full Subtractor

3.4. Binary Parallel Adder

An n-bit parallel adder can be constructed using n number of full adders are connected in parallel and hence; it is also known as parallel adder such that the previous carry or carry input for adder 0 is set to zero. The carry output of each adder is connected to the carry input of the next higher order adder. Hence, it is also known as carry propagate adder.

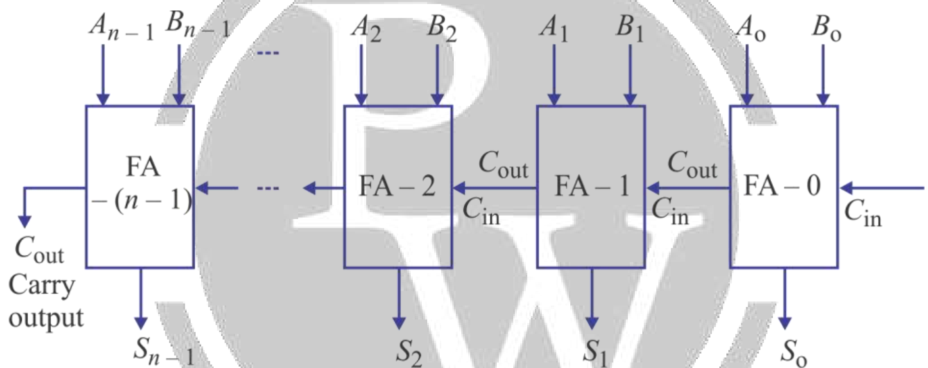


Fig. 3.12. n-bit Binary Adder

3.4.1. Propagation Delay in Parallel Adder:

Parallel adders suffer from propagation delay problem because higher bit additions depend on the carry generated from lower bit addition. In effect, carry bits must propagate or ripple through all stages before the most significant sum bit is valid. Thus, the total sum (the parallel output) is not valid until after the cumulative delay of all the adder.

3.5. Carry Look Ahead Adder

The look ahead carry adder speeds up the operation by eliminating this ripple carry delay. It examines all the input bits simultaneously and also generates the carry in bits for all the stages simultaneously. The method of speeding up the addition process is based on the two additional functions of the full adder called the carry generate and carry propagate functions.

3.5.1. Carry Generation

Carry is generated only if both the input bits are 1, that is, if both the bits A and B are 1's, a carry has to be generated in this stage regardless of whether the input carry C_{in} is a 0 or a 1. Let G as the carry generation function,

$$G = A \cdot B$$

Consider the present bit as the n^{th}, then

$$G_n = A_n \cdot B_n$$

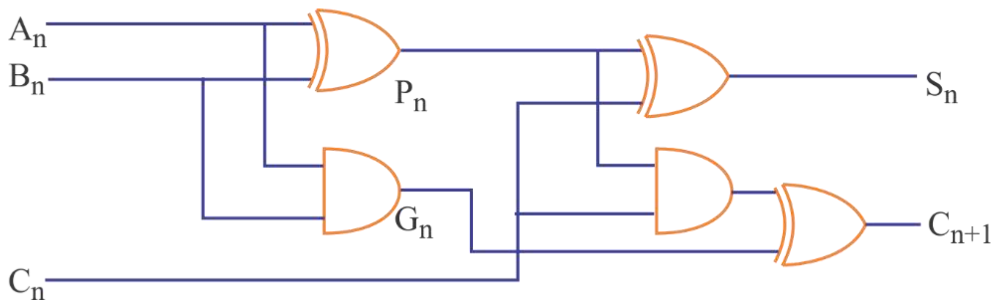


Fig. 3.13. Carry Look – ahead Generator Circuit

3.5.2. Carry Propagation

A carry is propagated if any one of the two input bits A or B is 1. If both A and B are 0, a carry will never be propagated. On the other hand, if both A and B are 1, then will not propagate the carry but will generate the carry. Let P as the carry – propagation function, then

$$P_n = A_n \oplus B_n$$

3.5.3. Look ahead Expressions

Let n[th] bit adder, the sum (S) and the carry out (C) for the n[th] bit may be expressed in terms of the carry generation function (G) and the carry propagation function (P) as

$$S_n = P_n \oplus C_n$$

$$C_{n+1} = G_n + P_n \cdot C_n$$

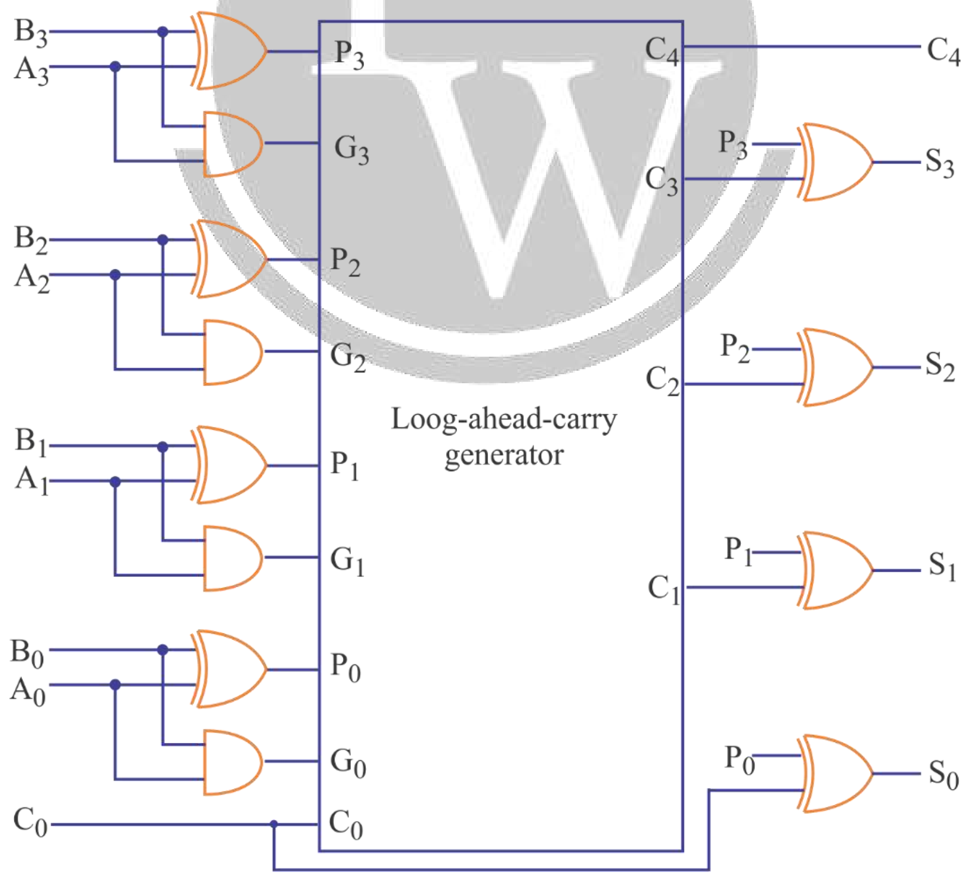


Fig. 3.14. 4-bit Full Adder with a look Ahead Carry Generator

Example: A full adder can be realized using half adder? Explain it in detail.

Solution: A full adder realization using half adder is given by:

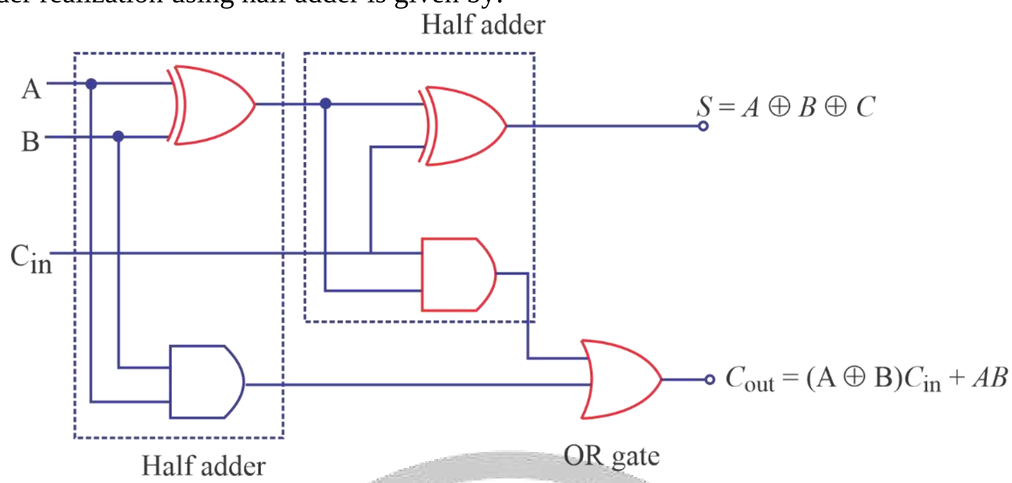


Fig. 3.15.

3.6. Comparator

The comparator is a combinational logic circuit. It compares the magnitude of two n-bit numbers and provides the relative result as the output. Let A and B are the two n-bit inputs. The comparator has three outputs namely A > B, A = B and A < B. Depending upon the result of comparison, one of these outputs will go high.

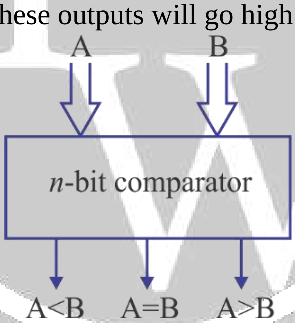


Fig. 3.16. Block diagram of digital comparator

3.6.1. 1-bit Magnitude Comparator

The 1-bit comparator is a combinational logic circuit with two inputs A and B and three outputs namely A < B, A= B and A > B.

Table : Truth Table of a 1-bit Comparator

Inputs		Outputs		
A	**B**	**X (A < B)**	**Y (A = B)**	**Z (A > B)**
0	0	0	1	0
0	1	1	0	0
1	0	0	0	1
1	1	0	1	0

Design of 1-bit Magnitude Comparator: We can write the expressions for the three outputs as under:

$$\text{For } (A < B), \qquad X = \bar{A}_0 B_0$$

$$\text{For } (A = B), \qquad Y = \bar{A}_0 \bar{B}_0 + A_0 B_0 = \overline{A_0 \oplus B_0}$$

$$\text{For } (A > B), \qquad Z = A_0 \bar{B}_0$$

Logic Diagram of 1-bit Comparator:

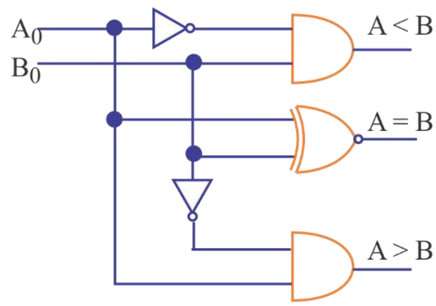


Fig. 3.17. Logic Diagram of 1-bit Comparator

Example: The circuit shown in given figure is:

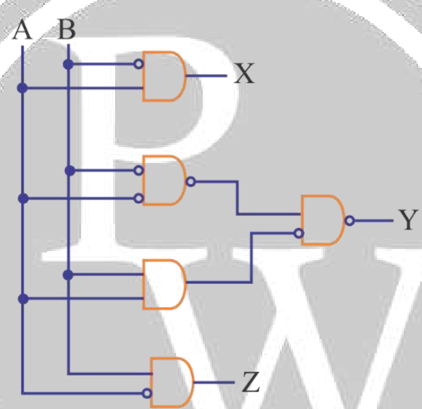


Fig. 3.18.

Solution: From the logic circuit shown in the figure, we obtain the following results,

$$X = A\bar{B}$$

$$Y = \overline{(\overline{\bar{A}\bar{B}})(\overline{AB})} = \bar{A}\bar{B} + AB = A \odot B$$

$$Z = \bar{A}B$$

So, we obtain the truth table for the above function as shown below.

From truth table, we deduce the following results.

It $A > B$, then $x = 1$

It $A = B$, the $y = 1$

It $A < B$, then $z = 1$

A	B	X	Y	Z
0	0	0	1	0
0	1	0	0	1
1	0	1	0	0
1	1	0	1	0

Therefore, it is a comparator circuit.

3.7. Multiplexer

A multiplexer, abbreviated as MUX, is a digital switch which selects one of the many inputs to a single output. A number of control lines determine which input data is to be routed to the output. If there are n select lines, then the number of maximum possible input lines is 2^n and the multiplexer is referred to as a 2^n–to-1 multiplexer or $2^n \times 1$ multiplexer.

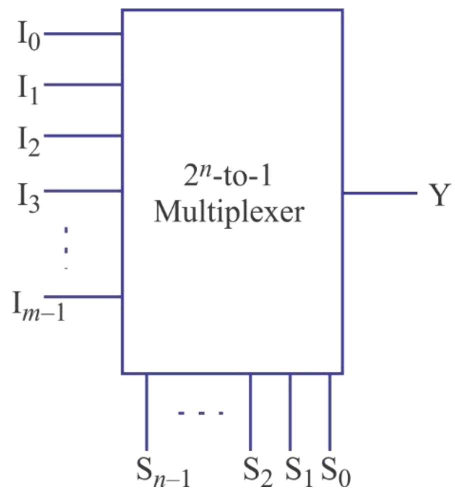


Fig. 3.19. Block diagram of a 2^n to 1 multiplexer

3.7.1. 2 × 1 MUX

A 2 to 1 multiplexer has 2 inputs. Since $2 = 2^1$, this multiplexer will have one control (select) line. It has two data inputs I_0 and I_1, one select input S, and one output.

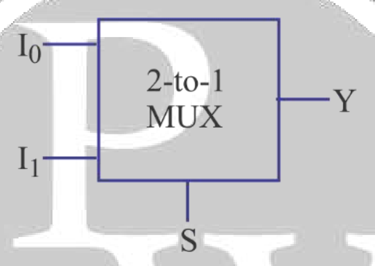


Fig. 3.20. Schematic block diagram of 2:1 Multiplexer

The truth table of this MUX is given below,

Select Line (S)	Output Y
0	I_0
1	I_1

Thus, the SOP expression for the output Y is,

$$Y = I_0\bar{S}_0 + I_1S_0$$

Realization of a 2:1 MUX using Logic Gates:

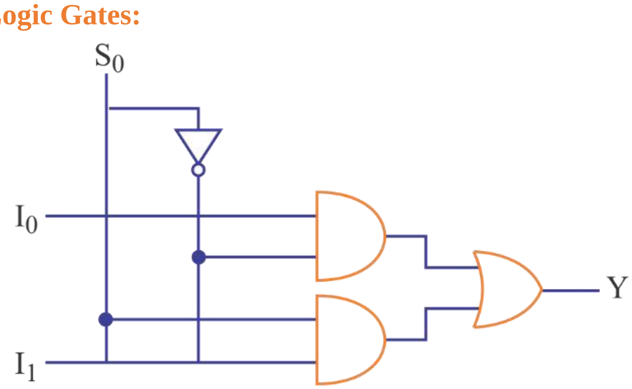


Fig. 3.21. Logic Diagram of a 2 x 1 Multiplexer

2.7.2. 4 × 1 MUX

A 4-to-1 multiplexer has 4 inputs and two select lines, where I_0 to I_3 are the four inputs to the multiplexer, and S_0 and S_1 are the select lines.

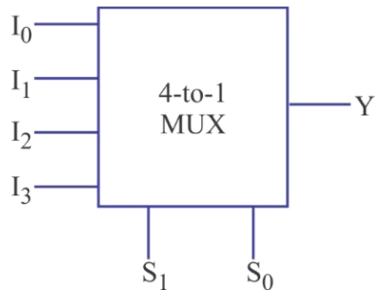


Fig. 3.22. Schematic block diagram of 4 x 1 MUX

Truth Table of a 4-to-1 Multiplexer

Select Inputs S_1	S_0	Output Y
0	0	I_0
0	1	I_1
1	0	I_2
1	1	I_3

Output Y for a 4-input multiplexer is

$$Y = I_0\bar{S}_1\bar{S}_0 + I_1\bar{S}_1S_0 + I_2S_1\bar{S}_0 + I_3S_1S_0$$

3.8. Implementation of Higher Order Mux Using Lower Order MUX

The methods for implementing higher order MUX using lower order MUX are

Step 1: If 2^n is the number of input lines in the available lower order multiplexer and 2^N is the number of input lines in the desired multiplexer, then the number of lower order multiplexers required to construct the desired multiplexer circuit would be $2^N – n$.

Step 2: From the knowledge of the number of selection inputs of the available multiplexer and that of the desired multiplexer, connect the less significant bits of the selection inputs of the desired multiplexer to the selection inputs of the available multiplexer.

Step 3: The most significant bits of the selection inputs of the desired multiplexer circuit are used to enable or disable the individual multiplexers so that their outputs when OR produce the final output.

Example: In realization of 32 : 1 MUX using 2 : 1 MUX, the required number of 2 : 1 MUX is ?

Solution: In realization of 2^n : 1 MUX using 2 : 1 MUX, the required number of 2 : 1 MUX is 2^n -1, since, we have to realize 32 : 1 MUX, so we have

$$n = 5$$

Hence, the required number of 2 : 1 MUX is

$$2^n - 1 = 2^5 - 1 = 31$$

3.9. Applications of Multiplexers

1. It is used as a data selector to select one out of many data inputs.
2. They are used in designing the combinational circuits.
3. They are used in digital-to-analog and analog-to-digital converters.
4. They can be used for simplification of logic design.
5. Multiplexers are also used in data acquisition systems.

3.10. Demultiplexer

The demultiplexer is a combinational logic circuit that performs the reverse operation of a multiplexer. The demultiplexer has one input line and m output lines. Again m = 2^n, so it requires n select lines. A demultiplexer with one input and m output is called a 1-to-m demultiplexer.

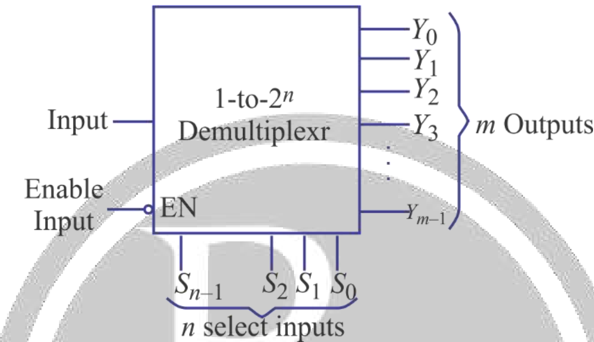


Fig. 3.23. Block Diagram of m-output Demultiplexer

The demultiplexer has one input line and m output lines. Again m = 2^n, so it requires n select lines. A demultiplexer with one input and m outputs is called a 1-to-m demultiplexer.

3.10.1. 1 × 2 Demultiplexer

A 1 to 2 demultiplexer has one input and two outputs. Since 2 = 2 × 1, it requires only one control (select) line.

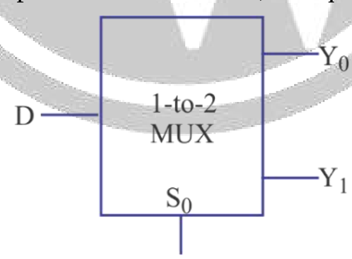


Fig. 3.24. Logic Diagram of 1 x 2 De-MUX

Truth table of a 1-to-2 demultiplexer

Table: Truth table of a 1-to-2 demultiplexer

Input S	Select input S_0	Output Y_1	Y_0
D	0	0	D
D	1	D	0

Thus, the Boolean expressions for the outputs can be written as

$$Y_0 = D\bar{S}_0 \ \& \ Y_1 = DS_0$$

Realization of a 1 x 2 Demultiplexer using Logic Gates:

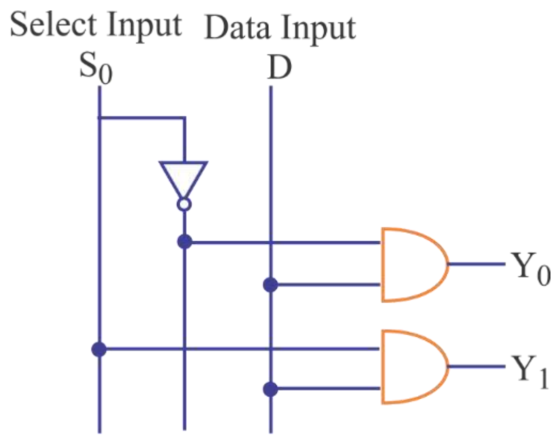


Fig. 3.25. Logic Diagram of 1 × 2 Demultiplexer

3.10.2. Applications of Demultiplexers

Demultiplexers are used in

1. Data transmission
2. Implementation of Boolean Functions
3. Combinational logic circuit design
4. Generate enable signals (enable one out of many). The application of enable signals in microprocessor systems are:
   (a) Selecting different banks of memory
   (b) Selecting different input/output devices for data transfer
   (c) Enabling different functional units
   (d) Enabling different rows of memory chips depending on address

3.11. Comparison Between Multiplexer and Demultiplexer

Table : Comparison between Multiplexer and Demultiplexer

S.No.	Parameter of comparison	Multiplexer	Demultiplexer
1.	Type of logic circuit	Combinational	Combinational
2.	Number of data inputs	m	1
3.	Number of select inputs	n	N
4.	Number of data output	1	M
5.	Relation between input/output lines and select lines	$m = 2^n$	$M = 2^N$
6.	Operation principle	Many to 1 or as data selector	1 to many or data distributor

3.12. Decoder

A decoder is a combinational circuit that converts an n-bit binary input data into 2^n output lines, such that each output line will be activated for only one of the possible combinations of inputs. Decoders are usually represented as n-to-2^n line decoders, where n is the number of input lines and 2^n is the number of maximum possible output lines.

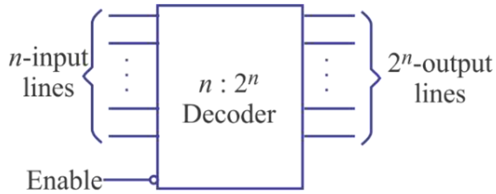


Fig. 3.26. Block Diagram of n-to-2^n Decoder

If there are some unused or 'don't care' combinations in the n-bit code, then there will be less than 2n output lines. In general, if n and m are respectively the numbers of input and output lines, then m ≤ 2n.

3.12.1. 2 to 4 Line Decoder

Consider a 2 to 4-line decoder, where *A* and *B* are two inputs whereas Y_0 through Y_3 are the four outputs.

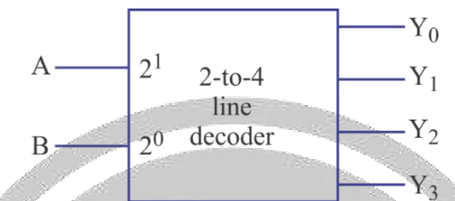


Fig. 3.27. Block Diagram of a 2 to 4 Line Decoder

Truth Table of a 2 to 4 Line Decoder

Table : Truth Table of a 2 to 4 Line Decoder

Inputs		Outputs			
A	B	Y_0	Y_1	Y_2	Y_3
0	0	1	0	0	0
0	1	0	1	0	0
1	0	0	0	1	0
1	1	0	0	0	1

The Boolean expressions for the four outputs is given as:

$$Y_0 = \bar{A}\,\bar{B} \text{ and } Y_1 = \bar{A}B$$

$$Y_2 = A\bar{B} \text{ and } Y_3 = AB$$

Realization of a 2 to 4 Line Decoder using Logic Gates

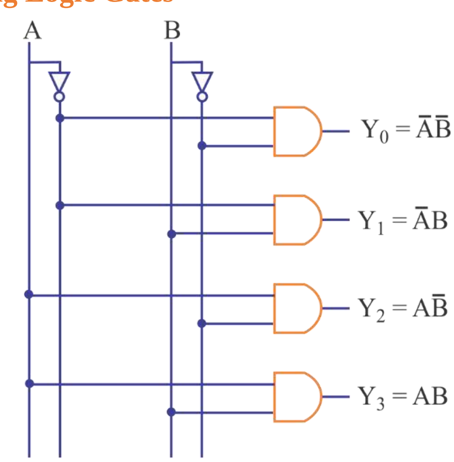


Fig. 3.28. Logic Diagram of a 2 to 4 Line Decoder

3.12.2. Applications of Decoder

Some of important applications of decoder are as follows:

1. When the decoder inputs come from a counter which is being continually pulsed, the decoder outputs will be activated sequentially. Hence, they can be used as timing or sequencing signals to turn devices on or off at specific times.
2. Decoder are use in memory system of a computer where they respond to the address code generated by the microprocessor to activate a particular memory location.
3. They are also used in computers for selection of external devices that include printers, modems, scanners, internal disk drives, keyboard, video monitor etc.

3.13. Encoders

An encoder is a combinational logic circuit that performs the inverse operation of a decoder. An encoder has 2^n (or fewer) input lines and n output lines.

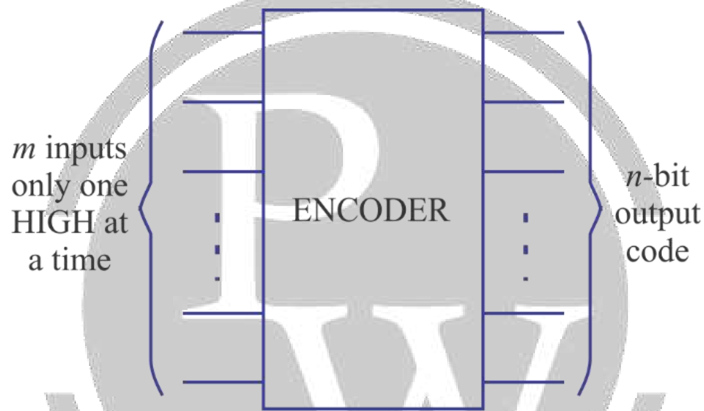


Fig. 3.29. Block Diagram of Encoder

3.13.1. Octal to Binary Encoder

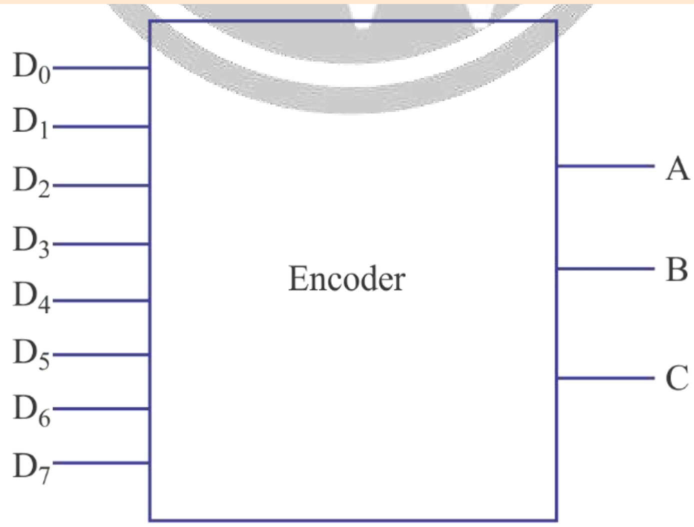


Fig. 3.30. Octal to Binary Encoder

Truth Table of an Octal to Binary Encoder:

Table: Truth Table of an Octal to Binary Encoder

Inputs								Outputs		
D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	A	B	C
1	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	1
0	0	1	0	0	0	0	0	0	1	0
0	0	0	1	0	0	0	0	0	1	1
0	0	0	0	1	0	0	0	1	0	0
0	0	0	0	0	1	0	0	1	0	1
0	0	0	0	0	0	1	0	1	1	0
0	0	0	0	0	0	0	1	1	1	1

The logical expressions for the outputs as follows:

$$A = D_4 + D_5 + D_6 + D_7$$
$$B = D_2 + D_3 + D_6 + D_7$$
$$C = D_1 + D_3 + D_5 + D_7$$

3.13.2. Octal to Binary Encoder

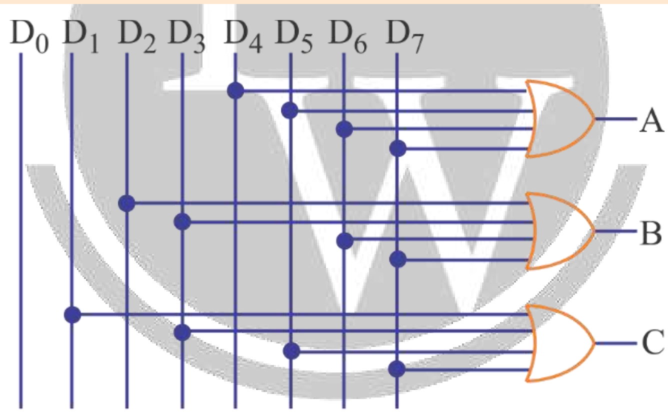


Fig. 3.31. Logic Diagram of Octal-to Binary Encoder

3.13.3. Decimal to BCD Encoder

This type of encoder has 10 inputs one for each decimal digit and 4 outputs corresponding to the BCD code.

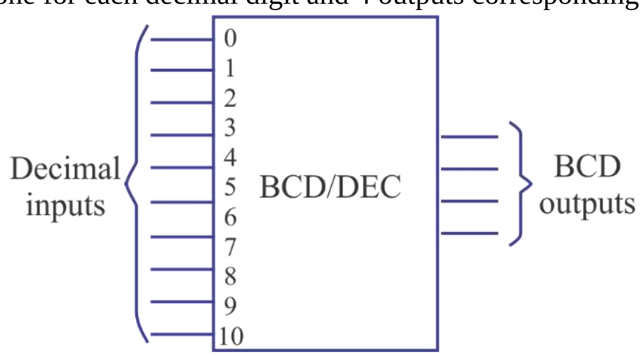


Fig. 3.32. Block Diagram of a Decimal-to-BCD Encoder

Truth Table of a Decimal to Binary Encoder:

Table : Truth Table of a Decimal to Binary Encoder

Input										Output			
0	1	2	3	4	5	6	7	8	9				
D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	A	B	C	D
1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	1
0	0	1	0	0	0	0	0	0	0	0	0	1	0
0	0	0	1	0	0	0	0	0	0	0	0	1	1
0	0	0	0	1	0	0	0	0	0	0	1	0	0
0	0	0	0	0	1	0	0	0	0	0	1	0	1
0	0	0	0	0	0	1	0	0	0	0	1	1	0
0	0	0	0	0	0	0	1	0	0	0	1	1	1
0	0	0	0	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	0	0	1	1	0	0	1

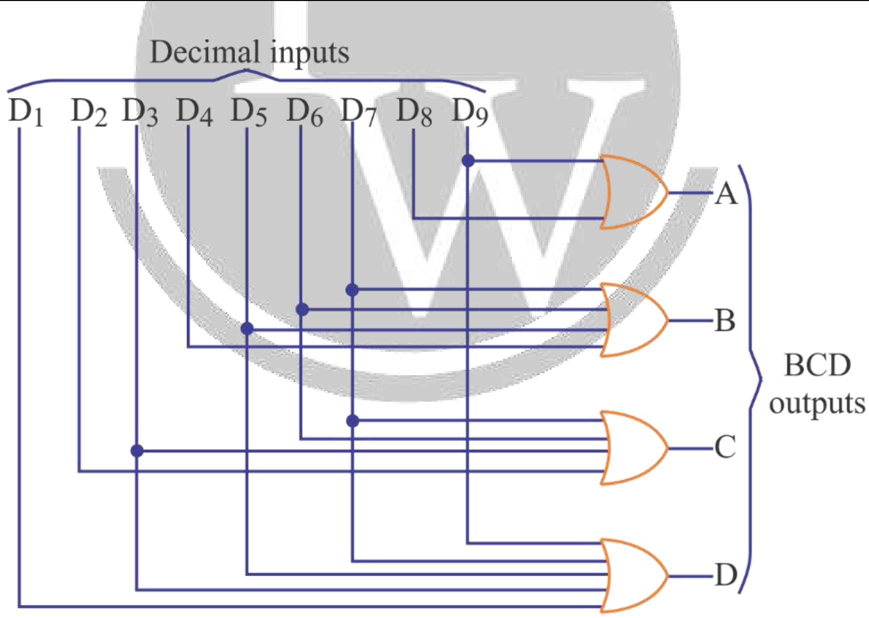


Fig. 3.33. Logic Diagram of Decimal-to-BCD encoder

The outputs of a decimal-to-BCD encoder:

$A = D_8 + D_9$

$B = D_4 + D_5 + D_6 + D_7$

$C = D_2 + D_3 + D_6 + D_7$

$D = D_1 + D_3 + D_5 + D_7 + D_9$

3.14. Priority Encoder

3.14.1. Truth Table of a Four Input Priority Encoder: (Taking LSB as priority)

Table: Truth Table of a Four Input Priority Encoder

Inputs				Outputs	
D_0	D_1	D_2	D_3	A	B
0	0	0	0	X	X
1	0	0	0	0	0
X	1	0	0	0	1
X	X	1	0	1	0
X	X	X	1	1	1

According to the truth table, the higher the subscript number, the higher the priority of the input.
The X's are don't care conditions indicating that the binary values they represent may be equal to 0 or 1.

3.15. Code Converters

A code converter is a combinational logic circuit which accepts the input information in one binary code, converts it and produces an output into another binary code.

3.15.1. The truth table for 4-bit Binary and its Equivalent BCD

Table: Truth table for 4-bit Binary and its Equivalent BCD

Decimal	Binary Input				BCD Output				
	A	B	C	D	B_4	B_3	B_2	B_1	B_0
0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	0	1
2	0	0	1	0	0	0	0	1	0
3	0	0	1	1	0	0	0	1	1
4	0	1	0	0	0	0	1	0	0
5	0	1	0	1	0	0	1	0	1
6	0	1	1	0	0	0	1	1	0
7	0	1	1	1	0	0	1	1	1
8	1	0	0	0	0	1	0	0	0
9	1	0	0	1	0	1	0	0	1
10	1	0	1	0	1	0	0	0	0
11	1	0	1	1	1	0	0	0	1
12	1	1	0	0	1	0	0	1	0
13	1	1	0	1	1	0	0	1	1
14	1	1	1	0	1	0	1	0	0
15	1	1	1	1	1	0	1	0	1

The minimized expression of outputs are as follows:

$$B_4 = AB + AC$$

$$B_1 = \bar{A}C + AB\bar{C}$$

$$B_2 = \bar{A}B + BC$$

$$B_3 = A\bar{B}\bar{C}$$

$$B_0 = D$$

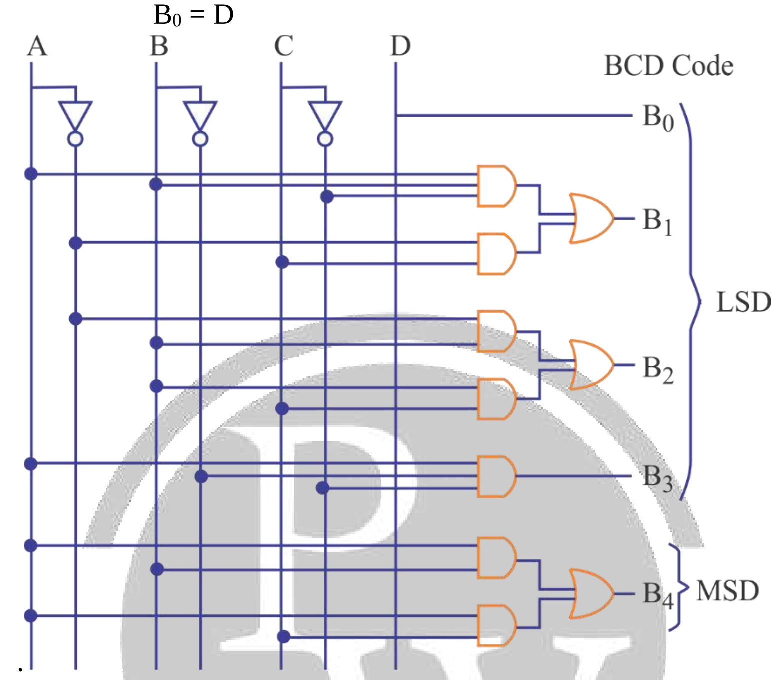


Fig. 3.34. Logic Diagram of a Binary-to-BCD Code Converter

3.16. Parity Generator

Parity generators are circuits that accept an (n-1) bit data stream and generate an extra bit that is transmitted with the bit stream. This extra bit is referred to as the parity bit. The parity added in binary message is such that the total number of 1's in the message can be either odd or even according to the type of parity used.

3.16.1. Even Parity Generator

The even parity generator is a combinational logic circuit that generates the parity bit such that the number of 1's in the message becomes even. The parity bit is '1' if there are odd number of 1's in the data stream and the parity bit is '0' if there are even number of 1's in the data stream.

Truth table for 4-bit data with Even Parity:

Table: Truth table for 4-bit data with Even Parity

4-bit data				Even Parity
A	B	C	D	P
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0

0	1	0	0	1
0	1	0	1	0
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	1
1	1	1	0	1
1	1	1	1	0

The minimized expression for even parity generator is

$$P = A \oplus B \oplus C \oplus D$$

The logic diagram for the even parity generator is given as

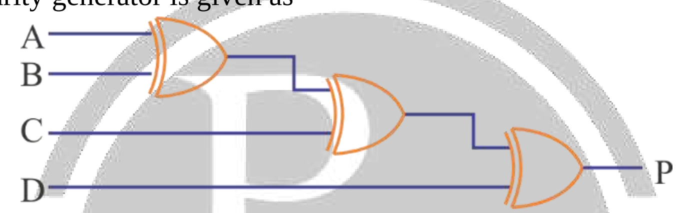


Fig. 3.35. Logic diagram of even parity generator

3.16.2. Odd Parity Generator

The odd parity generator is a combinational logic that generates the parity bit such that the number of 1's in the message becomes odd. The parity bit is '0' for odd number of 1's and '1' for even number of 1's in the bit stream.

Example: A parity generation circuit required to generator on odd parity bit may use ————?

Solution: Odd parity generation circuit consists of combination of EX-OR and EX-NOR gates, whereas even priority generator consists only EX-OR gates.

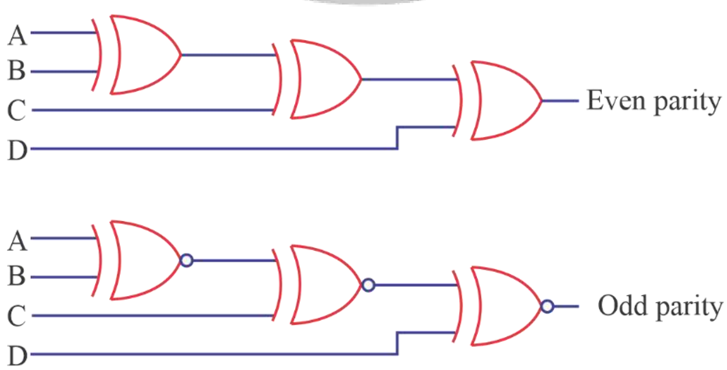


Fig. 3.36

∴ It is combination of EX-OR and EX-NOR gates.

❒❒❒

4 SEQUENTIAL LOGIC CIRCUITS

4.1. Sequential Logic Circuits

In sequential logic circuits, the output is a function of the present inputs as well as the inputs and outputs. Sequential circuit include memory elements to store past data. The flip-flop is a basic element of sequential logic circuits.

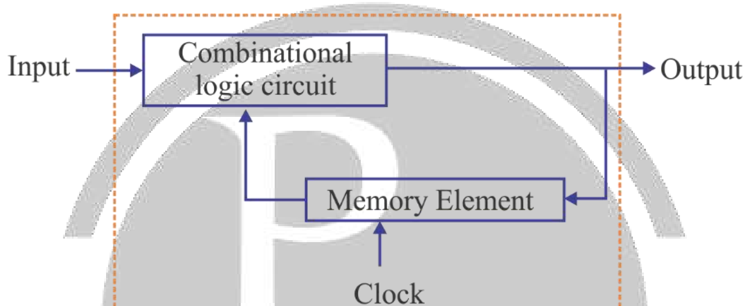


Fig. 4.1. General Block diagram of Sequential Logic Circuit

There are two types of sequential circuits:

4.1.1. Synchronous Circuits

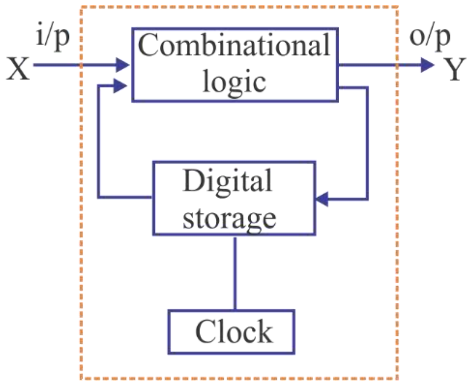
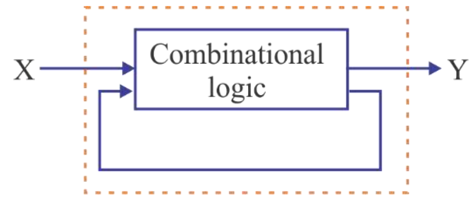
The sequential circuits which are controlled by a clock are called synchronous sequential circuits. These circuits get activated only when clock signal is present.

4.1.2. Asynchronous Circuits

The sequential circuits which are not controlled by a clock are called asynchronous sequential circuits, i.e. the sequential circuits in which events can take place any time the inputs are applied are called asynchronous sequential circuits.

4.2. Difference Between Synchronous and Asynchronous Sequential Circuits

S.No.	Synchronous Sequential Circuits	Asynchronous Sequential Circuits
1.	In synchronous circuits, the change in input signals can affect memory elements upon activation of clock signal.	In asynchronous circuits, change in input signals can affect memory elements at any instant of time.
2.	In synchronous circuits, memory elements are clocked FF's.	In asynchronous circuits, memory elements are either unlocked FF's or time delay elements.
3.	The maximum operating speed of the clock depends on time delays involved.	Since the clock is not present, asynchronous circuits can operate faster than synchronous circuits.
4.	They are easier to design.	More difficult to design.

5.	 Synchronous circuit	 Asynchronous circuit

4.3. Latches

Flip-flop is an electronic circuit or device which is used to store a data in binary form. Actually, flip-slop is a one-bit memory device and it can store either 1 or 0. Flip-flops is a sequential device that changes its output only when a clocking signal is changing. On the other hand, latch is a sequential device that checks all its inputs continuously and changes its outputs accordingly at any time independent of a clock signal. It refers to non-clocked flip-flops, because these flip-flops, because these flip-flops 'latch on' to a 1 or a 0 immediately upon receiving the input pulse.

4.3.1. General Block Diagram of a Latch or Flip-flop

Figure shown below is the general type of symbol used for a latch. In case of a flip-flop, a clock signal must be shown at input side. It has many inputs and two outputs, labelled Q and $\overline{Q}$. The Q output is the normal output of the latch and $\overline{Q}$ is the inverted output.

Note: A flip-flop is said to be in HIGH state or logic 1 state or SET state when Q = 1, and in LOW state or logic 0 state or RESET state or CLEAR state when Q = 0.

4.3.2. Difference between Latches and Flip-flops

S.No.	Latch	Flip-flop
1.	A latch is an electronic sequential logic circuit used to store information in an asynchronous arrangement.	A flip-flop is an electronic sequential logic circuit used to store information in a synchronous arrangement. It has two stable states and maintains its states for an indefinite period until a clock pulse is applied.
2.	One latch can store one-bit information, but output state changes only in response to data input.	One flip-flop can store one-bit data, but output state changes with clock pulse only.
3.	Latch is an asynchronous device and it has no clock input.	Flip-flop has clock input and its output is synchronised with clock pulse.
4.	Latch holds a bit value and it remains constant until new inputs force it to change.	Flip-flop holds a bit value and it remains constant until a clock pulse is received.
5.	Latches are level-sensitive, and the output tracks the input when the level is high. Therefore, as long as the level is logic level 1, the output can change if the input changes.	Flip-flops are edge sensitive. They can store the input only when there is either a rising or falling edge of the clock.

4.4. Latch

A latch is a type of bistable logic device or multivibrator that is most often used in applications that require data storage. The main characteristics of latch is that the output is not dependent solely on the on the present state of the input but also on the proceeding output state.

Latches are sometimes used for multiplexing data onto a bus. For example, data being input to a computer from a external source have to share the data bus with data from other sources. When the data bus becomes unavailable to external source, the existing data must be temporarily stored, and hence the latches are placed between the external source and data bus.

4.4.1. SR Latch

For the SR latch (S stands for set and R for reset). The logic circuit for SR latch is shown in figure below:

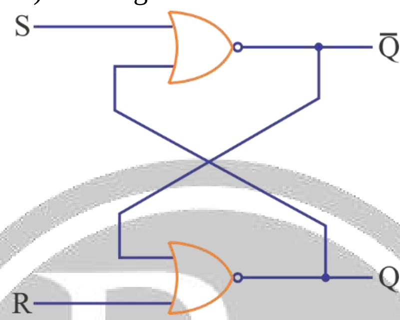


Fig. 4.1. Logic circuit of SR latch.

The state table for the SR latch is:

S	R	Q	Q^+	$\overline{Q^+}$
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	0	1
1	0	0	1	0
1	0	1	1	0
1	1	0	0	1
1	1	1	0	0

The symbol for SR Latch is:

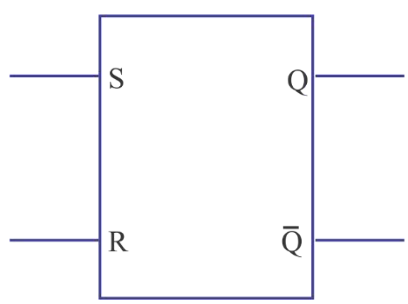


Fig. 4.2.

Obtaining the characteristic equations of the NOR gate based latch are; we get

$$Q^+ = \bar{R} \times S + \bar{R} \times Q = \bar{R} \times (S + Q) \text{ and } \bar{Q}^+ = \bar{S} \times R + \bar{S} \times \bar{Q} = \bar{S} \times (S + \bar{Q})$$

Note: It must be noted that the complementing Q^+ does not yield $\bar{Q}^+$.

Hence, the truth table for SR latch is

S	Q	Q^+	$\overline{Q^+}$	
0	0	Q	$\bar{Q}$	⇒ No change
1	1	0	1	⇒ Reset Q^+ to 0
1	0	1	0	⇒ Set Q^+ to 1
1	1	0	0	⇒ Forbidden state

However, the forbidden state (S = R = 1) is considered a don't care state.

Consider a Timing diagram for SR latch

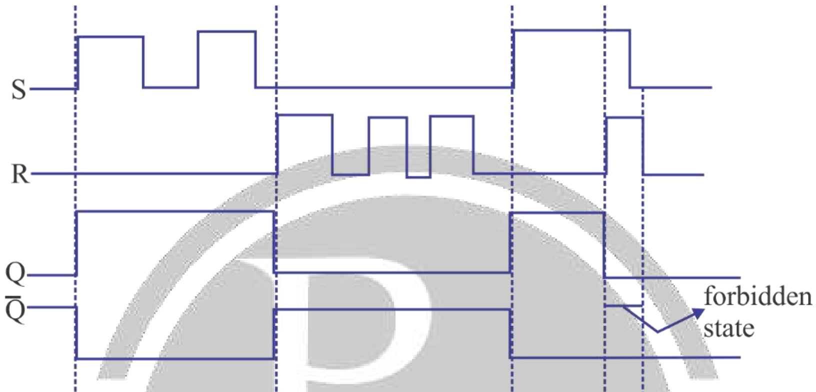


Fig. 4.3.

4. 4.2. $\bar{S}\bar{R}$ Latch:

An $\bar{S}\bar{R}$ latch can be implemented using NAND gates, as shown in figure below

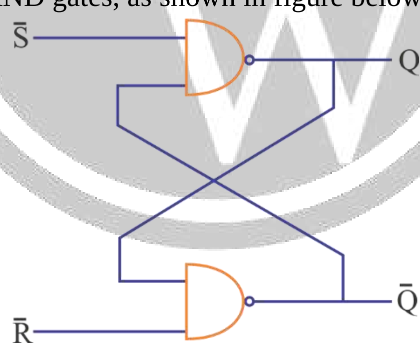


Fig. 4.4. Logic circuit for $\bar{S}\bar{R}$ Latch

The $\bar{S}\bar{R}$ latch is said to be set-dominant 1,

The symbol for $\bar{S}\bar{R}$ latch is shown below:

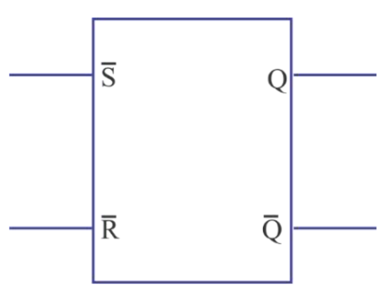


Fig. 4.5.

The truth table for $\bar{S}\bar{R}$ latch is given as:

$\bar{S}$	$\bar{R}$	Q^+	$\overline{Q^+}$	
1	1	Q	$\bar{Q}$	⇒ No change
1	0	0	1	⇒ Reset Q^+ to 0
0	1	1	0	⇒ Set Q^+ to 1
1	1	1	1	⇒ Forbidden state

Application of $\bar{S}\bar{R}$ latch: The application of $\bar{S}\bar{R}$ latch is in switch bouncing i.e. contact bounces of a push-button switch during its opening or closing can be eliminated by using $\bar{S}\bar{R}$ latch.

4.4.3. Gated SR Latch on Enable SR Latch or clocked SR Latch

A gated or level-sensitive SR latch uses a control signal C that can be used as a clock signal or can be used as enable input. The logic circuit diagram, symbol and truth is given as

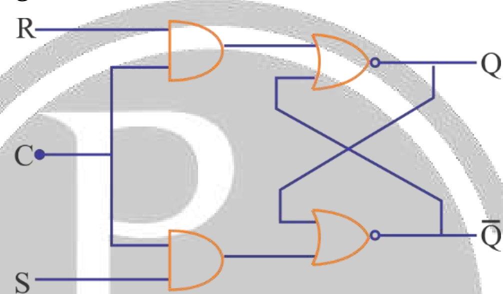


Fig. 4.6. Logic circuit of clocked SR latch.

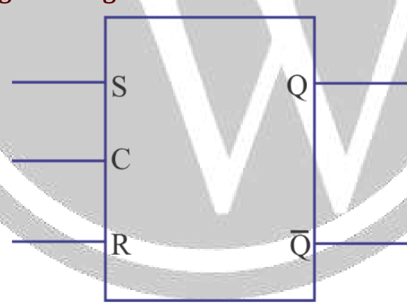


Fig. 4.7. Symbol for clocked SR latch

C	S	R	Q^+	$\overline{Q^+}$	
0	×	×	Q	$\bar{Q}$	} No change state
1	0	0	Q	$\bar{Q}$	
1	0	1	0	1	⇒ Reset
1	1	0	1	0	⇒ Set
1	1	1	0	0	⇒ Forbidden state

4.4.4. Gated $\bar{S}\bar{R}$ Latch or enable $\bar{S}\bar{R}$ Latch or clocked $\bar{S}\bar{R}$ Latch

Gated $\bar{S}\bar{R}$ latch is implemented using two NAND gates and an $\bar{S}\bar{R}$ latch.

The logic circuit diagram, symbol and truth is given as

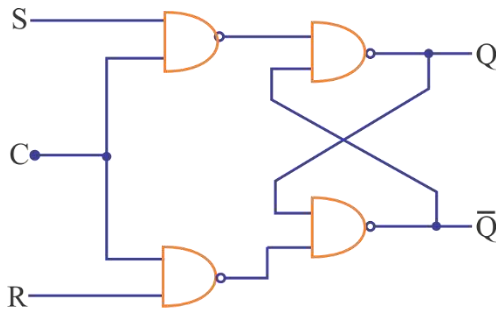


Fig. 4.8. Logic diagram of clocked $\bar{S}\bar{R}$ latch.

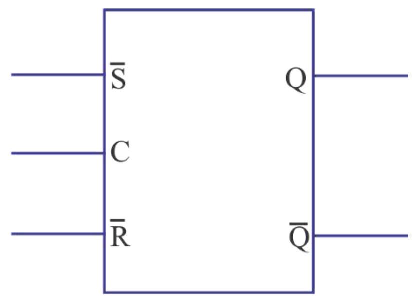


Fig. 4.9.

The truth table of the gated SR latch based on a $\bar{S}\bar{R}$ latch:

C	J	K	Q	$\overline{Q^+}$
O	X	X	Q	$\bar{Q}$
1	0	0	Q	$\bar{Q}$
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

The characteristic equation for SR flip-flop is given as

$$Q^+ = Q_{n+1} = S + \bar{R}Q_n = S + \bar{R}Q$$

4.5. Flip-Flops

Flip-flops are synchronous bistable devices also known as bistable multivibrator. Its output change its state only at a verified paint (i.e. leading or trailing edge) on the triggering input called the clock (CLK), i.e. changes in the output occur in synchronization with the clock.

Flip-flops are edge-triggered or edge-sensitive whereas gated latches are level-sensitive.

4.5.1. Edge-triggered flip-flop

An edge-triggered flip-flop changes its state either at positive edge (rising edge) or at negative edge (falling edge) of the check pulse.

There are two type of edge-triggered flip-flops. The key to identify an edge-triggered flip-flop is by its logic symbol by small triangle inside the block at the clock input C. This triangle is called the dynamic input indicator.

Positive edge triggered has no bubble at input C whereas negative edge triggered has bubble at input C.

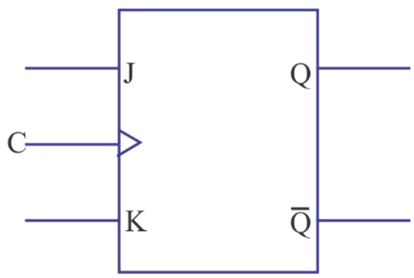


Fig. 4.10. Positive edge triggered flip-flop.

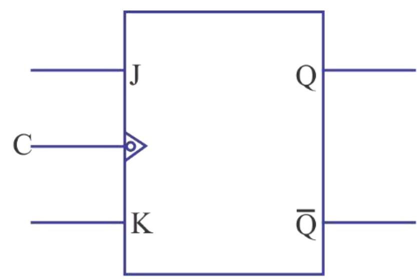


Fig. 4.11. Negative edge-triggered flip-flop.

4.5.2. Basic JK flip-flop

JK Flip-flop (J as a set input and K as a reset input) is the most versatile of the basic flip-flops.

The logic circuit of the gated JK flip-flop is shown in figure below:

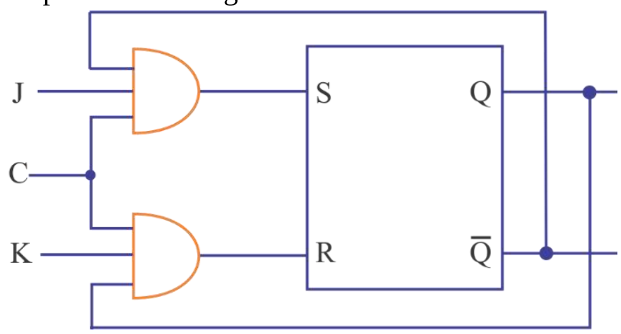


Fig. 4.12. Logic circuit diagram of clocked JK flip-flop.

The state table for the JK flip-flop is given as

C	J	K	Q	Q^+
O	X	X	X	Q
1	0	0	0	0
1	0	0	1	1
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	0

Hence, the truth table becomes,

C	S	R	Q^+	$\bar{Q}^+$	
0	X	X	Q	$\bar{Q}$	} No change state
1	0	0	Q	$\bar{Q}$	
1	0	1	0	1	⇒ Reset
1	1	0	1	0	⇒ Set
1	1	1	0	0	⇒ Forbidden state

Note: The forbidden state, inherent to SR flip-flop is eliminated by adding two feedback loops such that the output becomes 1 only if Q = 0 and reset to only if Q = 1.

It should also be noted that when the inputs (J & K) are set to 1 and clock signal change to 1, then the feedback value of Q & $\bar{Q}$ forced the flip-flop is toggle its value.

(i.e. to switch its state to its logical complement) hence, to ensure this operation in smooth fashion, the pulse width of the clock must be smaller than the propagation delay of the flip-flop.

The characteristic equation of the JK flip-flop

$$Q_{n+1} = J\bar{Q}_n + \bar{K}Q_n \text{ or } Q^+ = J\bar{Q} + \bar{K}Q$$

4.5.3. T-flip-flop

A JK flip-flop can be transformed into a T- flip-flop (T stands for Toggle). When T flip-flop is activated, its output changes its state at every time a pulse is applied to the input T.
The logic circuit of the gated T flip-flop is.

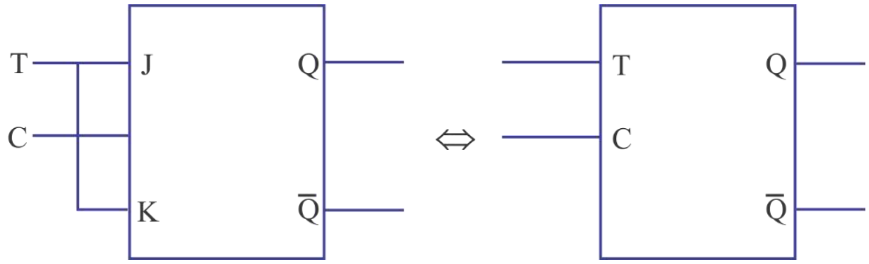


Fig. 4.13.

The state or characteristic table for T flip-flop is

C	T	Q	Q^+
O	X	X	Q
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

As J = K = T, we obtain the characteristic equation as

$$Q^+ = T \times \bar{Q} \times C + (\bar{T} + \bar{C}) \times Q$$

If C = 1, the characteristic the equation is reduced to

$$Q^+ = T \oplus Q$$

$$\text{If } C = 0,\ Q^+ = Q$$

Hence, the truth table of the T-flip flop is given as

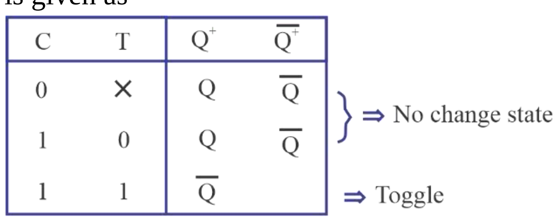


Fig. 4.14.

4.5.4. D Flip-Flop

D-flip-flop can be obtained by use of only two combinations of S-R or J-K flip-flop. It has only one input i.e. D-input or data input.

The logic symbol for D- flip-flop is given as

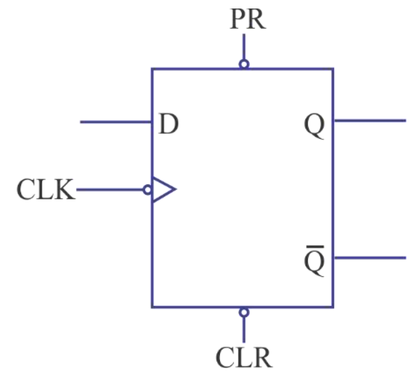


Fig. 4.15.

The truth table for D-flip-flop is

Input	**Output**
D	Q_{n+1}
0	0
1	1

The characteristic equation of D-flip-flop is:

$$Q_{n+1} = D$$

4.5.5. Excitation table of Flip-flops

The truth table of a flip-flop is sometimes referred as characteristic table as it specifies the operational characteristics of the flip-flop there may occurs some situations in which the present state and the next state of the circuit is desired and known. Then the designing of input conditions to as to fulfil the requirements of the circuit, there is a table called excitation table.
It is very important and useful design aid for sequential circuit.

The excitation table for flip-flops:

Present state	Next state	SR Flip-flop		JK Flip-flop		T Flip-flop	D Flip-flop
		S	R	J	K	T	D
0	0	0	×	0	×	0	0
0	1	1	0	1	×	1	1
1	0	0	1	×	1	1	0
1	1	×	0	×	0	0	1

4.6. Operating Characteristics of Flip-Flops

4.6.1. Propagation Delay Time:

Propagation delay time is the time interval required after an input signal has been applied for the resulting output change to occur.

There are four categories of propagation delay times which are as follows:

A. Propagation delay t_{PLH}, it is measured from the triggering edge of the clock pulse to Low-to-High transition of the output.

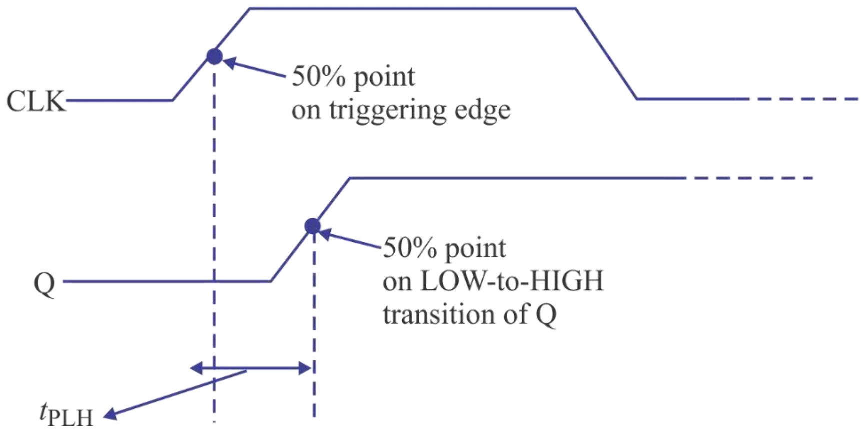


Fig. 4.16.

B. Propagation delay t_{PHL}, it is measured from the triggering edge of the clock pulse to HIGH-to-LOW transition of the output.

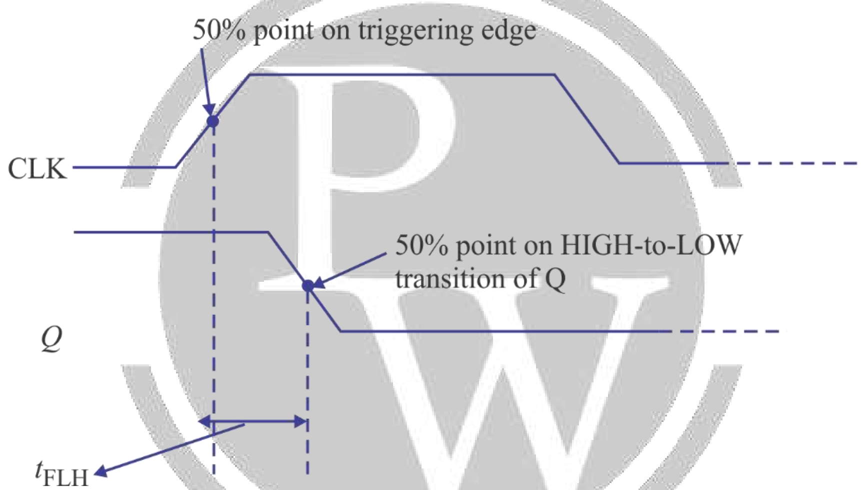


Fig. 4.17.

C. Propagation delay t_{PHL}, it is measured from the leading edge of the PRESET input to LOW-to-HIGH transition of the output.

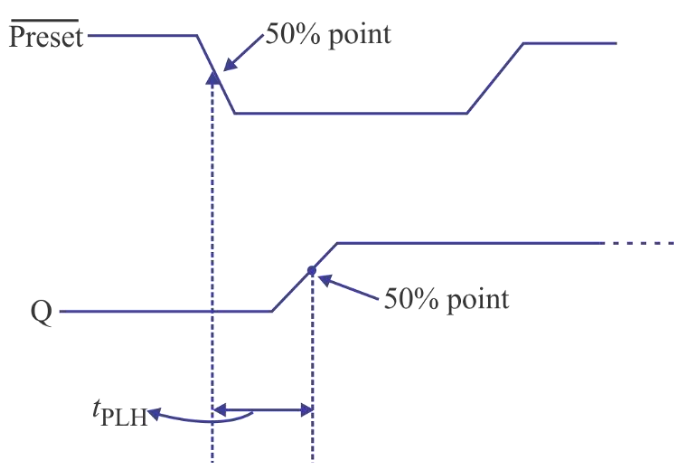


Fig. 4.18.

D. Propagation delay t_{PHL}, it is measured from the leading edge of the clear input to the HIGH-to-LOW transition of the output.

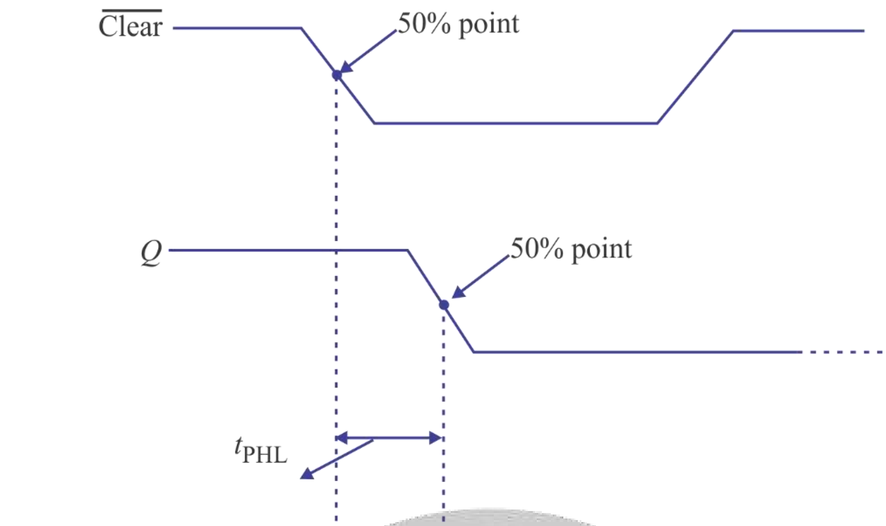


Fig. 4.19.

4.6.2. Set-up time (t_s)

It is the minimum time interval required for the logic levels (0 or 1) to be maintained constantly on the inputs (J, K or D) prior to the triggering edge of the clock pulse in order for the levels to be reliably clocked into the flip-flop.

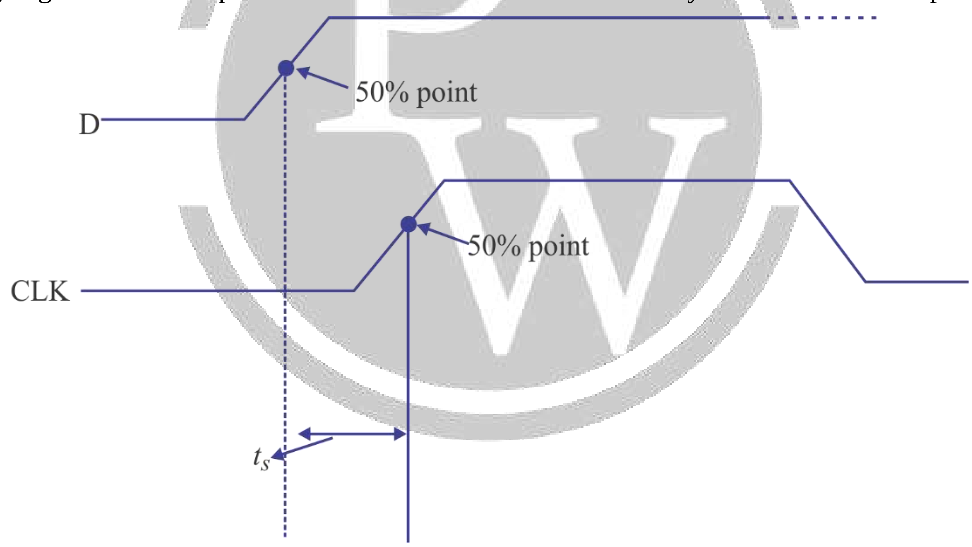


Fig. 4.20.

4.6.3. Hold time (t_h)

It is the time for which the data must remain stable after the triggering edge of the clock.

4.6.4. Clock-pulse width

The minimum time duration for which the clock pulse must remain HIGH and LOW which are designed by manufacturers. Failure to clock pulse width results in unreliable triggering.

4.6.5. Maximum clock frequency

The maximum clock frequency (f_{max}) is the highest rate at which flip-flop can be reliably operated.

4.7. Applications of Flip-Flops

Some of the common applications of flip-flops are as follows:

1. Switch bouncing.
2. Registers.
3. Counters.
4. Memory elements.

4.8. Race Around Condition

JK flip flop suffers from the problem of race around condition. When J = 1 & K = 1, is applied to the JK flip flop and JK flip flop is level triggered then output of the JK flip flop toggles so many times during the pulse width of the clock and output of the flip flop settled either at 1 or 0 depending upon the pulse width of the clock and propagation delay of the flip flop is called race around condition.

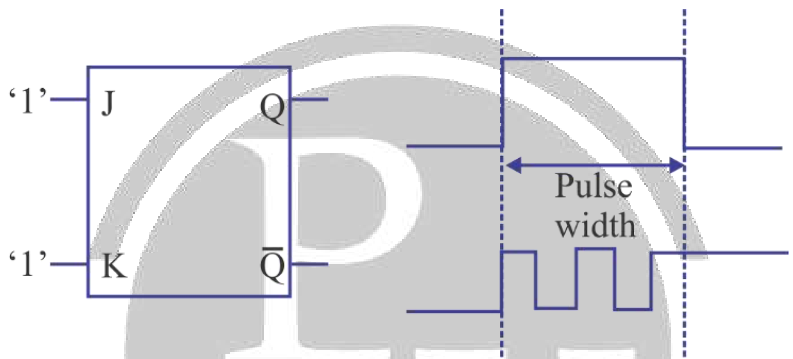


Fig. 4.21. Race Around Condition in JK Flip Flop

To avoid Race Around condition:

- $T_{\text{pulse-width}} < T_{\text{pd}} < T_{\text{clock}}$
- Master Slave flip flop

4.8.1. Master Slave Flip flop:

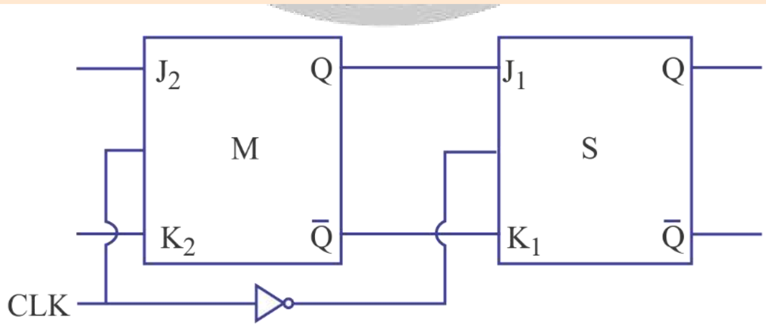


Fig. 4.22. Logic Diagram for Master Slave JK Flip Flop

(a) In master slave flip flop, inverted clock is given to the slave.

(b) Master slave flip flop is used to store single bit because output is taken only from slave flip flop.

(c) Here, master flip flop is level triggered while slave is negative edged triggered.

Note: JK flip flop is also known as Universal flip flop.

4.9. Designing of One Flip Flop by Other Flip Flop

The steps for designing of one flip flop or new flip flop using existing or same existing flip flop.

Step 1: Write the characteristic table for the designed flip flop.

Step 2: Write the excitation table for the available flip flop.

Step 3: Write the logical expression.

Step 4: Minimize the logical expression.

Step 5: Circuit Implementation.

4.10. Shift Registers

An array of flip-flops in required to store binary information, and the number of flip-flops required is equal to the number of bits used to store is referred as registers.
Examples of registers are general purpose registers flags, etc.

Now, the information or data can be stored or entered in serial form (one-bit at a time) or in parallel form (all the bits simultaneously) and can be retrieved like this manner too. The data will be entered or retrieved in serial form is known as temporal code and which is in parallel form is called special code.
Hence, registers can be classified into four categories depending upon the data being entered or retrieved.

4.10.1. SISO (serial-in, serial-out) Shift Register:

In serial-in, serial-out shift register, data input is in serial form and common clock pulse is applied to each of the flip-flop. After each clock pulse, data moves by one position. The output can be obtained in serial form, as shown in figure below:

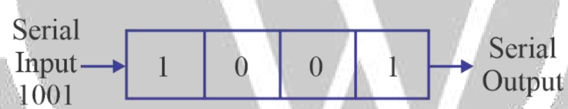


Fig. 4.23. SISO Shift Register

- It is the slowest shift register among all the shift registers.
- To store n-bits in a n-bit SISO register, then the minimum "n" clock pulses are required.
- To retrieve n-bits from a n-bit SISO register, then the minimum "(n-1)" clock pulses are required.

4.10.2. SIPO (serial-in, parallel-out) Shift Register

In serial-in, parallel-out shift register, data is applied at the input of register in serial form and the output can be obtained in parallel form after the completely shifting of data in register. Figure below shows the serial input data, and then parallel output.

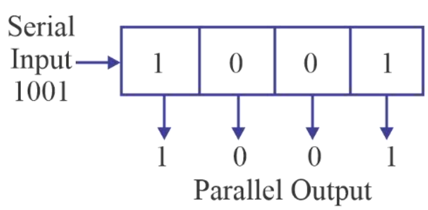


Fig. 4.24. SIPO Shift Register

- To store n-bits in a n-bit SIPO register, the minimum "n" clock pulses are required.
- To retrieve n-bits from a n-bit SIPO register, there is no pulse required.

4.10.3. PISO (parallel-in, serial out) Shift Register

In parallel-in, serial-out shift register, data is loaded into shift register in parallel form and the data output obtained will be serial form as shown in figure below:

- To store n-bit in a n-bit PISO register, a single clock pulse is required.
- To retrieve n-bit from n-bit PISO register, the minimum "(n-1)" clock pulses are required.

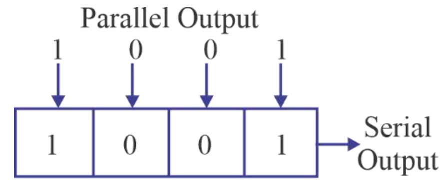


Fig. 4.25. PISO shift register

4.10.4. PIPO (parallel in, parallel out) Shift Register

In parallel-in, parallel-out shift register, data is loaded in parallel form and the data output obtained will be in parallel, as shown in figure below:

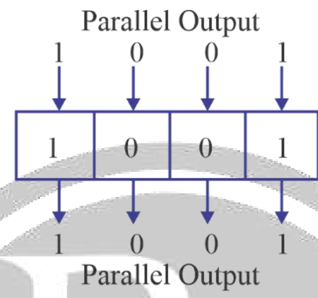


Fig. 4.26. PIPO Shift Register

- To store n-bit in n-bit PIPO register, only a single clock pulse is required.
- To retrieve n-bits from n-bit PIPO register, no clock pulse is required.

Serial Input: The data in the serial form is applied at the serial input after clearing the flip-flops using CLR.

The waveform of serial input shift register is shown below:

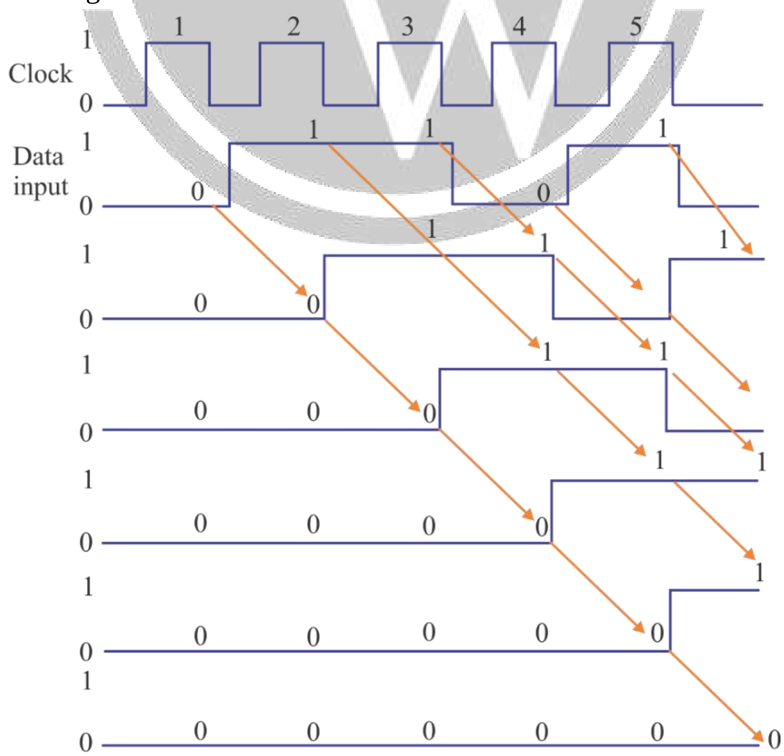


Fig. 4.27.

Parallel Input: Data can be entered in the parallel form making use of the pre-set inputs. Then after clearing the flip-flops, if the data lines are connected to the parallel lines and '1' is applied to the PRESET input.

4.10.5. Universal Shift Register

If the flip-flop outputs of a shift register are accessible, then information entered serially by shifting can be taking out in parallel from the outputs of the flip-flops. If a parallel

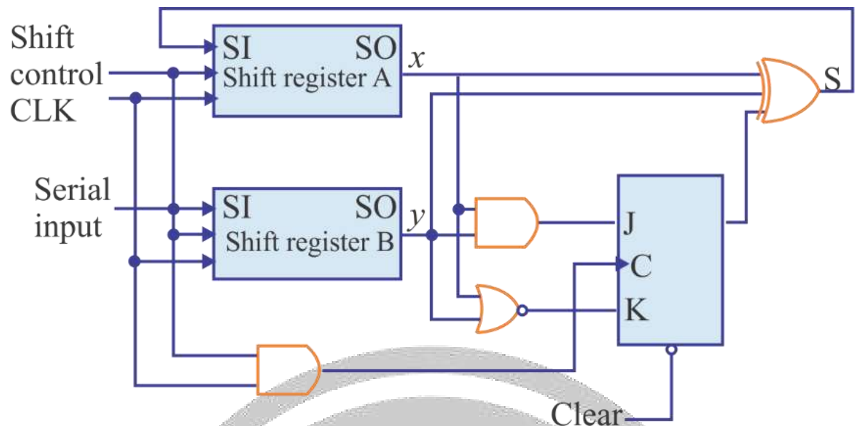


Fig. 4.28. Second form of serial adder

load capability is added to a shift register, then data entered in parallel can be taken out in serial fashion by shifting the data stored in the register. Some shift registers provide the necessary input and output terminals for parallel transfer. They may also have both shift-right and shift-left capabilities.

The most general shift register has the following capabilities:

1. A clear control to clear the register to 0.
2. A clock input to synchronize the operations.
3. A shift-right control to enable the shift-right operation and the serial input and output lines associated with the shift right.
4. A shift-left control to enable the shift-left operation and the serial input and output lines associated with the shift left.
5. A parallel-load control to enable a parallel transfer and the n input lines associated with the parallel transfer.
6. "n" parallel output lines.
7. A control state that leaves the information in the register unchanged in response to the clock.

Other shift registers may have only some of the preceding functions, with at least one. shift operation. A register capable of shifting in one direction only is a unidirectional shift register. One that can shift in both directions is a bidirectional shift register. If the register can shift in both directions and has parallel-load capabilities, it is referred to as a universal shift register. The block diagram symbol and the circuit diagram of a four-bit universal shift register is shown in figure below:

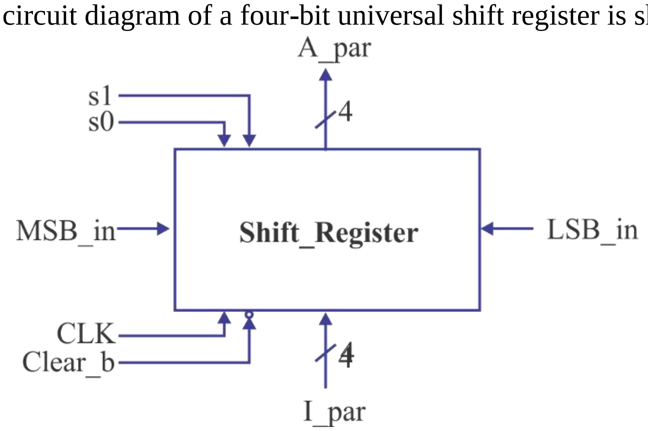


Fig. 4.29. 4-bit Universal Shift Register

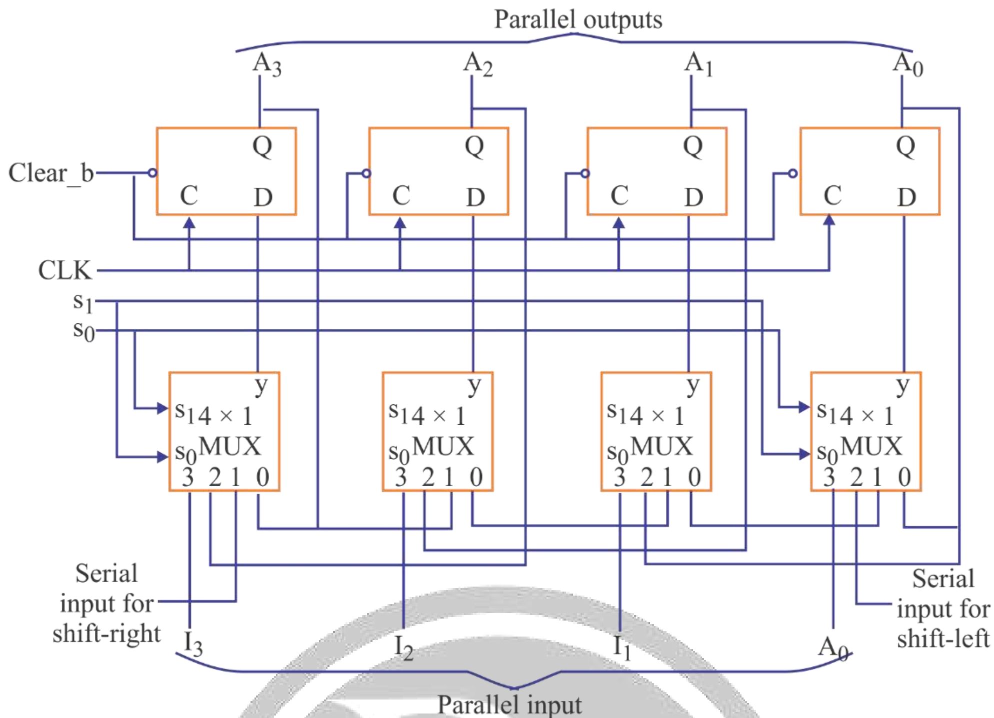


Fig. 4.30. Logic Diagram of 4-bit Universal shift register

The function for the Universal Shift Register is as follows:

Mode Control		Register Operation
S_1	**S_0**	
0	0	No change
0	1	Shift right
1	0	Shift left
1	1	Parallel load

Shift registers are often used to interface digital systems situated remotely from each other. For example, suppose it is necessary to transmit an n-bit quantity between two points. If the distance is far, it will be expensive to use n lines to transmit the its bits in parallel. It is more economical to use a single line and transmit the information serially, one bit at a time. The transmitter accepts the n-bit data in parallel into a shift register and then transmits the data serially along the common line. The receiver accepts the data serially into a shift register. When all n bits are received, they can be taken from the outputs of the register in parallel. Thus, the transmitter performs a parallel-to-serial conversion of data and the receiver does a serial-to-parallel conversion.

4.10.6. Applications of Shift Registers

(a) **Delay line:** A shift register can be used to introduce a delay (Δt) in signals

$$\boxed{\Delta t = N \times \frac{1}{f_c}}$$

Where N is number of stages & f_c is the clock frequency.

(b) Serial-to-parallel converter

(c) Parallel-to-serial converter

(d) Ring counter
(e) Twisted ring counter
(f) Sequence counter

4.11. Asynchronous Counter Or Ripple Counter

A circuit which is used for counting the numbers or pulses is known as counter. Counter is referred to as modulo-N (or divide by N), where the word modulo indicates the number of states in the counter.

4.11.1. 3- Bit Binary Counter

Consider a 3-bit binary counter which has total '8' number of states which require three flip-flops and Q_2, Q_1 and Q_0 are the outputs of those flip-flops.

The circuit diagram or logic circuit diagram for 3-bit binary counter,

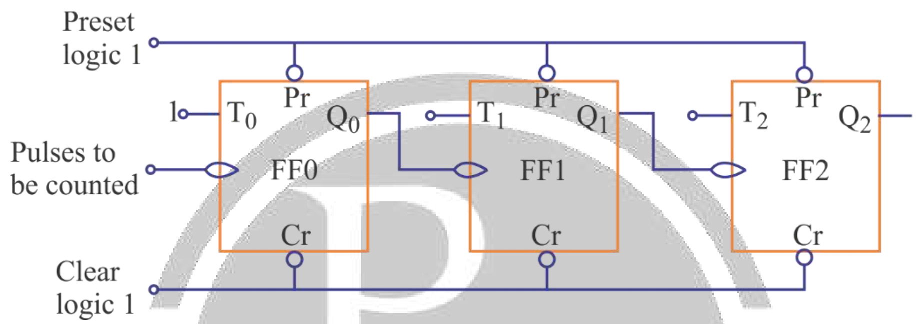


Fig. 4.31. A 3-bit Binary Counter

The truth table for 3-bit binary counter is given as:

Counter state	Count		
	Q_2	Q_1	Q_0
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1

Output waveforms of the above counter is:

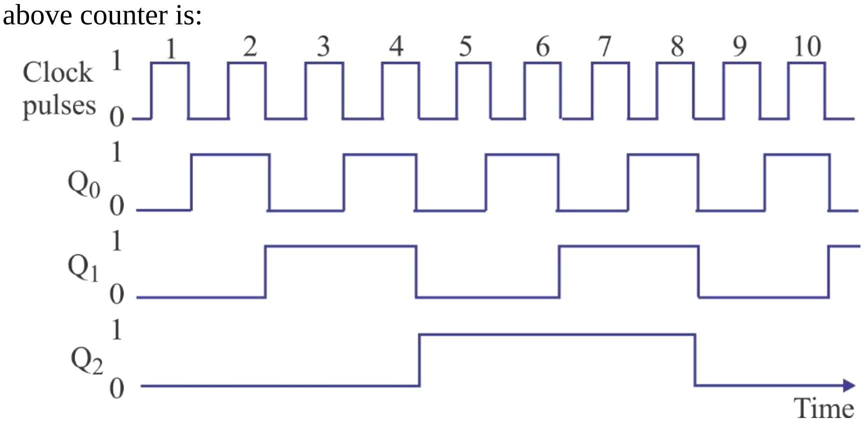


Fig. 4.32.

The frequency 'f' of clock pulses for reliable operation of the counter is given as

Where, N = number of flip-flops

t_{pd} = propagation delay of one flip-flop.

T_s = strobe pulse width.

If during the operation of counter, if some pulses are falsely operated for short duration, known as spikes, which change the state of the flip-flop. It may happen when the propagation delay of each flip-flop may vary and may happen that, all the flip-flops may not change their states or may be only one flip-flop changes its state during the pulse time.

This problem of spikes can be eliminated by using a strobe pulse with the help of strobe pulse, the state will charge only when flip-flops of the counter are in steady state.

Example: In a 4-stage ripple counter, the propagation delay of a flip-flop is 50n sec. If the pulse width of the strobe is 30n sec. Find the maximum frequency at which the counter operator reliably.

Solution: The maximum frequency is

$$f_{\max} = \frac{1}{nt_{pd} + t_s}$$

n = number of flip-flops or stage = 4

t_{pd} = propagation delay of each flip-flop = 50 nsec.

t_s = Strobe pulse width = 30 nsec

$$f_{\max} = \frac{1}{(4\times 50 + 30)\times 10^{-9}} = \frac{1000}{(200+30)}\text{ MHz} = \frac{1000}{230}\text{ MHz}$$

4.11.2. Modulo-6 Asynchronous Down Counter

Down counter is the counter which counters the values of pulses in descending order. Consider or Stable-8 counter ($2^3 = 8$), which uses three flip-flops.

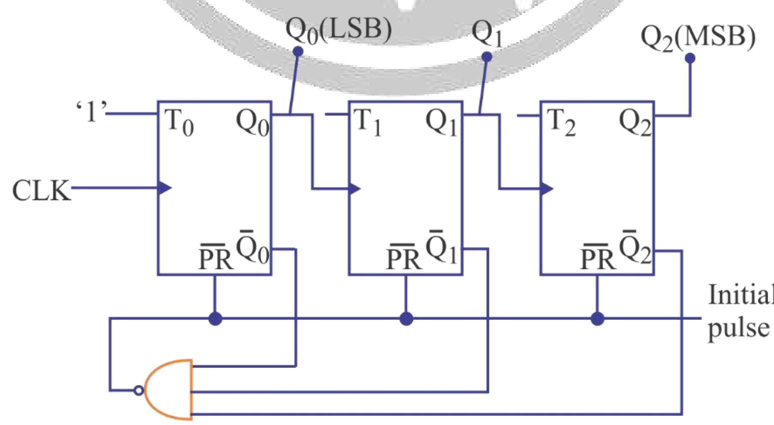


Fig. 4.33.

1. Only sequential counter can be designed. Random counter cannot be designed.
2. Glitch (undesirable state) would appear in case of asynchronous counter.
3. Speed of asynchronous counter is not fast.

4.11.5. BCD Ripple Counter

A decimal counter follows a sequence of 10 states and returns to 0 after the count of 9. Such a counter must have at least four flip-flops to represent each decimal digit, since a decimal digit is represented by a binary code with at least four bits. The sequence of states in a decimal counter is dictated by the binary code used to represent a decimal digit. If the BCD code is used, the sequence of states is as shown in the state diagram. A decimal counter is similar to a binary counter, except that the state after 1001 (the code for decimal digit 9) is 0000 (the code for decimal digit 0).

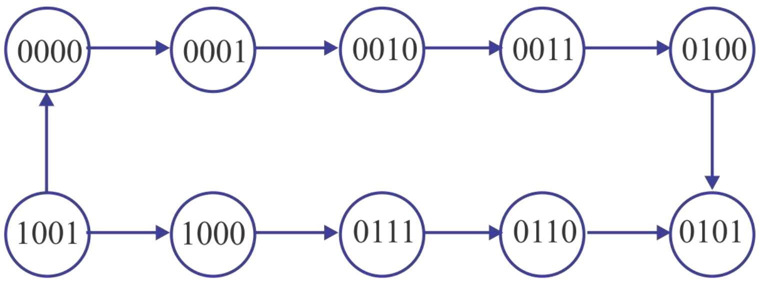


Fig. 4.34. State diagram of a decimal BCD counter.

The logic diagram of a BCD ripple counter using JK flip-flops is shown in figure below. The four outputs arc designated by the letter symbol Q, with a numeric subscript equal to the binary weight of the corresponding bit in the BCD code. Note that the output of Q_1 is applied to the C inputs of both Q_2 and Q_4 and the output of Q_2 is applied to the C and output of Q_2 and Q_3 applied to J through a two input AND gate.

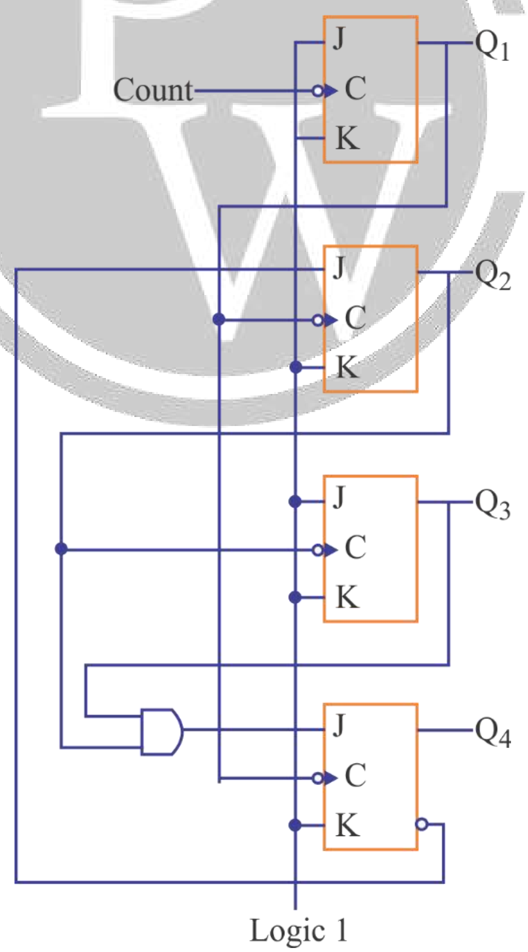


Fig. 4.35. BCD ripple counter

4.12. Synchronous Counter

The ripple counters have the advantage of simplicity (only FLIP-FLOP's are required) but their speed is low because of ripple action. The maximum time is required when the output changes from 111....1 to 00....0 and this limits the frequency of operation of ripple counters.

The speed of operation improves significantly if all the FLIP-FLOPS are clocked simultaneously. The resulting circuit is known as a synchronous counter. Synchronous counters can be designed for any count sequence (need not be straight binary).

The output Q_0 of the least-significant FLIP-FLOP changes for every clock pulse. This can be achieved by using a T-type FLIP-FLOP with $T_0 = 1$. The output Q_0 changes whenever Q_0 changes from 1 to 0. Therefore, if Q_0 is connected to T input (T_1) of the next FLIP-FLOP, Q_1 will change from 1 to 0 (or 0 to 1) when $Q_0 = 1$ ($T_1 = 1$) and will remain unaffected when $Q_0 = T_1 = 0$. Similarly, Q_2 changes whenever Q_1 and Q_0 are both "1". This can be achieved by making the T-input (T_2) of the most-significant FLIP-FLOP equal to $Q_1.Q_0$.

In addition to FF's, synchronous counters require some gates also. JK FLIP-FLOPS are the most commonly used FLIP-FLOP's for the design of synchronous counters. In this, each FLIP-FLOP has two control inputs (J and K) and circuit is required to be designed for each control input. Many programmable logic devices (PLDs) used for the design of digital systems utilise D FLIP-FLOPS for their memory elements, therefore, counter design using D FLIP-FLOPS will be useful for programming inside a PLD. It has only one control input which makes its design simpler than the design using J-K FLIP-FLOPS.

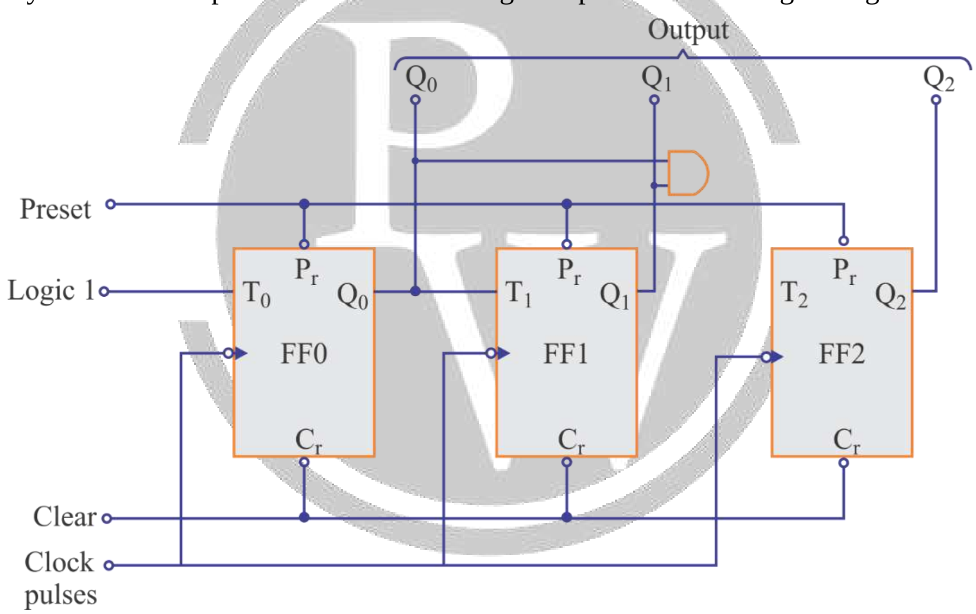


Fig. 4.36. A 3-bit Synchronous Counter

4.12.1. Synchronous Counter Design

Synchronous counters for any given count sequence and modulus can be designed in the following way:

1. Find the number of FLIP-FLOPs required.
2. Write the count sequence in the tabular form.
3. Determine the FLIP-FLOP inputs which must be present for the desired next state from the present state using the excitation table of the FLIP-FLOPS.
4. Prepare K-map for each FLIP-FLOP input in terms of FLIP-FLOP outputs as the input variables.
5. Simplify the K-maps and obtain the minimized expressions.
6. Connect the circuit using FLIP-FLOPS and other gates corresponding to the minimized expressions.

Example: Design a 3-bit synchronous counter using JK Flip-Flops.

Solution: The number of FLIP-FLOPs required is 3. Let the FLIP-FLOPs be FF0, FF1, FF2 and their inputs and outputs are given below:

FLIP-FLOP	Inputs	Outputs
FF0	J_0, K_0	Q_0
FF1	J_1, K_1	Q_1
FF2	J_2, K_2	Q_2

The count sequence and the required inputs of FLIP-FLOPs is shown below.

			FLIP-FLOP INPUTS					
Counter state			FF0		FF1		FF2	
Q_2	Q_1	Q_0	J_0	K_0	J_1	K_1	J_2	K_2
0	0	0	1	X	0	X	0	X
0	0	1	X	1	1	X	0	X
0	1	0	1	X	X	0	0	X
0	1	1	X	1	x	1	1	X
1	0	0	1	X	0	X	X	0
1	0	1	X	1	1	X	X	0
1	1	0	1	X	X	0	X	0
1	1	1	x	1	X	1	x	1
0	0	0						

$Q_0 \backslash Q_2Q_1$	00	01	11	10
0	1	1	1	1
1	x	x	x	x

$J_0 = 1$

(a)

$Q_0 \backslash Q_2Q_1$	00	01	11	10
0	x	x	x	x
1	1	1	1	1

$K_0 = 1$

(b)

$Q_0 \backslash Q_2Q_1$	00	01	11	10
0	0	x	x	0
1	1	x	x	1

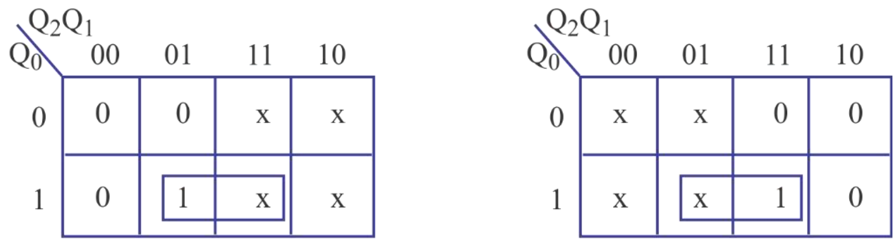
$J_1 = Q_0$

(c)

$Q_0 \backslash Q_2Q_1$	00	01	11	10
0	x	0	0	x
1	x	1	1	x

$K_1 = Q_0$

(d)



$J_2 = Q_0Q_1$ (e) $K_2 = Q_0Q_1$ (f)

Fig. 4.37. K-Maps of 3-bit Synchronous Counter

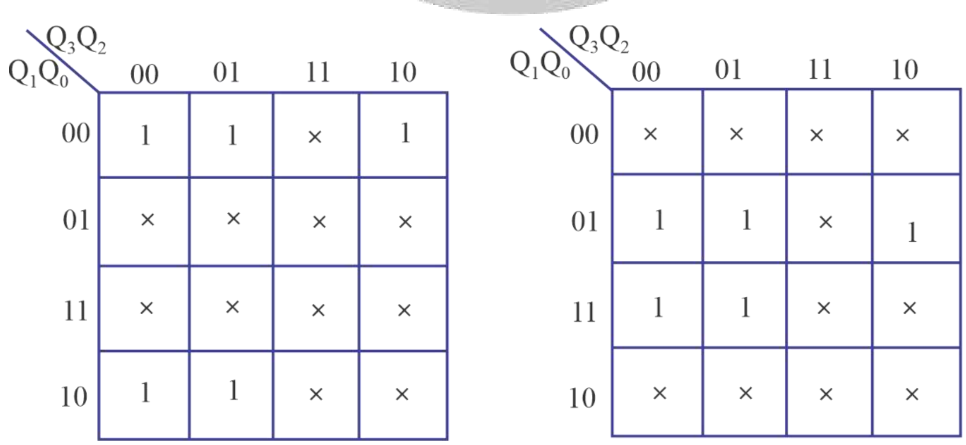
Example:

Design a natural binary sequence mod-8 synchronous counter using D FLIP—FLOPS.

Solution:

The number of FLIP-FLOPS required is 3. Let the FLIP—FLOPS be FF0, FF1 and FF2 with inputs D_0, D_1 and D_2, respectively. Their outputs are Q_0, Q_1, and Q_2 respectively

Counter State			FLIP-FLOP inputs		
Q_2	Q_1	Q_0	D_0	D_1	D_2
0	0	0	1	0	0
0	0	1	0	1	0
0	1	0	1	1	0
0	1	1	0	0	1
1	0	0	1	0	1
1	0	1	0	1	1
1	1	0	1	1	1
1	1	1	0	0	0



J_0 K_0

$Q_1Q_0 \backslash Q_3Q_2$	00	01	11	10
00	0	0	×	0
01	1_1	1	×	0
11	×	×	×	×
10	×	×	×	×

J_1

$Q_1Q_0 \backslash Q_3Q_2$	00	01	11	10
00	×	×	×	×
01	×	×	×	×
11	1	1	×	×
10	1	1	×	×

K_1

$Q_1Q_0 \backslash Q_3Q_2$	00	01	11	10
00	0	×	×	0
01	0	×	×	0
11	1	×	×	×
10	0	×	×	×

J_2

$Q_1Q_0 \backslash Q_3Q_2$	00	01	11	10
00	×	0	×	×
01	×	0	×	×
11	×	1	×	×
10	×	0	×	×

K_2

$Q_1Q_0 \backslash Q_3Q_2$	00	01	11	10
00	0	0	×	×
01	0	0	×	×
11	0	1	×	×
10	0	0	×	×

J_3

$Q_1Q_0 \backslash Q_3Q_2$	00	01	11	10
00	×	×	×	0
01	×	×	×	1
11	×	×	×	×
10	×	×	×	×

K_3

Fig. 4.38. K-Maps for 8-bit Synchronous counter

The K – maps for D_0, D_1 and D_2 are given as,

$Q_0 \backslash Q_2Q_1$	00	01	11	10
0	1	1	1	1
1	0	0	0	0

D_0

$Q_0 \backslash Q_2Q_1$	00	01	11	10
0	0	1	1	0
1	1	0	0	1

D_1

$Q_0 \backslash Q_2Q_1$	00	01	11	10
0	0	0	1	1
1	0	1	0	1

D_2

Fig. 4.39.

The minimised expressions for D0, D1 and D2 are:

$$D_0 = \bar{Q}_0$$

$$D_1 = Q_1\bar{Q}_0 + \bar{Q}_1 Q_0$$

$$D_2 = Q_2\bar{Q}_0 + Q_2\bar{Q}_1 + Q_2Q_1Q_0 = Q_2(\bar{Q}_0 + \bar{Q}_1) + \bar{Q}_2Q_1Q_0 = Q_2(\overline{Q_0.Q_1}) + \bar{Q}_2(Q_1Q_0)$$

$$= Q_2 \oplus Q_1.Q_0$$

The complete circuit of the synchronous counter using positive edge triggered D FLIP-FLOPs is shown in figure below as

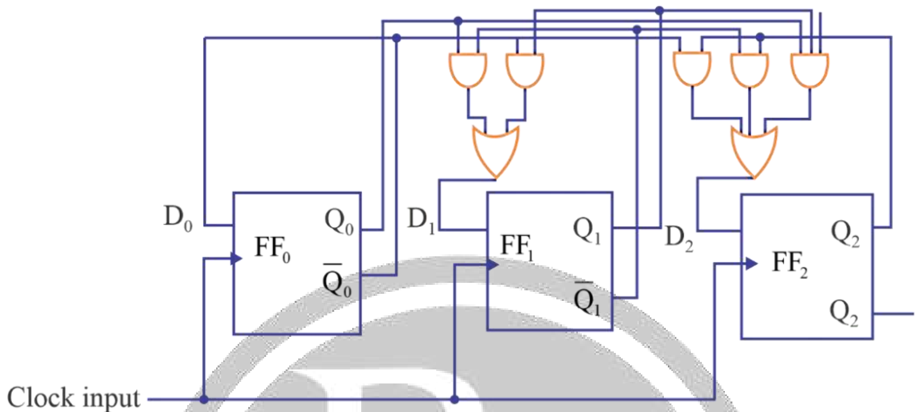


Fig. 4.40. 8-bit Synchronous Counter Circuit

4.12.2 Synchronous Sequential Circuit Models

A general block diagram of clocked sequential circuit is also known as finite stable machine (FSM). Depending upon the external outputs, there are two types of models of sequential circuits.

Mealy Model

In mealy model, the next of the function depends on present state as well as present inputs.

Moore Model

In more model, the next state depends on the present state. The block diagram of a Moore model is given as

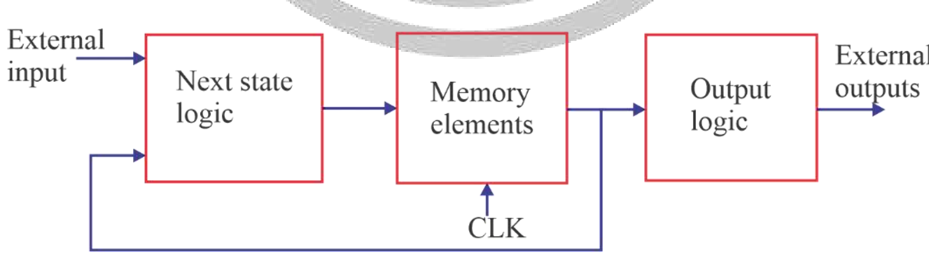


Fig. 4.41.

The systematic procedure for designing of clocked sequential circuit is based on the concept of 'state'. Hence the sequence of inputs, present & next states and output is represented by a state table or state diagram & if the procedure follows in the form of flow chart, it is known as algorithms state machine (ASM).

4.12.3. State Diagram

It is a directed graph, consisting of vertices (or nodes) and directed are between the nodes. Every state of the circuit is represented by a node in the graph. A node is represented by a circle with the name of the state written inside the circle. The directed are represents the state transitions.

With the circuit in may one state, at the occurrence of a clock pulse, there will be a state transition to the next state and there will be an output, corresponding to the requirement of the circuit. This state transition is represented by a directed line and we use each (/) for representing present state and the next state.

Example:

Draw the state diagram of D-flip-flop.

Solution:

The D flip-flop has only input (D) & two output states (Q = 0 & Q = 1).

Using the state table or characteristic table of the D-flip flop. The state diagram is given as

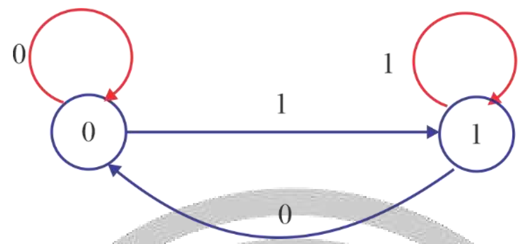


Fig. 4.42.

Example:

Draw the state diagram of a JK flip-flop.

Solution:

A JK flip-flop has inputs (J & K) and one clock input (CLK) and the two output states (Q = 0 & Q = 1).

Using the state table or characteristic table of the JK flip-flop, the state diagram is given as

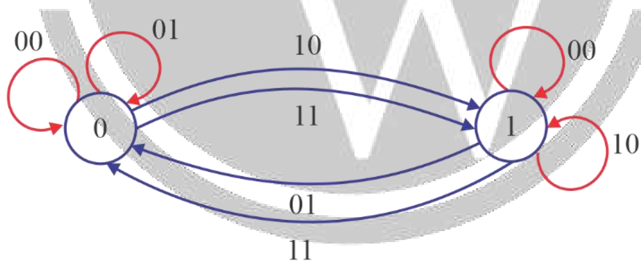


Fig. 4.43.

5 NUMBER SYSTEM

5.1. NUMBER SYSTEM

5.1.1. Base (Radix)

Total number of digit used in the system

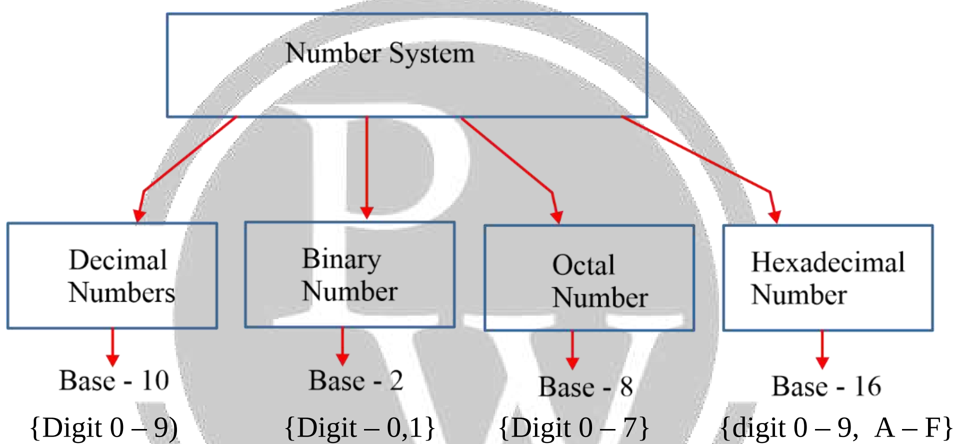


Fig. 5.1.

5.1.2. Decimal Number System

$\ldots \quad 10^4 \quad 10^3 \quad 10^2 \quad 10^1 \quad 10^0 \quad 10^{-1} \quad 10^{-2} \quad 10^{-3} \ldots$

$\ldots \quad a_4 \quad a_3 \quad a_2 \quad a_1 \quad a_0 \quad a_{-1} \quad a_{-2} \quad a_{-3} \ldots$

$a_i \rightarrow$ Coefficient of decimal number system

$10^i \rightarrow$ Weight of decimal number system

Example: - $(501.23)_{10}$

$10^2 \quad 10^1 \quad 10^0 \quad 10^{-1} \quad 10^{-2}$

$5 \quad 0 \quad 1 \quad 2 \quad 3$

Base	**Digit**
2	0, 1
3	0, 1, 2
4	0, 1, 2, 3
5	0, 1, 2, 3, 4
6	0, 1, 2, 3,4,5

7	0, 1, 2, 3, 4, 5, 6
8	0, 1, 2, 3, 4, 5, 6, 7
9	0, 1, 2, 3, 4, 5, 6, 7, 8
10	0, 1, 2, 3, 4, 5, 6, 7, 8, 9
11	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A
12	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B
13	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C
14	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D
15	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E
16	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F

5.1.3 Binary Number System (Base (Radix) = 2)

…	2^4	2^3	2^2	2^1	2^0	2^{-1}	2^{-2}	2^{-3} …
…	a_4	a_3	a_2	a_1	a_0	a_{-1}	a_{-2}	a_{-3} …

2^i → Weight of Binary number system

a_i → Coefficient of Binary number system {0, 1}

Example:- $(101.11)_2$

2^2	2^1	2^0	2^{-1}	2^{-2}
1	0	1	1	1

5.1.4. Octal Number System (Base (Radix) = 8)

…	8^3	8^2	8^1	8^0	8^{-1}	8^{-2}	8^{-3}…
…	a_3	a_2	a_1	a_0	a_{-1}	a_{-2}	a_{-3}…

8^i → Weight of Octal number system

a_i → Coefficient of Octal number system {0 -7}

Example:- $(728.64)_8$

8^2	8^1	8^0	8^{-1}	8^{-2}
7	2	8	6	4

5.1.5 Hexadecimal Number System (Base (Radix) = 16):

…	16^3	16^2	16^1	16^0	16^{-1}	16^{-2}	16^{-3}…
…	a_3	a_2	a_1	a_0	a_{-1}	a_{-2}	a_{-3}…

16^i → Weight of Hexadecimal number system

a_i → Coefficient of Hexadecimal number system {0 – 9, A–F}

Example: $(A2C.F)_{16}$

16^2	16^1	16^0	16^{-1}
A	2	C	F

5.1.6. In base conversion 2 key points are there:

(A) Any base to Decimal conversion
(B) Decimal to any other base conversion

(A) Any base to Decimal conversion:

$$\left(a_3\ \ a_2\ \ a_1\ \ a_0\ .\ a_{-1}\ \ a_{-2}\right) = (\quad)_{10}$$

$$\left(a_3 \times r^3 + a_2 \times r^2 + a_1 \times r^1 + a_0 \times r^0 + a_{-1} \times r^{-1} + a_{-2} \times r^{-2}\right)_{10}$$

Case (1) : Binary to Decimal conversion

Ex. $(1011.11)_2 = (\quad)_{10}$

$\Rightarrow$ $\left[(1\times 2^3)+(0\times 2^2)+(1\times 2^1)+(1\times 2^0)+(1\times 2^{-1})+(1\times 2^{-2})\right]$

$\Rightarrow$ $[8+0+2+1+0.5+0.25]_{10}$

$\Rightarrow$ $(11.75)_{10}$

Case (2) : Octal to Decimal conversion

Ex. $(721.4)_8 = (\quad)_{10}$

$\Rightarrow$ $\left[(7\times 8^2)+(2\times 8^1)+(1\times 8^0)+(4\times 8^{-1})\right]_{10}$

$\Rightarrow$ $[448+16+1+0.5]_{10}$

$\Rightarrow$ $(465.5)_{10}$

Case (3) : Hexadecimal to Decimal conversion

Ex. $(A2B.C)_{16} = (\quad)_{16}$

$\Rightarrow$ $\left[(A\times 16^2)+(2\times 16^1)+(B\times 16^0)+(C\times 16^{-1})\right]_{10}$

$\Rightarrow$ $\left[(10\times 256)+(2\times 16)+(11\times 1)+(12\times 16^{-1})\right]_{10}$

$\Rightarrow$ $[2560+32+11+0.75]_{10}$

$\Rightarrow$ $(2603.75)_{10}$

Case (4) : Base 5 to Decimal conversion

Ex. $(432.22)_5 = (\quad)_{10}$

$\Rightarrow$ $\left[(4\times 5^2)+(3\times 5^1)+(2\times 5^0)+(2\times 5^{-1})+(2\times 5^{-2})\right]_{10}$

$\Rightarrow$ $[100+15+2+0.4+0.08]_{10}$

$\Rightarrow$ $(117.48)_{10}$

(B) Decimal to any other Base conversion

$$(\underbrace{a_3\ \ a_2\ \ a_1\ \ a_0}_{\text{Before Decimal}}\ .\ \underbrace{a_{-1}\ \ a_{-2}\ \ a_{-3}}_{\text{After Decimal}})_{10} = (\quad)_r$$

r	a_3 a_2 a_1 a_0	Remainder
		b_0
		b_1
		b_2
		b_3

$$0\cdot a_{-1}a_{-2}a_{-3}\times r = x_0\cdot x_{-1}x_{-2}$$
$$0\cdot x_{-1}x_{-2}\times r = x_1\cdot x_{-3}x_{-4}$$
$$0\cdot x_{-3}x_{-4}\times r = x_2\cdot x_{-5}x_{-6}$$

$$(a_3a_2a_1a_0\cdot a_{-1}a_{-2}a_{-3})_{10} = (b_3b_2b_1b_0\cdot x_0x_1x_2)_r$$

Case (1) : Decimal to Binary Base conversion.

Ex.

$$(19.75)_{10} = (\quad)_2$$

Before Decimal R After Decimal

		R
2	19	1
2	9	1
2	4	0
2	2	0
	1	1

$0.75 \times 2 = 1.5$ 1
$0.5 \times 2 = 1.0$ 1

$$(19.75)_{10} = (10011.11)_2$$

Case (2) : Decimal to Octal Base conversion.

Ex.

$$(210.23)_{10} = (\quad)_8$$

Before Decimal Remainder After Decimal

		Remainder
8	210	2
8	26	2
	3	3

$0.23 \times 8 = 1.84$ 1
$0.84 \times 8 = 6.72$ 6
$0.72 \times 8 = 5.76$ 5

$$(210.23)_{10} = (322.165)_8$$

Case (3): Decimal to Hexadecimal Base conversion.

Ex.

$$(1228.55)_{10} = (\quad)_{16}$$

Before Decimal Remainder After Decimal

		Remainder
16	1228	12 (C)
16	76	12 (C)
	4	4

$0.55\times 16 = 8.8$ 8
$0.8 \times 16 = 12.8$ 12(C)

$$(1228.56)_{10} = (4CC.8C)_{16}$$

5.2. Some Special Case

Case (1): Binary to Octal base conversion

Example: $(10110111)_2 = (\quad)_8$

Octal → means base 8

$8 = 2^3$

Every three digits of binary represent one digit of octal

010 110 111

2 6 7

Hence $(10110111)_2 = (267)_8$

Case (2): Binary to Hexadecimal base conversion

Example: $(1011011)_2 = (\quad)_{16}$

Hexadecimal → means base 16

$16 = 2^4$

Every four digits of binary represent one digit of Hexadecimal.

0101 1011

5 11(B)

Hence $(1011011)_2 = (5B)_{16}$

5.1.2. BCD (Binary Coded Decimal)

- In this each digit of the decimal number is represented by its four-bit binary equivalent. It is also called natural BCD or 8421 code. It is weighted code.
- **Excess – 3 Code:** This is an non weighted binary code used for decimal digits. Its code assignment is obtained from the corresponding value of BCD after the addition of 3.
- **BCO (Binary Coded Octal):** In this each digit of the Octal number is represented by its three-bit binary equivalent.
- **BCH (Binary Coded Hexadecimal):** In this each digit of the hexadecimal number is represented by its four bit binary equivalent.

Decimal Digits	BCD 8421	Excess – 3	Octal digits	BCO	Hexadecimal Digits	BCH
0	0000	0011	0	000	0	0000
1	0001	0100	1	001	1	0001
2	0010	0101	2	010	2	0010
3	0011	0110	3	011	3	0011
4	0100	0111	4	100	4	0100
5	0101	1000	5	101	5	0101
6	0110	1001	6	110	6	0110
7	0111	1010	7	111	7	0111
8	1000	1011			8	1000

9	1001	1100			9 A B C D E F	1001 1010 1011 1100 1101 1110 1111

Don't care values or unused states in BCD code are 1010, 1011, 1100, 1101, 1110, 1111.

Don't care values or unused states in excess – 3 code are 0000, 0001, 0010, 1101, 1110, 1111.

The binary equivalent of a given decimal number is not equivalent to its BCD value.

Example: $25_{10} = 11001_2$.

The BCD equivalent of decimal number 25 = 00100101 from the above example the BCD value of a given decimal number is not equivalent to its straight binary value.

The BCO (Binary Coded Octal) value of a given Octal number is exactly equal to its straight binary value.

Example: $25_8 = 21_{10} = 010101_2$

The BCO Value of 25_8 is 010101.

From the above example, the BCO value of a given Octal number is same as binary equivalent of the same number.

The BCH (Binary Coded Hexadecimal) value of a given hexadecimal number is exactly equal to its straight binary.

Example: $25_{16} = 37_{10} = 100101_2$

The BCH value of hexadecimal number 25_{16} = 00100101.

From this example the above statement is true.

Complement	Binary r =2	Octal r = 8	Decimal r = 10	Hexadecimal r = 16
(r – 1)'s	1's	7's	9's	15's
r's Complement	2's	8's	10's	16's

Example: Add the two Binary numbers 101101_2.

Augned	101101
addend	100111
	1111
Sum	1010100

Example: Subtract the Binary number 100111_2 from 101101_2.

Minuend :	101101
Subtracted:	100111
Difference:	000110

Example: Multiple the Binary number 1011_2 from 101_2.

```
Multiplicand:   1011
Multiplier:     X101
----------------------
                1011
               0000
              1011
           +
----------------------
Product:      110111
```

While storing the signed binary numbers in the internal registers of a digital computer} most significant bit position is always reserved for sign bit and the remaining bits are used for magnitude.

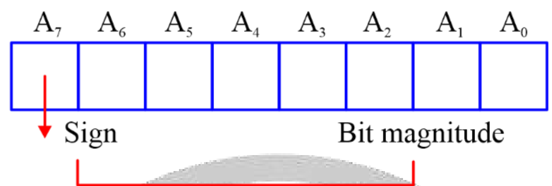


Fig. 5.2.

When the binary number is positive, the sign is represented by '0’. When the number is negative, the sign is represented by ' 1’.

5.2.2. Fixed-Point Representation and Floating-Point Representation;

The representation of the decimal point (ordinary point) in a register is complicated by the fact that it is characterized by a position between two flip- flops in the register.

There are two ways of specifying the position of the decimal point in a register.

(1) Fixed Point and

(2) Floating Point.

The fixed point method assumes that the decimal point (or binary point) is always fixed in one position. The two positions most widely used are (1) a decimal point in the extreme left of the register to make the stored number a fraction, and (2) a decimal point in the extreme right of the register to make the stored number an integer.

The floating-point representation uses a second register to store a number that designates the position of the decimal point in the first register.

Positive numbers are stored in the registers of digital computer in sign magnitude form only.

Negative number can be represented in one of three possible ways.

1. Signed – magnitude representation.
2. Signed – 1’s complement representation.
3. Signed – 2’s complement representation.

Example: +9 –9

Signed – magnitude 0 0001001 (a) 1 000 1001 signed – magnitude

(b) 1 111 0110 signed – 1’s complement

(c) 1 111 0111 signed – 2’s complement

The 2’s complement of a given binary number can be formed by leaving all least significant zeros and the first non-zero digit unchanged, and then replacing 1’s by 0’s and 0’s by 1’s in all other higher significant digits.

Example: The 2’s complement of 10011000_2 is 01101000.

Subtraction using 2’s complement: Represent the negative number in signed 2’s complement from, add the two numbers, including their sign bit, and discard any carry out of the most significant bit.

Since negative numbers are represented in 2’s compliment form, negative results also obtained in signed 2’s compliment form.

Example: 1’s complement:

+ 6 0000110	– 6 1111001	+ 6 0000110	– 6 1111001
+ 9 0001001	+ 9 0001001	– 9 1110110	– 9 1110110
+ 15 0001111	+3 (i) 0000010	– 3 1111100	– 15 (1) 1101111
	Carry + 1		Carry + 1
	+ 3 0000011		1110000
	carry		carry

The advantage of signed 2’s complement representation over the signed 1’s compliment from (and the signed – magnitude form) is that it contains only one type of zero.

The general form of floating – point number is mr^e. Where M = Mantissa, r = base, e = exponent.

Example: $+ 0.3574 \times 10^5$.

The mantissa can be a fixed point fraction or fixed point integer.

Normalization: Getting non-zero digit in the most significant digit position of the mantissa is called Normalization.

- If the floating point number is normalized, more number of significant digits can be stored, as a result accuracy can be improved.
- A zero cannot be normalized because it does not contain a non-zero digit. The hexadecimal code is widely used in digital systems because it is very convenient to enter binary data in a digital system using hexcode.
- The parity of a digital word is used for detecting error in digital transmission. Hollerith code is used for punched card data.
- In weighted codes, each position of the number has specific weight. The decimal value of a weighted code number is the algebraic sum of the weights of those positions in which 1's appears.
- Most frequently used weighted codes are 8421, 2421 code, 5211 code and 8421 code.
- **Reflective Code:** A code is called reflective or self-complimenting, if the code for 9 is the compliment for the code for 0, code for 8 is the compliment from 1 and so on. $2421, 842'1', 5211$ are examples for reflected codes.
- **Sequential Code:** A code is called sequential, if each successive code-is one binary number greater than its preceding code.

Example: 8421

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6 DIGITAL TO ANALOG CONVERTER

6.1. Basic concept

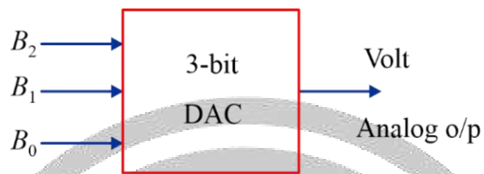

Fig. 6.1.

(i) Resolution/Step-Size

- Smallest change in analog output due to change in digital input

$$\left[\% \text{ Resolution} = \frac{\text{Step-size}}{\text{Full-Scale value}} \times 100\right]$$

$$\left[\% \text{ Resolution} = \frac{1}{2^n - 1} \times 100\right]$$

Note: Analog output = (Resolution or Step - size) × Decimal equivalent of binary input)

(ii) Accuracy

- It is specified in terms of full-scale error expressed as a percentage of full-scale output.
- Full-scale Error is the deviation of actual output from linear curve.

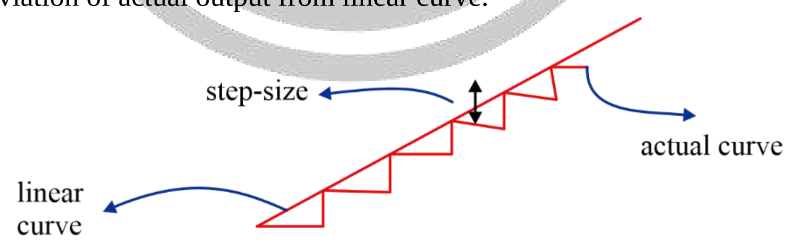

Fig. 6.2.

General tip to find resolution, Put LSB = 1 and other bits = 0.

The analog o/p obtained is resolution.

$$\left[\% \text{ Full-scale Error} = \frac{1}{2} \times (\% \text{ Resolution})\right]$$

(iii) Off-set Voltage

- Output of DAC at zero input is off-set voltage
- It occurs due to non-idealist of op-amp.

6.2. Types of DAC

(i) Weighted Resistor DAC

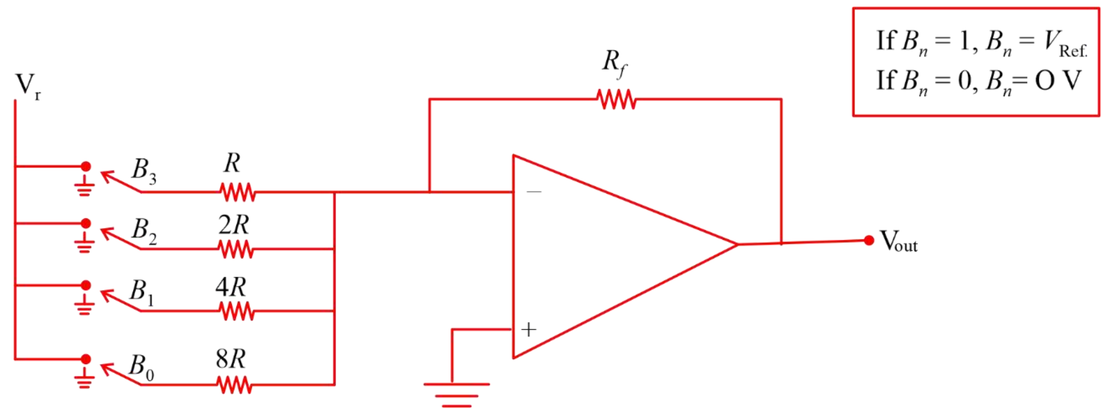

Fig. 6.3.

- Superposition Theorem is used to calculate output voltage
- Output voltage

$$\left[V_{out} = \frac{V_{Ref}}{2^{n-1}}\,\text{Resolution}\left(\frac{R_f}{R}\right) \times \text{Decimal equivalent of binary input}\right]$$

Where, n = no of bits of digital input

Resolution, $$\left[\text{Resolution} = \frac{V_{Ref}}{2^{n-1}}\left(\frac{R_f}{R}\right)\right]$$

- Full-Scale voltage, $$\left[V_{FSD} = -\frac{V_{Ref}}{2^{n-1}}\left(\frac{R_f}{R}\right)(2^{n-1})\right]$$

(ii) Accuracy R-2R Ladder network

(a) Non-Inverting configuration

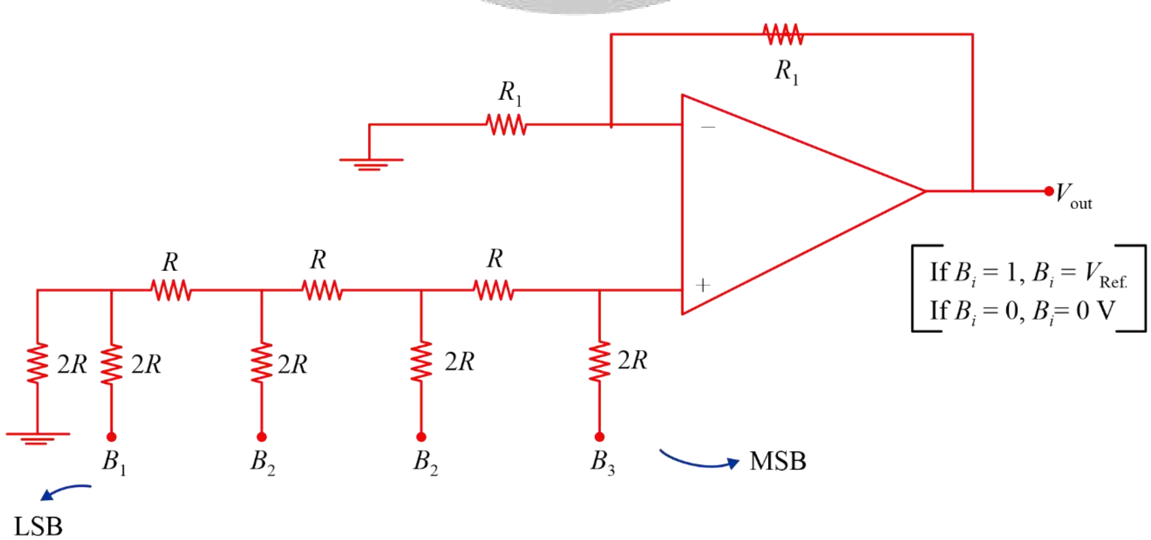

Fig. 6.4.

$$\left[V_{out} = \frac{V_{Ref}}{2^n} \text{ Resolution} \left(1 + \frac{R_f}{R_1}\right) \times \text{Decimal equivalent of binary input}\right]$$

Problem Solving Tip:

- Remember to simplify the ladder network using Thevenin Theorem is the circuit is modified. For modified circuit, above formula will fail.

(b) Inverting configuration

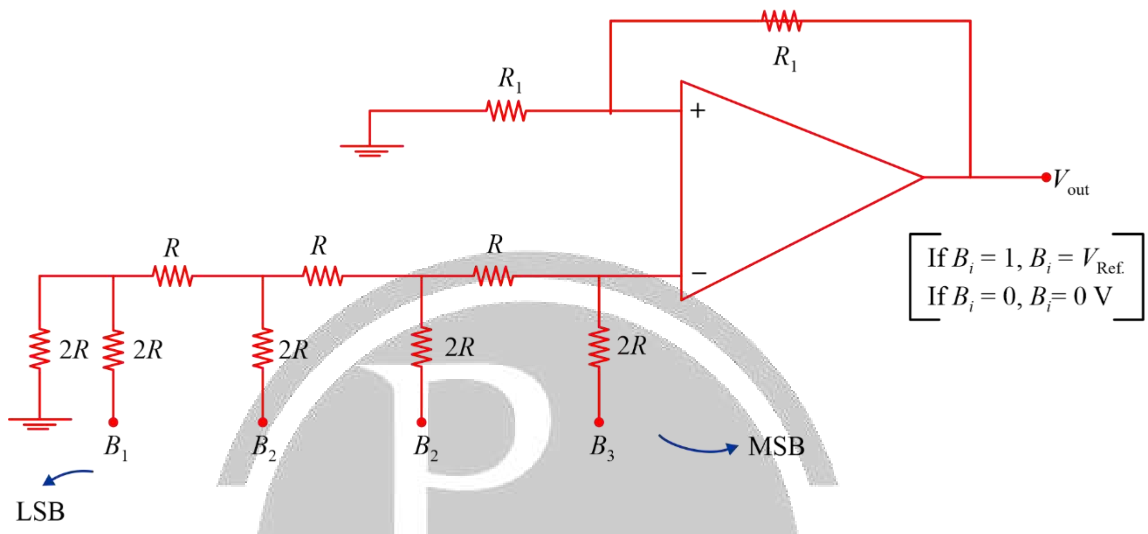

Fig. 6.5.

$$\left[V_{out} = \frac{-V_{Ref}}{2^n} \text{ Resolution} \left(1 + \frac{R_f}{R + R_1}\right) \times \text{Decimal equivalent of binary input}\right]$$

Problem Solving Tip

- Simplify ladder network using Thevenin Theorem and use virtual ground concept.

(iii) Current switched R-2R Ladder Network

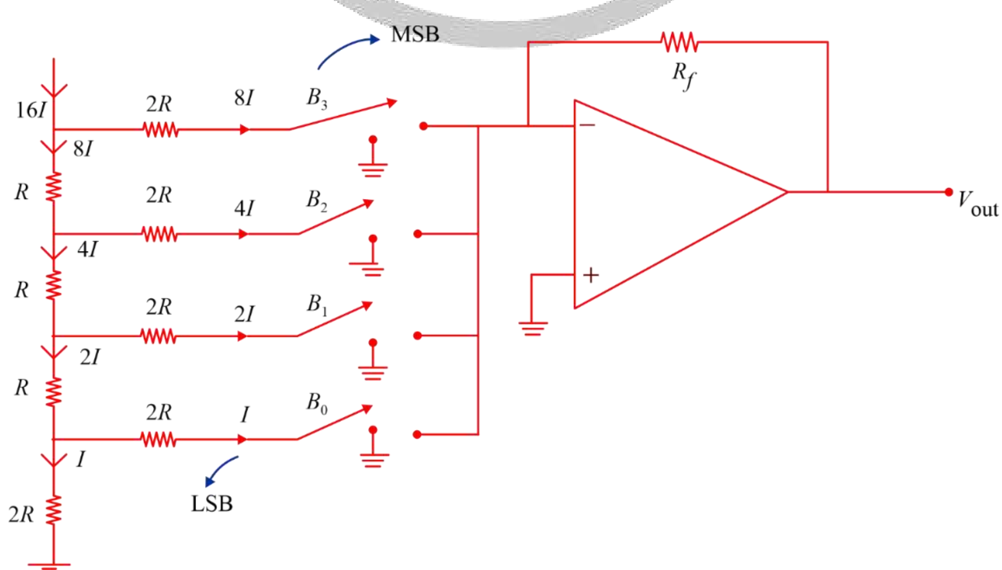

Fig. 6.6.

$$\left[V_{out} = \frac{-V_{Ref}}{2^n} \text{ Resolution} \left(\frac{R_f}{R}\right) \times \text{Decimal equivalent of binary input}\right]$$

Problem Solving Tip

- Use KCl and CDR to calculate total current 'I'.
- Calculate equivalent resistance across V_{Ref}.
  Output voltage is obtained as [volt = $-I R_f$]

6.4. Basic concept

(i) Counter-Type ADC/Ramp-Type ADC

- It is the time for which capacitor acquires the charge supplied by source.

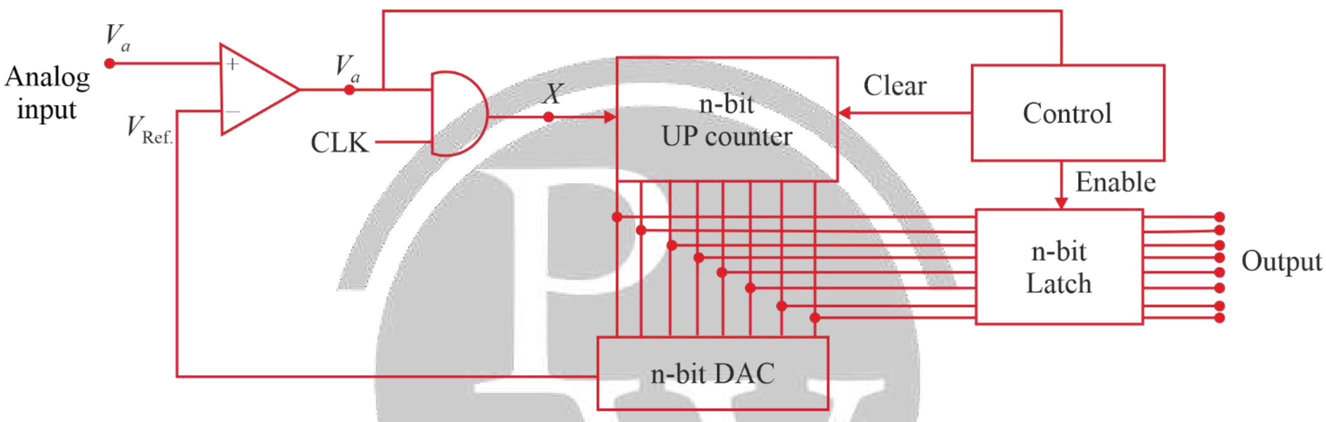

Fig. 6.7.

Important Points:

- Round-up count is greater than actual count.
- Error in quantized output is positive
- Total conversion time = T_{CLK} × Decimal equivalent of binary.
- Maximum conversion Time = (2^{n-1}) T_{CLK}.
  where, n = No. of bits of counter.
- Minimum conversion time = T_{CLK}.
- Average conversion Time = (2^{n-1}) T_{CLK}.
- Conversion time is directly proportional to analog input.

$$\left[\frac{V_{a1}}{V_{a2}} = \frac{T_1}{T_2}\right]$$

Working

Initially, counter o/p = 0, DAC o/p = 0, $V_{Ref.}$ = 0

- If $V_a > V_{Ref.}$, $V_1 = +V_{sat}$ (logic '1')

  $X = 1 \cdot$CLK = CLK, counter starts counting.

  Counter keeps on counting until $V_a < V_{Ref.}$

- If $V_a < V_{Ref.}$, $V_1 = -V_{sat}$ (logic '0')

  $X = 0 \cdot CLK = 0$, counter stops counting (const.)

- Counter enables latch such that counter o/p enters latch and clears counter o/p (set to 0)

  Output is obtained via latch.

  [Quantized analog = Resolution × Decimal equivalent of binary]

(ii) Successive Approximation Register ADC

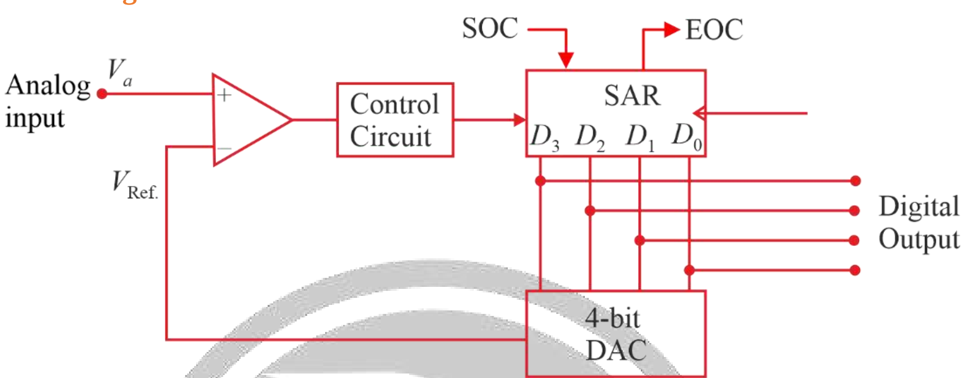

Fig. 6.8.

SOC → Start of Conversion

EOC → End of Conversion

Working

- If $V_a > V_{Ref.}$, SAR will set current bit at clock pulse.
- If $V_a > V_{Ref.}$, SAR will reset previous bit and set current bit at clock pulse.
- SAR continues to work until all bits are accessed one by one and once it happen, SAR makes EOC signal high to indicate end of conversion.

  e.g. – $V_a = 3.5\ V$, $V_{Ref.} = 5V$, 4-bit ADC Resolution of DA $= \frac{5}{2^4-1} = \frac{1}{3}$

  $T_{CLK} \to$ o/p = 1000 $\qquad V_{Ref.} = \frac{1}{3} \times 8 = 2.67 < V_a$

  $2T_{CLK} \to$ o/p = 1100 $\qquad V_{Ref.} = \frac{1}{3} \times 12 = 4 > V_a$

  $3T_{CLK} \to$ o/p = 1010 $\qquad V_{Ref.} = \frac{1}{3} \times 10 = 3.33 < V_a$

  $4T_{CLK} \to$ o/p = 1011 $\qquad V_{Ref.} = \frac{1}{3} \times 11 = 3.67 > V_a$

  Final output = 1010

Important Points:

Total conversion time = $n \times T_{CLK}$

Where, n = No. of bits in SAR

- Conversion time is independent of analog input
- Conversion time is directly proportional to no. of bits in SAR.

(iii) Flash Type ADC

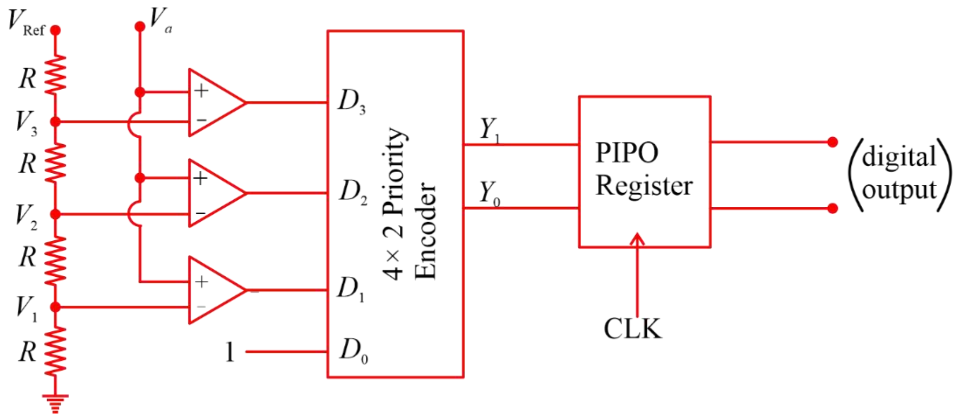

Fig. 6.9.

Important Points:

- It is the fastest *ADC* and doesn't require DAC.
- Conversion time is due to PIPO register.
- Conversion time is = $T_{CLK} + \Delta t_{pd}$

  Where, Δt_{pd} = Propagation delay of combinational circuit
- No. of registers required = 2^n
- No. of comparators required = $2^n - 1$

  Where, n = No. of bits of priority encoder

(iv) Dual Slope Integrating Type *ADC*

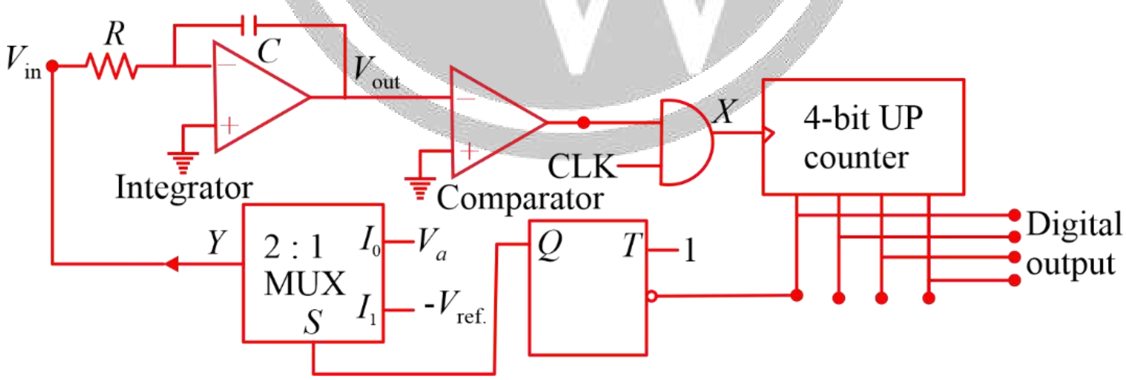

Fig. 6.10.

Working

Initially, counter and FE are cleared.

$S = 0$, $V_{in} = V_a$

$$V_{cut} = -\frac{V_a}{RC}t < 0 \Rightarrow V_1 = \text{logic '1'}$$

$X = 1 \cdot \text{CLK} = \text{CLK}$, counter starts counting.

At $t = 0$, $\quad V_{out} = \dfrac{-V_a}{RC}t = 0$, count = 0000

At $t = 15\ T_{CLK}$, $\quad V_{out} = \dfrac{-V_a}{RC}t < 0$, count = 1111

At $t = 16\ T_{CLK}, = T_1$ $\quad V_{out} = \dfrac{-V_a}{RC}T_1$, count = 0000 MSB provides –*ve* edge triggering to $T - FF$

$Q = S = 1$, $V_{in} = -V_{Ref}$.

$$V_{out} = \int_0^{T_1} \frac{-V_a}{RC}dt + \int_{T_1}^{T_2} \frac{V_{Ref}}{RC}dt$$

Counter keeps on counting until $V_{out} \to 0^+$

Such that V_1 = logic '*0*' $X = 0$, CLK = 0

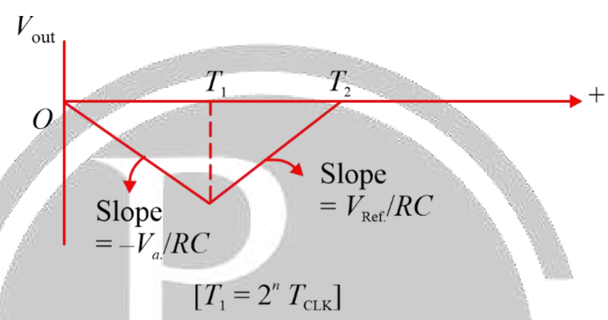

Fig. 6.11.

Important Points:

- It is the most accurate ADC.
- It is the slowest ADC
- Total conversion time, $T_2 = 2^n\left(1 + \dfrac{V_a}{V_{Ref.}}\right)T_{CLK}$
- Maximum conversion time, T_2 max, = $2^{n+1}\ T_{CLK}$
- Minimum conversion time, T_2 min, = $2^n\ T_{CLK}$
- Necessary condition for successful conversion,

$$[V_a \le V_{Ref.}]$$

Note:

- Speed of ADC'S → Flash – Type < SAR – Type > Dual Slope Integrating Type
- Speed of Ramp – Type ADC depends on analog input.

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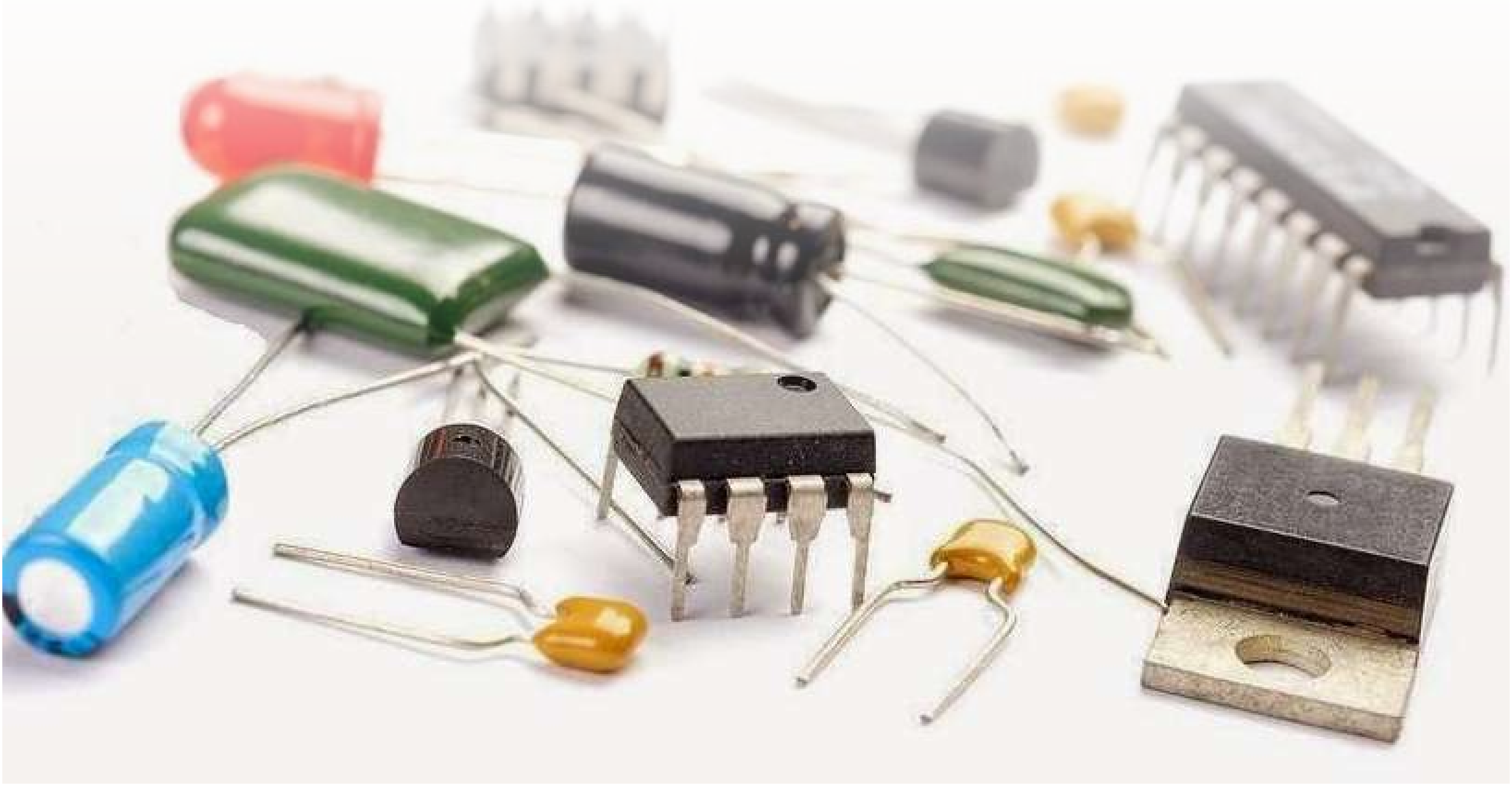


ELECTRONIC DEVICES

INDEX

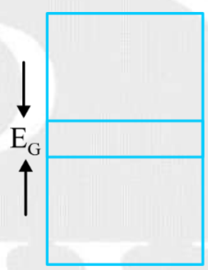
1 SEMICONDUCTOR DEVICE PHYSICS

1.1. Introduction

1.1.1. Energy Gap

Difference between the lower energy level of conduction band E_c and upper energy level of valence band E_v.

$$\boxed{E_G = E_{Go} - B_0 T \text{ eV}}$$



Where, E_{Go} = Energy Gap at 0K

E_G = Energy Gap at T = TK

B_0 = Material constant (eV/K)

T = Temperature in Kelvin

- Energy Gap decrease with increase in the temperature i.e. $\boxed{E_G \propto \frac{1}{\text{Temp}}}$
- For Si Energy Gap is $\boxed{E_{G(T)} = 1.21 - 3.6 \times 10^{-4} T}$
- For Ge Energy Gap is $\boxed{E_{G(T)} = 0.785 - 2.23 \times 10^{-4} T}$

1.1.2. Conductivity

- The current carrying capacity of any material is defined as its conductivity.
- It is the reciprocal of resistivity Unit :– $\frac{1}{\Omega \text{cm}} \rightarrow \frac{\mho}{\text{cm}}$
- Conductivity denotes current carrying capacity of material or device

$$\boxed{\text{Conductivity} = \text{carrier conc.} \times \text{charge} \times \text{mobility}}$$

- Conductivity depends on carrier conc, charge and mobility.
- For metal :– $\boxed{\sigma = nq\mu_n}$
- In metal conductivity decrease with increase in temperature.
- In metals as temperature increase, mobility of charge carrier decrease and therefor conductivity decreases.

For semiconductor : $\boxed{\sigma = nq\mu_n + pq\mu_p}$

- In semiconductor, conductivity increases with increase in temperature.
- In a semiconductor, conductivity mainly depends on carrier concentration.
- Semiconductor means by default it is intrinsic semiconductor

1.1.3. Effective Mass

- The effective mass is a quantity that is used to simplify band structure by modeling the behavior of a free particle with the mass.
- Effective mass takes into account the particle mass & also effects due to internal forces
- Consider an electron in an atom

$$\boxed{\vec{F}_{total} = F_{ext} + F_{internal}} \quad \text{where,}$$

$$F_{ext} = m * a$$

and $F_{internal}$ is due to scattering of charges in the structure

- The effective mass is parameter that relates the quantum mechanism results to the classical force equation

$$\boxed{\frac{1}{h^2}\frac{d^2E}{dK^2} = \frac{2C_1}{h^2} = \frac{1}{m*}}$$

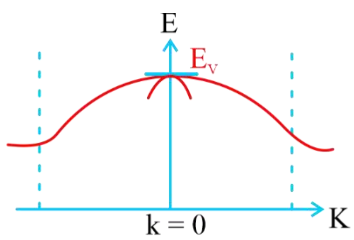
Where, $\hbar$, is plank's constant

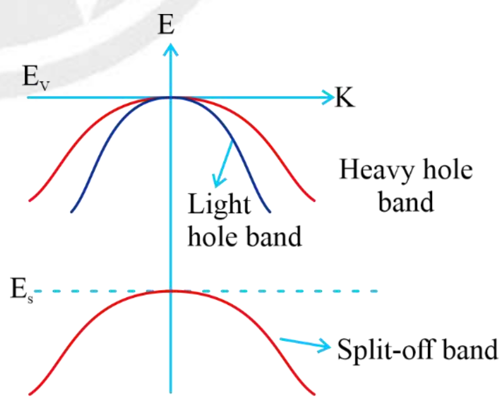
K is wave number

C_1, is constant

m^* is effective mass

- Higher the energy lower will be the effective mass





Where, mp* = effective mass of hole

$mp^* = \left(m_{lp}^{3/2} + m_{np}^{3/2}\right)^{2/3}$

mlp = effective mass of light hole

mnp = effective mass of heavy hole

For Si : mlp = 0.16 m_0, mnp = 0.49 m_0

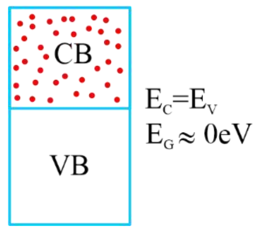
1.1.3. Energy Band Diagrams

Conductors:

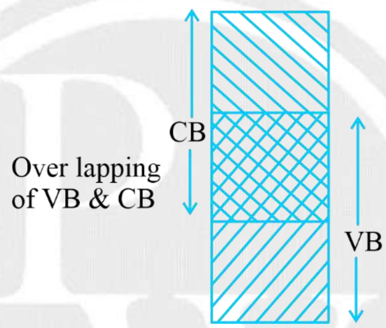
- All metals are very good conductors of current so they allow high flow of current through them.
- In metal current is only due to electrons i.e. metals are unipolar
- In metals free electron conc is independent of temperature

  Where CB = conduction Band and

  VB = valance Band



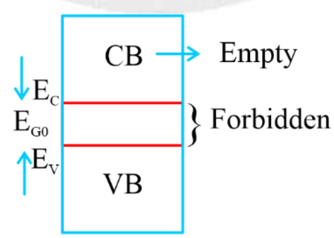
- **At 300 K :**



- When temperature is increased then over lapping of CB & VB also increase but there is free carrier concentration available in CB. Hence conductors have Finite amount of conductivity at 300K.

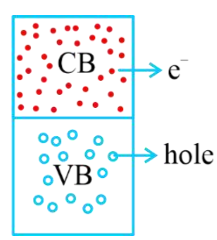
Semi-conductors:

- In semiconductor, the energy gap is small i.e., $E_G \rightarrow 0.7eV$ to $1.5eV$
- At T = 0K free carrier conc. are zero there by conduction band is empty hence conductivity is zero.



- All semiconductors are insulators at 0K.
- At T = 300 K, some covalent bonds are broken & free carriers are generated in CB & holes in VB.

  So, semiconductor has finite amount of conductivity at T = 300 K.

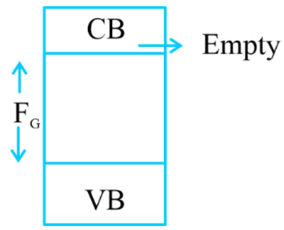


$$E_G \propto \frac{1}{\text{temp}}$$

- Common semiconductor elements are Si, Ge

1.3.4. Insulator

- Due to larger energy gap insulator requires high energy to produce conductivity.
- E_G required is larger
- For ideal insulator conductivity (σ) is zero



1.3.5. Mobility : (μ)

- Moving ability of the charge carriers is called as mobility

$$\boxed{\mu = \frac{V_d}{\vec{E}}}$$

where , V_d = drift velocity

And $\vec{E}$ = field intensity

Unit of μ = $cm^2/V\ sec$

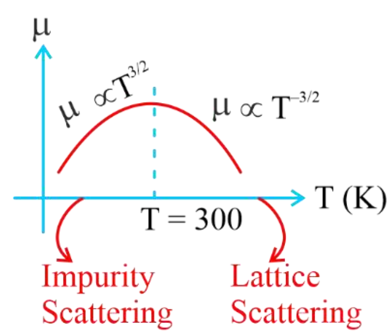
e^- mobility μ_n = $3800\ cm^2 / V$ sec for Ge

$1300\ cm^2 / V$ sec for Si

Hole mobility μ_p = $1800 cm^2 / V$ sec for Ge

$500\ cm^2 / V$ sec for Si

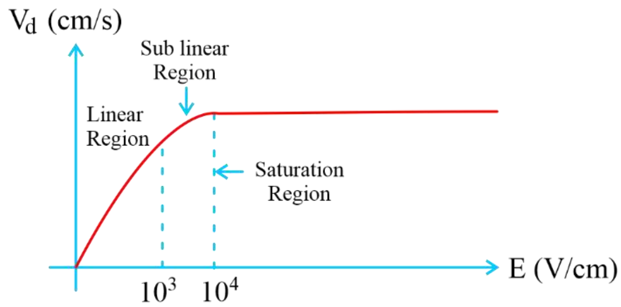
- e^- mobility is always greater then hole mobility so e^-'s can travel faster.
- **Variation of Mobility w.r.t temperature:**



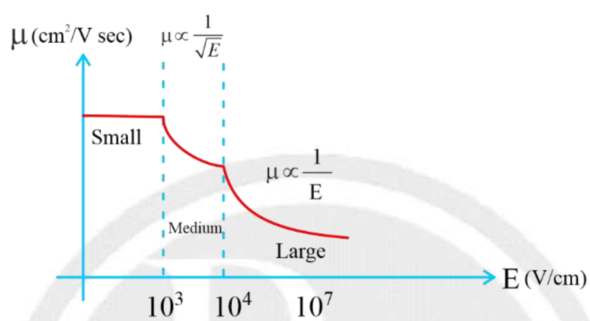
- Impurity scattering takes place till 300 K due to which carriers are generated and the carrier starts moving then drift velocity increases due to which mobility increases.
- Lattice scattering takes place at temp larger than 300 K. In which no. of collisions in e^-s generated increases & relaxation time decrease due to which mobility decreases.
- In general $\boxed{\mu \propto T^{-m}}$

In Ge, $m = 1.66$ for e^- and $m = 2.33$ for hole
In Si. $m = 2.5$ for e^- and $m = 2.7$ for hole

- Variation of drift velocity w.r.t $\vec{E}$ – field :–
- Here V_d (drift velocity) increases linearly then sublinearly then enters into saturation region.



- **Variation of mobility w.r.t $\vec{E}$ field :**



- **For small field Region:** $E\uparrow \rightarrow Vd\uparrow$ (linearly)

  $\uparrow V_d = \mu E\uparrow$

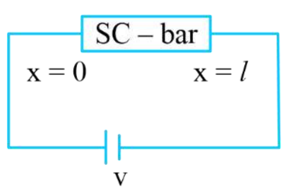
  μ = constant
- Mobility for charge carriers will remain constant
- Drift velocity linearly increases with the field $\left(\vec{E}\right)$
- For medium field region drift velocity (V_d) increases sublinearly and mobility decreases slowly

$$\boxed{\uparrow V_d = \mu\downarrow E\uparrow}$$

- For high field region drift velocity (V_d) remain constant and mobility decreases as increase of electric field intensity.

$$\text{Constant} \rightarrow \boxed{V_d = \downarrow\mu E\uparrow}$$

1.3.6. Electric field along the length of SC bar :–



- If length of semiconductor bar is ℓ and applying voltage V then,

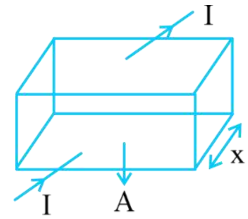
$$\boxed{V_{(x)} = V\left(1-\frac{x}{l}\right)}$$

And

$$\boxed{\vec{E} = \frac{V}{\ell}}$$

Current :

- The rate of flow of charge is called current.



$$\boxed{I = \frac{dQ}{dt}}$$

$$\boxed{I = nqV_dA}$$

where, $n \rightarrow$ no. of charge carrier per unit volume (carrier conc)

$V_d \rightarrow$ drift velocity

q = charge

A = Area

1.3.7. Variation of Conductivity w.r.t Temperature

Metal:

- The carrier conc is independent of temp but generally mobility decreases with increase in temp. so conductivity decreases.

$$T\uparrow \rightarrow \mu\downarrow$$

$$\boxed{T\uparrow \rightarrow \sigma\downarrow}$$

Semiconductors:

- In semiconductor as temperature increases, then free e^- concentration in CB & hole conc in VB ↑es by large amount and mobility of e^- & holes decreases wrt temperature by smaller amount. Thus overall the conductivity of SC increases with increases in temperature.

$$\boxed{\uparrow\sigma = \uparrow nq\mu_n + \uparrow pq\mu_p}$$

1.3.8. Resistivity : $\rho(\Omega - m)$

- It is the ability of material to resist the current that passes through it.

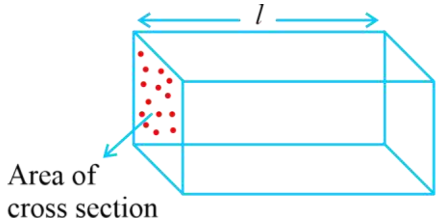
$$\boxed{\rho = \frac{1}{\sigma}}$$

- For metal : $\boxed{\rho = \frac{1}{nq\mu_n}}$

- For semiconductors $\boxed{\rho = \frac{1}{nq\mu_n + pq\mu_p}}$

Resistance : (R) $\boxed{R = \rho\frac{l}{A}}$

Where, ρ = Resistivity



- **For metal :** As increases of temp mobility decreases and Resistivity increases thus overall Resistance increases.
- **For semiconductor :** As increases of temp carrier concentration increases and mobility decreases thus overall resistance decreases.

1.2. Carrier Transport Phenomenon

1.2.1. Drift

- The migration of charge carrier under the influence of external forces which is electric field intensity is called as drift.
- Drift is not natural, It always requires external force.

Diffusion :

- The migration of charge carrier from their higher concentration to lower concentration is called diffusion.
- It is natural process no needs of external forces.
- It occurs due to the concentration gradient.

Drift current Density :– (J_d)

$$\boxed{J_d = \sigma E}$$

- For metal :– In metal drift current density in only due to e^-s

$$\boxed{J_d = nq\mu_n E}$$

- For semiconductor :– In SC drift current density is due to both electrons as well as concentration of holes

$$\boxed{J_d = nq\mu_n E + pq\mu_p E}$$

Where, n = no. of electrons present per cubic meter
p = no. of holes present per cubic meter
q = charge

Diffusion Current Density:

- The diffusion current density depends on diffusion constant or diffusivity
- Diffusion current flows only in semiconductors.
- e^- diffusion current density

where D_n = e^- diffusivity and $\frac{dn}{dx}$ = concentration gradient of e^-

$$\boxed{J_n = +qD_n \frac{dn}{dx}} A/cm^2$$

- Hole diffusion current density

$$\boxed{J_p = -qD_p \frac{dp}{dx}} A/cm^2$$

Where D_p = hole diffusivity

and $\frac{dp}{dx}$ = conc gradient of hole

Total current Density

- The total current density in a semiconductor

$$\boxed{J = J_n + J_p} \text{ A/cm}^2$$

Where Jn = Jn (Diff) + Jn (Drift)

$$\boxed{J_n = qD_n\frac{dn}{dx} + nq\mu_n E} \text{ A/cm}^2$$

$J_p = J_p(\text{drift}) + J_p(\text{Diff})$

$$\boxed{J_p = pq\mu_p E - qD_p\frac{dp}{dx}} \text{ A/cm}^2$$

1.2.2. Einstein's Relation

- It states that the ratio of diffusivity & mobility at a particular temp is always constant

$$\frac{D}{\mu} = \text{constant}$$

$$\boxed{\frac{D}{\mu} = V_T}$$

where V_T = thermal voltage

$$\boxed{V_T = \frac{T}{11600} Volts}$$

where T is temperature, at room temperature, V_T = 25.8mV $\simeq$ 26 mV

1.3. Mass Action Law

It states that, In a semiconductor under thermal equilibrium the product of electrons and holes will be always constant and is given by square of intrinsic concentration.

$$\boxed{np = n_i^2}$$

- This law is mainly used to calculate minority carrier conc.

For n-type semiconductor	For t-type semiconductor
$\boxed{p_n = \frac{n_i^2}{n_n}}$	$\boxed{n_p = \frac{n_i^2}{p_p}}$

Where, $n_n \Longrightarrow$ majority carrier are e⁻s

$p_p \Longrightarrow$ majority carrier are holes

$p_n \Longrightarrow$ minority carrier are holes

n_p = minority carrier are e⁻s

Intrinsic carrier conc : (n_i)

- It is the conc. available in the pure semiconductor at a given temperature

$$\boxed{n = p = n_i}$$

$$\boxed{n_i = \sqrt{N_c N_v}\ e^{-E_{Go}/2kT}}$$

Intrinsic concentration in a semiconductor depends on
(i) Temperature

(ii) Energy Gap

- In Ge intrinsic concentration is more as compare Si due to smaller energy gap at zero kelvin

For Ge, $n_i = 2.5 \times 10^{13}/\text{cm}^3$

For Si, $n_i = 1.5 \times 10^{10}$ / cm^3

Classification of semiconductors :

(i) Elemental & compounded type SC
(ii) Direct & Indirect band gap SC
(iii) Intrinsic & Extrinsic SC

1.3.1. Recombination

- In which an e^- after loosing its energy is migrated from conduction band to valence band & acquires a vacancy (hole) in a broken covalent bond & the covalent bond is reformed. This process is called Recombination.

Carrier Life Time (τ)

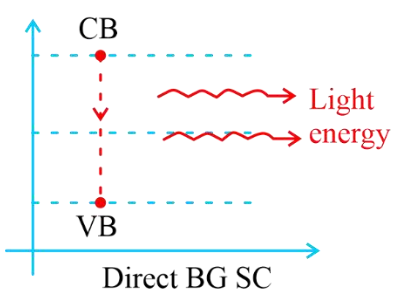
- It is the average lifetime of the charge carriers.
- It is the average time taken from generation to recombination.
- It is two types :

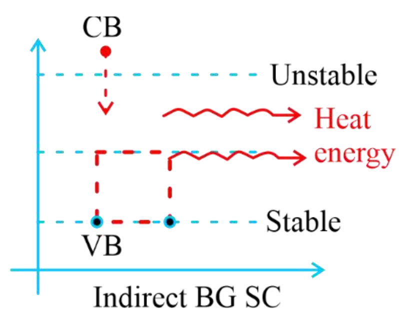
(i) e^- carrier life time

(ii) hole carrier life time

Direct and Indirect bandgap SC

- In Indirect bandgap SC, during recombination most of the energy is released in the form of heat because of collision of e^-s to each others.
- In direct band gap SC, during the migration, the value of K is fixed so no. of collision with another e– is very small, so most of energy released in the form of light.

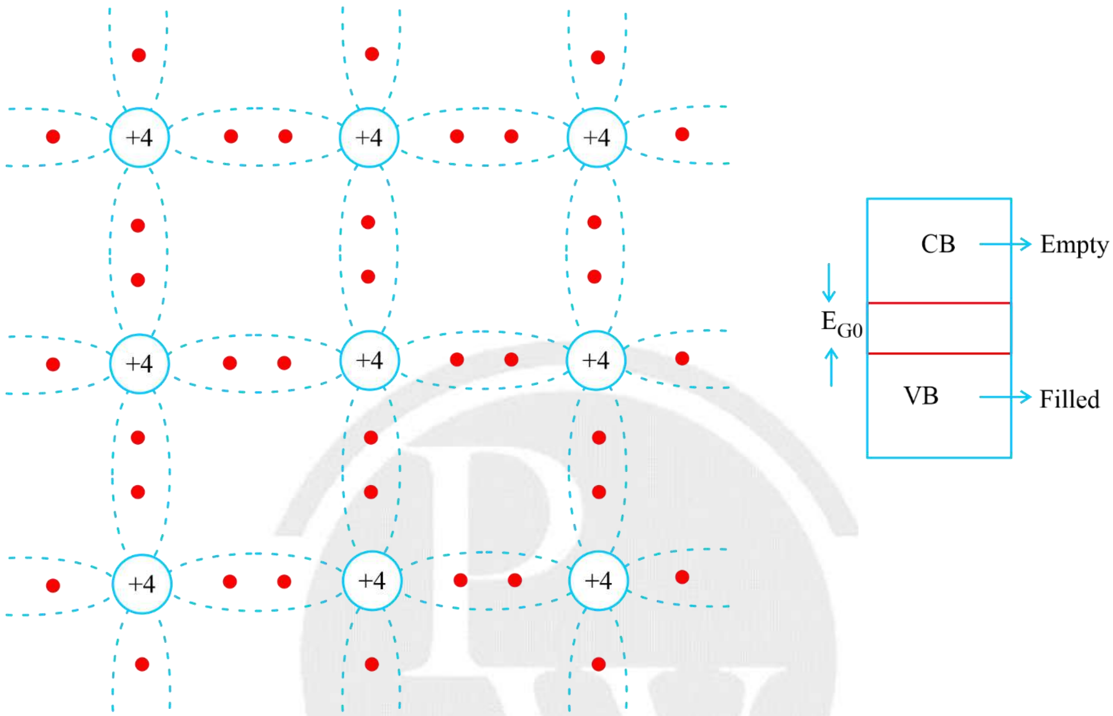




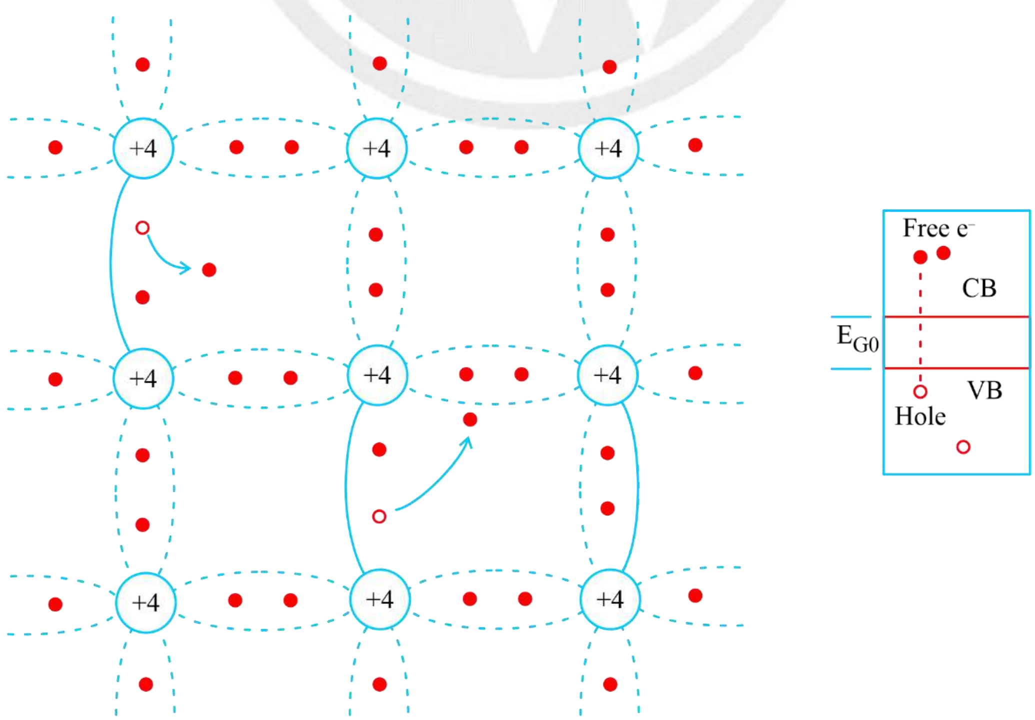
1.3.2. Intrinsic & Extrinsic SC

Intrnsic SC

- Also known as pure semiconductor or natural semiconductor or non-degenerate semiconductor
- The maximum no. of valency e⁻s are 8



- In one covalent bond there will be two valency electron
- At 0k all valency e's are imperfect covalent bonding
- Intrinsic semiconductor will act as insulator at 0k.



- When a covalent bond is broken, it will give one e^- & hole (e^- will be jump from VB to CB) and because a free e– and hole will remains in the valency band.
- For intrinsic semiconductor $\boxed{n = p = n_i}$
- Variation in intrinsic concentration w.r.t temperature :–

$$\boxed{\frac{dn_i}{dT} = \frac{1}{T}\left(\frac{E_G}{2KT} + \frac{3}{2}\right)n_i}$$

For Ge at 300 K

$$\boxed{\begin{array}{l} \dfrac{dn_i}{dT} = 0.076\ n_i \text{ at } 300\text{K} \\ \%\left.\dfrac{dn_i}{dT}\right|_{300K} = 7.6\%\ of\ n_i\ 300\text{K} \end{array}}$$

Note: In Si, n_i increases approximately 8% for 1° rising temp at 300K, As well as the conductivity increases by 8% (approx.) for 1° rising temp

Instrinsic conductivity : (σ_i)

$$\boxed{\sigma_i = n_i q(\mu_n + \mu_p)}\ ℧/\text{cm}$$

- With increases of temperature intrinsic conductivity will also increases

$$\boxed{\sigma_i \propto T^{3/2}}$$

Intrinsic Resistivity (ρ_i)

It is reciprocal of intrinsic conductivity

$$\rho_i = \frac{1}{\sigma_i}$$

$$\boxed{\rho_i = \frac{1}{n_i q\left[\mu_n + \mu_p\right]}}\ \Omega\ \text{cm}$$

Generation of e^- hole pair :

- The creation of e^- hole pair by Breaking of covalent bond is called generation of EHP (e– - hole pair)

Recombination :

- Pairing of free e– with hole is called as recombination
- During Recombination the free e–s holes will be lost in pair and covalent bond is created

Carrier life time :

- It is the avg lifetime of the charge carriers.
- It is the avg time taken from generation to recombination.

Doping :

- The process of adding impurities to the semiconductor is called doping
- Doping increases the carrier concentration and therefore increases the conductivity
- For an tetravalent atom, there are two types of dopants available

(i) Trivalent dopants : (B, Al, Ga, In) → Valency – 3

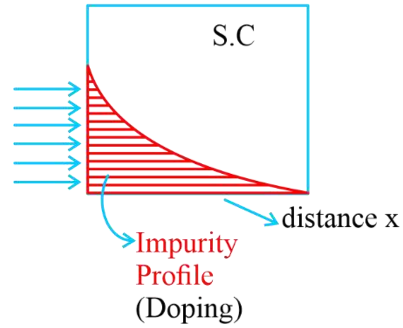
(ii) Pentavalent dopants : (P, As, Sn, Bi) → Valency – 5

Standard Doping concentration :

(i) Moderate Doping → 1 : [10^6 to 10^8] → P N

(ii) Lightly Doped → 1: 10^{11} → P^- N^-

(iii) Highly/Heavily doped → 1 : 10^3 → P^+ N^+

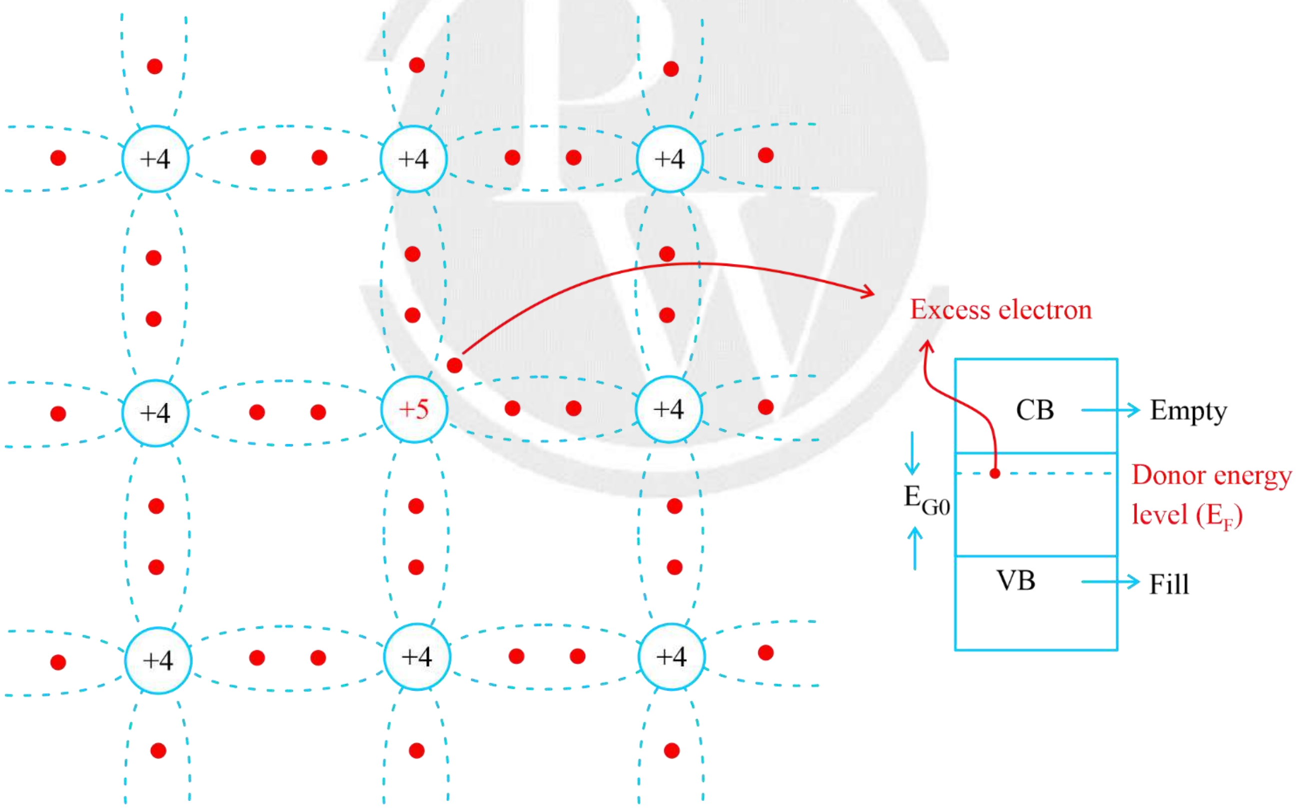


- A highly doped SC with 1 : 10^3 doping is called degenerate SC.
- Non-degenerate SC means moderately doped SC
- The impurity atoms are added to SC are called dopants or impurity profiles

1.3.4. Extrinsic SC

(i) n-type SC

- When pentavalent impurities or donor type impurities are added in the pure SC then it is called as n-type SC
  At T = 0K

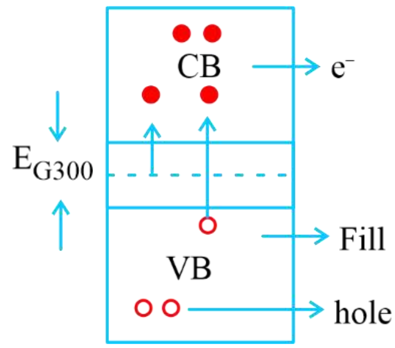


- DONOR energy level is a discrete energy level which is created Just below the conduction band.
- At 0K, the fifth e– of all the impurity atom will be existing in the donor energy level
- N-type semiconductor at 0K will be working as a insulator

$$5^{th}\ e^- = 5 \times 10^{22}\ \text{atoms/cm}^3 \times \frac{1}{10^8}$$

$$= 5 \times 10^{14}\ \text{atoms/cm}^3\ \text{(for Si)}$$

At T = 300 K



- As temperature increases from (0 to 300K) because of that Donor level ionization the 5th e– will be moving from donor energy level into CB and they become free e– and at the same time because of temp large no. of covalent bonds will be broken and (EHP) are generated.
- It overall results in larger e– concentration in CB & hole concentration in VB & the S.C will have finite amount of conductivity

For n-type SC, $N_A = 0$

$$n + N_A = p + N_D$$

$$\boxed{n = p + N_D}$$

For, n-type SC $\boxed{n > p}$

Where $$\boxed{p = \frac{n_i^2}{N_D}}$$

Hence, $$\boxed{n = \frac{N_D + \sqrt{N_D^2 + 4n_i^2}}{2}}$$

- Conductivity :– $$\sigma = \sigma_n + \sigma_p$$

$$\sigma = nq\mu_n + pq\mu_p$$

σ_p is very small for n-type

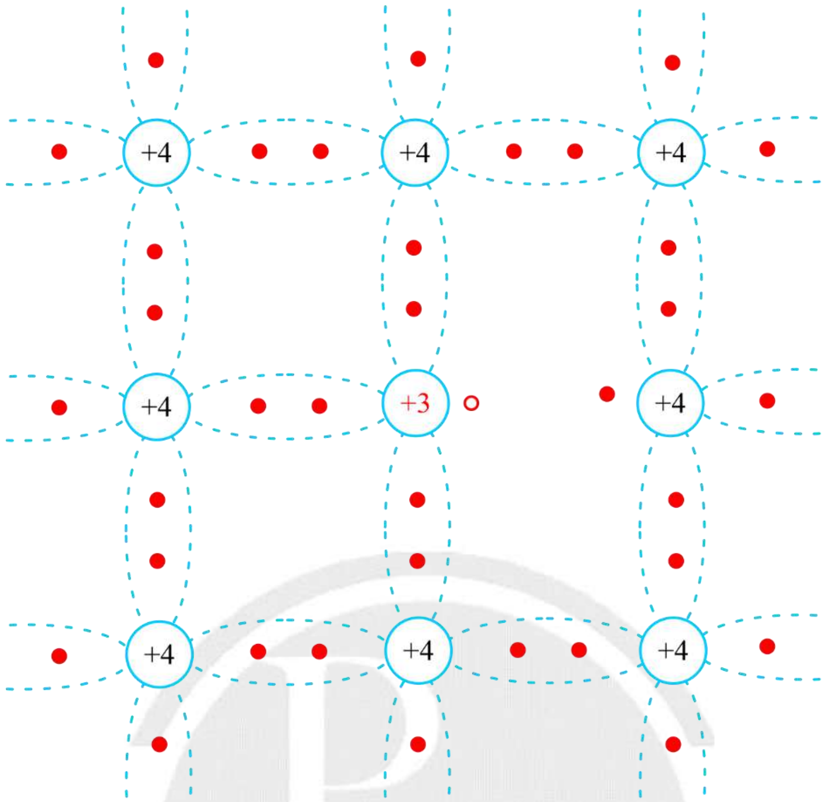
$$\boxed{\sigma = nq\mu_n}$$

If N_D is very large

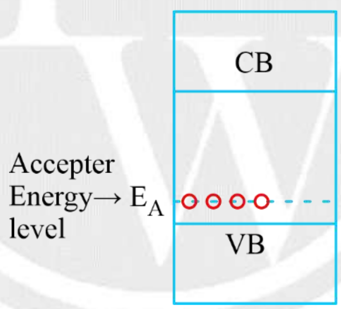
$$\boxed{\sigma = N_D\ q\mu_n}$$

(ii) p-type semi-conductor :

- When trivalent impurities are added into the pure SC then the SC will become p-type

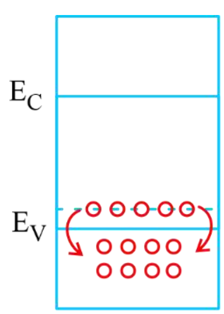


At T = 0 K



- Acceptor energy level is a discreate energy level created Just above the valence band.
- Acceptor energy level denotes energy level of all the trivalent atoms added to the pure semiconductor.
- P-type semiconductor at 0K will be work as a insulator.

  At T = 300 K



- At 300K Acceptor level ionization takes place & some covalent bonds are broken therefore, there is finite free e^- concentration in CB & hole concentration in the VB so SC will have finite conductivity at 300 K

  For p-type SC :– $N_D = 0$

$$\boxed{p = n + N_A}$$

Where $n = \dfrac{ni^2}{p}$

Hence, $p = \dfrac{N_A + \sqrt{N_A^2 + 4n_i^2}}{2}$

- Conductivity : $\sigma = \sigma_n + \sigma_p$

  Where σ_n is very small for p-type SC

$$\boxed{\sigma = \sigma_p}$$

$$\sigma = pq\mu_p$$

$$\boxed{\sigma = N_A q\mu_p} \quad \sigma \propto \text{doping concentration}$$

Note : Minimum conductivity of extrinsic SC, $\boxed{\sigma_{\min} = 2qn_i\sqrt{\mu_p\mu_n}}$

Doping Ratio or Doping Profile:

It is defined as the ratio of total no. of dopant atoms & total no. of atoms (atomic density) per unit volume in an extrinsic SC.

$$\text{Doping ratio} = \frac{\text{Doping concentration}}{\text{Atomic density}\left(\text{atoms/cm}^3\right)}$$

- $1 : 10^3 \rightarrow$ High Doping
- $1 : (10^6 \text{ to } 10^8) \rightarrow$ Moderate doping
- $1 : (10^{10} \text{ to } 10^{11}) \rightarrow$ Light doping

❑❑❑

2 COMPENSATED SEMICONDUCTORS

2.1. Introduction

Compensated semiconductor:-

- A semiconductor in which both donor & accepter impurities are added are called compensated SC.
- If trivalent atoms are added into n-type SC or pentavalent atoms are added into the p-type SC we get compensated SC.

In an intrinsic semiconductor:-

(i) If $N_A = N_D$ then it is general compensated semiconductor

(ii) If $N_D > N_A$ then it is n-type compensated semiconductor

$$\boxed{p_n = \frac{n_i^2}{n_n}} \Rightarrow \boxed{p_n = \frac{n_i^2}{N_D - N_A}}$$

(iii) If $N_A > N_D$ then it is p-type compensated SC

$$\boxed{n_p = \frac{n_i^2}{p_p} = \frac{n_i^2}{N_A - N_D}}$$

2.1.1. Procedure to calculate majority & minority concentration in the Compensated Semiconductor

(1) N-type – compensated SC

Condition – A:

- If ($N_D >> N_A$) or ($N_D > 10\ N_A$) or ($N_D - N_A$) >> n_i or is not given in the problem then,

minority carrier conc^n $\boxed{p_n = \frac{n_i^2}{n_n} \Rightarrow \frac{n_i^2}{N_D - N_A}}$

majority carrier conc^n. $\boxed{n_n \simeq N_D}$

Condition – B:

- If N_D is very close to n_i

  $N_D - N_A$ is very close to n_i

  N_D is very close to N_A

then, majority carrier conc^n. (n_n), will be

$$\boxed{n_n = \frac{N_D - N_A}{2} + \sqrt{\frac{(N_D - N_A)^2}{4} + n_i^2}}$$

and minority carrier conc^n will be

$$\boxed{p_n = \frac{n_i^2}{n_n}}$$

Note : Special Case: Let N_D is applied and $N_A = 0$

then $$\boxed{n_n = n = \frac{N_D}{2} + \sqrt{\frac{N_D^2}{4} + n_i^2}}$$

(2) p-type compensated SC :-

Condition A:-

- If $N_A >> N_D$ or $N_A > 10\ N_D$ or $(N_A - N_D) >> n_i$ or n_i is not given
  $p_p \simeq N_A - N_D$

$$\boxed{p_p \sim N_A}$$

and minority carrier conc^n.

$$\boxed{n_p = \frac{n_i^2}{p_p} = \frac{n_i^2}{N_A - N_D}}$$

Condition B:-

- If N_A is very close to n_i or $(N_A - N_D)$ is very close to n_i or N_A is very close to N_D then, majority carrier conc^n.

$$\boxed{p_p = \frac{N_A - N_D}{2} + \sqrt{\left(\frac{N_A - N_D}{2}\right)^2 + n_i^2}}$$

Note: Special case:
If N_A is applied & $N_D = 0$

then $$p_p = \boxed{p = \frac{N_A}{2} + \sqrt{\left(\frac{N_A}{2}\right)^2 + n_i^2}}$$

and $$\boxed{n_p = \frac{n_i^2}{p_p}}$$

Carrier Generation & Carrier Recombination:-

Generation is the process in which force electron & holes are generated and Recombination is the process in which electrons & holes are annihilated.

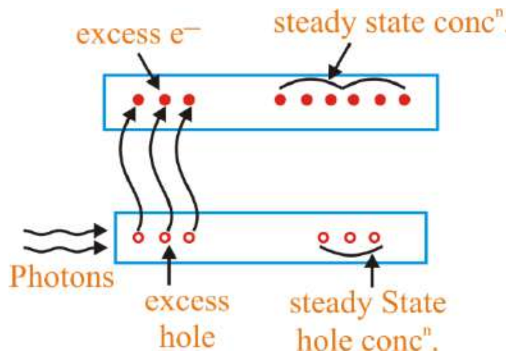
Carrier Generation & Carrier Recombination:

- In thermal equilibrium the conc^n of e^- & hole in CB & VB respectively are time independent & mass action law holds and also Recombination process annihilates both the e^- & hole.
- Since the net carrier conc^n. are independent of time in thermal equilibrium, the rate of e^- and hole generated & the rate of which they annihilates must be equal.
  In thermal equilibrium

$$\boxed{G_{no} = G_{po} = R_{no} = R_{po}}$$

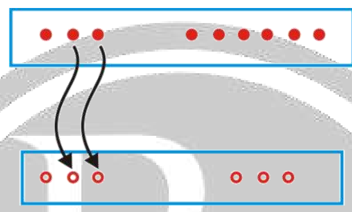
Excess carrier generation & Recombination:-

- When a high energy photons are incident on a SC. e^- in the valence band may be excited to the CB. At the same time, Q hole is also created in the VB. Thus EHP (electron – hole. pair) is generated.



$\boxed{g\,n' = g\,p'}$
↑
Excess carrier Generation

where gn´ = excess e^- generation rate
gp´ = excess hole generation rate



Excess carrier recombination $\boxed{Rn' = Rp'}$
Where, Rn´, Rp´ → Excess e^- & hole recombination rate

Case – I : Source applied for – ∞ to 0

The net rate of change of e^- concentration.

$$\boxed{\frac{d\ \Delta n(t)}{dt} = D_n \frac{d^2 \Delta n}{dx^2} - \frac{\Delta n}{\tau_n} + G_L}$$

Approximation

(i) $p_o >> n_o$ (ii) low level injection

then, $\boxed{\Delta_n(t) = \Delta_n(o)\, e^{-t/\tau\, n_o}}$

For n-type, excess minority carrier concentration.

$$\boxed{\Delta_p(t) = \Delta_n(o)\, e^{-t/\tau\, p_o}}$$

where, τ_{no}, τ_{po} → excess lifetime of e^- & hole.
Recombination rate of excess carrier:

- Recombination rate of excess carrier is equals to negative of net rate of change of excess carrier.
- For p-type, $\boxed{R_n' = \frac{\Delta n(t)}{\tau_n} = R_p'}$
- For n-type, $\boxed{R_n' = R_p' = \frac{\Delta p(t)}{\tau_p}}$

Case – II :

Consider a homogenous n-type SC with zero applied electric field, assume that for t < 0 the SC is in thermal equilibrium & that for t > 0, a uniform generation rate exist in the crystal.

$$\boxed{\Delta p(t) = g'\,\tau_{po}\,(1 - e^{-t/\tau_{po}})}$$

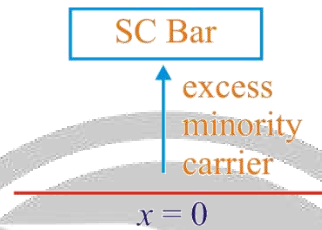
Δ p(t)/max = Δ p(∞)

$$\boxed{\Delta p(t)|\max = g'\,\tau_{po}}$$

for p-type, $$\boxed{\Delta n(t) = g'\tau_{no}\left[1 - e^{-t/\tau_{no}}\right]}$$

Case – III : Excess minority carrier concentration with length.

- In this we generate excess carrier at $x = 0$ so for $x < 0$ & $x > 0$
  It will be decayed (diffused).



- For at any 'x' excess minority carrier concentration.

  Let n-type :- $$\boxed{\Delta\, p(x) = \Delta\, p(o)\, e^{-\frac{x}{Lp}}}$$ For $(x \geq 0)$

  where, $L_p \to$ diffusion length for hole

  for $x < 0$ $$\boxed{\Delta\, p(x) = \Delta\, p(o)\, e^{x/Lp}}$$

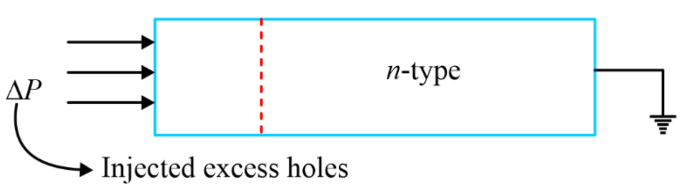
  Let p-type $$\boxed{\Delta\, n(x) = \Delta n(o)\, e^{-x/Ln}}$$ for $(x \geq 0)$

  $$\boxed{\Delta\, n(x) = \Delta n(o)\, e^{x/Ln}}$$ for $(x < 0)$

  where $L_n \to$ diffusion length for e^-

Case – II : Dielectric Relaxation time constant:-

- Consider a n-type S.C & let ΔP holes are injected in a portion of that n-type SC. Now we want to find behaviour of those injected excess holes inside the S.C.



$$\boxed{\delta_v(t) = \delta_v(0)e^{-\frac{\sigma}{\varepsilon}t}}$$

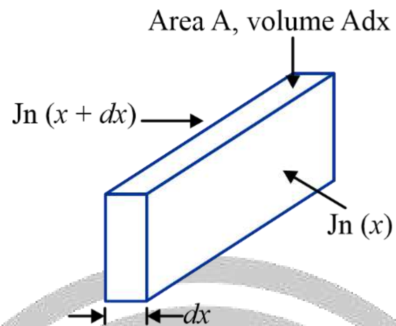
Let $\tau = \dfrac{\in}{\sigma}$ $\Rightarrow \boxed{\delta_v(t) = \delta_v(o)\, e^{-t/\tau}}$

Relation time constant for dielectric

Continuity Equation:

- Continuity equation describes the distribution of electrons and holes when there is excess carrier generation, recombination and carrier movement.
- As per law of conservation of charge, rate of charge of number of e^- inside the semiconductors is equal to no of e^- entering per second minus no. of e^- leaving per second plus no. of e^- generated for second by generation process minus no. of electrons lost per second by recombination process.

- Consider a volume in which carrier flux into/out – of



$$Adx\left(\frac{\partial n}{\partial t}\right)=-\frac{1}{q}\left[J_n(x)\ A-J_n\ (x+dx)A\right]+G_nAdx-R_n\ Adx$$

$$J_n\ (x+dx)=J_n(x)+\frac{\partial J_n(x)}{\partial x}\ dx$$

$$\Rightarrow \quad \frac{\partial n}{\partial t}=\frac{1}{q}\ \frac{\partial .\ J_n(x)}{\partial x}-R_n$$

$$\boxed{\begin{aligned}\frac{\partial n}{\partial t}&=\frac{1}{q}\ \frac{\partial\ J_n(x)}{\partial x}-\mathrm{R_n}+\mathrm{G_L}\\ \frac{\partial p}{\partial t}&=-\frac{1}{q}\ \frac{\partial\ J_p(x)}{\partial x}-\mathrm{R_p}+\mathrm{G_L}\end{aligned}}$$

Continuity equation.

- The minority carrier diffusion equations are derived from the continuity equation,

$$\frac{\partial\ \Delta n_p}{\partial t}=\mathrm{D_n}\ \frac{\partial^2 \Delta n_p}{\partial x^2}-\mathrm{R_n}+\ \mathrm{G_L}$$

$$\frac{\partial\ \Delta\ p_n}{\partial t}=D_p\ \frac{\partial^2\ \Delta p_n}{\partial x^2}-\mathrm{R_p}+\mathrm{G_L}$$

❒❒❒

3 FERMI LEVELS AND HEAT MEASUREMENT

3.1. Fermi Characteristic

(a) Fermi-Dirac Distribution: *f*(*E*)

$$f(E) = \frac{1}{1 + e^{\frac{E - E_F}{KT}}}$$

Where, E = Energy possessed by the e^- in eV

K = Boltzmann constant

E_F = Fermi energy level

- Indicates the probability of existing e^- in a given energy state
- It is also called as fermi – Dirac probability function
- At T = 0 K

(i) $E > E_F$, $f(E) = \frac{1}{1+e^{+\infty}} = \frac{1}{1+\infty} = 0$, this indicates no e^- are available in the SC with energy $E > E_f$

(ii) $E < E_F$, $f(E) = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$

Here probability is 1 it indicates at T = 0 K, elements are available in the semiconductor with energies $E < E_f$

- At T $\neq$ 0 K

$$E = E_F, \quad f(E) = \frac{1}{1+e^0} = \frac{1}{2} \text{ or } 50\%$$

- Fermi level is the characteristic level with 50% probability of being filled if no forbidden band exist
- For metal f(E) will be 1
- For SC if probability of e^- existing f(E) then probability of hole existing in SC will be [1 – f(E)]

(b) Fermi Energy (E_F)

- Fermi energy is defined as the maximum energy possessed by the e^- at 0K

$$E_F = \text{Max kinetic Energy}$$

$$E_F = \frac{1}{2} m v_{max}^2$$

Where, m = rest mass of e^- = 9.1×10^{-31} kg

and $$V_{max} = \sqrt{\frac{2E_F}{m}} \text{ m/s}$$

- Fermi energy is also defined as the energy possessed by fastest moving e^- at 0K

(c) Fermi Energy Level in Intrinsic semiconductor (E_{Fi})

- In Intrinsic SC, $n = p = n_i$

  Where, $n = N_c \cdot e^{\frac{-(E_c - E_{Fi})}{KT}}$

  $p = N_v \cdot e^{\frac{-(E_{Fi} - E_v)}{KT}}$

  $$N_c \cdot e^{\frac{-(E_c - E_{Fi})}{KT}} = N_v \cdot e^{\frac{-(E_{Fi} - E_v)}{KT}}$$

  by solving

  $$E_{Fi} = \frac{E_c + E_v}{2} + \frac{KT}{2} \ln\left(\frac{N_v}{N_c}\right)$$

  Where $\frac{E_c + E_v}{2} = E_{\text{midgap}}$

  and $N_v \propto (m_p^*)^{3/2}$

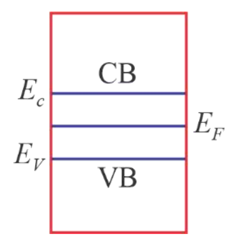
  $N_c \propto (m_n^*)^{3/2}$

  $$E_{Fi} = E_{\text{midgap}} + \frac{3}{4} KT \ \ln\left(\frac{m_p^*}{m_n^*}\right)$$

- If $m_p^* = m_n^*$ then $E_{Fi} = E_{\text{midgap}}$, Intrinsic fermi – energy level lies at the midgap
- If $m_p^* > m_n^*$ then $E_{Fi} > E_{\text{midgap}}$, E_{fi} lies just above midgap
- If $m_p^* < m_n^*$ then $E_{Fi} < E_{\text{midgap}}$, E_{fi} lies just below the midgap

Effect of Temperature:

1. AT T = 0 K, In intrinsic SC at 0 K, fermi level is existing at the centre of energy gap



$$E_F = \frac{E_c + E_c}{2}$$

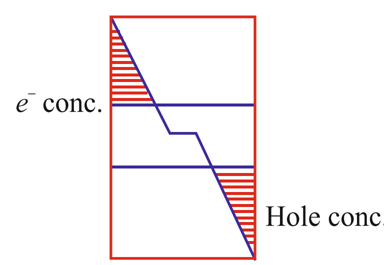
Note : Fermi level will be exactly at the centre of energy gap

(i) $m_n^* = m_p^*$

(ii) $N_c = N_v$

(iii) At T = 0 K

2. AT T = 300 K, In intrinsic semiconductor at room temperature, the fermi level will be passing through centre of energy gap.



$$E_F = \frac{E_c + E_v}{2} - \frac{KT}{2} \log_e\left(\frac{N_c}{N_v}\right)$$

here e^- conc. = hole conc.

(d) Fermi level in *n*–type semiconductor (E_{Fn}) :

- In n-type SC : $n \simeq N_D$

$$n \simeq N_D = N_c \cdot e^{-\frac{(E_c - E_{Fn})}{KT}}$$

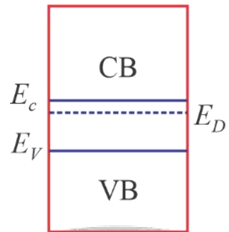
$$E_{Fn} - Ec = \text{KT}\ln\left(\frac{N_D}{N_C}\right)$$

- It indicates the position of fermi level below the conduction band
- In n-type semiconductor, fermi level is a function of temperature and doping conc.

Effect of Temperature:

(i) At T = 0 K, $\boxed{E_F = E_c}$ E_f is coincides with the edge of conduction band.

⇒ Donor energy level is always nearer to conduction band as compare to centre



(ii) At T = 300 K

- At room temperature, in n-type semiconductor fermi level exist just below CB Energy level
- At T = 0 K

  Semiconductor is non-degenerated i.e. ($N_D < N_C$)

  Here $E_{Fn} < E_c$, means fermi level lies below CB

  If semiconductor is degenerate, i.e. ($N_D > N_C$) here $E_{Fn} > E_c$, mean fermi level lies in conduction band.

(e) Effect of Doping :

- Shift in position of E_F due to Doping
- Shift in position of E_F w.r.t to E_{Fi}

$$\phi_n \text{ (Shift)} = +KT\ell n\frac{N_D}{n_i}eV$$

(f) Fermi -Energy level in p-type semiconductor (E_{Fp})

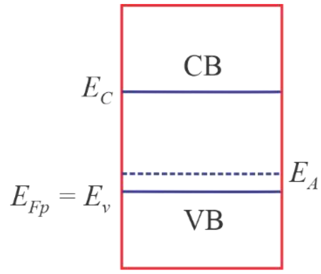
- on p-type SC, $p \simeq N_A$

$$p = N_A = N_v e^{\frac{-(E_{Fp}-E_v)}{KT}}$$

$$E_{Fp} = E_v + KT\ell n \frac{N_v}{N_A}$$

- It indicates the position of fermi level above the valence band in the p-type semiconductor
- Fermi level is a function of temperature and doping
- At T = 0 K, $\boxed{E_{Fp} = E_v}$, E_{Fp} coincides with the edge of VB

→ At 0 K, carrier conc. Are zero and therefore σ = 0, so p-type SC at 0 K behave as insulator.



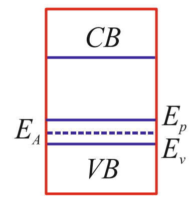
- At T = 300 K :

  $E_{Fp} = E_v + KT \log_e \dfrac{N_v}{N_A}$, In p-type semiconductor at 300 K fermi level exist just above the accepter energy level

at T > 0 K, if $N_A < N_v$ means $E_{Fp} > E_v$ here fermi energy level lies above VB (Non degenerated SC)

If $N_A > N_v$ means $E_{Fp} < E_v$ (generated SC)

Here fermi energy level lies above VB.



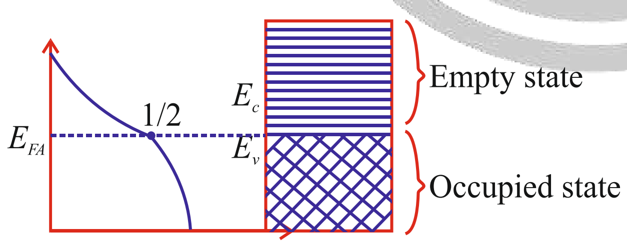
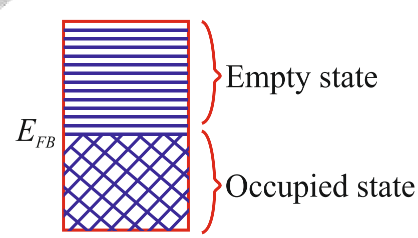
Effect of Doping :

- In p-type semiconductor as doping (N_A) increases then E_F moves towards the VB.
- As E_F moves away from the centre energy gap so σ increases with doping
- Shift in the position of E_F due to doping

$$\text{Shift} = -KT \ell n \frac{N_A}{n_i} \text{eV}$$

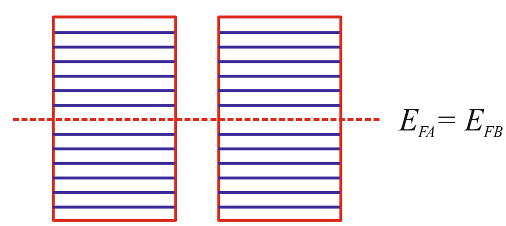
(g) Effect on fermi level when two different SC brought together:

In thermal equilibrium, the fermi energy level is constant through the system

Consider a S·C A Whose $e^{\ominus}$ are distributes in the energy state of an allowed band. 	**Consider a S·C B** Whose $e^{\ominus}$ are distributes are the energy state of an allowed band. 

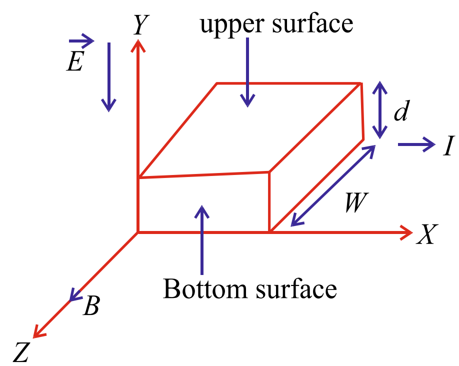
- It there two material are brought togethers into intimate contact then the $e^{\ominus}$ in *n*-type system will tend to seek the lowest possible energy level.
- In those above case, $e^{\ominus}$ will flow from material A to lowest energy state of material and until thermal equilibrium reach.
- Thermal equilibrium is only reached when distribution of electron is a function of energy is same in both material. It occour when fermi level will equal in both system

  Let $E_{FA} > E_{FB}$, In thermal equilibrium



3.2. Hall Effect

It states that if a specimen (metal or SC) currying the current I is placed in transverse magnetic field and an electric field intensity E is induces in perpendicular direction of both 'I' and 'B'



where, w = width of specimen

d = height of specimen

current in X – direction $\Rightarrow I_X$

Magnetic field in Z – direction = B_Z

Field intensity in Y – direction = E_Y

- The specimen must be square shape or rectangular shape.
- From the hall effect we can identify.

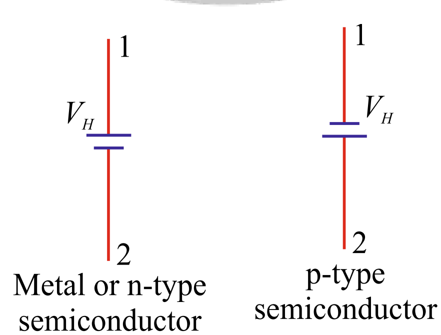
I. Whether the given specimen is a metal or SC (n-type or P-type)

II. The carrier concern in the specimen.

III. The mobility of charge caries.

IV. Magnetic field intensity 'H'

V. To measure the signal power in the electromagnetic wave.



- Hall voltage is induced voltage.
- Electric field intensity.

$$\boxed{|E| = \frac{|V_H|}{d}} \text{ V/m}$$

Hall voltage V_H:

$$\boxed{V_H = Ed} \text{ volt}$$

$$V_H = \frac{BI}{\rho w}$$

where, ρ is charge density.

$$\boxed{\frac{1}{\rho} = R_H}$$

where, R_H is hall coefficient.

$$\boxed{V_H = \frac{BI\,R_H}{W}} \textit{ volt}$$

Where, B → applied magnetic field
W → width of specimen

- By hall experiment, mobility is given by

$$\boxed{\mu = \frac{8}{3\pi}\sigma R_H}$$

Where, μ = mobility of charge carriers
σ = conductivity of metrical

Application:

I. Magnetic field meter.
II. Hall effect multiplier.

Hall voltage depends upon carrier concentration:

In metal, V_H = –ve

In n-type semiconductor, V_H = –ve

In P-type semiconductor, V_H = +ve

For intrinsic semiconductor, V_H = 0

Note : Charge density (ρ) = charge × carrier conc. C/m^3

Hall coefficient (R_H):

$$\boxed{R_H = \frac{1}{\rho} = \frac{1}{\text{charge} \times \text{carrier conc.}}}\ m^3/c$$

- In metal and n-type semiconductor R_H = –ve
- In p-type SC, R_H = +ve
- In intrinsic SC, R_H is very large
- In intrinsic semiconductor carrier concentration are very small and so Hall coefficient will by very large

$$\mu = \sigma R_H$$

So, $$R_H \propto \frac{\mu}{\sigma}$$

Since, $$V_H \propto R_H$$

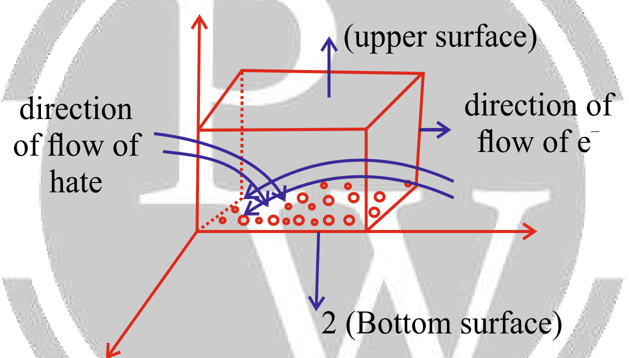
Since, $V_H \propto \frac{1}{\sigma}$

- In metal V_H is small because of in metal σ is very large.
- In semiconductor V_H is large because of semiconductor σ is small.
- In extrinsic SC, R_H is independent of temperature.

$$R_H = \frac{1}{q \times \text{carrier concentration}}$$

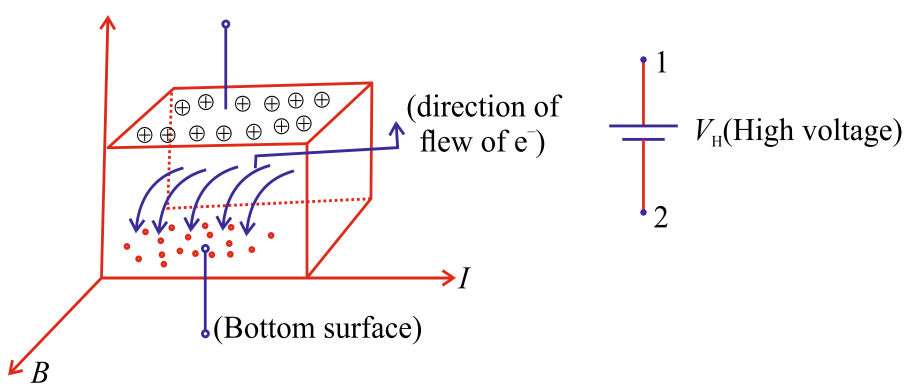
- $R_H = \frac{V_H W}{BI}$, since all parameter are constant so, hall coefficient is independent of temperature.
- $R_H = \frac{\mu}{\sigma}$, in intrinsic semiconductor, hall coefficient decreases with in temperature.
- $R_H = \frac{1}{q \times corrier\ conc}$, As in intrinsic semiconductor for carrier conc. increases with temperature so R_H decreases with temperature.

Hall coefficient (R_H):



- Increases of intrinsic SC, current will be produces by both e⁻ and holes because of no majority and minority. Increase of intrinsic SC due to which both change carrier will have magnetic force in (-gy) dissection deposited on bottom surface.

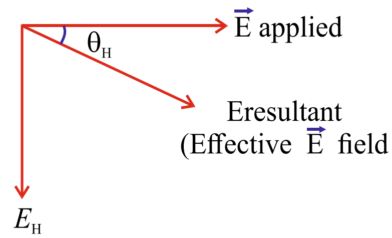
For n-type SC:



- For n-type SC bottom plate will be positively charge.

Half angle:

- Hall angle is defined as the angle made by the resultant $\vec{E}$ field with applied $\vec{F}$ field.



$$\tan(\theta_H) = \frac{E_H}{E_{applied}}$$

$$E_H = \frac{V_H}{d},\ J = \frac{I}{A} = E_{applied}.\sigma$$

So,

$$E_{applied} = \frac{I}{\sigma A}$$

$$\tan(\theta_H) = \frac{VH}{d.I} \times \sigma \times A$$

$$\tan(\theta_H) = \frac{R_H.IB \times \sigma A}{w.d \times I}$$

$$\tan(\theta_H) = R_H \times B \times \frac{\mu}{R_H}$$

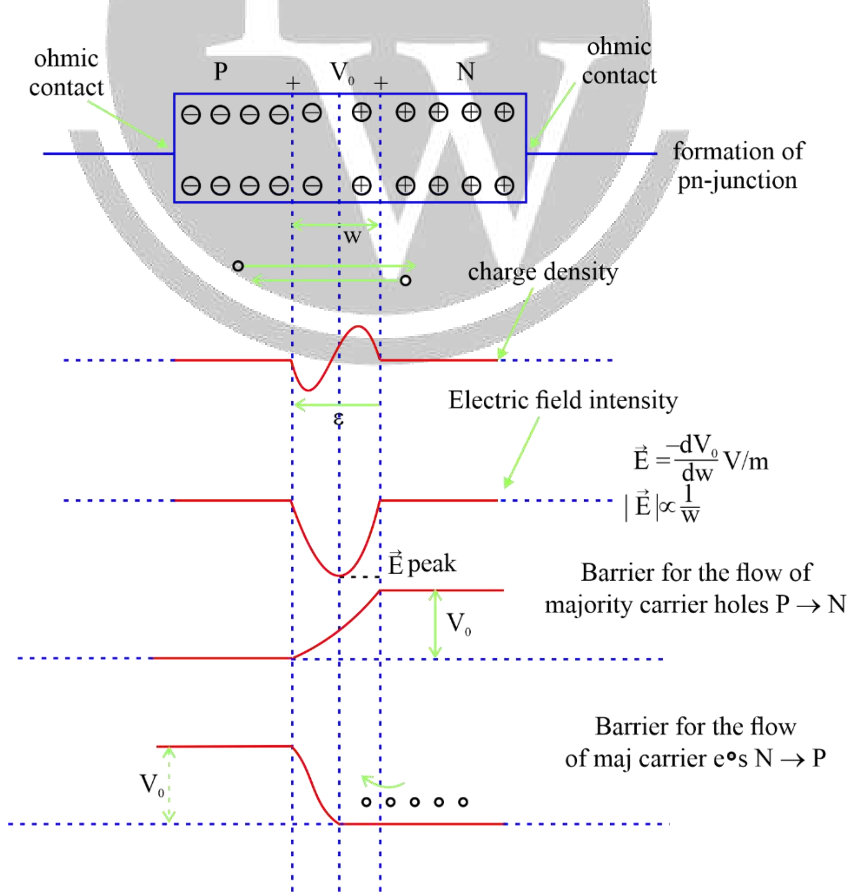
$$\boxed{\theta_H = \tan^{-1}(B\mu)}$$

4 PN JUNCTION

4.1. PN Junction

(a) PN Junction

- A pn – junction can be formed only when a bonding force is created in between p – type and n – type semiconductor
- Modern diodes are fabricated with using any one of the following methods
  - (a) Alloy – junction method
  - (b) Diffusion method
  - (c) Grown junction technique
  - (d) Epitaxial method



For Ge diode, V_0 = 0.1 v to 0.5 V

Typical value – 0.2 V

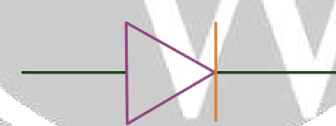
For Si diode, V_0 = 0.6 v to 0.98 v and typical value = 0.7 v

- Depletion layer is called as space charge region or transition region
- Depletion layer is created due to diffusion of majority carrier across junction
- In depletion layer mobile charge carrier are zero and contains immobile charge carrier
- Contains large no. of covalent bonds & Ions
- Depletion layer consist of (–ve) charges and positive charges on either side of the junction
- Depletion region not oppose minority carrier in crossing the junction

$$w \propto \frac{1}{\sqrt{\text{Doping concentration}}}$$

Where , w is width of depletion region

- By increasing doping concentration on both sides of the p – n junction or on one side of the junction, the width of depletion region will be reduced
- In Pn junction , v_0 is
  - Contact potential
  - Barrier potential
  - Diffusion voltage
  - Built in voltage
- In any type of PN junction field intensity is always directed from n – side to p – side
- In pn – junction field intensity is always maximum at the junction



Equation for width of depletion layer (ω)

$$w = \sqrt{\frac{2\varepsilon}{q}\left(\frac{1}{N_A}+\frac{1}{N_D}\right)v_0}$$

Where, ε = permittivity in F/m ⇒ ($\varepsilon = \varepsilon_0\varepsilon_r$)

ε_0 = permittivity in free space

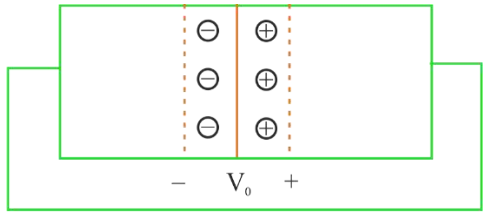
And ε_r = Absolute permittivity

Note: $\varepsilon_0 = 8.84 \times 10^{-12}$ F/m

ε_r(Si) = 11.7, ε_r(Ge) = 16

$\varepsilon = 11.7\ \varepsilon_0 = 8.85 \times 10^{-14}$ F/cm

Equation for contact potential of PN junction diode



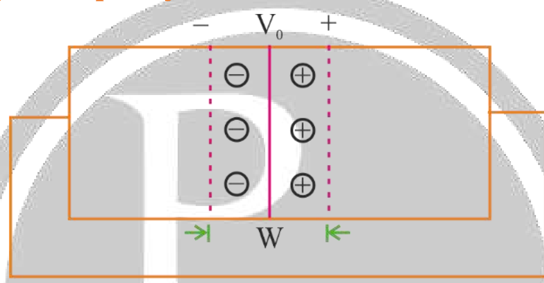
S.C. conditon

$$V_0 = V_{bi}$$

$$V_0 = V_T \ell n \frac{N_A N_D}{n_i^2}$$

- Contact potential is always (+v) for p – n junction diode
- Contact potential decreases with the temperature for increase in 1°C, v_0 is decrease by 2.5 mV

Equation for maximum field intensity in the p – n junction



unbiased

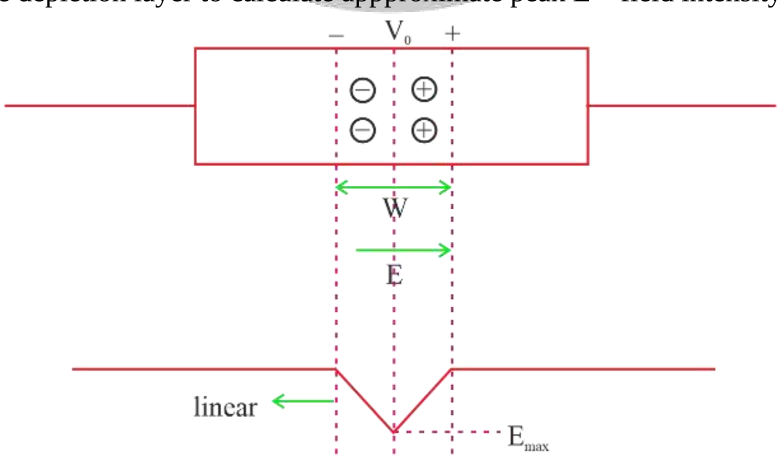
Maximum intensity is given by poison's equation

$$\vec{E}_{max} \text{ or } E_{peak} = \frac{-q}{\varepsilon} w_N N_D \quad = -\frac{q}{\varepsilon} w_p N_A$$

When w_n or x_n is width of n – side depletion layer and w_p or x_p is width of p – side depletion layer

Special Case:

Assuming $\vec{E}$ is linear with the depletion layer to calculate appproximate peak E – field intensity



$$E_{max} = \frac{-v_0}{W/2} = \frac{-2v_0}{W} \text{ v/m}$$

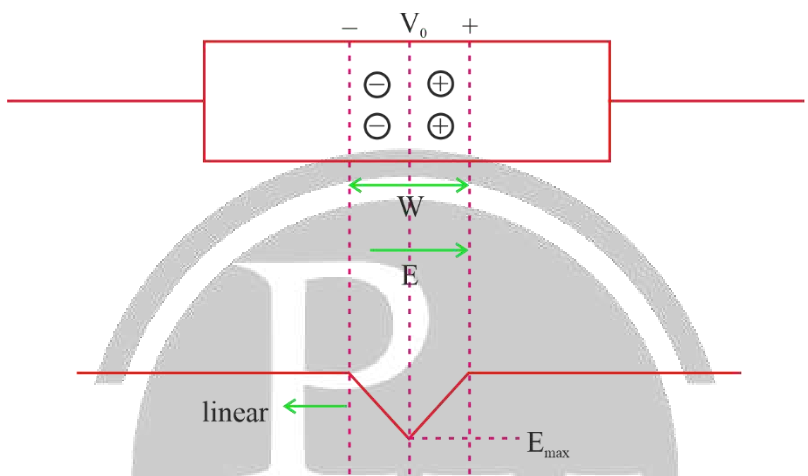
Equation for width of depletion layer (w) :

$$\text{Using } E_{max} = \frac{-q}{\varepsilon} w_n N_D \text{ or } -\frac{q}{\varepsilon} w_p N_A$$

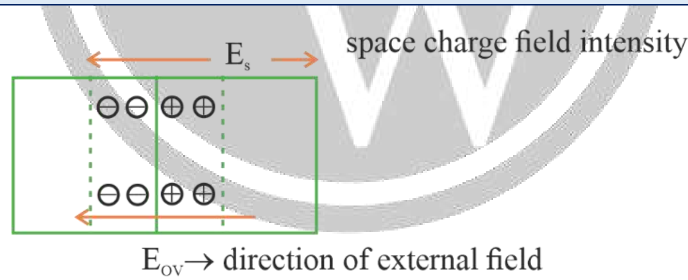
$$w_n = \frac{WN_A}{N_A + N_D} \qquad w_p = \frac{WN_D}{N_A + N_D}$$

$$w = \sqrt{\frac{2\varepsilon}{q}\left(\frac{1}{N_A} + \frac{1}{N_D}\right) V_0}$$

Current components in p – n junction diode



4.2. Law of Electrical Neutrality



In + qA $w_p N_P$ and –qA $w_n W D_D$ (Coulomb)

The charge density in the depletion region of N side and P side are $+qw_nN_D$ C/m^2 and – qw_pN_A C/m^2

Or

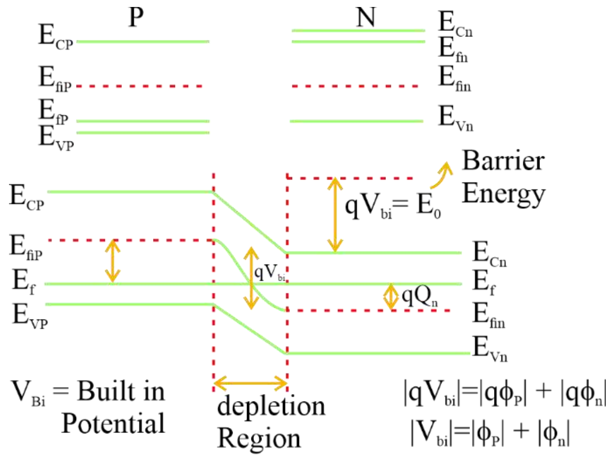
$$w_pN_A = w_nN_D$$

or,
$$\frac{w_p}{w_n} = \frac{N_D}{N_A}$$

Note :

- Ratio of depletion layer width of p – side & n – side depends on their doping concentration called charge equality equation
- Depletion layer will penetrate more into lightly doped region
- In a p – j junction if doping concentration are 10 : 1 then their depletion layer will be 1 : 10 ratio

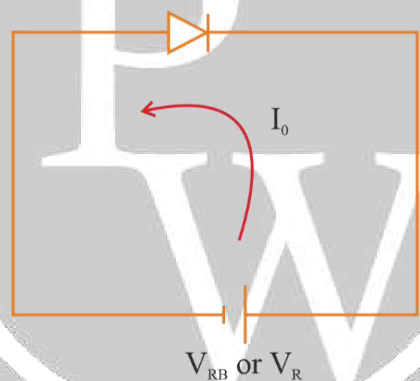
Energy Band diagram



Built in potential, $$V_{bi} = \frac{KT}{q}\ell n\left(\frac{N_A N_D}{n_i^2}\right)$$

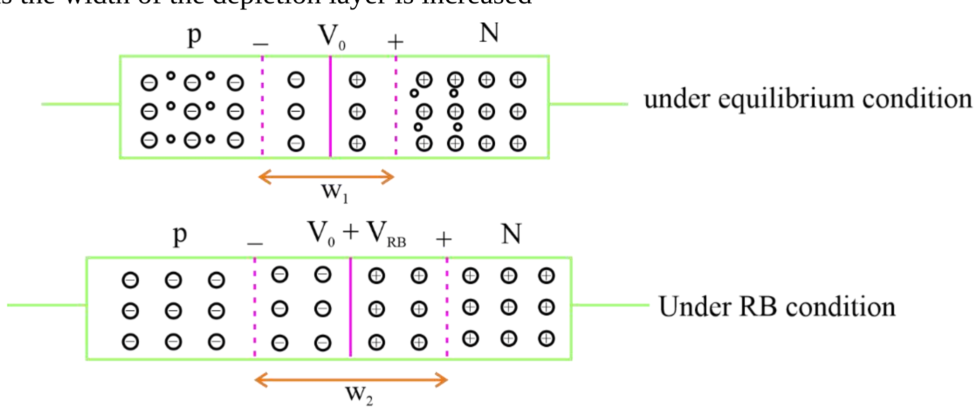
And Barrier energy $$E_0 = qV_{bi} = KT\ell n\left(\frac{N_A N_D}{n_i^2}\right)$$

Reverse Bias : or blocking bias



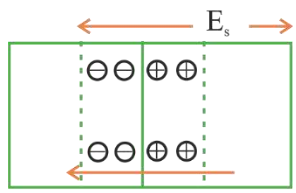
If we apply 0 potential difference between p & N region such that the n – side is at higher potential or +ve potential as compared to p – side, then the applied bias is called as reverse bias

Under the reverse bias the width of the depletion layer is increased



Here , ($w_2 > w_1$)

Here majority carriers of p – n junction will be moving away from the junction there by the region of immobile charges ↑ es. That width of depletion layer is increased

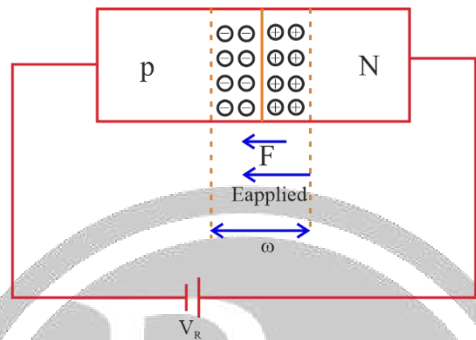


space charge field intensity

$E_{OV} \rightarrow$ direction of external field

- Both have same direction E_s & E_{ov}
- Under RB, the barrier voltage is increase by $|V_{RB}|$

4.2.1 Space Charge width & Electric Field



Where, $\vec{E} = \vec{E}$ Field that exist in depletion region already

On space charge region

$$\vec{E}_{net} = \vec{E} + \vec{E}_{app}$$

The $\vec{E}$ field originates from the +ve charges & terminates on – ve charges, this means the no. of +ve & –ve charge must ↑ ses if $\vec{E}$ field increases

w ↑ es with an increasing reverse voltage V_R

Here, $V_{total} = V_{bi} + V_R$

(i) $W = \sqrt{\frac{2\varepsilon}{q}(V_{bi} + V_R)\frac{N_A + N_D}{N_A N_D}}$

(ii) $x_n = \sqrt{\frac{2\varepsilon}{q}(V_{bi} + V_R)\frac{N_A}{N_D}.\frac{1}{(N_A + N_D)}}$

(iii) $x_P = \sqrt{\frac{2\varepsilon}{q}(V_{bi} + V_R)\frac{N_D}{N_A}.\frac{1}{(N_A + N_D)}}$

$\vec{E}$ field in depletion region

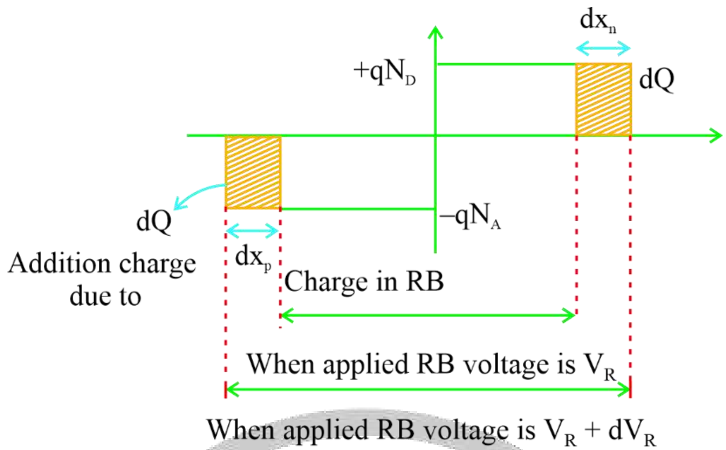
$$E = \frac{-qN_A}{\varepsilon}[x + x_P]; -x_p \le x \le 0$$

$$E = \frac{-qN_D}{\varepsilon}[x_n - x]; 0 \le x \le x_n$$

$$E_{max} = \frac{-2(V_{bi} + V_R)}{W}$$

4.3. Junction Capacitance

We have a separation of +ve & –ve changes in the depletion reason therefore a capacitance is associated with p – N junction, this capacitance is called as junction capacitance or depletion layer capacitance



$$C = qN_DA\sqrt{\frac{\varepsilon}{2q}, \frac{N_A}{N_D} \times \frac{1}{N_A + N_D} \times \frac{1}{(V_{bi} + V_R)}}$$

$$C \propto \frac{1}{\sqrt{V_{bi} + V_R}}$$

$$\frac{C}{A} = C' = \sqrt{\frac{\varepsilon q}{2}\frac{N_A N_D}{N_A + N_D} \times \frac{1}{V_{bi} + V_R}} \text{ (Capacitance per unit area)}$$

$$C' = \frac{E}{W} \text{ V/cm}^2 \ \& \ C = \frac{AE}{W} \text{ F}$$

Note : Where $V_R = 0 \Rightarrow C = C_0 \longrightarrow$ max depletion capacitance

$$\frac{C}{C_0} = \sqrt{\frac{1}{1 + V_R / V_{bi}}}$$

Under revers bias the current due to majority is low

- It is blocking the flow of majority carrier in R.B, so, the crossing of junction become block, hence reverse bias is also called blocking bias
- Under RB only minority carries will be falling from the barrier potential and they contribution majority carrier current (I_0)

I_0 = thermally generated or minority carrier current or reverse saturation current or leakage current

I_0 = µ A(for Ge)

I_0 = nA(For Si diode)

Highly sensitive to temp (I_0 inverse with temp. increase)

I_0 is independent of the applied reverse bias voltage that is I_0 is saturated w.r.t applied V_{RB} and hence called saturation current

I_0 is reverse current flow from n to p.

As temp increase minority carrier are generated

For 1°C increase in temp = I_0 increase by 7%

I_0 is a drift current because thus current is due to the field intensity, which is passing through junction

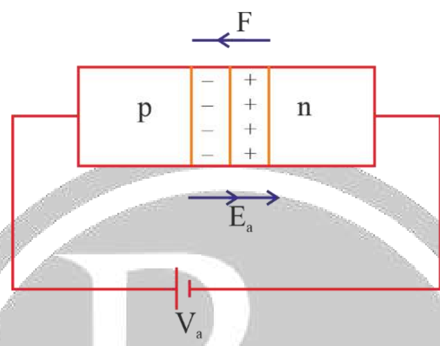
$$E_{Peak} = \frac{-V_j}{w/2} = \frac{-2V_j}{w}$$

$$E_{Peak} = \frac{2\left[V_0 + |V_{RB}|\right]}{W}$$

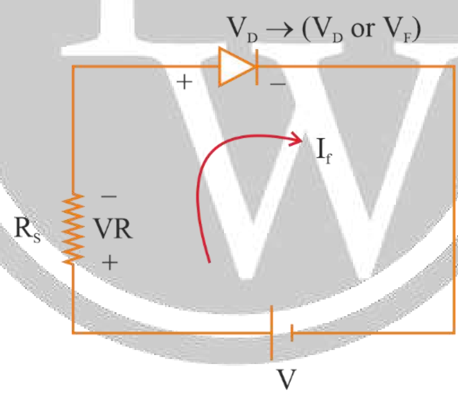
$$W = \frac{\varepsilon}{q}\left(\frac{1}{N_A} + \frac{1}{N_D}\right)|E_{peak}| \text{ depletion width in terms of } E_{Peak}$$

Forward Bias (FB)

When a positive potential is applied on p – side w.r.t *n* – side then applied bias is called as forward bias



When any type of p – n junction is forward biased limiting resistance must be connected in series with the diode, the limiting resistance and diode working as potential divider network



Forward voltage, across the diode is V_D

$$V_D \le 0.5 \text{ V for Ge}$$

$$V_D \le 0.9 \text{ V for Si}$$

$$V = V_R + V_D$$

$$V = I_f R_s + I_f R_f$$

Where, I_f = diode current/forward bias current

The majority carriers of P & n regions will be moving towards the junction and neutralize the immobile ions so that the region of immobile charges gets reduced that is width of the depletion layer is reduced

E_s and E_v are in opposite direction

Experimentally found resultant is directed from *n* to *p*.

The potential barrier is reduced by $|V_d|$ under forward bias

$$V_j = V_0 - |V_D|$$

Case I :

$V_D < V_0$, the diode is in forward biased and non conducting state and its is "Off state" or "zero state"

Case II :

$V_D = V_0$, effect of Barriers is "Nullified "

Case III:

More majority carriers will be crossing the junction and the junction and the forward current is large and the forward current will be exponentially increasing with V_d.

Excess Majority Carrier in Both Side

In normal equilibrium

$$n_{p_0} = n_{n_0} e^{-\frac{V_{bi}}{V_T}} \qquad p_n = p_{p_0} e^{-V_{bi}/V_T}$$

Where, $n_{n_0} \rightarrow$ e⁻ concentration on n – side (majority carrier)

$n_{p_0} \rightarrow$ e⁻ concentration of p – side (minority carrier)

After application of forward bias, as due to barrier potential

$$n_p = n_{p_0} e^{V_0/V_T} \quad p_n = P_{n_0} e^{-V_0/V_T}$$

Where, n_p = e^- concentration at the edge of depletion region on p – side

And p_n = hole concentration at the edge of depletion region on n – side

Forward Bias Current

$$I_f = \text{Diffusion current (P to n)}$$

$$I_f = I_0\left[e^{V_d/\eta V_T} - 1\right]$$

Where, V_d = voltage across diode (p to n) forward bias)

N = utility or ideality factor

V_T = thermal voltage ; KT/q

η = 1 = for large currents, eq : Ge

η = 2 = for small current , eq : Si

Default value of η = 1

$$I_0 = \text{Reverse saturation current (Drift current)}$$

$$I_0 = \left[\frac{AqD_p n_i^2}{L_P N_D} + \frac{AqD_n n_i^2}{L_n N_A}\right]$$

$$I_F = I_{diff} = \frac{AqD_p n_{i^2}}{L_P N_P}\left(e^{\frac{V_d}{\eta V_T}} - 1\right) + \frac{AqD_n n_i^2}{L_n N_A}\left(e^{\frac{V_d}{\eta V_T}} - 1\right)$$

$$I_{diff} = I_{p\,diff} + I_{n\,Diff}$$

$$I_{diff} = \frac{AqD_p n_i^2}{L_p N_D}\left(e^{V_d/\eta V_T} - 1\right) + \frac{qD_n n_i^2}{L_n N_A}\left(e^{V_d/\eta V_T} - 1\right)$$

Forward current is always independent of temperature because this current is carried by majority carrier & minority carrier concentration is always independent of temperature.

$$I_F = I_0\left[e^{V_d/\eta V_T} - 1\right]$$

$$V_d = \text{forward bias across p – n junction}$$

$$0 \leq V_d \leq V_{bi}$$

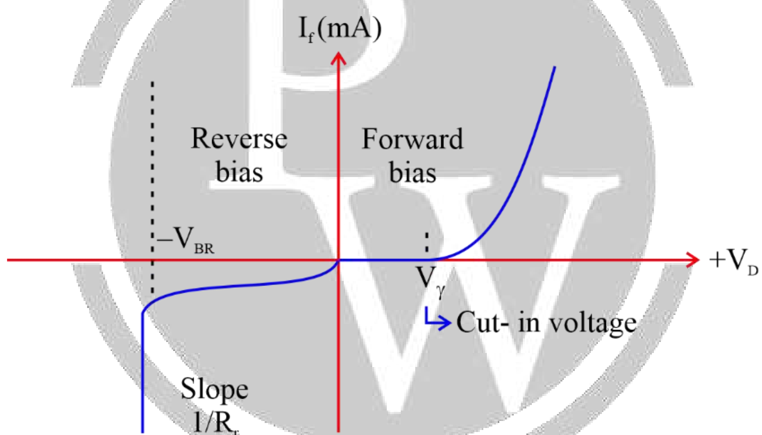
$$w \propto \sqrt{V_{bi} - V_a}$$

For $V_{bi} > V_a$, w will comes to be imaginary which is not possible

$$V_D = \eta V_T \ell n\left(\frac{I_f}{I_0}\right)$$

$$\frac{dV_d}{dt} = -2.5\,\text{mV}/°\text{C}$$

VI Characteristics of p – n junction diode :



When p – n junction is reverse biased the reverse voltage must be always less than breakdown voltage of the device otherwise the diode will be destroyer

$$\text{Breakdown voltage} = V_{BR} \text{ or } B_V$$

$$V_{BR} \propto \frac{1}{\text{Doping concentration}}$$

$$V_{BR} = \frac{\varepsilon E^2}{2q(\text{Doping concentration})}$$

Cut in Voltage (V_γ) :

- Also called offset voltage (in FET’s) or called threshold voltage (in tubes)
- Min voltage required so that the forward current just passes into diode

$$V_\gamma = 0.1 \text{ v to } 0.5 \text{ v [typical} = 0.2]$$

$$\min V_\gamma = 0.1\text{v or } 100 \text{ mv}$$

cut in voltage decrease with increase in temperature

v_y reduced by 2.5 volt for 1° c increase in temperature

4.4. Small Signal Equivalent Circuit of Diode

Diode Resistance

In reverse biased : Ideally current is zero then R = ∞ , open circuit practically $I = I_0$, R is very large

In forward biased

$$I_F = I_0\left[e^{\frac{V_d}{\eta V_T}-1}\right]$$

$$\frac{dI_F}{dV_d} = \frac{I_F}{\eta VT} \quad \text{admittance of diode (g)}$$

$$\frac{l}{g} = r_f = \frac{\eta V_T}{I_F} \quad \text{forward resistance or dynamic resistance}$$

Diffusion Capacitance (C_D)

C_D = Rate of change of injected minority charges w.r.t changes in forward voltage

C_D is due to a change in injected minority charge

C_D is the junction capacitance in a forward biased

$$C_D = C_j = A\varepsilon/W$$

$$C_D \propto A$$

$$C_D \propto 1/W$$

$$C_D \propto \sqrt{\text{Doping concentration}}$$

$$C_D = \tau \cdot g$$

Where g = dynamic conductance or reciprocal of dynamic resistance

If $g = 1/r_f = I_f/\eta VT$

Then, $C_D = \dfrac{\tau}{r_f}$ $\qquad C_D = \dfrac{\tau . I_f}{\eta v_T}$ forward $C_D \propto I_f$

Transition capacitance (C_T)

$$C_T = C_J = \frac{A}{\sqrt{\frac{2}{q\varepsilon}\left(\frac{N_A+N_D}{N_A N_D}\right)V_O\left(1+\left|\frac{V_{RB}}{V_0}\right|\right)}}$$

$$C_T = C_J = \frac{A\sqrt{\frac{q\varepsilon N_A N_D}{2V_0[N_A+N_D]}}}{\sqrt{1+\left|\frac{V_{RB}}{V_0}\right|}}$$

❐❐❐

5 SPECIAL DIODES

5.1. Varactor Diode

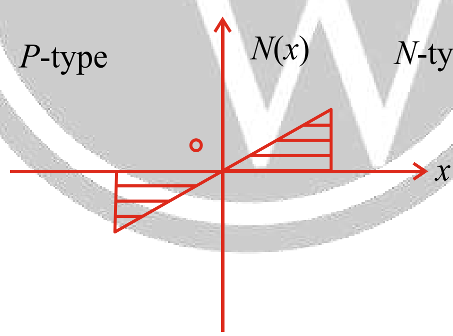
As the larger variations, in the diode capacitance. The varactor diode is always operates under reverse bias conditions. It is variable capacitance diode with linearly doped P and N regions. It is voltage variable capacitance.

$$C_T = \frac{C_{T_0}}{\left(1+\frac{V_{RB}}{V_{bi}}\right)^{1/n}}$$

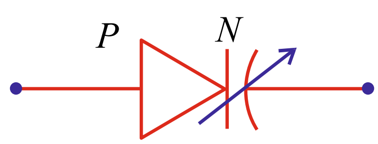
n = 2 for step graded Junction
n = 3 for lineally graded Junction.
In linearly graded junction the junction the doping profile is defined as :

$$\boxed{N_{(x)} = N_0 x}$$

For P region : $N_a(x) = N_{ao}x$
For N region : $N_d(x) = N_{do}x$



Symbol:



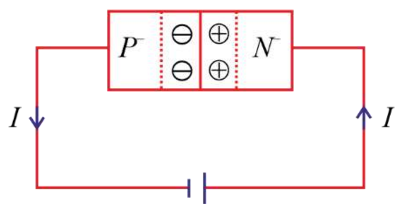
New for varactor diode : $\boxed{C_T = \frac{C_{T_0}}{\left(1+\frac{V_{RB}}{V_{bi}}\right)^{1/3}}}$

It is use in voltage-controlled oscillators (VCO), Radio frequency fitters and frequency and phase modulators.
It is also use in automatic frequency control devices self-balancing of AC bridges.

5.2. Photo Diode

It converts light energy into electrical energy. Photo diode has lightly doped P and N region, so the depletion region will have slightly larger depletion width (W), The advantage of larger W is, it will absorb larger number of photons from the light energy near the junction. So that both side by minority will be created near the junction.

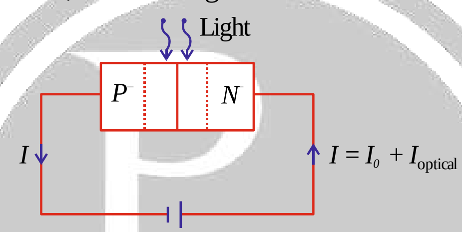
The falling photons will break the co-valent bonds and create minority carious of both side. Impact on the reverse bias current hence photo diode is always operates under reverse bias.



To absorbed more photons, photo sensitive material ZnS or CaS is coated over the depletion region.

In reverse bias :

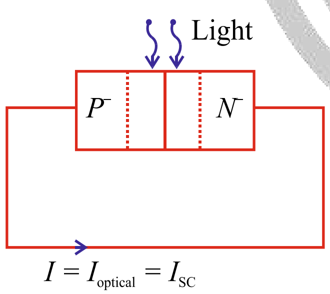
$$I = I_0\text{, when no light falls.}$$

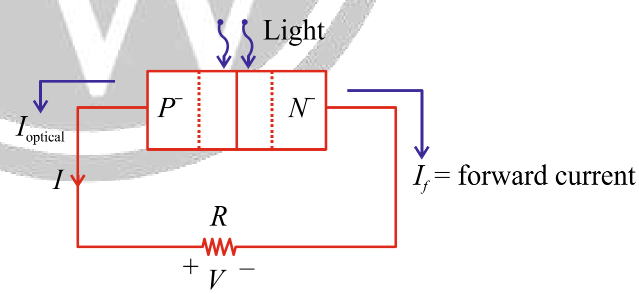


$I_{optical} \rightarrow$ Current due to excess carriers generated due to optical energy. (Diffusion current)

Short Circuit Current:

As there is no reverse voltage ($I_0 = 0$)





V = IR, The voltage V start bias the PN junction as forward bias, hence current I :

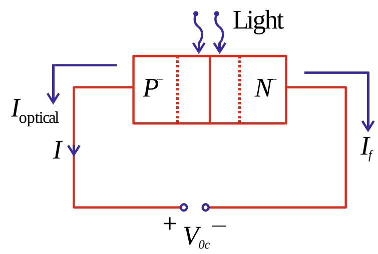
$$I = I_{optical} - I_f$$

$$\boxed{I = I_{optical} - I_o\left[e^{V/nV_T} - 1\right]}$$

Open Circuit Voltage:

$$I = I_{optical} - I_f = 0$$

$$\boxed{V = V_{oc}}$$

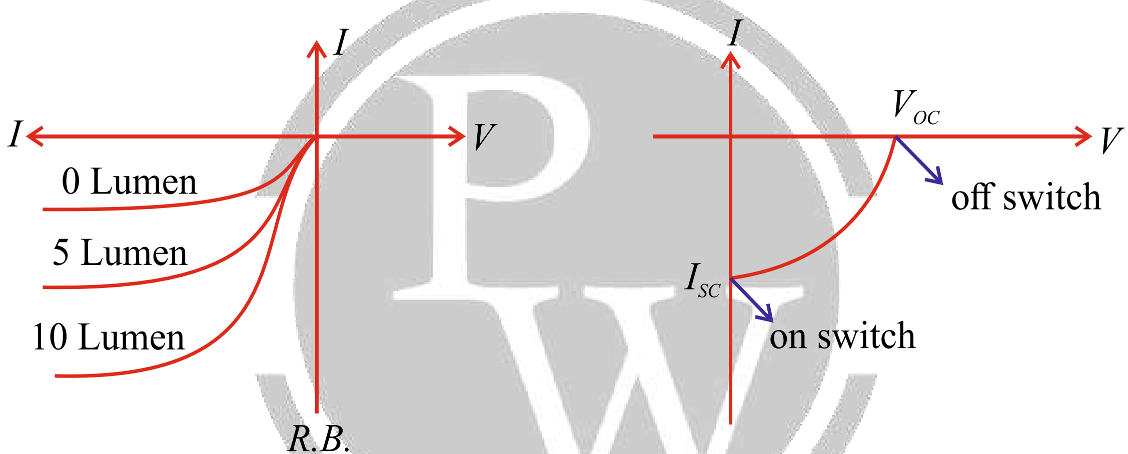


$$I_{optical} - I_0\left[e^{V_{oc}/\eta V_T} - 1\right] = 0$$

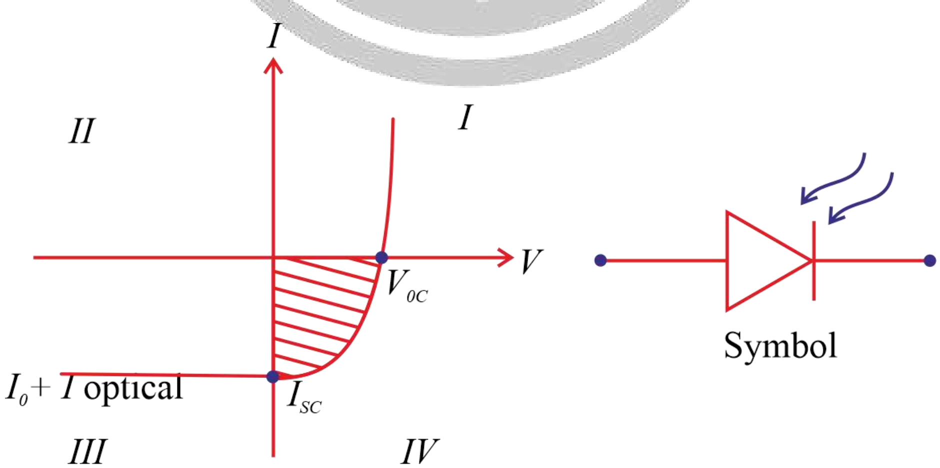
$$\boxed{V_{oc} = \eta V_T \ell n\left[\frac{I_{optical}}{I_o} + 1\right]}$$

$$\boxed{V_{oc} = \eta V_T \ell n\left[\frac{I_{sc}}{I_o} + 1\right]}$$

Characteristics :



It is a light operated switch with switching time is in μ seconds, and it is almost 1000 times faster than the normal diode.



As it converts optical energy into electrical energy, it is widely use in satellite communication as a transformer.
The only disadvantage of photo diode is, it has smaller power handing capacity. lubrication of PIN photo diode, which has larger power handling capacity.

Quantum Efficiency : [Q_e]

$$Q_e = \frac{\text{No. of electrons collected}}{\text{No. of incident photons}} = \frac{r_e}{r_p}$$

$$Q_e = \frac{I_{\text{optical}} \times \frac{hv}{q}}{P_0} \times 100\%$$

P_o = incident power.

$$\boxed{Q_e = R.\frac{hv}{q}}$$

Responsivity : [R]

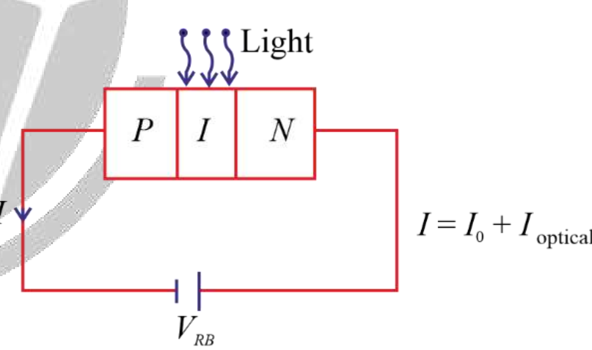
$$R = \frac{\text{output current}}{\text{incident power}} = \frac{\text{optical current}}{\text{Incident power}}$$

$$\boxed{R = \frac{I_P}{P_o} = \frac{I_{optical}}{P_o}} \ A/watt$$

$$\boxed{R = \frac{Q_e}{\frac{hv}{q}} = \frac{qQ_e}{hv} = \frac{qQ_e\lambda}{hc}}$$

3. PIN Photo Diode :

PIN diode has intrinsic region sandwiched between P and N region. Which offers high resistivity and provide larger power handling capacity.

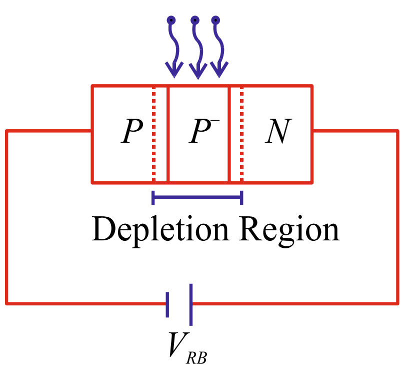


Because of intrinsic region ideal PIN Diode does not have any PN Junction.

To create PN Junction 'I" region is replaced by lightly doped P region. Which is called as π–region or I-region is replaced by lightly doped N-region. Which is called V-region.

In this overall π region is swap out and covered by depletion region.

$$\boxed{V_{RB} = V_{swap\ out}}$$



PVN Photon Diode :

In this overall v-region is swap out and covered by depletion region.

$$\boxed{V_{RB} = V_{swap\ out}}$$

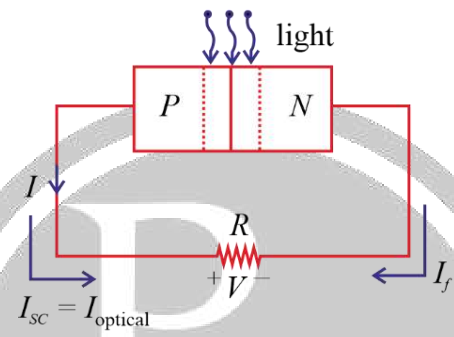
Both PπN and PγN diodes operate after swapping out of π and γ region in order to get larger current.

PIN diode is as faster switch than the photo diode and the switching time is in nS (nano seconds)

Due to larger power handling capacity PIN diode is used in microwave applications.

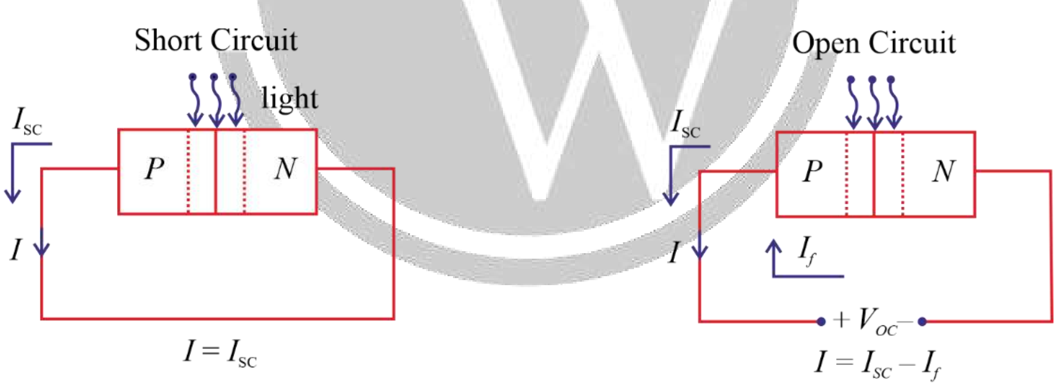
4. Solar Cell :

The working principle of solar cell is similar to the photo diodes.
Consider a PN Junction with resistive load.



This drop v will bias PN Junction, in forward bias.

$$\boxed{I = I_{optical} - I_f}$$



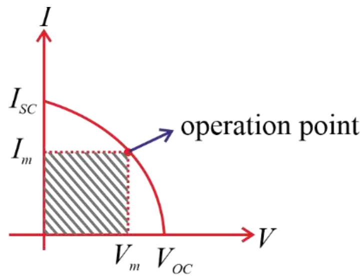
$I = I_{sc} - I_f$

$$V_{oc} = \eta V_T \ell n\left[\frac{I_{optical}}{I_O} + 1\right]$$

$$V_{oc} = \eta V_T \ell n\left[\frac{I_{sc}}{I_O} + 1\right]$$

$$\boxed{V_{oc_1} - V_{oc_2} = \eta V_T \ell n\left(\frac{I_{sc_1}}{I_{sc_2}}\right) = \eta V_T \ell n\left(\frac{J_{sc_1}}{J_{sc_2}}\right)}$$

Characteristics :



Maximum power delivered (p)

$$\boxed{P = I.V}$$

$$P = \left[I_{\text{optical}} - I_o\left(e^{V/\eta V_T} - 1\right)\right] \times V$$

$$\text{For } \frac{dP}{dV} = 0$$

At this, the value of V will give maximum power. That value of V and corresponding value of I is known as operating point, i.e. V_m and I_m.

$$\boxed{P_{\max} = \mathrm{I_m} . V_m}$$

Responsivity, $\boxed{R = \frac{I_{optical}}{P_{in}}}\ A/watt.$

Conversion efficiency : [η]

$$\eta = \frac{\text{Electrically genrated power}}{\text{Incidert optical power}}$$

$$\boxed{\eta = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{\mathrm{I_m} V_m}{P_{in}}}$$

$$\boxed{V_m = \frac{hv}{q}}$$

$$\boxed{\eta = \frac{I_{optical} \times hv}{P_{in} \times q} = R.\frac{hc}{q\lambda}}$$

$$\boxed{\eta = \frac{1.24\,R}{\lambda(in\ \mu m)}}$$

Fill factor : [FF]

It is the ratio of operating power and the maximum theoretical power.

$$\boxed{FF = \frac{\mathrm{I_m} V_m}{I_{sc} V_{oc}}}$$

5. LED (Light Emitting Diode) :

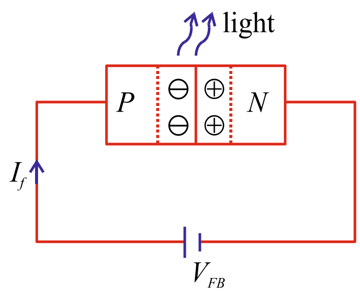
LED works on the principle of electro luminescence. It is the property by which light energy comes out due to electron hole recombination.

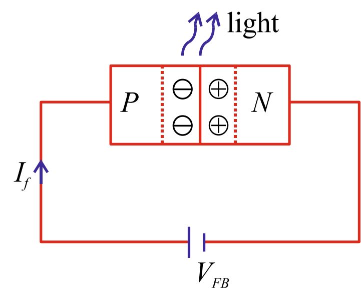
It always operates under forward bias, so that large recombination happen at the junction and produce light.

Light Intensity [I] $\propto I_f$. (forward current).

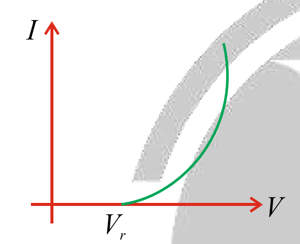
LEDs are fabricated by direct Band Gap semiconductors.

$$\begin{aligned} \text{GaAs} &\rightarrow \text{Infrared light} \\ \text{GaA} &\rightarrow \text{Blue light} \\ \text{GaAsP} &\rightarrow \text{Red or yellow light} \\ \text{GaP} &\rightarrow \text{Red or Green} \end{aligned}$$

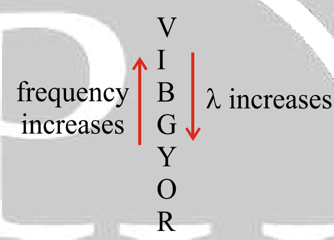




Characteristics:



cutin voltage



$$\boxed{\lambda = \frac{1.24}{E_g(ev)}\mu m.}$$

$$\boxed{\begin{aligned} \lambda_{max} &= \lambda_{red} \\ \lambda_{min} &= \lambda_{violet} \end{aligned}}$$

$\lambda_R > \lambda_0 > \lambda_Y > \lambda_G > \lambda_B > \lambda_I > \lambda_V$

$E_{gR} < E_{g0} < E_{gY} < E_{gG} < E_{gB} < E_{gI} < E_{gV}$

$V_{biR} < V_{biO} < V_{biY} < V_{biG} < V_{biB} < V_{biI} < V_{biV}$

Also $V_r \propto V_{bi}$ Built in voltage

V_r = Cutin voltage

V_{bi} = Built in voltage

Internal Efficiency : [η_{int}]

$$\boxed{\eta_{int} = \frac{\tau_r}{\tau}; \quad \frac{1}{\tau} = \frac{1}{\tau_r} + \frac{1}{\tau_{nr}}}$$

τ = total recombination carrier lifetime

τ_r = Radiative recombination carrier lifetime.

τ_{nr} = Non-radiative recombination carrier lifetime.

Internal Power : [P_{int}]

$$\boxed{P_{int} = \eta_{int} \cdot \frac{hCI}{\lambda q}} \text{ watt}$$

Emitting Power : [P_e]

$$Pe = \frac{P_{int} . F \eta^2}{4\eta_x^2}$$

$F \rightarrow$ Transmission factor between LED material and medium.

$\eta \rightarrow$ Refractive index of the medium.

$\eta_x \rightarrow$ Refractive index of the material.

External power Efficiency : η_{ep}

$$\eta_{ep} = \frac{Pe}{P} \times 100\%$$

P = input Power.

Power conversion or coupling power efficiency : [η_{cp}]

$$\eta_{cp} = \frac{P_c}{P} \times 100\%$$

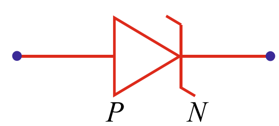
$$\boxed{\eta_{cp} = \frac{P_c}{IV} \times 100\%}$$

Zener Diode :

It is slightly highly doped diode, specially designed to operate under breakdown region.

Breakdown is a phenomena in which, large number of covalent bonds will be broken and this will change the minority carrier concentration significantly. Hence breakdown occurs in reverse bias. In forward bias only majority carriers will present and due to this breakdown is not possible in forward bias.

Symbol



Breakdown are of two types:

1. Zener breakdown
2. Avalanche breakdown

Zener Breakdown

In Zener breakdown both P and N region are highly doped, due to that barrier energy will be large and depletion width will be smaller and when we applied reverse bias voltage, the $\vec{E}$ -field in the depletion region will be increasing and at reverse bias voltage equals to the breakdown voltage, the electric field will become very large and called as critical $\vec{E}$ -field $\left(\vec{E}_{crit}\right)$ dut to that covalent bonds at the junction will be broken and current abruptly increases. It is negative temperature coefficient

- Zener breakdown is due to longer(E) field intensity in the depletion region
- Zener breakdown occurs at low voltages.
- When temp will increase covalent bonds will be broken current will start increasing due to that the breakdown occurs at low voltages.

$$\frac{dv_z}{dt} = \ominus ve \rightarrow \ \ominus \text{ve temp. coefficient}$$

This indicates Zener breakdown is $\ominus$ve temp. coefficient

5.3. Avalanche Breakdown

Avalanche lockdown occurs in lightly doped diode due to that the barrier is very small and depletion width will be larger due that $e^{\ominus}$ will spend more time when move for p region $\rightarrow n^{-}$ region and due to larger K.E. e^{-} will be strike on the covalent bond will be broken and one $e^{\ominus}$ will be generated again the unmuted $e^{\ominus}$ will impact on the another covalent bond and further $e^{\ominus}$ will be generated. After impact the $e^{\ominus}$ will be migrated into the *n*-region.

Breaking of covalent bond due to impact energy in called as impact ionization which gives $e^{\ominus}$ multiplication

$$T\uparrow \rightarrow \mu\downarrow \rightarrow V_d\downarrow \rightarrow \frac{1}{2}mV_d^2\downarrow$$

$$\Downarrow$$

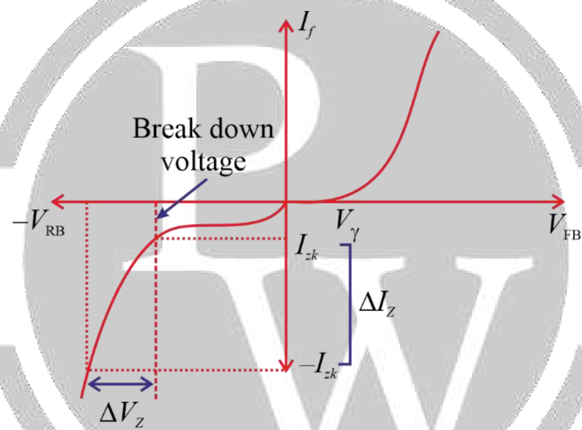
$$\text{For } qVRB\uparrow \rightarrow \frac{1}{2}mV_d^2\uparrow$$

when temp will increase mobility will decrease due to lattice scattering which make V_d smaller and the K.E. of $e^{\ominus}$ will be small and for breakdown of the diode equal high R.B voltage when temp (↑) breakdown voltage in avalanche breakdown(↑)

$$\frac{dV_z}{dt} = \oplus ve$$

This indicates ⊕ve temp coefficient of avalanche breakdown.

Characteristics : (Zener Diode)



Dynamic resistance, $r_z = \frac{\Delta V_z\uparrow}{\Delta I_z\uparrow}$ = dynamic resistance of Zener diode.

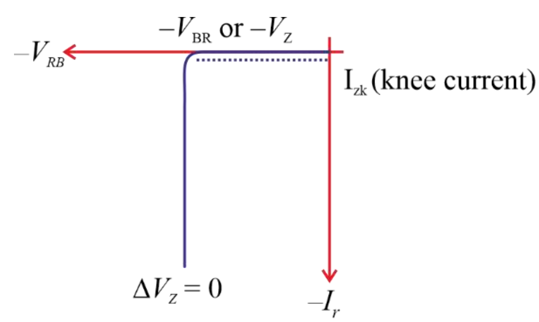
$V_\gamma = 0.7$ V (Zener Diode is always laid of Si)

$r_z \rightarrow$ very small $(1\Omega - 10\Omega)$

The min current required for a Zener diode to be operate in R.B. Region is known current

$$\because \quad r_z = \frac{\Delta V_z}{\Delta I_z} = 0$$

$$\Rightarrow \quad r_z = 0$$



If $V_{RB} < V_z$

$\Downarrow$

Normal diode in RB

If $V_{RB} > V_z \Rightarrow$ Zener diode in Breakdown region (conducting)

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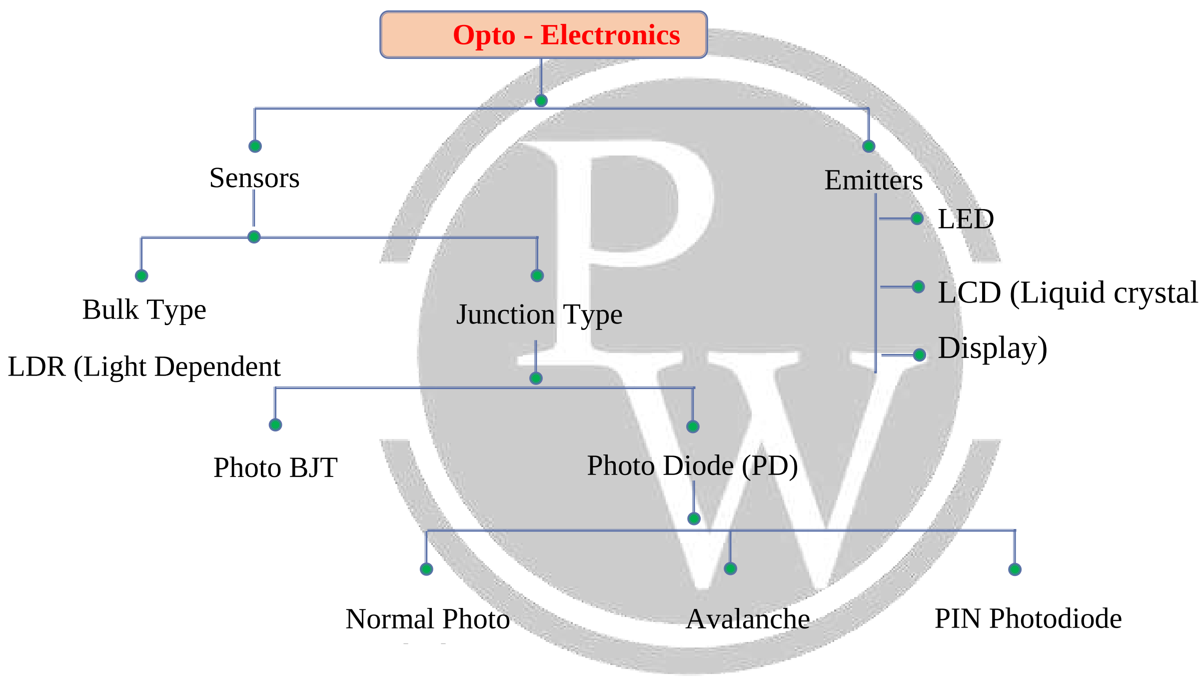
6 Opto-Electronics and Quasi Fermi Levels

6.1. Opto-Electronics and Quasi Fermi Levels



When Light falls at

Junction type of sensors or bulk type of sensors then due to absorption of photon. The Covalent bonds will be broken and Electron – Hole pairs will generate. Multiple photons will generate multiple EHP's

To generate EHP

$$\text{Photon energy} = h\nu$$

h = planks constant = 6.62×10^{-34} J – sec.

ν = frequency (Hz)

$$\nu = \frac{c(\text{Speed of light})}{\lambda(\text{wave length})}\text{Hz.}$$

$$h\nu \geq E_g \text{ (energy band gap)}$$

$$\frac{hc}{\lambda} \geq E_g$$

$$\lambda \leq \frac{hc}{E_g}$$

$$h = \frac{6.62\times10^{-34}}{q} \text{ ev-sec.} = \frac{6.62\times10^{-3}}{1.6\times10^{-19}} \text{ ev - sec}$$

$$c = 3 \times 10^8 \text{ m/sec.}$$

$$\lambda \leq \frac{1.24}{E_g(ev)} \mu\text{m}$$

Critical wave length $\lambda_c = \dfrac{1.24}{E_g(ev)} \mu m$

Generation rate (G_L) due to irradiation of light :

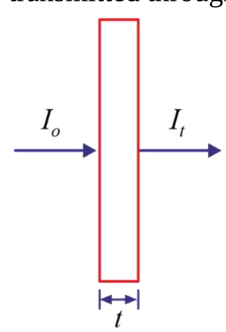
$$G_L = \frac{\text{Excess minority carrier concentration}}{\text{Minority carrier life time}}$$

When light falls equal number of e^- and holes will be created

Beer-Lambert Law

This states that, if light passes through the sample then some part of light will get absorbed and rest will be transmitted through the sample. This transmitted part of light depends on the absorption coefficient (α).

$$I_t = I_o e^{-\alpha t}$$



$I_t \rightarrow$ Transmitted Intensity

$I_o \rightarrow$ Incident Intensity

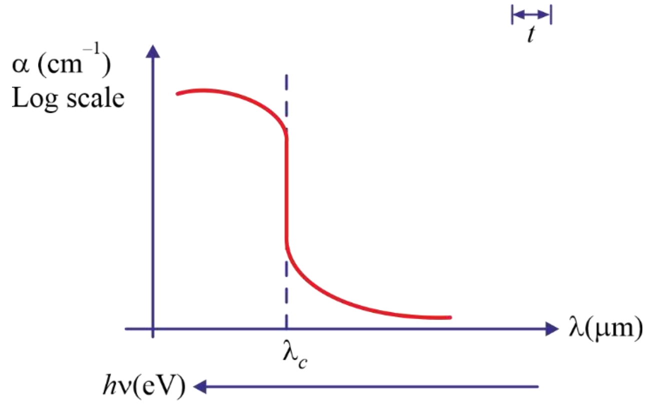
$\alpha \rightarrow$ Absorption coefficient (cm^{-1})

$t \rightarrow$ thickness of the sample

$$\boxed{\text{Absorbed Intensity} = I_o - I_t}$$

Absorption coefficient (α), depends on the material and associated wavelength (λ) of light.

$$\boxed{\lambda_c = \frac{1.24}{E_g(eV)} \mu m}$$



Let n – type SC :

Before illumination : $n_{no} = N_D$

$$p_{n_0} = \frac{n_i^2}{n_{n_0}} = \frac{n_i^2}{N_D}$$

After illumination :

$$n_n = n_{n_0} + \Delta n_n$$

$$p_n = p_{n_0} + \Delta p_n$$

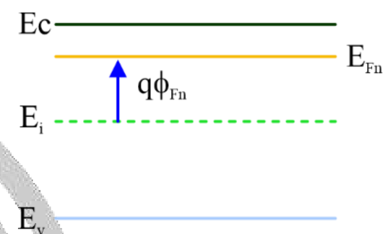
$$\Delta n_n = \Delta p_n$$

$\Delta n_n \rightarrow$ Excess e^- concentration

$\Delta P_n \rightarrow$ Excess hole concentration

$$G_L = \frac{\Delta n_n}{\tau_{P_0}} = \frac{\Delta P_n}{\tau_{P_0}} \text{ m}^3\text{/sec or cm}^{-3}\text{ s}^{-1}$$

Before Illumination:



$\Phi_{fn} \rightarrow$ Fermi potential for electrons

$$\phi_{fn} = \frac{KT}{q}\ell n\left(\frac{n_{no}}{n_i}\right) = \frac{KT}{q}\ell n\left(\frac{N_D}{n_i}\right)$$

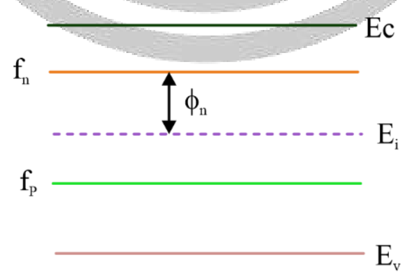
$$q\phi_{fn} = KT\ell n\left(\frac{N_D}{n_i}\right) eV$$

f_n = Quasi fermi level of e^-

f_P = Quasi fermi level of hole

When we starts illuminating light over the semiconductor (n – type), due to EHP generation. There will be significant change in hole concentration and almost insignificant change in e^- concentration. Due to that that there will be change in Quasi fermi level of holes (f_p) and almost no change in quasi fermi levels of e^- (f_n)

After illumination :



More illumination gives more shift in F_P

$$f_n - E_i = KT\ell n\left(\frac{n_n}{n_i}\right) eV$$

$$E_i - f_p = KT\ell n\left(\frac{P_n}{n_i}\right) eV$$

f_n and ϕ_{fn} is consider to be negative as it goes upwards than E_i(Fermi level of intrinsic) and f_p and ϕ_{FP} is consider to be positive as it goes down wards than E_i.

Similarly for P – type SC:

Before illumination :

$$p_{p0} = N_A$$

$$n_{p_0} = \frac{n_1^2}{p_{p_0}} = \frac{n_i^2}{N_A}$$

After illumination :

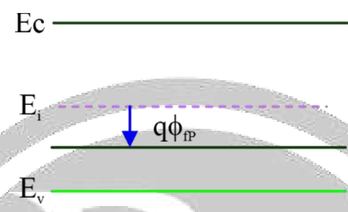
$$p_p = p_{p0} + \Delta p_p$$

$$n_P = n_{p0} + \Delta n_p$$

$$= n_p = \Delta p_p$$

$$G_L = \frac{\Delta n_p}{\tau_{no}} = \frac{\Delta P_p}{\tau_{no}} \text{ cm}^{-3}\text{s}^{-}1$$

Before Illumination :



$\Phi_{fP} \longrightarrow$ fermi potential for holes

$$\phi_{f_P} = \frac{KT}{q}\ell n\left(\frac{P_{PO}}{n_i}\right) = \frac{KT}{q}\ell n\left(\frac{N_A}{n_i}\right)$$

$$q\phi_{f_P} = KT \ln\left(\frac{N_A}{n_i}\right) eV$$

Here due to EHP generation the change in e^- concentration will be significant and change in hole concentration will be insignificant. Hence change in f_n will be significant and f_P will be insignificant

After illumination:

More illumination gives more shift in f_n.

$$f_n - E_i = KT\ell n\left(\frac{n_P}{n_i}\right) eV$$

$$E_i - f_P = KT\ell n\left(\frac{p_P}{n_i}\right) eV$$

Conductivity due to illumination:

$$\sigma_{opt} = \Delta nq\mu_n + \Delta Pq\mu_p \ \ \mho/\text{cm}$$

Where, Δn = change in e^- concentration

$\Delta P \longrightarrow$ change in hole concentration

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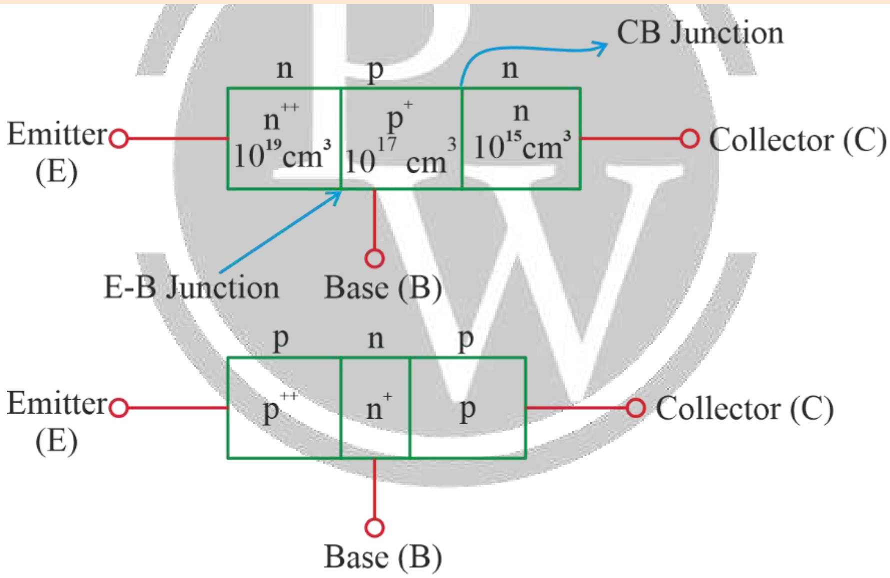
7 BIPOLAR JUNCTION TRANSISTOR

7.1. BJT (Bipolar Junction Transistor)

- Transistor is a multijunction semiconductor device that together with other circuit element is capable of current gain, voltage gain and signal power gain, therefore it is an active device whereas diode is passive device.
- The basic transition action is the control of current at one terminal by the voltage applied across other two terminal of the device.

7.1.1. Basic Structure of BJT



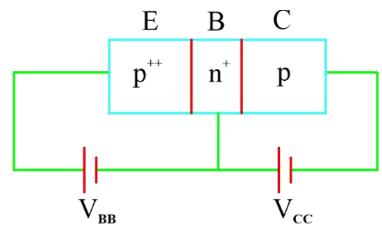
- BJT has 3 separately doped region and 2 p-n Junction, 3-terminal connection are emitter, base and collector.
- The width of the base region is small as compared to the minority carrier diffusion length.
- The emitter region has the largest doping concentration region and base region has the smallest.
- Bipolar Junction transistor is not a symmetrical device.

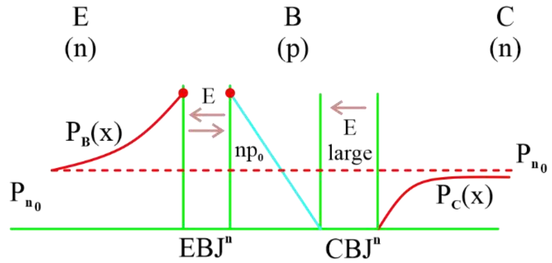
7.1.2. Basic Operation of BJT

- Forward active mode

  Where, EB Junction is forward Bias

  And CB junction is in reverse Bias





- Distribution of minority carriers in npn transistor in forward active mode.
- The BE junction is forward biased and CB junction is reverse biased. This configuration is called the forward active operating mode.
- The electron from the emitter are injected across the base-emitter junction into the base. These injected electrons create an excess concentration of minority carrier in base.
- Similarly holes from the base injected across the base-emitter junction into the emitter.
- These injected electrons and holes create an excess concentration of minority carrier in the base and emitter respectively.
- The base collector junction is reverse biased so the minority carrier electron concentration at the edge of base collector voltage and the two close enough junction are said to be interacting p-n junction.

7.3. Simplified Current Relation in BJT

(1) Collector Current:

- Assuming ideal linear electron distribution in base, then collector current can be written as diffusion current.

$$I_c = qDn\frac{dn}{dx}\cdot A_{BE}$$

$$I_c = I_s \cdot e^{V_{BE}/V_T}$$

where, $$I_s = \frac{qD_n A_{BE} n_{BO}}{x_B}$$

$A_{BE} \rightarrow$ Base emitter junction Area

$N_{BO} \rightarrow$ Thermal equilibrium electron concentration in base region

$x_B \rightarrow$ Neutral base width (not considering space charge region)

(2) Emitter Current:

- I_{E_1} = It is due to the flow of electron injected from emitter to base, thus ideally it is equal to collector current.
- I_{E_2} = Base – Emitter junction is forward based majority carrier holes of base injected across the base emitter junction into the emitter.

$$I_{E_2} = I_{S_2} \cdot e^{V_{BE}/V_T}$$

Where $I_{S_2} \rightarrow$ it includes minority carrier hole parameters in emitter region.

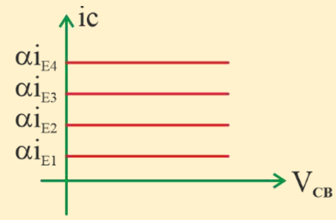
- Now, $$I_E = I_{E_1} + I_{E_2}$$
$$I_E = \left(I_S + I_{S_2}\right)e^{V_{BE}/V_T}$$

$\frac{I_C}{I_E} = \alpha \rightarrow$ Common-Base current gain factor

$I_C = \alpha I_E$ as, $I_E > I_C$ so $\alpha < 1$

Note:

- Since I_{E_2} is not a part of the basic transistor action we would like to be this component as small as possible in order to make common base current gain close to unity.
- Ideally I_C is independent of V_{CB} as long as collector base junction is reverse bias so we can say BJT act as constant current source
- Ideal BJT I-V characteristics in common base configuration.



(3) Base Current:

- The emitter current I_{E_2} is a base emitter junction current so that this current is also a component of base current.
- Since majority carrier holes in the base are disappearing, they must by resupplied by a flow of positive charge into the base terminal, this flow of charge is I_{Bb}.
- The total base current is the sum of I_{Ba} and I_{Bb}.

$$I_B = I_{Ba} + I_{Bb}$$

- The ratio of i_c and i_b is also a constant

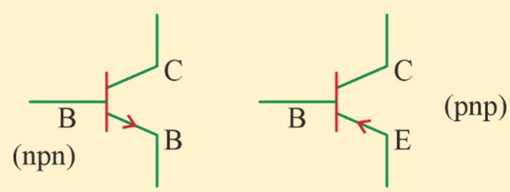
$$\frac{I_C}{I_B} = \beta \Rightarrow I_C = \beta I_B$$

Where, β = common-emitter current gain factor.

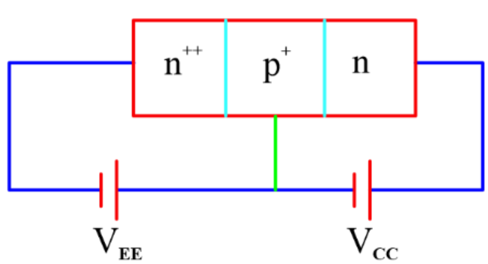
Note: Applying kCL

$$I_E = I_C + I_B$$

- $\frac{I_C}{I_B} = \beta \Rightarrow I_C = \beta I_B$

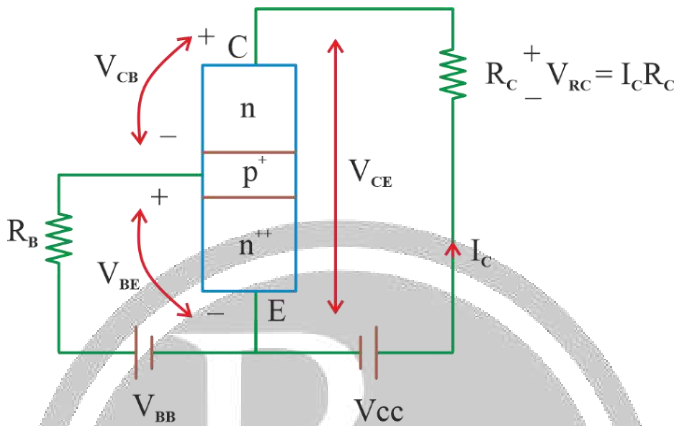


7.3.1. Other Operating Models of BJT



BE Junction	CB Junction
FB	RB $\rightarrow$ Forward active mode
RB	RB $\rightarrow$ Cut-off mode
FB	FB $\rightarrow$ Saturation mode
RB	FB $\rightarrow$ Inverse Active mode

Common Emitter Configuration:



If $V_{CB} > 0$, then Reverse Bias

IF $V_{BE} > 0$, then Forward Bias

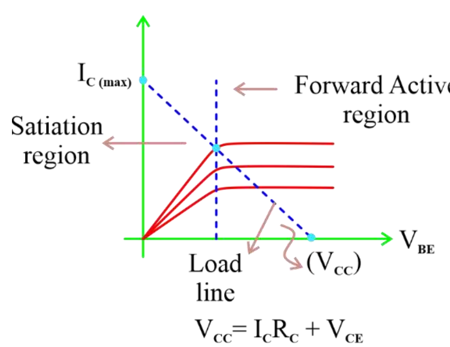
$$V_{CC} - I_C R_C - V_{CE} = 0$$

$$V_{CC} = V_{CE} + V_{RC}$$

$$V_{CE} = V_{CB} + V_{BE}$$

If $V_{BE} \uparrow$es $\rightarrow I_C\uparrow \rightarrow V_{RC}\uparrow \rightarrow V_{CE}\downarrow \rightarrow V_{CB}\downarrow$es

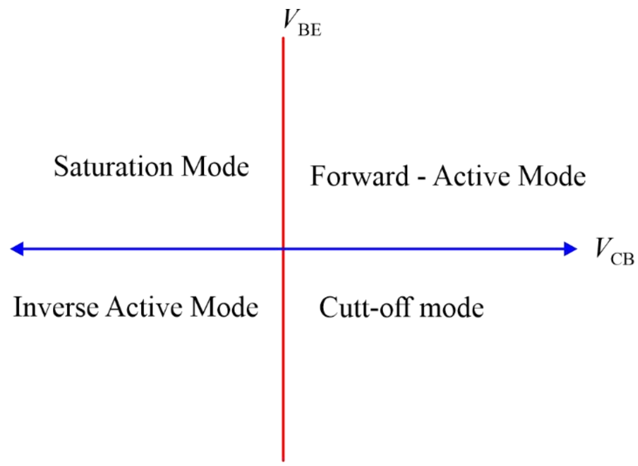
- If V_{CC} is large enough and V_{RC} is small enough the $V_{CB} > 0$ therefore base collector junction is RB and BE junction is FB already, this condition is of forward active region of operation.
- At a certain point of V_{BE}, I_C become large enough to make $V_{CB} = 0$, Beyond this point if I_C is slightly increased the V_{CB} become negative and CB junction becomes FB and IC is no longer controlled by BE voltage i.e., the relation $I_C = \beta I_B$ will no longer hold.
- O/P characteristics



$$V_{CC} = I_C R_C + V_{CE}$$

Work as an amplifier or current source

For npn:



Expression for 'β': (Current gain factor)

$$\beta = \left[\frac{D_P}{D_n}\cdot\frac{N_B}{N_E}\cdot\frac{x_B}{L_P} + \frac{x_B^2}{2\tau_{no}p_n}\right]^{-1}$$

Here x_B is very small and $N_E \gg N_B$

- The current gain factor β, depends on two factors
  (i) Effective Base width (x_B)
  (ii) Ratio of doping concentration in Base and Emitter Region (N_B/N_E)
- Prismatic and non-prismatic

Relation between α and β:

$$\alpha = \frac{\beta}{\beta+1} \text{ and } \beta = \frac{\alpha}{1-\alpha}$$

β increases by large amount for small increase in 'α'.

7.3.2. Reverse Current (I_{CBO} and I_{CEO}): [Leakage Current]

- The current I_{CBO} is the reverse current flowing from collector to base with emitter open circuit.
- The current I_{CEO} is the collector to emitter leakage current when base is open circuited.

Input	Output	Common Terminal
B	C	Emitter (Common Emitters)
B	E	Collector (Common Collector)
E	C	Base (Common Base)

For common base:

$$I_C = \alpha I_E + I_{CBO}$$
$$I_D = \beta I_S + (\beta+1)I_{CBO}$$

For Common Emitter:

$$\frac{I_E}{I_B} = \gamma \text{ common collector current gain factor}$$

$$I_E = v_{IB} \Rightarrow I_E = v\,(I_E - I_C)$$

$$\frac{I_C}{I_E} = \frac{\gamma - 1}{\gamma} = \alpha$$

$$\gamma = \frac{1}{1-\alpha} = 1 + \beta$$

Current gain contributing factors:

1. DC common base current gain:-

$$\alpha_0 = \frac{I_C}{I_E} = \frac{J_{nc} + J_G + J_{PCO}}{J_{nE} + J_{PE} + J_R}$$

where, J_{nC}: It is due to the diffusion of minority carrier $e^{\ominus}$ in the base at $x = x_b$

J_G: It is due to the generation of carriers in the R_B in the base – collector junction

J_{PCO}: It is due to the flow of minority carries in the R_B in the base collector junction

J_{nE} : due to diffusion of minority carries $e^{\ominus}$ in the base at $x = 0$

J_{PE} : due to diffusion of minority carrier hole in the base at $x' = 0$

J_R : due to recombination of carrier in the F_B base-emitter junction.

For AC sinusoidal i/p :

$$\alpha = \frac{\partial I_C}{\partial I_E} = \frac{J_{nC}}{J_{nE} + J_{PE} + J_R}$$

Change in I_C corresponding to change in I_f and J_G & J_{PCO} are independent of emitter current

$$s = \gamma \propto T^{\delta}$$

where, $\gamma = \dfrac{J_{nE}}{J_{nE} + J_{PE}} \rightarrow$ Emitter injection efficiency

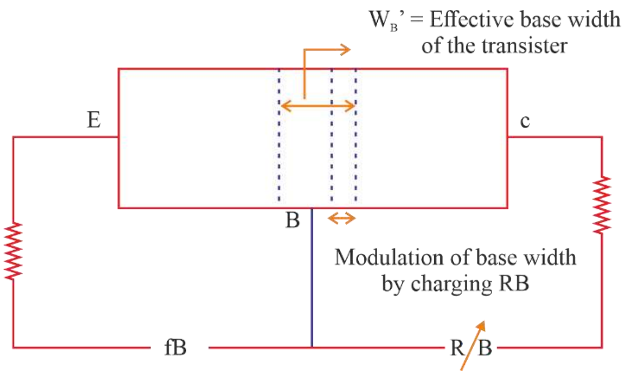
$$\alpha T = \frac{J_{nC}}{J_{nE}} \rightarrow \text{Base transport factor}$$

$$\delta = \frac{J_{nE} + J_{PE}}{J_{nE} + J_{PE} + J_R} \rightarrow \text{Recombination factor}$$

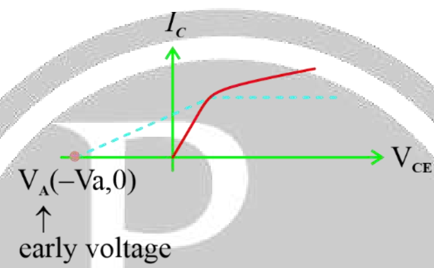
- α_T, γ & s are contributing factors, ideally all values are 1 but partially its close to 1.
- Distribution of minority carriers in npn transistor in forward active mode.

Base Width Modulation:

- The process where the effective base width of the transistor is altered by varying collector junction voltage is called base width modulating.
- It is also called as early effect



- As the base-callector RB voltage ↑es which reduced the neutral base width W'B by increasing the base, collector space charge region width.
- As W'B reduces, It will increase β
- The early effect produces a non-zero slope in o/p chrematistics & gives rise to a finite o/p conductance



Slope, $g_0 = \frac{\partial I_C}{\partial V_{CE}}$

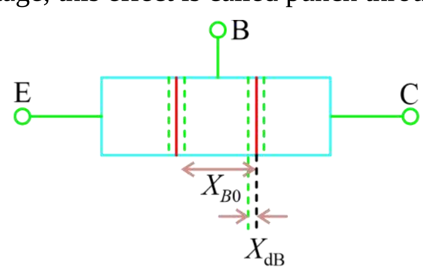
$I_C = g_o\,(V_{CE} + V_A)$

Where, $g_0 = \frac{I_C}{V_{CE} + V_A}$

7.3.4. Breakdown phenomena's in BJT

(1) Punch through breakdown:

- As the reverse bias base-collector voltage increases, the base-collector space charge region widens & enters further into the neutral base, It is possible for the base-collector depletion region to penetrate complete the base & reach the base emitter space charge region, the effect called punch through.
- The lowering of the potential barrier at the base-Emmiter junction, produces a large ↑ing current with the very small decreasing in the collector base voltage, this effect is called punch through breakdown phenomenan.



Let the depletion width for BE junction is negligible for breakdown
total neutral base width = X_{B0}

$$x_{dB} = x_{BO}$$

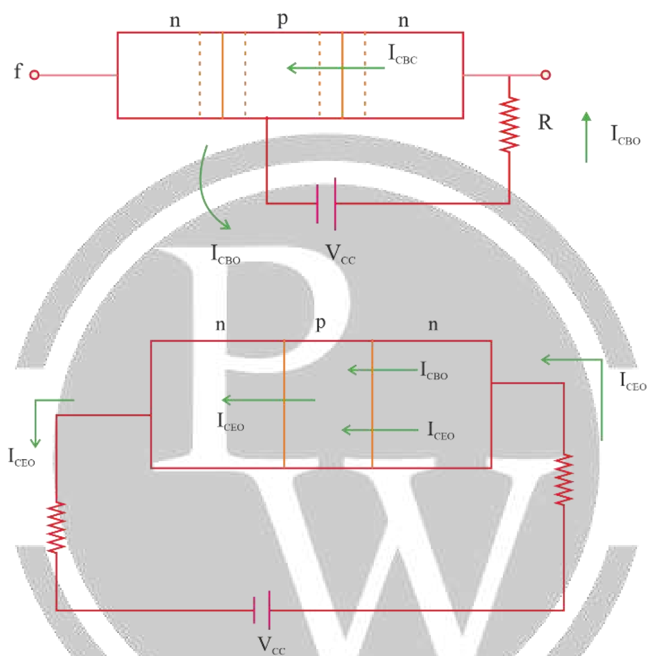
and, $$x_{dB} = \sqrt{\frac{2\varepsilon}{q}(v_{bi} + V_{CB}).\frac{N_C}{N_B} \times \left(\frac{1}{N_C + N_B}\right)}$$

At break down :- $x_{dB} = x_{BO}$ & $V_{CB} = V_{PT}$ (negligible V_{bi})

Then, $$V_{PT} = \frac{x_{dB}^2 \, q N_B (N_C + N_B)}{2\varepsilon N_C}$$

(2) Avalanche Breakdown:

- When a high reverse voltage is applied across the diode then avalanche breakdown takes place.
- When emitter is open then current (I_{CBO}) will flow from the collector to base & then by V_{CC} by applying V_{CC}.

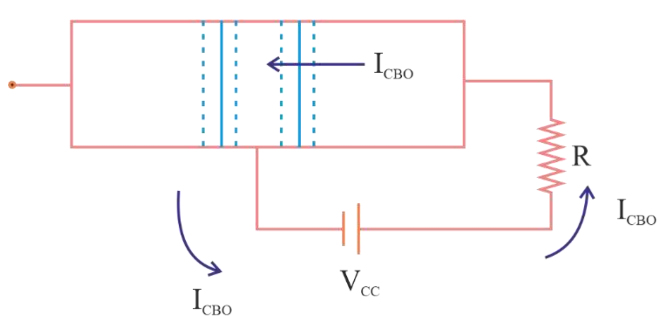


In B – C RB minority carrier of collector will flow to base then base will positively charged the E – B is FB.

$$I_{CEO} = I_{CBO} + \alpha I_{CEO}$$

$$I_{CEO} = \frac{I_{CBO}}{I - \alpha} \Rightarrow I_{CEO} = (1 + B)I_{CBO}$$

$$\boxed{I_{CEO} = (1+\beta)I_{CBO}}$$



Breakdown occurs when $I = MI_{CBO}$

Where, $M \rightarrow$ multiplication factor

Empirical formula for M

$$\boxed{M = \frac{1}{1-\left(\frac{V_{CB}}{BV_{CBO}}\right)^{n}}}$$

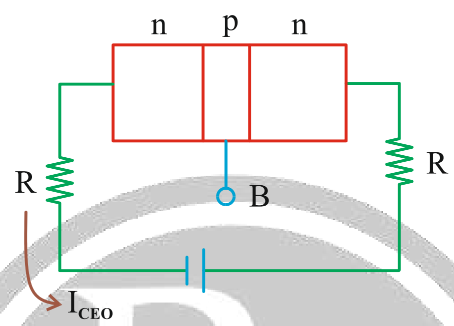
Where, $BV \rightarrow$ breakdown voltage

$n \rightarrow$ emphatical constant

BV_{CBO} = Breakdown voltage (V_{CB}) in open Emitter configuration

i.e For Breakdown $V_{CD} = BV_{CBO}$

7.3.5. In Open Base Configuration



$$I_{CEO} = M[I_{CBO} + \alpha I_{CEO}]$$

$$I_{CEO} = \frac{M}{1-M\alpha} \cdot I_{CBO}$$

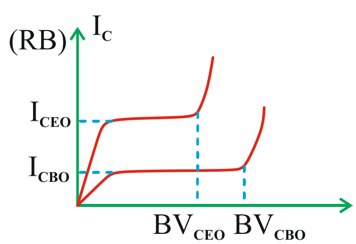
For B.D : $M = 1$

$$M = \frac{1}{1-\left(\frac{V_{CB}}{BV_{CBO}}\right)^{n}} \text{, at } V_{CB} = V_{CE} \quad M = \frac{\alpha}{1-\left(\frac{BV_{CEO}}{BV_{CB}}\right)^{n}} = 1$$

$$BV_{CEO} = BV_{CB}\,(1-\alpha)^{1/n}$$

$$BV_{CEO} = \frac{BV_{CEO}}{(\beta)^{1/n}}$$

The breakdown voltage in the open base configuration is smaller then the actual avalanche junction breakdown voltage.



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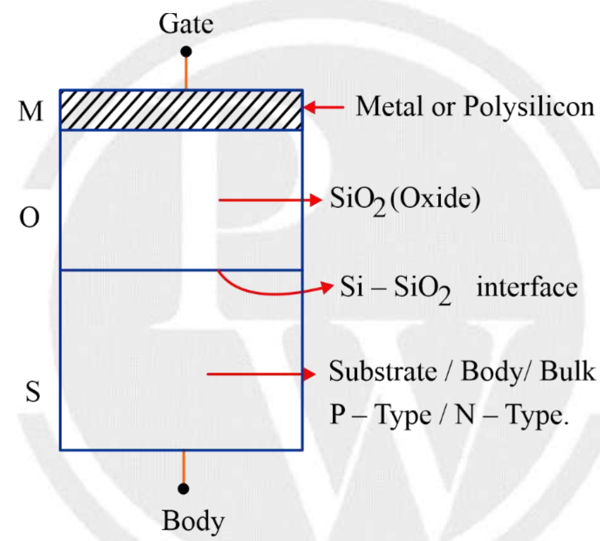
8 MOS CAPACITOR

8.1. MOS Capacitor

MOS → Metal oxide semiconductor

Oxide used is (SiO_2) →Insulator.



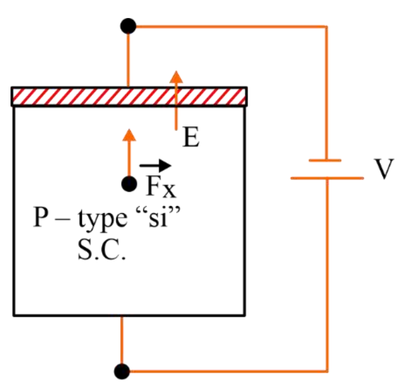
MOS capacitor Operates in 3 – modes:

(1) Accumulation.

(2) Depletion.

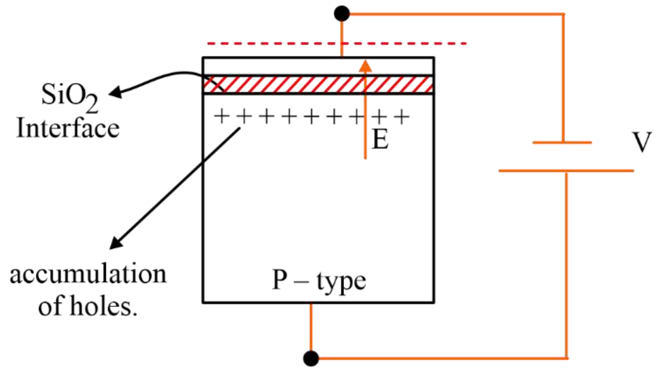
(3) Inversion.

(1) Accumulation.

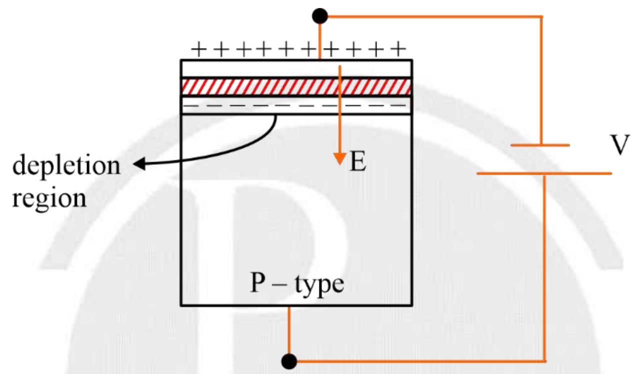


This E field is very strong so that it penetrates in p – type S.C.
Also, the holes of S.C. P – type get forced and accumulate at surface.

(2) Depletion mode:



(3) Inversion Process:



Inversion process:

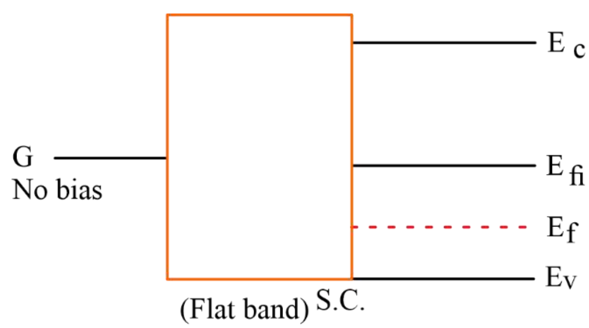
- In this on metal the ions are there and in p – type holes are forced in the direction of $\vec{E}$ field and this is created a depletion region.
- When a MOS capacitor with a p-type SC substrate is biased such that the top metal gate is at a (–ve) voltage w.r.t. the S.C. substrate.
- An accumulation layer of hole is created at the oxide S.C. interface corresponds to the (+V) charge on the bottom plate of the MOS capacitor.

Strong Inversion

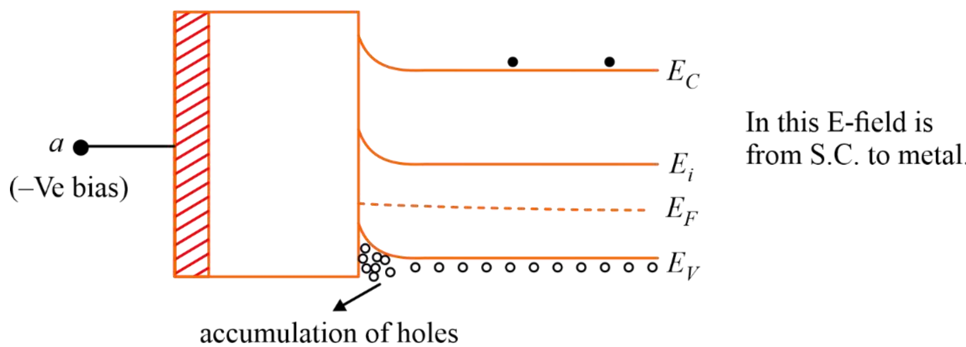
- Now consider the same MOS capacitor in which the polarity of applied voltage is opposite of 1’st case.
- A positive case charge exists on the top’ of metal plate and the induced E-field is in the direction from metal to S.C.

Energy Band Diagram: (p-type S.C substrate)

(1) No gate bias:

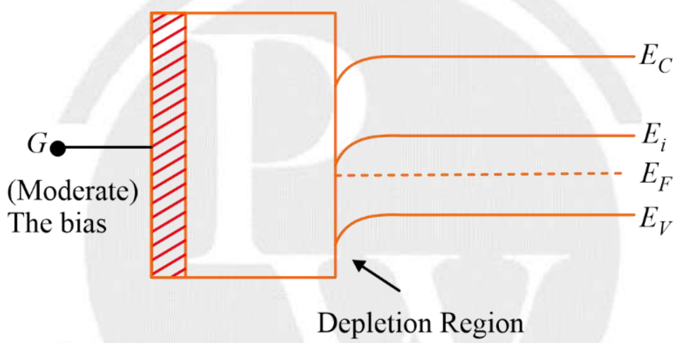


(2) Gate at Negative Voltage:



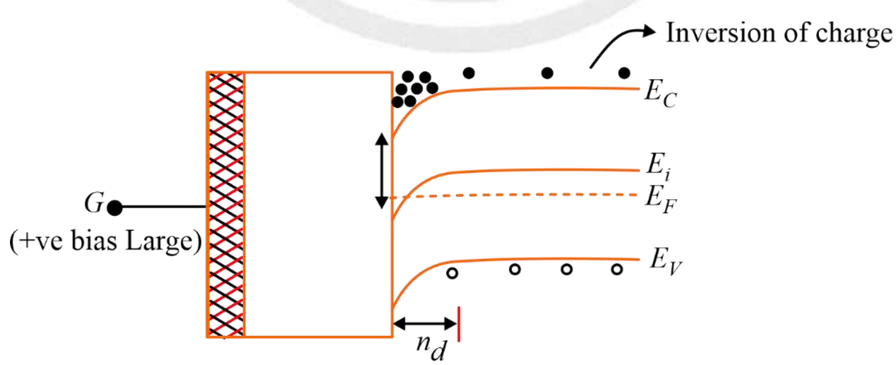
- Here, V_B is more nearer to fermi level than bulk so it is more p-type surface then bulk.
- "E_f" is constant, because it is in thermal equilibrium.

(3) Gate at positive voltage: (moderate)



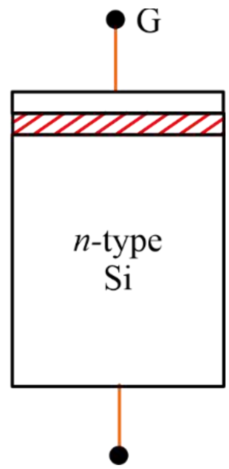
- In this case bulk is more p-type then surface and holes are depleted towards bulk and there is depletion region at surface. (Thermal equilibrium is also there) surface starts to become n-type.

(4) Inversion:

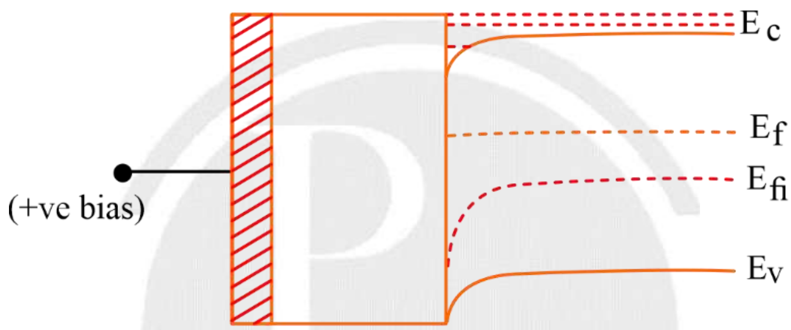


- – ve charge in a MOS capacitor implies a larger induced space charged region and more bonding
- The intrinsic fermi level at the surface is now below the fermi level.

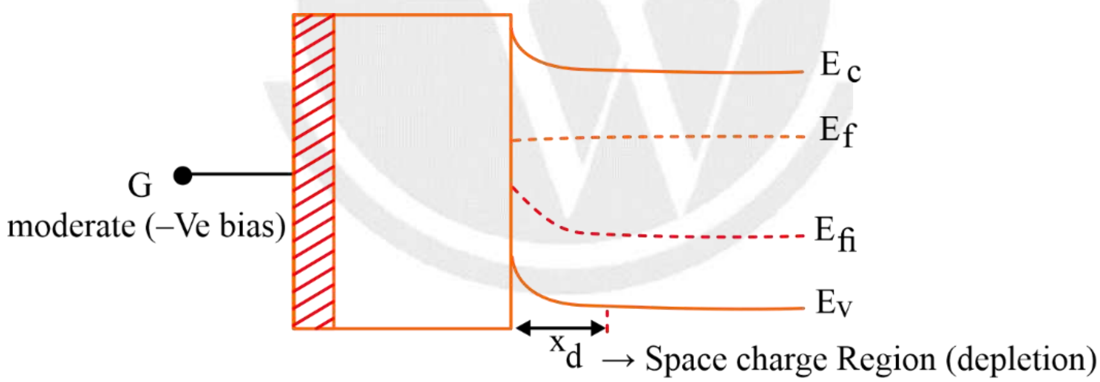
n-type: (Energy Band diagram)



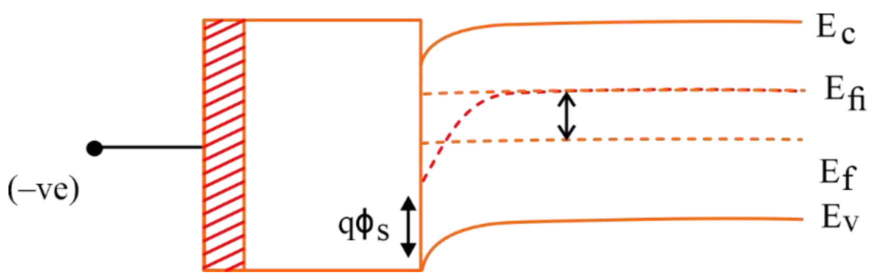
(1)



(2)



(3)



ϕs = surface potential

$q\phi_s = E_{fi}$ Bulk – E_{fi} surface.

- At inversion point → Surface is as much n-type as bulk is p-type.

$$n = N_c\, e^{-\left(E_c - E_f\right)/KT} = n_i\, e^{-\left(E_{fi} - E_f\right)/KT}$$

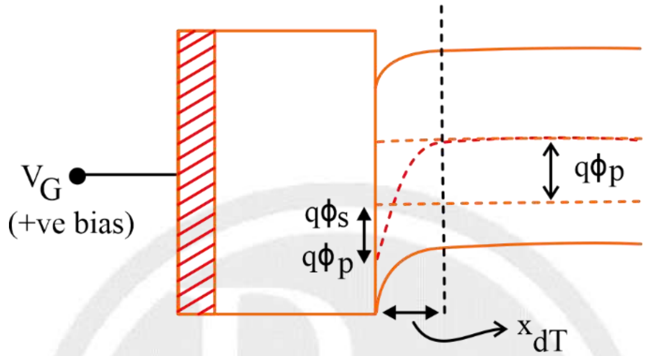
$$p = N_v e^{-(E_f - E_v)/KT} = n_i\, e^{-(E_f - E_{fi})/KT}$$

Electron concentration at bulk S.C.

$$p_e = N_i\, e^{-(E_f - E_{fi})/KT}. \qquad \text{for } n_s = p_B$$

$$\left[E_{fis} - E_f = E_f - E_{fiB}\right] \rightarrow \text{inversion}$$

for Inversion: $\phi_s = 2\phi_{fp}$



Maximum depletion region thickness.

$$\phi_{fp} = V_T \ln\left(\frac{N_A}{n_i}\right)$$

$N_A \longrightarrow$ Acceptor concentration in p-type substrate.

$\phi_s \longrightarrow$ surface potential (potential deft. Across depletion region)

$$x_d = \left(\frac{2\varepsilon}{q}\frac{\phi_s}{N_A}\right)^{1/2} \longrightarrow \text{Like a one sided.}$$

Maximum depletion region width occurs at

$$\phi_s = 2\phi_{fp}.$$

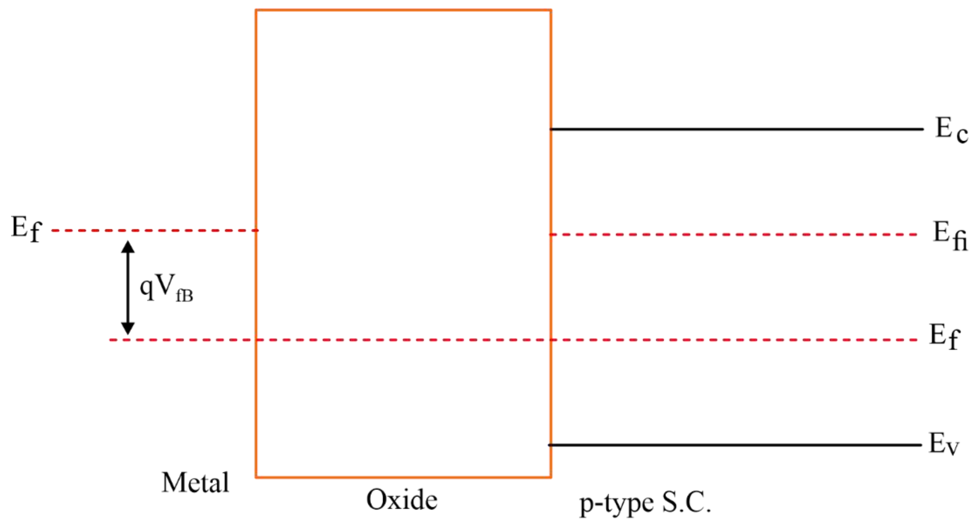
$$x_{dT} = \left(\frac{4\varepsilon\phi_{fn}}{q\, N_D}\right)^{1/2}$$

$$\phi_{fn} = V_T \ln\left(\frac{N_D}{n_i}\right)$$

- At the surface, potential $\phi_s = 2\phi_{fp}$
- The fermi level at the surface is as for above the intrinsic level in the bulk S.C.
- This condition is known as the 'threshold inversion point'.
- The applied gate voltage operating this condition is known as the threshold voltage.

Flat Band Voltage:

This is a applied gate voltage such that is no band bending in the S.C. and as a result there is zero net space charge in this region.



$$V_G = \Delta V_{ox} + \Delta\phi_s$$

At flat – Band,

$$V_G = V_{FB}$$

$$V_{FB} = (V_{ox} - V_{oxo}) + (\phi_s - \phi_{so})$$

We know:

$$(V_{oxo} - \phi_{so}) = \phi_{ms}$$

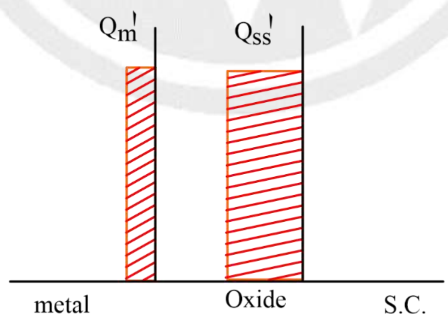
$$V_{fB} = \phi_{ms} + V_{ox} + \phi_s$$

Also, at flat bond

$$\phi_s = 0;$$

$$V_{fB} = \phi_{ms} + V_{ox}\,.$$

Charge distribution in MOS cap at FB voltage



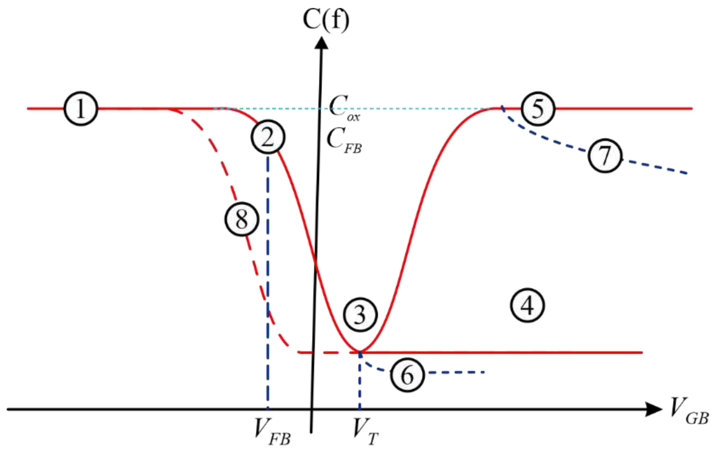
For conservation of charge:

$$\phi_m^{'} + Q_{ss}^{'} = 0 \quad ; \quad Q_m^{'} = -Q_{ss}^{'}$$

Across oxide voltage drop = V_{OX} → Voltage drop across capacitance.

$$V_{ox} = \frac{Q_m^{'}}{C_{ox}} = \frac{-Q_{ss}^{'}}{C_{ox}}$$

$$V_{FB} = \phi_{ms} + \frac{-Q_{ss}^{'}}{C_{ox}}$$



① Accumulation
② Flat band
③ Inversion Start
④ High Frequency
⑤ Low Frequency
⑥ Deep Depletion
⑦ Poly Depletion Effect
⑧ Positive Oxide Charges Introduced

$V_{fB} \longrightarrow$ flat Band voltage.

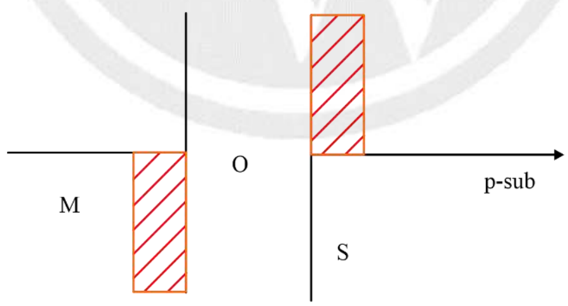
$V_G \longrightarrow$ Gate Voltage.

$V_T \longrightarrow$ Threshold voltage.

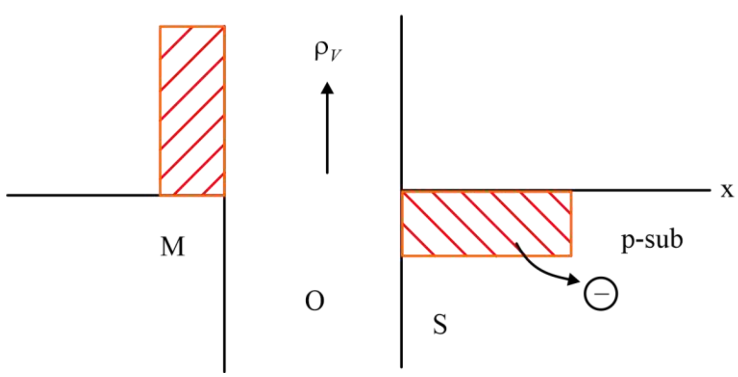
MOS Capacitor can operate in 3-modes Accumulation, Depletion, Inversion, which is explained earlier.

$$V_T = \pm Q_{msi} \pm \frac{Q_{ox}}{C_{ox}} \pm \frac{Q_d}{C_{ox}} \pm 2\phi_f$$

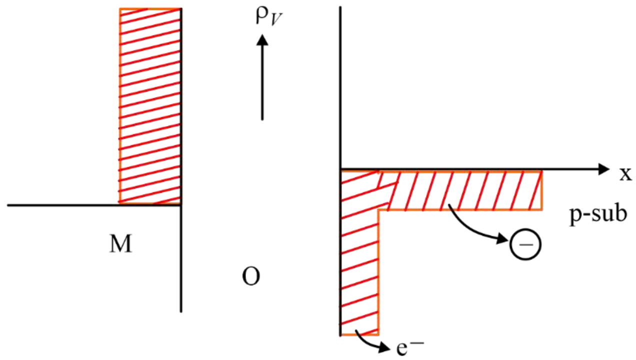
(I) Accumulation



(II) Depletion



(III) Inversion



Boundary Condition:

$$\bar{D}_{N_1} = \bar{D}_{N_2}$$

$\Rightarrow$ $$\epsilon_{ox} E_{ox} = \epsilon_{si} E_s$$

Where, $\epsilon_{si} = 11.7\epsilon_o$

$\epsilon_{ox} = 3.9\epsilon_o$

Then, $$E_s = \frac{\epsilon_{ox}}{\epsilon_{si}} E_{ox}$$

$$\bar{E}_{ox} = \frac{|Q_s|}{\epsilon_{ox}} c/cm^2$$

$$\bar{E}_s = \frac{|Q_s|}{\epsilon_{si}} c/cm^2$$

$$E_s = \frac{2\phi_s}{w_d}$$

$$V_{ox} = E_{ox} t_{ox}$$

Flat band capacitance:

$$C_{FB} = \frac{1}{\frac{1}{C_{ox}} + \frac{L_D}{\epsilon_s}}$$

L_D = Debye length

$$L_D = \sqrt{\frac{\varepsilon_S \phi_t}{qN_A}}$$

$\phi_t \rightarrow$ Thermal voltage

$$C'_{FB} = \frac{\varepsilon_{ox}}{t_{ox} + \frac{\varepsilon_{ox}}{\varepsilon_s}\sqrt{\frac{\varepsilon_s \phi_t}{qN_A}}} \text{ F/cm}^2$$

The flat band capacitance of the MOS structure at flat band is obtained by calculating the series connection of the oxide capacitance and the capacitance of the semiconductor

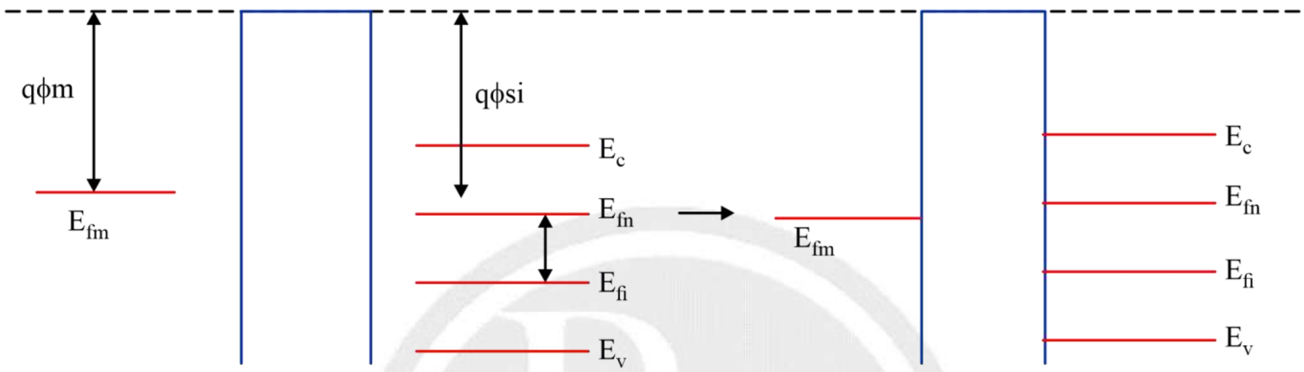
Energy Band Diagram:

For N-sub MOS capacitor:

Ideal case: (1) $\phi_m Si = 0$

(2) $Q_{ox} = 0$

(i) Flat band: ($V_G = 0$)



(ii) Accumulation → $V_G > 0$

(iii) Depletion mode $V_G < 0$

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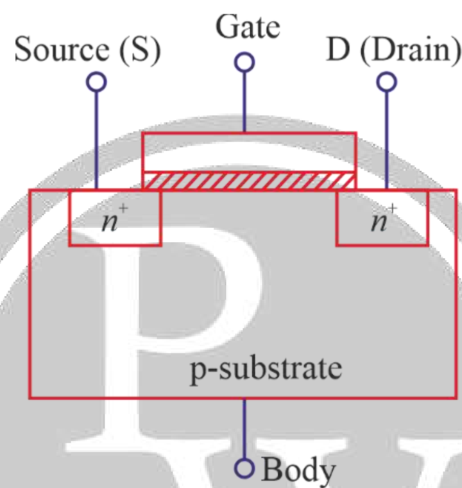
9 MOSFET
(Metal Oxide Semiconductor Field Effect Transistor)

9.1. MOSFET



- **Source :** terminal through which majority charge carrier enters into the SC bar.
- **Drain :** terminal through which majority charge carriers leaves the SC bar.
- **Channel:** path between source & drain by which majority carrier travel from source to drain
- **Gate :** Terminal to control the flow of charge carrier from source to drain

MOSFET

Enhancement type MOSFET

- No initial channel between ‘S’ & ‘D’ at zero gate voltage

(a) n - channel (N-mos)

(b) p - channel (P-mos)

Depletion type MOSFET

- Channel is initially present in between ‘S’ & ‘D’ at zero gate voltage

(a) n - channel

(b) p - channel

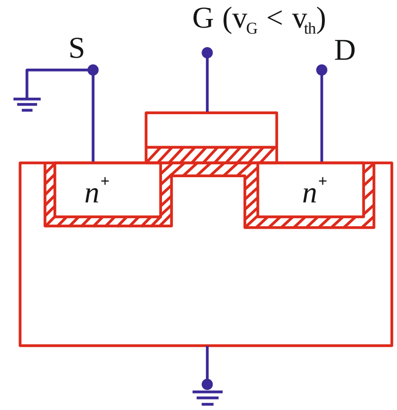
- Depletion type MOSFET, the already created channel is either of following two types

  (i) Internally doped

  (ii) A mos device with applied threshold voltage.

9.2. Operation of Enhancement Mode N-MOS :

(I)

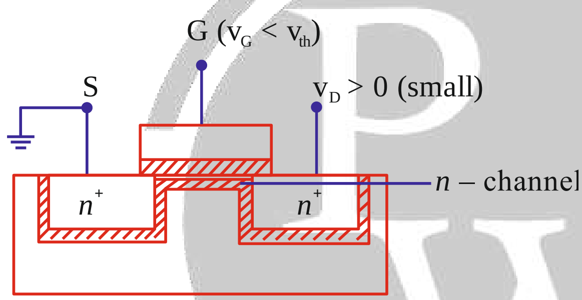


- There is no flow of 'S' carrier to drain

  Here, v_{th} = threshold voltage   +ve for N-channel MOS

  –ve for P-channel MOS

(II)



$g_l \Rightarrow$ channel conductance

- Source carrier (e^-) will flow from S to D, therefore channel & a current will flow from 'D' to 'S'
- For small V_D, $\boxed{I_{DS} = g_d v_{DS}}$

  $g_d \propto |Q_n'|$,   where $Q_n \to$ Inversion charge

  $$\boxed{|Q_n'| \propto v_{ox} - v_{th}}$$

  Potential difference across oxide layer

  $$|Q_n'| = \left[v_{GS} - v_{(x)} - v_{TH}\right]$$

  Where, $v_{(x)} \to$ potential at any point in channel at source : $x = 0$

  $$v_{(o)} = ov$$

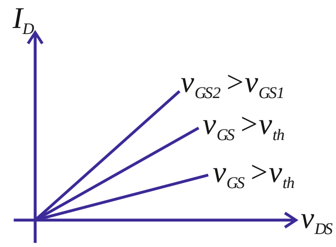
  and $v_{(L)} = v_{DS}$ $\left(o \le v_{(x)} \le v_{DS}\right)$

- For v_{DS} very small

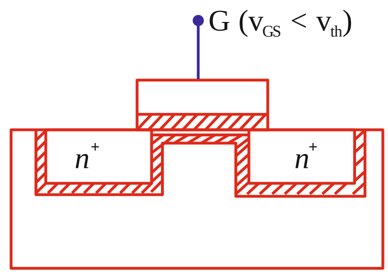
  $$\boxed{v_{ox} \simeq v_{GS} - v_{TH}}$$

  Then $|Q_n'| \propto v_{GS} - v_{TH}$

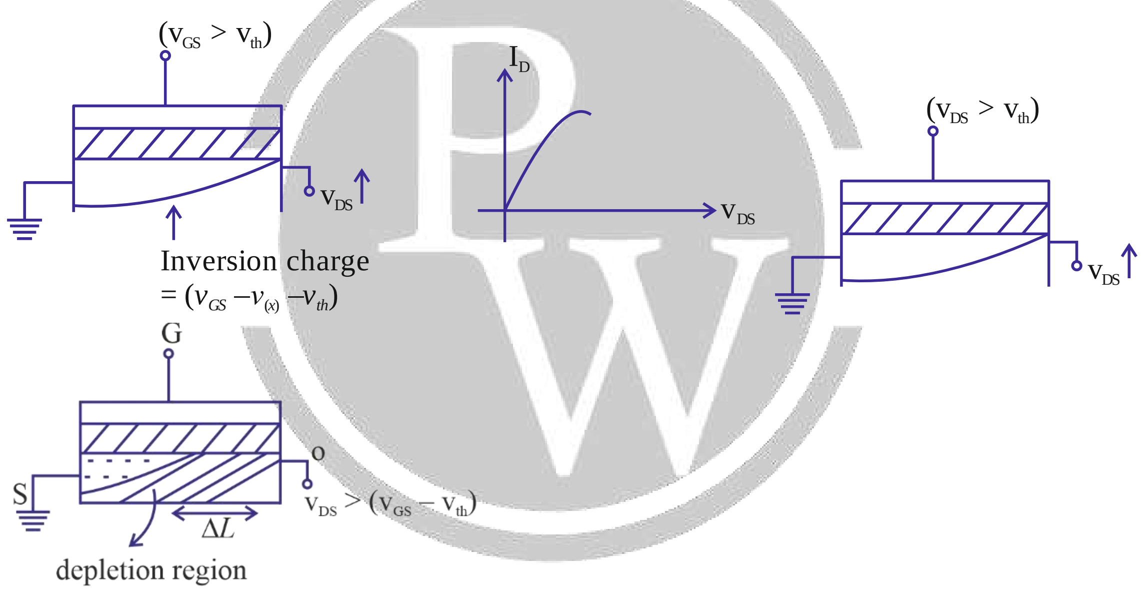
  $\boxed{g_d \propto v_{GS}} \to$ Basic MOSFET action (modulation of channel cond by applying gate val)

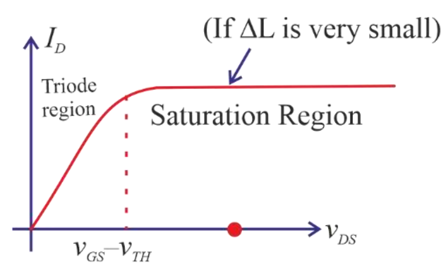


(III)



- At small $v_{DS}, |Q_n'| \propto v_{GS} - v_{TH}$
- Inversion charge density at the channel, *i* const.





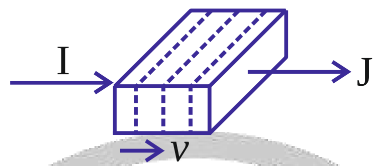
For $v_{GS} > v_{TH}$:

- For small value of v_{DS} the charge density is const along the entire channel length therefore drain current ↑es linearly with v_{DS}.
- If v_{DS} ↑es, the voltage drop across one oxide layer near the drain terminal decreases, which means the incremental conductance of the channel at the drain decreases which means slope of I_D vs V_{DS} will decreases
- When v_{DS} increase to ($v_{GS} - v_{TH}$) then the slope of I_D v/s v_{DS} curve will zero.

9.2.1. In depletion mode MOSFET

- If the n-channel region is actually induced an inversion layer of e^- created by the metal SC work function diff. & the fixed charge in the oxide, the current voltage characteristics is same as above, only except v_{th} is negative quantity.
- If the n-channel region is actually an n-type SC region & Q $-ve$ gate voltage will induced a depletion region under the oxide, reducing the thickness of the n-channel region, reduced thickness decreases the channel conductance which reduces the drain current.
- In order to be able of to turn the device off, the channel thickness must be less than the maximum charge width (xdT)

Derivation of I-V characteristics :

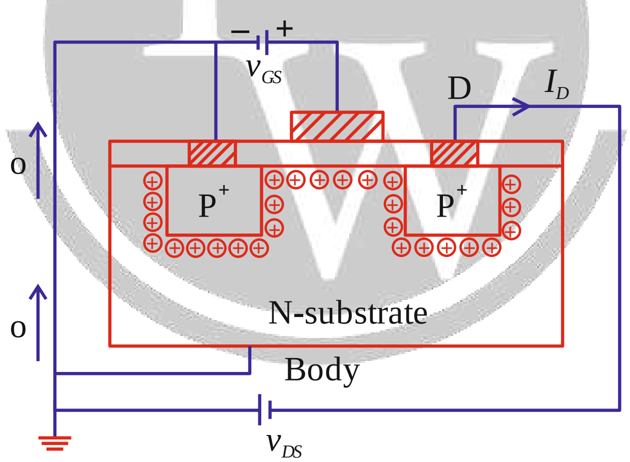


$Q_d \rightarrow$ Charge carrier/length along the direction of flow of current

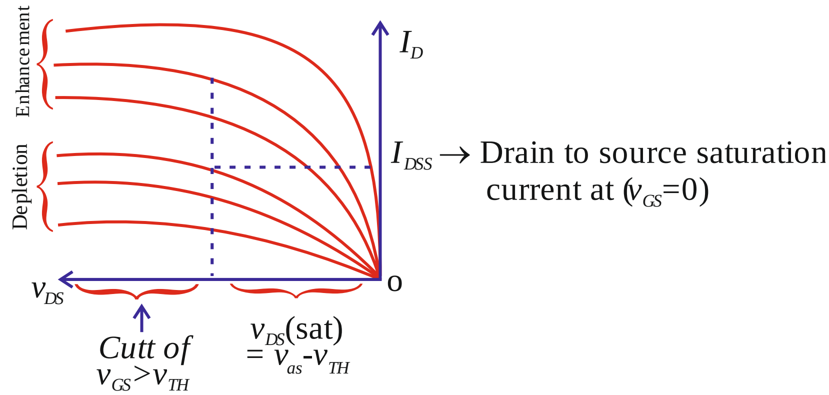
$v \rightarrow$ Velocity of charge carrier

Total charge = Qdv

P-channel Depletion Type MOSFET :



Output Characteristic :



- If $v_{GS} > 0$ & v_{DS} (vary)
  - Depletion charge Q_D increases
  - Less holes available in channel i.e. I_D decrease
  - Channel depleted
- If $V_{GS} < 0$ & V_{DS} (vary)
  - Depletion charge Q_b in creases
  - More holes available i.e. I_D will increases
  - Channel – Enhance
- In linear region :

$$\boxed{I_D = \frac{1}{2}\mu_n Cox\left(\frac{W}{L}\right)(v_{GS} - v_{TH})^2}$$

Symbol :

- N-channel Enhancement type MOSFET :

$$\boxed{I_D = \frac{1}{2}\mu_n Cox\frac{W}{L}\left[(v_{GS} - v_{TH})v_{DS} - \frac{v_{DS}{}^2}{2}\right]}$$

MOSFET works as voltage variable resistor

Trans Conductance :

- It is a figure of merit indicates that how well a transistor convert the voltage to the current

$$g_m = \frac{\partial I_D}{\partial V_{GS}}\bigg|V_{DS} = \text{const.}$$

- In triode region :

$$\frac{\partial I_D}{\partial V_{GS}}\bigg|(V_{DS} = \text{const.}) = g_m = \mu_n Cox\frac{W}{L}v_{DS}$$

- In saturation region :

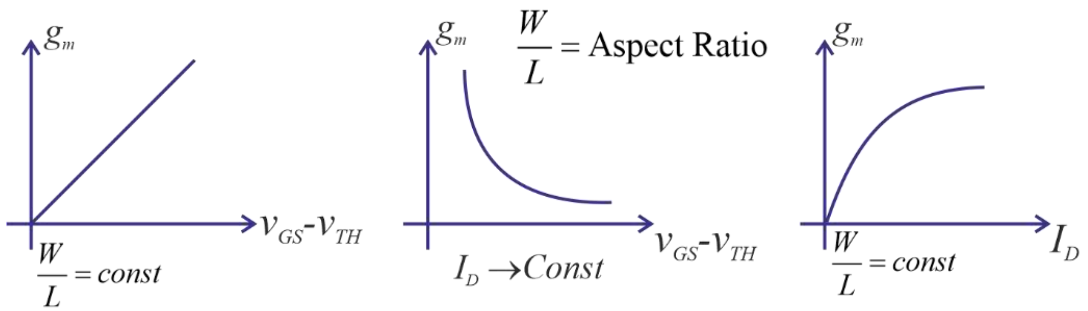
$$v_{DS} = v_{GS} - v_{TH}$$

$$\boxed{I_{D\,\text{sat}} = \frac{1}{2}\mu_n C_{ox}\frac{W}{L}[v_{DS} - v_{TH}]^2}$$

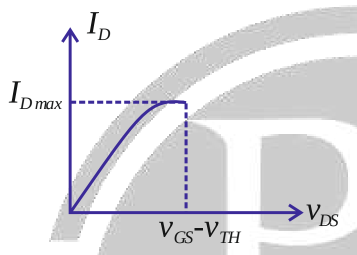
and $$g_m = \mu_n C_{ox}\frac{W}{L}[v_{GS} - v_{TH}]$$

$$g_m = \frac{2I_{D\,sat}}{v_{GS} - v_{TH}}$$

$$g_m = \sqrt{2\mu_n C_{ox}\left(\frac{W}{L}\right) I_D}$$



- When, $v_{DS} < v_{GS} - v_{TH}$ (the Triode region)



$v_{GS} < v_{TH} = v_{ov}$ (over drive voltage)

- The channel is pinched off at $v_{DS} = v_{ov}$. The drain current reaches its maximum value & if we further increases v_{DS}, a const current flow because of the pinched off channel side $Q_d \rightarrow o$ & velocity of carrier increases.

In triode Region :

$$I_D = \mu_n C_{ox}\frac{W}{L}\left[(v_{GS} - v_{TH})v_{DS} - \frac{v_{DS}^{\;2}}{2}\right]$$

Let $2(v_{GS} - v_{TH}) >> v_D$

Then it is deep triode region $\Rightarrow$ Linear Region

$$I_D I \cdot \mu_n C_{ox}\frac{W}{L}(v_{GS} - v_{TH})v_{DS}$$

$$I_D \propto v_{DS} \Rightarrow I_D = g_d v_{DS} \text{ (channel conductivity)}$$

$$\boxed{g_d = \mu_n C_{ox}\frac{W}{L}(v_{GS} - v_{TH})}$$

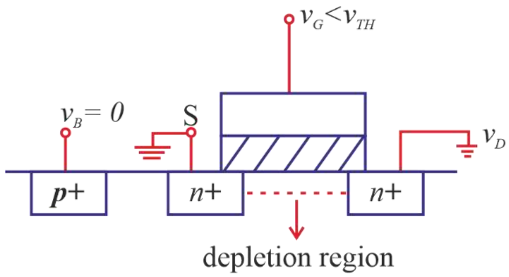
$$\boxed{R_{CH} = \frac{1}{\mu_n C_{ox}\frac{W}{L}(v_{GS} - v_{TH})}}$$

In deep triode region MOSFET work as a linear resistor which can be varied by v_{GS} therefore in deep triode region, we can say that it is working as a voltage variable resistor.

2nd order effects :

1. Body - Bias effect : (N-MOS)

- If $v_B < 0$



Then, attracts more holes towards body terminal hence depletion charge increase before on set of inversion.

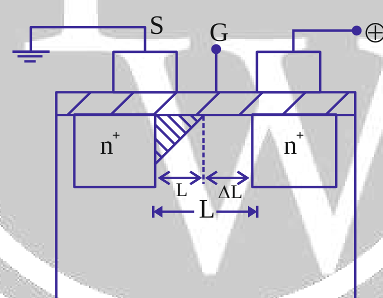
- The gate charge must mirror Q_d before an inversion layer is formed. Therefore, v_{GS} is a function of total charge in depletion region.
- If v_B is increased, then Q_o will increase and then v_m will increase

$$v_{TH} = v_{THo} + \gamma\sqrt{2C_{ox} + v_{SB}} - \sqrt{2\phi_F}$$

$$\boxed{v = \frac{\sqrt{2 \in qN_{sub}}}{C_{ox}}}$$

Body effect coefficient

2. Channel length Modulation:



Where, L $\rightarrow$ Actual length of channel

ΔL $\rightarrow$ Depleted channel length

L' = L – ΔL $\rightarrow$ Un-depleted length or effective channel length

$$\boxed{I_D = \frac{1}{2}\mu_n Cox\left(\frac{W}{L}\right)(v_{GS} - v_{TH})^2\left[1 + \lambda\left[v_{DS} - v_{DS}(sat)\right]\right]}$$

If $v_{DS} >> v_{DS(sat)}$

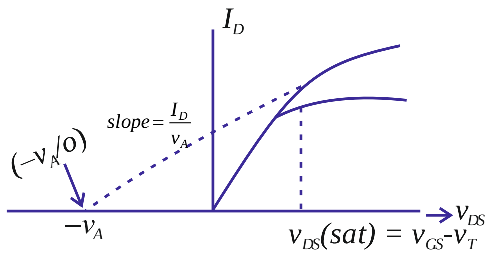
$$\boxed{I_D = \frac{1}{2}\mu_n Cox\left(\frac{W}{L}\right)(v_{GS} - v_{TH})^2\left[1 + \lambda v_{DS}\right]}$$

Where, $\lambda \rightarrow$ channel length modulation parameter

unit = volt^{-1}

$$\boxed{\lambda = \frac{1}{v_A}(\text{volt}^{-1})}$$

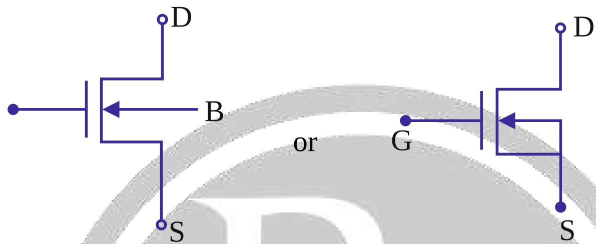
Where, $v_A \rightarrow$ Early voltage



- r_o (or) $rd = \frac{1}{slope} = \frac{V_A}{I_D} = \frac{1}{\lambda I_D}$

$$\boxed{r_o = \frac{1}{\lambda I_D} = \frac{V_A}{I_D}} \text{ output resistance}$$

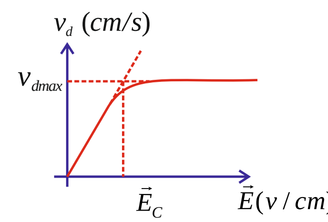
- Symbol N-channel enhancement type (NMOS) :



9.3. Short Channel Effects

The MOSFET is consider to be short channel device when the channel length is in the same order of the depletion width of source and drain junction.

1. Velocity Saturation
2. Drain-induced barrier lowering (DIBL)
3. Impact Ionization
4. Hot electron



$\vec{E}_C \rightarrow$ Critical Electric Field

1. Velocity Saturation:

By using previous study we can state:

$$v_d = \mu\vec{E}\left(\vec{E} << \vec{E}_C\right)$$

$$v_d = v_{d\max}\left(\vec{E} >> \vec{E}_C\right)$$

But the variations in v_d near the critical $\vec{E}$ field cannot be modelled using the above equation. This is the region in which v_d changes with $\vec{E}$, sub linearly.

Proposed model for velocity saturation.

$$\boxed{v_d = v_{d\max}\frac{\vec{E}/\vec{E}_C}{1+\frac{\vec{E}}{\vec{E}_C}} \text{cm/s}}$$

This model gives the actual v_d versus $\vec{E}$.

2. Drain Induced Barrier Lowering

When we apply positive voltage at the drain in n-channel MOSFET having short channel, then its threshold voltage (v_T) will decrease due to depletion region of drain region. This effect can be reduce by increasing substrate doping.

3. Impact Ionization

In short channel devices the $\vec{E}$ field in the channel will be high and due to that the e$^-$ velocity will be high which impact on Si atoms and generate electron hole pair, this is called as impact Ionization.

4. Hot Electron

This is also due to high $\vec{E}$ field in the channel which give rise to high energy of e$^-$ and that can enter into the oxide and will be treated as trapped oxide charge. This will increase the threshold voltage (v_T).

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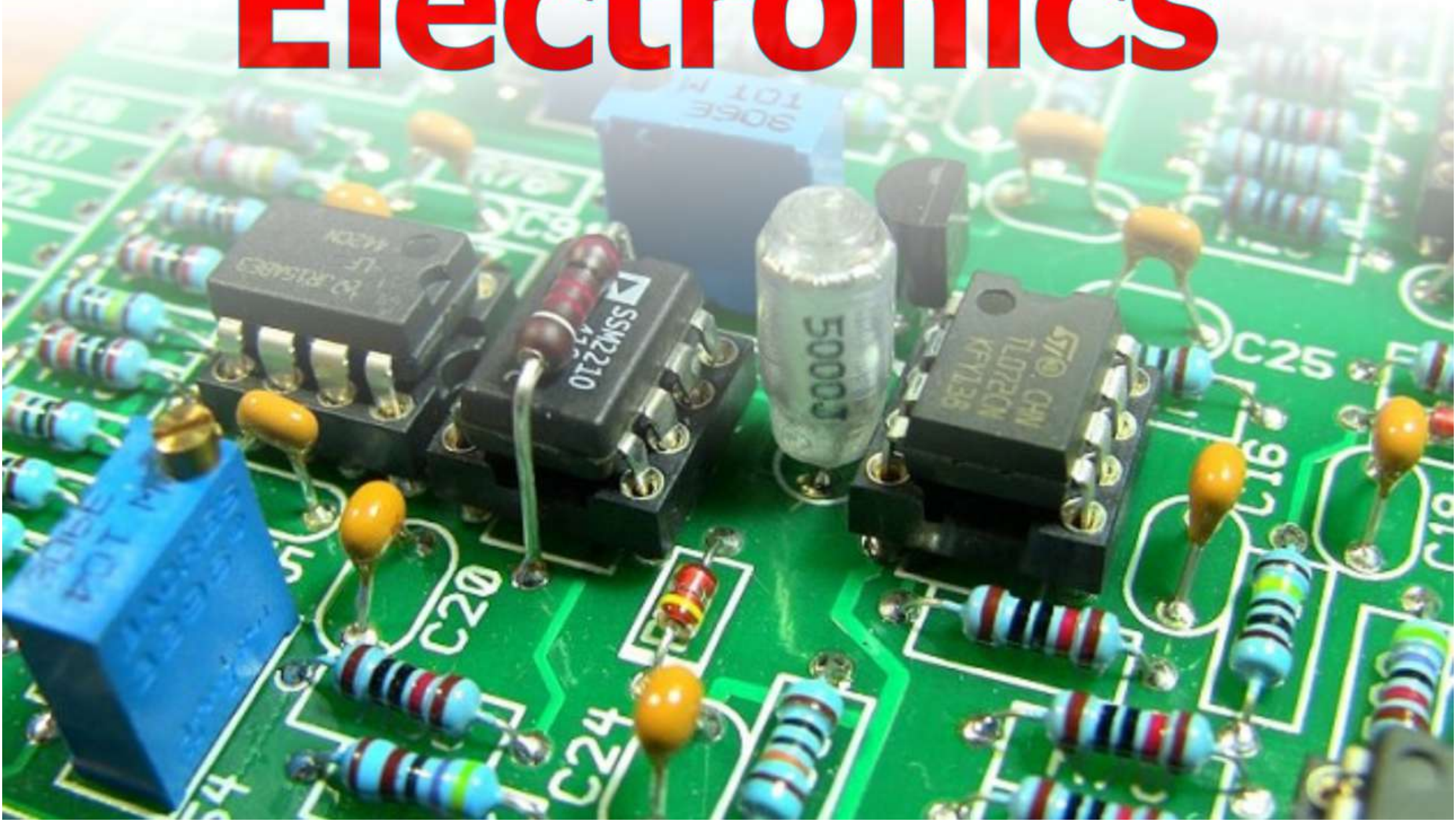
Analog Electronics



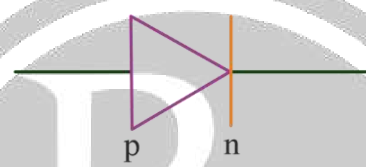
ANALOG ELECTRONICS

INDEX

1 DIODE CIRCUITS & APPLICATIONS

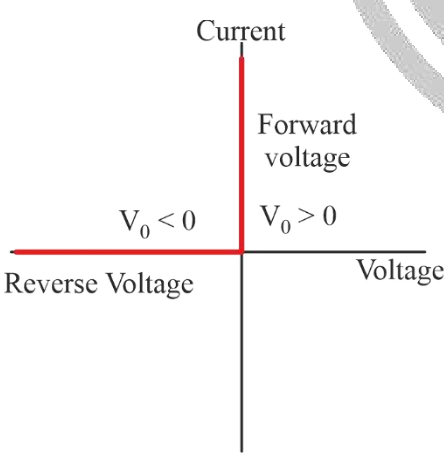
1.1. Diode

- A two terminal semiconductor device with PN junction is called a diode
- A PN junction diode is a two terminal device formed by doping with acceptor and dopant impurities at different regions.

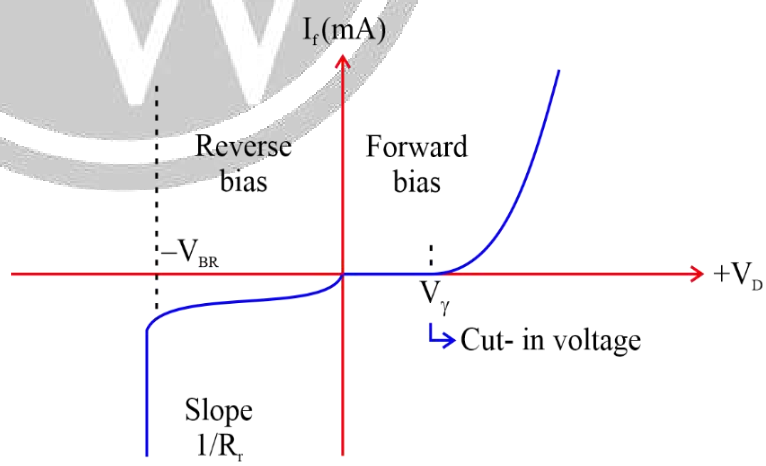


1.2. VI Characteristics

- In ideal condition, a diode works as a short circuit at during forward bias and open circuit during reverse bias, i.e. like an ideal switch
- In practical condition a diode works as low resistance, and high resistance at reverse bias, i.e. like a practical switch.



Ideal Characteristics



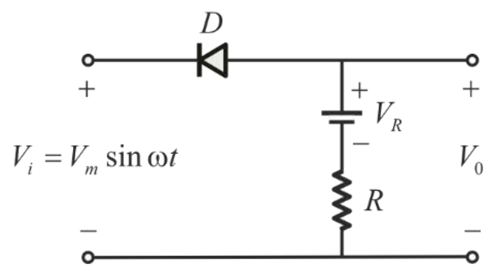
Practical Characteristics

1.2.1. Application of Diode in Clipper circuit

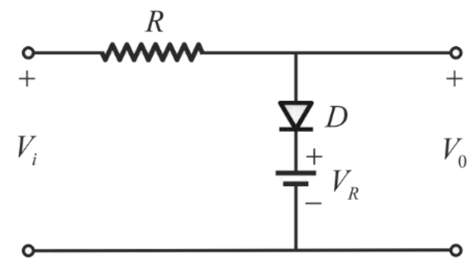
- The Circuit clips a portion of the input signal
- On the basis of which part the circuit clips, the circuit is named as Positive or Negative clipper
- A series or shunt clipper is named according to the placement of diode in the circuit.

Positive Clipper

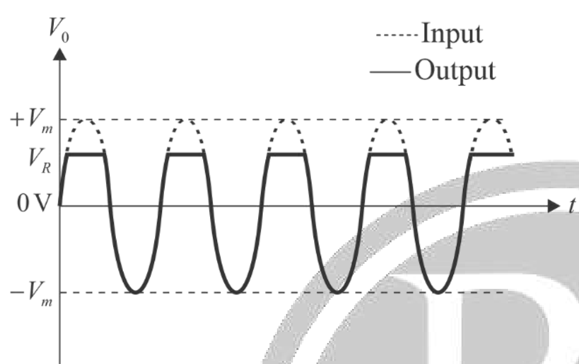
Clips positive portion of the input signal.



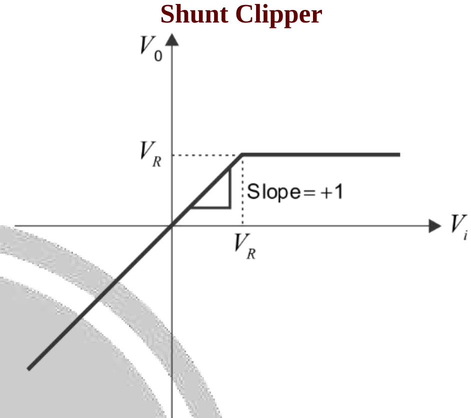
Series Clipper



Shunt Clipper



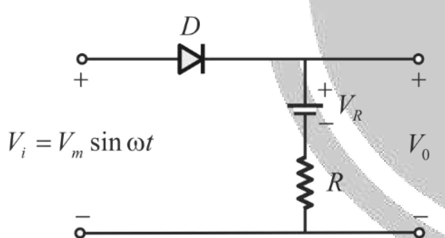
Response



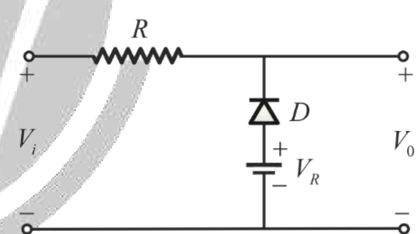
Transfer Characteristics

Negative Clipper

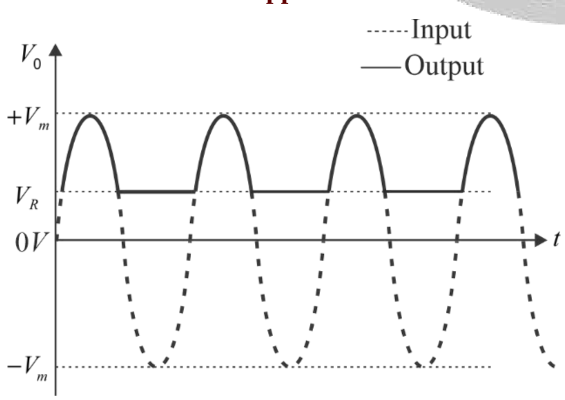
Clips negative portion of the input signal.



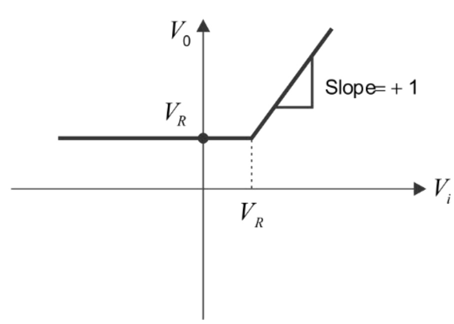
Series Clipper



Shunt Clipper



Response



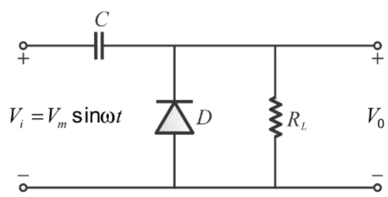
Transfer Characteristics

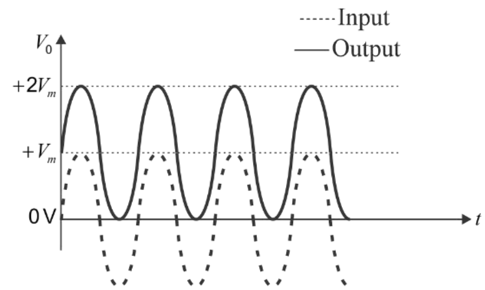
Application of Diode in Clamper Circuit

- A clamper circuit clamps or adds a DC shift to the input signal.
- Based on the polarity of shift, a Positive or Negative clamper is named.

Positive Clamper Circuit

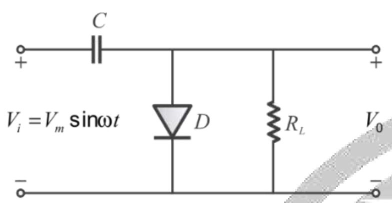
Adds a positive DC shift to the input signal.

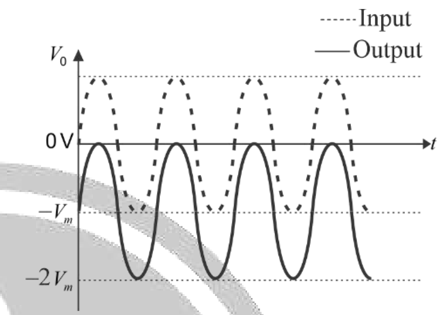




Negative Clamper Circuit

Adds a negative DC shift to the input signal.





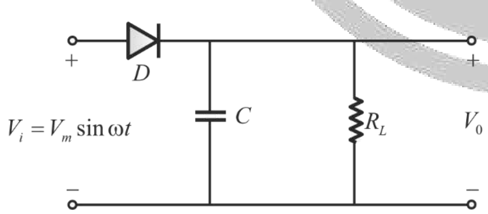
Note: For Proper functioning of clamper circuit, time constant of circuit should be much greater than time period of input signal ($R_L C >> T$)

1.3. Application of Diode in Peak Detector

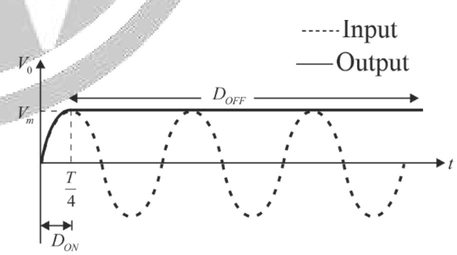
A peak detector detects the peak of the input signal.

Positive Peak Detector

Detects the positive peak of input signal.

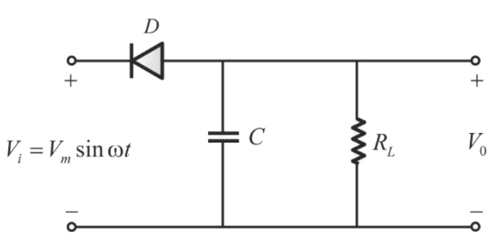


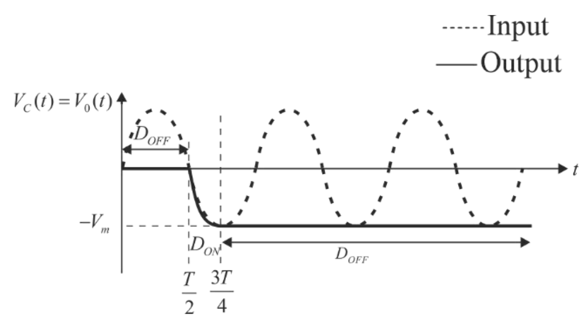
$R_L.C$=high



Negative Peak Detector

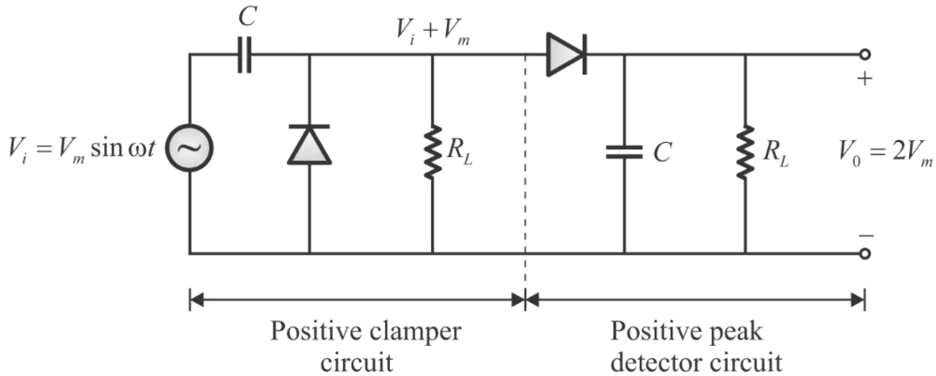
Detects the negative peak of input signal





1.4. Application of Diode in Voltage Doubler

- Voltage doubler gives the output as double of the input signal.
- A level shifter or clamper followed by a peak detector gives a Voltage Doubler



$R_L C >> T$

1.5. Application of Diode in Voltage Multiplier

Using multiple stages of a set of a clamper and peak detector gives voltage multiplier, also called as Cockroft-Walton Voltage Multiplier.

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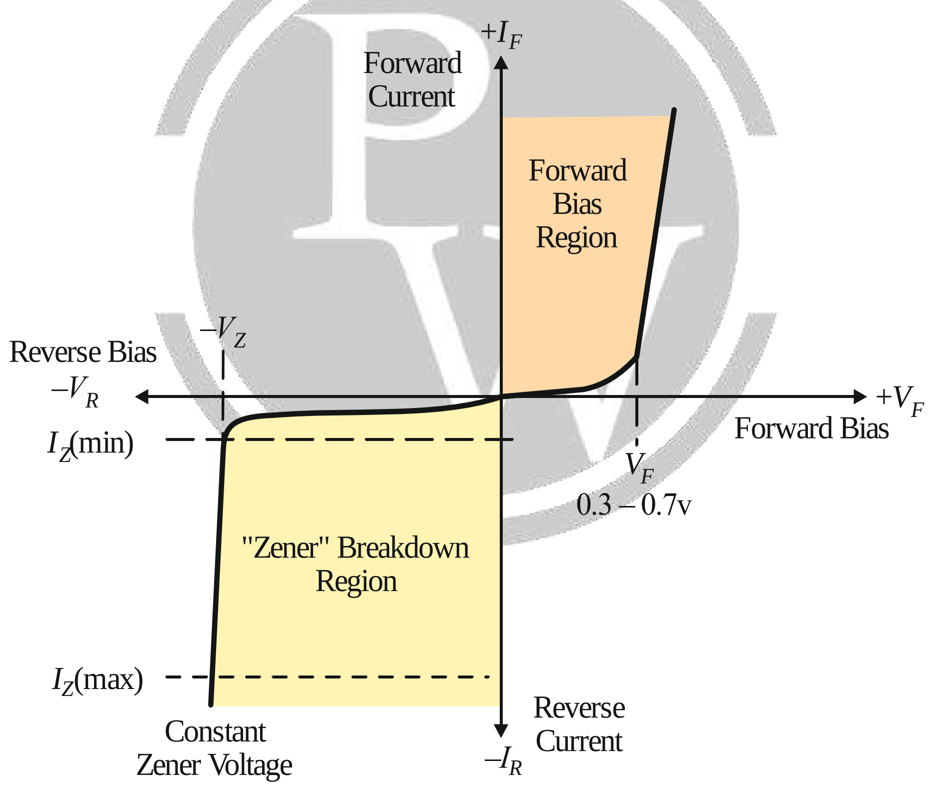
2 ZENER DIODES REGULATOR CIRCUIT

2.1. Zener Diode VI Characteristics

- A "reverse biased" diode blocks current in the reverse direction, but will suffer from premature breakdown or damage if the reverse voltage applied across it is too high.
- Zener Diode are basically the same as the standard PN junction diode but are specially designed to have a low predetermined Reverse Breakdown Voltage that takes advantage of this high reverse voltage.



Zener Diode in Forward Bias:

Works same as a normal PN diode i.e short circuit ideally and a low resistance practically.

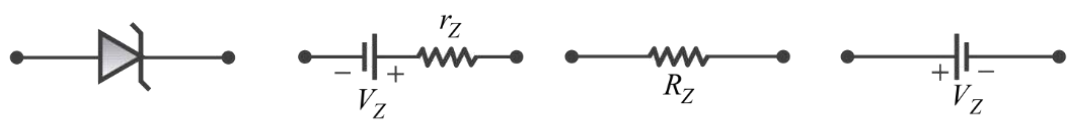
Zener Diode in Reverse Bias:

1. V < V_Z , works as a normal PN diode in reverse bias i.e open circuit ideally and a high resistance practically.
2. V > V_Z, works as a voltage regulator i.e becomes a source of constant voltage for the connected load.

Ideal Zener Diode

Zener Diode | Reverse Biased (with Breakdown) | Reverse Biased (without Breakdown) | Forward Biased

Practical Zener Diode



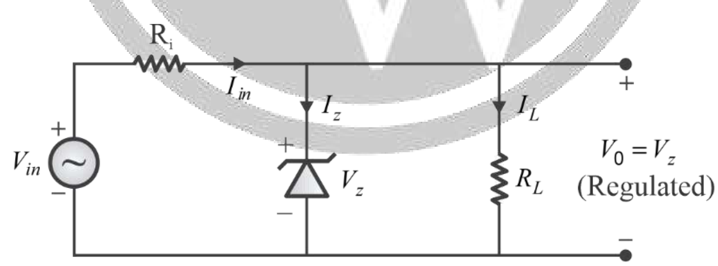
Zener Diode | **Reverse Biased (with Breakdown)** | **Reverse Biased (without Breakdown)** | **Forward Biased**

Operating Conditions for Zener diode to maintain break down characteristics:

1. Current through Zener diode, $I_{Z(\text{knee})} \leq I \leq I_{Z(\max)}$
2. The magnitude of open circuit reverse voltage across the Zener diode should be greater than or equal to V_Z.
   - I_Z (knee) is the minimum current required for the Zener diode to work as a voltage regulator
   - I_Z (max) is the maximum current the Zener diode can operate without damaging the device. It is specified by manufacturer.

Zener Diode as Voltage Regulator

Zener Diodes are used to produce a stabilised voltage output with low ripple under varying load current conditions.



Output voltage :

$$V_0 = V_z = \text{Zener voltage}$$

- Input current : $I_{in} = \dfrac{V_i - V_0}{R}$
- Source resistance : $R = \dfrac{V_i - V_0}{I_Z + I_L}$
- Zener power dissipation : $P_{Z(\max)} = I_{Z(\max)} V_Z$

Note: Zener diodes can also be connected together in series along with normal silicon diodes to produce a variety of different reference voltage output values

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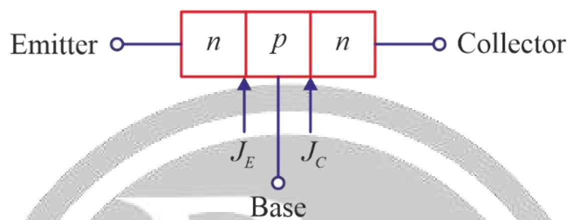
3 BJT BIASING AND REGION OF OPERATION

3.1 Operating Region of BJT



where, J_E = Base-Emitter Junction

J_C = Collector-base Junction

Normal Active Region

In this region, J_E operating in forward bias while J_C operating in reverse bias. It is used as amplifier.

Saturation Region

Both J_E and J_C are operating in forward bias and is used as switch (ON).

Cut-OFF Region

Both J_E and J_C are operating in reverse bias and is used as switch (OFF).

Reverse Active Region

In this region, J_E in reverse bias and J_C in forward bias. It is used as attenuator.

Some Standard values for *npn* transistor.

	Si	Ge
V_{BE} (Active Region)	0.7 V	0.2
V_{BE} (Saturation region)	0.8 V	0.3 V
V_{CE} (Saturation region)	0.2 V	0.1 V
V_{BE} (Cut-off region)	0 V	– 0.1 V

Note : If nothing is mentioned then we will assume transistor is silicon type.

3.2. Different Methods used to Identify Operating Region of BJT

Method-I

Assume transistor in Saturation Region

1. $I_C \neq \beta I_B$
2. $V_{CE} = V_{CE}(\text{sat})$
3. $I_{B(\min)} = \dfrac{I_C(\text{sat})}{\beta_{dC}}$ $\quad (\beta_{dC} = h_{fe})$
4. If $I_B \geq I_B$ (min) then transistor will work in saturation region otherwise in active region.

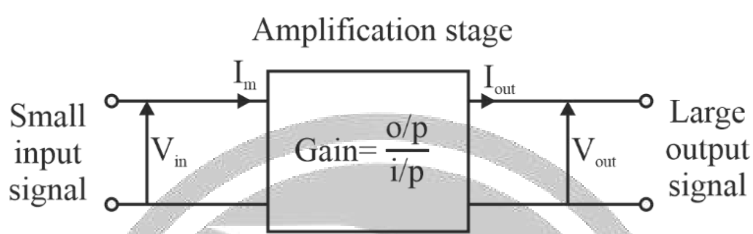
Method-II

Assume transistor in active region.

1. $I_C = \beta_{dc} I_B$
2. $I_E = I_C + I_B = (I + \beta_{dc})I_B$
3. For active region = $\begin{cases} V_{CB} > 0 \quad for \quad npn \ \text{transistor} \\ V_{CB} < 0 \quad for \quad pnp \ \text{transistor} \end{cases}$

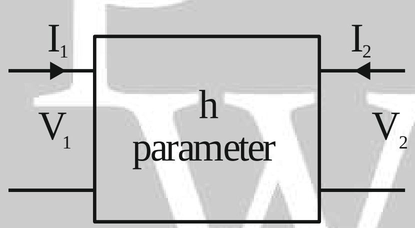
4 Low Frequency BJT Amplifier

4.1. Amplifiers



4.1. Small Signal Modelling of BJT

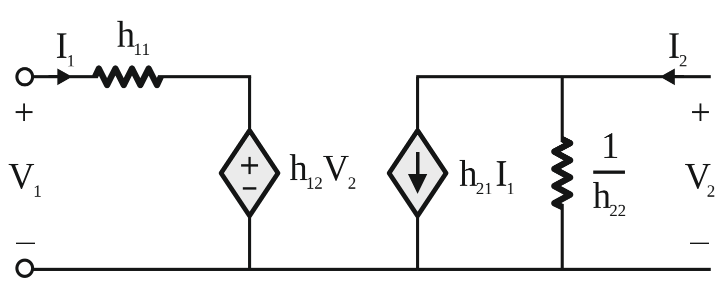
1. ***h*-parameter modelling:**



$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

$$V_1 = h_{11} I_1 + h_{12} V_2$$

$$V_2 = h_{21} I_1 + h_{22} V_2$$



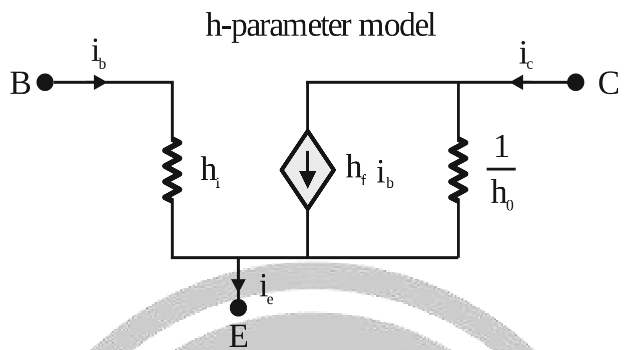
$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \text{input impedance } = h_i$$

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \text{reverse voltage gain } = h_r$$

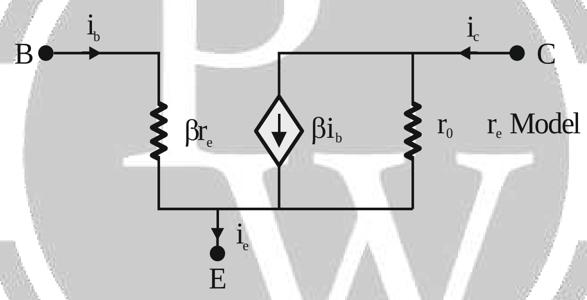
$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \text{forward current gain } = h_f$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \text{ output admittance } = h_0$$

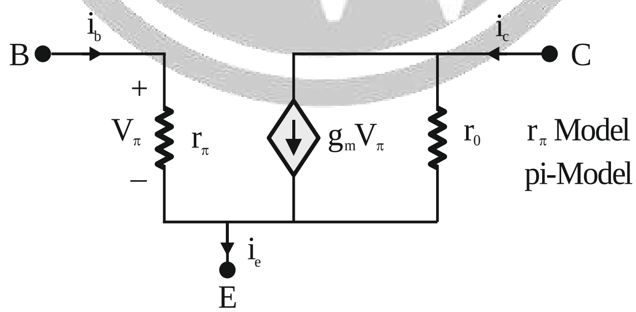
2. Approximated h-model



2. r_e modelling



3. r_π modelling



Relation between small signal modelling parameters

1. $h_i = \beta r_e = r_\pi$

2. $h_f = \beta$

3. $\frac{1}{h_0} = r_0 = V_A / I_{CQ} = \infty$ if no early effect

4. $\beta i_b = g_m V_\pi$

$= g_m r_\pi i_b$

$= g_m \cdot \beta r_e i_b$

$= g_m \cdot \beta r_e i_b$

$\Rightarrow g_m = \frac{1}{r_e}$

r_e = emitter dynamic resistance

$$r_e = \frac{\eta V_T}{I_{EQ}}.$$

Note: Practically $r_0 = 0$, and $1/h_0 = \infty$, if not mentioned explicitly.

4.2. Procedure of AC analysis

Step 1: Do the mid frequency analysis

(a) $C_{c_1}, C_{c_2}, C_E \rightarrow$ short circuit

$C_T, C_D, C_{sh} \rightarrow$ Open circuit

(b) all DC independent voltage source → Short circuit

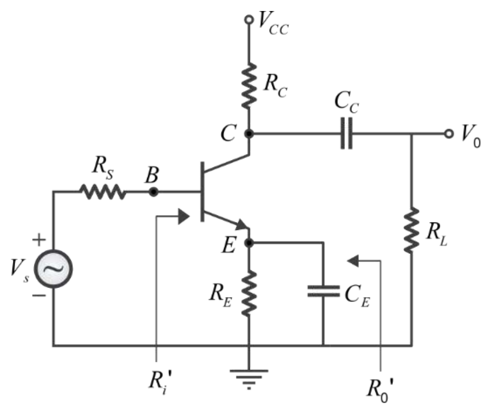
All independent current source = open circuit

Step 2: Replace BJT with small signal equivalent.

Internal Performance Parameter of an Amplifier

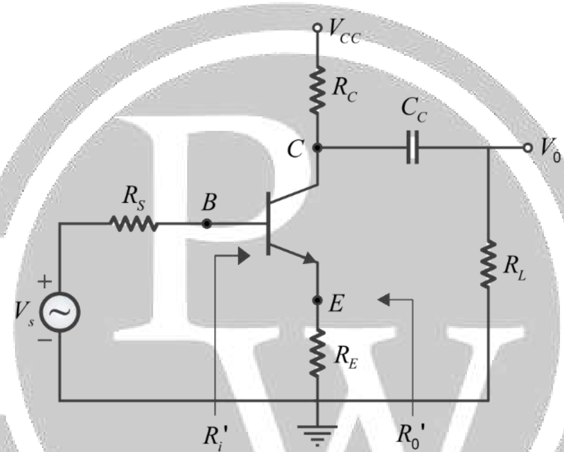
1. Current Gain (A_i) = – I_2 / I_1
2. Input Resistance (R_i) = V_1/I_1
3. Voltage Gain (A_v) = V_2/V_1
4. Output Resistance (R_o) = 1/Output Admittance ($1/y_o$) = $I_2/V_2 \mid V_S = 0$

CE Amplifier without R_E



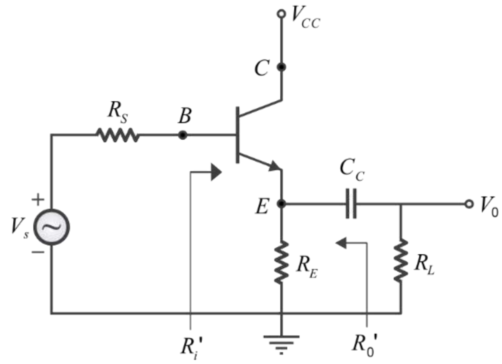
Parameter	Current Gain	Input Resistance	Voltage Gain	Output Resistance
h-model	$A_I' = -h_{fe}$	$R_i' = h_{ie}$	$A_V' = A_I' \times \frac{R_L'}{R_i'}$ $R_L' = R_L \parallel R_C$	$R_0' = \infty$
r_e model	$A_I' = -\beta$	$R_i' = r_\pi = \frac{\beta}{g_m}$	$A_V' = A_I' \times \frac{R_L'}{R_i'} = \frac{-\beta \times R_L'}{r_\pi}$ $A_V' = -g_m R_L' = \frac{-R_L'}{r_e}$ $R_L' = R_L \parallel R_C$	$R_0' = r_0 = \left\lvert\frac{V_A}{I_C}\right\rvert [V_A = \infty]$ $R_0' = r_0 = \infty \quad [V_A = \infty]$

CE Amplifier with R_E



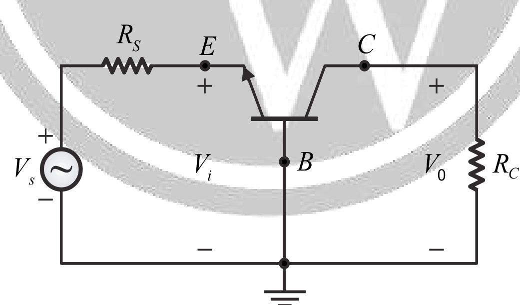
Parameter	Current Gain	Input Resistance	Voltage Gain	Output Resistance
h-model	$A_I' = -h_{fe}$	$R_i' = h_{ie} + (1 + h_{fe}) R_E$	$A_V' = A_I' \times \frac{R_L'}{R_i'}$ $A_V' \approx \frac{-R_L'}{R_E}$ $R_L' = R_C \parallel R_L$	$R_0' = \infty$
r_e model	$A_I' = -\beta$	$R_i' = r_\pi + (1+\beta) R_E$	$A_V' = A_I' \times \frac{R_L'}{R_i'}$ $A_V' = \frac{-\beta R_L'}{r_\pi + (1+\beta) R_E} \approx \frac{-R_L'}{R_E}$ $[(1+\beta) R_E \gg r_\pi]$ $R_L' = R_C \parallel R_L$	$R_0' = \infty \quad [V_A = \infty]$

CC Amplifier



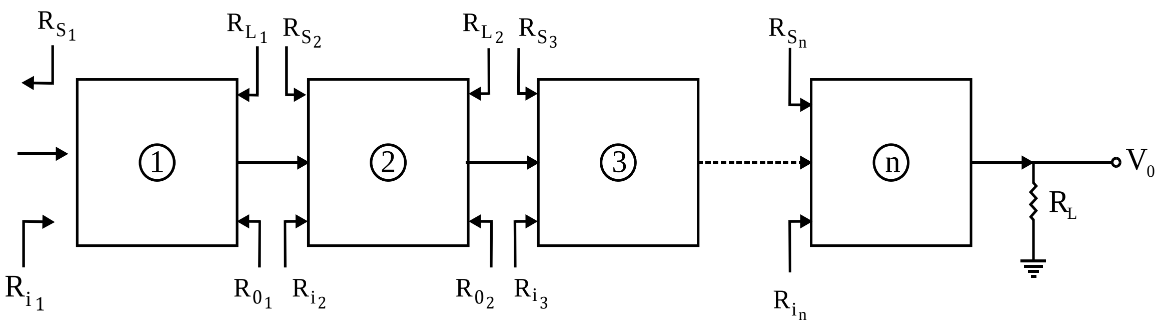
Parameter	Current Gain	Input Resistance	Voltage Gain	Output Resistance
h-model	$A_i' = 1 + h_{fe}$	$R_i' = h_{ie} + (1 + h_{fe}) R_L'$	$A_V' = \frac{A_i' R_L'}{R_i'} \approx 1$ $R_L' = R_E \parallel R_L$	$R_0' = \frac{R_S' + h_{ie}}{1 + h_{fe}}$ R_S' = Effective source impedance
r_e model	$A_i' = 1 + \beta$	$R_i' = r_\pi + (1+\beta) R_L'$	$A_V' = \frac{(1+\beta)R_L'}{r_\pi + (1+\beta)R_L'} \approx 1.0$ $R_L' = R_E \parallel R_L$	$R_0' = \frac{R_S' + r_\pi}{1+\beta}$ If $r_\pi > R_S'$ $R_0' = \frac{r_\pi}{1+\beta} \approx \frac{r_\pi}{\beta}$ $R_0' = \frac{r_\pi}{\beta} = \frac{1}{g_m}$

CB Amplifier



Parameter	Current Gain	Input Resistance	Voltage Gain	Output Resistance
h-model	$A_i' = \frac{h_{fe}}{1+h_{fe}} \approx 1$	$R_i' = \frac{h_{ie}}{1+h_{fe}}$	$A_V' = \frac{A_i' R_L'}{R_i'}$ $R_L' = R_C$	$R_0' = \infty$
r_e model	$A_i' = \frac{\beta}{1+\beta} \approx 1$	$R_i' \approx \frac{r_\pi}{1+\beta} \approx \frac{r_\pi}{\beta} = \frac{1}{g_m}$	$A_V' = \frac{1 \times R_L'}{1/g_m} = g_m R_L'$ $R_L' = R_C$	$R_0' = \infty$ $[V_A = \infty]$

Cascade Amplifier (Multistage Effect)



1. $R_{L_1} \neq \mathrm{f}\left(R_{01}, R_{sa}\right)$

   $R_{L_1} = f\left(R_{i2}\right)$

2. $R_{s_2} \neq f\left(R_{L1}, R_{i2}\right)$

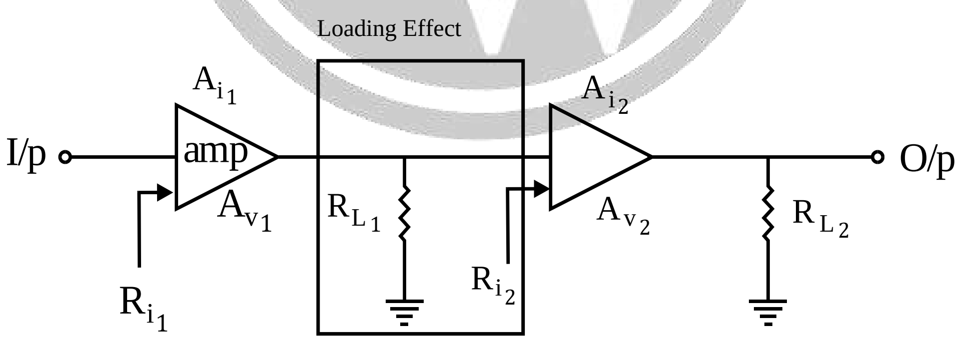
   $R_{s2} = f\left(R_{01}\right)$

3. R_i [cascade] = R_i (1^{st} stage)
4. o/p R_0 (cascade) = R_{0N} (last stage)
5. $A_v = A_{v1} \times A_{v2} \times A_{v3} \times \ldots\ldots A_N$

   (for proper impedance matching)
6. $A_I = A_{i1} \times A_{i2} \times A_{i3} \times \ldots\ldots A_{iN}$

   (for proper impedance matching)

Loading Effect

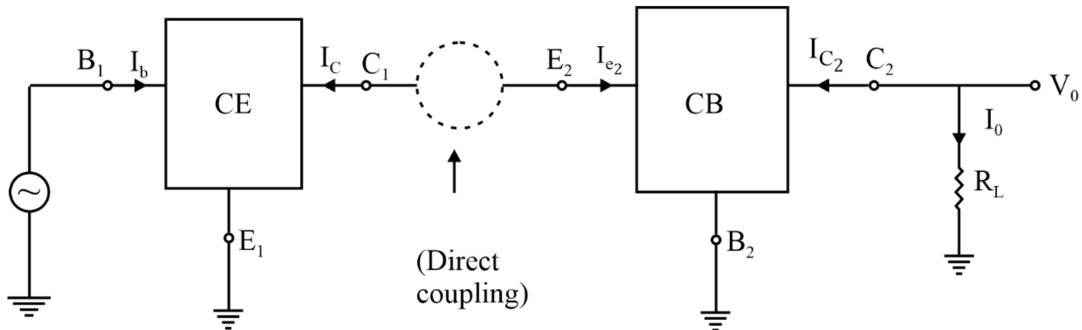


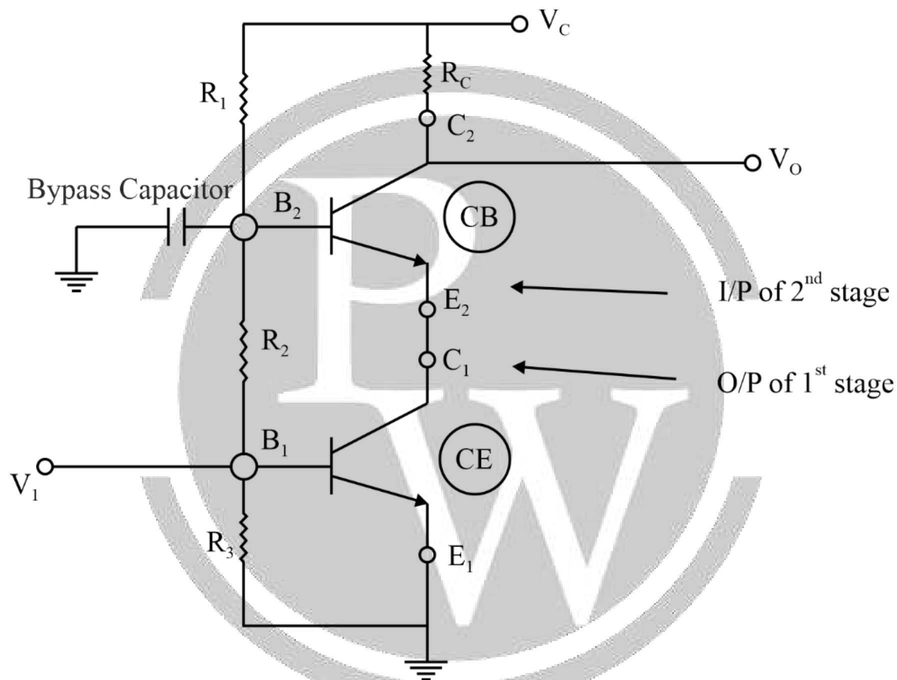
1. The decrease in the gain of first stage due to low value of input impedance of second stage is called Loading effect
2. Loading effect occurs in BJT amplifiers, and not in JFET and MOSFET amplifiers because they have a very high input impedance.

Cascode Amplifier

- A combination of CE followed by CB is referred as Cascode amplifier
- CE acts as input stage and CB acts as output stage.

- Cascode amplifier can amplify both voltage and current
- Also known as direct coupled amplifier because output of CE configuration is directly connected to input of CB configuration.



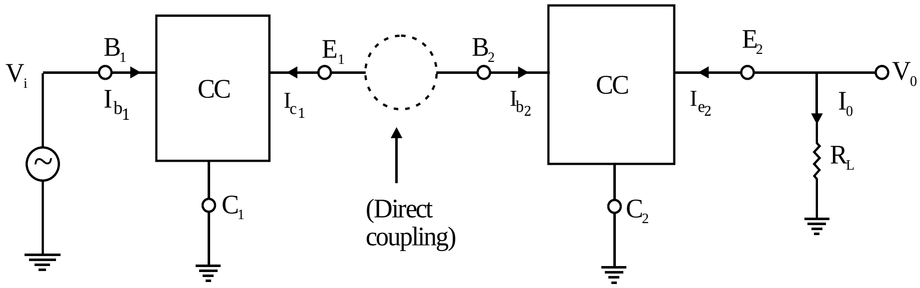


Cascode Amplifier Parameters

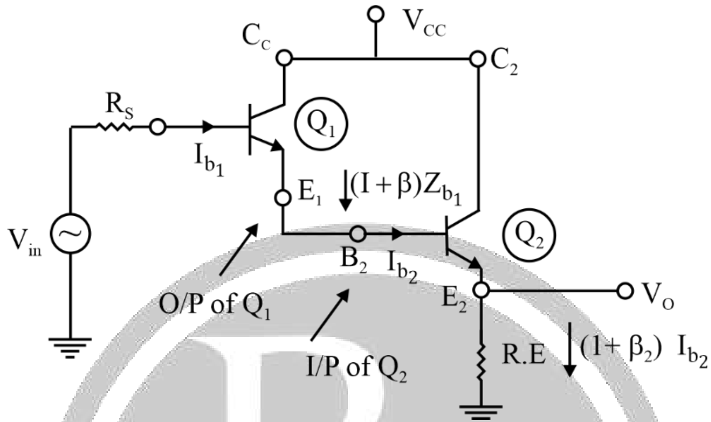
1. Transconductance (g_m) = A_I[CB] × g_m[CE]
2. Input Impedance (R_i) = R_i [CE]
3. Output Impedance (R_O) = R_O [CB]
4. Current Gain (A_I) = A_I [CB] × A_I [CB] > A_I [CE]
5. Voltage Gain (A_V) = A_V [CB] × A_V [CE] > – A_V [CB]

Darlington Pair

- Series combination of two CC configurations.
- Also referred as direct coupled amplifier



4. Circuit diagram of Darlington pair -

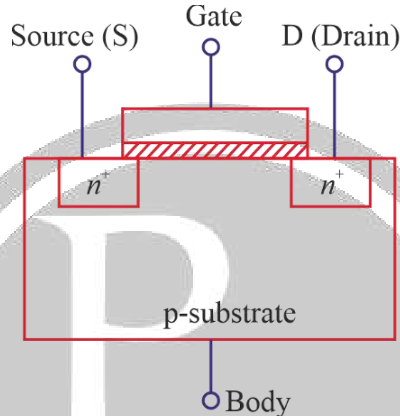


Darlington Amplifier Parameters

1. Transconductance (g_m) = A_i [2^{nd} stage] × g_m[1^{st} stage]
2. Input Impedance (R_i) = R_i [1^{st} stage]
3. Output Impedance (Ro) = R_O [2^{nd} stage]
4. Current Gain (A_i) = A_i [1^{st} stage] × A_i [2^{nd} stage]
5. Voltage Gain (A_v) = 1

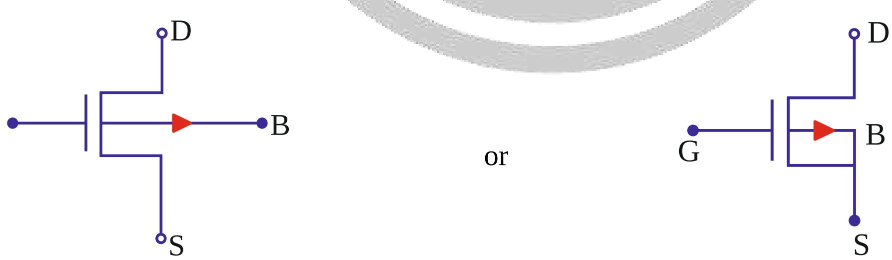
5 MOSFET AMPLIFIER WITH BIASING

5.1. MOSFET

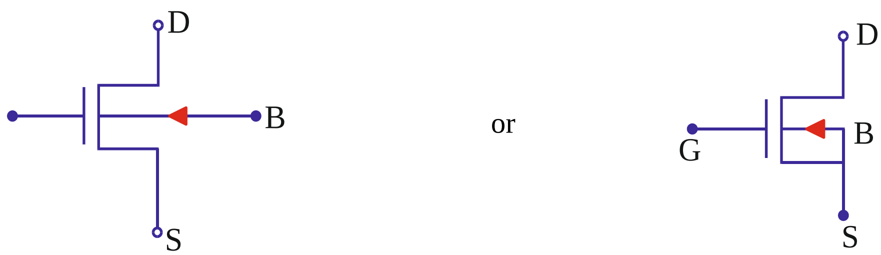


- **Source :** terminal through which majority charge carrier enters into the SC bar.
- **Drain :** terminal through which majority charge carriers leaves the SC bar.
- **Channel:** path between source & drain by which majority carrier travel from source to drain
- **Gate :** Terminal to control the flow of charge carrier from source to drain

Symbol :



- P-channel Enhancement type MOSFET :



- N-channel Enhancement type MOSFET.

Channel Current

The current flowing from source to drain via the channel. It is a function of aspect ratio of MOSFET and applied voltages V_{GS}, and V_{DS}.

$$\boxed{I_D = \frac{1}{2}\mu_n C_{ox}\frac{W}{L}\left[(v_{GS} - v_{TH})v_{DS} - \frac{v_{DS}^{\;2}}{2}\right]}$$

Trans Conductance :

It is a figure of merit indicates that how well a transistor convert the voltage to the current

$$g_m = \frac{\partial I_D}{\partial V_{GS}}\bigg| V_{DS} = \text{const.}$$

In cut-off Region:

MOSFET works as an OFF switch.

$$V_{GS} < V_T$$
$$I_D = 0$$

In Triode Region:

MOSFET works as a resistor

$$V_{GS} > V_T$$
$$V_{DS} > (V_{GS} - V_T)$$

$$I_D = \frac{1}{2}\mu_n Cox\frac{W}{L}\left[(v_{GS} - v_{TH})v_{DS} - \frac{v_{DS}^{\;2}}{2}\right]$$

$$\frac{\partial I_D}{\partial V_{GS}}\bigg|(V_{DS} = \text{const.}) = g_m = \mu_n Cox\frac{W}{L}v_{DS}$$

In Saturation region :

MOSFET works as on ON switch

$$V_{GS} >> V_T$$

$$v_{DS} = v_{GS} - v_{TH}$$

$$\boxed{I_{D\,\text{sat}} = \frac{1}{2}\mu_n C_{ox}\frac{W}{L}[v_{GS} - v_{TH}]^2}$$

and
$$g_m = \mu_n C_{ox}\frac{W}{L}[v_{GS} - v_{TH}]$$

$$g_m = \frac{2I_{D\,sat}}{v_{GS} - v_{TH}}$$

$$g_m = \sqrt{2\mu_n C_{ox}\left(\frac{W}{L}\right)I_D}$$

MOSFET Biasing

- Biasing is a process of applying the operating point of device.
- Three operating regions.

  Cut-off $V_{GS} < V_T$

  Linear: $V_{GS} > V_T \mid V_{DS} < V_{GS} - V_T$

  Saturation: $V_{GS} > V_T \mid V_{DS} < V_{GS} - V_T$

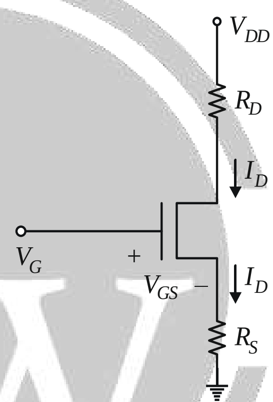
- There are three types of biasing

  (1) Fixed bias

  (2) Drain to base bias

  (3) Potential divider bias

1. Fixed Bias Configuration:



by kVL

$$V_G - V_{GS} - I_D R_S = 0$$

$\Rightarrow$ $$V_{GS} = V_G - I_D R_S$$

Assume operating in saturation mode

Find I_D from standard drain current equation

kVL in output loop

$$V_{DD} - I_D R_D - V_{DS} - I_D R_s = 0$$

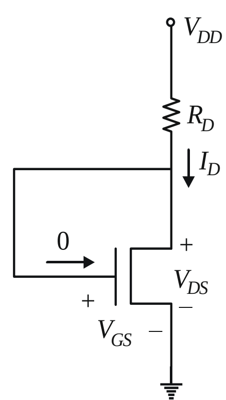
$\Rightarrow$ $$V_{DS} = V_{DD} - I_D (R_D + R_S)$$

Now checks VDS & $V_{GS} - V_T$ condition for saturation.

True $\rightarrow$ assumption correct

False $\rightarrow$ assumption false, assume linear & solve again.

2. Drain to Gate Bias



As $$V_{DS} = V_{GS}$$

$$V_{DS} > V_{GS} - V_T$$

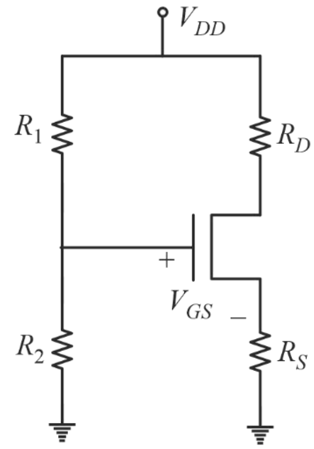
MOS biased in saturation region.

Applying KVL $$V_{DD} - I_D R_D - V_{DS} = 0$$

$$V_{DS} = V_{DD} - I_D R_D$$

as saturation, $$I_D = k[V_{GS} - V_T]^2$$

3. Potential Divider Bias



By voltage division $$V_G = \frac{V_{DD} \times R_2}{R_1 + R_2}$$

$$I_S = I_D$$

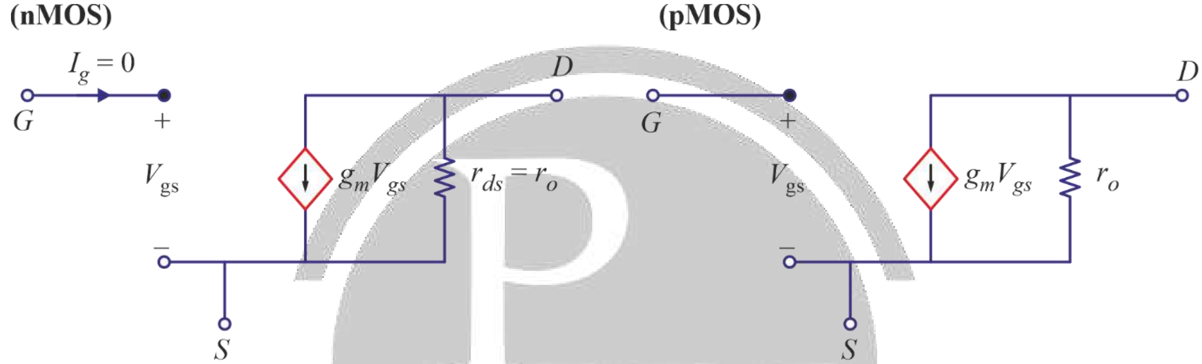
$$V_{GS} = V_G - I_D R_S$$

$$V_{DS} = V_{DD} - I_D(R_D + R_S)$$

Assume saturation, $I_D = k[V_{GS} - V_T]^2$ find I_D, V_{GS}, V_{DS}

$V_{DS} > V_{GS} - V_T$ saturation.
otherwise assume active.

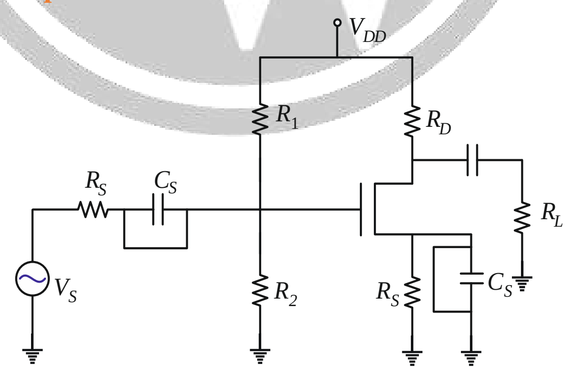
Small Signal (or) AC Analysis

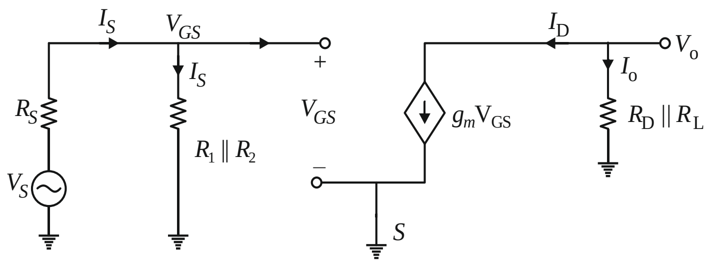


If no channel length modulation ($\lambda = 0$).

$$V_A = \frac{1}{\lambda} = \infty; \quad r_{ds} = r_o = \frac{V_A}{I_D} = \infty$$

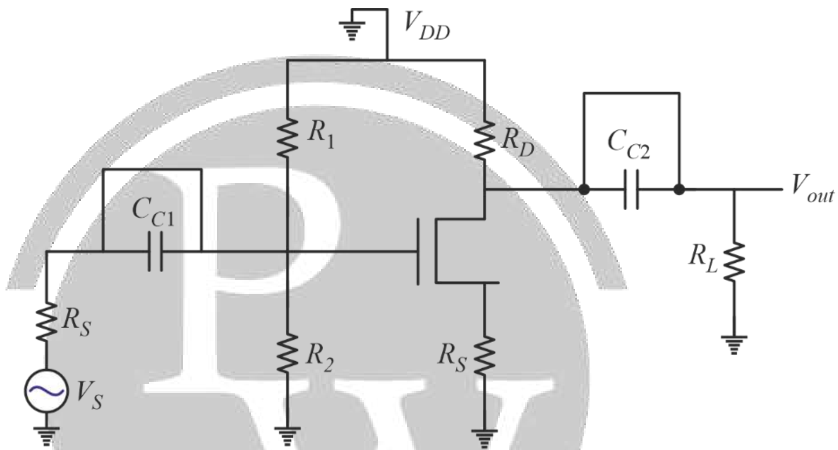
(a) RC Coupled amplifier with Coupling Capacitor

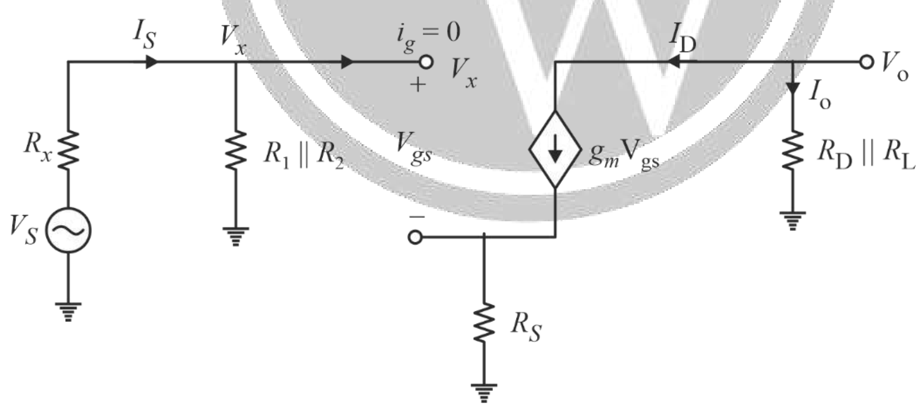




1. $I_S = \dfrac{V_{GS}}{R_1 \parallel R_2}$
2. $I_0 = -g_m V_{GS}$
3. $A_I = -g_m \left[R_1 \parallel R_2\right]$
4. $R_{in} = R_s + [R_1 \parallel R_2]$
5. $V_S = I_S \cdot R_{in}$
6. $V_0 = -g_m V_{GS} \ (R_D \parallel R_L)$
7. $A_V = \dfrac{V_0}{V_S} = -\dfrac{g_m(R_D \parallel R_L)}{R_{in}}[R_1 \parallel R_2]$
8. $R_{out} = R_D$(Load open)

(b) RC coupled amplifier without bypass capacitor:





$$i_o = -g_m v_{gs}$$

$$V_o = -[R_L \parallel R_D]\, g_m V_{gs}$$

$$R_{in} = R_x + [R_1 \parallel R_2]$$

$$Vx = V_{gs} + g_m V_{gs} R_s$$

$$= (1 + g_m R_s) V_{gs}$$

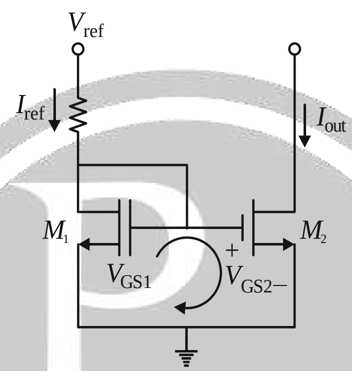
$$V_x = \frac{R_1 \| R_2}{(R_1 \| R_2 + R_2)}$$

$$\Rightarrow \quad Vx = \left[\frac{R_1 \| R_2 + R_x}{R_1 \| R_2}\right](1 + g_m R_s)V_{gs}$$

$$\text{Voltage gain} = \frac{V_o}{V_s}$$

Current Mirror

- It copies the ref current flowing at the input of the system to the output.
- It is also known as practical current source.
- It is designed by using IC technology.



$$V_{GS1} = V_{GS2} = V_{GS}$$

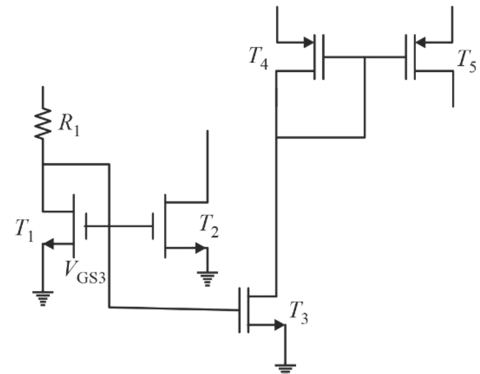
$$I_{ref} = \frac{1}{2}\mu_x C_{ox}\left(\frac{W}{L}\right)_1 (V_{GS} - V_T)^2 \quad \text{...(1)}$$

$$I_{out} = \frac{1}{2}\mu_x C_{ox}\left(\frac{W}{L}\right)_2 (V_{GS} - V_T)^2 \quad \text{...(2)}$$

$$\frac{I_{out}}{I_{ref}} = \frac{(W/L)_2}{(W/L)_1}$$

as now output current is independent of output resistance, if $\left(\frac{W}{L}\right)_1 = \left(\frac{W}{L}\right)_2 \rightarrow \boxed{I_{out} = I_{ref}}$

MOS Current Steering Circuit:



T_1 and T_2 are in current mirror.

$\Rightarrow$
$$I_{D_2} = I_{\text{ref}}\left[\frac{(W/L)_2}{(W/L)_1}\right]$$

T_1 and T_3 are in current mirror.

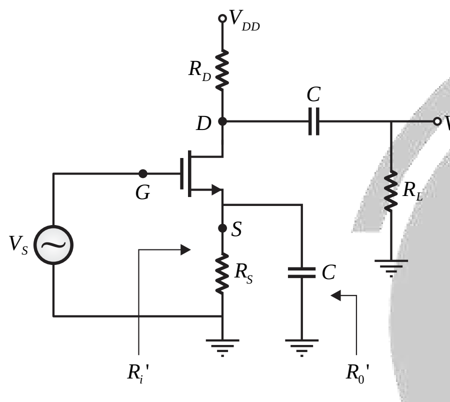
$$I_{D_3} = I_{\text{ref}}\left[\frac{(W/L)_3}{(W/L)_1}\right]$$

T_4 and T_5 are in current mirror $\rightarrow$ pMOS

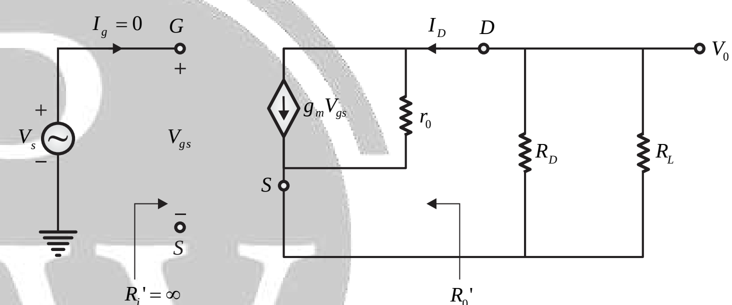
$$I_{D_5} = I_{D_4}\left[\frac{(W/L)_5}{(W/L)_4}\right] = I_{D_3}\left[\frac{(W/L)_5}{(W/L)_4}\right]$$

5.2. MOSFET Amplifiers

Common Source MOSFET Amplifier without R_S and its AC Equivalent circuit



(a) CS MOSFET amplifier without R_S



(b) AC equivalent of CS MOSFET amplifier without R_S

Common Source MOSFET Amplifier with R_S and its AC Equivalent circuit

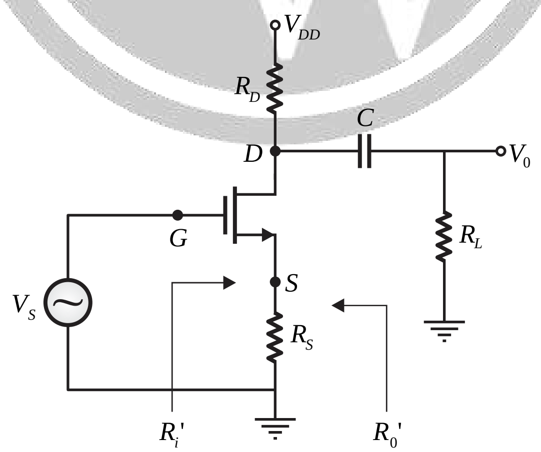


Fig. (a) CS MOSFET amplifier with R_S

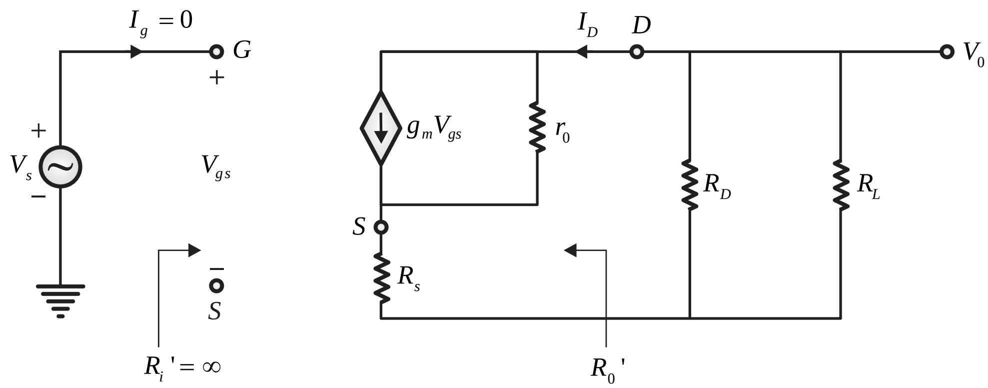
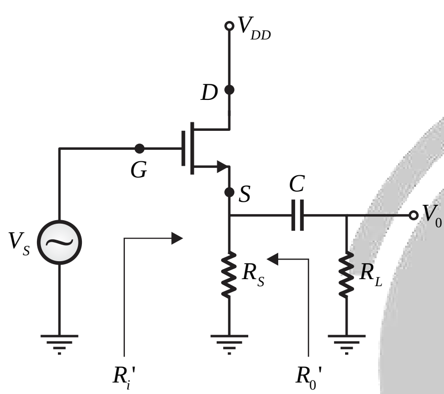
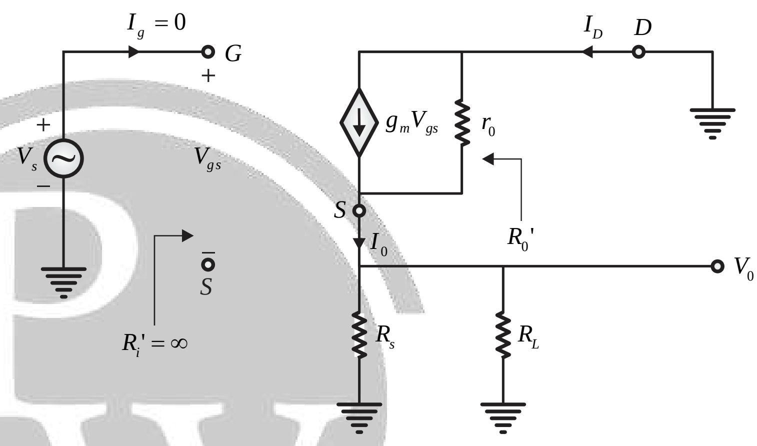


Fig. (b) AC equivalent of CS MOSFET amplifier with R_S

Common Drain MOSFET Amplifier and its AC Equivalent circuit



(a) CD MOSFET amplifier (n-channel enhancement)



(b) AC equivalent of CD MOSFET amplifier

Common Gate MOSFET Amplifier and its AC Equivalent circuit

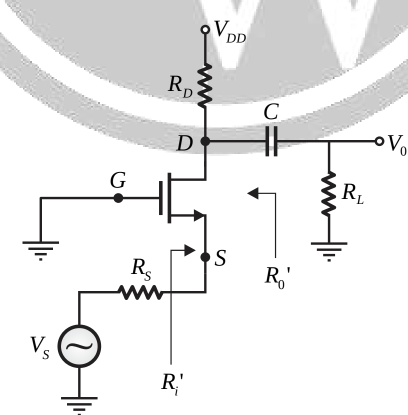


Fig. (a) CG MOSFET amplifier (n channel enhancement)

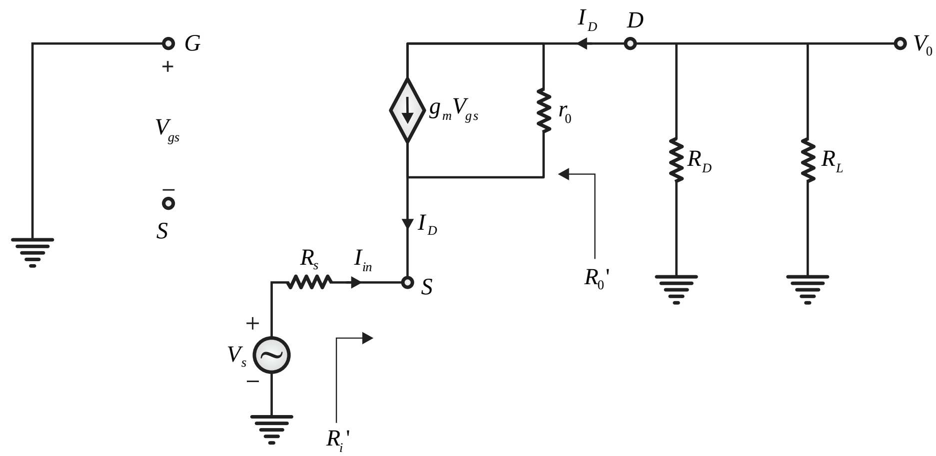


Fig. (b) AC equivalent of CG MOSFET amplifier

MOSFET Amplifiers AC Parameters

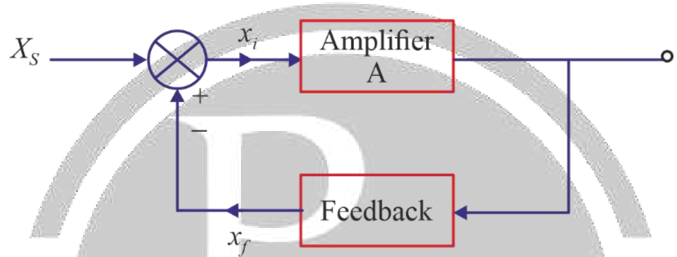
AC parameter	CS without R_S	CS with R_S	Common drain	Common Gate
Input resistance	$R_i' = \infty\ [\because\ I_g = 0]$	$R_i' = \infty\ [\because\ I_g = 0]$	$R_i' = \infty$	$R_i' \approx \frac{1}{g_m}$
Output resistance	$R_0' = r_0\ [\lambda \neq 0]$ $R_0' = \infty\ [\lambda = 0]\ \left[r_0 = \frac{1}{\lambda I_D}\right]$	$R_0' = r_0 + (1+\mu)R_S$ $R_0' = \infty\ [\text{if } \lambda = 0 \Rightarrow r_d = \infty]$	$R_0' \approx \frac{1}{g_m}$	$R_0' = r_0 + R_S(1+\mu)$ $R_0' = \infty\ [\text{if } \lambda = 0 \Rightarrow r_0 = \infty]$
Voltage gain	$A_V' = -g_m R_L''$ $(R_L'' = r_0 \parallel R_D \parallel R_L)$	$A_V' = \frac{-\mu R_L'}{R_L' + r_0 + R_S(1+\mu)}$ $(R_L' = R_D \parallel R_L)$	$A_V' = \frac{g_m R_L'}{1 + g_m R_L'}$ $(R_L' = R_s \parallel R_L)$	$A_V' = g_m R_L''$ $(R_L'' = r_0 \parallel R_D \parallel R_L)$
Current gain	Does not exist $[I_g = 0\text{ A}]$	Does not exist $[I_g = 0\text{ A}]$	Does not exist $[I_g = 0\text{ A}]$	$A_I' = 1.0$

6 FEEDBACK AMPLIFIERS

6.1. Introduction

6.1. Feedback

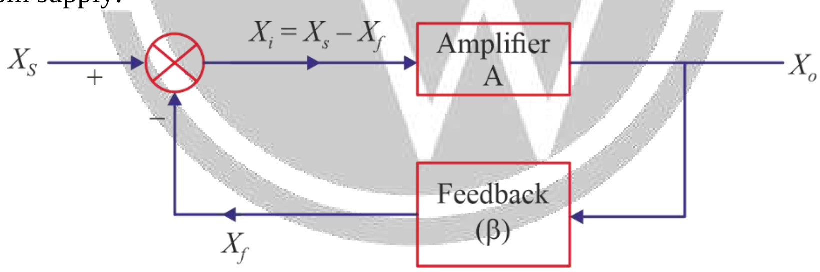
It is a process of taking sample from output and mix with input.



According to type of mixing two types of feedback.

I. Negative Feedback

If sample gets subtracted from supply.



1. Overall Gain:

$$A_F = \frac{A}{1 + A\beta}$$

gain reduced by $(1 + A\beta)$

if $A\beta >> 1 \quad \Rightarrow \quad \boxed{A_f = \frac{1}{\beta}}$

2. Bandwidth:

$$\text{Gain} \times \text{B.W.} = \text{Constant}$$

$$\Rightarrow \quad (\text{B.W.})_f = (\text{BW})(1 + A\beta)$$

so, bandwidth increased by $(1 + A\beta)$

3. Noise and Distortion

without feedback:

$$V_o = AV_i + V_N + V_o$$

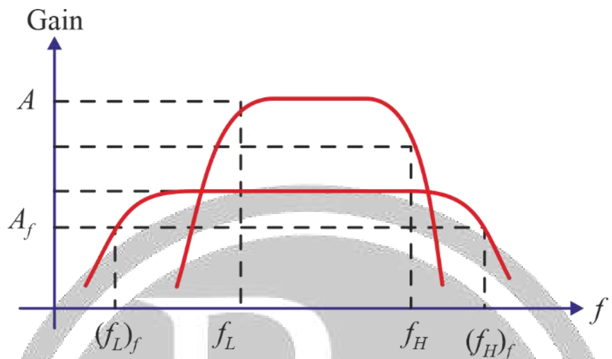
with Feedback

$$V_o = A(V_s - \beta V_o) + V_N + V_D$$

$$\Rightarrow \quad V_o = \frac{AV_s}{1+A\beta} + \frac{V_N}{1+A\beta} + \frac{V_D}{1+A\beta}$$

Hence, noise and distortion reduced by $(1 + A\beta)$

4. Frequency Response:



5. Gain Sensitivity:

$$S_g = \frac{\partial A_f / A_f}{\partial A / A} = \frac{A}{A_f} \times \frac{\partial A_f}{\partial A}$$

$$= \frac{A}{\left(\frac{A}{1+A\beta}\right)} \times \frac{\partial}{\partial A}\left[\frac{A}{1+A\beta}\right]$$

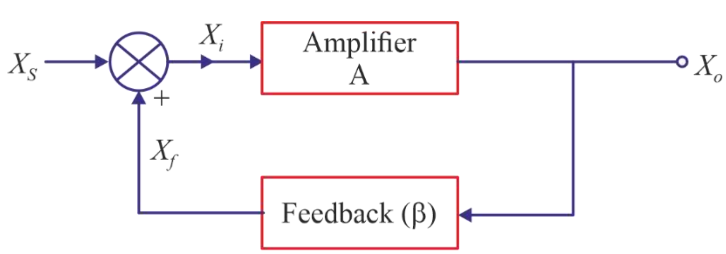
$$= (1+A\beta)\left[\frac{(1+A\beta) - A\beta}{(1+A\beta)^2}\right] = \frac{1}{1+A\beta}$$

6. Desensitivity

$$D = \frac{1}{S_g} = (1+A\beta)$$

II. Positive Feedback

If sampled signal gets added with input then it is called as positive feedback.



(a) Gain:

$$A_f = \frac{X_o}{X_s} = \frac{X_i A}{X_i - X_f} = \frac{X_i A}{X_i - A\beta X_i}$$

$$\Rightarrow \quad X_o = \left[\frac{A}{1-A\beta}\right] X_S$$

$\Rightarrow$ $\boxed{A_F = \frac{A}{1-A\beta}}$ practically $|A\beta| < 1$.

Effects:

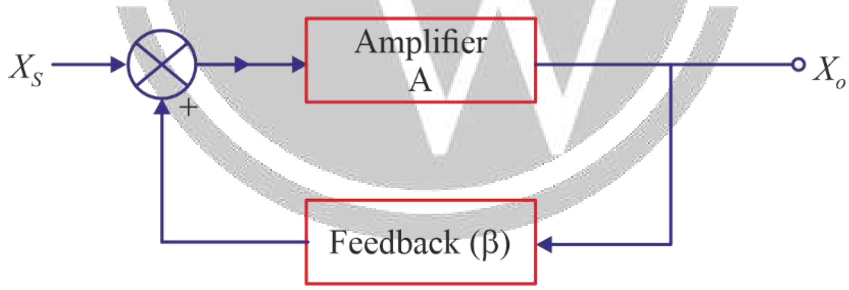
1. Reduces bandwidth of system.
2. Increase noise, as well as distortion of system.

Negative feedback amplifiers classificed into 4-types.

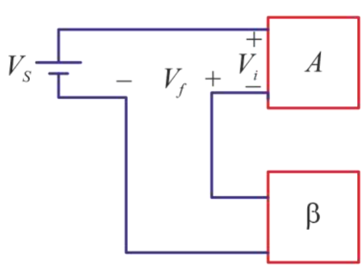
1. Series Series Feedback
2. Series Shunt Feedback
3. Shunt Series Feedback
4. Shunt Shunt Feedback

6.2. Classification of Amplifier

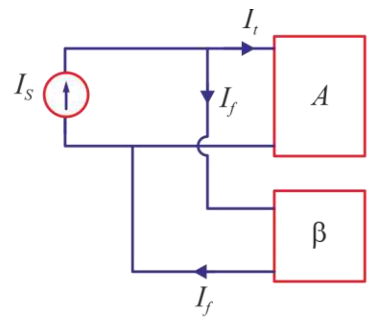
In the feedback amplifier



A Mixter Network

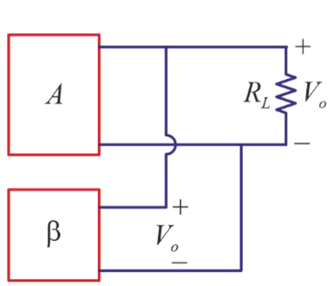


Voltage Mixing
$V_i = V_s - V_f$
R_i increase

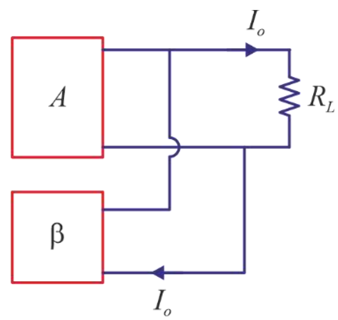


Current Mixing
$I_i = I_s - I_f$
R_{in} decrease

B-Sampling Network

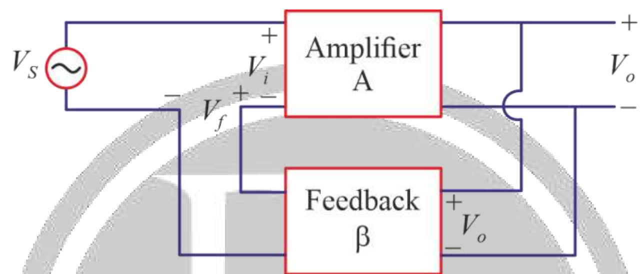


Voltage Sampling
R_o decrease



Current Sampling
R_o increase

(1) Series-Shunt Feedback: (Voltage Amplifiers)



Observations:

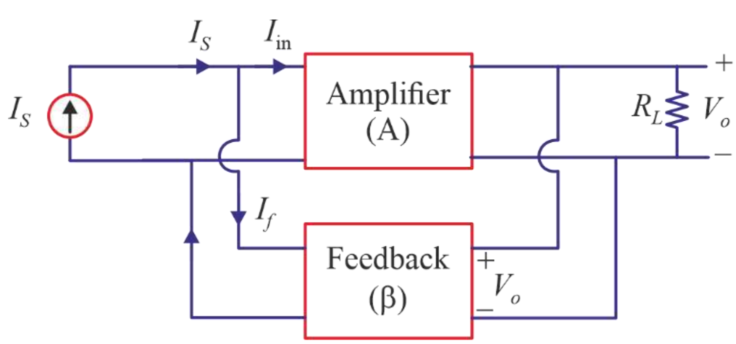
$$\text{Gain} = \frac{A_V}{1+A_V\beta}$$

$$\text{Bandwidth} = BW\left[1+A_V\beta\right]$$

$$(R_{\text{in}})f = R_i[1+A_V\ \beta]$$

$$(R_o)f = \frac{R_o}{1+A_V\beta}$$

(2) Shunt-Shunt Feedback : (Trans-resistance Amplifiers)



Gain:

$$I_f = \beta V_o$$

$$I_{\text{in}} = \left(I_f - \beta V_o\right)\cdot\frac{R_s}{R_s+R_{in}}$$

$$V_o = R_M I_{in}\left[\frac{R_L}{R_L+R_o}\right] = R_m\left[\frac{R_L}{R_L+R_o}\right]\left[\frac{R_S}{R_S+R_{in}}\right] = (R_m)f(I_s-\beta V_o)$$

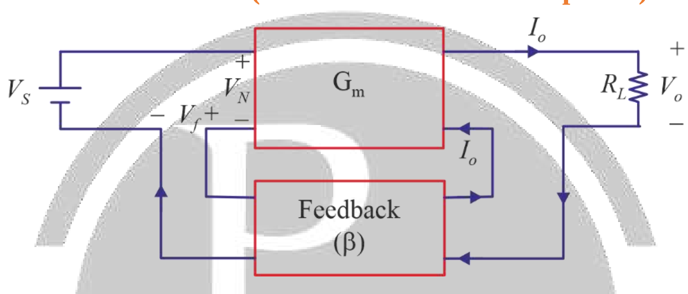
$$V_o[1+\beta(R_m)] = I_S R_{ms}$$

$$\Rightarrow \quad = \boxed{\frac{V_o}{I_s}=\frac{R_{ms}}{1+\beta R_{ms}}}$$

Observations:

1. Gain $(R_m)_f = \dfrac{R_{ms}}{1+R_{ms}\beta}$ decreased.
2. $(R_i)_f = \dfrac{R_i}{1+R_{ms}\beta}$ decreased
3. $(R_o)_f = \dfrac{R_o}{1+R_{ms}\beta}$ decreased.

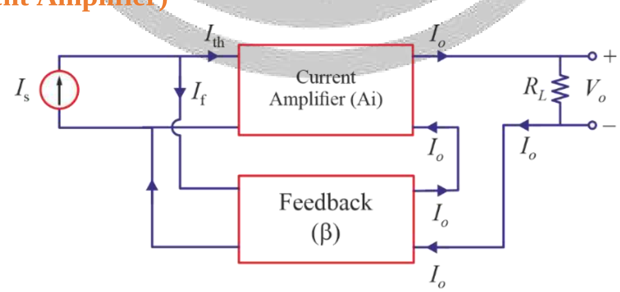
3. Series-Series Feedback Current Series Feedback (Trans-Conductance Amplifier)



Observations:

1. $\dfrac{I_o}{V_s}=\dfrac{G_{ms}}{1+G_{ms}\beta}$
2. $(R_{in})_f = R_{in}(1+G_m\beta)$
3. $(R_o)_f = R_o(1+G_m\beta)$

4. Shunt-Series Feedback : (Current Amplifier)



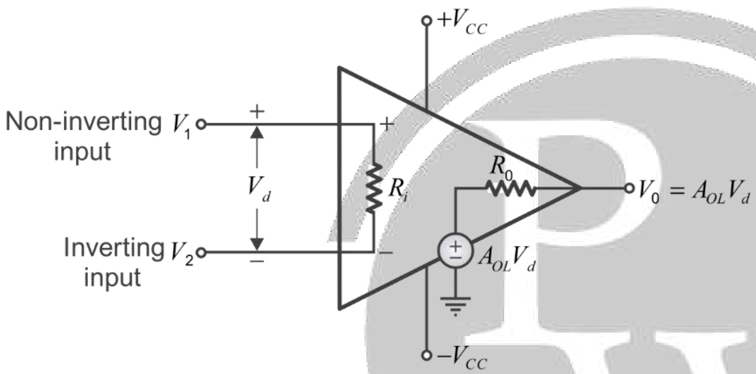
Observations:

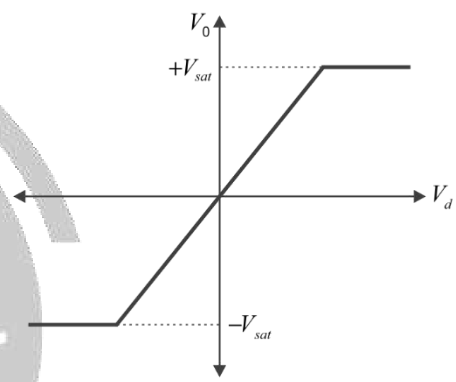
1. $\dfrac{I_o}{I_s}=\dfrac{A_I}{1+A_I\beta}$
2. $(R_{in})_f = \dfrac{R_{in}}{1+A_I\beta}$
3. $(R_0)_f = R_0[1+A_I\beta]$

7 OPERATIONAL AMPLIFIERS

7.1. Op-amp

A difference amplifier (amplifies the difference between two inputs)





where, $V_d = V_1 - V_2$

Op-amp Characteristics/Parameters

1. Open Loop Voltage Gain $\left(A = \frac{V_o}{V_i}\right)$**:** The internal gain of op-amp without any feedback.

   Ideally = ∞; Practically = very high

2. Gain Bandwidth Product (GBW)**:** Unity gain bandwidth product i.e. GBW = 1 refers to be behavior that the gain reduces at the same rate as frequency increases.

3. Input Resistance $\left(R_i = \frac{V_i}{I_i}\right)$**:** It is the internal input impedance of the op-amp.

   Ideally = ∞ ; practically = very high

4. Output Resistance $\left(R_o = \frac{V_o}{I_o}\right)$: Ideally = 0 ; practically = very small

5. CMRR (Common Mode Rejection Ratio) : Metric used to quantify the ability of the device to reject common-mode signals.

   Ideally CMRR is infinite, Practically very high.

$$\text{CMRR} = \left|\frac{A_d}{A_c}\right|$$

- CMRR in dB $= 20\log\left|\frac{A_d}{A_c}\right| = (A_d)_{\text{dB}} - (A_c)_{\text{dB}}$

6. Slew Rate :

   The maximum rate of change in output voltage per unit of time

$$SR = \left|\frac{dV_0}{dt}\right|_{\max} = \left|\frac{dV_0}{dV_i}\right|_{\max} \times \left|\frac{dV_i}{dt}\right|_{\max} = |A_{CL}| \times \left|\frac{dV_i}{dt}\right|_{\max}$$

Non-Idealities in Op-amp

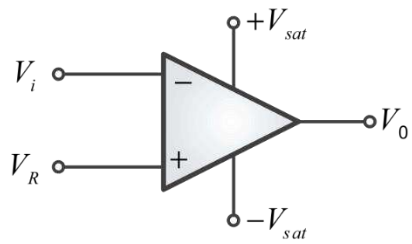
- Offset voltage (V_{OS}): If the two transistors are not perfectly matched, an offset will show up as a non-zero DC offset at the output.
- Bias current (I_{bias}): The transistor inputs actually do draw some current. The bias current is defined to be the average of the currents of the two inputs.
- Offset current (I_{OS}): The difference between the input bias currents.

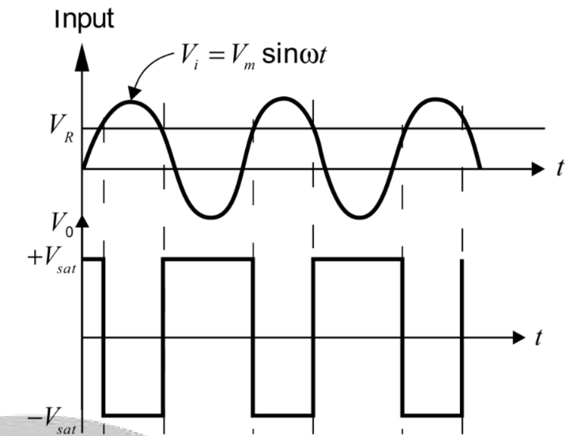
Ideal Op-amp Characteristics

Parameter	Symbol	Ideal
Open loop voltage	A	∞
Unity gain frequency	F_{unity} (GBW)	∞
Input resistance	R_{in}	∞
Output resistance	R_{out}	Zero
Input bias current	I_{bias}	Zero
Input offset current	$I_{\text{in(OS)}}$	Zero
Input offset voltage	$V_{\text{in(OS)}}$	Zero
Slew rate	S_R	∞
Common mode rejection ration	CMRR	∞

7.2. Common Op-amp Circuits

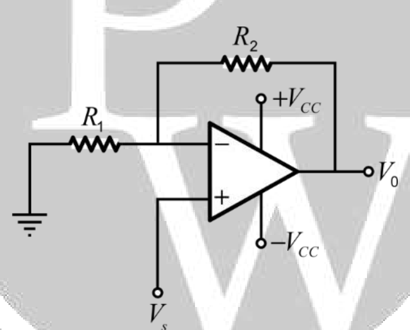
Comparator





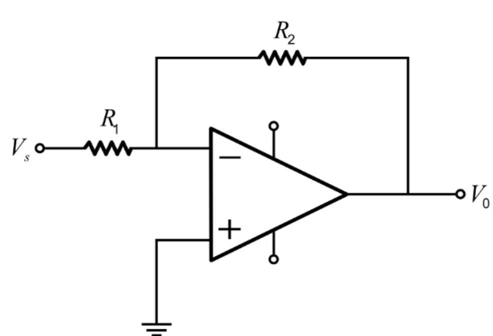
$$V_0 = V_{sat} \;;\; V_i > V_R$$
$$V_0 = -V_{sat} \;;\; V_i < V_R$$

Non Inverting Op-amp



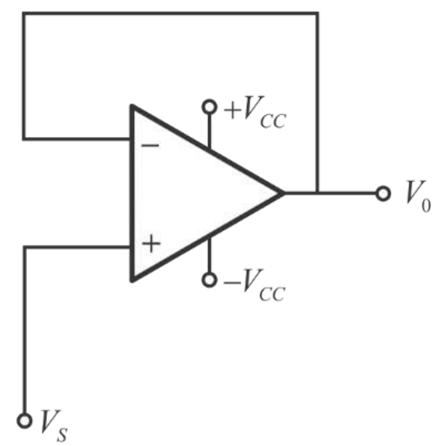
$$\frac{V_0}{V_s} = A_V = \left(1 + \frac{R_2}{R_1}\right)$$

Inverting Op-amp



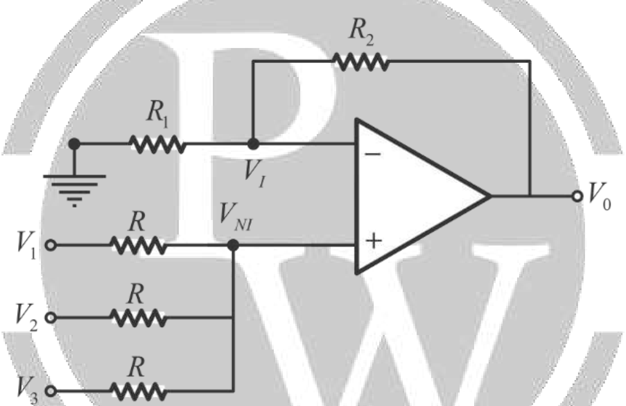
$$\frac{V_0}{V_s} = A_v = \frac{-R_2}{R_1}$$

Voltage Follower



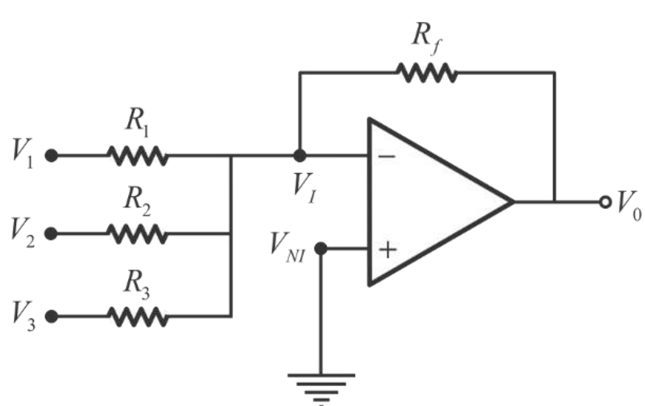
$$\boxed{V_0 = V_S}$$

Non Inverting Summing Amplifier



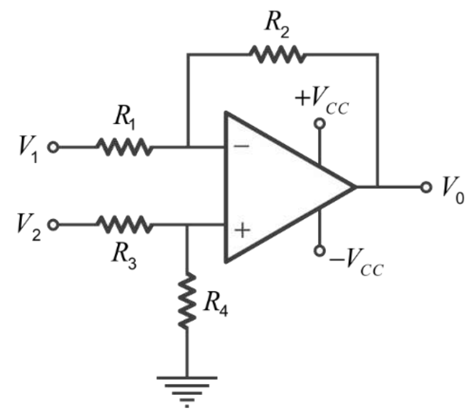
$$\boxed{V_0 = \frac{1}{3}(V_1 + V_2 + V_3)\left(1 + \frac{R_2}{R_1}\right)}$$

Inverting Summer



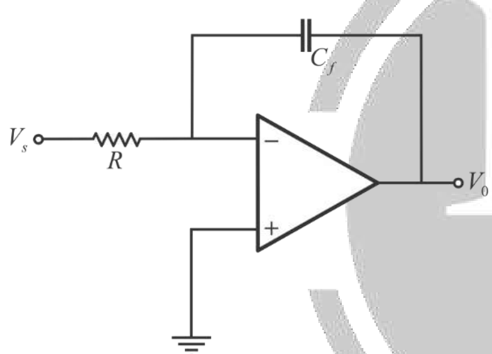
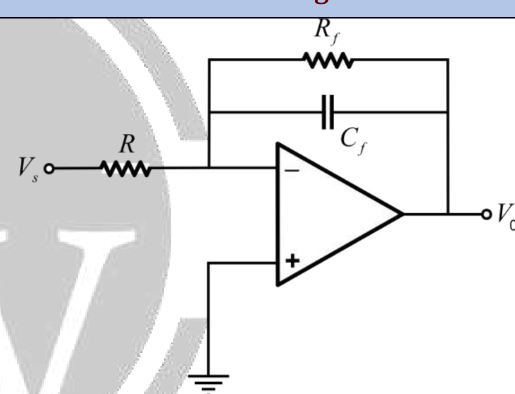
$$\boxed{V_0 = -R_f\left(\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}\right)}$$

Difference/Subtractor

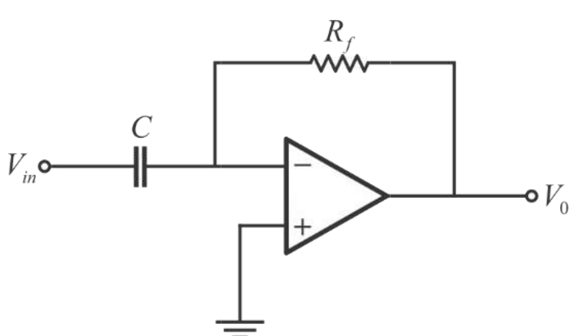
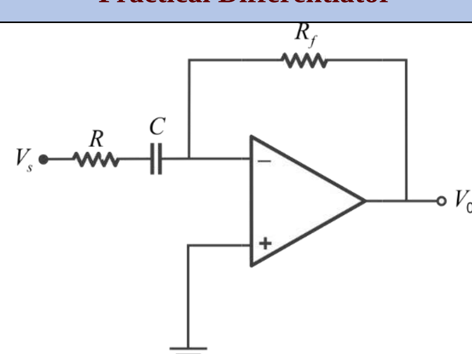


$$V_0 = V_2\left(\frac{R_4}{R_3+R_4}\right)\left(\frac{R_1+R_2}{R_1}\right) - V_1\left(\frac{R_2}{R_1}\right)$$

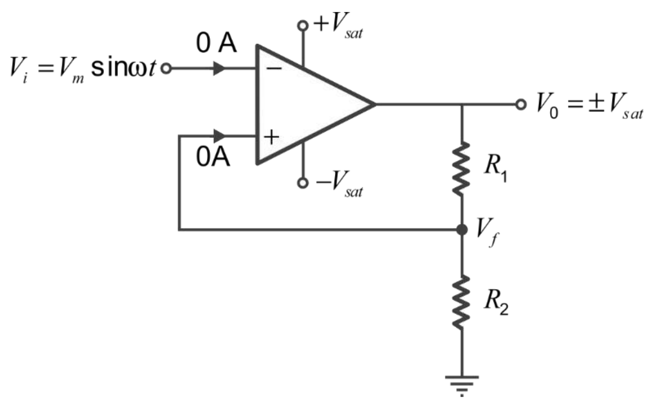
Integrator Circuit

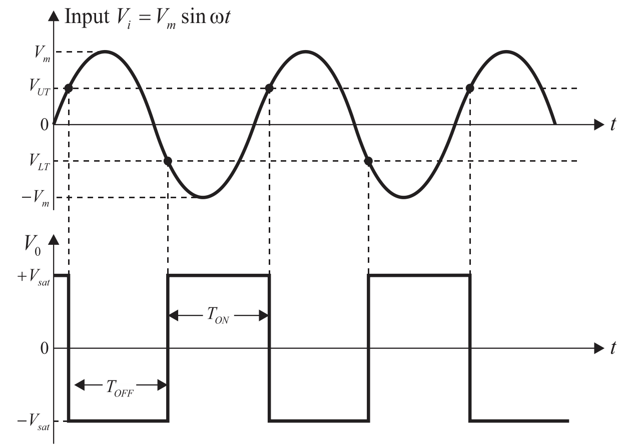
Ideal Integrator	Practical Integrator
 $V_0 = -\frac{1}{RC_f}\int V_s dt$	 $\frac{V_o}{V_S} = \frac{-R_f/R}{sR_fC_f+1}$

Differentiator Circuit

Ideal Differentiator	Practical Differentiator
 $V_0 = -R_f C\frac{d}{dt}V_{in}$	 $\frac{V_o}{V_S} = \frac{-sR_fC}{1+sRC}$

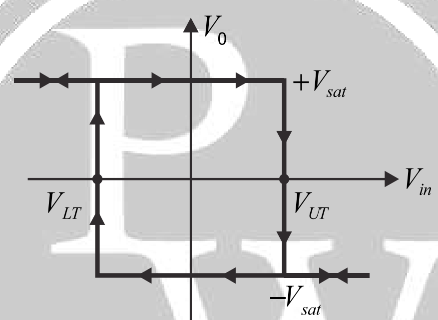
Schmitt Trigger





$$\boxed{V_{UT} = \frac{V_{sat} R_2}{R_1 + R_2},\ V_{LT} = \frac{-V_{sat} R_2}{R_1 + R_2}}$$

Hysteresis Curve:



$$\boxed{\begin{aligned} V_H &= V_{UT} - V_{LT} \\ V_H &= \frac{2V_{sat} R_2}{R_1 + R_2} \end{aligned}}$$

Communication System

COMMUNICATION SYSTEM

INDEX

1 AMPLITUDE MODULATION

1.1. Introduction

Band limiting : Bandun limited to bandlimited (LPF)

Base band signal : Message signals, low cut off *fre* = 0 Hz or very close to 0 Hz.

Bandpass signal : By shifting baseband signal to very high freq.

- Wideband signal : $\frac{f_H}{f_L} >>> 1$ (Base band signal)
- Narrowband signal : $\frac{f_H}{f_L} \approx 1$ (Bandpass signal)

Modulated Signal:

$$C(t) = A_c \cos(\omega_c t + \phi) = A_c \cos \omega_c t$$

Carrier signal

(carrier before modulation)

$S(t) = A(t)\cos[\omega_c t + \phi(t)]$ Modulated signal

Instantaneous amplitude — $A(t)$; Instantaneous frequency — $\omega_c t$; Instantaneous phase — $\phi(t)$

Amplitude Modulation:

DSB-FC (Double side band full carrier)

$C(t) = A_c \cos \omega_c t$ carrier before modulation

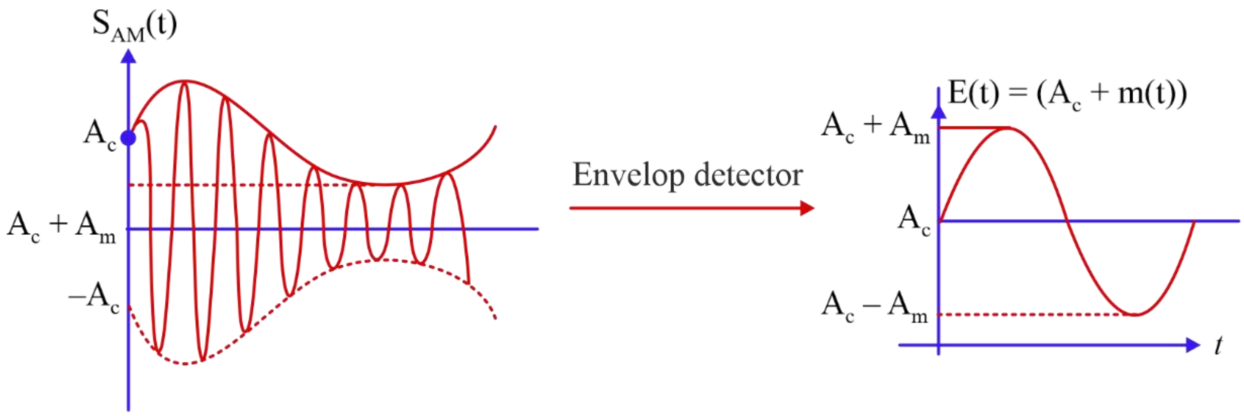
$S_{AM}(t) = A_c \cos \omega_c t + m(t) \cos \omega_c t \Rightarrow S_{AM}(t) = [A_c + m(t)] \cos \omega_c t$ carrier after modulation

Modulation Index $\mu = \frac{[m(t)]}{A_c}$ max

(1) $\mu < 1$ (under modulation)

$$\mu = \frac{A_m}{A_c} < 1$$

$$\mu = \frac{[E(t)]_{max} - [E(t)]_{min}}{[E(t)]_{max} + [E(t)]_{min}}$$



$\mu > 1,\ A(t) > 0,\ E(t) = A(t)$

Recovery through E, D possible.

$S(t)_{max} = E(t)|_{max} = A_c(1+\mu)$

$S(t)_{min} = E(t)|_{min} = A_c(1-\mu)$

(2) Critical Modulation:- $\mu = 1, A(t) \geq 0$, $E(t) = A(t)$, m(t) can be recovered with envelope detector .

(3) Over modulation: $\mu > 1, A(t) \not> 0, E(t) = |A(t)|$, not possible by E.D

Frequency Related Parameters

(1) m(t) → B.W $= f_m$; $f_{max} = f_m$

(2) $C(t) \to f_{max} = f_m$

(3) S(t) → B.W $= 2f_m$, $f_{max} = f_c + f_m$, $f_{min} = f_c - f_m$ (2 times of freq. of m(t))

➢ $P_{AM} = P_C + P_{SB}$

$$P_{USB} = P_{LSB} = \frac{P_m}{4}$$

Modulation efficiency

$$\eta = \frac{P_{SB}}{P_{AM}} = \frac{P_m/2}{P_c + \frac{P_m}{2}}$$

Share of sideband power in total power

➢ k_a [Amplitude sensitivity of amplitude modulator]

$$k_a = \frac{1}{A_c} \text{ (per volt),}$$

$A(t) = A_c[1 + k_a m(t)]$

A (t) > 0, E. D. Applicable

DSB – FC
- $\rightarrow [A_c + m(t)]\cos 2\pi f_c t \rightarrow \mu = \dfrac{|m(t)|_{max}}{A_c}$
- $\rightarrow A_c[1 + k_a m(t)]\cos 2\pi f_c t \rightarrow \mu = k_a\,|m(t)|_{max}$

For single tone sinusoidal signal

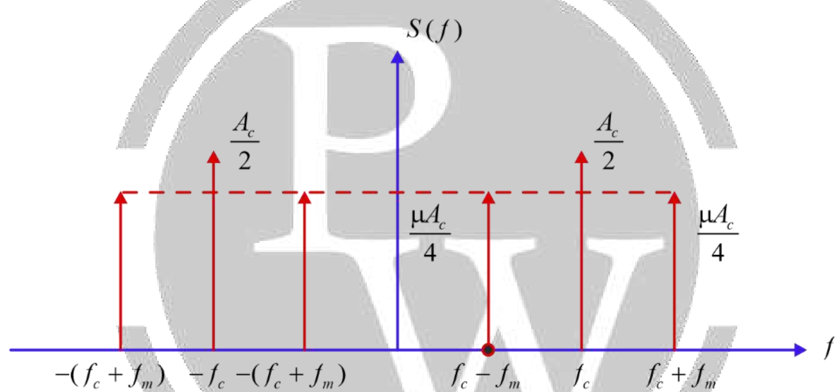
$$f_{max} = f_m, BW = 0\text{Hz}, P_m = \frac{A_m^{\ 2}}{2} \longrightarrow \text{for message signal}$$

$$\mu = \frac{A_m}{A_c}$$

$$S_{AM}(t) = [A_c + A_m\cos 2\pi f_m t]\cos 2\pi f_c t$$

$$= A_c[1 + \mu\cos 2\pi f_m t]\cos 2\pi f_c t$$

$$S_{AM}(t) = \underset{\substack{\downarrow \\ carrier}}{A_c\cos 2\pi f_c t} + \frac{\mu A_c}{2}\cos[2\pi(\underset{\substack{\downarrow \\ USB}}{f_c + f_m})t] + \frac{\mu A_c}{2}\cos[2\pi(\underset{\substack{\downarrow \\ LSB}}{f_c - f_m})t]$$



$$P_{AM} = P_c + \frac{P_m}{2} = \frac{A_c^2}{2}\left[1 + \frac{\mu^2}{2}\right] = \underset{\substack{\downarrow \\ P_c}}{P_c} + \frac{P_c\mu^2}{2} \rightarrow P_{SB}$$

$$\eta = \frac{\dfrac{P_c\mu^2}{2}}{P_c + \dfrac{P_c\mu^2}{2}} \Rightarrow \%\eta = \frac{\mu^2}{2 + \mu^2} \times 100\%$$

- If k_a given $\longrightarrow \mu = k_a\,|m(t)|_{max}$

  If k_a not given $\longrightarrow \mu = \dfrac{|m(t)|_{max}}{A_c}$

Important Points:

$\mu = 0$	$\mu = 1$	% Change
$P_{AM} = P_c$	$P_{AM} = 1.5P_c$	50 %
$\eta = 0$	$\eta = \dfrac{1}{3} = 33.33\ \%$	0 % to 33.33 %

(2) $\mu \uparrow \rightarrow \eta \uparrow$

(3) $P_{AM} \rightarrow$ Will be constant if $P_c \uparrow$ and $\mu \downarrow$

If $m(t)$ is multiple single tone signal-

$$S(t) = A_c\left[1 + \frac{A_{m_1}}{A_c}\cos 2\pi f_{m_1}t + \frac{A_{m_2}}{A_c}\cos 2\pi f_{m_2}t\right]\cos 2\pi f_c t$$

$$\mu_1 = \frac{A_{m_1}}{A_c}, \mu_2 = \frac{A_{m_2}}{A_c} - \mu_1 > \mu_2$$

$$m(t) \rightarrow f_{m_1}, f_{m_2} \rightarrow f_{\max} = f_{m_2}$$

$f_{m_2} > f_{m_1}$ $\quad$ $BW = f_{m_2} - f_{m_1}$

$S(t) = f_c$ $\quad$ BW = 2 × Max. Freq. component of $m(t)$

$f_c \pm f_{m_1}$ → $f_c + f_{m_1}$ USB$_1$

$f_c \pm f_{m_1}$ → $f_c - f_{m_1}$ LSB$_1$

$f_c \pm f_{m_2}$ → $f_c + f_{m_2}$ USB$_2$

$f_c \pm f_{m_2}$ → $f_c - f_{m_2}$ LSB$_2$

Power Related Parameters

$$P_m = \frac{A_{m_1}^2}{2} + \frac{A_{m_2}^2}{2}$$

$$P_{AM} = P_c + \frac{P_m}{2} = \frac{A_c^2}{2}\left[1 + \frac{\mu_1^2}{2} + \frac{\mu_2^2}{2}\right]$$

$$P_{USB_1} = P_{LSB_1} = \frac{P_c\mu_1^2}{2}, P_{USB_2} = P_{LSB_2} = \frac{P_c\mu_2^2}{2}$$

$$P_{USB_1} = P_{LSB_1} = \frac{P_c\mu_1^2}{4}, P_{USB_2} = P_{LSB_2} = \frac{P_c\mu_2^2}{4}$$

$$\eta = \frac{P_{SB}}{P_{AM}} = \frac{\mu_T^2}{2 + \mu_T^2}$$

$$\mu_T = \mu_1^2 + \mu_2^2 + \mu_3^2 + ----$$

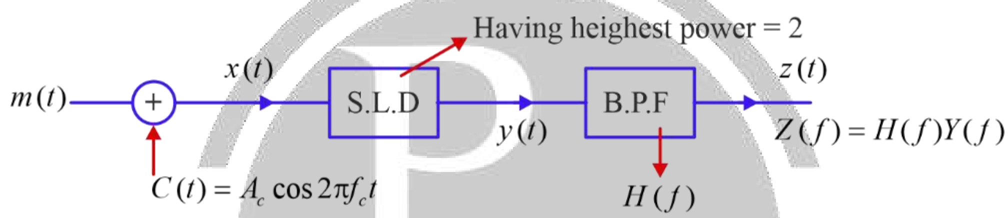
Important Points:

$m(t)$ (volt)		P_{AM}	P_{rod}
(1)	Sinusoidal	$P_c\left(1+\frac{\mu^2}{2}\right)$	$\frac{P_c}{R}\left(1+\frac{\mu^2}{2}\right)$
(2)	Square wave	$P_c(1+\mu^2)$	$\frac{P_c}{R}(1+\mu^2)$
(3)	Triangular wave	$P_c\left(1+\frac{\mu^2}{3}\right)$	$\frac{P_c}{R}\left(1+\frac{\mu^2}{3}\right)$

$$V_{AM} = V_c\sqrt{1+\frac{\mu^2}{2}},\ I_{AM} = I_c\sqrt{1+\frac{\mu^2}{2}} \text{ for sinusoidal}$$

DSB- FC [AM] Modulator

(1) Square law Modulator:



$$y(t) = a_0 m(t) + a_0 A_c \cos 2\pi f_c t + a_1 m^2(t) + \frac{a_1 A_c^2}{2} + \frac{a_1 A_c^2}{2}\cos 4\pi f_c t + 2m(t)A_c \cos 2\pi f_c t$$

(1) (2) (3) (4) (5) (6)

- Only (2) and (6) are desirable

$$Z(t) = a_0 A_c \cos 2\pi f_c t\left[1+\frac{2a_1}{a_0}m(t)\right] \text{ DSB –FC}$$

$$Z(t) = A_c'[1+k_a m(t)]\cos 2\pi f_c t \text{ only when } f_c >>> 3 f_m\ f_c >>> (2+1) f_m$$

$$A_c' = a_0 A_c,\ k_a = \frac{2a_1}{a_0},\ \mu = k_a\,|m(t)|_{\max}$$

(2) Switching Modulator:

$$Z(t) = \frac{A_c}{2}\left[1+\frac{4}{\pi A_c}m(t)\cos 2\pi f_c t\right] DSB-FC$$

$$Z(t) = A_c'[1+k_a m(t)]\cos 2\pi f_c t$$

$$A_c' = \frac{A_c}{2},\ k_a = \frac{4}{\pi A_c} \qquad \mu = k_a\,|m(t)|_{\max}$$

DSB- FC Demodulator-

(1) Square law demodulator-

$$Y(t) = a_1 A_c A_m \cos 2\pi f_m t + \frac{a_1 A_m^2}{4} + \frac{a_1 A_m^2}{4} \cos 4\pi f_m t$$

$$Y(t) = B_0 + B_1 \cos\omega_0 t + B_2 \cos 2\omega_0 t$$

- 2nd harmonic distortion $D_2 = \left|\frac{B_2}{B_1}\right| = \frac{\mu}{4}$

  $(D_2)_{\max}\% = 25\%$

- Practically not used
- $\left(\frac{S}{I}\right)_{\min} = \frac{2}{\mu}$

Envelope Detector:

$$\sqrt{A^2 + B^2}\cos\omega_c t \rightarrow \boxed{E.D} \rightarrow \sqrt{A^2 + B^2}$$

(1) $x(t) = A\cos\omega_0 t + B\sin\omega_0 t \rightarrow E(t) = \sqrt{A^2 + B^2}$

(2) $x(t) = A\cos(\omega_0 t + \theta) + B\sin\omega_0 t \rightarrow E(t) = \sqrt{A^2 + B^2 - 2AB\sin\theta}$

(3) $x(t) = A(t)\cos\omega_c t \rightarrow E(t) = |A(t)|$

(4) $x(t) = (A_c + m(t))\cos\omega_c t \rightarrow E(t) = |A_c + m(t)|$

Important Points:

- Used only when $\mu \le 1$
- $T_c = R_S C <<, \frac{1}{f_c}$ (charging time constant)
- $T_c = R_S C >>> \frac{1}{f_c} \rightarrow$ Peaks are not detected.
- Diagonal clipping $\rightarrow R_L C = \frac{1}{f_m}$
- To avoid diagonal clipping $R_L C <<< \frac{1}{f_m}, R_L C \le \frac{\sqrt{1-\mu^2}}{\omega_m \mu}$
- $T_a = R_L C \approx \frac{1}{f_c}$ fluctuation is output
- To remove fluctuation $R_L C >>> \frac{1}{f_c}$
- Proper choice of discharging time constant $R_L C$ -

No fluctuation $\frac{1}{f_c} <<< R_L C <<< \frac{1}{f_m}$ No diagonal clipping

(7) m(t): Multitone $f_m \to f_{\max} =$ Max. freq. component of $m(t)$

1.2. Synchronous Detector

$\Delta\omega$	$\Delta\phi$	$\Delta\phi$	Recovery
$= 0$	$\neq 0$	$\pm(2n+1)\frac{\pi}{2}$	✓
$= 0$	$\neq 0$	$=(2n+1)\frac{\pi}{2}$	Q.N.E
$\neq 0$	$= 0$	$= 0$	×
$= 0$	$= 0$	$= 0$	✓

DSB-SC :

$$S_{DSB-SC}(t) = m(t)A_c(\cos 2\pi f_c t) \quad E(t) = |A(t)| = |A_c m(t)|$$

- $B.W = 2 \times$ max. freq. component of $m(t)$
- $P_{DSB} = P_m P_c = P_{SB} \to P_{USB} = P_{LSB} = \frac{P_{SB}}{2} + \frac{P_m P_c}{2}$

$$P_{DSB} = \frac{P_c \mu^2}{2} = P_{SB}$$

- Single tone modulation.

$$S(t) = \frac{A_c A_m}{2}\cos[2\pi(f_c + f_m)t] + \frac{A_c A_m}{2}\cos[2\pi(f_c - f_m)t]$$

$$P_{DSB} = P_m P_c = \frac{A_c^2 A_m^2}{4}$$

Multiline $P_{DSB} = P_c P_m = \frac{A_c^2}{2}\left[\frac{A_{m_1}^2}{2} + \frac{A_{m_2}^2}{2}\right]$

Square wave - $P_{DSB} = P_c P_m = \left(\frac{A_c^2}{2}\right) A_m^2$

Triangular wave $P_{DSB} = P_c P_m = \left(\frac{A_c^2}{2}\right)\left(\frac{A_m^2}{3}\right)a$

Saw-toothed wave- $P_{DSB} = P_c P_m = \left(\frac{A_c^2}{2}\right)\left(\frac{A_m^2}{3}\right)$

(1) Balanced Modulator- $S_{DSB}(t) = 2A_c k_a m(t)\cos f_c t$

(2) Ring Modulator- $y(t) \propto m(t)\cos\omega_c t$

$$\Delta\omega = 0,\ \Delta\phi \neq 0,\ y(t) = 0 \quad \text{QNE}$$

$$\Delta\omega \neq 0,\ \Delta\phi = 0,\ y(t) = \frac{A_c A_C'}{2} m(t)\cos(\Delta\omega t) \rightarrow \text{distorted } m(t)$$

$$\Delta\omega = 0,\ \Delta\phi = 0,\ y(t) = \frac{A_c A_C'}{2} m(t) \rightarrow \text{Attenuated}$$

Hilbert Transformation.

$$h(t) = \frac{1}{\pi t},$$

$$m(t) \longrightarrow \underset{h(t)}{\boxed{H.T.}} \longrightarrow m_h(t)$$

$$mh(t) = m(t) * \frac{1}{\pi t}$$

$$H(\omega) = -j\,\text{sgn}(\omega) = \begin{cases} -j & \omega > 0 \\ 0 & \omega = 0 \\ j & \omega < 0 \end{cases}$$

- $M_h(f) = M(f)[-j\,\text{sgn}(f)]$
- $H.T[\cos\omega(t)] = \sin\omega(t) \xrightarrow{H.T} -\cos\omega(t)$
- Non causal LTI system.
- $x(t) \xleftrightarrow{H.T.} x_h(t)$

  $x_h(t) \xleftrightarrow{H.T.} -x(t)$
- Magnitude spectrum of x(t) and $x_h(t)$ will be same
- If x(t): Band limited then $x_h(t)$ is also bandlimited.
- If x(t) is non periodic then $x_h(t)$ is also non periodic
- x(t) and $x_h(t)$ are orthogonal signal.

Drawback of DSB-SC

- 2 sideband Txed.
- If receiver is designed in such a way that it may recover the complete message signal from single SB then DSB-SC S/S becomes impractical.

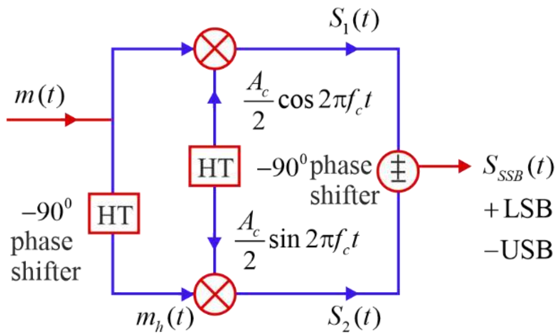
SSB- SC (Single sideband suppressed carrier)

(1) Point to point communication

(2) Two methods of generation
- Phase derserimination
- Frequency discrimination

(a) Phase Discrimination:

$$S(t)_{SSB} = \frac{A_c m(t)}{2}\cos 2\pi f_c t \pm \frac{A_c m_h(t)}{2}\sin 2\pi f_c t \quad \begin{matrix} + \Rightarrow LSB \\ - \Rightarrow USB \end{matrix}$$

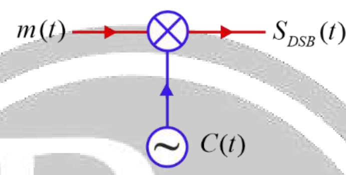


Problem Solving

(1) Identify the phase descrimination setup

(2) Phase descrimination setup

(3) Phase descrimination setup:

$P =$ 

$+ \Rightarrow S_{DSB}(t) \to LSB$

$- \Rightarrow S_{DSB}(t) \to USB$

Spectral gap in D.S.B	BPF	Signal
0 Hz	Ideal	SSB-SC
0 Hz	Practical	VSB-SC
$\neq$ 0Hz	Ideal	SSB -SC
$\neq$ 0Hz	Practical	depends on practical BPF $\to V_{SB}-SC$ / $SSB-SC$

➢ SSB- SC can be demodulated by Synchronous detection.

(1) $\Delta\omega = 0, \Delta\phi \neq 0, m(t)$ recovery not possible $\to$ freq. synchronization

(2) $\Delta\omega \neq 0 \;\; \Delta\phi = 0, m(t)$ recovery not possible $\to$ Phase synchronization

(3) Perfect syne, $\Delta\phi = 0, \Delta\omega = 0$ can be recovered

(4) $\Delta\omega = 0, \Delta\phi = \frac{\pi}{2} \to$ No QNE

Note:

(1) When video signal is transmitted through SSB- SC modular VSB- SC is generated.

(2) Synchronous detector can not recover $m(t)$ video signal from the above generated VSB- SC.

Percentage Power Saved

(1) % power saved in DSB- SC as compare to DSB-FC.

$$\%\ P_{saved} = \frac{P_{saved}}{P_{Total}} \times 100\%$$

$$\%\ P_{saved} = \frac{P_c}{P_c\left[1+\frac{\mu^2}{2}\right]} = \frac{2}{2+\mu^2} = (1-\eta)$$

(2) % power saved in SSB- SC as compare to DSB-FC-

$$\%\ P_{saved} = \frac{4+\mu^2}{4+2\mu^2}$$

(3) % power saved in SSB-SC as compared to DSB-SC.

$$\%\ P_{saved} = 50\%$$

Modulation		B.W	Power	Application
(1)	DSB-FC	$2f_{max}$	$P_C + P_{SB}$	Broadcatting
(2)	DSB- SC	$2f_{max}$	P_{SB}	$\times$
(3)	SSB-SC	f_{max}	$\frac{P_{SB}}{2}$	Point to point voice communication
(4)	VSB-SC	$f_{max} < f < 2f_{max}$	$\frac{P_{SB}}{2} < P_{VSB} < P_{SB}$	Point to point video communication.

Pre envelope and Complex Envelope

(1) Pre Envelope calculated for both baseband and bandpass signal.

Let $x(t)$ is real signal.

$$x_+(t) = \text{Pre envelope of } x(t)$$

$$x_+(t) = x(t) + j\hat{x}(t)$$

$$\hat{x}(t) = HT\,[x(t)]$$

$$x_+(f) = x(f)[1+\text{sgn}(f)]$$

Complex Envelope: For bandpass only but result in low pass only

$x(t) \to$ Bandpass signal.

Step-1. Calculate $x_+(t) = x(t) + j\hat{x}(t)$

Step 2. $\dfrac{x_c(t) = x_+(t)e^{-j\omega_c t}}{X_c(f) = X_+(f+f_c)}$ left shift of pre envelope by f_c

2 ANGLE MODULATION

2.1. Introduction

	Signal = *x*(*t*)	**|*x* (t)|$_{max}$**
(1)	$A\cos\omega_0 t + B\cos\omega_0 t$	$\lvert A+B\rvert$
(2)	$A\sin\omega_0 t + B\sin\omega_0 t$	$\lvert A+B\rvert$
(3)	$A\sin\omega_0 t + B\cos\omega_0 t$	$\sqrt{A^2+B^2}$
(4)	$A\cos\omega_1 t + B\cos\omega_2 t$	$\lvert A+B\rvert$
(5)	$A\sin\omega_1 t + B\sin\omega_2 t$	$\lvert A+B\rvert$
(6)	$A\cos\omega_1 t + B\sin\omega_2 t$	$\lvert A+B\rvert$ if $A=B$ $<\lvert A+B\rvert$ if $A\neq B$

2.1.1. Instantaneous Angle and Instantaneous frequency-

$$S(t) = A_c\cos[\theta_i(t)]$$

$\theta_i(t) \rightarrow$ Instantaneous angle (rad)

$$\frac{d\theta_i(t)}{dt} = \omega_i(t) \rightarrow \text{instantaneous angular frequency.}$$

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt} \text{ or } f_i(t) = \frac{\omega_i(t)}{2\pi}$$

$$\theta_i(t) = \int_{-\infty}^{t} \omega_i(t)$$

- **Angle Modulation :**
  - Frequency Modulation
  - Phase Modulation
- **Frequency Modulation :**

  $S_{angle}(t) = A_c\cos[\omega_c t + \Delta\phi(t)]$

If angle Modulation is FM, $\frac{d\Delta\phi(t)}{dt} \propto m(t)$

$$\frac{d\Delta\phi(t)}{dt} = K_f m(t),\ K_f = \text{ frequency sensitivity of frequency modulator}$$

$$\omega_i(t) = \omega_c + K_f m(t) \quad \Rightarrow \quad \omega_i(t) = \omega_c + \underbrace{\Delta\omega(t)}_{\substack{\text{frequency}\\ \text{deviation}}}$$

$$\theta_i(t) = \theta_c + \int_{-\infty}^{t} K_f m(t)\, dt \qquad \Delta\omega(t) = K_f m(t)$$

$$\Delta\omega(t) = \frac{d\Delta\phi(t)}{dt}$$

Few Important Results

For Important Results	For $K_f : \frac{\text{rad}}{\text{V-sec}}$	$K_f : \frac{\text{Hz}}{\text{Volt}}$
1. Instantaneous frequency	$\omega_i(t) = \omega_c t + K_f m(t)$	$f_i(t) = f_c + K_f m(t)$
2. Instantaneous frequency deviation	$\Delta\omega(t) = K_f m(t)$	$\Delta f(t) = K_f m(t)$ Hz
3. Frequency deviation in +ve direction	$[\Delta\omega(t)]_{\max} = K_f [m(t)]_{\max}$	$[\Delta f]_{\max} = K_f [m(t)]_{\max}$
4. Frequency deviation in –ve direction	$[\Delta\omega(t)]_{\min} = K_f [m(t)]_{\min}$	$[\Delta f(t)]_{\min} = K_f [m(t)]_{\min}$
5. Maximum value of instantaneous frequency	$[\omega_i(t)]_{\max} = \omega_c + [\Delta\omega(t)]_{\max}$	$[f_i(t)]_{\max} = f_c + [\Delta f(t)]_{\max}$
6. Minimum value of instantaneous frequency	$[\omega_i(t)]_{\min} = \omega_c + [\Delta\omega(t)]_{\min}$	$[f_i(t)]_{\min} = f_c + [\Delta f(t)]_{\min}$
7. Peak to peak frequency deviation	$[\Delta\omega]_{p-p} = [\omega_i(t)]_{\max} - [\omega_i(t)]_{\min}$	$[\Delta f]_{p-p} = [f(t)]_{\max} - [f(t)]_{\min}$
8. Maximum frequency deviation		
9. Modulation index or deviation ratio of FM	$\lvert\Delta\omega(t)\rvert_{\max} = K_f [m(t)]_{\max}$	$\lvert\Delta f(t)\rvert_{\max} = K_f [m(t)]_{\max}$
$B_{FM} = \frac{\text{Maximum frequency deviation}}{\text{Maximum frequency component of } m(t)}$	$B_{FM} = \frac{K_f \lvert m(t)\rvert_{\max}}{\omega_{\max}}$	$B_{FM} = \frac{K_f \lvert m(t)\rvert_{\max}}{f_{\max}}$

Important Phase Calculation	$K_f \left(\frac{\text{rad}}{\text{V-sec}}\right)$	$K_f : \frac{\text{Hz}}{\text{Volt}}$
1. Instantaneous phase deviation in FM	$\Delta\phi(t) = K_f \int_{-\infty}^{t} m(\tau)\, d\tau$	$\Delta\phi(t) = 2\pi K_f \int_{-\infty}^{t} m(\tau)\, d\tau$
2. Maximum phase deviation in FM	$\lvert\Delta\phi(t)\rvert_{\max} = K_f \left\lvert \int_{-\infty}^{t} m(\tau)\, d\tau \right\rvert$	$2\pi K_f \left\lvert \int_{-\infty}^{t} m(\tau)\, d\tau \right\rvert_{\max}$

General expression for FM

$$K_f : \frac{\text{rad}}{\text{V-sec}}$$

$$S_{angle}(t) = A_c \cos\left[\omega_c t + \int_{-\infty}^{t} K_f m(\tau)\, d\tau\right]$$

For $K_f : \frac{\text{Hz}}{\text{Volt}}$

$$S_{FM} = A_c \cos\left[\omega_c t + 2\pi K_f \int_{-\infty}^{t} m(\tau)\, d\tau\right]$$

For $m(t) = A_m \cos 2\pi f_m t$ –

$$f_{\max} = f_m, [m(t)]_{\max} = +A_m,\ [m(t)]_{\min} = -A_m,\ |m(t)|_{\max} = A_m$$

$$S_{FM}(t) = A_c \cos[\omega_c(t) + B_{FM} \sin(2\pi f_m t)]$$

For $m(t) = A_{m_1} \cos 2\pi f_{m_1} t + A_{m_2} \cos 2\pi f_{m_2} t$ –

$$f_{\max} = (f_{m_1}, f_{m_2})_{\max}$$

$$S_{FM}(t) = A_c \cos\left[\omega_c(t) + B_1 \sin 2\pi f_{m_1} t + B_2 \sin 2\pi f_{m_2} t\right]$$

$$B_1 = \frac{K_f A_{m_1}}{f_{m_1}}, B_2 = \frac{K_f A_{m_2}}{f_{m_2}}$$

Phase Modulation –

$$\Delta\phi(t) \propto m(t)$$

$$\underset{\text{rad}}{\underset{\downarrow}{\Delta\phi(t)}} = K_p \underset{\text{Volt}}{\underset{\downarrow}{m(t)}}$$

K_p : Phase sensitivity of phase modulator

$$K_p = \frac{\text{rad}}{\text{Volt}}$$

Phase Calculation :

K_p : rad/Volt

$$\theta_i(t) = \omega_c t + K_p m(t)$$

1. Instantaneous phase deviation $= \Delta(t) = K_p m(t)$
2. Maximum phase deviation $= |\Delta\phi(t)|_{\max} = K_p |m(t)|_{\max}$

Frequency Calculation

$$\omega_i(t) = \omega_c + \Delta\omega t$$

- $$\Delta\omega(t) = K_p \frac{dm(t)}{dt}$$
- $$|\Delta\omega(t)|_{\max} = K_p \left|\frac{dm(t)}{dt}\right|_{\max}$$
- $$|\Delta\omega(t)|_{\min} = K_p \left|\frac{dm(t)}{dt}\right|_{\min}$$
- $$|\Delta\omega_i(t)|_{\max} = \omega_c + [\Delta\omega(t)]_{\max}$$
- $$|\Delta\omega_i(t)|_{\min} = \omega_c + [\Delta\omega(t)]_{\min}$$
- $$\Delta\omega_{p-p} = [\omega_i(t)]_{\max} - [\omega_i(t)]_{\min}$$
- $$|\Delta\omega(t)|_{\max} = K_p \left|\frac{dm(t)}{dt}\right|_{\max}$$
- $$\beta_{FM} = \frac{|\Delta\omega(t)|_{\max}}{\omega_{\max}} = \frac{K_p \left|\frac{dm(t)}{dt}\right|_{\max}}{\omega_{\max}}$$

$$S_{FM}(t) = A_c \cos\left[\omega_c(t) + K_{PM}(t)\right]$$

When $m(t) = A_m \cos 2\pi f_m t$

$$|m(t)|_{\max} = A_m,\ f_{\max} = f_m,\ \Delta\omega(t) = -K_p A_m \omega_m \sin\omega_m t$$

- $$[\Delta\omega(t)]_{\max} = K_p A_m \omega_m$$
- $$\{\omega(t)\}_{\min} = -K_p A_m \omega_m$$
- $$[\omega_i(t)]_{\max} = \omega_c + K_p A_m \omega_m,\ [\omega_i(t)]_{\min} = \omega_c - K_p A_m \omega_m$$
- $$(\Delta\omega)_{p-p} = 2K_p A_m \omega_m$$
- $$|\Delta\omega(t)|_{\max} = K_p A_m \omega_m$$
- $$\beta = K_p A_m = |\Delta\phi(t)|_{\max}$$

$$S_{PM}(t) = A_c \cos\left[\omega_c(t) + \beta_{PM} \cos 2\pi f_m t\right]$$

$$S_{PM}(t) = A_c \cos\left[\omega_c(t) + \beta_1 \cos 2\pi f_{m_1} t + \beta_2 \cos 2\pi f_{m_2} t\right]$$

Types of FM –

- Narrow Band $(\beta <<< 1)$
- Wide Band

$$S_{FM}(t) = \underset{\text{Carrier}}{A_c \cos 2\pi f_c(t)} + \underset{\text{USB}}{\frac{A_c\beta}{2}\cos[2\pi(f_c + f_m)t]} - \underset{\text{LSB}}{\frac{A_c\beta}{2}\cos[2\pi(f_c - f_m)t]}$$

$$S_{FM}(t) = S_{NBFM}(t)$$

- B.W $= 2f_m$
- $P_{NBFM} = P_C\left(1+\frac{\beta^2}{2}\right) \quad \beta <<< 1, \beta^2 <<< 1$

$$P_{NBFM} \approx P_C = \frac{A_c^2}{2}$$

Relation between DSB-FC and NBFM –

$$S_{AM}(t) = A_c \cos 2\pi f_c(t) + \frac{A_c\mu}{2}\cos[2\pi(f_c + f_m)t] + \frac{A_c\mu}{2}\cos[2\pi(f_c - f_m)t]$$

$$S_{NBFM}(t) = A_c \cos 2\pi f_c(t) + \frac{A_c\beta}{2}\cos[2\pi(f_c + f_m)t] - \frac{A_c\beta}{2}\cos[2\pi(f_c - f_m)t]$$

1.

Frequency Component	Strength AM	Strength NBFM
f_c	$\frac{A_c}{2}$	$\frac{A_c}{2}$
$f_c + f_m$	$\frac{\mu A_c}{4}$	$\frac{\beta A_c}{4}$
$f_c - f_m$	$\frac{\mu A_c}{4}$	$\frac{-\beta A_c}{4}$

2. $S_{NBFM}(t) + S_{AM}(t) = \text{SSB-SC} \rightarrow \text{USB-FC}$

   $S_{AM}(t) - S_{NBFM}(t) = \text{SSB-SC} \rightarrow \text{LSB-FC}$

3. LSB in NBFM is 180^0 inverted w.r.t to LSB in AM

- $S_{FM}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\cos[2\pi(f_c + nf_m)t]$

  For aby value of β

- $J_n(\beta) = (-1)^n J_n(\beta)$

- $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$
- $J_0(\beta) = 0, \beta = 2.4, 5.5, 8.6, 11.8$
- as $n \uparrow, \rightarrow J_n(\beta) \downarrow$

  $\beta <<< 1: S(t) \rightarrow$ 1 Carrier + 2 SB   NB Angle Modulation

  If $\beta >>> 1: S(t):$1 Carrier + Infinite SB   Wide Band Angle Modulation

  Ideal BW of WBFM $= \infty$

Carson's Rule –

$$BW = (\beta+1)2f_m \quad \beta_{PM} \text{ for PM}$$

β_{FM} for FM

- Power of Carrier before modulation $= \dfrac{A_c^2}{2} = P_c$
- Power of Carrier after modulation $P = P_c\left[J_0^2\beta + 2(J_1^2(\beta) + J_2^2(\beta) + \ldots\right]$
- Power of Carrier component in modulated signal $= P_c J_0^2(\beta)$
- $P_{SB} = 2P_c\left[J_1^2(\beta) + J_2^2(\beta) + \ldots\right]$
- $\eta = \dfrac{P_{SB}}{P_{Total} \rightarrow P_c\left[J_0^2(\beta) + 2(J_1^2(\beta) + J_2^2(\beta) + \ldots)\right]}$

  If $J_0(\beta) = 0$ then $\eta = 100\%$
- For Infinite sidebands $P_{WB} = P_c$

For Non sinusoidal –

$$S_{FM} = A_c \sum_{n=-\infty}^{\infty} |C_n| \cos\left[2\pi(f_c + nf_m)t + \angle C_n\right]$$

$m(t)$		BW
Singletone sinusoidal	$\longrightarrow$	$(\beta+1)2f_m$
Non sinusoidal periodic signal	$\longrightarrow$	$(\beta+1)2f_m$, f_m = fundamental frequency
Other Cases	$\longrightarrow$	$(\beta+1)2f_{\max}$

$\mathrm{BW} = (1+\beta)2f_{\mathrm{m}}$ or $2(\Delta f + f_{\max})$

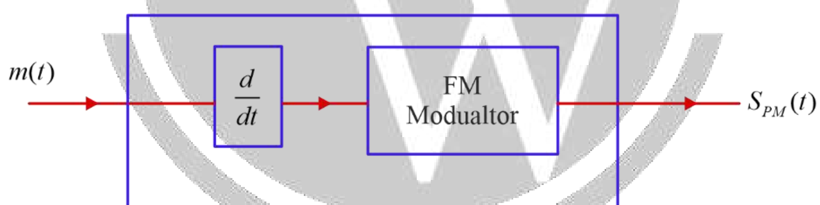
Frequency Mixture and Multiplier

Mixture/Multiplier Input	Mixture Output	(Multiplied by *n*) Multiplier Output
A_c	$A_c^{'}$	$A_c^{'}$
f_c	$\lvert f_C - f_L \rvert$ or $f_C + f_L = f_C^{'}$	nf_c
β	β	$n\beta$
f_m	f_m	f_m
Δf	Δf	$n\Delta f$
BW	BW	$(n\beta+1)2f_m$
Spactral spacing	f_m	f_m
Frequency components	$f_c^{'}, f_c^{'} \pm f_m, f_c^{'} \pm 2f_m$	$nf_c, nf_c \pm f_m, nf_c \pm 2f_m$

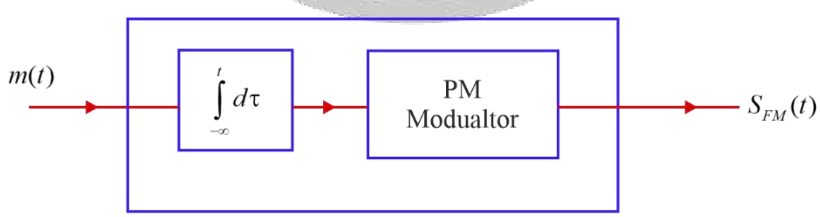
Wideband Angle Modulation generation –

$$PM[m(t)] = FM\left[\frac{dm(t)}{dt}\right] \text{ If } K_p = K_f = K$$

$$FM[m(t)] = PM\left[\int_{-\infty}^{t} m(\tau)\,d\tau\right]$$



Phase Modualtor



Frequency Modualtor

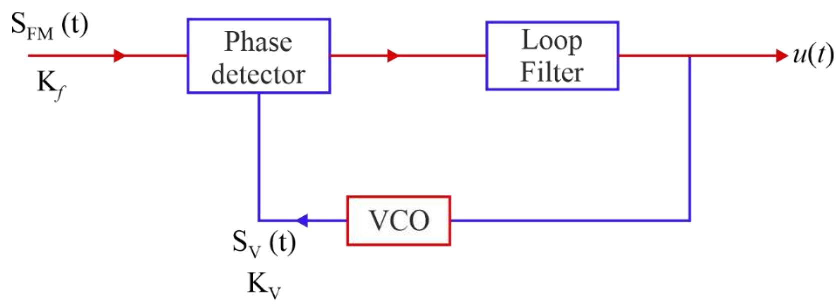
Wideband FM Generation Methods

1. Armstrong Method (Indirect Method)
2. Direct Method
   - VCO (Voltage Controlled Oscillator is used). It is modified version of Hartley oscillator

$$\frac{\Delta\omega}{\omega_c} = \frac{\Delta C}{2C_0}$$

FM Demodulator

1. Theoretical method
2. Practical method. PLL (Phase Locked Loop)



(1) $v(t) = \dfrac{K_f}{K_V} m(t)$

(2) Lock mode $\longrightarrow$ Frequency lock

Capture mode $\longrightarrow$ Phase lock

(3) $\text{L.R} \geq \text{C.R}$

Super Hetrodyne Receiver

f_L = Local oscillator frequency

f_S = Desired frequency

f_{Si} = Frequency of image station

Case 1 : If relation between f_l and f_s is not mentioned.

Assume : $f_l > f_s$

1. $f_l = f_s + IF$
2. $f_{Si} = f_l + IF$
3. $f_{Si} = f_s + 2IF$

Case 2 : When relation between f_i and f_s is given

If $f_{Si} < f_l < f_s$ If $f_S < f_s < f_{sI}$, then Case 1

then
1. $f_s = f_l + IF$
2. $f_l = f_{Si} + IF$
3. $f_s = f_{Si} + 2IF$

Image Rejection Ratio

$$\text{IRR} = \sqrt{1 + P^2Q^2}$$

Q : Quality factor of Oscillator

$$P = \frac{f_{Si}^2 - f_s^2}{f_{Si} f_s} \qquad f_{Si} > f_s \qquad P^2Q^2 >>> 1$$

$$\text{IRR} = PQ$$

3 RANDOM VARIABLE AND RANDOM PROCESS

3.1. Introduction

Random variable → Real and complex

- R.V. is a function performing mapping from sample space of R.E. to real line.
- $X(\lambda)$: Random variable
- Domain of R.V. $\rightarrow \lambda$ (Sample point)
- Range of R.V. $\rightarrow$ Subset of real line
- One to one or many to one mapping
- $P\{X \le a\} \rightarrow$ Probability of set in which all the comes satisfy $x(\lambda) \le a$.

CDF of R.V.

Let random variable X, $x \rightarrow$ Values taken by R.V.

(1) $F_X(x) = P\{X \le x\} = 1 - P\{X > x\}$

(2) $F_X(a) = P\{X \le a\} = 1 - P\{X > a\}$

(3) $F_{|X|}(y) = P\{|X| \le y\} = P\{-y \le X \le y\}$

Properties

(1) $F_X(\infty) = 1$

(2) $F_X(-\infty) = 0$

(3) $F_X(\infty) + F_X(-\infty) = 1$

(4) $F_X(x) = P\{X \le x\} \Rightarrow 0 \le F_X(x) \le 1$

(a) CDF always non negative.

(b) Lower bound: $F_X(x) = 0$, upper Bound = 1

(5) CDF is monotonically non decreasing function of $x\left(\dfrac{dF_X(x)}{dx} \ge 0\right)$

(6) Graph of CDF is always amplitudes continuous from right.

- **Key point :**

(1) $P\{a < x \le b\} = F_X(b^+) - F_X(a^+)$

(2) $P\{a \le X \le b\} = F_X(b^+) - F_X(a^-)$

(3) $P\{a < X < b\} = F_X(b^-) - F_X(a^+)$

(4) $P\{a \le X < b\} = F_X(b^-) - F_X(a^-)$

$P(x = a)$

CDF is continuous at $x = a$

$P\{x = a\} = F_X(a^+) - F_X(a^-)$

$P\{x = a\} = 0$

CDF is continuousat $x = a$

$P\{x = a\} = F_X(a^+) - f_X(a^-)$

$\neq 0$

$P\{x = a\} =$ Size of Jump

Probability Density Function

Random variable X

$x \rightarrow$ Variable taken by R.V.

$f_X(x) \rightarrow$ Symbol

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$$f_X(x) = \int_{-\infty}^{x} f_X(x)\,dX$$

$$f_X(x) = \int_{-\infty}^{a} f_X(\lambda)\,d\lambda$$

Properties :

(1) $f_X(x) \ge 0 \rightarrow$ Non negative

(2) $0 \le f_x(x) < \infty \longrightarrow$ Upper bound

$0 \downarrow$ Lower bound

(3) $F_X(\infty) = \int_{-\infty}^{\infty} f_X(x)\,dx = 1$

(4) Graph of PDF can be even or NENO but cannot be odd.

(5) $P\{-\infty < X \le x\} = \int_{-\infty}^{x} f_X(\lambda)\,d\lambda$

(6) $P\{a < X \le b\} = \int_{a^+}^{b^+} f_X(x)\,dx$

(7) $P\{a \le X \le b\} = \int_{a^-}^{b^+} f_X(x)\,dx$

(8) $P\{a < X < b\} = \int_{a^+}^{b^-} f_X(x)\,dx$

$P(x = a)$

Graph having impulse at $x = a$

$P\{x = 0\} = F_X(a^+) - F_X(a^-)$

$P\{x = a\} = \int_{a^-}^{a^+} f_Y(x)\,dx$

= Total area

No impulse at $x = a$

$P\{x = a\} = \int_{a^-}^{a^+} f_Y(x)\,dx$

$P\{x = a\} = 0$

Discrete Random Variable:

(1) PDF should have impulses only.

(2) CDF should have staircase only.

(1) **Probability mass function of DRV :** Let X is D.R.V.

$P_X(x) = P(X = x)$ probability such that $X = x$

- $0 \le P_X(x) = 1$
- $\sum_x P_X(x) = 1$

(2) **PDF of a D.R.V :** Let X is O.R.V.

$$f_X(x) = \sum_i P_X(x_i)\delta(x - x_i) = \sum_i P(x = x_i)\delta(x - x_i)$$

(3) **CDF of a D.R.V. :** Let X is D.R.V.

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$F_X(x) = \sum_x P\{X = x_i\}u\{x - x_i\}$$

(4) $P\{X = a\}$ may or may not be zero.

Continuous Random Variable

Maps sample point to continuous range of values on real axis.

(1) PDF of C.R.V should not contain impulses at all.

(2) CDF of C.R.V

- Should not contain jump type discontinuity
- It should be amplitude continuous every where

(3) PMF not defined for C.R.V because for CRV $P\{X = a\}$ will always be zero.

$$P(A/B)=\frac{P(A\cap B)}{P(B)} \rightarrow \text{Conditional probability of A given B.}$$

$$P(A\cap B)=P(B)P(A/B)=P(A)P\left(\frac{B}{A}\right)=\text{Joint probability.}$$

Expectation operator : Performs operations on R.V. only.

Linear Operator

$$E[C]=C,\ E[C^2]=C^2 \qquad E(X)=\begin{cases}\int_{-\infty}^{\infty} xf_X(x)\,dx & X:CRV\\ \sum_i x_i P\{X=x_i\} & X:DRV\end{cases}$$

$$E[aX]=aE[X]$$

$$E[aX+b]=aE[X]+E[b]$$

$$E[ag(X)+bH(y))=aE[g(x)]+bE[H(y)]$$

$$E[X^2]=\int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$E[g(x)]=\int_{-\infty}^{\infty} g(x)\,f_X(x)\,dx$$

Gaussian Random Variable

CRV X is having Gaussian or random distribution.

X is having Gaussian PDF, X is called G.R.V.

$$E[X]=\mu_X,\ E[(X-\mu_x)^2]=\text{Variance }=\sigma_X^2$$

$$X\sim N\{\mu_X,\sigma_X^2\} \qquad f_X(x)=\frac{1}{\sqrt{2\pi\sigma_X^2}}e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}} \qquad -\infty<x<\infty$$

Key Point :

(1) $$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma_X^2}}e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}}dx=1$$

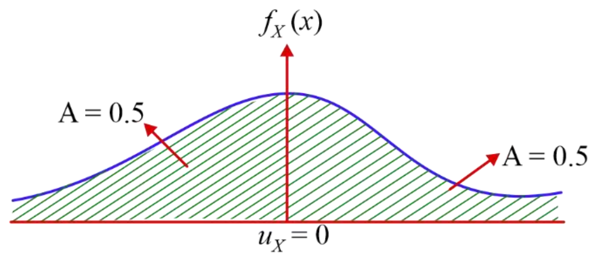
(2) $$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma_X^2}}e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}}dx=\mu_X=E[X]$$

(3) $$\int_{\mu_X}^{\infty}\frac{1}{\sqrt{2\pi\sigma_X^2}}e\frac{-(x-\mu_X)^2}{2\sigma_X^2}dx=\int_{-\infty}^{\mu_X}\frac{1}{\sqrt{2\pi\sigma_X^2}}e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}}dx=\frac{1}{2}$$

Zero mean Gaussian distribution-

$$X \neg N(\mu_X, \sigma_X^2) \Rightarrow X \sim N[0, \sigma_X^2] \Rightarrow E[X] = 0$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{\frac{-x^2}{2\sigma_X^2}}$$



Zero Mean, unit variance :

$$X \sim N(0.1) \quad f_y(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, \quad \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = \frac{1}{2}$$

Q- function :

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-z^2/2} dz \qquad \text{as } x\uparrow, Q(x)\downarrow$$

$$Q(\infty) = 0,\ Q(-\infty) = 1,\ Q(0) = 0.5,\ Q(x) + Q(-x) = 1$$

$$P[X > z] = Q(P) = Q\left[\frac{z - \mu_X}{\sigma_X}\right]$$

$$f_X(z) = P(X \le z) = 1 - P(X > z) = 1 - Q\left[\frac{z - \mu_X}{\sigma_X}\right]$$

Statistical averages of a R.V.

n^{th} order moment about origin-

$$E[(X-0)^n] = E[X^n] = \begin{cases} \int_{-\infty}^{\infty} x^n f_X(x)\,dx & X : CRV \\ \sum_i x_i^n P\{X = x_i\} & X : DRV \end{cases}$$

1st order moment about origin

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \sum_i x_i P\{X = x_i\}$$

$$E[X] = \bar{X} = \mu_X = m_i \rightarrow dc \text{ value, avg. value Mean value}$$

$[E[X]]^2 \rightarrow d.c.$ power

2^{nd} order moment about origin-

$$E[(X-0)^2] = E[X^2] = \overline{X}^2 \begin{cases} \int_{-\infty}^{\infty} x^2 f_X(x) & X : CRV \\ \sum_i x_i P\{X = x_i\} & X : DRV \end{cases}$$

$E[X^2] =$ Mean square value of R.V. X = Total power of R.V. x

1^{st} order moment about mean - $E[(X-\mu_X)] = 0$

2^{nd} order moment about mean $-E[(X-\mu_X)^2] = E[X^2] - \mu_X^2$

$$\underset{\substack{\downarrow \\ \text{A.C.} \\ \text{Power}}}{\sigma_X^2} = \underset{\substack{\downarrow \\ \text{Total} \\ \text{Power}}}{E[X^2]} - \underset{\substack{\downarrow \\ \text{dc} \\ \text{Power}}}{u_X^2}$$

Important point:

(1) $\sigma_X^2 \geq 0,\ E[X^2] \geq \mu_X^2$

(2) If X is zero mean R.V.

$$E[X^2] = \sigma_X^2,\ \text{MSV}(X) = Var(X)$$

(3) Standard deviation

$$\sqrt{\text{Variance}} = \sqrt{\sigma_X^2} = \pm\sigma_X$$

(4) $Y = aX + b$

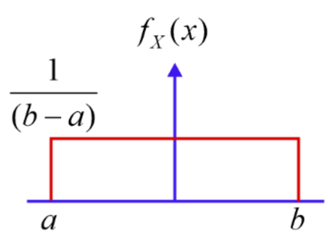
$$E[Y^2] = a^2 E[X^2] + b^2 + 2ab\,E[X]$$

$$\sigma_Y^2 = a^2 \sigma_X^2$$

Standard Distribution of R.V.

(1) Uniform distribution $X \sim U[a,b]$

$$f_X(x) = \begin{cases} \dfrac{1}{(b-a)} & a \leq X \leq b \\ 0 & \text{otherwise} \end{cases}$$

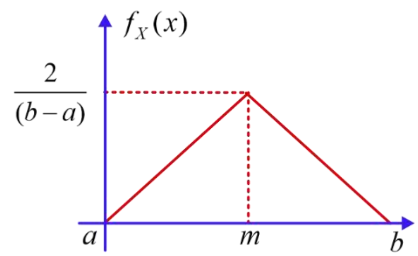


$$E[X]=\frac{a+b}{2},\ E[X^2]=\frac{a^2+b^2+ab}{3},\quad \sigma_X^2=\frac{(b-a)^2}{12}$$

(2) Triangular distribution

$X \sim$ tri (a,m,b)

$$E[X]=\frac{a+m+b}{3}$$



(3) Rayleigh Distribution

$X \to CRV$

$$f_X(x)=\begin{cases}\frac{x}{\sigma_X^2}e^{\frac{-x^2}{2\sigma_X^2}} & x\ge 0\\ 0 & \text{else}\end{cases}$$

$$\int_0^\infty \frac{x}{\sigma_X^2}e^{\frac{-x^2}{2\sigma_X^2}}dx=1$$

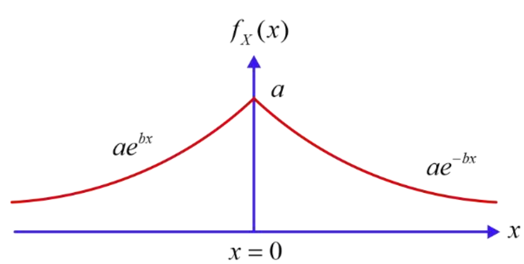
If X and Y are two G.R.V. Then $Z=\sqrt{X^2+Y^2}$ will have reyleigh distribution.

(4) Exponential Distribution : If CRV has exponential distribution then it will have PDF

$$f_X(x)=\begin{cases}\lambda e^{-\lambda x} & x\ge 0\\ 0 & x>0\end{cases}\qquad \int_0^\infty \lambda e^{-\lambda x}dx=1$$

Laplacian Distribution

$X \to CRV$



$$f_X(x)=ae^{-b|x|}\quad -\infty<x<\infty$$

If $\frac{2a}{b}=1,\ \ a>0, b>0$

$$f_X(x) = \begin{cases} ae^{bx} & x<0 \\ ae^{-bx} & x>0 \end{cases}$$

Discrete Random variable-Binomial, Position distribution

Binomial distribution necessary condition:-

(1) The no of trials n showed be finite.

(2) Trials are independent

(3) Each trials should result in 2 outcomes success or failure.

(4) Prob of success in each trial should be constant.

PMF: $$P\{X = r \text{ success}\} = n_{c_r} p_r q^{n-r}$$

$$E[X] = \sum_i x_i p\{X = x_i\} = n_p \quad \sigma_X^2 = npq$$

$$E[X^2] = npq + (np)^2$$

Std deviation $\sigma_X = \pm\sqrt{npq}$

Position Distribution

Specific type of binomial distribution where $n \to \infty$

$n \to$ very large, $p \to$ very small, $np \to$ finite $\lambda = np$

$$p\{X = r\} = \frac{\lambda^r e - \lambda}{r!} \text{ probability of } X = r \text{ (success)}$$

$$E[X] = \lambda, \sigma_X^2 = \lambda$$

Monotonic Transformation

Linear | Non-Linear

If $Y = g(X)$ is having monotonic T_X.

Given $X \xrightarrow{PDF} f_X(x)$,

$$f_Y[y] = \left\{ f_X[x] \left|\frac{dx}{dy}\right| \right\} \text{function of y}$$

$Y = aX + b \to$ X:CDF $f_X(x)$

$Y = aX + b \to$ X:PDF $f_X(x)$

(1) case $a > 0$

$$F_Y[y] = F_X\left(\frac{y-b}{a}\right), f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

(2) $Y = -aX + b \quad a > 0$

$$F_Y(y) = 1 - F_X\left(\frac{y-b}{-a}\right),\ f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{-a}\right)$$

Monotonic linear T_X:

$$y = aX + b$$

$$X \sim U[m_1, m_2] \rightarrow Y \sim U[am_1 + b, am_2 + b]$$

$$X \sim \Delta[m_1, m_2, m_3] \rightarrow Y \sim \Delta[am_1 + b, am_2 + b, am_3 + b]$$

$$X \sim N[\mu_X, \sigma_X^2] \rightarrow Y \sim N[\mu_Y, \sigma_y^2]$$

$$Y \sim N[a\mu_X + b, a^2\sigma_X^2]$$

Monotonic Non-Linear T_X:

$$X \rightarrow f_X(x)$$

$$Y \rightarrow X^3,\ f_Y(y) = ?$$

$$f_Y(y) = \left\{ f_X(x)\left|\frac{dx}{dy}\right| \right\}$$

$$f_Y(y) = \frac{1}{3y^{2/3}} f_X(y^{1/3})$$

Non - Monotonic T_X:

$$Y = y, g(X) = y, X = g^{-1}(y)$$

$$\begin{cases} \rightarrow x_1 \\ \rightarrow x_2 \\ \rightarrow x_3 \end{cases}$$

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right| + \text{----}$$

2D Random variable :

$$(X, Y) \rightarrow 2\,\text{DR.V.} \begin{cases} \rightarrow F_{X,Y}(x, y) = \text{Joint CDF} \\ \rightarrow f_{X,Y}(x, y) = \text{Joint PDF} \\ \rightarrow P_{XY}(x_i, y_i) = \text{Joint PMF} \end{cases}$$

If A and B are independent

$$P\left(\frac{A}{B}\right) = P(A),\ \ P\left(\frac{B}{A}\right) = P(B),\ \ P(A \cap B) = P(A)P(B)$$

Marginal CDF of Y

$$F_{X,Y}(x,y) = F_X(x)F_{\frac{Y}{X}}\left(\frac{y}{x}\right) = F_Y(y)F_{\frac{X}{Y}}\left(\frac{x}{y}\right)$$

Conditional CDF of X given Y

Marginal CDF of X

Conditional CDF of Y given X

If X and Y are independent R.V.

$$F_{XY}(x,y) = f_X(x)f_Y(y)$$

- $$f_{XY}(x,y) = f_X(x)f_{\frac{Y}{X}}\left(\frac{y}{x}\right) = f_Y(y)f_{\frac{X}{Y}}\left(\frac{x}{y}\right)$$

If X and Y are independent R.V.

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

- $$P_{XY}(x_i,y_j) = P_X(x_i)P_{\frac{Y}{X}}\left(\frac{y_j}{x_i}\right) = P_Y(y_j)P_{\frac{X}{Y}}\left(\frac{x_i}{y_j}\right)$$

If X and Y are independent R.V.

$$P_{XY}(x_i,y_j) = P_X(x_i)P_Y(y_j)$$

Joint CDF = Let (X,Y) are BIVARIATE R.V.

$$F_{XY}(x,y) = P\{X \le x) \cap (Y \le y) = P\{X \le x; Y \le y\}$$

Properties:

(1) $0 \le F_{XY}(x,y) \le 1$

(2) $F_{XY}(-\infty, y) = P\{(X \le -\infty) \cap (Y \le y)\} = 0$

(3) $F_{XY}(x, -\infty) = 0$

(4) $F_{XY}(-\infty, -\infty) = 0$

(5) $F_{XY}(\infty, \infty) = 1$

(6) $F_{XY}(x_1, y_1) = P\{(X \le x_1) \cap (Y \le y_1)\}$

(7) $P\{(x_1 < X \le x_2) \cap (y_1 < Y \le y_2)\}$

$$= F_{XY}(x_1^+ . y_1^+) + F_{XY}(x_2^+, y_2^+) - F_{XY}(x_1^+, y_2^+) - F_{XY}(x_2^+, y_1^+)$$

(8) $F_{XY}(x,y) = F_X(x)F_{\frac{Y}{X}}\left(\frac{y}{x}\right) = F_Y(y)F_{\frac{X}{Y}}\left(\frac{x}{y}\right)$

(9) X and Y are independent R.V.

$$F_{XY}(x,y) = F_X(x)F_Y(y)$$

(10) $F_X(x,y) = F_{XY}(x,\infty)$, $\quad F_Y(y) = F_{XY}(\infty, y)$

Conditional CDF

$$F_{\frac{X}{Y}}\left(\frac{x}{y}\right)=\frac{F_{XY}(x,y)}{F_Y(y)} \quad \text{of} \quad F_Y(y)\neq 0$$

$$F_{\frac{X}{Y}}\left(\frac{x}{y}\right)=\frac{P[(X\le x)\cap(Y\le y)]}{P[(X\le\infty)\cap(Y\le y)]}$$

Joint PDF

$$f_{XY}(x,y)=\frac{\partial^2 F_{XY}(x,y)}{\partial X\,\partial Y}$$

$$F_{XY}(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)\,du\,dv$$

$$F_{XY}(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy=1$$

Marginal PDF

(1) $f_X(x)=\int_{-\infty}^{\infty} f_{XY}(x,y)dy,\ \ f_Y(y)=\int_{-\infty}^{\infty} f_{XY}(x,y)dx$

If X and Y are independent $f_{XY}(x,y)=f_X(x)\,f_Y(y)$

$$f_{XY}(x,y)=f_X(x)\,f_{\frac{Y}{X}}\left(\frac{y}{x}\right)=f_Y(y)\,f_{\frac{X}{Y}}\left(\frac{x}{y}\right)$$

Conditional PDF

$$f_{XY}(x,y)=\frac{f_{XY}(x,y)}{f_Y(y)}=\frac{\int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y)dxdy}{\int_{-\infty}^{\infty} f_{XY}(x,y)dx}$$

Probability Calculation in 2-D region

Given Joint PDF

R_2
Region in which PDF if defined

$$f_{XY}(x,y)=\begin{cases}\text{define} & [(x_1<X\le x_2)\cap(y_1<Y\le y_2)]\\ 0 & \text{else}\end{cases}$$

$P\{(a<X\le b)\cap(c<y\le d)\}=?$

R_1 : Region in which probability has to be calculated.

Method:

$$P(X,Y\in R_1)=\iint_R f_{XY}(x,y)dxdy \qquad (R=R_1\cap R_2)$$

(1) X and Y not independent R.V.

$$P(X,Y\in R_1)=\iint_{R_r} f_{X,Y}(x)f_Y(Y)dxdy \qquad R=(R_1\cap R_2)$$

(Central Limit Theorem)

If X and Y are D.R.V

$$\sum_i\sum_j P_{XY}(x_i,y_j)=1$$

$$P_{XY}(x_i,y_j)=P\{(X=x_i)\cap(Y=y_j)\}$$

Joint PMF

Marginal PMF :

$$P_X(x_i)=\sum_j P_{XY}(x_i,y_j)$$

$$P_Y(y_j)=\sum_i P_{XY}(x_i,y_j)$$

Minimum of 2 independent R.V.

X, Y are two I.R.V

$$\min(X,Y)>Z=(X>Z)\cap(Y>Z)$$

$$P[\min(X,Y)>Z]=P[X>Z]P[Y>Z]=\iint_R f_{XY}(x,y)dxdy$$

$$P[\min(X,Y)\le Z]=1-P[\min(X,Y)>Z]$$

$$\underbrace{P[\min(X,Y)>Z]}_{R}=\iint_R f_X(x)f_Y(y)dxdy$$

Let $Z=\text{Max }(X,Y)\to\text{R.V.}$

CDF of Z $\quad F_Z(Z)=F_X(Z)\cdot F_Y(Z)$

PDF of Z $\quad f_Z(Z)=F_X(Z)f_Y(Z)+F_Y(Z)f_X(Z)$

Let $Z=\min[X,Y]\to\text{R.V.}$

CDF of Z $\quad F_Z(Z)=f_X(Z)+g_Y(Z)+f_Y(Z)F_X(Z)$

PDF of Z $\quad f_Z(Z)=f_X(Z)+f_Y(Z)-F_X(Z)f_Y(Z)-F_Y(Z)f_X(Z)$

Statistical parameters of 2D R.V.

(1) $(k,r)^{th}$ order joint moment about origin $E[X^kY^r]$ $(1,1)^{st}$ order joint moment about origin.

$E[X^1,Y^1]=E[XY]=R_{XY}\to$ Cross correlation between R.V. X and V.

- $E[XY]=R_{XY}=0 \rightarrow$ R.V. X and Y are orthogonal.

(2) $(k,r)^{th}$ order joint moment about mean-

$E[(X-\bar{X})^k(Y-\bar{Y})^r]$

$(1,1)^{st}$ order joint Moment about mean-

$E[(X-\bar{X})(Y-\bar{Y})]=E[XY]-\bar{X}\bar{Y}=\text{cov}(X,Y)$

$\text{cov}(X,Y)=\sigma_{XY}=E[XY]-E[X]E[Y]=R_{XY}-\mu_x\mu_y$

When 2 R.V. X and Y are uncorrelated-

$\text{cov}(X,Y)=0$, $E[X,Y]=E[X]E[Y]$

- $E[X^kY^r]=E[X^k]E[Y^r]=X,Y$ are independent.
- If 2 R.V. are independent then they has to be uncorrelated but converse is not necessarily true.

One function of two R.V.

$$W=aX+bY$$

(1) $E[W]=aE[X]+bE[Y]$

(2) $E[W^2]=a^2E[X^2]+b^2E[Y^2]+2ab\,R_{XY}$

(3) $\sigma_W^2=a^2\sigma_X^2+b^2\sigma_Y^2+2ab\,\text{cov}(X,Y)$

One function of Three R.V. $W=aX_1+bX_2+cX_3$

(1) $E[W]=a\mu_{X_1}+b\mu_{X_2}+a\mu_{X_3}$

(2) $E[W^2]=a^2X_1^{\,2}+bX_2^2+c^2X_3^2+2abX_1X_2+2bcX_2X_3+2caX_1X_3$

(3) $\sigma_W^2=a^2\sigma_{X_1}^2+b^2\sigma_{X_2}^2+c^2\sigma_{X_3}^2+2ab\,\text{cov}(X_1,X_2)+2bc\,\text{cov}(X_2,X_3)+2ca\,\text{cov}(X_1,X_3)$

Var (X + Y) = Var (X – Y)

Only when X, Y are $\rightarrow$ uncorrelated and independent

Correlation coefficient

$$\rho(X,Y)=\frac{\sigma_{XY}}{\sigma_X\sigma_Y}=\frac{\text{cov}(X,Y)}{(\text{Std.dev.of }X)\times(\text{std.dev.of }Y)}$$

- $-1\le\rho\le1$
- $\rho(X,X)=1,\quad \rho(X,-X)=-1$
- X, Y are independent $\rho(X,Y)=0$

Let X, Y are two R.V.

$$E\left[\frac{g(Y)}{X=x}\right]=\int_{-\infty}^{\infty} g(y)\, f_{\frac{Y}{X}}\left(\frac{y}{x}\right)dy$$

$$E\left[\frac{g(X)}{Y=y}\right]=\int_{-\infty}^{\infty} g(x)\, f_{\frac{X}{Y}}\left(\frac{x}{y}\right)dx$$

Calculation of probability in n-D region

Theorem -1

If $X_1, X_2, X_3 --- X_n$ are statistically independent random variables.

Let $Z = X_1 + X_2 ---- X_n$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ f_{X_1}(z) & f_{X_2}(z) & f_{X_n}(z) \end{array}$$

$$f_Z(z) = f_{X_1}(z) * f_{X_2}(z) ---- * f_{X_n}(z)$$

When and only when all the R.V. are statistically independent.

- R.V. are linearly combined.

Theorem-2

$X_1, X_2, X_3 --- X_n$ are statically independent non Gaussian R.V.

$Z = X_1 + X_2 + X_3 ---- X_n$

$f_Z(z) = f_{X_1}(z) * f_{X_2}(z) ---- * f_{X_n}(z)$

If $n \to \infty$ $f_Z(z) =$ Gaussian irrespective of nature of $(X_i)_{i=1}^{n}$

Theorem-3

$X_1, X_2, X_3 --- X_n$ are statistically independent G.R.V.

$Z = X_1 + X_2 --- + X_n$

$f_Z(z) = f_{X_1}(z) + f_{X_2}(z) + ---- f_{X_n}(z)$

$n \to$ Finite| infinite, $\to$ Z: GRV

Problem Solving Technique :

Case 1 : $X_1, X_2 ---- X_n$ are statistically independent G.R.V.

$$P[X_1 + X_2 + X_3 > a] = \underset{\text{Non GRV}}{\underset{\downarrow}{P(Z > a)}} = \int_a^{\infty} f_Z(z)dz = 1 - \int_{-\infty}^{a} f_Z(z)dz$$

Where $Z = X_1 + X_2 + X_3$

$$f_Z(z) = f_{X_1}(z) \times f_{X_2}(z) \times f_{X_3}(z)$$

Case 2 : If X_1, X_2, X_3 are statistically independent G.R.V.

$$P(X_1 + X_2 + X_3 > a) = P[Z > a] = Q\left[\frac{a - \mu_z}{\sigma_z}\right]$$

$$Z = X_1 + X_2 + X_3$$

$$\mu_2 = \mu_{X_1} + \mu_{X_2} + \mu_{X_3}, \quad \sigma_z^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \sigma_{X_3}^2$$

> **Note :** If $X_1, X_2, X_3 ----X_n$ are I.I.D. random variables
>
> P (one of them is largest) $= \dfrac{1}{n}$
>
> P (one of them is smallest) $= \dfrac{1}{n}$

Random Process

$X(\lambda, t) = \{X(\lambda_1, t), X(\lambda_2, t)\} \longrightarrow$ Collection of sample function of Euemble of sample function

$\downarrow$ Random process or Random signal or stochastic signal

$X(\lambda_1, t_1) \rightarrow$ sample value, values taken by R.V. When R.P. is observed at $t = t_1$

C.T.R.P $\rightarrow$ It maps the sample points onto continuous time sample function, collection of continuous time sample function.

$$\underset{\text{C.T.R.P}}{X(t) = A\cos(\omega_0 t + \phi)} \xrightarrow{t=t_1} \underset{\text{R.V., D.R.V.}}{X(t_1) = A\cos(\omega_0 t_1 + \phi)}$$

t $\rightarrow$ Continuous time, A $\rightarrow$ Constant, $\omega_0 \rightarrow$ constant

$\phi \sim U[-\pi, \pi] \rightarrow$ CRV

Any typical R.P can be understood as $x(t) = f(t, \phi)$

(Function of time and R.V.) $\quad x(n) = f(n, \phi)$

Statistical parameter of R.P.

Case 1 : $X(t) \xrightarrow[t = t_0]{} X(t_0) - \text{CRV}$

$$E[X(t_0)] = \int_{-\infty}^{\infty} x\, f_{X(t_0)}(x)\, dx$$

$$E[x^2(t_0)] = \int_{-\infty}^{\infty} x^2\, f_{X(t_0)}(x)\, dx, \qquad \sigma_{X(t_0)}^2 = E[X^2(t_0)] - 9E(X(t_0))^2]$$

Case 2 : $\underset{\text{CTRP}}{X(t)} \xrightarrow[t = t_0]{} X(t_0) - \text{DRV}$

$$E[X(t_0)] = \sum_i x_i P_{X(t_0)}(x_i) = \sum_i x_i P\{X(t_0) = x_i\}$$

$$E[x^2(t_0)] = \sum_i x_i^2 P_{X(t_0)}(x_i) = \sum_i x_i^2 P\{X(t_0) = x_i\}$$

$$\sigma^2_{X(t_0)} = E[X^2(t_0)] - (E[X[t_0])^2$$

Case 3 : DTRP $\rightarrow$ CRV

$E[X(n_0)], E[x^2(n_0)], \sigma^2_{X(n_0)}$ $\rightarrow$ Some as case 1, replace to by n_0

Case 4 : DTRP $\rightarrow$ DRV

$E[X(n_0)], E[x^2(n_0)], \sigma^2_{X(n_0)}$ $\rightarrow$ Same as case-2, Replace to by n_0

CTR.P

$$X(t) \xrightarrow{t=t_1} X(t_1)$$
$$X(t) \xrightarrow{t=t_2} X(t_2)$$

$$E[X(t_1)X(t_2)] = R_{X(t_1)X(t_2)} = R_{XX}(t_1, t_2)$$

Auto correlation of RP X(t)

$$X(t) \rightarrow X(t)$$
$$X(t) \rightarrow X(t+\tau)$$

Then $E[X(t_1)X(t+\tau)] = R_{XX}(t, t+\tau)$

Cov $(X(t_1)X(t_2)) = E[X(t_1)X(t_2)] - E[X(t_1)]E[X(t_2)]]$

$\sigma_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_{X(t_1)}\mu_{X(t_2)}$

Auto covariance of R.P. $X(t)$

$\sigma_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - \mu_{X(t)}\mu_{X(t+\tau)}$

Cross Correlation

$$\underset{R.P}{X(t)} \xrightarrow[t=t_1]{} \underset{R.V}{X(t_1)}, \ \underset{R.P}{Y(t)} \xrightarrow[t=t_2]{} \underset{R.V}{Y(t_2)}$$

$$E[X(t_1)Y(t_2)] = R_{XY}(t_1, t_2)$$

(1) If $R_{XY}(t_1, t_2) = 0 \ \forall \ t_1 \in \text{TR}$ $X(t)$ and $Y(t)$ $R.P.$ will

$t_2 \in \text{TR}$ Become orthogonal.

(2) $\text{Cov}[X(t_1), Y(t_2)] = R_{XY}(t_1, t_2) - \mu_{X(t_1)}\mu_{X(t_2)}$

$= 0 \ \forall \ t_1 \in \text{TR}$
$\quad\quad t_2 \in \text{TR}$

RP $X(t)$ and $Y(t)$ are uncorrelated.

If $X(t_1)$ and $X(t_2)$ are independent

$$E[X(t_1)X(t_2)]\begin{cases} E[X(t_1)E[X(t_2)] & t_1 \neq t_2 \\ E[X^2(t_1)] & t_1 = t_2 \end{cases}$$

Same for DRV, replace t by n.

Types of R.P.

(1) Strict sense stationary R.P. $\rightarrow$ R.P. should be independent of time shift

$$X(t) \rightarrow \frac{X(t_1)X(t_2)---X(t_k)}{kR.V.}$$

K^th order Joint PDF-

$$(x_1, x_2 --- x_k) \qquad \ldots\text{(i)}$$

$$f_X(t_1)X(t_2)---X(t_k)$$

$$X(t): \quad \frac{X(t_1+\tau), X(t_2+\tau)---X(t_k+\tau)}{k \text{ R.V.}}$$

K^th order Joint PDF-

$$(x_1, x_2 --- x_k) \qquad \ldots\text{(ii)}$$

$$f_{X(t_1+\tau)X(t_2+\tau)---X(t_k+\tau)}$$

(i) = (ii) $\rightarrow X(t)$ is solid to be SSSRP.

$$f_{X(t_1)}(x) = f_{X(t_2+\tau)}(x) \quad \text{independent of time}$$

2nd order joint PDF is independent of time shift.

$$f_{X(t_1)X(t_2)}(x_1, x_2) \rightarrow f_{X(0)X(t_2-t_1)}(x_1, x_2)$$

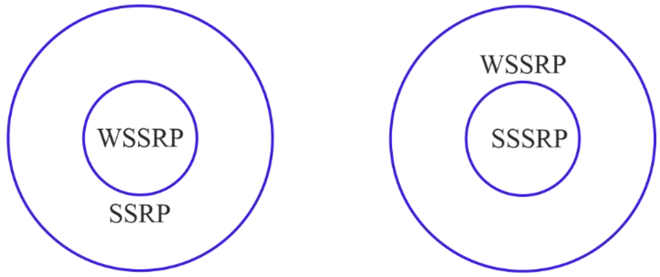
- Does not depend on individual sampling instances t_1 and t_2
- Depends on time difference between sampling instances t_1 and t_2

$$E[X(t_1)X(t_2)] = E[X(0)X(t_2-t_1) = R_{XX}(t_1, t_2)$$

$$E[X(0)X(t_2-t_1)] = R_{XX}(0, t_1-t_2) = R_{XX}(t_1 \sim t_2)$$

$$\sigma_{XX}(t_1 \sim t_2) = R_{XX}(t_1 \sim t_2) - \mu_X^2$$

WSSRP $\rightarrow$ There are stationary RP which are stationary at least upto 2nd order.



(1) ✓ (2) → Not necessarly ×

(1) $E[X(t)] = \mu_X$ Constant

(2) $E[X^2(t)] = \text{Constant}$

(3) $\sigma^2_{X(t)} = \text{Constant}$

$\text{E[RP]} = \text{E[RV]}$

$\text{MSV(RP)} = \text{MSV(RV)}$

$\text{Var(RP)} = \text{Var(RV)}$

(4) $E[X(t_1)X(t_2)] = R_{XX}(t_1 \sim t_2)$

$E[X(t+\tau)X(t)] = R_{XX}(-\tau)$

$E[X(t)X(t+\tau)] = R_{XX}(\tau)$

$R_{X_X}(\tau) = R_{XX}(-\tau)$

$\sigma^2_{X(t)} = R_{XX}(0) - \mu_X^2$

Cov $[X(t)X(t+\tau)] = R_{X_X}(\tau) - \mu_X^2$

(5) $E[X^2(t)] = R_{XX}(0) \qquad \tau = 0$

(6) $E[X(t)X(t+\tau)] = R_{XX}(\tau)\text{ACF} = \begin{cases} E[X(t)]E[X(t+\tau)] = \mu_X^2 & (\tau \neq 0) \\ E[X^2(t)] = R_{XX}(0) & (\tau = 0) \end{cases}$

(7) $\text{cov}[X(t)X(t+\tau)] = R_{XX}(t, t+\tau) - \mu_X(t)\mu_X(t+\tau)$

$$C_{XX}(\tau) = R_{XX}(\tau) - \mu_X^2 = \begin{cases} 0 & \tau \neq 0 \\ R_{XX}(0) - \mu_X^2 & \tau = 0 \end{cases}$$

$$R_{X_X}(\tau) = \begin{cases} \mu_X^2 & \tau \neq 0 \\ R_{X_X}(0) & \tau = 0 \end{cases}$$

Important point :

(1) If X(t) is zero mean WSSRP.

E [X (t) = 0

$$\sigma^2_{X(t)} = E[X^2(t)]$$

$$\text{Var }[X(t)] = \text{MSV}\{X(t)\}$$

$$\text{Var }\{X(t=t_1)\} = \text{MSV}\{X(t=t_1)\}$$

(2) If X (k) is zero mean WSSRP

$$E[X(k)] = 0 \quad \sigma^2_{X(k)} = E[X^2(k)]$$

$$\text{Var }[X(k)] = \text{MSV}[X(k)]$$

- $X(t): \text{WSSRP} + \text{IIDRP}$

$$E[X(t)] = \mu_X,\ E[X(t+\tau)] = \mu_X,\ E[X^2(t)] = R_{XX}(0) = \text{Constant}$$

$$\sigma^2_{X(t)} = \text{Constant}$$

Let $Y(t) = X(at+b)$

$$E[Y(t) = \mu_X,\ E[y^2(t)] = R_{XX}(0)$$

$$\sigma^2_{Y(t)} = R_{XX}(0) - \mu_X{}^2$$

$$E[Y(t)Y(t+\tau)] = R_{XX}(a\tau) = R_{YY}(\tau)$$

$$\text{Cov }[Y(t)Y(t+\tau)] = C_{YY}(\tau) = R_{XX}(a\tau) - \mu_X^2$$

$$\text{For } \ Y(t) \rightarrow \left[\begin{array}{l} \rightarrow \mu_Y = \mu_X = \text{Constant} \\ R_{YY}(\tau) = R_{XX}(a\tau) \end{array}\right., Y(t) \rightarrow \text{WSSRP}$$

- Time shift, Time reversal, time scaling does not affect stationary nature of R.P.

Let $Y(t) = aX(t) + b,\ X(t)$ is WSSRP

$$E[Y(t)] = a\mu_X + b = \text{Constant}$$

$$E[Y^2(t)] = a^2 R_{XX}(0) + b^2 + 2ab\mu_X = \text{Constant}$$

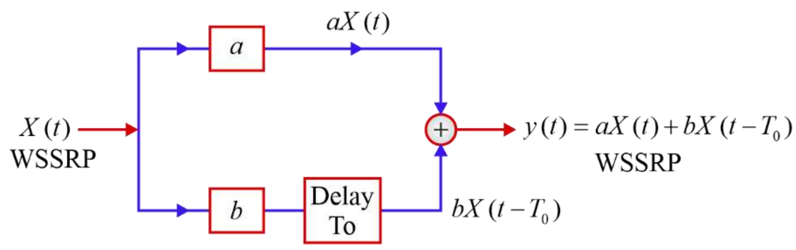
$$\sigma^2_{Y(t)} = a^2 \sigma^2_{X(t)}$$

$$\text{Cov }[Y(t)Y(t+\tau)] = R_{YY}(\tau) - \mu_Y^2$$

$$E[Y(t)Y(t+\tau)] = a^2 R_{XX}(\tau) + 2ab\mu_X + b^2 = R_{YY}(\tau)$$

$$y(t) \rightarrow \text{WSSRP}$$

- Linear transformation of WSSRP does not change its stationarity.
- If WSSRP passed through LTI system, output is also a WSSRP.



$$E[Y(t)] = (a+b)\mu_X$$

$$E[Y^2(t)] = a^2R_{XX}(0) + b^2R_{XX}(0) + 2abR_{XX}(\tau_0)$$

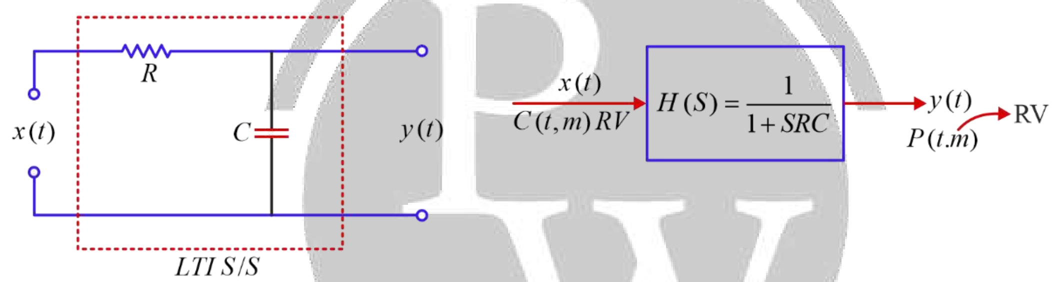
$$\sigma^2_{Y(t)} = (a^2+b^2)\sigma^2_{X(t)} + 2ab[R_{XX}(T_0) - \mu^2_X]$$

$$R_{YY}(\tau) = (a^2+b^2)R_{XX}(\tau) + abR_{XX}(\tau - T_0) + abR_{XX}(\tau + T_0)$$

$$C_{YY}(\tau) = a^2C_{XX}(\tau) + b^2C_{XX}(\tau) + abR_{XX}(\tau - T_0) + abR_{XX}(\tau + T_0) - 2ab\mu^2_X$$

$$E[X(n)X(n+k)] = \delta[k] = R_{XX}(k) \text{ IIDRP}$$

$$E[X(n)X(n+k)] = E[X^2(n))(k=0)$$



A, $\omega_0 \to$ constant, $\theta \sim U(0, 2\pi)$ OR $\theta \sim U[-\pi, \pi]$

$$X(t) = A\cos(\omega_0 t + \theta)$$

$$E[A\cos(\omega_0 t + \theta)] = 0$$

$$E[A\cos(\omega_0 t + \theta + \infty)] = 0$$

$$E[X^2(t)] = \frac{A^2}{2}, \sigma^2_{X(t)} = \frac{A^2}{2}$$

$$E[X(t)X(t+\tau)] = \frac{A^2}{2}\cos\theta\omega_0\tau = R_{XX}(\tau)$$

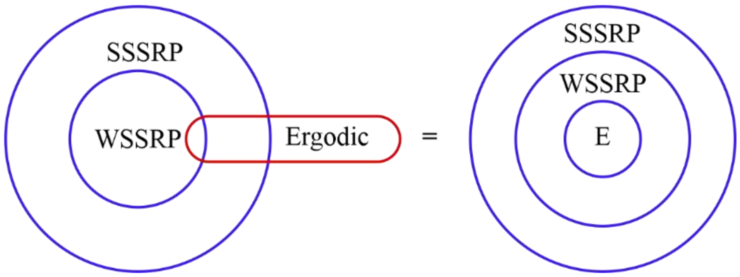
$$\text{Cov }[X(t)X(t+\tau)] = \frac{A^2}{2}\cos\omega_0\tau$$

$X(t)$: WSSRP + periodic with $\tau_0 \to R_{XX}(\tau)$ will also be periodic with same T.P.

ERGODIC Random Process :

Time Avg = Statistical Aug.

$$\frac{1}{T}\int X(t)\,dt = E[X(t)]$$



Auto Correlation and its properties

Similarity between 2 Samples

Let $X(t)$ is WSSRP, $X(t)$ is observed τ duration apart

(1) $E[X(t)X(t+\tau) = R_{XX}(\tau)$

(2) $R_{XX}(-\tau) = R_{XX}(\tau)$: Even

(3) $|R_{XX}(\tau)| \le R_{XX}(0)$

(4) $\{R_{XX}(\tau)\}_{\max} = R_{XX}(0) =$ Maximum similarity

(5) $\int_{-\infty}^{\infty} R_{XX}(\tau)d\tau = 2\int_{0}^{\infty} R_{XX}(\tau)d\tau = 2\int_{-\infty}^{0} R_{XX}(\tau)d\tau$

(6) $x(t)$
- Penodic $\rightarrow R_{xx}(\tau)$: Periodic
- Non Penodic $\rightarrow R_{xx}(\tau)$: Non Periodic

(7) $\text{E}[x^2(t)] = MSV[x(t)] = R_{XX}(0)$
- Energy of R.P. $\rightarrow$ Energy Signal $x(t)$
- Power of R.P $\rightarrow$ Power Signal $x(t)$

(8) $X(t)$ is power signal

$$R_{XX}(0) = E[X^2(t)] = \sigma^2_{X(t)} + \sigma^2_{X(t)}$$

Total power of R.P — A.C power of R.P — D.C power of R.P

(9) If $X(t)$ is ergodic and WSSRP, it has no periodic component

$E[X(t) = \mu_X \ne 0$

$\mu_X^2 = \lim_{\tau\to\infty} R_{XX}(\tau) = \lim_{|\tau|\to\infty} R_{XX}(\tau)$

If not ergodic but WSSRP then $R_{XX}(0) = E[X^2(t)]$, $R_{XX}(\infty) \ne \mu_X^2$

Important point:

$X(t) \leftrightarrow X(\omega)$ $\qquad$ $X(t) \leftrightarrow X(f)$

$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ $\qquad$ $X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}dt$ $\qquad$ $x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}dt$

$X(0) = \int_{-\infty}^{\infty} x(t)dt$ $\qquad$ $x(0) = \int_{-\infty}^{\infty} X(f)df$

$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)d\omega$ $\qquad$ $X(0) = \int_{-\infty}^{\infty} x(t)dt$

3.2. Parserval Theorem

$$E_{x(t)} = \int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

Density Function

Energy spectral Density (ESD): $\boxed{\begin{matrix} G_{XX}(f) \\ G_{XX}(\omega) \end{matrix}}$ $\frac{\text{Joule}}{\text{Hz}}$

Power spectral Density (PSD): $\boxed{\begin{matrix} S_{XX}(f) \\ G_{XX}(\omega) \end{matrix}}$ $\frac{\text{Watt}}{\text{Hz}}$

Energy spectral density $X(t) \rightarrow$ WSSRP, Engery

ESD

$x(t) \xrightarrow{F.T} X(\omega) \Rightarrow |X(\omega)|^2 = G_{XX}(\omega)$

$x(t) \longrightarrow X(f) \Rightarrow |X(f)|^2 = G_{XX}(f)$

ESD of $x(t)$

$$E[X(t)X(t+\tau)] = R_{XX}(\tau) \xrightarrow{F.T.} G_{XX}(\omega)$$

$$R_{XX}(\tau) \xrightarrow{F.T.} G_{XX}(f)$$

$$\text{ACF}(X(t)) \xleftarrow{F.T} \text{ESD}[x(t)]$$

$$G_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau)d\tau = 2\int_{0}^{\infty} R_{XX}(\tau)d\tau$$

Zero freq. value of ESD = Area under ACF

$$R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_{XX}(\omega)d\omega = \frac{\text{Area under ESDG}_{XX}(\omega)}{2\pi}$$

$$\underset{E[X^2(t)]}{\downarrow}$$

$$\int_{-\infty}^{\infty} G_{XX}(f)df = \text{Area under ESD} \qquad \text{G}_{XX}(f)$$

Energy Calculation :

$$E_X(t) = \int_{-\infty}^{\infty} |X(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(f)|^2 df$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_{XX}(\omega)d\omega = \int_{-\infty}^{\infty} G_{XX}(f)df$$

$$G_{XX}(\omega) = G_{XX}(-\omega)$$

Power spectral density – (PSD)

$X(t) \rightarrow$ Power signal, WSSRP

$$X(t) \xleftrightarrow{PSD} S_{XX}(\omega) = \lim_{T\to\infty}\frac{1}{T}|X_T(\omega)|^2$$

$$X(t) \xleftrightarrow{PSD} S_{XX}(f) = \lim_{T\to\infty}\frac{1}{T}|X_T(f)|^2$$

(1) $E[X(t)X(t+\tau)] = R_{XX}(\tau)$ $\qquad S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau$

(2) $\text{ACF}[X(t)] \xleftrightarrow{F.T.} \text{PSD}[X(t)]$

$R_{XX} \xleftrightarrow{F.T.} \tau_{XX}(\omega)$ $\qquad S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j2\pi f\tau}d\tau$

(3) $S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau)d\tau = 2\int_{0}^{\infty} R_{XX}(\tau)d\tau$

Zero freq. value of = Area under ACF

PSD

(4) $R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega = \int_{-\infty}^{\infty} S_{XX}(f)e^{j\omega\tau}df$

$$R_{XX}(0) = \begin{cases} \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega \\ \int_{-\infty}^{\infty} S_{XX}(f)df \end{cases}$$

(5) $R_{XX}(\tau) = R_{XX}(-\tau),\ S_{XX}(\omega) = S_{XX}(-\omega)$

(6) Calculation of power

$$E[X^2(t)] = R_{XX}(0) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega = \dfrac{\text{Area under PSD}}{2\pi} \\ \displaystyle\int_{-\infty}^{\infty} S_{XX}(f)df = \text{Area under PSD} \end{cases}$$

$$E[X^2(t)] = R_{XX}(0) = \frac{1}{\pi}\int_0^{\infty} S_{XX}(\omega)d\omega = 2\int_0^{\infty} S_{XX}(f)df$$

Total power

$$\text{A.C. Power} = \sigma^2 X(t), \text{D.C.Power} = \mu^2{}_X(t)$$

Mean or Aug value

$$E[X(t)] = \sqrt{\frac{1}{2\pi}\int_{0^-}^{0^+} S_{XX}(\omega)d\omega}$$

$$E[X(t)] = \mu^2_{X(t)} = \begin{cases} \dfrac{1}{2}\displaystyle\int_{0^-}^{0^+} S_{XX}(\omega)d\omega \\ \displaystyle\int_{0^-}^{0^+} S_{XX}(f)df \rightarrow \end{cases}$$

Is non-zero only when impulse is not prelent at zero frequency

$$\sigma^2_{X(t)} = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{0^-} S_{XX}(\omega)d\omega + \dfrac{1}{2\pi}\int_{0^+}^{\infty} S_{XX}(\omega)d\omega \\ \displaystyle\int_{-\infty}^{0^-} S_{XX}(f)df + \int_{0^+}^{\infty} S_{XX}(f)df \end{cases}$$

- If X (t) is real, PSD is also real.
- PSD is even.
- PSD is non-negative, $S_{XX}(\omega) \geq 0;\ \ S_{XX}(f) \geq 0$

ESD of Modulated Signal (Band Pass Signal)

$$\underset{\text{Base band R.P.}}{\underset{\downarrow}{X(t)}} \xleftrightarrow{ESD} G_{XX}(f)$$

$$\underset{\text{Band pass R.P.}}{\underset{\downarrow}{Y(t)}} = X(t).A_c \cos 2\pi f_c t \qquad Y(f) = \frac{A_c}{2}[X(f+f_c) + X(f-f_c)]$$

Or $\xleftrightarrow{ESD} G_{YY}(f) = |Y(f)|^2$

$$X(t).A_c \sin 2\pi f_c t \xleftrightarrow{ESD} G_{YY}(f) = |Y(f)|^2$$

$$G_{YY}(f) = \frac{A_c^2}{4}[G_{XX}(f - f_c) + G_{XX}(f + f_c)] \qquad f_c >>> f_m$$

$$R_Y(\tau) = \frac{A_c^2}{2} R_X(\tau)\cos 2\pi f_c T$$

PSD of Modulated Signal (Bandpass Signal)

$X(t) \rightarrow$ power single

$$Y(t) = A_c \cos(2\pi f_c t).X(t) \rightarrow S_{YY}(f) \rightarrow \text{PSD}$$

$$S_{YY}(f) = \lim_{T\to\infty} \frac{|Y_T(f)|^2}{T}$$

$$S_{YY}(f) = \frac{A_c^2}{4}\left\{\lim_{T\to\infty} \frac{|X_T(f - f_c)|^2}{T} + \lim_{T\to\infty} \frac{|X_T(f + f_c)|^2}{T}\right\}$$

$$S_{YY}(f) = \frac{A_c^2}{4}[S_{XX}(f - f_c) + S_{XX}(f + f_c)]$$

$$R_{YY}(\tau) = \frac{A_c^2}{4} R_{XX}(\tau)\cos 2\pi f_c \tau$$

$x(t)$: WSSRP

Mixing with deterministic sinusoidal signal	Mixing with deterministic Random process
$R_Y(\tau) = \frac{A_c^2}{2} R_X(\tau)\cos\omega_c\tau$	$R_Y(\tau) = \frac{A_c^2}{2} R_X(\tau)\cos\omega_c\tau$
$S_Y(f) = \frac{A_c^2}{2}[S_X(f - f_c) + S_X(f + f_c)]$	$S_Y(f) = \frac{A_c^2}{4}[S_X(f - f_c) + S_X(f + f_c)]$

3.3. Cross Correlation

$X(t)$; WSSRP, $Y(t)$: WSSRP

$$E[X(t)Y(t+\tau)] = R_{XY}(t, t+\tau) = R_{XY}(\tau)$$

$$E[Y(t+\tau)X(t)] = R_{YX}(-\tau)$$

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

- $R_{XY}(\tau) \pm R_{YX}(-\tau)$

- $R_{XY}(\tau) = R_{XY}(f) \rightarrow$ May/may not be
- $R_{XY}(\tau) \leq \sqrt{R_{XX}(0)R_{YY}(0)}$
- $|R_{XY}(\tau)| = \frac{1}{2}R_{XX}(0) + R_{YY}(0)$

Cross Covariance

$$C_{XY}(\tau) = R_{XY}(\tau) - \mu_X \mu_Y$$

$$C_{YX}(\tau) = R_{YX}(\tau) - \mu_Y \mu_X$$

Cross Spectral Density

$$R_{XY}(\tau) \xleftrightarrow{F.T} S_{XY}(f)$$

$$R_{YX}(\tau) \xleftrightarrow{F.T} S_{YX}(f)$$

If RP X (t) and Y (t) are orthogonal-

$$E[X(t)Y(t+\tau)] = R_{XY}(\tau) = 0 = R_{YX}(-\tau)$$

$$E[Y(t)X(t+\tau)] = R_{YX}(\tau) = 0 = R_{XY}(-\tau)$$

- X(t) and Y(t) are uncorrelated and atleast one of them have zero mean-

  $E[X(t)] = 0 \text{ or } E[Y(t)] = 0$

  $\text{cov}[X(t)Y(t+\tau)] = 0$

  $C_{XY}(\tau) = R_{XY}(\tau) - \mu_X \mu_Y = 0$

  $R_{XY}(\tau) = 0$

  $R_{XY}(\tau) = R_{YX}(\tau) = 0$

- X(t) and Y(t) are independent R.P. and atleast one of them have zero mean.

  Cov $[X(t)Y(t+\tau)] = 0$

  $C_{XY}(\tau) = R_{XY}(\tau) - \mu_X \mu_Y = 0$

  $R_{XY}(\tau) = 0$

  $R_{YX}(\tau) = 0 = R_{XY}(\tau)$

Combination of WSSRP

$$Z(t) = X(t) \pm Y(t)$$

$$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{XY}(\tau) \pm R_{YX}(\tau) \pm R_{YY}(\tau)$$

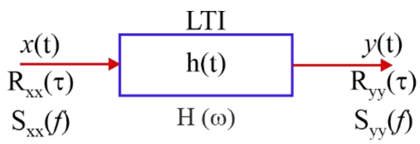
$$S_{ZZ}(f) = S_{XX}(f) + S_{YY}(f) \pm S_{XY}(f) \pm S_{YX}(f)$$

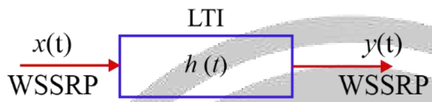
If orthogonal [X (t) and Y (t)] then

$$R_{YX}(\tau) = R_{XY}(\tau) = 0$$

$$S_{YX}(f) = S_{XY}(f) = 0$$

Transmission of WSSRP through in LTI system





$$E[X(t)] = \mu_X \qquad E[Y(t)] = \mu_X [H(\omega)]_{\omega=0}$$

$$\mu_Y = \mu_X H(0)$$

If x(t) is zero mean WSSRP then y(t) is also zero mean WSSRP

$$R_{XY}(\tau) = R_{XX}(\tau) * h(\tau) \text{ and } S_{XY}(f) = S_{XX}(f)H(f)$$

$$R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau) \text{ and } S_{YX}(f) = S_{XX}(f)H(-f)$$

$$R_{YY}(\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)$$

$$S_{YY}(t) = S_{XX}(f)H(f)H(-f) \qquad \text{if h(t) = real}$$

$$\underset{\text{PSD of O/P}}{\underset{\downarrow}{S_{YY}(f)}} = \underset{\text{PSD of i/p}}{\underset{\downarrow}{S_{XX}(f)}}|H(f)|^2 \qquad H(f) = H(-f)$$

Power of Y(t)

$$E[Y^2(t)] = \int_{-\infty}^{\infty} S_{YY}(f)df = \int_{-\infty}^{\infty} |H(f)|^2 S_{XX}(f)df$$

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4 DIGITAL COMMUNICATION

4.1. Sampling

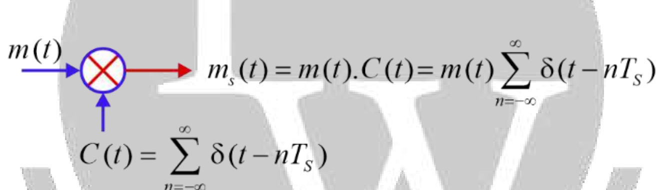
Sampling converts C.T.S into D.T.S, it retains analog or digital nature of signal.

Sampling
- → Base band (Low pass sampling)
- → Bandpass sampling

$C(t)$: Impulse Train – Instantaneous or Ideal sampling

$C(t)$: Rectangular Pulse Train : Natural sampling or Flat Top sampling.

Ideal instantaneous sampling



$$m_s(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t - nT_s)$$

$$m(t) = M(f) \,|\, M(\omega)$$

$$M_s(\omega) = f_s \sum_{n=-\infty}^{\infty} M(\omega - n\omega_s)$$

$$M_s(f) = f_s \sum_{n=-\infty}^{\infty} M(f - nf_s)$$

1. If a low pass signal is sampled at $f_s > 2f_m$ then it can be recovered from its samples, when
   $(f_s > 2f_m) \cap (f_m \le f_c \le f_s - f_m)$
   $(PBG = T_s)$
   $f_c =$ cut off free of ideal LPF at RX
2. $f_s = 2f_m$, the sampled signal $m_s(t)$ can be recovered into $m(t)$ if,
   $(f_s = 2f_m) \cap (f_c = f_m)$ Ideal LPF
   $(PBG = T_s)$

3. $f_s < 2f_m$, under sampling,

T_X : Replace generation with ALIASING

R_X : Recovery not possible

- ALIASING is overlapping of adjacent replica's in sampled signal.

4.1.1. Low pass Sampling Theorem

A low pass sampling signal band limited to $f_{\max}$ Hz, can be sampled and recovered from its samples when and only when

$f_s \geq 2f_{\max}$ at T_X Proper LPF at R_X

No Aliasing Recovery

Nyquist Rate and Nyquist Internal

Let $m(t)$ is lowpass signal bandlimited to $f_{\max}$ Hz.

$$f_{NY} = 2f_{\max} \qquad T_{NY} = \frac{1}{f_{NY}} = \frac{1}{2f_{\max}}$$

$$\boxed{(f_s)_{\min} = 2f_{\max} \qquad \text{min} \rightarrow \text{sampling rate which ensure to aliasing}}$$

$$S(t) = m(t)\cos\omega_c(t) = (f_c + \underset{\substack{\uparrow \\ f_s(\max)}}{f_m})$$

$$N_R = 2(f_c + f_m) = f_{NY}$$

$$N_J = \frac{1}{2(f_c + f_m)} = T_{NY}$$

Combination of Two signals –

$$x_1(t) \rightarrow f_{\max} \rightarrow f_1, x_2(t) \rightarrow f_{\max} \rightarrow f_2$$

(1) $\pm x_1(t) \pm x_2(t)$
- $\rightarrow f_{\max} = \max(f_1, f_2)$
- $\rightarrow f_{Ny} = 2f_{max} = \max \text{ of } (2f_1, 2f_2)$

(2) $x_1(t) \cdot x_2(t)$
- $\rightarrow f_{\max} = (f_1 + f_2)$
- $\rightarrow f_{Ny} = 2f_{max} = (2f_1 + 2f_2)$

(3) $x_1(t) * x_2(t)$
- $\rightarrow f_{\max} = \text{Min}(f_1, f_2)$
- $\rightarrow f_{Ny} = 2f_{max} = \text{Min}(2f_1, 2f_2)$

$$m(t): A_m \cos\omega_m t, A_m \sin\omega_m t \rightarrow f_m$$

$$c(t): \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \rightarrow 0, f_s, 2f_s, 3f_s$$

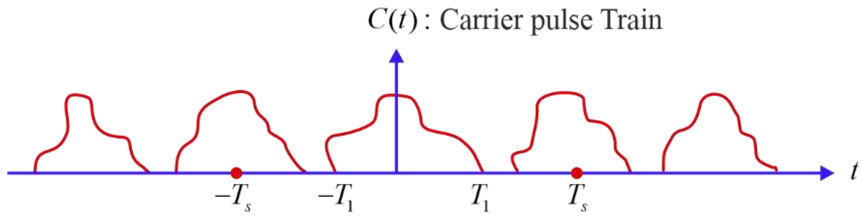
$$m_s(t) = m(t)c(t) = |0 \pm f_m|, |f_s \pm f_m|, |2f_s \pm f_m|, |3f_s \pm f_m| \ldots$$

Sampling of signal by using general carrier pulse train

$m(t) \longrightarrow M(f)$ or $M(\omega)$

$$c(t) \longrightarrow c(f) = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_s), C(\omega) = \sum_{n=-\infty}^{\infty} 2\pi C_n \delta(\omega - n\omega_s)$$

$$M_s(t) = M_s(f) = \sum_{n=-\infty}^{\infty} C_n M(f - nf_s),\ M_s(\omega) = \sum_{n=-\infty}^{\infty} C_n M(\omega - n\omega_s)$$

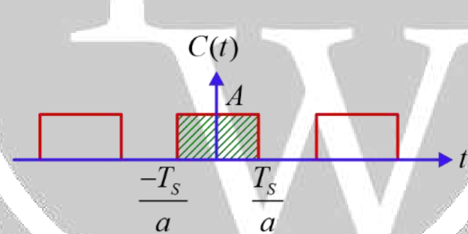


$$C_0 = \frac{\text{Area in } T_s}{T_s}$$

Recovery – T_X : $f_s \geq 2 f_m$ R_X : LPF = Proper f_c

$$y(t) = m(t) \qquad PBG = \frac{1}{C_0}$$

If $c(t)$ is rectangular pulse



- $m(t)$ is lowpass –

$$m_s(t) = c(t)m(t) \rightarrow M_s(f) = \sum_{n=-\infty}^{\infty} \frac{2A}{a} \sin C\left(\frac{2n}{a}\right) M(f - nf_s)$$

Recovery-$(f_s > 2 f_m) \cap (f_m \leq f_c \leq f_s - f_m)$

PBG of LPF	$y(t)$
1	$C_0 m(t)$
$\frac{1}{C_0}$	M(t)
K	$KC_0\, m(t)$

$$C_0 = \frac{\text{Area in } T_s}{T_s}$$

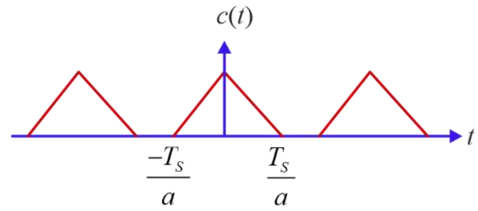
- m(t) is sinusoidal –

$$m(t) = A_m \cos 2\pi f_m t \longrightarrow f_m$$

$$c(t) = 0,\ f_s, 2f_s, \ldots \text{ except } \ n = \frac{Ka}{2} \quad K \in I, K \neq 0$$

$$m_s(t) = 0 \pm f_m, f_s \pm f_m, 2f_s \pm f_m, \ldots$$

If c(t) is Triangular-Frequency absent $n = Ka \quad K \neq 0, K \in I$



- Rest of the things same as Rectangular pulse

Bandpass Sampling :

m(t) is lowpass signal.

$$f_s = \frac{2f_H}{K} \qquad K = \frac{f_H}{f_H - f_L} \qquad NR = 2f_H$$

Previous Integer

f_H = Maximum frequency component of Bandpass signal

Natural Sampling

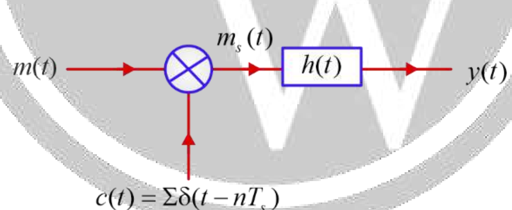
***m*(t):** **L**ow pass sampling

c(t): Train of finite duration pulse or rectangular pulse

$$M_s(f) = \sum_{n=-\infty}^{\infty} C_n M(f - nf_s)$$

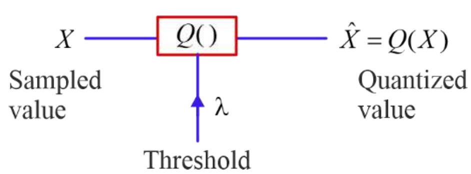
Flat Top Sampling

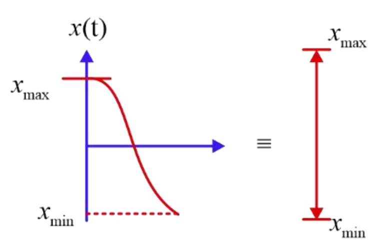
Instantaneous sampling followed by a filter.



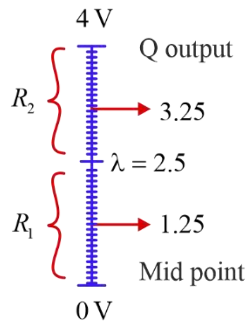
$$y(t) = \sum_{n=-\infty}^{\infty} m(nT_s) h(t - nT_s)$$

Quantizer





Discretizes amplitude axis, analog to digital signal.
Dynamic Range of $x(t) = (x_{max} - x_{min})$
Q input : $\qquad Q$: output : $\hat{x}(t)$



$$\hat{x}(t) = \begin{cases} 1.25 & 0 \le x(t) \le 2.5 \\ 3.25 & 2.5 \le x(t) \le 4 \end{cases}$$

- Many to one circuit.

Uniform Quantizer

$$\Delta = \frac{\text{DR of } Q}{L} = \frac{m_{L+1} - m_1}{L}$$

$\Delta_1 = \Delta_2 = \Delta_3 \ldots$

L = Number of quantization level of Q.

Non uniform Quantizer

1. $\Delta_1 = \Delta_2 \neq \Delta_3 \ldots \Delta_i = \Delta_{i+1} \ldots \neq \Delta_L$
2. $\Delta_1 \neq \Delta_2 \neq \Delta_3 \ldots \Delta_i \neq \Delta_{i+1} \ldots \neq \Delta_L$

Quantizer Error –

$$X \longrightarrow Q(\,) \longrightarrow \hat{X} = Q(X)$$

$$Q \cdot E = X_{QE} = X - \hat{X} = X - Q(X)$$

R.V Sampled Value Quantized Value

Q.E.P $E\left[X_{QE}^2\right] = \int_{-\infty}^{\infty} x_{QE}^2 f_{QE}[qE]\, dqE$

When PDF of QE is given.

$$E\left[X_{QE}^2\right] = E\left[(X - \hat{X})^2\right] = \int_{-\infty}^{\infty} (x - \hat{x})^2 f_X(x)\, dx$$

When PDF of R.V at input of Quantizer (X) is given.

(a) PDF is uniform

$$E\left[X_{QE}^2\right] = \frac{\Delta_1^2}{12} \times A_1 + \frac{\Delta_2^2}{12} \times A_2 + \frac{\Delta_3^2}{12} \times A_3 + \ldots$$

$$= \frac{\Delta_1^2}{12} \text{ [Area of region in which step size is 1]} + \frac{\Delta_2^2}{12} \text{ [Area of region in which step size is 2]} + \ldots$$

If quantization is uniform $E\left[X_{QE}^2\right]=\frac{\Delta^2}{12}$

(b) PDF is stair case $E\left[X_{QE}^2\right]=\frac{\Delta_1^2}{12}\times A_1+\frac{\Delta_2^2}{12}\times A_2+\ldots$

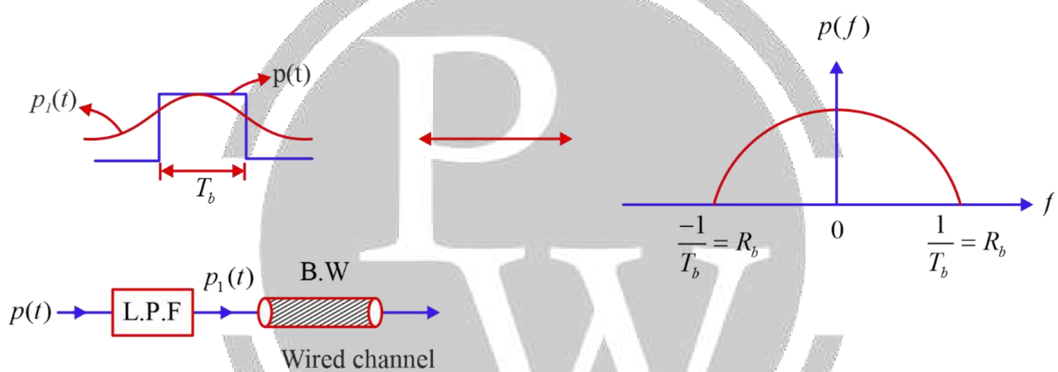
(c) PDF is non uniform $E\left[X_{QE}^2\right]=\frac{\Delta_1^2}{12}A_1+\frac{\Delta_2^2}{12}A_2+\ldots$

$$SQNR=\frac{\text{Signal power}}{Q\cdot E\cdot P}=\frac{E[X^2]}{E\left[X_{QE}^2\right]}$$

$$(SQNR)_{dB}=10\log_{10}SQNR$$

4.2. Pulse Transmission

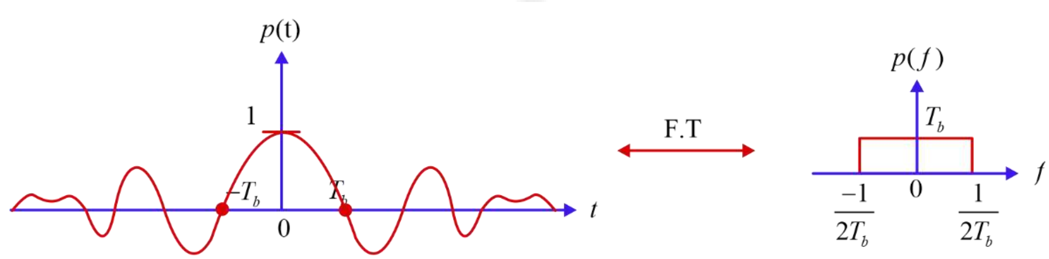
1. Rectangular Pulse



- Bandwidth of wired channel for T_X of single pulse of duration T_b :

  $(BW)\geq R_b$

2. Since pulse



- BW of wired channel for T_X of single $\sin C$ pulse having zero cross over or integer multiple of T_b.

  $(BW)\geq\frac{R_b}{2}$

  Tb: Bit interval

  R_b : Bit rate $\rightarrow$ Bit/sec

- Minimum transmission BW of a wired channel for baseband transmission is $=\frac{R_b}{2}$

  Number of levels $L \le 2^n$ $\quad L_{\max} = 2^n$

  n is used to represent binary power quantization level.

M-ary Scheme

$$M = 2^N$$

M $\Rightarrow$ Number of different symbols of duration NT_b each.

$$T_s = NT_b : \text{Symbol duration}$$

$N =$ Number of bits combined in binary sequence at a time

Pulse Code Modulation

Bit rate, $\quad R_b = nf_s$ bits/sec

$$\frac{-\Delta}{2} \le Q \cdot E \le \frac{\Delta}{2}, \quad E(y)^2 = \frac{\Delta^2}{12}$$

If Mid point Mapping is used.

$$\Delta = \frac{DR \text{ of signal}}{L} = \frac{DR \text{ of } Q}{L}$$

$$L \le 2^n \qquad \frac{-\Delta}{2} \le Q_e \le \frac{\Delta}{2} \qquad Q_e\big|_{\max} = \frac{\Delta}{2} \qquad \Delta_{\min} = \frac{DR \text{ of signal}}{2^n}$$

$$P_{QE} = E[X_{QE}^2] = E[y^2] = \int y^2 f_Y(y)\,dy$$

When PDF of Q. Eis given.

$$P_{QN} = P_{QE} = \sum_{i=L}^{L} \int_{m_i}^{m_{i+1}} (x - \hat{x}_i)^2 f_X(x)\,dx$$

When PDF of X is given.

If $Q_e \sim U\left[\frac{-\Delta}{2}, \frac{\Delta}{2}\right] = P_{QE} = \frac{\Delta^2}{12}$

$$SQNR = \frac{12}{\Delta^2} P_s$$

Bit Interval $\quad T_b = \frac{1}{R_b}$

$\text{B.W} \ge R_b \rightarrow$ Rectangular Pulse

$\text{B.W} \ge \frac{R_b}{2} \rightarrow$ Since Pulse

$(\text{B.W})_{\min} = (\text{BW})_{PCM} = \frac{R_b}{2}$

Signal to Quantization Noise Power

$m(t) \to$ Single tone sinusoidal

$m(t) = A_m \cos \omega_m t$

1. $P_s = \overline{m^2(t)} = \dfrac{A_m^2}{2}$

2. $P_s = \overline{m^2(t)} = \dfrac{A_m^2}{3L^2}$

3. $\Delta = \dfrac{2A_m}{L}$

4. $SQNR = \dfrac{3}{2}L^2$

5. $(SQNR)_{dB} = (1.76 + 20\log_{10} L)\, dB$

6. $(SQNR)_{\max} = \dfrac{3}{2}4^n$

7. $(SQNR)_{\max}\, dB = (1.76 + 6n)\, dB$

$x(t)$ is uniformly distributed $[-A_m, A_m]$

1. $P_s = \dfrac{A_m^2}{3}$

2. $P_{QE} = \dfrac{A_m^2}{3L^2}$

3. $\Delta = \dfrac{2A_m}{L}$

4. $SQNR = L^2$

5. $(SQNR)_{dB} = 20\log_{10} L$

6. $(SQNR)_{dB} \le 6n\,\text{dB}$

7. $(SQNR)_{\max}\,\text{dB} = 6n\,\text{dB}$

Key point:

1. $SQNR = \dfrac{3}{2}L^2$

2. $(SQNR)_{\max} = \dfrac{3}{2}4^n$

   If $n \to n \pm k$

   $(SQNR)_{\max} \to 4^{\pm k}$ $\qquad R_b = nf_s$

3. $(SQNR)_{\max} = (1.76 + 6n)\,dB$

   $n \to n \pm k$

   $(SQNR)_{\max} \to \pm 6\,dB$

4. n given : $L = 2^n$

5. L given:
   - L = Binany Power : $L = 2^n$
   - L $\pm$ Binany Power : $L \leq 2^n$

6. $SQNR$:
   - L=Binany Power : $(SQNR) = \frac{3}{2}L^2 = (SQNR)_{\max}$
   - L $\neq$ Binany Power : $(SQNR)_{\max} = \frac{3}{2}4^n$

7. n calculation : $n_{\min}$

8. Default $m(t) =$ Sinusoidal

Drawback of PCM

$$\text{BW} = \frac{nf_s}{2}, P_{QE} = \frac{\Delta^2}{12}$$

$$n\uparrow \longrightarrow L\uparrow \longrightarrow \Delta\downarrow \longrightarrow PQE\downarrow \longrightarrow BW\uparrow$$

$$n\downarrow \longrightarrow L\downarrow \longrightarrow \Delta\uparrow \longrightarrow \underbrace{PQE\uparrow \longrightarrow BW\downarrow}$$

4.3. DPCM (Differential Pulse Code Modulation)

4.3.1. PCM vs DPCM

1. Δ fix for Both Q –

   $(\text{BW})_{PCM} > (\text{BW})_{DPCM}$

   $(SQNR)_{PCM} = (SQNR)_{DPCM}$

   D.R at input of Q of PCM is greater than DPCM.

2. L fix for Both Q –

   $(\text{BW})_{PCM} = (\text{BW})_{DPCM}$

   $(SQNR)_{PCM} < (SQNR)_{DPCM}$

3. In case of DPCM the difference between current sample and predicted value of current sample is Quantized, Encoded, Line Coded and Wired T_{Xed}.

Delta Modulator

- The recovered signal is "stair-case" approximation of t$_{Xed}$ original analog message signal.
- Stairs are added or substracted of sampling instance.
- Size of each stair is Δ = Step size of stairs

Tracking Error in DM.

1. Slope overload Error

$$\left|\frac{dm(t)}{dt}\right|_{\max} >> \frac{\Delta}{T_s} \quad \text{Occurance of S.O.E}$$

➢ To avoid SOE, $\Delta\uparrow\uparrow\uparrow$ by keeping T_s constant such that –

$$\left|\frac{d}{dt}m(t)\right|_{\max} \le \frac{\Delta}{T_s} \Rightarrow \text{For sinusoidal } m(t) = A_m \cos\omega_n t$$

$$A_m \le \frac{\Delta f_s}{\omega_m}$$

2. Granular Error

It occurs when Δ is large.

➢ To remove it Δ→ small

If $SOE\uparrow$, G.E $\downarrow$ and vice versa.

SQNR in DM

$$P_{QE} = E\left[X_{QE}^2\right] = \frac{\Delta^2}{3}$$

1. $SQNR = \dfrac{P_s}{P_{QE}} = \dfrac{3P_s}{\Delta^2}$ $\qquad f_H$ = cut off frequency of LPF

2. $(SQNR)_D = \dfrac{3P_s}{\Delta^2} \times \dfrac{f_s}{f_m} = \dfrac{3P_s}{\Delta^2} \times \dfrac{f_s}{f_H}$

3. $(SQNR)_{\max} + m(t)$ is sinusoidal + SOE avoid $= \dfrac{3}{80}\left(\dfrac{f_s}{f_m}\right)^2$

4. $[(SQNR)_D]_{\max} + m(t)$ is sinusoidal + SOE avoid $= \dfrac{3}{80}\left(\dfrac{f_s}{f_m}\right)^3 \to$ By default.

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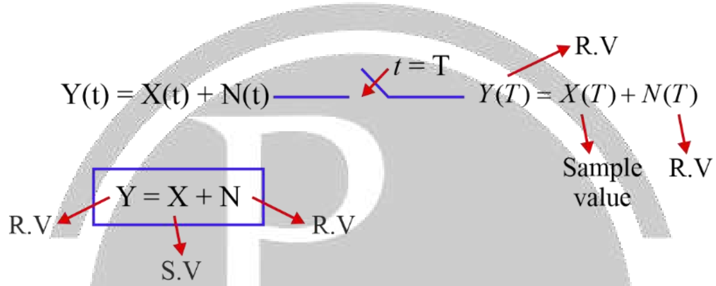
5 DIGITAL RECEIVER

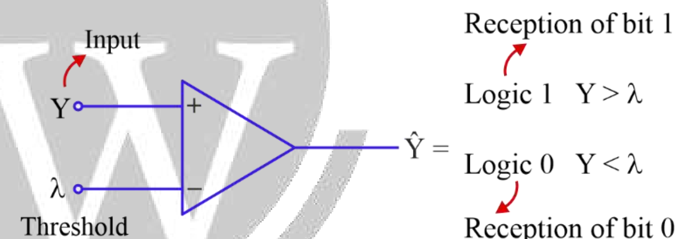
5.1. Introduction

$X(t) \rightarrow$ Deterministic signal process

$N(t) \rightarrow$ random signal process



N	Y = N + X
Uniform PDF	Uniform PDF
Gaussian PDF	Gaussian PDF
Triangular PDF	Triangular PDF



1. If but 1 is taxed $\rightarrow$ Receiver must recover but '1'.
2. If but 0 is taxed $\rightarrow$ Receiver must recover but '0'.

Output of Sampler:

$$Y = S_{01} + N_0 = \begin{cases} S_{01} + N_{01} & 1\,T_X \\ S_{02} + N_{02} & 0\,T_X \end{cases}$$ Channel noise is signal dependent

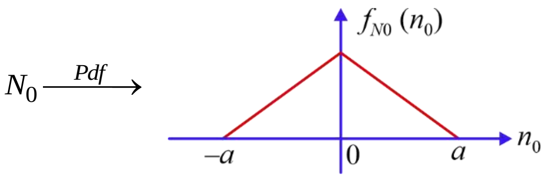
$$Y = S_0 + N_0 = \begin{cases} S_{01} + N_0 & 1\,T_X \\ S_{02} + N_0 & 0\,T_X \end{cases}$$ Channel noise is signal independent

BER Calculation :

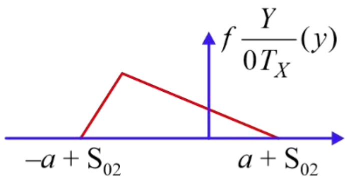
$$P(1\,T_X) \Rightarrow \quad P(S_1(t) : T_X) = P(S_{01}(t) : \text{Reception}) = p$$

$$P(0\,T_X) \Rightarrow \quad P(S_2(t) : T_X) = P(S_{02}(t) : \text{Reception}) = (1 - p)$$

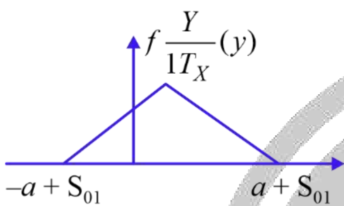
At the input of decision device a condition R.V. is obtained



$Y = S_0 + N_0$ → $Y = S_{02} + N_0$
$Y = S_0 + N_0$ → $Y = S_{01} + N_0$



→ Conditional PDF of Y when 0 is Txed by T_x



→ Conditional PDF of Y when 1 is Txed by T_x

$Y > \lambda$: Decide in favour of 1, or decides that bit 1 would have been Txed by Txer

$Y < \lambda$: Decide in favour of 0, or decides the bit 0 would have been Txed by Txer.

Average Bit Error Rate :

$$\boxed{P_e = \underbrace{P(1\ T_X}_{\text{1 Txed}} \cap \underbrace{0)}_{\text{decide in favour of "0"}} + \underbrace{P(0\ T_X}_{\text{0 Txed}} \cap \underbrace{1)}_{\text{decide in favour of "1"}}}$$

$$\boxed{P_e = P(1\ T_X)\, \underbrace{P\left(\frac{0}{1\ T_X}\right)}_{\text{decides in favour of "0" provided "1" was txed}} + P(0\ T_X)\, \underbrace{P\left(\frac{1}{0\ T_X}\right)}_{\text{decides in favour of "1" provided "0" was txed}}}$$

Problem Solving Technique:

Case 1: When PDF of Noise [noise R.V. at i/p of D.D] is given

- (1) λ : Given
  (2) Calculate P_e =?
- (1) Calculate optimum threshold (λ_{opt})
  (2) Calculate min prob. of error
  → Min probability of error is calculated at λ_{opt}.

(a) λ is given

$$P_e = P(0\,T_X)P\left(\frac{1}{0\,T_X}\right) + P(1\,T_X)P\left(\frac{0}{1\,T_X}\right)$$

$$P\left(\frac{1}{0\,T_X}\right) = P[Y(0\,T_X) > \lambda] = P(S_{01} + N_0 > \lambda) = P(N_0 > \lambda - S_{01}) = \int_{\lambda - S_{01}}^{\infty} f_{N_0}(no)dno$$

$$P\left(\frac{0}{1\,T_X}\right) = P[Y(1\,T_X) < \lambda] = P(S_{02} + N_0 < \lambda) = P(N_0 < \lambda - S_{02}) = \int_{-\infty}^{\lambda - S_{02}} f_{N_0}(no)dno$$

(b) $\lambda_{opt} \rightarrow$ calculate, $P_{e\min} \rightarrow$ calculate

Steps 1 :

1. Identify conditional PDF of conditional R.V. from the noise R.V. pdf (pdf of No.)

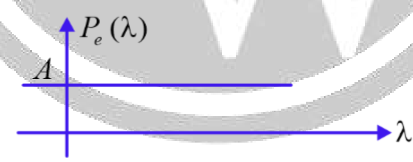
$$Y = \begin{cases} S_{01} + N_0 & 1\,T_X \\ S_{02} + N_0 & 0\,T_X \end{cases}$$

2. Plot the conditional PDF one over another
3. Identify the overlapping or common region and decide range of λ

   $(\lambda_1 \le \lambda \le \lambda_2)$

4. Choose any arbitrary λ in the above range and calculate $P_e = P_e(\lambda)$
5. $P_e(\lambda)$ Vs λ

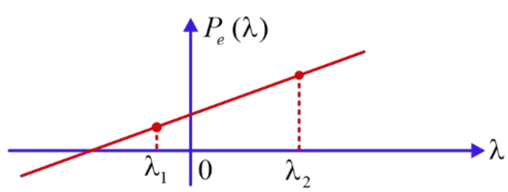
   (i) $P_e(\lambda)$ Vs λ : Independent of λ



   (a) $\lambda_1 \le \lambda \le \lambda_2 \rightarrow$ optimum λ is every $\lambda \Rightarrow \lambda \in (\lambda_1 : \lambda_2)$

   (b) $P_{e(\min)} = A$

   (ii) $P_e(\lambda)$ Vs λ: Linear



$\lambda_1 \le \lambda \le \lambda_2 \rightarrow \lambda_{opt} = \lambda_1$

$P_e(\lambda)_{\min} = P_e(\lambda_{opt})$

(iii) $Pe(\lambda)$ Vs λ: Non linear

$$\boxed{\frac{d}{d\lambda}P_e(\lambda)=0} \rightarrow \lambda_{opt} \qquad \boxed{P_e(\lambda=\lambda_{opt})=P_e|_{\min}}$$

$P_e(\lambda=\lambda_{opt})=P_e|_{\min}$

(iv) If no overlapping region b/w conditional PDF

$P_e(\lambda)=0 \rightarrow$ BER is 0

Case 2: When PDF of conditional R.V. Y is gives at the input of the decision device

$\lambda \rightarrow$ Given
$P_e \rightarrow$ Calculate

$\lambda_{opt} \rightarrow$ Calculate
$P_e|_{\min} \rightarrow$ Calculate

(a) $$P_e=P(0\,T_X)P\left(\frac{1}{0\,T_X}\right)+P(1\,T_X)P\left(\frac{0}{1T_X}\right)$$

$$P\left(\frac{1}{0\,T_X}\right)=P(Y(0\,T_X)>\lambda)=\int_{\lambda}^{\infty}\frac{f_Y}{0\,T_X}(y)dy$$

$$P\left(\frac{0}{1\,T_X}\right)=P[(1\,T_X)<\lambda]=\int_{-\infty}^{\lambda}\frac{f_Y}{1T_X}(y)dy$$

(b) Same as case 1 (b)

Calculate of min prob. of error

↓

Calculate of optimum threshold (λ_{opt})

↓

$\frac{d}{dt}P_e(\lambda)=0$ $\equiv$ MAP Analysis & ML Analysis

$\lambda_{opt}=\checkmark$

MAP Analysis (Maximum A posteriori Analysis)

- MAP receiver always calculate min P_e.
- Calculation of $\lambda_{opt} \Rightarrow$ using

$$\frac{d}{d\lambda}P_e(\lambda)=0$$

MAP Analysis

$$P\left(\frac{1\,T_X}{Y}\right)\underset{"0"}{\overset{"1"}{\gtrless}}P\left(\frac{0\,T_X}{Y}\right)$$

$$P(1\,T_X)\,f\frac{y}{1\,T_X}(y)\underset{"0"}{\overset{"1"}{\gtrless}}P(0\,T_X)\,f\frac{y}{0\,T_X}(y)\equiv y\underset{"0"}{\overset{"1"}{\gtrless}}\lambda \rightarrow \lambda_{opt}=\lambda$$

ML Analysis : (Maximum Likelihood Analysis)

It is same as MAP analysis When $P(0\ T_X) = P(1\ T_X) = \frac{1}{2}$

If noise is independent of signal then PDF of conditional R.V at input of decision device will have.

$\frac{f_Y}{0\,T_X}(y)$ and $\frac{f_Y}{1T_X}(y)$ → Same Width, → Same Height + Shifted to different parameters

Key point : $P(0\ T_X) \neq P(1\ T_X)$

1. Noise is signal dependent/independent $\lambda_{\min} \leq \lambda \leq \lambda_{\max}$,

$$\bar{\lambda} = \frac{\lambda_{\min} + \lambda_{\max}}{2}$$

$$P(0\ T_X) < P(1\ T_X) \qquad \lambda_{opt} < \bar{\lambda}$$

$$P(1\ T_X) < P(0\ T_X) \qquad \lambda_{opt} > \bar{\lambda}$$

2. If noise is signal dependent

$$Y = \begin{cases} S_{01} + N_0 & 1\ T_X \\ S_{02} + N_0 & 0\ T_X \end{cases}$$

$$P(0\ T_X) = P(1\ T_X) = \frac{1}{2};\ \lambda_{opt} = \bar{\lambda}$$

Only when Y is having Non uniform PDF.

$$P(0\ T_X) \neq P(1\ T_X) = \frac{1}{2};\ \lambda_{opt} < \bar{\lambda};\ P(0\ T_X) < P(1\ T_X)$$

$$\lambda_{opt} > \bar{\lambda};\ P(0\ T_X) > P(1\ T_X)$$

When channel noise is Gaussian Random Process

$$P_e = P(0\ T_X)P\left(\frac{1}{0\ T_X}\right) + P(1\ T_X)P\left(\frac{0}{1\ T_X}\right)$$

Method 1 :

$$P\left(\frac{1}{0\ T_X}\right) = P[Y[0\ T_X] > \lambda] = Q\left[\frac{\lambda - \mu_y[0\ T_X]}{\sigma_y[0\ T_X]}\right]$$

$$P\left(\frac{0}{1\ T_X}\right) = 1 - Q\left[\frac{\lambda - \mu_y(1\ T_X)}{\sigma_y[1\ T_X]}\right]$$

Method 2 :

$$P\left(\frac{1}{0\,T_X}\right) = P\left[y[0\,T_X] > \lambda\right] = P\left[S_{02} + N_0 > \lambda\right] = Q\left[\frac{(\lambda - S_{02}) - \mu_{N_0}}{\sigma_{N_0}}\right]$$

$$P\left(\frac{0}{1\,T_X}\right) = P\left[y[1\,T_X] < \lambda\right] = P\left[S_{01} + N_0 < \lambda\right] = 1 - Q\left[\frac{(\lambda - S_{01}) - \mu_{N_0}}{\sigma_{N_0}}\right]$$

If PDF of noise at the input of D.D is given along with λ–

$$P\left(\frac{1}{0\,T_X}\right) = P\left[y[0\,T_x] > \lambda\right] = \int_{\lambda}^{\infty} \frac{f_Y}{0\,T_X}(y)dy = Q\left[\frac{y - \mu_2}{\sigma}\right]$$

$$P\left(\frac{0}{1\,T_X}\right) = P\left[y[1\,T_x] < \lambda\right] = \int_{-\infty}^{\lambda} \frac{f_Y}{1\,T_X}(y)dy = 1 - Q\left[\frac{y - \mu_2}{\sigma}\right]$$

λ optimum

1. Differentiation : $P_e = Q(\lambda)$, $\frac{d}{d\lambda}P_e(\lambda) = 0 \rightarrow \lambda_{opt}$

2. Map Analysis : $\frac{d}{d\lambda}P_e(\lambda) = 0 \rightarrow \lambda_{opt}$

$$\lambda_{opt} = \left(\frac{\mu_1 + \mu_2}{2}\right) + \frac{\sigma y^2}{(\mu_1 - \mu_2)} \ln \frac{P(0\,T_X)}{P(1\,T_X)}$$

Channel noise is Gaussian, signal and channel noise are independent

$$P_e(\lambda) = P_e(\lambda_{opt}) = P_e\big|_{\min}$$

3. ML Analysis : $P(0\,T_X) = P(1\,T_X) = \frac{1}{2}$

$$\lambda_{opt} = \frac{\mu_1 + \mu_2}{2}$$

$$P_e\big|_{\min} = Q\left[\frac{\mu_1 - \mu_2}{2\sigma_y}\right]$$

Schwartz Inequality

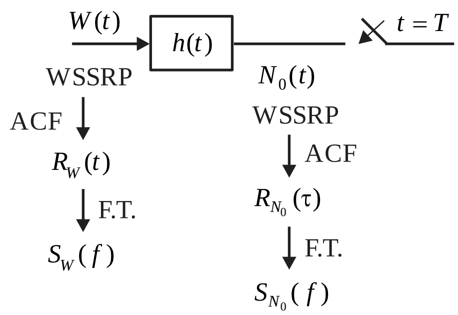
$$\left|\int_{-\infty}^{\infty} X_1(f)X_2(f)df\right|^2 \le \int_{-\infty}^{\infty} |X_1(f)|^2\,df \int_{-\infty}^{\infty} |X_2(f)|^2 df$$

$$\boxed{P(S_0)_{\max} = E_{s(t)} \times E_{h(t)}}$$

Max. signal power at sampling instance | Signal energy at input of filter | Energy of $h(t)$

$$E_{s(t)} = \int_{-\infty}^{\infty} |S(f)|^2 df$$

$$\int_{-\infty}^{\infty} |H(f)|^2 df = E_{h(t)}$$



$$P_{N_0(t)} = E\left[N_0^{\,2}(t)\right] = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df$$

$$E\left[N_0^2(t)\right] = \frac{N_0}{2} \times E_{h(t)}$$

$$(SNR)_{\max} = \frac{E_{s(t)}}{(N_0/2)}$$

Only when $H(f) = [S(f)e^{j2\pi fT}]^*$

$$(NSR)_{\max} \text{ at } t = T = \frac{\text{Energy of i/p pulse}}{\text{PSD of i/p white noise}}$$

For General Noise

$$H(f) = \left[S(f)e^{j2\pi fT}\right]^*$$

When $$(SNR)_{\max} = \frac{[P_{S0}]_{\max}}{P_{N_0}} = \frac{E_{s(t)} \times E_{h(t)}}{\int_{-\infty}^{\infty} S_{N_0}(f)df}$$

$$S_{N_0}(f) = |H(f)|^2 S_N(f)$$

5.2. Optimum Filter

$$H(f) = e^{-j2\pi fT} S^*(f) = e^{-j2\pi fT} S(-f)$$

$$h(t) = S(T - t)$$

T = Sampling instance

= Duration of incoming pulse

Unit impulse response of optimum filter

Optimum filter = matched filter $\Rightarrow$ Maximizes signal power at sampling instances.

Properties of MF

S(t) is an energy pulse of duration *T*.

1. $h(t) = S(T - t)$
2. $S(t) = h(T - t)$
3. $S_0(t) = S(T) * h(t)$
4. $S(t),\ h(t),\ S_0(t)$ are energy signal
5. $E_S(t) = E_h(t) = |S_0(t)|_{\max}$
6. $(SNR)_{\max}$ at $t = T = \dfrac{E_s(t)}{(N_0/2)}$
7. $E_{S_1}(t) > E_{s_2}(t)$ then $Pe_1 < Pe_2$
8. $S_0(f) = |S(f)|^2 e^{-j2\pi fT}$

$$S(t) \leftrightarrow S(f) \leftrightarrow |S(f)|^2 = G_s(f)$$

$$R_s(\tau) \xleftrightarrow{F.T.} G_s(f)$$

$$ACF[S(t)] \xleftrightarrow{F.T.} PSD[S(t)]$$

$$R_s(\tau) \xleftrightarrow{F.T.} |S(f)|^2$$

$$S_0(\tau) = R_s(\tau - T) \xleftrightarrow{F.T.} |S(f)|^2 e^{-j2\pi fT} = S_0(f)$$

Matched Filter Output

$$y(t) = S_0(t) + N_0(t) = \begin{cases} a_1(t) + N_0(t) & : 1\,T_X \\ a_2(t) + N_0(t) & : 0\,T_X \end{cases}$$

Channel Noise is White

Let $P(0\,T_X) = P(1\,T_X) = \dfrac{1}{2}$

$$[Pe]_{\min} = Q\left[\frac{a_1 - a_2}{2\sigma_y}\right] = Q\left(\frac{x}{2}\right) \quad x \uparrow \Rightarrow Q\left(\frac{x}{2}\right)\downarrow$$

Maximization of $|x|^2$

$$|x|^2_{\max} = \frac{\int_{-\infty}^{\infty} |S_1(f) - S_2(f)|^2\, df}{(N_0/2)}$$

When $$H(f) = \left[[S_1(f) - S_2(f)]e^{j2\pi fT}\right]^*,\ E_d = \int_{-\infty}^{\infty} |S_1(f) - S_2(f)|^2 df$$

$$x_{max} = \sqrt{\frac{2E_d}{N_0}}$$

$$h(t) = S_1(T-t) - S_2(T-t) \to x : x_{max} = \sqrt{\frac{2E_d}{N_0}}$$

$$P_e|_{min} = Q\left(\frac{a_1 - a_2}{2\sigma_y}\right)$$

$$P_e|_{min} = Q\left(\frac{x}{2}\right) \longrightarrow \underset{h(t)}{\boxed{M.F}} \longrightarrow P_e|_{min}|_{min} = Q\left(\frac{x_{max}}{2}\right)$$

$$P_e|_{min}|_{min} = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$$

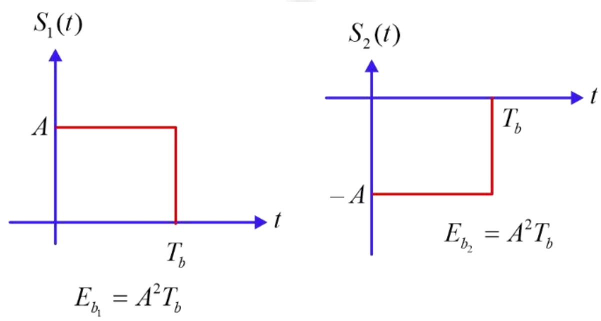
- Only when, $P(0\,T_X) = P(1\,T_X) = 1/2$
- AWGN, $\lambda \to \lambda_{opt}$
- M.F

For K noise R.V

$$P_e|_{min} = Q\left[\sqrt{K}\left\{\frac{(a_1 - a_2)}{2\sigma}\right\}\right]$$

$$P_e|_{min}|_{min} = Q\left[\sqrt{K}\sqrt{\frac{E_d}{2N_0}}\right]$$

1.



$$E_d = \int_{-\infty}^{\infty} d^2(t)dt = 4A^2T_b$$

$$d(t) = S_1(t) - S_2(t) = 2\text{ A}$$

$$0 \le t \le T_b$$

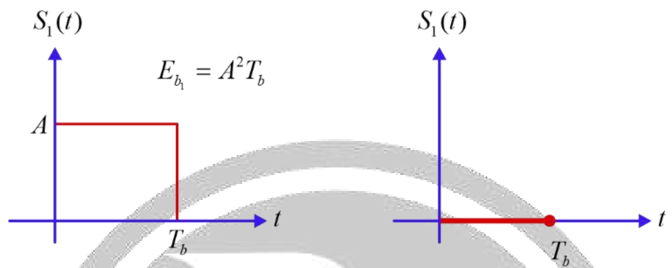
$$P_e = Q\left[\sqrt{\frac{2A^2T_b}{N_0}}\right]$$

$$P_e = Q\left[\sqrt{\frac{2(E_b)_{avg}}{N_0}}\right]$$

$$(E_b)_{avg} = A^2T_b$$

$$E_b = A^2T_b$$

2.



$$P_e = Q\left[\sqrt{\frac{E_b}{N_0}}\right]$$

$$(E_b)_{avg} = \frac{E_b}{2}$$

$$P_e = Q\left[\sqrt{\frac{A^2T_b}{2N_0}}\right]$$

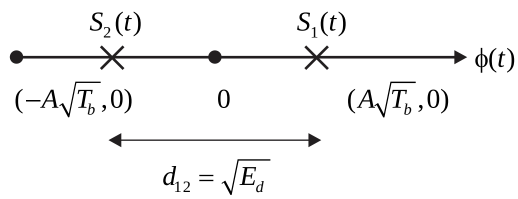
M-ary Base Bond Signaling

1. Bit rate : R_b
2. Bit interval $= T_b = 1/R_b$
3. Symbol duration $= T_s = NT_b$
4. Symbol rate or Baud or Baud rate $R_s = \frac{1}{T_s} = \frac{1}{NT_b} = \frac{R_b}{N}$
5. T_X Bandwidth $(BW) \geq R_s \rightarrow$ Rectangular, $(BW) \geq \frac{R_s}{2} \rightarrow \sin C$

$$(BW)_{\min} = \frac{1}{2}\left(\frac{R_b}{N}\right) = \frac{R_s}{2}$$

M-ary PAM (2-Any PAM)

1. $M = 2, (N \leq M)$, $s(t) = \begin{cases} S_1(t) = A & 0 \leq t \leq T_b \\ S_2(t) = A & 0 \leq t \leq T_b \end{cases}$ NRZ coding.



$$P_e|_{\min}|_{\min} = Q\left[\sqrt{\frac{E_d}{2N_0}}\right]$$

$$E_d = (d_{12})^2$$

$$P_e = Q\left[\sqrt{\frac{2A^2T_b}{N_0}}\right] \text{ for NRZ}$$

$$(E_s)_{avg} = A^2T_b$$

- Distance of each point from origin $= \sqrt{\text{Energy of that point}}$
- Distance between two point $= \sqrt{\text{Difference energy}} = d_{12}$
- $d_{12} \uparrow \rightarrow Q(\,) \downarrow \rightarrow P_e \downarrow$

$$P_e = Q\left[\sqrt{\frac{A^2T_b}{2N_0}}\right] \text{ for RZ}$$

Bandpass Sampling:

(a) Binary ASK : (m-ary ASK), For '1'→ A, '0', →0

$$P_e = Q\left[\sqrt{\frac{E_d}{2N_0}}\right] = Q\left[\sqrt{\frac{A^2T_b}{4N_0}}\right]$$

$$(E_b)_{avg} = p_1E_1 + p_2E_2 = \frac{1}{2}\times\left(\frac{A^2T_b}{2}\right)$$

$$P_e = Q\left[\sqrt{\frac{(E_b)_{avg}}{N_0}}\right]$$

$$P_e = Q\left[\sqrt{\frac{A^2T_b\cos^2\phi}{4N_0}}\right]$$

Correlator Based :

$$\phi(t) = \sqrt{\frac{2}{T_s}}\cos 2\pi f_C t$$

$$0 \le t \le T_s$$

$$"1" = A\cos\omega_C t \qquad 0 \le t \le T_b$$

$$"0" = 0 \qquad 0 \le t \le T_b$$

(b) BPSK : $p(t)=\begin{cases} A & 0 \le t \le T_b \; : \; 1T_X \\ -A & 0 \le t \le T_b \; : \; 0\,T_X \end{cases} \rightarrow \text{Baseband}$

$$s(t)=\begin{cases} A\cos 2\pi f_C t & 0 \le t \le T_b \; : \; 1T_X \\ A\cos(2\pi f_C t+\pi) & 0 \le t \le T_b \; : \; 0\,T_X \end{cases}$$

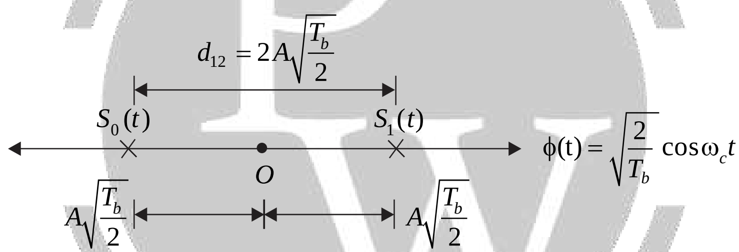
$$P_e = Q\left[\sqrt{\frac{A^2 T_b}{N_0}}\right]$$

Orthonormal Basis Function

$$\phi(t)=\sqrt{\frac{2}{T_b}}\cos 2\pi f_c t \qquad 0 \le t \le T_b$$

$$P_e = Q\left[\sqrt{\frac{A^2 T_b \cos^2\theta}{N_0}}\right]$$

$$P_e = Q\left[\sqrt{\frac{d_{12}^2}{2N_0}}\right]$$



M-Ary PSK (Quadrature PSK):

$$M=4 \qquad S_k(t)=A\cos\left(2\pi f_c t+\frac{2\pi}{M}K\right) \qquad 0 \le t \le T_s,(T_s = NT_b)$$

$$N=2 \qquad S_k(t)=A\cos\left(2\pi f_c t+\frac{\pi}{2}K\right) \qquad T_s = N\,T_b$$

$K=0,1,2,3$ $\qquad d_{\min}=2d_0\sin\left(\frac{\phi}{2}\right) \qquad d_0=\sqrt{E_s},\quad \phi=\frac{2\pi}{M}$

$$d_{12}=2d_0\sin\left(\frac{\phi}{2}\right)$$

M-ary PSK

$$M=(2^N)$$

1. Bit Interval $=T_b$, Bit rate $=R_b$, symbol duration $(T_s = NT_b)$

2. Baud rate or symbol rate $R_s=\frac{1}{T_s}=\frac{R_b}{N}$

3. Bit energy $\Rightarrow E_b = \frac{A^2}{2} T_b$

4. Symbol energy $E_s = NE_b$

5. Radius of constellation : $d_0 = \sqrt{E_s}$

6. Area of constellation circle $= \pi d_0^2 = \pi E_s$

7. $d_{\min} = 2d_0 \sin\left(\frac{\phi}{2}\right), \left(\phi = \frac{2\pi}{M}\right)$

Binary FSK

$$SFSK(t) = \begin{cases} A\cos 2\pi f_1 t : 0 \le t \le T_b : 1T_X \\ A\cos 2\pi f_2 t : 0 \le t \le T_b : 0T_X \end{cases} \qquad (f_1 >>> f_2)$$

BFSK

Same phase ← Coherent BFSK

Phase shift keying ← Non Coherent 8FSK

$$S(t) = \begin{cases} S_1(t) = A\cos(\omega_1 t + \phi_1); 0 \le t \le T_b \\ S_2(t) = A\cos(\omega_2 t + \phi_2); 0 \le t \le T_b \end{cases}$$

$$S(t) = \begin{cases} S_1(t) = A\cos\omega_1 t : 0 \le t \le T_b \\ S_2(t) = A\cos\omega_2 t : 0 \le t \le T_b \end{cases} \qquad S(t) = \begin{cases} S_1(t) = A\cos(\omega_1 t + \phi_1)\ 0 \le t \le T_b \\ S_2(t) = A\cos(\omega_2 t + \phi_2)\ 0 \le t \le T_b \end{cases}$$

Coherent BFSK

1. $\phi = 0,\ R_b = \text{HCF}[mR_b, nR_b] = \text{HCF}[2(f_1 + f_2), 2(f_1 - f_2)]$

2. $(\phi \ne 0)\ R_b = \text{HCF}[mR_b, nR_b] = \text{HCF}[(f_1 + f_2), 2(f_1 - f_2)]$

3. Non-Coherent – $R_b = \text{HCF}[mR_b, nR_b] = \text{HCF}[(f_1 + f_2), (f_1 - f_2)]$

Condition for Orthogonality

Coherent FSK

- $d = 0 (f_1 + f_2) = \frac{mR_b}{2}, (f_1 - f_2) = \frac{nR_b}{2}$
- $d \ne 0 (f_1 + f_2) = mR_b, (f_1 - f_2) = nR_b$

$R_b = \text{HCF}[mR_b, nR_b]$

Non-Coherent FSK

ϕ_1, ϕ_2

$(f_1 + f_2) = mR_b, (f_1 - f_2) = nR_b$

$R_b = \text{HCF}(mR_b, nR_b)$

➢ $P(OT_X) = P(1T_X) = \frac{1}{2}$

➢ Channel Noise : White (AWGN)

- λ_{opt}

- Filter Method → $f_1 = k \mid T_b$; → $f_2 = m \mid T_b$

$$Pe = Q\left[\sqrt{\frac{E_d}{2N_0}}\right] = Q\left[\sqrt{\frac{A^2T_b}{2N_0}}\right]$$

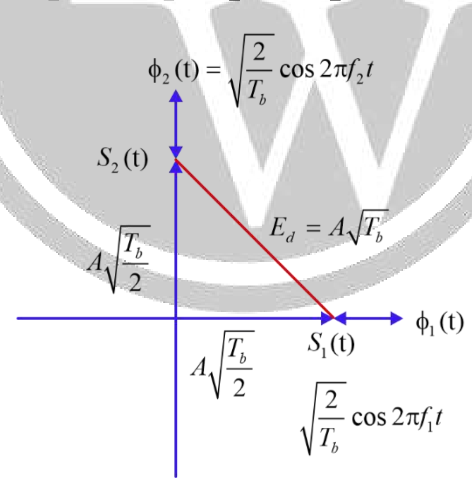
Orthogonal FSK :

$$Pe = Q\left[\sqrt{\frac{0.5A^2T_b}{N_0}}\right]$$

Non-orthogonal FSK :

$$Pe = Q\left[\sqrt{\frac{0.6A^2T_b}{N_0}}\right]$$

$$Pe = Q\left[\sqrt{\frac{(d_{12})^2}{2N_0}}\right] = Q\left[\sqrt{\frac{A^2T_b}{2N_0}}\right]$$



M-ary FSK

N bits are grouped together so that $M = 2^N$ symbols or sinusoids of duration $T_s = NT_b$ are generated having

- Same amplitude, same frequency, different frequency.

$$f_k = \frac{n}{T_s} \quad E_{s_0} = E_{s_1} = \ldots = E_{s_{M-1}} = \left(\frac{A^2}{2} \times T_s\right)$$

$$T_s = NT_b \qquad R_s = \frac{1}{T_s} = \frac{1}{NT_b} = \frac{R_b}{N}$$

Scheme		***Pe***	**For *K***
BASK	$\longrightarrow$	$P_e = Q\left[\sqrt{\dfrac{A^2T_b}{4N_0}}\right]$	$\Rightarrow P_e = Q\left[\sqrt{\dfrac{KA^2T_b}{4N_0}}\right]$
BPSK	$\longrightarrow$	$P_e = Q\left[\sqrt{\dfrac{A^2T_b}{N_0}}\right]$	$\Rightarrow P_e = Q\left[\sqrt{\dfrac{KA^2T_b}{N_0}}\right]$
BFSK	$\longrightarrow$	$P_e = Q\left[\sqrt{\dfrac{A^2T_b}{2N_0}}\right]$	$\Rightarrow P_e = Q\left[\sqrt{\dfrac{KA^2T_b}{2N_0}}\right]$
		$P_e = Q\left(\dfrac{\mu_1-\mu_2}{2N_0}\right)$	$\Rightarrow Q\left[\sqrt{K}\left(\dfrac{\mu_1-\mu_2}{2N_0}\right)\right]$

For K AWGN identical independent.

Amplitude phase shift keying (APSK)

$$S_i(t) = r_i\cos[2\pi f_c t + \theta_i] \qquad (0 \le t \le T_s) \qquad i = 0 \text{ to} \ldots M-1$$

$$T_s = NT_b$$

Case 1: $r_i =$ constant

$\theta_i =$ variable

Case 2: $r_i =$ variable

$\theta_i =$ constant

$$\begin{cases} S_i(t) = r_i\cos(2\pi f_c t + \theta_i) \rightarrow \text{M-Ary PSK} \\ S_i(t) = r_i\cos(2\pi f_c t + \theta_i) \rightarrow \text{M-Ary ASK} \end{cases}$$

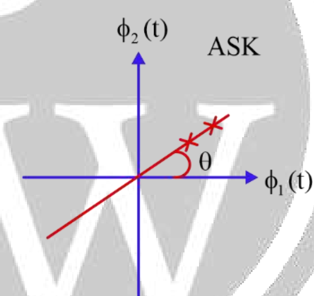


Fig. ASK

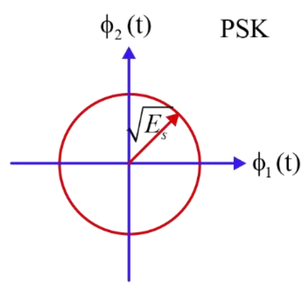
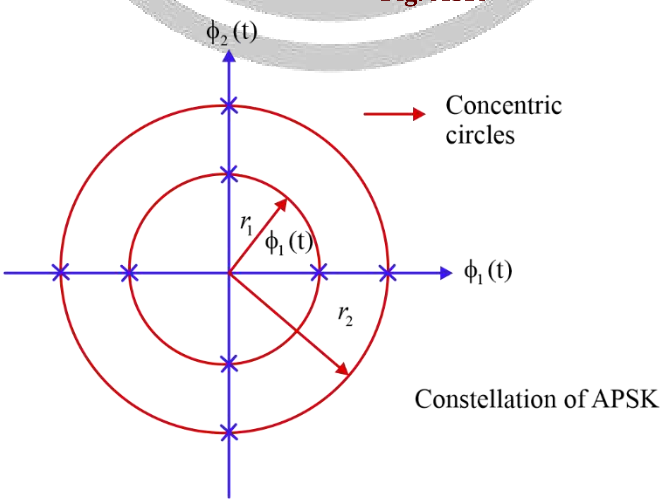


Fig. PSK



8 point APSK = 8 point QAM

6 INFORMATION THEORY

6.1. Introduction

Information in Event $(X = x_i)$	Base	Unit
$I[X = x_i] = -\log_b p\{X = x_i\}$	2	Bits
	10	Decit
	e	Nat

6.1.1. Properties of Digital Information

1. $I[X = x_i] = -\log_2 P[X = x_i]$
2. $P[X = x_i] > P[X = x_2] \Leftrightarrow I[X = x_2] < I[X = x_2]$
3. $P[X = 1] = \log_2 1 = 0$ bits
4. $P[X = 0] = -\log_2 0 = \infty$ bits
5. $0 \le P[X = x_i] \le 1 \;\Leftrightarrow 0$ bits $\le I[X = x_i] < \infty$ bits
6. For any event $[X = x_i], I[X = x_i] \ge 0$
7. $I\left[(X = x_i) \cap (X = x_2)\right] = I[X = x_1] + I[X = x_2]$

Average information of source X = Entropy of source X

$$H[X] = -\sum_{i=1}^{M} P[X = x_i] \log_2 P[X = x_i] \text{ bits/symbol}$$

$$H[X] = -\sum_{i=1}^{M} P_i \log_2 P_i$$

Case 1. All M events are equiprobable –

$H[X] = \log_2 M$ bits/symbol $\Rightarrow$ Maximum entropy

Case 2. Out of M events only 1 event is certain.

$H[X] = 0$

$$0 \le H[X] \le \log_2 M \qquad M = \frac{1}{P}$$

Information Rate – Symbol rate = r symbols/sec

Entropy = H(X) bits/symbol

Information Rate R = r H (X) bits/sec

1. If $r = f_s$ and all event equiprobable, $L = 2^n$, $H(X) = \log_2 L$

   $R = nf_s$

Source Coding

1. Reduces the redundancy of bits.
2. Two types of source coding
   (a) Fixed length source coding
   (b) Variable length source coding
      (i) Shannon Fano coding
      (ii) Huffman coding

Key Point :

(a) Average code length $= L_{avg} = \sum_{i=1}^{K} n_i p_i$

(b) Entropy of source $= H(X)$

(c) Code efficiency $\eta = \dfrac{H(X)}{L_{avg}}$

η should be as high as possible.

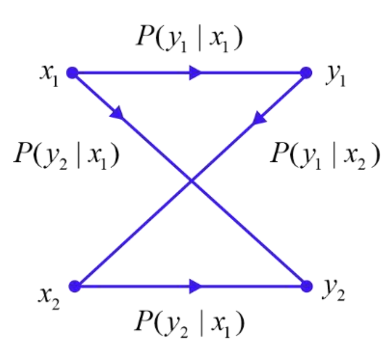
(d) Code redundancy $\lambda = (1 - \eta)$

Discrete channel : A channel is called as discrete if X and Y are having finite size.

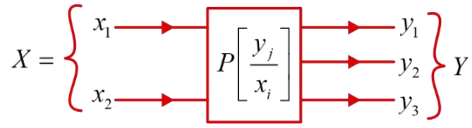
Memoryless channel : Each present output symbol depends on present input symbol.

$$x = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \rightarrow \boxed{P(y_j / x_i)} \rightarrow \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = y$$

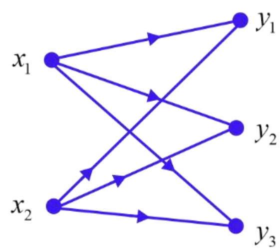
Binary Channel : (2 input & 2 output)



Non Binary Channel :



Binary Channel :



*Sum of elements of row in channel matrix is always ‘1’.

Joint Channel Matrix

$$[P(x;y)] = \begin{array}{c} \\ x_1 \\ \vdots \\ \vdots \\ x_n \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & \cdots & y_m \end{array} \\ \begin{bmatrix} P(x_1 \cap y_1) & \cdots & P(x_1 \cap y_m) \\ P(x_2 \cap y_1) & \cdots & P(x_2 \cap y_m) \\ & & \\ P(x_n \cap y_1) & \cdots & P(x_n \cap y_m) \end{bmatrix}_{n \times m} \end{array}$$

$$[P(x \cap y)] = P(X)\,P\left(\frac{Y}{X}\right)$$

Condition Channel Matrix

$$[P\left(\frac{y}{x}\right) = \begin{array}{c} x_1 \\ \vdots \\ \vdots \\ x_n \end{array} \begin{bmatrix} P\left(\frac{y_1}{x_1}\right) & \cdots & P\left(\frac{y_m}{x_1}\right) \\ \vdots & \vdots & \vdots \\ P\left(\frac{y_1}{x_n}\right) & \cdots & P\left(\frac{y_m}{x_n}\right) \end{bmatrix}_{n \times m}$$

$$[P(y)] = P(X)\,P\left(\frac{Y}{X}\right)$$

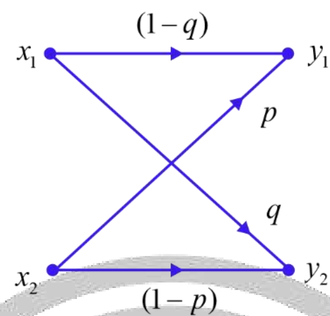
$$P(y_1) = P(x_1 \cap y_1) + P(x_2 \cap y_1) + \ldots + P(x_n \cap y_1)$$

$$[P(y)]_{1 \times m} = \left[P(x_1), P(x_2) \ldots P(x_n)\right]_{1 \times n}$$

$$[P(y)]_{1 \times m} = [P(x)]_{1 \times n} \left[P\left(\frac{Y}{X}\right)\right]_{n \times m}$$

$$\begin{bmatrix} P\left(\frac{y_1}{x_1}\right) & \cdots & P\left(\frac{y_m}{x_1}\right) \\ \vdots & & \vdots \\ P\left(\frac{y_1}{x_n}\right) & \cdots & P\left(\frac{y_m}{x_n}\right) \end{bmatrix}_{n\times m}$$

Binary Non-symmetrical channel



Cross over probabilities are different

$$P\left(\frac{Y}{X}\right) = \begin{array}{cc} \begin{array}{cc} y_1 & y_2 \end{array} & \\ \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix} & \begin{array}{c} x_1 \\ x_2 \end{array} \end{array}$$

$$= \begin{bmatrix} P\left(\frac{y_1}{x_1}\right) & P\left(\frac{y_2}{x_1}\right) \\ P\left(\frac{y_1}{x_2}\right) & P\left(\frac{y_2}{x_2}\right) \end{bmatrix}_{2\times 2}$$

- $P(x_1) + P(x_2) = 1$
- $P(y_1) + P(y_2) = 1$
- $P\left(\frac{y_1}{x_1}\right) + P\left(\frac{y_2}{x_2}\right) = 1$
- $P\left(\frac{y_1}{x_2}\right) + P\left(\frac{y_2}{x_2}\right) = 1$
- $P(y_1) = P(x_1 \cap y_1) + P(x_2 \cap y_1)$
- $P(y_2) = P(x_1 \cap y_2) + P(x_2 \cap y_2)$

Aposteriori Probabilities

$$P\left(\frac{x_1}{y_1}\right) = \frac{P(x_1)P\left(\frac{y_1}{x_1}\right)}{P(y_1)}$$

$$P\left(\frac{x_2}{y_2}\right)=\frac{P(x_2)P\left(\frac{y_2}{x_2}\right)}{P(y_2)}$$

Map Analysis

(a) At r_0 :

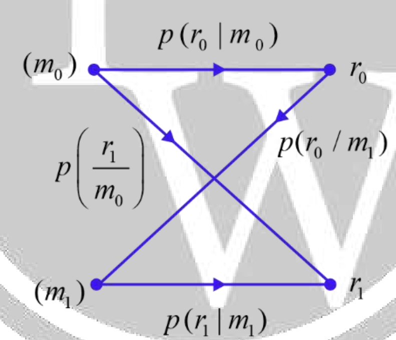
$$P\left(\frac{m_0}{r_0}\right)\underset{m_1}{\overset{m_0}{\gtrless}}P\left(\frac{m_1}{r_0}\right)$$

$$P(m_0)P\left(\frac{r_0}{m_0}\right)\underset{m_1}{\overset{m_0}{\gtrless}}P(m_1)P\left(\frac{r_0}{m_1}\right)$$

(b) At r_1:

$$P\left(\frac{m_0}{r_1}\right)\underset{m_1}{\overset{m_0}{\gtrless}}P\left(\frac{m_1}{r_1}\right)$$

$$P(m_0)P\left(\frac{r_1}{m_0}\right)\underset{m_1}{\overset{m_0}{\gtrless}}P(m_1)P\left(\frac{r_1}{m_1}\right)$$



- After MAP application receiver will be optimum.

Probability of Correctness

$$P_c = P(m_0 \cap r_0) + P(m_0 \cap r_1)$$

$$P_c = P(m_0)P\left(\frac{r_0}{m_0}\right) + P(m_0)P\left(\frac{r_1}{m_0}\right)$$

$$P_e = 1 - P_c$$

Binary Symmetrical Channel

Cross over probabilities are same.

- $P\left(\frac{x_1}{y_2}\right) =$ Probability that x_1 was transmitted given than y_2 received

$$P\left(\frac{x_1}{y_2}\right)+P\left(\frac{x_2}{y_2}\right)=1$$

$$P\left(\frac{x_1}{y_1}\right)+P\left(\frac{x_2}{y_1}\right)=1$$

Joint Entropy

$$H(XY)=-\sum_{i=1}^{n}\sum_{j=1}^{m}P(x_i,y_j)\log_2 P(x_i,y_j)$$

$$H(XY)=-\sum_{i=1}^{n}\sum_{j=1}^{m}P\{(X=x_i)\cap(Y=y_j)\}\log_2 P\{(X=x_i)\cap(Y=y_j)\}$$

$$H(XY)=H(YX)$$

Conditional Entropy

$$H\left(\frac{X}{Y}\right)=-\sum_{i=1}^{n}\sum_{j=1}^{m}P\{(X=x_i)\cap(Y=y_j)\}\log_2 P\left\{\frac{X=x_i}{Y=y_j}\right\}$$

Conditional entropy of
X given Y

- Similarly can write $H\left(\frac{Y}{X}\right)$

Important point :

1. $H(XY)=H(X)+H\left(\frac{Y}{X}\right)$
2. $H(XY)=H(Y)+H\left(\frac{X}{Y}\right)$
3. If X and Y statically, independent $H(XY)=H(X)+H(Y)$

$$H\left(\frac{Y}{X}\right)=H(Y), H\left(\frac{X}{Y}\right)=H(X)$$

For B.S.C – C_s = Channel capacity

$$C_s=1+P\log_2 P+(1-P)\log_2(1-P)$$

$$C_s=\{I(X;Y)\}_{\max}$$

$$I(X;Y)=H(Y)-H\left(\frac{Y}{X}\right)$$

$$I(X;Y)=H(X)-H\left(\frac{X}{Y}\right)$$

$$H\left(\frac{Y}{X}\right) = -\sum\sum P(x_i, y_j)\log_2 P\left(\frac{y_i}{x_i}\right)$$

$$C_s = \log_2 n$$

$I(X;Y) = H(X)$ loss less channel

Lossless Channel :

1. Single non zero element in each column.
2. Channel matric should be DMC type
3. Summation of each now must be 1.
4. $H\left(\frac{X}{Y}\right) = 0 \quad I(X;Y) = H(X),\ C_s = I[X;Y]_{\max} = [H(X)]_{\max} = \log_2 n$

   $n =$ number of input symbol.

6.2. Average Mutual Information

$$I(X;Y) = I(X) - I\left(\frac{X}{Y}\right)$$

$$I(X;Y)_{Avg} = H(X) - H\left(\frac{X}{Y}\right) \text{ bit/symbol}$$

$$I(X;Y)_{Avg} = H(Y) - H\left(\frac{Y}{X}\right)$$

$$I(X;Y) = I(Y;X)$$

$$I(X;Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P\{x_i, y_j\}\log_2\left[\frac{P(x_i \mid y_j)}{P(x_i)}\right]$$

$$I(X;Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\log_2\left[\frac{f_{\frac{X}{Y}}\left(\frac{x}{y}\right)}{f_X(x)}\right]dxdy$$

$$P_{XY}(x_i, y_j) = P_X(y_j)P\left(\frac{x_i}{y_j}\right)$$

$$P_{XY}(x_i, y_j) = P(x_i)P\left(\frac{y_j}{x_i}\right)$$

If R.VS are Independent then $I(x;y) = 0$

6.2.1. Channel Capacity

Maximum Average Mutual Information

$$C_s = \{I(x;y)\}_{\max}$$

$$I(x;y) = H(Y) + P\log_2 P + (1-P)\log_2(1-P)$$

$$C_s = 1 + P\log_2 P + (1-P)\log_2(1-P)$$

B.S.C

P $\rightarrow$ Cross over probability

- Input are equiprobable.

Determine Channel :

1. Number of rows in each row must be single.
    - In each row angle element must be 1.
    - Summation of each row will become 1.
    - $H\left(\frac{Y}{X}\right) = 0, I(X;Y) = H(Y) - H\left(\frac{Y}{X}\right)$

      $I(X;Y) = H(Y)$

      $C_s = [H(Y)]_{\max} = \log_2 m$ bit/symbol

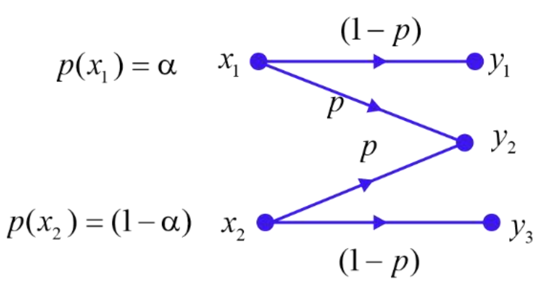
Noise Less Channel :

1. Deterministic + Lossless
2. Each row $\rightarrow$ Single element
3. Each column $\rightarrow$ Single element

   Mult be 1
4. $H\left(\frac{X}{Y}\right) = 0, H\left(\frac{Y}{X}\right) = 0$
5. $I(X;Y) = H(X) = H(Y)$
6. $C_s = [I(X;Y)]_{\max} = [H(X)]_{\max} = [H(y)]_{\max} = \log_2 m = \log_2 n$ bits/symbol

   $m = n$

Binary Erasure Channel



$$I(X;Y) = (1-P)H(X)$$

$$C_s = I[(X;Y)]_{\max}$$

$$= (1-P)\log_2 n \qquad n = 2 \text{ for BEC}$$

$$C_s = (1-P)$$

➢ $C_s = I(X;Y) = \frac{1}{2}\log_2\left[1+\left(\frac{\sigma_x^2}{\sigma_N^2}\right)\right]$ Bits/symbol

$\sigma_N^2 = N_0 B$

(i) X is zero mean R.V $\quad E[X^2] = \sigma_x^2 = S$

(ii) Noise is zero mean R.V $\quad E[N^2] = \sigma_N^2 = N$

Channel Capacity of AWGN Channel

$$C_s = \frac{1}{2}\log_2\left(1+\frac{S}{N}\right) \text{ Bit/symbol}$$

$$C = B\log_2\left(1+\frac{S}{N}\right) \text{ bit/sec}$$

Channel capacity for AWGN

$$\underset{\text{Channel Capacity}}{C_s} \geq \underset{\text{Information rate}}{R}$$

- For X = zero mean R.V, $N = N_0 B$

  $B \to \infty$

$$C_s = 1.44\frac{S}{N_0} \quad \text{Finite value}$$

For loss less transmission.

6.3. Continuous Source and Differential Entropy

X : DRV, H(X) $= -\sum_i p[x = x_i]\log_2 p\{x = x_i\}$

X : CRV, H(X) = $-\int_{-\infty}^{+\infty} Fx(x)\log_2 f_x(x)dx \to$ Differential entropy

Y : DRV, H(Y) $= -\sum_j p\{y = y_j\}\log_2 p\{y = y_j\}$

Y : CRV, H(Y) = $-\int_{-\infty}^{-\infty} f_y(y)\log_2 f_y(y)dy \to$ Differential entropy

6.3.1. Channel capacity

(1) For error less | distortion less transmission

(i) If all quantization level are not eauiprobable :

$$C \geq R$$

$$C \geq rH(X)$$

$$C \geq f_S H(X)$$

(ii) If all quantization level are eauiprobable :

$$C \geq R$$

$$C \geq rH(X)$$

$$C \geq f_s \log_2 L$$

$$C \geq nf_s$$

$$\boxed{C \geq R_b}$$

(2) For AWGN channel- Y: GRV,

Y = X + N, Let X and N are independent

$$\sigma_y^2 = \sigma_x^2 + \sigma_N^2$$

$$H(y) = \frac{1}{2}\log_2[2\pi\sigma_y^2 e]$$

$$\boxed{H(\text{y}) = \frac{1}{2}\log_2[2\pi\text{e}[\sigma_N^2 + \sigma_X^2]} \quad \text{Maximum}$$

P_x = Power of signal X

$$\boxed{C_S = \frac{1}{2}\log_2\left(1+\frac{\sigma_X^2}{\sigma_N^2}\right)} \frac{\text{bits}}{\text{symbol}}$$

P_x = Noise Power

$$C = C_2 \times f_s$$

$$\sigma_N^2 = N_0 B$$

$$\boxed{C = B\log_2\left(1+\frac{\sigma_X^2}{\sigma_N^2}\right)} \frac{\text{Bits}}{\text{sec}} \quad \frac{N_0}{2} \rightarrow \text{PSD of white Noire}$$

$B \rightarrow$ B.W of channel

$$\boxed{C = B\log_2(1+\text{SNR})}$$
$\downarrow$
Not in dB

$$\boxed{\delta_{NR} = \frac{P_X}{P_N} = \frac{\sigma_X^2}{\sigma_N^2}}$$

$$\boxed{C = B\log_2\left(1+\frac{E_b R_b}{N_0 B}\right)}$$

$$\boxed{C = 1.44\frac{P_X}{N_0}}$$

For infinite Bandwidth $B \to \infty$

Information in bits | symbol

$$H\left[\frac{y}{x_0}\right] = I\left[\frac{y_0}{x_0}\right]P\left[\frac{y_0}{x_0}\right]P(x_0) + I\left[\frac{y_1}{x_0}\right]P\left[\frac{y_1}{x_0}\right]P(x_0)$$

7 MISCELLANEOUS

7.1. FDMA (Frequency Division Multiplexing)

- Multiple signals are multiplexed and simultaneously transmitted through channel.

  K = Number of signals are multiplexed

  B.W ≥ K [B.W of modulation scheme] + (K – 1) [BW of guard Band]

  TDMA (Time division Multiplexing)

  T_s = Frame rate or sampling interval or time taken by commentator to completer its 1 rotation

  (Band limited to same freq.)

  $T_s = nT_b \times N$

  $\boxed{T_s = NnT_b}$ N = Number of signals being multiplexed

  n = of bits/sample

  $T_b = 1$ Bit duration

$$\boxed{R_b = Nnf_s}$$

$$\text{Speed of commentator } = f_s \frac{\text{rotation}}{\text{seocnd}} = f_s \times 60\,\text{rpm}$$

$$(\text{BW})_{\min} = \frac{R_b}{2} = \frac{Nnf_s}{2}$$

- When x number of synchronization $\dfrac{\text{bits}}{\text{frame}}$ are added – (Band limited to same freq.)

$$\boxed{T_s = (Nn + x)T_b} \qquad \boxed{R_b = (Nn + x)\, f_s}$$

- $x\,\text{bit/frame} : Ts = (Nn + x)T_b$

  $x\,\text{bit/2frame} : Ts = \left(Nn + \dfrac{x}{2}\right)T_b$

- y% (Total of y%) synchronization bits are added –(Band limited to same freq.)

$$T_s = \left[Nn + \frac{Nn \times y\%}{100}\right]T_b$$

$$R_b = \left[Nn + \frac{Nn \times y\%}{100}\right] f_s$$

- When N signals are band limited to different freq.

$$R_b = nf_{s_1} + nf_{s_2} + ...nf_{s_n}$$

COMA (Code division Multiple Aces)

Processing gain of CDMA $\Rightarrow G = \left(\frac{R_c}{R_h}\right)$

- Each user is assigned with unique code

Noise

(1) PSD of thermal noise is Gaussian in nature. Also known as Johnson noise

(2) Thermal Noise power $P_n = 4KTBR = \overline{V_n^2} = (\mathrm{V}_n)^2 \,\mathrm{rms}$

Thermal Noise voltage

$$(V_n)\mathrm{rms} = \sqrt{4KTBR}$$

$$(I_n)\mathrm{rms} = \frac{(V_n)\mathrm{rms}}{R}$$

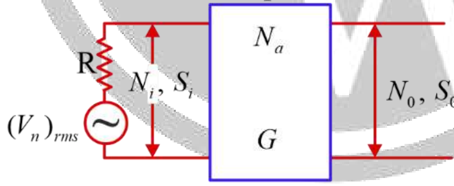
- Max. Power which could be deliver to amplifier = KTB
- Noise figure (F) or Noise factor

$$F(dB) = 10\log_{10} F$$

$$N_1 = \mathrm{kTB}$$

$$N_o = N_i G + N_a = KTBG + N_a$$

Amplifier G: Power gain



Te: Equivalent Temp of amplifier

$$\boxed{F = \frac{(N_i G + N_a)}{GN_i}}$$

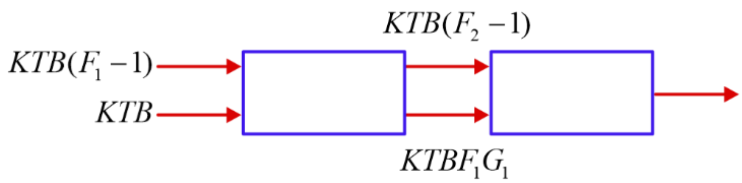
$$\mathrm{F} = \frac{\text{Output Noise including Noisy amplifier}}{\text{Output Noise Considering noiseless amplifier}}$$

$$\boxed{T_e = \frac{Na}{\mathrm{KBG}}} \qquad \boxed{F = \frac{(\mathrm{SNR})_{i/\mathrm{p}}}{(\mathrm{SNR})_{o/p}}} = 1 + \frac{T_e}{T}$$

$\boxed{T_e = (f-1)T}$ $\qquad$ N_0 (output Noise power) = KTBGF

$$N_0 = K\,BG(T + T_e)$$

Cascaded Amplifier



Output Noise with noisy $amp^r = [KTB(F_2-1) + KTBF_1G_1]G_2$

$$F = F_1 + \frac{F_2-1}{G_1}$$

$$F = F_1 + \frac{F_2-1}{G_1} + \frac{(F_3-1)}{G_1G_2} + \dots\dots\dots\dots$$

$$T_e \text{ (equivalent Temp.) } = T_{e_1} + \frac{T_{e_2}}{G_1} + \frac{T_{e_3}}{G_1G_2} + \dots\dots\dots\dots$$

$$xdBW = (x+30) \text{ dBm}$$

Noise performance of Analog Signal

$$FOM = \frac{(\text{SNR})_0}{(\text{SNR})_i} = \frac{\text{SNR at the output of } R_X}{(\text{SNR}) \text{ in m}} = \gamma$$

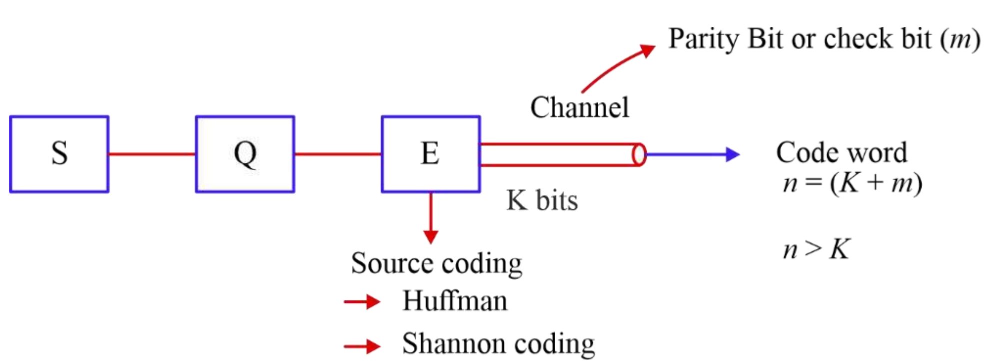
For DSB-Se $\boxed{\gamma = 1}$, $(SNR)_0 = \frac{P_m}{2N_0B}, (SNR)_i = \frac{P_m}{2N_0B}$

For DCB-FC $\boxed{\gamma = \frac{p_m}{A_c^2 + p_m}} = \eta \rightarrow \text{efficiency}$

For F.M $\gamma = \frac{3}{4\pi^2}\frac{K_f^2 p_m}{B^2} = \frac{3}{2}\beta_{FM}^2$ (For sinusoidal)

For PM $\gamma = K_p^2 P_m = \frac{\beta_{PM}^2}{2}$ (for sinusoidal)

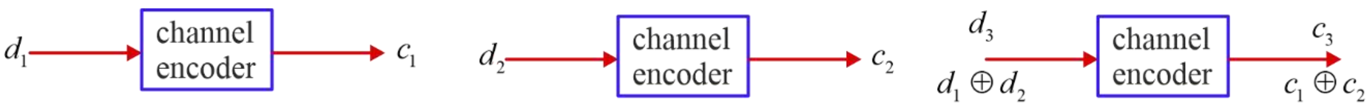
Channel Coding



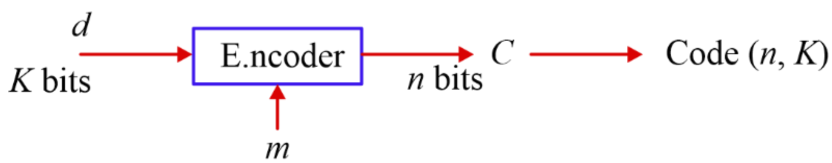
Linear Block Code

$\oplus$ modulo 2 sum $\rightarrow$ EXOR

(1)



(2)



(3) Different data words (message word) with K bits = 2^K

(4) Each data word will have m parity bits attached to generate 2^K code words.

(5) Total no of arrangements with n bits at output of encoder will be $\rightarrow 2^n$ out of which only 2^K code words are valid.

(6) Rate efficiency = code efficiency = code rate = K / n

Hamming Weight

Number of 1's present in L, B,C

C (7.4)

Example: C : 1110001, H.W = 4

Hamming distance

It represent bit change at respective position

$$\begin{aligned} X &= 1\,1\,0\,1\,0\,1\,1\,1 \\ Y &= 0\,1\,1\,1\,0\,1\,0\,1 \quad d(x,y) = 3 \end{aligned}$$

Minimum Hamming Distance (d_{min}):

Method 1 d_{min} = Min hamming weight of 2^K codes except codes having 0 weight.

Combination: $^{2^K}C_2$ crosscheck

Method 2 $d\min \le n - K + 1$

Method 3 "Minimum no of columns in parity check matrix [H] Which makes zero sum (module 2)."

Error detection L.B.C

$\boxed{d_{\min} \ge t+1}$ can detect t errors

Error correction $\boxed{d_{\min} \ge 2t+1}$

Code Generation at $T_X - [C]_{1\times n} = [D]_{1\times K}[G]_{K\times n}$ $\boxed{C = DG}$

$[G]_{K\times n} \rightarrow$ Generator Matrix

$[G]_{K\times n} = [I_K; p]_{K\times n}$ or $[p; I_K]_{K\times n}$ I_k = Identity Matrix of order K.

Parity check Matrix – $[H] = [P^T;\ I_n - K]_{(n-K)\times n}$

Or

$[H] = [I_{n-K};\ P^T]_{(n-K)\times n}$

Note – $[C]\,[H^T] = 0$

Correction at Receiver

$c \to r$

$r = C$ (No error)

$r \neq C$ (Error)

- r will given
- Calculate syndrome : $S = r[H]^T$
- Observe the syndrome: S matches with i^{th} row of $[H]^T$ which Means i^{th} bit from left has error.

Non systematic L.B.C

Hamming Code

(1) It is a L.B.C

(2) $d_{\min} = 3$

(3) Detect upto 2 bit error

(4) Correct upto 1 bit error

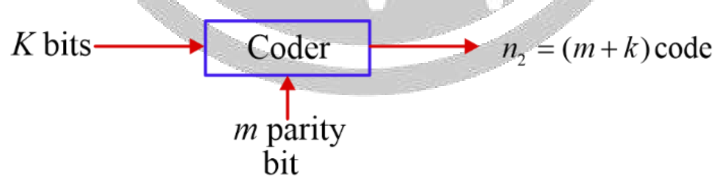
(5) K bit data, m bits parity $\Rightarrow n = (m + K)$ bits code

(6) Parity bit no. is calculated $2^m \geq (m + K + 1)$

$m = ?$

(7) Placing of parity bits ate at $2^0, 2^1, 2^2, \ldots\ldots..$ locations

Cyclic redundancy check code (CRC-Code)



Problem solving Technique:

(i) $d = K$ bits msg

(ii) divisor polynomial

$x^3 + x + 1 = x^3 + 0x^2 + x + 1 \to (1011)$

Step 1. K msg bits are given

From (K + m) message bits

$\neq m \to$ addition of m zeros(append)

- Highest order of divisor polynomial (or) (number of bits in divisor polynomial)-1

Step 2. Modulo 2 division

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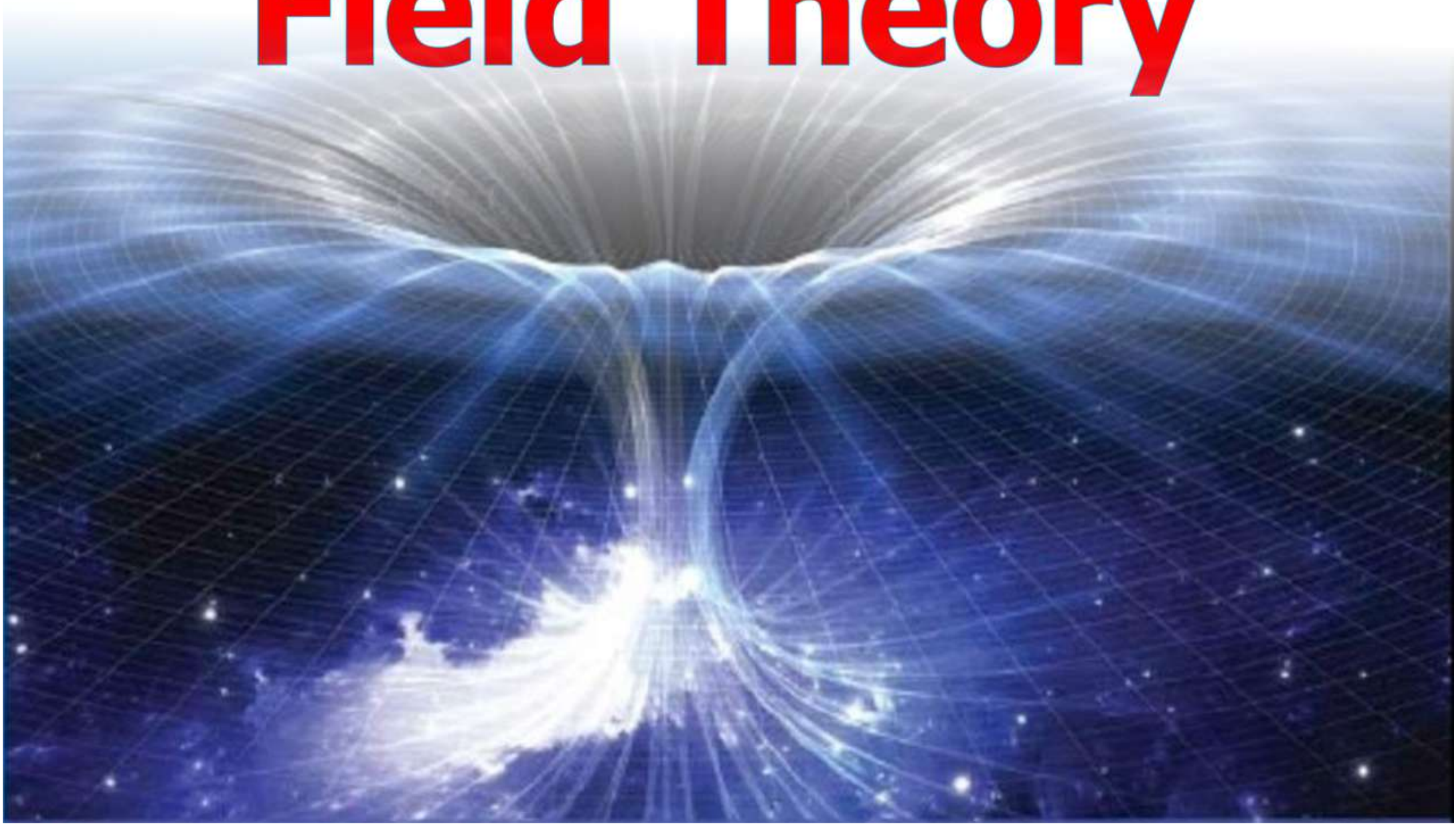
Electromagnetic Field Theory



ELECTROMAGNETIC FIELD THEORY

INDEX

1 CO-ORDINATE SYSTEM

1.1. Introduction

1. **Vector :** It has magnitude and direction. In addition, it follows Vector Law of Addition.
   e.g. :- Electric field, Magnetic Field, Force etc.

2. **Scalar :** It has magnitude and no direction. It does not follow Vector Law of Addition.
   eg :– Current, Distance, Potential etc.

3. **Tensor :** It has magnitude and direction. It does not follow Vector Law of Addition. It shows different values in different directions at the same point.
   e.g.:– Conductivity, Resistivity, Refractive Index.

4. **Unit Vector:** It is the vector which has unit magnitude and directed along increasing direction of parameters.

1.2. Equation of line in 3-dimentional

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}=constant$$

Or
$$\frac{x-x_2}{x_2-x_1}=\frac{y-y_2}{y_2-y_1}=\frac{z-z_2}{z_2-z_1}=constant$$

Equation of Plane In 3-Dimentional.

$$ax+by+cz=d$$

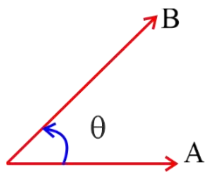
Eg. $$3x+4y=5$$

Equation of line in two-dimensional. However, equation of plane in three-dimensional.

1. X = Constant.
   (a) A plane is parallel to Y and Z-axis.
   (b) Y and Z-axis is tangential component.
   (c) X axis is normal component.
2. Y = Constant.
   (a) A plane is parallel to X and Z-axis.
   (b) X and Z axis is tangential component.
   (c) Y axis is normal component

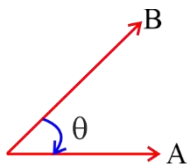
3. Z = Constant
   (a) A plane is parallel to X and Y-axis.
   (b) X and Y-axis is tangential components.
   (c) Z axis is normal components.

1.2.1. Cross Product



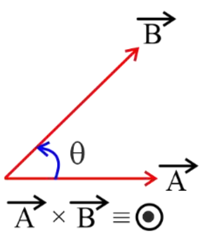
$$\vec{A}\times\vec{B}\equiv AB\sin\theta\odot$$

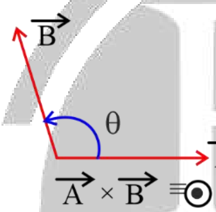
$\odot\equiv$ Outward direction (Anticlockwise)

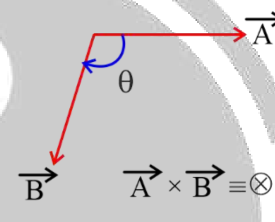


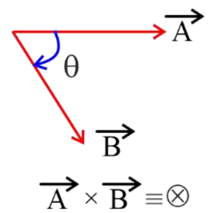
$$\vec{B}\times\vec{A}\equiv AB\sin\theta\otimes$$

$\otimes\equiv$ Inward direction







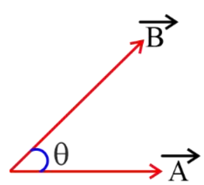


$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

1.2.2. Dot Product

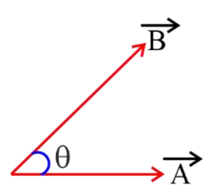
$$\vec{A}\bullet\vec{B}=a_1b_1+a_2b_2+a_3b_3$$

1. Projection of B along Vector $\vec{A}$



A||ar = B cos θ

2. Projection of vector $\vec{B}$ along Vector $\vec{A}$.



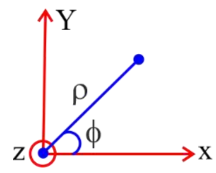
$\vec{A}$||ar = B cos θ $\hat{A}$

3. Projection of vector $\vec{B}$ perpendicular to Vector $\vec{A}$.

$\vec{A}_{\perp r}=\vec{B}-(B\cos\theta)\hat{A}$.

Point Conversion

1. Cartesian to cylindrical

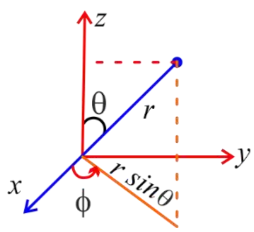


$$\rho = \sqrt{X^2 + Y^2},\ \ \phi = \tan^{-1}\left(\frac{Y}{X}\right),\ \ Z = Z$$

2. Cylindrical to Cartesian

$$X = \rho\cos\phi,\ Y = \rho\sin\phi,\ \ \mathrm{Z} = \mathrm{Z}$$

3. Cartesian to spherical



$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

4. Spherical to Cartesian

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

5. Cylindrical to Spherical

$$r = \sqrt{\rho^2 + z^2},\ \theta = \cos^{-1}\left(\frac{z}{\sqrt{\rho^2 + z^2}}\right), \phi = \phi$$

6. Spherical to Cylindrical

$$\rho = r\sin\theta,\ \phi = \phi,\ z = r\cos\theta.$$

Unit vector Conversion

1. Cartesian to Cylindrical

$$\begin{bmatrix} \hat{a}_\rho \\ \hat{a}_\phi \\ \hat{a}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{a}_x \\ \hat{a}_y \\ \hat{a}_z \end{bmatrix}$$

2. Cylindrical to Cartesian

$$\begin{bmatrix} \hat{a}_x \\ \hat{a}_y \\ \hat{a}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{a}_\rho \\ \hat{a}_\phi \\ \hat{a}_z \end{bmatrix}$$

3. Cartesian to Spherical

$$\begin{bmatrix} \hat{a}_r \\ \hat{a}_\theta \\ \hat{a}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{a}_x \\ \hat{a}_y \\ \hat{a}_z \end{bmatrix}$$

4. Spherical to Cartesian

$$\begin{bmatrix} \hat{a}_x \\ \hat{a}_y \\ \hat{a}_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{a}_r \\ \hat{a}_\theta \\ \hat{a}_\phi \end{bmatrix}$$

5. Spherical to Cylindrical.

$$\begin{bmatrix} \hat{a}_\rho \\ \hat{a}_\phi \\ \hat{a}_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{a}_r \\ \hat{a}_\theta \\ \hat{a}_\phi \end{bmatrix}$$

6. Cylindrical to Spherical

$$\begin{bmatrix} \hat{a}_r \\ \hat{a}_\theta \\ \hat{a}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{a}_\rho \\ \hat{a}_\phi \\ \hat{a}_z \end{bmatrix}$$

1.3. Differential Length, Area and Volume

Parameter			Coefficient		
u	v	w	h_1	h_2	h_3
x	y	z	1	1	1
ρ	ϕ	z	1	ρ	1
r	θ	ϕ	1	r	$r\sin\theta$

1. Differential Length

It is a vector quantity and directed along tangential direction.

In general form.

$$\vec{dl} = h_1 du\hat{a}_u + h_2 dv\hat{a}_v + h_3 dw\hat{a}_w$$

(a) Cartesian co-ordinate system.

$$\vec{dl} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z$$

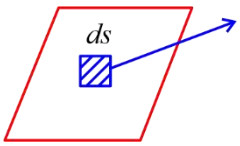
(b) Cylindrical Co-ordinate system.

$$\vec{dl} = d\rho\hat{a}_\rho + \rho d\phi\hat{a}_\phi + dz\hat{a}_z$$

(c) Spherical co-ordinate system.

$$\vec{dl} = dr\hat{a}_r + rd\theta\hat{a}_\theta + r\sin\theta d\phi\hat{a}_\phi$$

2. Differential Area



$$d\vec{s} = ds\hat{a}_n$$

$$\hat{a}_n = \text{unit normal to the surface}$$

It is a vector quantity and directed normal to the surface.
In general Form

$$d\vec{S} = h_2h_3dvdw\hat{a}_u + h_1h_3dudw\hat{a}_v + h_1h_2dudv\hat{a}_v$$

(a) Cartesian co-ordinate system.

$$d\vec{S} = dydx\hat{a}_x + dxdz\hat{a}_y + dxdy\hat{a}_z$$

(b) Cylindrical co-ordinate system.

$$d\vec{S} = \rho d\phi dz\hat{a}_\rho + d\rho dz\hat{a}_\phi + \rho d\rho d\phi\hat{a}_z$$

(c) Spherical co-ordinate system3.

$$d\vec{S} = r^2\sin\theta d\theta d\phi\hat{a}_r + r\sin\theta dr d\phi\hat{a}_\theta + r dr d\theta\hat{a}_\phi$$

3. Differential Volume

In general form

$$dv = h_1h_2h_3dudvdw$$

(a) Cartesian co-ordinate system :–

$$dv = dxdydz$$

(b) Cylindrical co-ordinate system :–

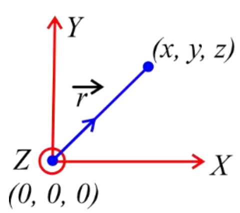
$$dv = \rho d\rho d\phi dz$$

(c) Spherical co-ordinate system.

$$dv = r^2\sin\theta dr d\theta d\phi$$

1.3.1. Position Vector

(a) Cartesian co-ordinate system



$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

(b) Cylindrical Co-ordinate system.

$\vec{r} = \rho\hat{a}_\rho + z\hat{a}_z$ → ‘Z’ is an axis

$\vec{r} = \rho\hat{a}_\rho + y\hat{a}_y$ → ‘Y’ is an axis

$\vec{r} = \rho\hat{a}_\rho + x\hat{a}_x$ → ‘X’ is an axis

(c) Spherical Co-ordinate system :

$$\vec{r} = r\hat{a}_r$$

(d) Position vector in 2D

$$\vec{\rho} = \rho\hat{a}_\rho = (x_2 - x_1)\hat{a}_x + (y_2 - y_1)\hat{a}_y \rightarrow 'Z'axis$$

$$\vec{\rho} = \rho\hat{a}_\rho = (y_2 - y_1)\hat{a}_y + (z_2 - z_1)\hat{a}_z \rightarrow 'X'axis$$

$$\vec{\rho} = \rho\hat{a}_\rho = (x_2 - x_1)\hat{a}_x + (z_2 - z_1)\hat{a}_z \rightarrow 'Y'axis$$

❒❒❒

2 VECTOR CALCULUS

2.1. Introduction

I. Differential Form

1. Gradient of scalar function
2. Divergence.
3. Curl.
4. Laplacian operator.

II. Integral Form

1. Open line integration
   (a) Path independent
   (b) Path dependent
2. Closed Line integration.
3. Closed Surface integration.
4. Volume integration.

Gradient of scalar function 'f': –

1. **Definition**: It is a vector quantity which gives maximum rate of change of scalar function 'f' and is directed normal to surface 'f' or scalar function 'f'.

$$\vec{\nabla} f = \left.\frac{df}{dl}\right|_{\max} \hat{a}_n$$

2. **Formulae**:

$$\vec{\nabla} f = \frac{1}{h_1}\frac{\partial f}{\partial u}\hat{a}_u + \frac{1}{h_2}\frac{\partial f}{\partial v}\hat{a}_v + \frac{1}{h_3}\frac{\partial f}{\partial w}\hat{a}_w$$

(a) Cartesian Co-ordinate system.

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{a}_x + \frac{\partial f}{\partial y}\hat{a}_y + \frac{\partial f}{\partial z}\hat{a}_z$$

(b) Cylindrical Co-ordinate system.

$$\vec{\nabla} f = \frac{\partial f}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{a}_\phi + \frac{\partial f}{\partial z}\hat{a}_z$$

(c) Spherical Co-ordinate system.

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{a}_\phi$$

3. Physical Significance:

(a) Maximum rate of change of scalar function 'f' will be given by gradient of scalar function 'f'.

$$\left.\frac{df}{dl}\right|_{\max} = \left|\vec{\nabla} f\right|$$

(b) To find directional derivate.

$$D.D = \vec{\nabla} f . \hat{A}$$

(c) It gives unit normal on surface.

$$\hat{a}_n = \frac{\vec{\nabla} f}{\left|\vec{\nabla} f\right|}$$

(d) To find angle between the surfaces.

(e) $\vec{A} = \vec{\nabla} f$

$$'f' \equiv \text{Scalar function of vector} \vec{A}$$

4. Properties of Gradient

(a) $\vec{\nabla}(f+g) = \vec{\nabla} f + \vec{\nabla} g$

(b) $\vec{\nabla}(fg) = f\vec{\nabla} g + g\vec{\nabla} f$

(c) $\vec{\nabla}\left(\frac{f}{g}\right) = \frac{g\vec{\nabla} f - f\vec{\nabla} g}{g^2}$

5. Application:

(a) $\vec{\nabla} r = \hat{a}_r$

(b) $\vec{\nabla}\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$

(c) $\vec{\nabla}\left(r^n\right) = \left(nr^{n-2}\right)\vec{r}$

(d) $\vec{\nabla}(lnr) = \frac{\vec{r}}{r^2}$

(e) $\vec{\nabla}\left(r^2 + lnr\right) = \left(2r + \frac{1}{r}\right)\hat{a}_r$

(f) $\vec{\nabla}\left(r^2 lnr\right) = r(2 + \ln r)\hat{a}_r$

(h) $\vec{\nabla}\left(\frac{lnr}{r^2}\right) = \left(\frac{r - 2rlnr}{r^4}\right)\hat{a}_r$

Divergence:

1. Definition: It gives total outward flux per unit volume.

$$\vec{\nabla}.\vec{A} = \frac{\oiint \vec{A}.d\vec{S}}{\lim\limits_{\Delta v \to 0} \Delta v}$$

(a) $\vec{\nabla}.\vec{A} \equiv$ *Divergence at a point.*

(b) $\oiint \vec{A} \bullet d\vec{S} \equiv$ *Divergence in a range.*

(c) $\oint\!\!\oint \vec{A} \bullet d\vec{S} = \int_{v} (\vec{\nabla} \bullet \vec{A}) dv \Rightarrow$ *Divergence theorem.*

2. Formulae:

$$\vec{\nabla} \bullet \vec{A} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_2 h_3 A_u)}{\partial u} + \frac{\partial(h_1 h_3 A_v)}{\partial v} + \frac{\partial(h_1 h_2 A_w)}{\partial w}\right)$$

(a) Cartesian Co-ordinate System

$$\vec{\nabla} \bullet \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

(b) Cylindrical Co-ordinate System.

$$\vec{\nabla} \bullet \vec{A} = \frac{1}{\rho}\left(\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{\partial A_\phi}{\partial \phi} + \frac{\partial(\rho A_z)}{\partial z}\right)$$

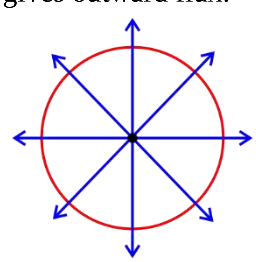
(c) Spherical Co-ordinate system.

$$\vec{\nabla} \bullet \vec{A} = \frac{1}{r^2 \sin\theta}\left(\frac{\partial(r^2 \sin\theta A_r)}{\partial r} + \frac{\partial(r \sin\theta A_\theta)}{\partial \theta} + \frac{\partial(r A_\phi)}{\partial \phi}\right)$$

3. Physical Significance:

It gives outward flux.

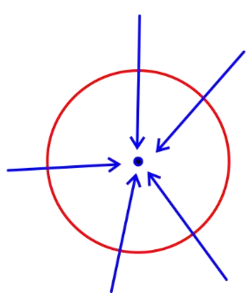
(a)



Outward Flux ≠ 0
Inward Flux = 0
Net Flux ≠ 0

$\vec{\nabla} \bullet \vec{A} > 0$

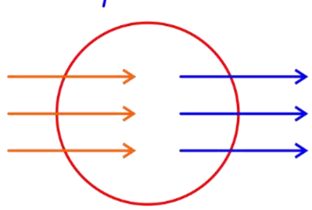
(b)



Outward Flux = 0
Inward Flux ≠ 0
Net Flux ≠ 0

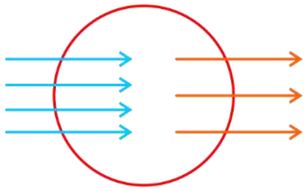
$\vec{\nabla} \bullet \vec{A} < 0$

(c)



Outward Flux
= Inward Flux
Net Flux = 0

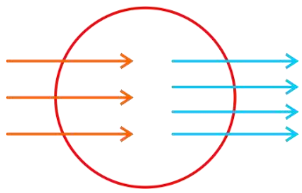
$\vec{\nabla} \bullet \vec{A} = 0$

(d) 

Outward Flux < Inward Flux

Net Flux ≠ 0

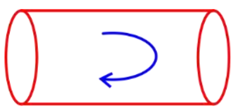
$\vec{\nabla}\cdot\vec{A}<0$

(e) 

Outward Flux > Inward Flux

Net Flux ≠ 0

$\vec{\nabla}\cdot\vec{A}>0$

(f) 

Outward Flux = 0
Inward Flux = 0
Net Flux = 0

$\vec{\nabla}\cdot\vec{A}=0$

4. Properties

(a) $\vec{\nabla}\cdot\left(\vec{A}+\vec{B}\right)=\vec{\nabla}\cdot\vec{A}+\vec{\nabla}\cdot\vec{B}$

(b) $\vec{\nabla}\cdot\left(f\,\vec{A}\right)=f\left(\vec{\nabla}\cdot\vec{A}\right)+\vec{A}\cdot\left(\vec{\nabla}f\right)$

(c) $\vec{\nabla}\cdot\left(\frac{\vec{A}}{f}\right)=\frac{f\left(\vec{\nabla}\cdot\vec{A}\right)-\vec{A}\cdot\left(\vec{\nabla}f\right)}{f^2}$

5. Application

(a) $\vec{\nabla}\cdot\vec{r}=3$

(b) $\vec{\nabla}\cdot\left(r^n\hat{a}_r\right)=(n+2)r^{n-1}$

Curl:

1. Definition:

$$\vec{\nabla}\times\vec{A}=\frac{\oint\vec{A}\cdot\vec{dl}}{\lim\limits_{\Delta S\to 0}\Delta S}\hat{n}$$

Curl gives total Motive Force due to $\vec{A}$ per unit area. And it is directed normal to the rotatory plane.

(a) $\vec{\nabla}\times\vec{A}=$ Curl at a point

(b) $\oint\vec{A}\cdot\vec{dl}$ = Curl in a range

(c) $\oint\vec{A}\cdot\vec{dl}=\iint\left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S}\Longrightarrow$ Stoke's theorem

2. Formulae:

$$\vec{\nabla}\times\vec{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{a}_u & h_2\hat{a}_v & h_3\hat{a}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1A_u & h_2A_v & h_3A_w \end{vmatrix}$$

(a) Cartesian Co-ordinate system

$$\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z\end{vmatrix}$$

(b) Cylindrical Co-ordinate System.

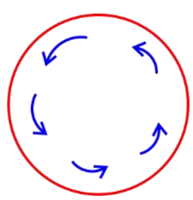
$$\vec{\nabla}\times\vec{A}=\frac{1}{\rho}\begin{vmatrix}\hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z\end{vmatrix}$$

(c) Spherical Co-ordinate system

$$\vec{\nabla}\times\vec{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix}$$

3. Physical Significance: – It gives rotation.

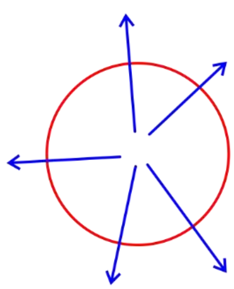
(a)



$\vec{\nabla}\times\vec{A}\neq\vec{0}$

$\vec{\nabla}\bullet\vec{A}=0$

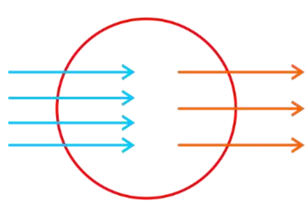
(b)



$\vec{\nabla}\times\vec{A}=\vec{0}$

$\vec{\nabla}\bullet\vec{A}\neq 0$

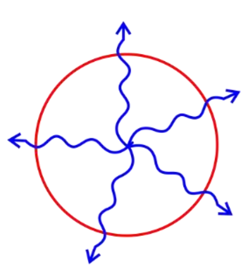
(c)



$\vec{\nabla}\times\vec{A}=\vec{0}$

$\vec{\nabla}\bullet\vec{A}\neq 0$

(d)



$\vec{\nabla}\times\vec{A}\neq\vec{0}$

$\vec{\nabla}\bullet\vec{A}\neq 0$

4. Properties:

(a) $\vec{\nabla}\times\left(\vec{A}+\vec{B}\right)=\vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$

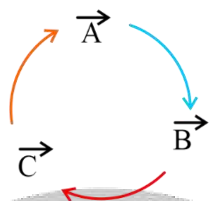
(b) $\vec{\nabla}\times\left(f\,\vec{A}\right)=f\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\times\left(\vec{\nabla}f\right)$

(c) $\vec{\nabla}\times\left(\dfrac{\vec{A}}{f}\right)=\dfrac{f\left(\vec{\nabla}\times\vec{A}\right)+\vec{A}\times\left(\vec{\nabla}f\right)}{f^2}$

Mixed Product:

1. Scalar Product:

(a) $\vec{A}\bullet\left(\vec{B}\times\vec{C}\right)=\vec{B}\bullet\left(\vec{C}\times\vec{A}\right)=\vec{C}\bullet\left(\vec{A}\times\vec{B}\right)$



$$\vec{A}\bullet\left(\vec{B}\times\vec{C}\right)=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(b) $\vec{A}\bullet\left(\vec{B}\times\vec{C}\right)=0\Rightarrow\left[\vec{A}\vec{B}\vec{C}\right]=0$

Then $\vec{A},\vec{B}\,\&\,\vec{C}$ are independent vectors and they do not lie in a single plane.

(c) Volume of parallelepiped

$\left|\vec{A}\bullet\left(\vec{B}\times\vec{C}\right)\right|$

2. Vector Product:

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\left(\vec{A}\bullet\vec{C}\right)\vec{B}-\left(\vec{A}\bullet\vec{B}\right)\vec{C}$$

3. Mixed product with Del operator.

(a) $\vec{\nabla}\left(\vec{\nabla}\ f\right)=Doesnot\ exist$

(b) $\vec{\nabla}\bullet\left(\vec{\nabla}\ f\right)=\nabla^2 f$

(c) $\vec{\nabla}\times\left(\vec{\nabla}\ f\right)=\vec{0}$

(d) $\vec{\nabla}\left(\vec{\nabla}\bullet\vec{A}\right)=exist$

(e) $\vec{\nabla}\bullet\left(\vec{\nabla}\bullet\vec{A}\right)=does\ not\ exist$

(f) $\vec{\nabla}\times\left(\vec{\nabla}\bullet\vec{A}\right)=does\ not\ exist$

(g) $\vec{\nabla}\left(\vec{\nabla}\times\vec{A}\right)=does\ not\ exist$

(h) $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}\bullet\vec{A}\right)-\nabla^2\vec{A}$

(i) $\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{A}\right)=0$

Some Important Identities

1. $\vec{\nabla}\left(f+g\right)=\vec{\nabla}f+\vec{\nabla}g$
2. $\vec{\nabla}\left(fg\right)=f\vec{\nabla}g+g\vec{\nabla}f$

3. $\vec{\nabla}\left(\frac{f}{g}\right)=\frac{g\vec{\nabla}f-f\vec{\nabla}g}{g^2}$
4. $\vec{\nabla}\bullet\left(\vec{A}+\vec{B}\right)=\vec{\nabla}\bullet\vec{A}+\vec{\nabla}\bullet\vec{B}$
5. $\vec{\nabla}\bullet\left(f\vec{A}\right)=f\vec{\nabla}\bullet\vec{A}+\vec{A}\bullet\vec{\nabla}f$
6. $\vec{\nabla}\left(\frac{\vec{A}}{f}\right)=\frac{f\left(\vec{\nabla}\bullet\vec{A}\right)-\vec{A}\bullet\vec{\nabla}f}{f^2}$
7. $\vec{\nabla}\bullet\left(\vec{A}\times\vec{B}\right)=\left(\vec{\nabla}\times\vec{A}\right)\bullet\vec{B}-\left(\vec{\nabla}\times\vec{B}\right)\bullet\vec{A}$
8. $\vec{\nabla}\times\left(\vec{A}+\vec{B}\right)=\vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$
9. $\vec{\nabla}\times\left(f\,\vec{A}\right)=f\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\times\left(\vec{\nabla}f\right)$
10. $\vec{\nabla}\times\left(\frac{\vec{A}}{f}\right)=\frac{f\left(\vec{\nabla}\times\vec{A}\right)+\vec{A}\times\left(\vec{\nabla}f\right)}{f^2}$
11. $\vec{\nabla}\bullet\left(\vec{\nabla}f\right)=\nabla^2 f$
12. $\vec{\nabla}\times\left(\vec{\nabla}f\right)=\vec{0}$
13. $\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{A}\right)=0$
14. $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}\bullet\vec{A}\right)-\nabla^2\vec{A}$

Laplacian Operator:

1. Laplacian operator with scalar function ‘f’

$$\nabla^2 f=\vec{\nabla}\bullet\left(\vec{\nabla}f\right)$$

$$=\frac{1}{h_1h_2h_3}\left(\frac{\partial\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u}\right)}{\partial u}+\frac{\partial\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial v}\right)}{\partial v}+\frac{\partial\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial w}\right)}{\partial w}\right)$$

(a) Cartesian co-ordinate system

$$\nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$

(b) Cylindrical co-ordinate system.

$$\nabla^2 f=\frac{\partial^2 f}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial f}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}+\frac{\partial^2 f}{\partial z^2}$$

(c) Spherical co-ordinate system :–

$$\nabla^2 f=\frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial f}{\partial r}+\frac{1}{\left(r\sin\theta\right)^2}\frac{\partial^2 f}{\partial\phi^2}$$

Note: When scalar function '*f*' is function of 'r' only. Then, $\nabla^2 f(r) = \frac{1}{r}\frac{df(r)}{dr} + \frac{d^2 f(r)}{dr^2}$

(d) (i) $\nabla^2 r^n = n(n+1)r^{n-2}$ (ii) $\nabla^2 r = \frac{2}{r}$

(iii) $\nabla^2\left(\frac{1}{r}\right) = 0$ (iv) $\nabla^2 lnr = \frac{1}{r^2}$

2. Laplacian operator with vector $\vec{A}$

$$\nabla^2\vec{A} = \frac{1}{h_1^2}\frac{\partial^2\vec{A}}{\partial u^2} + \frac{1}{h_2^2}\frac{\partial^2\vec{A}}{\partial V^2} + \frac{1}{h_3^2}\frac{\partial^2\vec{A}}{\partial w^2}$$

(a) Cartesian co-ordinate system.

$$\nabla^2\vec{A} = \frac{\partial^2\vec{A}}{\partial x^2} + \frac{\partial^2\vec{A}}{\partial y^2} + \frac{\partial^2\vec{A}}{\partial z^2}$$

(b) Cylindrical co-ordinate system

$$\nabla^2\vec{A} = \frac{\partial^2\vec{A}}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\vec{A}}{\partial\phi^2} + \frac{\partial^2\vec{A}}{\partial z^2}$$

(c) Spherical co-ordinate system.

$$\nabla^2\vec{A} = \frac{\partial^2\vec{A}}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\vec{A}}{\partial\theta^2} + \frac{1}{(r\sin\theta)^2}\frac{\partial^2\vec{A}}{\partial\phi^2}$$

Some Important points:

(a) $\vec{\nabla} r = \hat{a}r$

(b) $\vec{\nabla}\bullet\vec{r} = 3$

(c) $\vec{\nabla}\times\vec{r} = \vec{0}$

(d) $\nabla^2 r = \frac{2}{r}$

(e) $\nabla^2\vec{r} = \vec{0}$

(f) $\vec{\nabla} f(r) = \frac{df(r)}{dr}\hat{a}r$

(g) $\nabla^2 f(r) = \frac{2}{r}\frac{df(r)}{dr} + \frac{d^2 f(r)}{dr^2}$

(h) $\vec{\nabla}\bullet\vec{A} = \vec{0}$ then $\vec{A}$ is solenoidal and diversion less.

(i) $\vec{\nabla}\times\vec{A} = \vec{0}$ then $\vec{A}$ is irrational, conservative and path independent vector.

(j) $\nabla^2\phi = 0$ (Laplacian Equation).

2.3. Path dependent open Line Integral.

1. $\int \vec{P}\cdot\vec{dl}$

Condition for path dependent open Line Integral is given as.

$$\vec{\nabla}\times\vec{P}\neq\vec{0}$$

Then the above integrals will be solved using parameterization process.

Example:

The Line Integral of the Vector Field

$\vec{F}=5xz\hat{i}+(3x^2+2y)\,j+x^2zk$ along a path from (0, 0, 0) to (1, 1, 1) parameterized by (t, t^2, t) is …...

Solution:

$x = t \Rightarrow dx = dt$

$y = t^2 \Rightarrow dy = 2tdt$

$z = t \Rightarrow dz = dt$

$$\int \vec{F}\cdot\vec{dl}=\int 5xzdx+\int(3x^2+2y)dy+\int x^2zdz$$

$$=\int_0^1 5t^2dt+\int_0^1(3t^2+2t^2)(2tdt)+\int_0^1 t^3dt$$

$$=\frac{5}{3}+\frac{10}{4}+\frac{1}{4}=\frac{20+33}{12}=\frac{53}{12}=4.41$$

2. $\int \vec{P}\cdot\vec{dl}$

Condition for path independent open Line Integral is given by $\vec{\nabla}\times\vec{P}=\vec{0}$

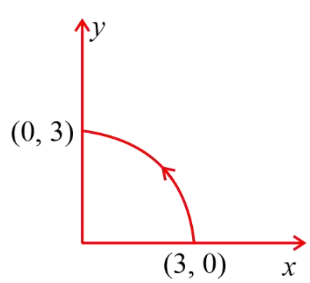
So, open line integral is performed through straight line.

Question:

As shown in the figure, C is the arc from the point (3, 0) to the point (0, 3) on the circle $x^2 + y^2 = 9$. The value of the integral.

$$\int_C (y^2+2yx)dx+(2xy+x^2)dy$$

is …….



Solution:

Method I: - Basic Method

$$x^2 + y^2 = 9$$

$$x = \pm\sqrt{9-y^2}\ ,\ y = \pm\sqrt{9-x^2}$$

Since, the curve lies in 1st quadrant of xy plane. Hence,

$$x = \sqrt{9-y^2}\ , y = \sqrt{9-x^2}$$

$$I=\int_3^0 (9-x^2)+2\left[\left(\sqrt{9-x^2}\right)x\right]dx+\int_0^3\left[2\left(\sqrt{9-y^2}\right)y+(9-y^2)\right]dy = 0$$

Method II: - Parameterization Process.

$$x^2 + y^2 = 9 \ \Rightarrow x = 3\cos\theta,\ y = 3\sin\theta$$

$$x = 3 \text{ to } 0 \ \Rightarrow \theta = 0 \text{ to } \frac{\pi}{2}$$

$$y = 0 \text{ to } 3 \ \Rightarrow \theta = 0 \text{ to } \frac{\pi}{2}$$

$$I=\int_3^0 (y^2+2yx)dx+\int_0^3 (2xy+x^2)dy$$

$$=3\int_0^{\frac{\pi}{2}}(9\sin^2\theta+18\sin\theta\cos\theta)(-\sin\theta d\theta)+3\int_0^{\frac{\pi}{2}}(9\cos^2\theta+18\sin\theta\cos\theta)(\cos\theta d\theta) = 0$$

Method III: - To check path independent or dependent.

$$\int (y^2+2yx)dx+(2xy+x^2)dy$$

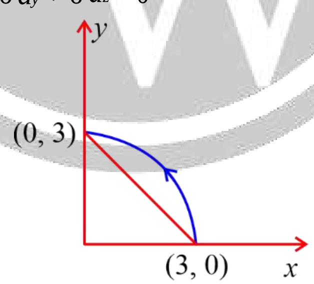
$$=\int\left((y^2+2yx)a_x+(x^2+2xy)a_y\right)\bullet\left(dxa_x+dya_y\right)$$

$$=\int \vec{P}\bullet\vec{dl}$$

$$\vec{V}\times\vec{P}=\begin{vmatrix} a_x & a_y & a_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2+2xy & 2xy+x^2 & 0 \end{vmatrix}$$

$$= (0-0)\,a_x - (0-0)\,a_y + (2x + 2y - 2y - 2x)\,a_z$$

$$= 0\,a_x + 0\,a_y + 0\,a_z = \vec{0}$$



So, the open line integral will be through straight line.

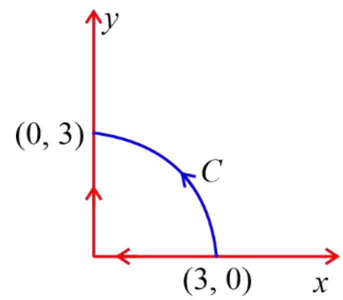
$$\frac{x-3}{3-0}=\frac{y-0}{0-3} \Rightarrow x = -y + 3 \Rightarrow y = 3 - x$$

$$I=\int (y^2+2xy)dx+(2xy+x^2)dy$$

$$\int_3^0\left[(3-x)^2+2x(3-x)\right]dx+\int_0^3\left[2(3-y)y+(3-y)^2\right]dy = 0$$

Method IV:

Open line integral is not performed through curve 'C' but through along x-axis and then along y-axis.



Along x-axis, $y = 0$, $x = 3$ to 0, $dx \neq 0$, $dy = 0$

$$I_1 = \int\left[(0)^2 + 2x \times 0\right]dx + \int\left[2x(0) + x^2\right](0) = 0$$

Along y-axis, $x = 0$, $y = 0$ to 3, $dx = 0$, $dy \neq 0$

$$I_2 = \int_0^3\left[2y(0) + y^2\right](0) + \int_3^0\left[(0)^2 + 2(0)y\right]dy = 0$$

$\therefore$ $I = I_1 + I_2 = 0 + 0 = 0$

3. $\oint \vec{A} \cdot \vec{dl}$ = closed line integral

To solve above integral, we use closed line integral or open surface integral.

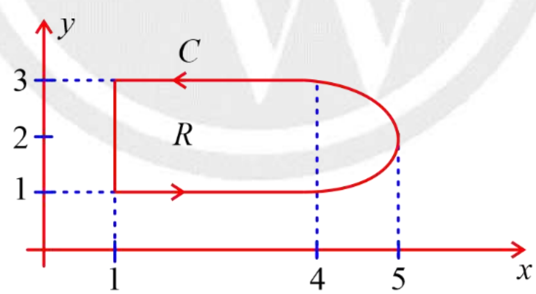
$$\oint \vec{A} \cdot \vec{dl} = \iint \left(\vec{\nabla} \times \vec{A}\right) \cdot \vec{ds}$$

Question:

Consider the line integral

$$\int_C (xdy - ydx)$$

The integral being taken in a counter clock-wise direction over the closed curve 'C' that forms the boundary of the region 'R' shown in the figure below. The region 'R' is the area enclosed by the union of a 2 × 3 rectangle and a semicircle of radius 1. The line integral evaluates to –



Solution:

Method I: Using closed line integral

Path 1. $x = 1$ to 4, $y = 1$, $dx \neq 0$, $dy = 0$

$$I_1 = \int_1^4 -y\,dx\Big|_{y=1} = -3$$

Path 2. Along semicircle centre = (4, 2)

$$(x - 4)^2 + (y - 2)^2 = (1)^2$$

$$x = \cos\phi + 4,\ y = \sin\phi + 2$$

$$y = 1 \text{ to } 3 \Rightarrow 2 + \sin\phi = 1 \text{ to } 3$$

$$\sin\phi = -1 \text{ to } 1 \Rightarrow \phi = -\frac{\pi}{2} \text{ to } \frac{\pi}{2}$$

$$x = 4 + \cos\phi \Rightarrow dx = -\sin\phi\, d\phi$$

$y = 2 + \sin\phi \Rightarrow dy = \cos\phi d\phi$

$$I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4+\cos\phi)(\cos\phi d\phi) - \int (2+\sin\phi)(-\sin\phi d\phi)$$

$$I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos\phi d\phi + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sin\phi d\phi \ = 8 + \pi$$

Path 3.

$x = 4$ to 1, $y = 3$, $dx \neq 0$, $dy \neq 0$

$$I_3 = \int_4^1 -y\,dx\Big|_{y=3} = 3 \times 3 = 9$$

Path 4.

$x = 1$, $y = 3$ to 1, $dx = 0$, $dy \neq 0$

$$I_4 = \int_3^1 x\,dy\Big|_{x=1} = -2$$

$\therefore$ $I = I_1 + I_2 + I_3 + I_4$

$= -3 + (\pi + 8) + 9 - 2 = \pi + 12$

Method 2: - Using Stoke's theorem or Green's theorem

$$\oint \vec{A}\bullet\vec{dl} = \iint \left(\nabla\times\vec{A}\right)\bullet\vec{ds}$$

$$\oint (xdy - ydx) = \oint (-ya_x + xa_y)\bullet(dxa_x + dya_y)$$

$$\vec{A} = -ya_x + xa_y$$

$$\nabla\times\vec{A} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$= [1 - (-1)]\,a_z = 2\,a_z$

Since, the given curve line in XY plane. Hence differential area will be written as

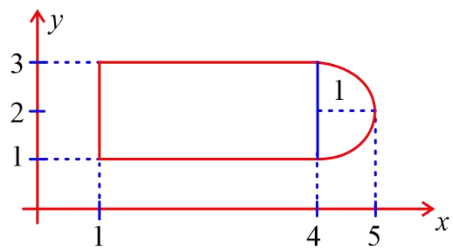
$$d\vec{s} = dx\,dy\,a_z\Big|_{z=0}$$

$$\oint (-y\,a_x + x\,a_y)\bullet(dx\,a_x + dy\,a_y)$$

$$= \iint 2a_z \bullet dx\,dy\,a_z = 2\iint dx\,dy$$

$= 2$ (Area of Curve)

$= 2$ (Area of rectangle + Area of semicircle)



$$=2\left(2\times 3+\frac{\pi(1)^2}{2}\right)=\pi+12$$

Note: As we have seen that open surface integral is simpler method than closed line integral. So, we generally use Stoke's or Green theorem to solve closed line integral.

4. Closed Surface Integral or volume integrals.

$$\oiint \vec{A}\bullet d\vec{s}=\int_{\upsilon}\left(\vec{\nabla}\bullet\vec{A}\right)d\upsilon$$

⇒ Divergence theorem

Volume integral is easier than closed surface integral.

Question:

Consider a closed surface S surrounding a volume V. If $\vec{r}$ is the position vector of a point inside S, with n the unit normal on 'S', the value of the integral

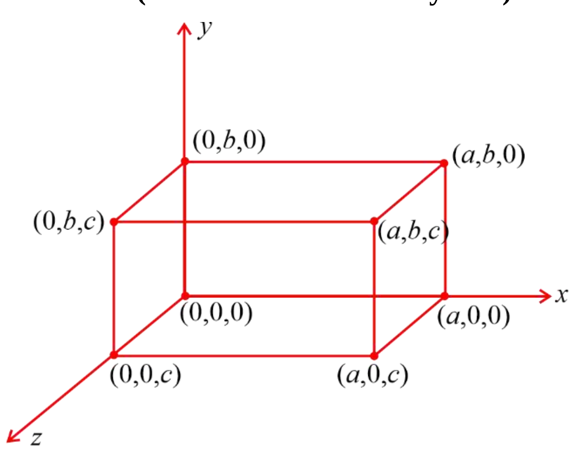
$$\oiint 5\vec{r}\bullet nds \text{ is ....}$$

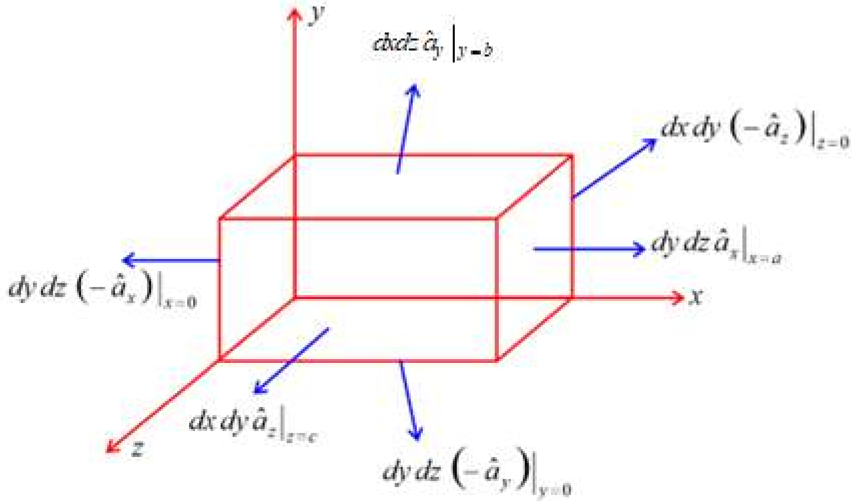
(A) 3 V **(C)** 10 V

(B) 5 V **(D)** 15 V

Solution:

Method I: - Assuming closed as a cuboid (cartesian co-ordinate system).





$$\oiint 5\vec{r}.d\vec{s}$$

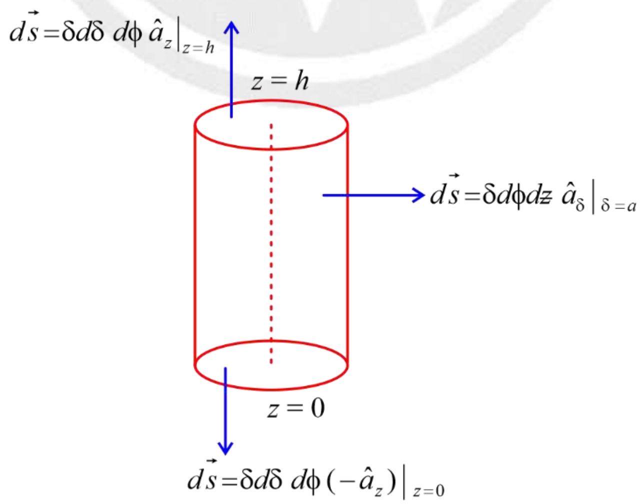
$$= \iint 5(x\hat{i}+yj+zk).dy\,dz\,a_x\Big|_{x=a} + 5\iint (x\hat{i}+yj+zk).dy\,dz\left(-a_x\right)\Big|_{x=0} + 5\iint (x\hat{i}+yj+zk).dx\,dz\,a_y\Big|_{y=b}$$

$$+\ 5\iint (x\hat{i}+yj+zk).dx\,dz\left(-a_y\right)\Big|_{y=0} + 5\iint (x\hat{i}+yj+zk).dx\,dy\,a_z\Big|_{z=c} + 5\iint (x\hat{i}+yj+zk).dx\,dy\left(-a_z\right)\Big|_{z=0}$$

$$= 5\int_0^b\int_0^c x\,dy\,dz\Big|_{x=a} - 5\int_0^b\int_0^c x\,dy\,dz\Big|_{x=0} + 5\int_0^a\int_0^c y\,dx\,dz\Big|_{y=b} - 5\int_0^a\int_0^c y\,dx\,dz\Big|_{y=0} + 5\int_0^a\int_0^b z\,dx\,dy\Big|_{z=c} - 5\int_0^a\int_0^b z\,dx\,dy\Big|_{z=0}$$

$$= 5abc - 0 + 5abc - 0 + 5abc - 0 = 15abc = 15\text{V}$$

Method II: - Assuming closed surface as a cylinder (cylindrical co-ordinate system).



$$\oiint 5\vec{r}\bullet d\vec{s}$$

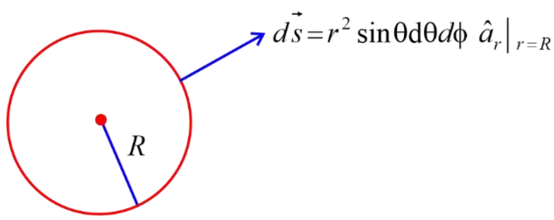
$$= 5\iint (\rho a_\rho + z a_z)\bullet \rho\, d\phi\, dz\, a_\rho\Big|_{\rho=a} + 5\iint (\rho a_\rho + z a_z)\bullet(-\rho\, d\rho\, d\phi\, a_z)\Big|_{z=0} + 5\iint (\rho a_\rho + z a_z)\bullet(\rho\, d\rho\, d\phi\, a_z)\Big|_{z=h}$$

$$= 5\int_0^{2\pi}\int_0^h \rho^2 d\phi\, dz\Big|_{\rho=a} + 5\int_0^a\int_0^{2\pi} \rho z\, d\rho\, d\phi\Big|_{z=h} - 5\int_0^a\int_0^{2\pi} \rho z\, d\rho\, d\phi\Big|_{z=0}$$

$$= 5\,a^2 2\pi h + 5h\,\frac{a^2}{2}2\pi - 0$$

$$= 10\pi a^2 h + 5\pi a^2 h = 15\pi a^2 h = 15V$$

Method III: - Assuming closed surface as a sphere (Spherical co-ordinate system).



$$\oint\!\!\oint 5\vec{r}\bullet d\vec{s} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} 5(ra_r)\bullet(r^2\sin\theta d\theta d\phi a_r)\Big|_{r=R}$$

$$= 5R^3(2)\,(2\pi) = 20\pi R^3$$

$$= 15\left(\frac{4\pi R^3}{3}\right)$$

$$= 15V$$

Method IV: - Consider a cuboid as volume.

$$I = \oint\!\!\oint 5\vec{r}\bullet d\vec{s} = \int_V \left(\vec{\nabla}\bullet 5\vec{r}\right) dv$$

$$\vec{r} = x\hat{i} + yj + zk$$

$$\vec{\nabla}\bullet\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1 = 3$$

$$I = 5\int_V \left(\vec{\nabla}\bullet\vec{r}\right) dv$$

$$= 5 \times 3\int_v dv$$

$$= 15\int_0^a\int_0^b\int_0^c dx\,dy\,dz$$

$$= 15abc$$

Method V: - consider a cylindrical as volume.

$$\vec{r} = \rho a_\rho + z a_z$$

$$\vec{\nabla}.\vec{r} = \frac{1}{\rho}\left(\frac{\partial(\rho\bullet\rho)}{\partial\rho} + \frac{\partial(0)}{\partial\phi} + \frac{\partial(\rho\bullet z)}{\partial z}\right)$$

$$= \frac{1}{\rho}(2\rho + 0 + \rho) = 3$$

$$I = \int_v \left(\vec{\nabla} \bullet 5\vec{r}\right) dv$$

$$= 5\int_v \left(\vec{\nabla} \bullet \vec{r}\right) dv$$

$$= 5 \times 3 \int_0^a \int_0^{2\pi} \int_0^h \rho d\rho d\phi dz$$

$$= 15\ (\pi a^2 h)$$

$$= 15V$$

Method VI: - Consider a spherical as volume.

$$\vec{r} = r a_r,\ \vec{\nabla} \bullet \vec{r} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial (r^2 \sin\theta \bullet \mathrm{r})}{\partial r} + 0 + 0\right] = 3$$

$$I = \int_v \vec{\nabla} \bullet (5\vec{r})\, dv$$

$$= 15\int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 15\left(\frac{4\pi R^3}{3}\right)$$

$$= 15V$$

Note:
(a) In all three-co-ordinate system, closed surface integration in spherical co-ordinate is easier.
(b) Volume integral is easier than closed surface integration.

❒❒❒

3 MAXWELL EQUATION

3.1. Type of Medium

(A) Free space/air.

- Volume charge density (ρ_v) = 0
- Conductivity (σ) = 0 (perfect insulator)
- Dipole moment (P) $\neq$ 0
- Displacement current density (J_d) $\neq$ 0
- $\in=\in_o=\frac{1}{36\pi}\times10^{-9}\text{F/m} = 8.85\times10^{-12}\text{F/m}$
- $\mu=\mu_o=4\pi\times10^{-7}\text{H/m}$
- $\epsilon_r = 1$, $\mu_r = 1$

(B) Lossless Dielectric/perfect dielectric: -

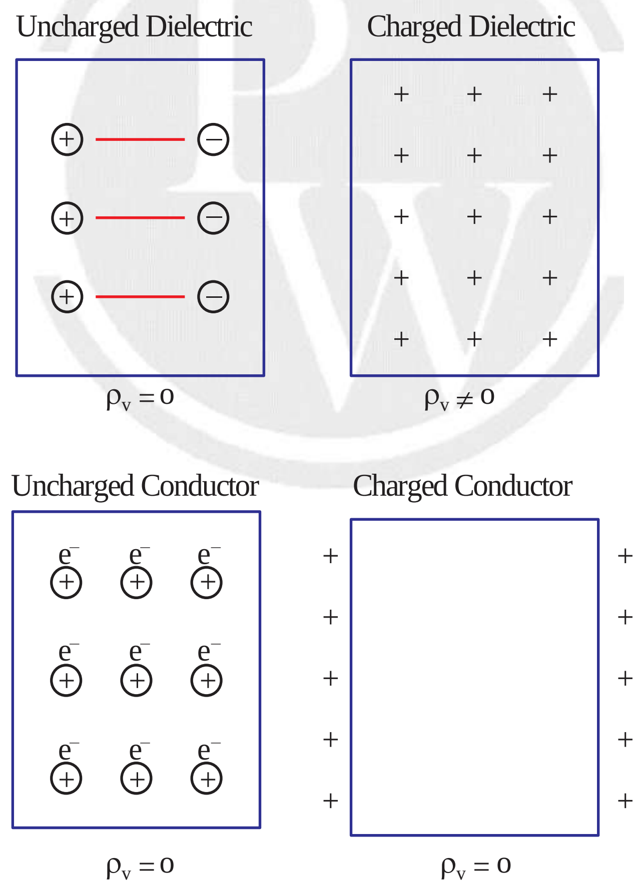
- Dielectric is a type of insulator in which it has dipole.
- Perfect dielectric means perfect insulator.
- $\rho_v = 0$,
- Conductivity of dielctric (σ_d) = 0
- Dipole moment (P) $\neq$ 0
- Displacement current (J_d) $\neq$ 0
- $\in=\in_0\in_r$
- $\mu = \mu_o \mu_r$
- In general $\epsilon_r > 1$ and $\mu_r = 1$ (Non-magnetic).

(C) Lossy Dielectric/Imperfect dielectric: -

- $\rho_v = 0$
- $\sigma_d \neq 0$ (It gives conductor loss)

- $\in = \in_0 \in_r$
- $\mu = \mu_o \mu_r$
- In general $\epsilon_r > 1$ and $\mu_r = 1$ (Non-magnetic).
- Dipole moment (P) $\neq 0$
- $J_d \neq 0$ (due to dipoles)
- Conduction current density (J_c) $\neq 0$

  $\rightarrow$ due to conductivity of lossy dielectric
- $\dfrac{\sigma_d}{\omega \in} < \dfrac{1}{100} \Rightarrow$ Low loss dielectric
- $\dfrac{1}{100} < \dfrac{\sigma_d}{\omega \in} < 100 \Rightarrow$ Medium-loss dielectric
- $\dfrac{\sigma_d}{\omega \in} > 100 \Rightarrow$ *High*-loss dielectric



but $\rho_s \neq 0$ (surface charge density).

(D) Conductor:

- $\rho_v = 0$, $\sigma_c \neq 0$ (Very high)

- σ_c = conductivity of conductor.
- Conductor is very high loss dielectric.
- Perfect conductor: - $\sigma_c = \infty$ (Very-very high)
  eg: - Gold, Silver etc.
- Good conductor: - σ_c = very high
  eg: - Brass, conductor etc.
- Poor conductor: - σ_c = high
  eg: - Aluminium

3.2. Types of Conductors

(A) Perfect Electric Conductor (PEC)

E = 0, H = maximum

(B) Perfect Magnetic Conductor (PMC)

E = maximum, H = 0

(C) Super Conductor

E = 0, H = 0

3.3 Properties of Medium

ϵ, μ & σ are known as consecutive property.

(A) Linear: If consecutive property of any medium does not depends upon strength of field, then that medium is linear.

$\sigma, \mu, \epsilon \neq f(E, H) \rightarrow$ Linear Medium $\sigma, \mu, \epsilon = f(E, H) \rightarrow$ Non-Linear Medium

$\epsilon_r = 5 \rightarrow$ Linear $\quad \epsilon_r = \frac{E}{10} + \frac{E^2}{1023} + \frac{H^3}{20319} + \ldots.. \Rightarrow$ Non-Linear

(B) Homogeneous Medium: If consecutive property of any medium does not depends on the point in space, them that medium is homogeneous.

$\epsilon_r = 5 \rightarrow$ Homogeneous

$\epsilon_r = 10x(x + 9) \rightarrow$ Inhomogeneous

$\epsilon_r = f(10f + 1) \rightarrow$ Inhomogeneous

$\epsilon_r, \mu_r, \sigma = f(x, y, z, \delta, \phi, z\ r, \theta, \phi) \rightarrow$ Inhomogeneous

$\epsilon_r, \mu_r, \sigma \neq f(x, y, z, \delta, \phi, z\ r, \theta, \phi) \rightarrow$ Inhomogeneous

(C) Isotropic Medium: If consecutive property of any medium does not depends upon direction, them that medium is isotropic.

Case I: Isotropic Medium

$$\vec{D} = \epsilon \vec{E}, \quad \epsilon_r = 3$$

$$\vec{D} = 3\epsilon_o \vec{E}, \Rightarrow \left(D_x\hat{i} + D_y\hat{j} + D_z\hat{k}\right) = 3\epsilon_o\left(E_x\hat{i} + E_y\hat{j} + E_z\hat{k}\right)$$

$$D_x = 3\epsilon_o E_x, D_y = 3\epsilon_0 E_y, D_z = 3\epsilon_o E_z$$

Since Er is same in all direction. Hence medium is isotropic.

$$\begin{bmatrix} Dx \\ Dy \\ Dz \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} E_o \begin{bmatrix} Ex \\ Ey \\ Ez \end{bmatrix}$$

↳ Scalar Matrix ⟶ Isotropic Medium

Case II: Uni-isotropic Medium

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \epsilon_o \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

↳ Diagonal Matrix

$D_x = 3\epsilon_o E_x$, $D_y = 4\epsilon_o E_y$, $D_z = 5\epsilon_o E_z$

Case III: Anisotropic Medium

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \epsilon_o \begin{bmatrix} 2 & 3 & 4 \\ 5 & 7 & 9 \\ 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$D_x = 2\epsilon_o E_x + 3\epsilon_o E_y + 4\epsilon_o E_z$

$D_y = 5\epsilon_o E_x + 7\epsilon_o E_y + 9\epsilon_o E_z$

$D_z = 6\epsilon_o E_x + 3\epsilon_o E_y + \epsilon_o E_z$

Note: All homogeneous are isotropic and all isotropic are homogeneous.

$$H \rightleftharpoons I$$

(D) Non-Dispersive: - If consecutive property of any medium does not depend upon frequency then that medium is non-dispersive.

$\sigma = 2 \longrightarrow$ Non-dispersive

$\sigma = 5\omega\epsilon \longrightarrow$ dispersive

3.4. Electric Gauss Law

Total electric flux through closed surface is equal to algebraic sum of charges enclosed.

$$\oiint \vec{D} \cdot \vec{ds} = Q_{enc}$$

(a) $\vec{\nabla} \cdot \vec{D} = \rho_V \rightarrow$ Point Form of Gauss law

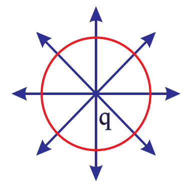
(b) $\oiint \vec{D} \cdot \vec{ds} = \iiint (\vec{\nabla} \cdot \vec{D}) dv = \iiint \rho_V dv$

Gauss theorem or divergence theorem

(c) $\oiint \vec{D} \cdot \vec{ds} \equiv$ Total Flux through Closed Surface.

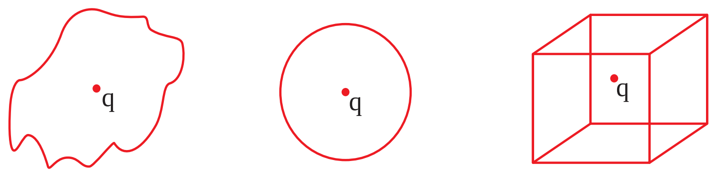
(d) $\oiint \vec{E} \cdot \vec{ds} \equiv$ Total number of electric field lines.

(e) Gaussian surface must be closed.



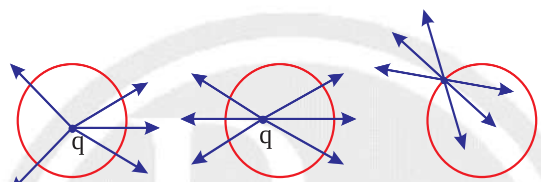
$\Psi = q$

(f) Gaussian surface may be irregular/regular.



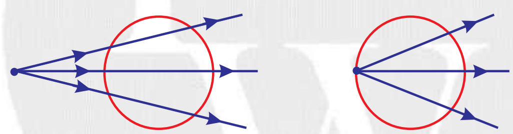
$\Psi = q$ $\Psi = q$ $\Psi = q$

(g) The charge will be placed anywhere inside the Gaussian surface.



at center $\Psi = q$ | Anywhere $\Psi = q$ | Just outside $\Psi = q$

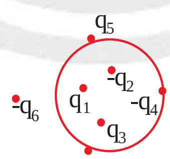
(h) When charge is placed outside the Gaussian surface.



outside $\Psi = 0$ | Just outside $\Psi = 0$

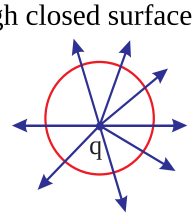
(outward flux = inward flux)

(i) $Q_{enc} \equiv$ algebraic sum of the charges enclosed.



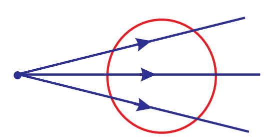
$Q_{enc} = q_1 - q_2 + q_3 - q_4$

(j) Total flux through closed surface

(i) 

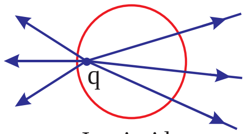
at center

$\Psi = q$

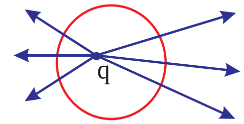
(ii) 

outside

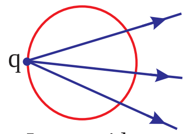
$\Psi = 0$

(iii) 

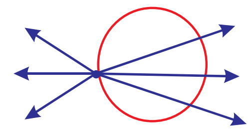
Just inside
$\Psi = q$

(iv) 

Anywhere inside
$\Psi = q$

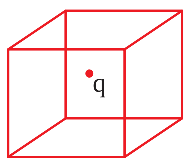
(v) 

Just outside
$\Psi = 0$

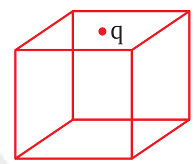
(vi) 

at the Surface
$\Psi = \frac{q}{2}$

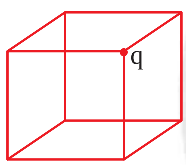
(k) Total flux through cube, when charge is placed at

(i) 

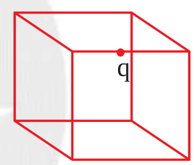
$\Psi = q$
(Flux through one cube).

(ii) 

$\Psi = \frac{q}{2}$
(Flux through one cube).

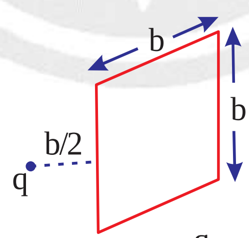
(iii) 

$\Psi = \frac{q}{8}$
(Flux through one cube).

(iv) 

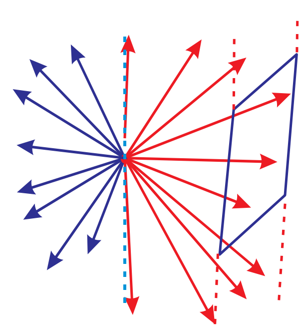
$\Psi = \frac{q}{4}$
(Flux through one cube).

(l) Flux through the plane



$\Psi = \frac{q}{6}$

(m) Flux through infinite length plane



$\Psi = \frac{q}{2}$

3.5. Magnetic Gauss Law

Total magnetic flux through any closed surface is zero.

$$\oiint \vec{B} \bullet d\vec{s} = 0$$

(a) Point form $\Rightarrow \vec{\nabla} \bullet \vec{B} = 0$

(b) Integral form $\Rightarrow \vec{B} \bullet d\vec{s} = 0$

(c) $\oiint \vec{B}.d\vec{s} = 0$, $\vec{\nabla}.\vec{B} = 0$
- Divergericeless
- Solenoidal
- Magnetic monopole does not exist
- Originating and terminating point are not defined

3.6. Electric Field Conservative/KVL.

$$\oint \vec{E}.d\vec{l} = 0,\ \vec{\nabla} \times \vec{E} = \vec{0}$$

Work done in a closed path is zero.

$\oint \vec{E}.d\vec{l} = 0$, $\vec{\nabla} \times \vec{E} = \vec{0}$
- Irrotational
- Path independent
- Conservative
- KVL
- Does not exist in a closed loop

3.7. Ampeare Circuital Law

Total magneto motive force in a loop is equal to algebraic sum of current enclose.

$$\oint \vec{H}.d\vec{l} = I_{enc}, \vec{\nabla} \times \vec{H} = \vec{J}.$$

(a) If loop is closed in anticlockwise or counter clockwise direction, then

(i) Outward current is taken as positive.

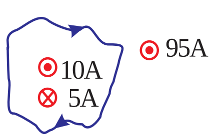
(ii) Inward current is taken as negative.

(b) If loop is closed in clockwise direction then

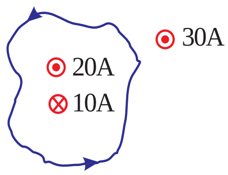
(i) Inward current is taken as negative.

(ii) Outward current is taken as positive.

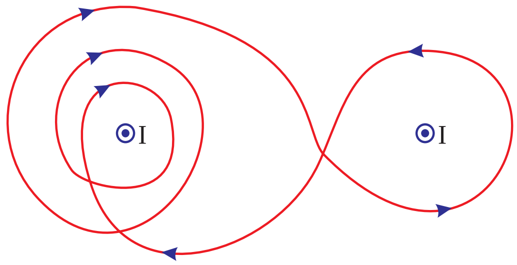
(c)



$$\oint \vec{H}.\vec{dl} = Ienc$$
$$= -10 + 5 = 5A$$

(d) 

$$\oint \vec{H}.\vec{dl} = I_{enc}$$
$$= -20 - 10 = 10A$$

(e) 

$$\oint \vec{H}.\vec{dl} = I_{enc} = -3I - I = -2I$$

3.8. Maxwell Equation in Statics

Integral Form | **Point Form**

(a) $\oint \vec{D}.d\vec{l} = Q_{enc}$ — $\vec{\nabla} \times \vec{D} = \rho_V$

Electric Gauss Law

(b) $\oint \vec{E}.\vec{dl} = 0$ — $\vec{\nabla} \times \vec{E} = \vec{0}$

Electric field conservative

(c) $\oiint \vec{B}.d\vec{s} = 0$ — $\vec{\nabla} \times \vec{B} = 0$

Magnetic Gauss Law

(d) $\oint \vec{H}.\vec{dl} = I_{enc}$ — $\vec{\nabla} \times \vec{H} = \vec{J}$

Ampeare Circuital Law

3.9. Maxwell Equation in Ideal and Practical Medium

		Ideal	**Practical**
(a)	$\oiint \vec{D}.d\vec{s} = Q_{enc}$	✓	✓
(b)	$\vec{\nabla} \cdot \vec{D} = \rho_V$	✓	×
(c)	$\oiint \vec{E} \cdot d\vec{s} = \frac{Qenc}{\epsilon}$	✓	×

(d)	$\vec{\nabla}\times\vec{E}=\frac{\rho_v}{\epsilon}$	✓	×
(e)	$\oint\vec{E}\cdot\vec{dl}=0$	✓	✓
(f)	$\vec{\nabla}\times\vec{E}=\vec{0}$	✓	✓
(g)	$\oint\vec{D}\cdot\vec{dl}=0$	✓	×
(h)	$\vec{\nabla}\times\vec{D}=\vec{0}$	✓	×
(i)	$\oiint\vec{B}\cdot\vec{ds}=0$	✓	✓
(j)	$\vec{\nabla}\cdot\vec{B}=0$	✓	✓
(k)	$\oiint\vec{H}\cdot\vec{ds}=0$	✓	×
(l)	$\vec{\nabla}\cdot\vec{H}=0$	✓	×
(m)	$\oint\vec{H}\cdot\vec{dl}=I_{enc}$	✓	✓
(n)	$\vec{\nabla}\times\vec{H}=\vec{J}$	✓	✓
(o)	$\oint\vec{B}\cdot\vec{dl}=\mu I_{enc}$	✓	×
(p)	$\vec{\nabla}\times\vec{B}=\mu\vec{J}$	✓	×

- **Ideal Medium:** Linear, Homogeneous, and Isotropic
- **Practical Medium:** Non-linear, Inhomogeneous and an isotropic.

3.10. Maxwell Equation in Different types of Medium

(a) Charge Free Medium

$$Q = 0,\ \rho_L = 0,\ \rho_s = 0,\ \rho_v = 0$$

- $\oiint \vec{D} \cdot d\vec{s} = 0$ $\quad \vec{\nabla} \cdot \vec{D} = 0$
- $\oint \vec{E} \cdot d\vec{l} = 0$ $\quad \vec{\nabla} \times \vec{E} = \vec{0}$
- $\oiint \vec{B} \cdot d\vec{s} = 0$ $\quad \vec{\nabla} \cdot \vec{B} = 0$
- $\oint \vec{H} \cdot d\vec{l} = I_{enc}$ $\quad \vec{\nabla} \times \vec{H} = \vec{J}$

(b) Current Free Medium

$I = 0, K = 0, J = 0$

- $\oiint \vec{D} \cdot d\vec{s} = Qenc$ $\quad \vec{\nabla} \cdot \vec{D} = \rho_V$
- $\oint \vec{E} . d\vec{l} = 0$ $\quad \vec{\nabla} \times \vec{E} = \vec{0}$
- $\oiint \vec{B} \cdot d\vec{s} = 0$ $\quad \vec{\nabla} \cdot \vec{B} = 0$
- $\oint \vec{H} . d\vec{l} = 0$ $\quad \vec{\nabla} \times \vec{H} = \vec{0}$

(c) Source Free Medium

$Q = 0, \rho_L = 0, \rho_S = 0, \rho_V = 0$

$I = 0, K = 0, J = 0$

- $\oiint \vec{D} \cdot d\vec{s} = 0$ $\quad \vec{\nabla} \cdot \vec{D} = 0$
- $\oint \vec{E} \cdot d\vec{l} = 0$ $\quad \vec{\nabla} \times \vec{E} = \vec{0}$
- $\oiint \vec{B} \cdot d\vec{s} = 0$ $\quad \vec{\nabla} \cdot \vec{B} = 0$
- $\oint \vec{H} \cdot d\vec{l} = 0$ $\quad \vec{\nabla} \times \vec{H} = \vec{0}$

3.11. To Find Unknowns of $\vec{E}$, $\vec{H}$ & V.

(a) $\vec{\nabla} \cdot \vec{B} = 0 \rightarrow$ Unknown of $\vec{B}$

(b) $\vec{\nabla} \times \vec{E} = \vec{0} \rightarrow$ Unknown of $\vec{E}$

(c) $\nabla^2 V = 0 \rightarrow$ Unknown of V

(d) $\rho_V = \vec{\nabla} \cdot \vec{D}$

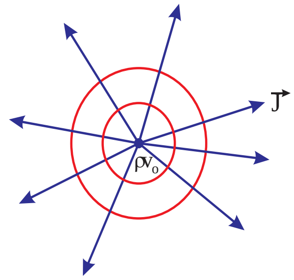
(e) $\vec{J} = \vec{\nabla} \times \vec{H}$

3.12. Continuity Equation/KCL

- It gives flow of charge in a medium.

- Continuity Equation in point form.

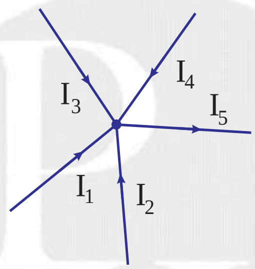
$$\vec{\nabla} \cdot \vec{J} = \frac{-\partial \rho_V}{\partial t}$$



- Continuity Equation in integral form

$$\oiint \vec{J} \cdot \vec{ds} = -\iiint_{\vartheta} \left(\frac{\partial \rho_v}{\partial t} \right) dv$$

- KCL

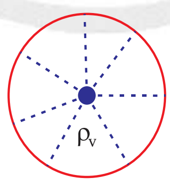


At node

$$\sum I_i = 0$$

$$\left.\begin{aligned} &\Rightarrow \vec{\nabla} . \vec{J} = 0 \\ &\Rightarrow \oiint \vec{J} . \vec{ds} = 0 \end{aligned}\right\} \mapsto \text{KCL}$$

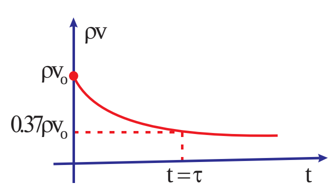
- Equation of flow of charge in a medium



(a) $\rho_V = \rho_{Vo} e^{-\frac{\sigma}{\varepsilon} t}$ (b) $\rho_S = \rho_{So} e^{-\frac{\sigma}{\varepsilon} t}$

(c) $\rho_L = \rho_{Lo} e^{-\frac{\sigma t}{\varepsilon}}$ (d) $Q = Q_o e^{-\frac{\sigma t}{\varepsilon}}$

Where $\frac{\varepsilon}{\sigma} = \tau$ (Relaxation time constant)

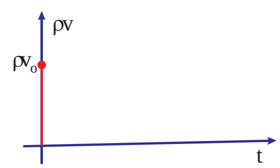


- Flow of charge in conductor.

  σ is very high

  So, τ (Relaxation time constant) = 0

  Hence, $\rho_V = \rho_{vo} e^{-\frac{\sigma}{\varepsilon} t} \simeq 0$



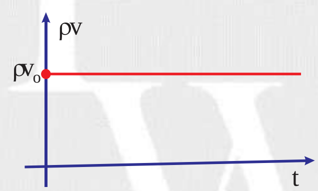
Very fast flow of charge in a conductor. So, charge does not reside inside conductor.

- Flow of charge inside insulator.

  $\sigma_d = 0$ (very low)

So, τ (Relaxation time constant) = ∞

$$\rho_V = \rho_{vo} e^{-\frac{\sigma}{\varepsilon} t} = \rho_{vo}$$



There is no flow of charge inside perfect insulator.

3.13. Laplace's Equation/Poisson Equation

(a) Poisson Equation in point form

(i) For Ideal Medium

$$\nabla^2 V = \frac{-\delta_V}{\varepsilon}$$

(ii) For Practical Medium

$$\vec{\nabla}.\left(\varepsilon\left(\vec{\nabla}V\right)\right) = -\delta_V$$

$$\varepsilon\left(\nabla^2 V\right) + \left(\vec{\nabla}_V\right).\left(\vec{\nabla}_\varepsilon\right) = -\delta_V$$

(b) Laplace Equation

$\delta_V \rightarrow$ Free space/air/vacuum/uncharged/ dielectric/conductor/singular charge distribution/cavity/source free medium charge free medium.

$\nabla^2 V = 0 \Rightarrow$ Laplacian Equation

3.14. Difference between Laplacian & Poisson Equation

	Laplace's Equation		Poisson Equation
(a)	$\nabla^2 V = 0$	(i)	$\nabla^2 V = \frac{-\rho_V}{\varepsilon}$
(b)	$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$	(ii)	$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{-\rho_V}{\varepsilon}$
(c)	Complementary function	(iii)	Complementary function + Particular integral
(d)	Unique solution and follow uniqueness theorem	(iv)	Does not follow uniqueness theorem
(e)	Linear Equation	(v)	Non- Linear Equation
(f)	Homogeneous Equation	(vi)	Non- Homogeneous Equation

3.15. Magnetic Force

$$\vec{F}_m = I\left(\vec{l} \times \vec{B}\right)$$

$\vec{l} \equiv$ length of wire and directed along current direction.

3.16. Magnetic Energy Density

$$\mu_m = \frac{1}{2}\vec{B}.\vec{H} = \frac{1}{2}\mu H^2 = \frac{B^2}{2\mu}$$

$$\mu_m = \frac{1}{2}\vec{J}.\vec{A} \text{ where } \vec{A} \equiv \text{Magnetic vector potential}$$

3.17. Faraday's Law

(a) Faraday's 1st Law (Qualitative Analysis):

When magnetic field lines cuts conductor, then an electromotive force will be developed, which is known as induced electromotive force.

(b) Faraday's 2nd Law (Quantitative Analysis): -

The induced electromotive force is directly proportional to time rate of charge of flux.

$$e \alpha \frac{d\psi m}{dt} \Rightarrow \text{Faraday's Law}$$

$$e = \underbrace{-}_{\text{Lenz's Law}} \underbrace{\frac{d\psi\, m}{dt}}_{\text{Faraday's Law}}$$

(C) Faradays and Lenz's Law in integral form

$$\boxed{\oint \vec{E} \bullet d\vec{l} = \iint -\frac{\partial \vec{B}}{\partial t} \bullet d\vec{s}}$$

(d) Faraday's and Lenz's Law in point form

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

(e)

- ψ_m = Magnetic Flux

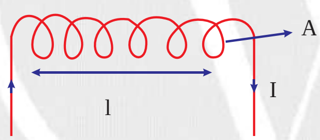
  N = Number of turns

  l = length of solenoid

  ϕ_m = Magnetic flux due to one turn

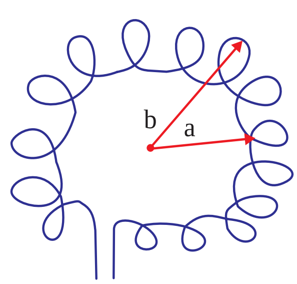
  A = cross-sectional area of solenoid

(f)



- l = (N-1)d
- $H = \frac{NI}{l}$
- $B = \mu H = \frac{\mu NI}{l}$
- $\phi_m = BA = \frac{\mu NIA}{l}$
- $\psi_m = LI$ (Weber)
- $e = \frac{-d\psi_m}{dt} = \frac{-\mu N^2 A}{l}\left(\frac{di}{dt}\right) = -L\frac{dI}{dt}$
- $V = -e = +L\frac{dI}{dt}$

(g) Toroid

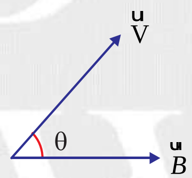


- $r = \dfrac{a+b}{2}$
- $l = 2\pi r$
- $H = \dfrac{NI}{2\pi r}$
- $B = \mu H = \dfrac{\mu NI}{2\pi r}$
- $\phi_m = \left(\dfrac{\mu NIA}{2\pi r}\right)$
- $\psi_m = N\phi_m = \left(\dfrac{\mu N^2 A}{2\pi r}\right) I = LI$
- $V = -e = +\dfrac{d\psi_m}{dt} = L\dfrac{dI}{dt}$
- $L = \dfrac{\mu N^2 A}{2\pi r}$

Lorentz's Force

- $\vec{F}_{net} = q\left(\vec{V} \times \vec{B}\right) + q\vec{E}_{in}$
- $\vec{E}_{in} = \vec{B} \times \vec{V}$
- $e = BVL \sin\theta$

$$e = \left(\vec{V} \times \vec{B}\right).\vec{l}$$



3.18. Modified Ampere's Circuital Law:

(a) Conduction current: - It is due to flow of electron. It is due to conductivity.

- Since, conductivity of conductor is very high. Hence, conduction current is very high.
- Since, conductivity of dielectric is very low. Hence, conduction current is very low. (Leakage Current)

(b) Displacement current: - It is due to rotation of dipoles.

- Since, number of dipoles inside conductor is very low. Hence, displacement current is very low.
- Since, number of dipoles inside dielectric is very high. Hence, displacement current is very high.

(c)

Conductor	Dielectric
$I_c \gg I_d$	$I_c \ll I_d$

(d) Ampeare's circuit law is modified by maxwell

$$\oint \vec{H} \bullet \vec{dl} = \mathrm{I}_{\mathrm{enc}} \Rightarrow \text{In statics}$$

$$\oint \vec{H} \bullet \vec{dl} = \mathrm{I}_{\mathrm{c}} + \mathrm{I}_{\mathrm{d}} \Rightarrow \text{In time varrying}$$

(e) Modified Ampeare's circuital low in integral low

$$\oint \vec{H} \bullet \vec{dl} = \mathrm{I}_{\mathrm{c}} + \mathrm{I}_{\mathrm{d}}$$

(f) Modified Ampeare's circuital law in point form

$$\vec{\nabla} \times \vec{H} = \overrightarrow{\mathrm{J}_{\mathrm{c}}} + \overrightarrow{\mathrm{J}_{\mathrm{d}}}$$

(g) $\overrightarrow{\mathrm{J}_{\mathrm{c}}}$ **= Conduction current density =** $\sigma \vec{\mathrm{E}}$

$$\overrightarrow{J_d} = \frac{\partial \vec{D}}{\partial t} = \varepsilon \frac{\partial \vec{E}}{\partial t} = \text{Displacement current density}$$

(h) $$\mathrm{I}_{\mathrm{c}} = \iint \vec{\mathrm{J}}_{\mathrm{c}} . d\vec{s}$$

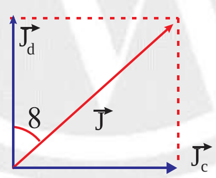
$$\mathrm{I}_{\mathrm{d}} = \frac{\partial \psi_e}{dt} \equiv \text{Time rate of change of electric flux per unit time.}$$

(i)

$$J = \sqrt{J_c^2 + J_d^2}$$

$$\tan \delta = \frac{J_c}{J_d}$$

$$\mathrm{I} = \mathrm{I}_{\mathrm{c}} + \mathrm{I}_{\mathrm{d}}$$



(j) $$\vec{\nabla} \times \vec{H} = \vec{J}_c + \vec{J}_d = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} = (\sigma + jw\varepsilon) \vec{E}$$

(k) $$\oint \vec{H} \bullet \vec{dl} = \iint \left(\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \right) \bullet d\vec{s} = \iint (\sigma + j\omega\varepsilon) \vec{E} \bullet d\vec{s}$$

3.19. Maxwell's Equation in time Varying Fields

(a) $$\oiint \vec{D} \bullet d\vec{s} = \mathrm{Q}_{\mathrm{enc}}$$ (i) $$\vec{\nabla} \bullet \vec{D} = \rho_{\mathrm{V}}$$

Electric Gauss Law

(b) $$\oiint \vec{B} \bullet d\vec{s} = 0$$ (ii) $$\vec{\nabla} \bullet \vec{B} = 0$$

Magnetic Gauss Law

(c) $\oint \vec{E} \bullet \vec{dl} = -\iint \frac{\partial \vec{B}}{\partial t} \bullet d\vec{s}$ (iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -jw\mu\vec{H}$

Faraday's and Lenz's Law

(d) $\oint \vec{H} \bullet \vec{dl} = I_c + I_d$ (iv) $\vec{\nabla} \times \vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial \vec{E}}{\partial t} = (\sigma + jw\varepsilon)\vec{E}$

Modified Ampeare Circuital

3.20. Magnetic Vector Potential and Magnetic scaler Potential

(a) Magnetic scalar potential: -

$$\vec{J} = \vec{0} \Rightarrow \text{Volume current density} = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{0} \Rightarrow \vec{H} = -\vec{\nabla}V_m$$

$$\boxed{V_m = -\int \vec{H} \cdot \vec{dl}} \rightarrow \text{Magnetic scalar potential}$$

(b) Magnetic Vector Potential

- $\vec{B} = \vec{\nabla} \times \vec{A}$ ($\vec{A} \equiv$ Magnetic Vector Potential)
- $\vec{A} = \frac{\mu}{4\pi}\int \frac{I\vec{dl}}{R}$
- $\vec{A} = \frac{\mu}{4\pi}\iint \frac{\vec{k}ds}{R}$
- $\vec{A} = \frac{\mu}{4\pi}\iiint \frac{\vec{J}dv}{R}$
- $\oint \vec{A} \bullet \vec{dl} = \iint (\vec{\nabla} \times \vec{A}) \bullet d\vec{s} = \iint \vec{B}.d\vec{s} = \psi_m = \text{Magentic flux}$
- $\vec{\nabla}(\vec{\nabla} \bullet \vec{A}) - \nabla^2\vec{A} = \mu\vec{J}$

(a) In statics $\boxed{\vec{\nabla} \bullet \vec{A} = 0}$ $\therefore \nabla^2\vec{A} = -\mu\vec{J}$

(b) In time varying $(\vec{\nabla}(\vec{\nabla} \bullet \vec{A})) = \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2}$

$$\nabla^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J}$$

❑❑❑

4 ELECTROMAGNETIC WAVE

4.1. Time Harmonic Equation

(a) Lossless Medium

$$\nabla^2 \vec{H} = \mu \in \frac{\partial^2 \vec{H}}{\partial t^2} \Rightarrow \nabla^2 \vec{H} = \frac{1}{v_P^2}\frac{\partial^2 \vec{H}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \mu \in \frac{\partial^2 \vec{E}}{\partial t^2} \Rightarrow \nabla^2 \vec{E} = \frac{1}{v_P^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

Where, $v_P =$ Phase velocity in a medium.

(b) Lossy Medium

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu \in \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{H} = \mu\sigma \frac{\partial \vec{H}}{\partial t} + \mu \in \frac{\partial^2 \vec{H}}{\partial t^2}$$

Loss : $\mu\sigma \frac{\partial \vec{E}}{\partial t}, \quad \mu\sigma \frac{\partial \vec{H}}{\partial t}$

Harmonic : $\mu \in \frac{\partial^2 \vec{E}}{\partial t^2}, \quad \mu \in \frac{\partial^2 \vec{H}}{\partial t^2}$

4.2. Wave Equation/Helmholtz Equation/Telegraph Equation

(a) Lossy Medium

$$\nabla^2 \vec{E} = \gamma \vec{E}, \ \nabla^2 \vec{H} = \gamma^2 \vec{H}$$

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$$

$$\gamma = \alpha + j\beta$$

$$\alpha \equiv \text{ attenuation constant } \left(\frac{\text{Neper}}{\text{m}}\right)$$

$$\beta \equiv \text{ Phase constant } \left(\frac{\text{radian}}{\text{m}}\right)$$

$$\gamma \equiv \text{ Propagation constant } \left(\frac{1}{\text{m}}\right)$$

(b) Lossless medium

$$\nabla^2\vec{E}-\left(\omega^2\mu\in\right)\vec{E}=\vec{0}$$

$$\nabla^2\vec{H}-\left(\omega^2\mu\in\right)\vec{H}=\vec{0}$$

$$\gamma=j\omega\sqrt{\mu\in},\alpha=0,\beta=\omega\sqrt{\mu\in}$$

4.3. Equation of Electric Field and Magnetic Field

- $\vec{E}=E_0e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\,a_x$ (Complex Form)
- $\vec{H}=H_0e^{-\alpha z}e^{-j\beta z}e^{j\omega t}a_y$ (Complex Form)
- $\vec{E}=E_0e^{-\alpha z}\cos(\omega t-\beta z)\widehat{a}_x$
- $\vec{H}=H_0e^{-\alpha z}\cos(\omega t-\beta z)\widehat{a}_y$
- $E_0=\text{Amplitude of Electric Field}\left(\frac{V}{m}\right)$
- $\alpha=\text{Attenuation constant}\left(\frac{\text{Neper}}{m}\right)$
- $e^{-\alpha z}=\text{Attenuation Factor}$
- $H_0=\text{Amplitude of Magnetic Field}\left(\frac{A}{m}\right)$
- $\beta=\frac{2\pi}{\lambda}\Rightarrow\text{Phase Constant/Wave Number}$

4.4. Intrinsic Impedance/Wave Impedance.

- It relates conversion from Electric Field to Magnetic Field or vice versa.
- It is impeadance offered by medium during conversion Electric Field to Magnetic Field or vice versa.
- $\eta\equiv$ Intrinsic Impedance (Ω)

Lossy Medium

$$\eta=\frac{|E|}{|H|}=\frac{E_0^+}{H_0^+}=\frac{-E_0^-}{H_0^-}=\frac{j\omega\mu}{\gamma}=\frac{\gamma}{\sigma+j\omega\in}=\sqrt{\frac{j\omega\mu}{\sigma+j\omega\in}}$$

Lossless Medium

$$\eta=\frac{\omega\mu}{\beta}=\frac{\beta}{\omega\in}=\sqrt{\frac{\mu}{\in}}$$

Derivation of Intrinsic Impeadance.

$$\left.\begin{aligned}\vec{E}&=E_0^+e^{-\gamma z}e^{j\omega t}a_x\\ \vec{H}&=H_0^+e^{-\gamma z}e^{j\omega t}a_y\end{aligned}\right\}\text{Wave travelling in +z direction}$$

(a) $\vec{\nabla}\times\vec{H}=(\sigma+j\omega\in)\vec{E}$ (Modified Ampeare Circuital Law)

$$\frac{E_0^+}{H_0^+}=\frac{\gamma}{\sigma+j\omega\in}$$

(b) $\vec{\nabla}\times\vec{H}=-j\omega\mu\vec{H}$ (Faraday's + Lenz's Law)

$$\frac{E_0^+}{H_0^+}=\frac{j\omega\mu}{\gamma}$$

$$\left.\begin{aligned}\vec{E}&=E_0^- e^{\gamma z}e^{j\omega t}\hat{a}_x\\ \vec{H}&=H_0^- e^{\gamma z}e^{j\omega t}\hat{a}_y\end{aligned}\right\}\text{ Wave travelling in -z direction.}$$

(a) $\vec{\nabla}\times\vec{E}=-j\omega\mu\vec{H}$ (Faraday's + Lenz's Law)

$$\frac{-E_0^-}{H_0^-}=\frac{j\omega\mu}{\gamma}$$

(b) $\vec{\nabla}\times\vec{H}=(\sigma+j\omega\in)\vec{E}$

$$\frac{-E_0^-}{H_0^-}=\frac{\gamma}{\sigma+j\omega\in}\text{(Modified Ampeare circuital Law)}$$

4.5. Phase Velocity and Group Velocity

(a) Phase Velocity: The velocity which is along the propagation direction and defined at single frequency.

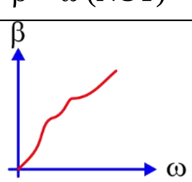
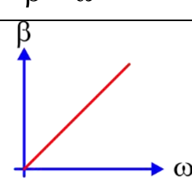
$$\vec{v}_p=\frac{\omega}{\beta}a_p$$

(b) Group Velocity: The velocity which is along the propagation direction and defined for multiple frequency.

$$\vec{v}_g=\frac{d\omega}{d\beta}a_p$$

* $a_p\equiv$ unit vector directed along propagation direction.

(c) Phase Velocity and Group Velocity in Dispersive and Non-Dispersive Medium.

Dispersive	Non-Dispersive
(a) $v_p=f(\omega)$ Phase velocity depends upon frequency of operation.	(a) $v_p\neq f(\omega)$ Phase velocity does not depend upon frequency of operation.
(b) $v_p\neq v_g$	(b) $v_p=v_g$
(c) $\beta\propto\omega$ (NOT)	(c) $\beta\propto\omega$
(d) 	(d) 

(d) $v_g = \dfrac{v_p}{1 - \dfrac{\omega}{v_p}\dfrac{dv_p}{d\omega}}$

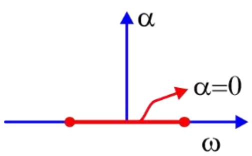
(i) $v_p \neq f(\omega) \Rightarrow \dfrac{dv_p}{d\omega} = 0 \Rightarrow v_g = v_p$

(ii) $v_p = f(\omega)$

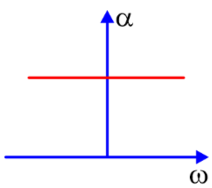
Case I: $\dfrac{dv_p}{d\omega} > 0 \Rightarrow v_g > v_p$

Case II: $\dfrac{dv_p}{d\omega} < 0 \Rightarrow v_g < v_p$

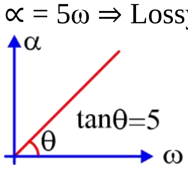
(e)
- $\propto = 0 \Rightarrow$ Lossless and Distortionless.



- $\propto =$ Constant $\Rightarrow$ Lossy and Distortionless.



- $\propto = 5\omega \Rightarrow$ Lossy and Distortive.



- $\propto = 0 \Rightarrow$ Lossless, $\propto = f(\omega) \Rightarrow$ Distortive
  $\propto \neq 0 \Rightarrow$ Lossy, $\propto \neq f(\omega) \Rightarrow$ Distortionless

* **All lossline medium is distortionless medium but all distortionless medium is not lossless medium.**

4.6. Wave Parameters of Different Medium

	Free Space or Air	Lossless or Perfect Dielectric	Lossy or Imperfect dielectric	High Loss or Conductor
α	0	0	$\dfrac{\sigma_d}{2}\sqrt{\dfrac{\mu}{\in}}$	$\sqrt{\pi\mu f\,\sigma_c}$
	Lossless and Distortionless	Lossless and Distortionless	Lossy and Distortionless	Lossy and Distortive
β	$\beta_0 = \omega\sqrt{\mu_0 \in_0}$	$\omega\sqrt{\mu\in} = \beta_0\sqrt{\mu_r \in_r}$	$\omega\sqrt{\mu\in} = \beta_0\sqrt{\mu_r \in_r}$	$\sqrt{\pi f\,\mu\sigma_C}$
	Non-Dispersive	Non-Dispersive	Non-Dispersive	Dispersive
λ	$\lambda_0 = \dfrac{2\pi}{\beta_0}$	$\dfrac{2\pi}{\beta} = \dfrac{\lambda_0}{\sqrt{\mu_r \in_r}}$	$\dfrac{2\pi}{\beta} = \dfrac{\lambda_0}{\sqrt{\mu_r \in_r}}$	$\dfrac{2\pi}{\beta}$

v_p	$\frac{1}{\sqrt{\mu_0 \in_0}} = C_0$	$\frac{1}{\sqrt{\mu \in}} = \frac{C_0}{\sqrt{\mu_r \in_r}}$	$\frac{1}{\sqrt{\mu \in}} = \frac{C_0}{\sqrt{\mu_r \in_r}}$	$\sqrt{\frac{2\omega}{\mu\sigma_C}}$
v_g	v_p	v_p	v_p	$2v_p$
$v_p.v_g$	C_0^2	$\frac{C_0^2}{\mu_r \in_r}$	$\frac{C_0^2}{\mu_r \in_r}$	$\frac{4\omega}{\mu\sigma_C}$
η	$\sqrt{\frac{\mu_0}{\in_0}} = 120\pi = 377$ (Resistive)	$\sqrt{\frac{\mu}{\in}} = 120\pi\sqrt{\frac{\mu_r}{\in_r}} =$ $377\sqrt{\frac{\mu_r}{\in_r}}$ (Resistive)	$\sqrt{\frac{\mu}{\in}} + \frac{j\sigma_d\sqrt{\mu}}{2\omega \in^{1.5}} = \text{R+jX}$ (Complex and Inductive)	$\sqrt{\frac{\omega\mu}{2\sigma_C}} + j\sqrt{\frac{\omega\mu}{2\sigma_C}} = R_S + jX_S$ (Complex and Inductive)

4.7. Wave in Space

(a) $\vec{\beta} = \beta_x \hat{a}_x + \beta_y \hat{a}_y + \beta_z \hat{a}_z$

$\beta = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2}$

(b) $\frac{1}{\lambda^2} = \frac{1}{\lambda_x^2} + \frac{1}{\lambda_y^2} + \frac{1}{\lambda_z^2}$

(c) $\frac{1}{V_p^2} = \frac{1}{V_{p_x}^2} + \frac{1}{V_{p_y}^2} + \frac{1}{V_{p_z}^2}$

4.8. Tangent Loss

(a) Lossless medium (σ = 0)

$$\vec{\nabla} \times \vec{H} = \vec{J}_d = \in \frac{\partial \vec{E}}{\partial t} = j\omega \in \vec{E}$$

$$= j\omega \in_0 \in_r \vec{E}$$

Where $\in_r$ is real quantity.

(b) Lossy medium

$$\vec{\nabla} \times \vec{H} = \underset{\text{Loss}}{\underset{\Downarrow}{\vec{J}_c}} + \underset{\text{Storage}}{\underset{\Downarrow}{\vec{J}_d}}$$

$$\vec{\nabla} \times \vec{H} = (\sigma + j\omega \in)\vec{E} = j\omega \in_0 (\in_r - j\frac{\sigma}{\omega \in_0})\vec{E}$$

$$\vec{\nabla} \times \vec{H} = j\omega_0 \in_0 \in_r^* \vec{E}$$

$\in_r^*$ = Complex form of relative permittivity

$$\in_r^* = \in_r' - j\in_r'' = \in_r - j\frac{\sigma}{\omega \in_0}$$

Where $\epsilon_r' = \epsilon_r$ gives storage

$$\epsilon_r'' = \frac{\sigma}{\omega \epsilon_0} \text{ gives loss}$$

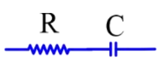
(c) Tangent Loss or Loss Tangent

$$\tan\delta = \frac{J_c}{J_d} = \frac{\text{Loss}}{\text{Storage}}$$

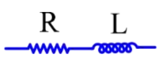
$$\tan\delta = \frac{J_c}{J_d} = \frac{I_c}{I_d} = \frac{\sigma}{\omega \epsilon_0 \epsilon_r} = \frac{\epsilon_r''}{\epsilon_r'}$$

(d) Tangent Loss related to network Circuit.

(i) $\tan\delta = \frac{R}{X_C} = \frac{V_R}{V_C} = \frac{P}{Q_C}$

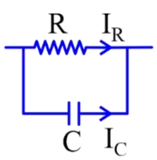


(ii) $\tan\delta = \frac{R}{X_L} = \frac{V_R}{V_L} = \frac{P}{Q_L}$



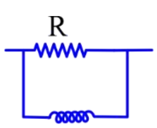
(iii) $Y = G + jB_C$

$$\tan\delta = \frac{G}{B_C} = \frac{X_C}{R} = \frac{P}{Q_C} = \frac{I_R}{I_C}$$



(iv) $Y = G - jB_L$

$$\tan\delta = \frac{G}{B_L} = \frac{X_L}{R} = \frac{P}{Q_L} = \frac{I_R}{I_L}$$



(e) $y = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$

$2\alpha\beta = \omega\mu\sigma$

$\beta^2 - \alpha^2 = \omega^2\mu\epsilon$

$$\tan\delta = \frac{\sigma}{\omega\epsilon} = \frac{2\alpha\beta}{\beta^2 - \alpha^2}$$

(f) $\eta = \sqrt{\dfrac{j\omega\mu}{\sigma + j\omega\in}} \Rightarrow$ Intrinsic Impedance

$$\theta_\eta = \frac{\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega\in}{\sigma}\right)}{2}$$

$$\Rightarrow \frac{\omega\in}{\sigma} = \cot(2\theta_\eta)$$

$$\Rightarrow \frac{\sigma}{\omega\in} = \tan(2\theta_\eta)$$

$$\Rightarrow \tan\delta = \tan(2\theta_\eta)$$

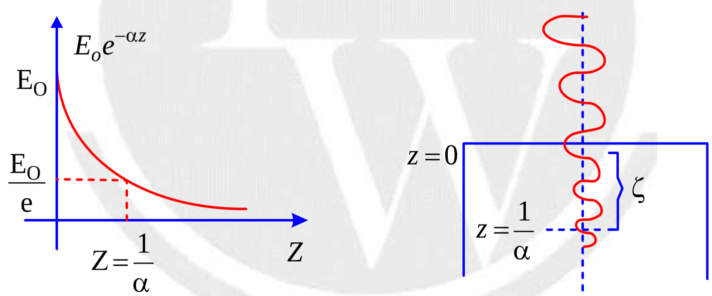
$$\therefore \delta = 2\theta_\eta = 2(\angle E - \angle H)$$

4.9. Skin depth and Depth of Penetration.

Skin depth $\Rightarrow$ µm to cm

Depth of Penetration $\Rightarrow$ cm to m.

$$\vec{E} = \underbrace{E_0 e^{-\alpha z}}_{\text{decreases with distance}} \underbrace{\cos(\omega t - \beta z)}_{\text{Constant.}} a_x$$



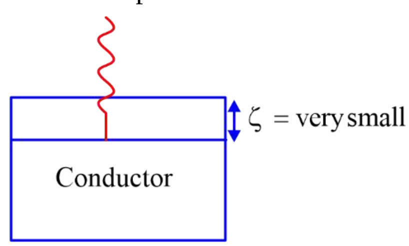
(a) Skin Depth: The distance at which the amplitude of electric field becomes $\dfrac{1}{e}$ times of its initial value.

$$E_0 \xrightarrow{z=d} E_0 e^{-\alpha d} = \frac{E_0}{e} \Rightarrow \alpha d = 1 \Rightarrow d = \frac{1}{\alpha}$$

$\therefore$ $\xi \equiv$ Skin depth $\Rightarrow$ $\xi = \dfrac{1}{\alpha}$

For high loss medium or conductor, attenuation constant is very high (10^6 to 10^8).

Hence, ξ is very small (µm). So, it is called as skin depth.



Electric field remains at the surface of conductor. It will not penetrate inside conductor.

$$\alpha = \beta = \frac{2\pi}{\lambda} = \sqrt{\pi f \mu \sigma_C}$$

$$\eta = R_S + jX_S; \;\; R_S = X_S = \sqrt{\frac{\omega\mu}{2\sigma_C}}$$

$$\xi = \frac{1}{\alpha} = \frac{1}{\beta} = \frac{1}{\sqrt{\pi f \mu \sigma_C}} = \frac{\lambda}{2\pi} = \frac{V_p}{\omega} = \frac{1}{\sigma_C R_S}$$

(b) Depth of Penetration:

For Low Lossy medium, attenuation constant is very low. Hence, electric field penetrates very high to that medium.

$$\alpha = \frac{\sigma_d}{2}\sqrt{\frac{\mu}{\in}}$$

$$\xi = \frac{2}{\sigma_d}\sqrt{\frac{\in}{\mu}}$$

(c) Lossless and Perfect Dielectric:

$$\sigma_d = 0 \Rightarrow \alpha = 0 \Rightarrow \text{Depth of penetration} = \infty.$$

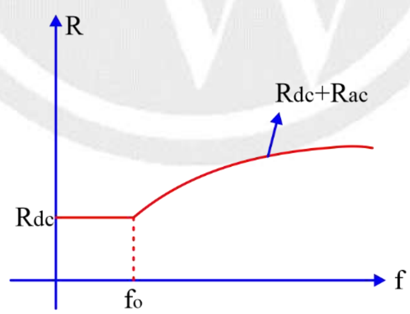
(d) Perfect Conductor

$$\sigma_c = \infty \Rightarrow \alpha = \infty \Rightarrow \text{Skin depth} = 0$$

(e) Surface Resistance

It is found at high frequency.

$$R_S = \sqrt{\frac{\mu\omega}{2\sigma_C}}$$



$$R_{dc} = \frac{\rho l}{A}, \qquad R_{ac} = \sqrt{\frac{\omega\mu}{2\sigma_C}}$$

Relation between Neper and dB Scale.

Attenuation Factor = AF = $e^{-\alpha d}$

$$AF|_{dB} = -20\log_{10} e^{-\alpha d} = 20\alpha d \log_{10} e = 8.68\alpha d \, dB$$

$$\frac{AF|_{dB}}{d} = +8.68\alpha\left(\frac{dB}{m}\right) = \alpha\left(\frac{\text{Neper}}{m}\right)$$

$1\,\text{Neper} = 8.68\,dB$

4.10. Angle of Wave Impedance or Intrinsic Impedance.

$$\theta_\eta = \frac{90° - \tan^{-1}\left(\frac{\omega \in}{\sigma}\right)}{2}$$

$$\eta = \frac{E}{H} \Rightarrow \theta_\eta = \angle E - \angle H$$

(a) Lossless Dielectric / Free Space / Air

$$\sigma = 0 \Rightarrow \theta_\eta = 0$$

$\therefore \angle E - \angle H = 0 \Rightarrow \angle E = \angle H$

Hence, Phase of Electric Field and Magnetic Field have same.

(b) Conductor / High Loss Medium

$$\theta_\eta = 45°$$

$\Rightarrow \angle E - \angle H = 45°$

Hence, Electric Field leads Magnetic Field by $\frac{\pi}{4}$

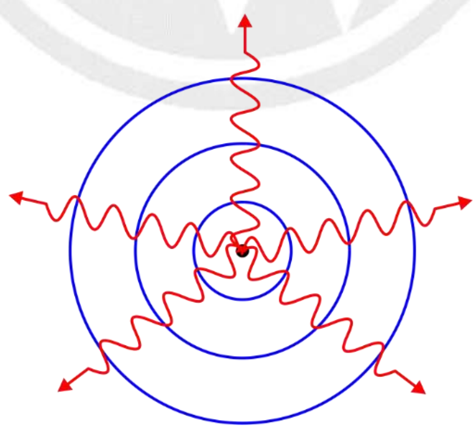
Magnetic Field lags Electric Field by $\frac{\pi}{4}$

(c) Low Loss / Medium Loss Dielectric

$0 < \theta_\eta < 45°$

4.11. Pointing Vector

(a) Propagation Loss $= -\oiint \left(\vec{E} \times \vec{H}\right) . d\vec{S}$



(b) Conductor Loss or Ohmic Loss $\equiv -\iiint \left(\sigma E^2\right) dv$

(c) $\frac{\partial}{\partial t}\left(\int_v (\mu_e + \mu_m) dv\right) = -\oiint_S \left(\vec{E} \times \vec{H}\right) \bullet d\vec{S} - \int_v \sigma E^2 dv$

The above equation represents flow of Electromagnetic energy in any medium.

(d) Poynting Vector $\equiv \vec{E}\times\vec{H}\left(\frac{\text{Watt}}{m^2}\right)$

Instantaneous Poynting Vector $\equiv \vec{P}(t)=\vec{E}(t)\times\vec{H}(t)$

Instantaneous Power Density $\equiv \vec{E}(t)\times\vec{H}(t)$

Complex Poynting Vector $\equiv \vec{E}(t)\times\vec{H}(t)$

(e) $\text{Re}(\vec{P})=\text{Re}(\vec{E})\times\text{Re}(\vec{H})$

$\text{Re}(\vec{P})\equiv$ Real part of Poynting Vector

(f) Average Poynting Vector. $\left(\vec{P}_{avg}\right)$

(i) Lossy Medium

$$|\eta|=\frac{E_{rms}}{H_{rms}}, \eta = \text{Complex + inductive}$$

$$E_{rms}=E_0e^{-\alpha z}, H_{rms}=H_0e^{-\alpha z}$$

$$\vec{P}_{avg}=\frac{E_0H_0}{2}e^{-2\alpha z}\cos\theta_\eta a_p \qquad =\frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos\theta_\eta a_p$$

$$=\frac{|\eta|H_0^2}{2}e^{-2\alpha z}\cos\theta_\eta a_p \qquad =E_{rms}H_{rms}\cos\theta_\eta a_p$$

$$=\frac{E_{rms}^2}{|\eta|}\cos\theta_\eta a_p \qquad =|\eta|H_{rms}^2\cos\theta_\eta a_p$$

(ii) Lossless Medium $\left(\sigma=0,\alpha=0,\theta_\eta=0\right)$

$$\vec{P}_{avg}=\frac{E_0H_0}{2}a_p=\frac{E_0^2}{2\eta}a_p=\frac{\eta H_0^2}{2}a_p=E_{rms}H_{rms}a_p=\frac{E_{rms}^2}{\eta}a_p=\eta H_{rms}^2 a_p$$

(iii) $\vec{P}_{avg}=\frac{1}{2}\text{Re}\left(\vec{E}\times\vec{H}^*\right)$

(iv) Power (W) = $\iint \vec{P}_{avg}\bullet d\vec{S}\,(\text{Watt})$

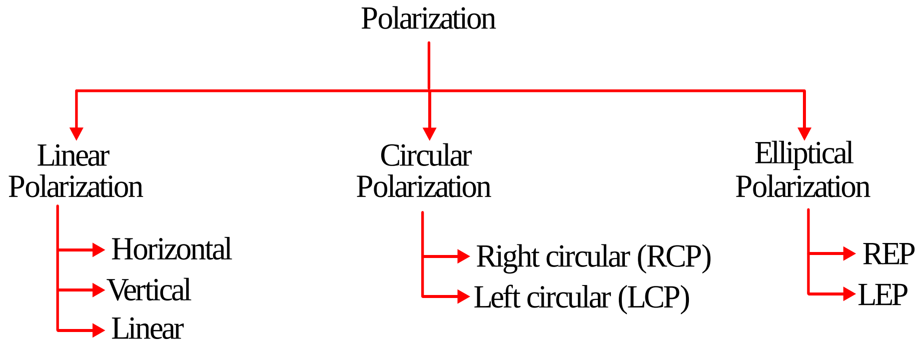
Total Power $(W_0)=\oiint \vec{P}_{avg}\bullet d\vec{S}\,(\text{Watt})$

(v) $E\to\alpha, \omega, \beta, \lambda, V_p, f=\frac{\omega}{2\pi}$

$H\to\alpha, \omega, \beta, \lambda, V_p, f=\frac{\omega}{2\pi}$

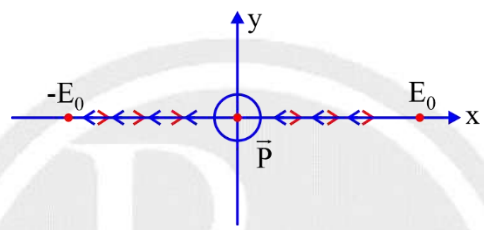
$P\to 2\alpha, 2\omega, 2\beta, 0.5\lambda, V_p, f=\frac{\omega}{\pi}$

4.12. Polarization of Electromagnetic Wave



(a) Horizontal Polarization

$$\vec{E} = E_0 \cos(\omega t - \beta z) a_x$$

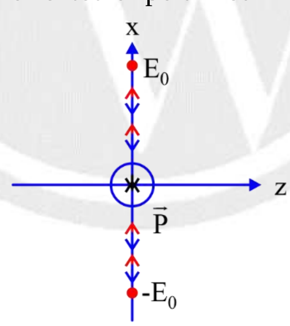


Electric field is propagating in (z) direction and oriented or polarized in (x) direction.

(b) Vertical Polarization.

$$\vec{E} = E_0 \cos(\omega t + \beta y)(-a_x)$$

Electric field is propagating in (-y) direction and oriented or polarized in (-x) direction.



(c) Linear Polarization.

$$\vec{E} = E_1 \cos(\omega t - \beta z) a_x + E_2 \cos(\omega t - \beta z) a_y$$

Wave is linearly Polarized in X and Y, propagating in +z direction.

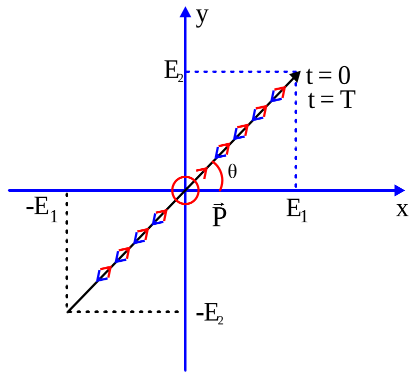
Tilt angle (θ) :- The angle subtend by Electromagnetic Wave at z = 0, t = 0 with

X – Axis ≡ (XY Plane)

Y – Axis ≡ (YZ Plane)

Z – Axis ≡ (ZX Plane)

$$\tan\theta = \frac{E_2}{E_1}$$



(d) Circular Polarization $\left(E_1 = E_2, \Delta\phi = \pm 90°\right)$

Step 1. Thumb represent direction of Propagation.

Step 2. Remaining Finger gives rotation.

Step 3. Right hand gives Right Circular Polarization where as Left hand gives Left Circular Polarization.

(i) Right Circular Polarization

$$\vec{E} = E_0 \cos(\omega t - \beta z) a_x + E_0 \sin(\omega t - \beta z) a_y$$

$$Z = 0 \Rightarrow \vec{E} = E_0 \cos(\omega t) a_x + E_0 \sin(\omega t) a_y$$

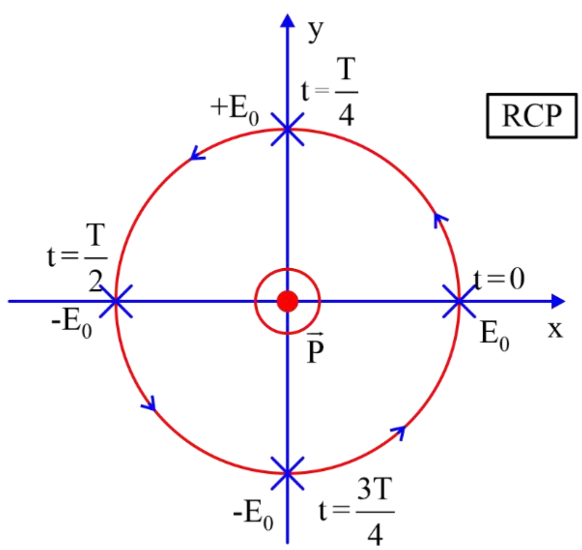
$$t = 0 \Rightarrow \vec{E} = E_0 a_x$$

$$t = \frac{T}{4} \Rightarrow \vec{E} = E_0 a_y$$

$$t = \frac{T}{2} \Rightarrow \vec{E} = -E_0 a_x$$

$$t = \frac{3T}{4} \Rightarrow \vec{E} = -E_0 a_y$$

$$t = T \Rightarrow \vec{E} = E_0 a_x$$



(ii) Left Circular Polarization.

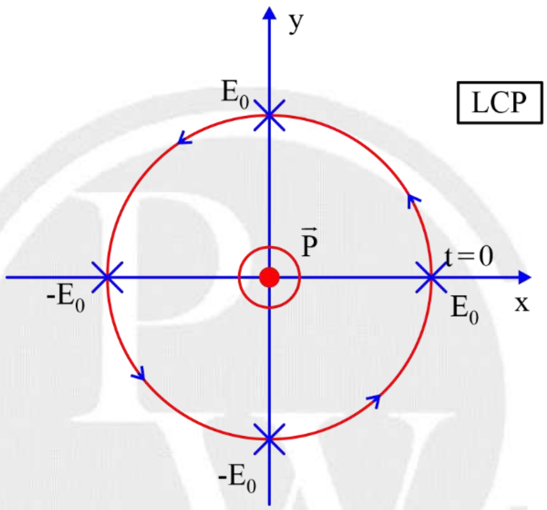
$$\vec{E} = E_0 \cos(\omega t - \beta z) a_x - E_0 \sin(\omega t - \beta z) a_y$$

$$t = 0 \Rightarrow \vec{E} = E_0 a_x$$

$$t = \frac{T}{4} \Rightarrow \vec{E} = -E_0 a_y$$

$$t = \frac{T}{2} \Rightarrow \vec{E} = -E_0 a_x$$

$$t = \frac{3T}{4} \Rightarrow \vec{E} = E_0 a_y$$

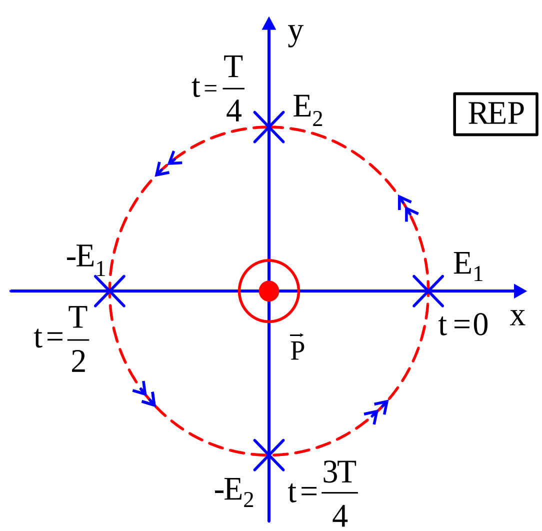


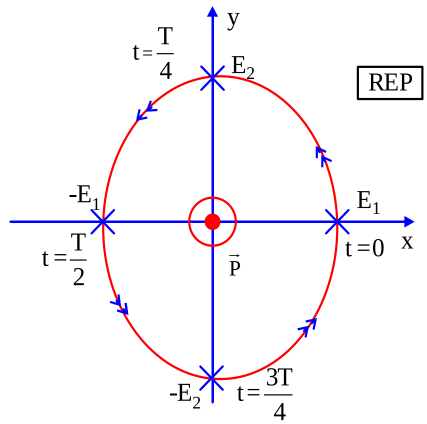
(e) Elliptical Polarization

$E_1 = E_2 \;\; \Delta\phi \neq \pm 90°, \pm 180° \quad E_1 \neq E_2, \;\; \Delta\phi \neq \pm 180° \quad E_1 \neq E_2 \;\; \Delta\phi = \pm 90°$

(i) Right Elliptical Polarization $(E_1 \neq E_2)$

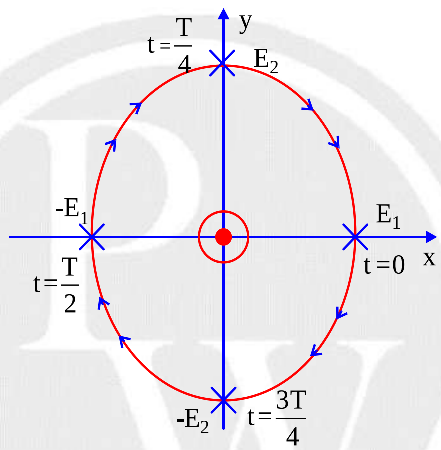
$$\vec{E} = E_1 \cos(\omega t - \beta z) a_x + E_2 \sin(\omega t - \beta z) a_y$$



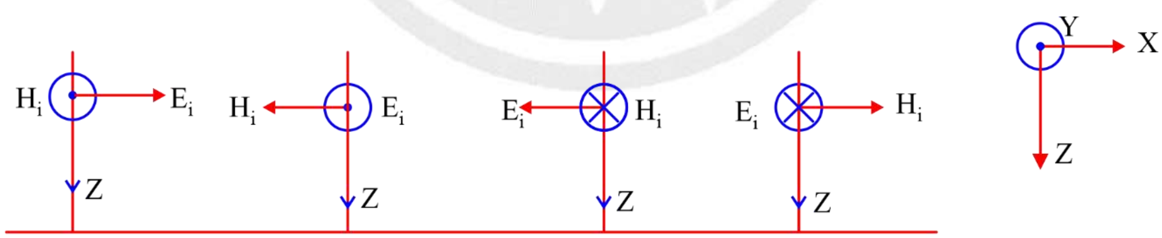


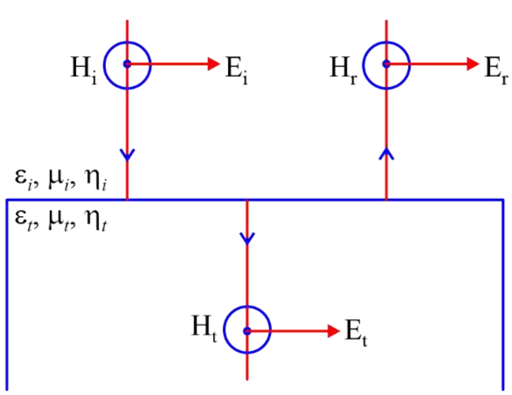
(ii) Left Elliptical Polarization $(E_1 \neq E_2)$

$$\vec{E} = E_1 \cos(\omega t - \beta z) a_x - E_2 \sin(\omega t - \beta z) a_y$$



4.13. Normal Incidence





(a) Formulae

(i) $\Gamma_E = -\Gamma_H$

(ii) $\tau_E = 1 + \Gamma_E$

(ii) $\tau_H = 1 + \Gamma_H = 1 - \Gamma_E$

(iv) $\eta_t = \eta_0 \sqrt{\dfrac{\mu_t}{\in_t}}, \ \eta_i = \eta_0 \sqrt{\dfrac{\mu_i}{\in_i}}$

(v) $\Gamma_E = \dfrac{\eta_t - \eta_i}{\eta_t + \eta_i}$

(vi) $\Gamma_E = \dfrac{\sqrt{\mu_t \in_i} - \sqrt{\mu_i \in_t}}{\sqrt{\mu_t \in_i} + \sqrt{\mu_i \in_t}}$

Non-Magnetic Material $\mu_i = \mu_t = 1$

(vii) $\Gamma_E = \dfrac{\sqrt{\in_i} - \sqrt{\in_t}}{\sqrt{\in_i} + \sqrt{\in_t}}$

(viii) $\beta \propto \sqrt{\in_r} \Rightarrow \Gamma_E = \dfrac{\beta_i - \beta_t}{\beta_i + \beta_t}$

(ix) $\lambda \propto \dfrac{1}{\sqrt{\in_r}} \Rightarrow \Gamma_E = \dfrac{\lambda_t - \lambda_i}{\lambda_t + \lambda_i}$

(x) $V_P \propto \dfrac{1}{\sqrt{\in_r}} \Rightarrow \Gamma_E = \dfrac{V_{P_t} - V_{P_i}}{V_{P_t} + V_{P_i}}$

(xi) $\eta \propto \dfrac{1}{\sqrt{\in_r}} \Rightarrow \Gamma_E = \dfrac{\eta_t - \eta_i}{\eta_t + \eta_i}$

(xii) $n \propto \sqrt{\in_r} \Rightarrow \Gamma_E = \dfrac{n_i - n_t}{n_i + n_t}$ (n =Refractive Index)

Magnetic – Magnetic Material $\in_t = \in_i = 1$

(xiii) $\Gamma_E = \dfrac{\sqrt{\mu_t} - \sqrt{\mu_i}}{\sqrt{\mu_t} + \sqrt{\mu_i}}$

(xiv) $\beta \propto \sqrt{\mu_r} \Rightarrow \Gamma_E = \dfrac{\beta_t - \beta_i}{\beta_t + \beta_i}$

(xv) $\lambda \propto \dfrac{1}{\sqrt{\mu_r}} \Rightarrow \Gamma_E = \dfrac{\lambda_i - \lambda_t}{\lambda_i + \lambda_t}$

(xvi) $V_P \propto \dfrac{1}{\sqrt{\mu_r}} \Rightarrow \Gamma_E = \dfrac{V_{P_i} - V_{P_t}}{V_{P_i} + V_{P_t}}$

(xvii) $\eta \propto \sqrt{\mu_r} \Rightarrow \Gamma_E = \dfrac{\eta_t - \eta_i}{\eta_t + \eta_i}$

(xviii) $n \propto \sqrt{\mu_r} \Rightarrow \Gamma_E = \dfrac{n_t - n_i}{n_t + n_i}$

(xix) $\vec{P}_r = -|\Gamma_E|^2 \vec{P}_i$

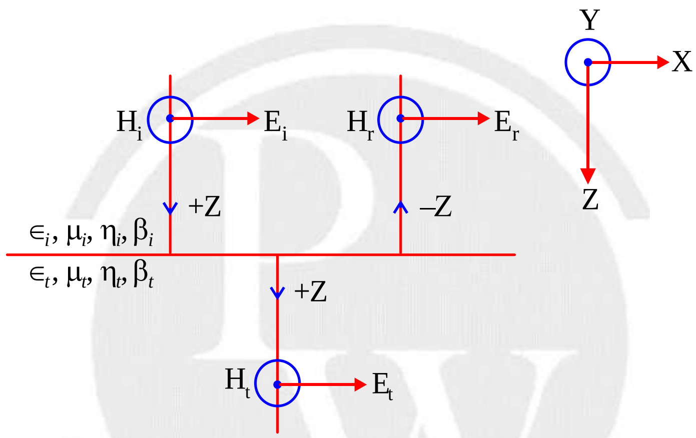
(xx) $\vec{P}_t = -\left(1 - |\Gamma_E|^2\right)\vec{P}_i$

(b) Representation of electric field and Magnetic field.

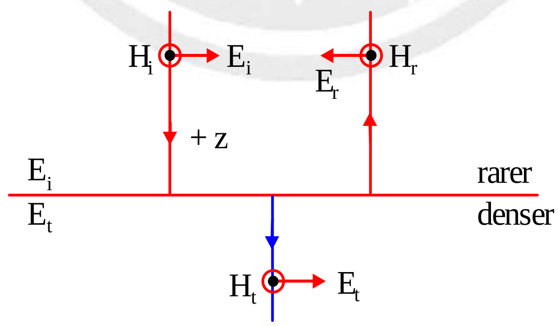
$$\vec{E}_i = E_o \cos(wt - \beta_i z)\hat{a}_x \qquad \vec{E}_r = \Gamma_E E_o \cos(wt + \beta_i z)\hat{a}_x$$

$$\vec{E}_t = \tau_E E_o \cos(wt - \beta_t z)\hat{a}_x \qquad \vec{H}_i = H_o \cos(wt - \beta_i z)\hat{a}_y$$

$$\vec{H}_r = \Gamma_H H_o \cos(wt + \beta_i z)\hat{a}_y \qquad \vec{H}_t = \tau_H H_o \cos(wt - \beta_t z)\hat{a}_y$$

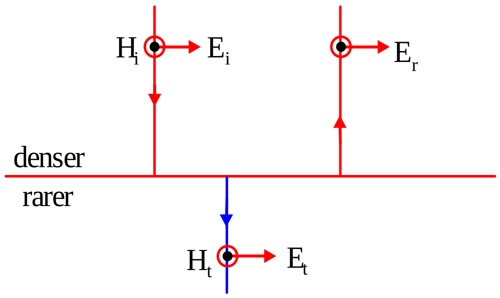


(c) When wave travels from rarer to denser medium.



- $n_i < n_t \Rightarrow \epsilon_i < \epsilon_t \qquad \left(\because n \propto \sqrt{\epsilon_r}\right)$
- $\Gamma_E = \dfrac{\sqrt{\epsilon_i} - \sqrt{\epsilon_t}}{\sqrt{\epsilon_i} + \sqrt{\epsilon_t}} < 0 \Rightarrow \Gamma_H = -\Gamma_E > 0$
- $\dfrac{E_r}{E_i} = \Gamma_E < 0 \Rightarrow \dfrac{E_r}{E_i} < 0 \Rightarrow$ E_r & E_i are of opposite sign.
- $\dfrac{H_r}{H_i} = \Gamma_H > 0 \Rightarrow \dfrac{H_r}{H_i} > 0 \Rightarrow$ H_r & H_i are of same sign.

(d) When wave travels from denser to rarer medium.



- $n_i > n_t \Rightarrow \epsilon_i > \epsilon_t \quad \left(\because n \alpha \sqrt{\epsilon_r}\right)$
- $\Gamma_E = \dfrac{\sqrt{\epsilon_i} - \sqrt{\epsilon_t}}{\sqrt{\epsilon_i} + \sqrt{\epsilon_t}} > 0 \Rightarrow \Gamma_H = -\Gamma_E < 0$
- $\dfrac{E_r}{E_i} = \Gamma_E > 0 \Rightarrow \dfrac{E_r}{E_i} > 0 \Rightarrow$ E_r & E_i are of same sign.

  $\dfrac{H_r}{H_i} = \Gamma_H < 0 \Rightarrow \dfrac{H_r}{H_i} < 0 \Rightarrow$ H_r & H_i are of opposite sigh.

(e) Range of Reflection and Transmission coefficient.

- $\Gamma \in [-1, 1]$
- $\tau \in [0, 2]$

(f) When wave travels from air to conductor.

- $\eta_i = \eta_o \sqrt{\dfrac{\mu_r}{\epsilon_r}}$

  $\eta_t = \eta_o \sqrt{\dfrac{j\omega\mu}{\sigma_c + j\omega\epsilon}}$

Since σ_c is very high. Hence η_t value will be in the range of milli ohm.

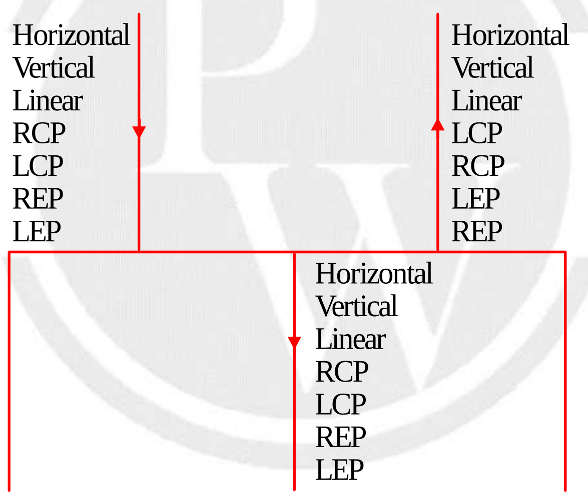
- $\eta_i >>>> \eta_t$
- $\Gamma_E = \dfrac{\eta_t - \eta_i}{\eta_t + \eta_i} = -1 \Rightarrow \Gamma_H = -\Gamma_E = 1$
- $\tau_E = 1 + \Gamma_E = 0 \quad \& \quad \tau_H = 1 + \Gamma_H = 2$
- $\dfrac{E_r}{E_i} = -1 \Rightarrow E_r = -E_i$
- $\dfrac{E_t}{E_i} = 0 \Rightarrow E_t = 0 \rightarrow$ Minima

- $\dfrac{H_r}{H_i} = 1 \Rightarrow H_r = H_i$
- $\dfrac{H_t}{H_i} = 2 \Rightarrow H_t = 2H_i \rightarrow$ Maxima
- Reflected and Incident Electric Field are out of phase.
- Electric Field at the surface of conductor is '0' (minima).
- Magnetic Field at the surface is maximum.
- Hence, Electric Field and Magnetic Field on the surface of conductor is having 90° phase difference,

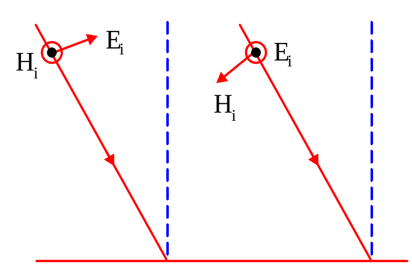
i.e. Electric Field at the surface of conductor lags Magnetic Field at the surface of conductor by 90°.

- Inside Conductor, Electric Field leads Magnetic Field by $\dfrac{\pi}{4}$.
- $\vec{P}_r = -\vec{P}_i$
- $\vec{P}_t = \vec{0}$

(g) Effect of polarization during normal incidence

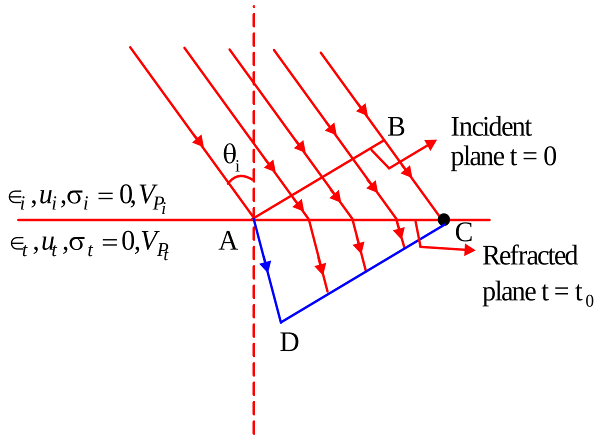


4.14. Oblique Incidence



(I) Parallel Polarization (A) Perpendicular polarization
(II) Vertical (B) Horizontal
(III) P - polarization (C) S - polarization

(a) Snell' law:



(i) $\sqrt{\mu_r \in_t}\sin\theta_t = \sqrt{\mu_i \in_i}\sin\theta_i \to$ Lossless Medium

Non-magnetic $\mu_i = \mu_t = 1$

(ii) $\sqrt{\in_t}\sin\theta_t = \sqrt{\in_i}\sin\theta_i$ (iii) $\beta \propto \sqrt{\in_r} \Rightarrow \beta_t \sin\theta_t = \beta_i \sin\theta_i$

(iv) $\lambda \propto \dfrac{1}{\sqrt{\in_r}} \Rightarrow \dfrac{\sin\theta_t}{\lambda_t} = \dfrac{\sin\theta_i}{\lambda_i}$ (v) $V_P \propto \dfrac{1}{\sqrt{\in_r}} \Rightarrow \dfrac{\sin\theta_t}{V_{pt}} = \dfrac{\sin\theta_i}{V_{pi}}$

(vi) $\eta \propto \dfrac{1}{\sqrt{\in_r}} \Rightarrow \dfrac{\sin\theta_t}{\eta_t} = \dfrac{\sin\theta_i}{\eta_i}$ (vii) $n \propto \sqrt{\in_r} \Rightarrow n_t \sin\theta_t = n_i \sin\theta_i$

Magnetic – Magnetic $\in_i = \in_t = 1$

$$\sqrt{\mu_i}\sin\theta_i = \sqrt{\mu_t}\sin\theta_t$$

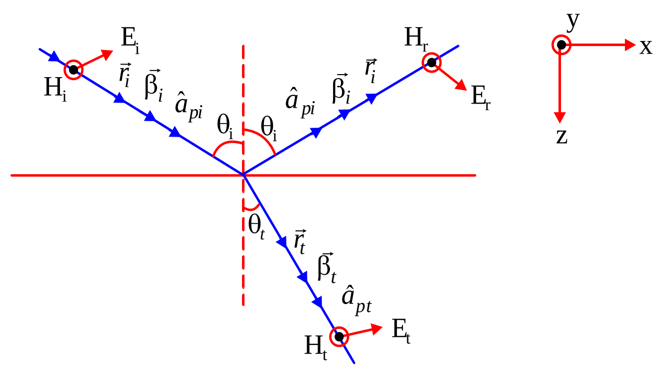
(viii) $\beta \propto \sqrt{\mu_r} \Rightarrow \beta_i \sin\theta_i = \beta_t \sin\theta_t$ (ix) $\lambda \propto \dfrac{1}{\sqrt{\mu_r}} \Rightarrow \dfrac{\sin\theta_i}{\lambda_i} = \dfrac{\sin\theta_t}{\lambda_t}$

(x) $V_P \propto \dfrac{1}{\sqrt{\mu_r}} \Rightarrow \dfrac{\sin\theta_i}{V_{P_i}} = \dfrac{\sin\theta_t}{V_{P_t}}$

(xi) $\eta \propto \sqrt{\mu_r} \Rightarrow \eta_i \sin\theta_i = \eta_t \sin\theta_t$

(xii) $n \propto \sqrt{\mu_r} \Rightarrow n_i \sin\theta_i = n_t \sin\theta_t$

4.15. Parallel/Vertical/P-Polarization



- $\vec{r}_i = x\hat{i} + z\hat{k}$
- $\vec{r}_r = x\hat{i} - z\hat{k}$
- $\vec{r}_t = x\hat{i} + z\hat{k}$
- $\hat{a}_{P_i} = \sin\theta_i \hat{i} + \cos\theta_i \hat{k}$
- $\hat{a}_{P_r} = \sin\theta_i\ \hat{i} - \cos\theta_i\ \hat{k}$
- $\hat{a}_{P_t} = \sin\theta_t\ \hat{i} + \cos\theta_t\ \hat{k}$
- $\vec{\beta}_i = \beta_i \sin\theta_i\ \hat{i} + \beta_i \cos\theta_i \hat{k}$
- $\vec{\beta}_r = \beta_i \sin\theta_i\ \hat{i} + \beta_i \cos\theta_i \hat{k}$
- $\vec{\beta}_t = \beta_t \sin\theta_t \hat{i} + \beta_t \cos\theta_t \hat{k}$
- $\beta_i = \dfrac{\omega}{c_o}\sqrt{\mu_i \in_i}$
- $\beta_t = \dfrac{\omega}{c_o}\sqrt{\mu_t \in_t}$
- $\eta_i = \eta_o\sqrt{\dfrac{\mu_i}{\in_i}}$
- $\eta_t = \eta_o\sqrt{\dfrac{\mu_t}{\in_t}}$

Summary:

(a) $\Gamma_H = -\Gamma_E$

(b) $\tau_H = 1 + \Gamma_H = 1 - \Gamma_E$

(c) $\tau_E = (1 + \Gamma_E)\left(\dfrac{\cos\theta_i}{\cos\theta_t}\right)$

(d) $\Gamma_E = \dfrac{\eta_t \cos\theta_t - \eta_i \cos\theta_i}{\eta_t \cos\theta_t + \eta_i \cos\theta_i}$

(e) $\Gamma_E = \dfrac{\sqrt{\mu_t \in_i}\cos\theta_t - \sqrt{\mu_i \in_t}\cos\theta_i}{\sqrt{\mu_t \in_i}\cos\theta_t + \sqrt{\mu_i \in_t}\cos\theta_i}$

For non-magnetic ($\mu_i = \mu_t = 1$)

(f) $\Gamma_E = \dfrac{\sqrt{\in_i}\sec\theta_i - \sqrt{\in_t}\sec\theta_t}{\sqrt{\in_i}\sec\theta_i + \sqrt{\in_t}\sec\theta_t}$

(g) $\beta \propto \sqrt{\in_r} \Rightarrow \Gamma_E = \dfrac{\beta_i \sec\theta_i - \beta_t \sec\theta_t}{\beta_i \sec\theta_i + \beta_t \sec\theta_t}$

(h) $\lambda \propto \dfrac{1}{\sqrt{\in_r}} \Rightarrow \Gamma_E = \dfrac{\lambda_t \cos\theta_t - \lambda_i \cos\theta_i}{\lambda_t \cos\theta_t + \lambda_i \cos\theta_i}$

(i) $V_P \propto \dfrac{1}{\sqrt{\in_r}} \Rightarrow \dfrac{V_{P_t} \cos\theta_t - V_{P_i} \cos\theta_i}{V_{P_t} \cos\theta_t + V_{P_i} \cos\theta_i}$

(j) $\eta \propto \dfrac{1}{\sqrt{\in_r}} \Rightarrow \Gamma_E = \dfrac{\eta_t \cos\theta_t - \eta_i \cos\theta_i}{\eta_t \cos\theta_t + \eta_i \cos\theta_i}$

(k) $n \propto \sqrt{\in_r} \Rightarrow \Gamma_E = \dfrac{n_t \cos\theta_t - n_i \cos\theta_i}{n_t \cos\theta_t + n_i \cos\theta_i}$

For Magnetic medium $(\in_i = \in_t = 1)$

(l) $\Gamma_E = \dfrac{\sqrt{\mu_t} \cos\theta_t - \sqrt{\mu_i} \cos\theta_i}{\sqrt{\mu_t} \cos\theta_t + \sqrt{\mu_i} \cos\theta_i}$

(m) $\beta \propto \sqrt{\mu_r} \Rightarrow \Gamma_E = \dfrac{\beta_t \cos_t - \beta_i \cos\theta_i}{\beta_t \cos_t + \beta_i \cos\theta_i}$

(n) $\lambda \propto \dfrac{1}{\sqrt{\mu_r}} \Rightarrow \Gamma_E = \dfrac{\lambda_i \sec\theta_i - \lambda_t \sec\theta_t}{\lambda_i \sec\theta_i + \lambda_t \sec\theta_t}$

(o) $V_P \propto \dfrac{1}{\sqrt{\mu_r}} \Rightarrow \Gamma_E = \dfrac{V_{P_i} \sec\theta_i - V_{P_t} \sec\theta_t}{V_{P_i} \sec\theta_i + V_{P_t} \sec\theta_t}$

(p) $\eta \propto \sqrt{\mu_r} \Rightarrow \Gamma_E = \dfrac{\eta_t \cos\theta_t - \eta_i \cos\theta_i}{\eta_t \cos\theta_t + \eta_i \cos\theta_i}$

(q) $n \propto \sqrt{\mu_r} \Rightarrow \Gamma_E = \dfrac{n_t \cos\theta_t - n_i \cos\theta_i}{n_t \cos\theta_t + n_i \cos\theta_i}$

(r) Representation of Electric Field and Magnetic Field

$\odot\ \vec{E}_i = E_o\left(\cos\theta_i\ \hat{i} - \sin\theta_i \hat{k}\right) e^{jwt}\ e^{-j\vec{\beta}_i \bullet \vec{r}_i}$

$\vec{r}_i = x\hat{i} + z\hat{k}$

$\vec{\beta}_i \bullet \vec{r}_i = (\beta_i \sin\theta_i)x + (\beta_i \cos\theta_i)z$

$\odot\ \vec{E}_r = \Gamma_E E_0\left(\cos\theta_i \hat{i} + \sin\theta_i \hat{k}\right) e^{jwt}\ e^{-j\vec{\beta}r \bullet \vec{r}_r}$

$\odot\ \vec{E}_r = \tau_E E_0\left(\cos\theta_t \hat{i} - \sin\theta_t \hat{k}\right) e^{jwt}\ e^{-j\vec{\beta}_t \bullet \vec{r}_t}$

$\odot\ \vec{H}_i = \dfrac{E_o}{\eta_i} e^{jwt}\ e^{-j\vec{\beta}_i \bullet \vec{r}_i}$

$\odot\ \vec{H}_r = \Gamma_H \left(\dfrac{E_o}{\eta_i}\right) e^{jwt}\ e^{-j\vec{\beta}_r \bullet \vec{r}_r}$

$\odot \vec{H}_t = \tau_H \left(\dfrac{E_o}{\eta_i}\right) e^{jwt} e^{-j\vec{\beta}_t \cdot \vec{r}_t}$

(s) Power Density

$\odot\ \vec{P}_i = \dfrac{E_0^2}{2\eta_i} \hat{a}_{P_i}, \odot\ \vec{P}_r = \dfrac{|\Gamma_E|^2 E_o^2}{2\eta_i} \hat{a}_{P_r}$

$\odot\ \vec{P}_i = \left(1 - |\Gamma_E|^2\right) \dfrac{\cos\theta_i}{\cos\theta_t} \left(\dfrac{E_o^2}{2\eta_i}\right) \hat{a}_{P_t}$

$\odot\ \dfrac{\mathrm{P_r}}{P_i} = |\Gamma_E|^2 \quad \odot\ \dfrac{\mathrm{P}_t}{P_i} = \left(1 - |\Gamma_E|^2\right)\left(\dfrac{\cos\theta_i}{\cos\theta_t}\right)$

(t) When electromagnetic wave travels from air to conductor.

$\odot \eta_i = \eta_o \sqrt{\dfrac{\mu_r}{\epsilon_r}} \ \odot \eta_t = \sqrt{\dfrac{jw\mu}{\sigma_c + jw\varepsilon}}$

$\odot$ Since, σ_c is very high. Hence, η_t value will be found in the range of milli ohm.

$\odot\ \eta_i >>> \eta_t$

$\odot\ \Gamma_E = \dfrac{\eta_t \cos\theta_t - \eta_i \cos\theta_i}{\eta_t \cos\theta_t + \eta_i \cos\theta_i} = -1$

$\Rightarrow \Gamma_H = -\Gamma_E = 1 \quad \Rightarrow \tau_E = 1 + \Gamma_E = 0 \ \ \& \tau_H = 1 + \Gamma_H = 2$

$\odot \tau_E = 1 + \Gamma_E = 0 \ \ \& \tau_H = 1 + \Gamma_H = 2$

$\odot\ \dfrac{E_r}{E_t} = -1 \Rightarrow E_r = -E_i$

$\odot\ \dfrac{E_r}{E_t} = 0 \Rightarrow E_t = 0 \rightarrow Minima$

$\odot\ \dfrac{H_r}{H_i} = 1 \Rightarrow H_r = H_i$

$\odot\ \dfrac{H_t}{H_i} = 2 \Rightarrow H_t = 2H_i \rightarrow Maxima$

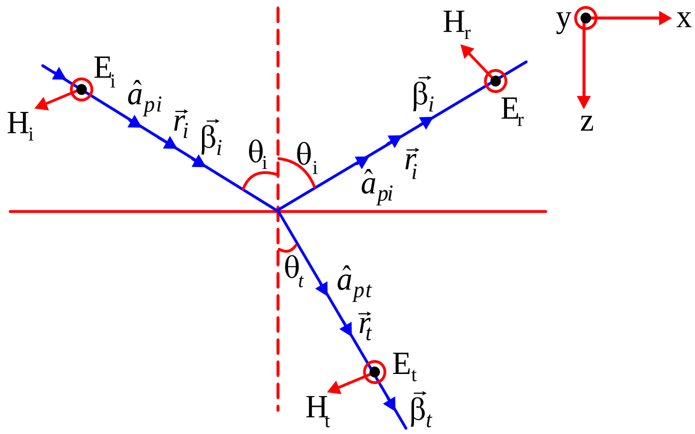
$\odot$ Reflected and Incident Electric Field are out of phase.

$\odot$ Hence, Electric Field and Magnetic Field or the surface of conductor is having 90° phase difference.
i.e. Electric Field at the surface of conductor lags Magnetic Field at the surface of conductor by 90°.

$\odot$ Inside conductor, Electric Field leads magnetic field by $\dfrac{\pi}{4}$

$\odot \vec{P}_r = -\vec{P}_i \quad \odot \vec{P}_t = \vec{0}$

4.16. Perpendicular polarization/Horizontal polarization/S-Polarization



- $\vec{r}_i = x\hat{i} + z\hat{k}$
- $\vec{r}_r = x\hat{i} - z\hat{k}$
- $\vec{r}_t = x\hat{i} + z\hat{k}$
- $\hat{a}_{P_i} = \sin\theta_i \hat{l} + \cos\theta_i \hat{k}$
- $\hat{a}_{P_r} = \sin\theta_i \; \hat{i} - \cos\theta_i \; \hat{k}$
- $\hat{a}_{P_t} = \sin\theta_t \; \hat{i} + \cos\theta_t \; \hat{k}$
- $\vec{\beta}_i = \beta_i \sin\theta_i \; \hat{i} + \beta_i \cos\theta_i \hat{k}$
- $\vec{\beta}_r = \beta_i \sin\theta_i \; \hat{i} + \beta_i \cos\theta_i \hat{k}$
- $\vec{\beta}_t = \beta_t \sin\theta_t \hat{i} + \beta_t \cos\theta_t \hat{k}$
- $\beta_i = \dfrac{\omega}{c_o}\sqrt{\mu_i \in_i}$
- $\beta_t = \dfrac{\omega}{c_o}\sqrt{\mu_t \in_t}$
- $\eta_i = \eta_o \sqrt{\dfrac{\mu_i}{\in_i}}$
- $\eta_t = \eta_o \sqrt{\dfrac{\mu_t}{\in_t}}$

Summary:

(a) $\Gamma_H = -\Gamma_E$

(b) $\tau_H = (1+\Gamma_H)\left(\dfrac{\cos\theta_i}{\cos\theta_t}\right)$

(c) $\tau_E = 1 + \Gamma_E$

(d) $\Gamma_E = \dfrac{\eta_t \sec\theta_t - \eta_i \sec\theta_i}{\eta_t \sec\theta_t + \eta_i \sec\theta_i}$

(e) $\Gamma_E = \dfrac{\sqrt{\mu_t \in_i}\sec\theta_t - \sqrt{\mu_i \in_t}\sec\theta_i}{\sqrt{\mu_t \in_i}\sec\theta_t + \sqrt{\mu_i \in_t}\sec\theta_i}$

For Non-Magnetic ($\mu_i = \mu_t = 1$)

(f) $\Gamma_E = \dfrac{\sqrt{\epsilon_i}\cos\theta_i - \sqrt{\epsilon_t}\cos\theta_t}{\sqrt{\epsilon_i}\cos\theta_i + \sqrt{\epsilon_t}\cos\theta_t}$

(g) $\beta \propto \sqrt{\epsilon_r} \Rightarrow \Gamma_E = \dfrac{\beta_i\cos\theta_i - \beta_t\cos\theta_t}{\beta_i\cos\theta_i + \beta_t\cos\theta_t}$

(h) $\lambda \propto \dfrac{1}{\sqrt{\epsilon_r}} \Rightarrow \Gamma_E = \dfrac{\lambda_t\sec\theta_t - \lambda_i\sec\theta_i}{\lambda_t\sec\theta_t + \lambda_i\sec\theta_i}$

(i) $V_P \propto \dfrac{1}{\sqrt{\epsilon_r}} \Rightarrow \Gamma_E = \dfrac{V_{P_t}\sec\theta_t - V_{P_i}\sec\theta_i}{V_{P_t}\sec\theta_t + V_{P_i}\sec\theta_i}$

(j) $\eta \propto \dfrac{1}{\sqrt{\epsilon_r}} \Rightarrow \Gamma_E = \dfrac{\eta_t\sec\theta_t - \eta_i\sec\theta_i}{\eta_t\sec\theta_t + \eta_i\sec\theta_i}$

(k) $n \propto \sqrt{\epsilon_r} \Rightarrow \Gamma_E = \dfrac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$

For magnetic medium $(\epsilon_i = \epsilon_t = 1)$

(l) $\Gamma_E = \dfrac{\sqrt{\mu_t}\sec\theta_t - \sqrt{\mu_i}\sec\theta_i}{\sqrt{\mu_t}\sec\theta_t + \sqrt{\mu_i}\sec\theta_i}$

(m) $\beta \propto \sqrt{\mu_r} \Rightarrow \Gamma_E = \dfrac{\beta_t\sec_t - \beta_i\sec\theta_i}{\beta_t\sec_t + \beta_i\cos\theta_i}$

(n) $\lambda \propto \dfrac{1}{\sqrt{\mu_r}} \Rightarrow \Gamma_E = \dfrac{\lambda_i\cos\theta_i - \lambda_t\cos\theta_t}{\lambda_i\cos\theta_i + \lambda_t\cos\theta_t}$

(o) $V_P \propto \dfrac{1}{\sqrt{\mu_r}} \Rightarrow \Gamma_E = \dfrac{V_{P_i}\cos\theta_i - V_{P_i}\cos\theta_t}{V_{P_i}\cos\theta_i + V_{P_i}\cos\theta_t}$

(p) $\eta \propto \sqrt{\mu_r} \Rightarrow \Gamma_E = \dfrac{\eta_t\sec\theta_t - \eta_i\sec\theta_i}{\eta_t\sec\theta_t + \eta_i\sec\theta_i}$

(q) $n \propto \sqrt{\mu_r} \Rightarrow \Gamma_E = \dfrac{n_t\sec\theta_t - n_i\sec\theta_i}{n_t\sec\theta_t + n_i\sec\theta_i}$

(r) Representation of Electric Field and Magnetic Field

$\odot\ \vec{E}_i = E_o e^{jwt} e^{-j\vec{\beta}_i \cdot \vec{r}_i}\hat{a}_y$

$\odot\ \vec{E}_r = \Gamma_E E_o e^{jwt} e^{-j\vec{\beta}_r \cdot \vec{r}_r}\hat{a}_y$

$\odot\ \vec{E}_t = \tau_E E_o e^{jwt} e^{-j\vec{\beta}_t \cdot \vec{r}_t}\hat{a}_y$

$\odot\ \vec{H}_i = \dfrac{E_o}{\eta_i}\left(-\cos\theta_i\,\hat{i} + \sin\theta_i\hat{k}\right)e^{jwt} e^{-j\vec{\beta}_i \cdot \vec{r}_i}$

$\odot\ \vec{H}_r = \Gamma_E \dfrac{E_o}{\eta_i}\left(-\cos\theta_i\,\hat{i} - \sin\theta_i\,\hat{k}\right)e^{jwt} e^{-j\vec{\beta}_r \cdot \vec{r}_r}$

$\odot\ \vec{H}_t = \tau_H \dfrac{E_o}{\eta_i}\left(-\cos_t\,\hat{i} + \sin\theta_t\hat{k}\right)e^{jwt} e^{-j\vec{\beta}_t \cdot \vec{r}_t}$

(s) Power Density

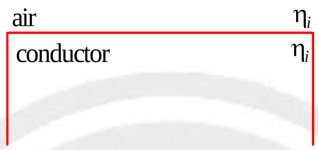
$\odot\ \vec{P}_i = \dfrac{E_0^2}{2\eta_i}\hat{a}_{P_i}$

$\odot\ \vec{P}_r = \dfrac{|\Gamma_E|^2 E_o^2}{2\eta_i}\hat{a}_{P_t}$

$\odot\ \vec{P}_t = \left(1-|\Gamma_E|^2\right)\dfrac{E_o^2}{2\eta_t}\hat{a}_{P_t}$

$\odot\ \dfrac{P_r}{P_i} = |\Gamma_E|^2 \quad \odot\ \dfrac{P_t}{P_i} = \left(1-|\Gamma_E|^2\right)\left(\dfrac{\cos\theta_i}{\cos\theta_t}\right)$

(t) When electromagnetic wave travels from air to conductor.



$\odot\ \eta_i = \eta_o\sqrt{\dfrac{u_r}{\epsilon_r}}$

$\odot\ \eta_t = \eta_o\sqrt{\dfrac{jwu}{\sigma_c + jw\epsilon}}$

Since, σ_c is very high. Hence, η_t value will be found in the range of milli ohm.

$\eta_i >>> \eta_t$

$$\Gamma_E = \frac{\eta_t\sec\theta_t - \eta_i\sec\theta_i}{\eta_t\sec\theta_t + \eta_i\sec\theta_i} = -1$$

$\Rightarrow \Gamma_H = -\Gamma_E = 1$

$\odot\ \tau_E = 1+\Gamma_E = 0$

$\odot\ \tau_E = (1+\Gamma_E)\left(\dfrac{\cos\theta_i}{\cos\theta_t}\right) = \dfrac{2\cos\theta_i}{\cos\theta_t}$

$\odot \dfrac{E_r}{E_i} = -1 \Rightarrow E_r = -E_i$

$\odot \dfrac{E_t}{E_i} = 0 \Rightarrow E_t = 0 \rightarrow Minima$

$\odot \dfrac{H_r}{H_i} = 1 \Rightarrow H_r = H_i$

$\odot \dfrac{H_t}{H_i} = \dfrac{2\cos\theta_i}{\cos\theta_t} \Rightarrow H_t = \left(\dfrac{2\cos\theta_i}{\cos\theta_t}\right)H_i \rightarrow Maxima$

$\odot$ Reflected and Incident Electric Field are out of phase.

$\odot$ Electric Field at the surface of conductor is ‘0’ (Minima).

⊙ Magnetic Field at the surface is maximum.

⊙ Hence, Electric Field and Magnetic Field on the surface of conductor is having 90° phase difference.

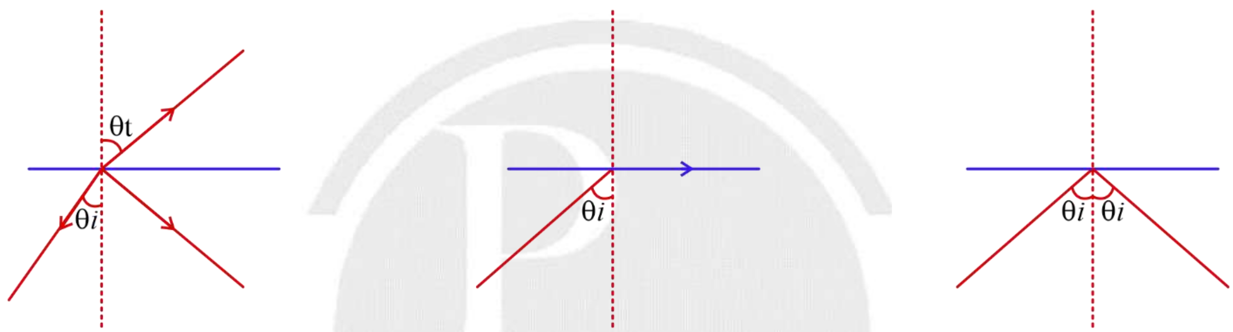
⊙ Electric Field at the surface of conductor lags Magnetic Field at the surface of conductor by 90°.

⊙ Inside conductor, Electric Field loads Magnetic Field by $\frac{\pi}{4}$.

⊙ $\vec{P}_r = -\vec{P}_i$ ⊙ $\vec{P}_t = \vec{0}$

4.17. Critical angle and total integral reflection (TIR)

When wave travels from denser to rarer medium, then at some angle, the electromagnetic wave starts gazing or travelling along the surface. So that angle is known as critical angle.

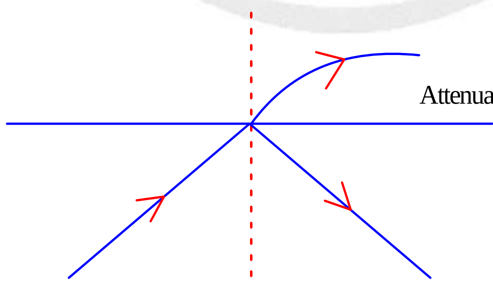


$\theta_i < \theta_c$ — Transmission + Reflection

$\theta_i = \theta_c$ — Wave travels along the surface

$\theta_i > \theta_c$ — No Transmission + Only Reflection (TIR)

- $\theta_i > \theta_c \Rightarrow \theta_t =$ Imaginary angle or does not exist.



- In general medium $\sin\theta_c = \sqrt{\frac{\mu_t \in_t}{\mu_i \in_i}}$
- In non-magnetic medium $\sin\theta_c = \sqrt{\frac{\in_t}{\in_i}}$
- In magnetic medium $\sin\theta = \sqrt{\frac{\mu_t}{\mu_i}}$

4.18. Brewster's Angle

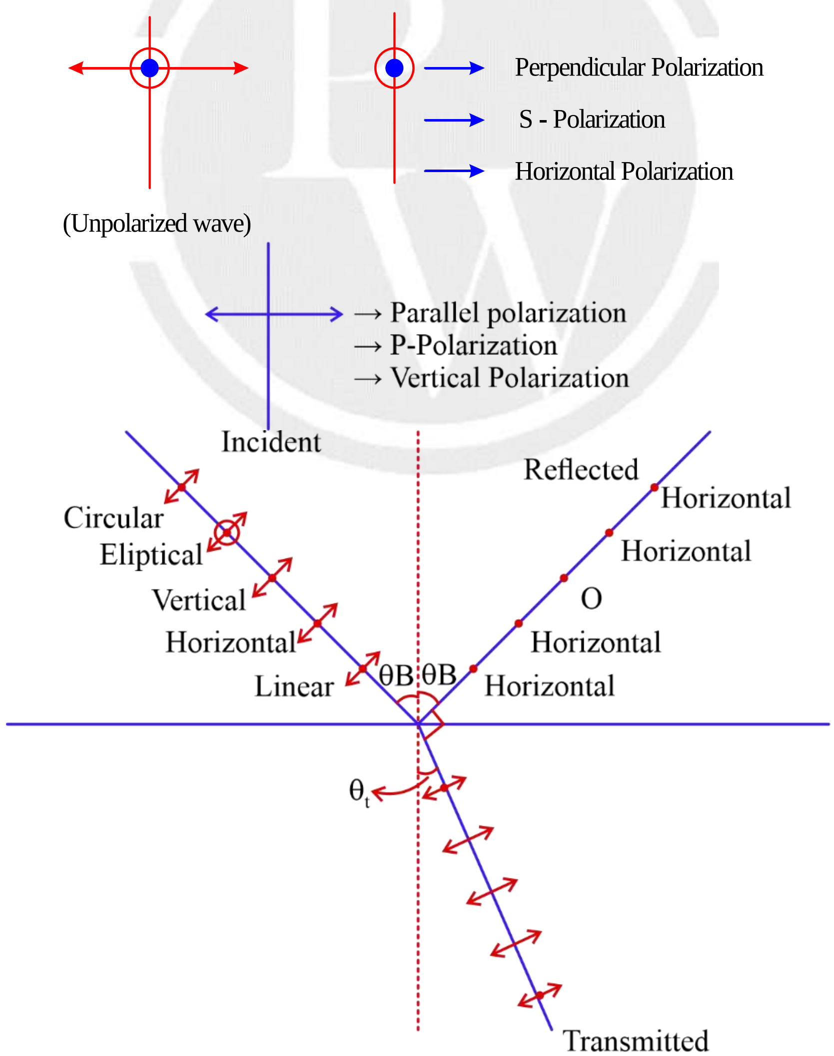
$$\theta_i = \theta_B$$

- No Reflection only Transmission
- The angle of incidence at which Electromagnetic wave is only transmitted but not reflected.

Case I : For Non-magnetic Medium

- $\sqrt{\epsilon_i} \sin\theta_B = \sqrt{\epsilon_t} \sin\theta_t$
- at $\theta_i = \theta_B, \Gamma_E = 0, \tau_E = 1$
- For S-polarization $\theta_i = \theta_B$ (does not exist)
- For P-Polarization, $\theta_i = \theta_B$ exists.

$$\tan\theta_B = \sqrt{\frac{\epsilon_t}{\epsilon_i}}$$



- $\tan\theta_B = \sqrt{\frac{\epsilon_t}{\epsilon_i}}$, For P-Polarization only, not for S-Polarization.
- $\theta_B + \theta_t = 90°$ i.e., Reflected and Transmitted wave are perpendicular to each other.

Case II : For General Medium

- $\sqrt{\mu_i \epsilon_i}\sin\theta_B = \sqrt{\mu_t \epsilon_t}\sin\theta_t$
- For S-polarization, $\sin\theta_B = \sqrt{\frac{(\mu_t \epsilon_i - \mu_i \epsilon_t)\epsilon_t}{\mu_i\left(\epsilon_i^2 - \epsilon_t^2\right)}}$
- For P-Polarization, $\sin\theta_B = \sqrt{\frac{(\mu_i \epsilon_t - \mu_t \epsilon_i)\mu_t}{\left(\mu_i^2 - \mu_t^2\right)\epsilon_i}}$
- For general medium, Brewster's angle will exist for both S-and P-Polarization
- Their Brewster's angles are different for same medium.
- For non-magnetic medium Brewster's angle will exist for S-polarization. Whereas it will not exist for P-polarization.

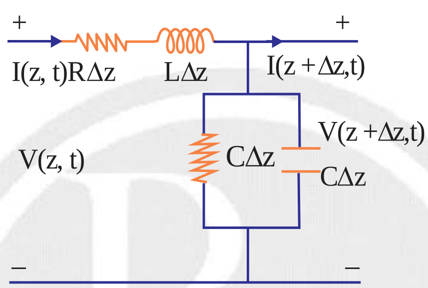
4.19. Difference between Brewster's angle and Critical angle.

	Critical Angle		Brewster's Angle
(a)	$\theta_i > \theta_c \rightarrow$ Total internal reflection	(a)	$\theta_i = \theta_B \rightarrow$ Brewster's Angle
(b)	$\theta_i > \theta_c \rightarrow$ Many angle	(b)	$\theta_i = \theta_B \rightarrow$ Only one angle
(c) (d)	Only Reflection No Transmission. Wave should travel from denser to rarer medium	(c) (d)	Only Transmission No Reflection No constrain for EM wave transmission
(e)	$\sin\theta_c = \sqrt{\frac{\epsilon_t}{\epsilon_i}}$	(e)	$\tan\theta_B = \sqrt{\frac{\epsilon_t}{\epsilon_i}}$
(f)	Critical angle exist for both P-polarization and S-polarization.	(f)	For non-magnetic medium, Brewster's angle exist only for S-polarization only
(g)	$\lvert\Gamma\rvert = 1, \tau = 0$	(g)	$\lvert\Gamma\rvert = 0, \tau = 1$

❑❑❑

5 TRANSMISSION LINE

5.1. Introduction



- $\frac{-dV(z)}{dz} = (R + j\omega L)I(z)$
- $\frac{-dI(z)}{dz} = (G + j\omega C)V(z)$
- $\left.\begin{array}{l} \frac{d^2I(z)}{dz^2} = y^2 I(z) \\ \frac{d^2V(z)}{dz^2} = y^2 V(z) \end{array}\right\} VI -$ Wave equation.
- $\left.\begin{array}{l} \frac{d^2I(z,t)}{dz^2} = \left(\frac{1}{V_p}\right)^2 \frac{d^2I(z,t)}{dt^2} \\ \frac{d^2V(zt)}{dz^2} = \left(\frac{1}{V_p}\right)^2 \frac{d^2V(z,t)}{dt^2} \end{array}\right\}$ Time harmonic Equation.

Here, V_p = phase velocity.

1. Transmission Line Parameter:

(a) R = Resistance per unit length

(i) It is distributed along the transmission line.

(ii) It is due to conductor loss.

(iii) It is in series.

(iv) It gives attenuation in the voltage.

(b) G = Conductance per unit length.

(i) It is distributed in between the transmission line.

(ii) It is due to dielectric loss.

(iii) It is in parallel.

(iv) It gives attenuation in the current.

(c) L = Inductance per unit length.

(i) It is distributed along the transmission line.

(ii) It is due to storage of magnetic field.

(iii) It is in series.

(iv) It gives phase shift in voltage.

(d) C = Capacitance per unit length.

(i) It is distributed in between the transmission line.

(ii) It is due to storage of electric field.

(iii) It is in parallel.

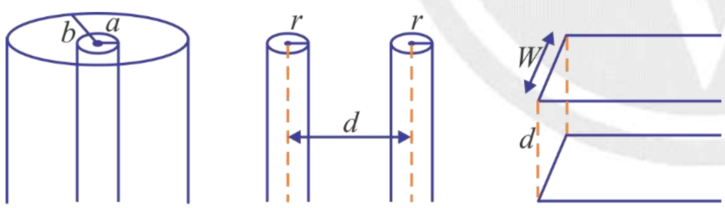
(iv) It gives phase shift in current.

(e) $R \xleftarrow{\alpha} V \ \angle\theta \xrightarrow{\beta} L$

$G \xleftarrow{\alpha} I \ \angle\theta \xrightarrow{\beta} C$

(f) $LC = \mu_E$

(g) $\dfrac{G}{C} = \dfrac{\sigma d}{E}$



Co-axial cable | Twin-wire | Semi infinite parallel plate

(h)

(i)

	Semi-Infinite Parallel Plate	Co-axial cable	Twin wire
R	$\dfrac{2Rs}{W}$	$\dfrac{Ra}{2\pi a} + \dfrac{Rs}{2\pi b}$	$\dfrac{2Rs}{2\pi r}$
C	$\dfrac{WE}{d}$	$\dfrac{2\pi E}{\ln\left(\dfrac{b}{a}\right)}$	$\dfrac{\pi E}{\ln\left(\dfrac{d}{r}\right)}$
L	$\dfrac{\mu d}{W}$	$\dfrac{2\mu}{2\pi}\ln\left(\dfrac{b}{a}\right)$	$\dfrac{2\mu}{2\pi}\ln\left(\dfrac{d}{r}\right)$

G	$\frac{W\sigma_d}{d}$	$\frac{2\pi\sigma_d}{\ln\left(\frac{b}{a}\right)}$	$\frac{2\pi\sigma_d}{\ln\left(\frac{d}{r}\right)}$
Z_o	$\left(\sqrt{\frac{\mu}{E}}\right)\left(\frac{d}{W}\right)$	$\frac{60\sqrt{\mu_r}}{\sqrt{\epsilon_r}}\ln\left(\frac{b}{a}\right)$	$\frac{120\sqrt{\mu_x}}{\sqrt{\epsilon_r}}\ln\left(\frac{d}{r}\right)$

2. Intrinsic Impedance: -

(a) $Z_o = \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} = \frac{R+jwL}{\gamma} = \frac{\gamma}{G+j\omega C} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$

(b) It is defined at every point of transmission line.

(c) It is independent from length of the transmission line.

(d) Z_0 = depends upon σ_d, σ_c, μ, $\angle$ (medium) and geometry of transmission line.

3. Lossless Transmission Line:

R = 0 $\rightarrow$ Perfect conductor $\rightarrow \sigma_c \rightarrow \infty$

G = 0 $\rightarrow$ Perfect dielectric $\rightarrow \sigma_d = 0 \rightarrow \tan\delta = 0$

(a) $\gamma = \sqrt{(R+j\omega L)(G+j\omega L)} = \sqrt{(R+j\omega L)(G+j\omega L)} = j\omega\sqrt{LC}$

$\alpha = 0 \Rightarrow$ Lossless and Distortionless

$\beta = \omega\sqrt{LC} \Rightarrow$ Non-Dispersive

$\beta = \omega\sqrt{LC} = \omega\sqrt{\mu\varepsilon} = \beta_0\sqrt{\mu_r\varepsilon_r}$

(b) $\lambda = \frac{\lambda_0}{\sqrt{\mu_r\varepsilon_r}}$

(c) $V_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c_o}{\sqrt{\mu_r\varepsilon_r}}$

(d) $V_g = \frac{1}{\sqrt{LC}}$

(e) $V_p \cdot V_g = \frac{1}{LC} = \frac{c_0^2}{\mu_r\varepsilon_r}$

(f) $Z_o = \sqrt{\frac{L}{C}}$

(g) $\alpha, \beta, \lambda, V_p, V_g \rightarrow$ Medium dependent

$Z_0 \rightarrow$ medium and geometry depend

4. Distortionless Transmission Line

$$LG = RC$$

(a) $\alpha = \sqrt{RG} = G\sqrt{\frac{L}{C}}$ → Lossy and Distortionless

$\beta = \omega\sqrt{LC} = \omega\sqrt{\mu\varepsilon} = \beta_0\sqrt{\mu_r\varepsilon_r}$ → Non-dispersive

(b) $V_P = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c_o}{\sqrt{\mu_r\varepsilon_r}}$

(c) $V_g = \frac{1}{\sqrt{LC}}$

(d) $V_p.\ V_g = \frac{1}{\sqrt{LC}} = \frac{c_o^2}{\mu_r\varepsilon_r}$

(e) $Z_o = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}$

5. Some important formulae

(a) $\Gamma v = \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$

(b) $\Gamma_I = \frac{Z_0 - Z_L}{Z_0 + Z_L}$

(c) $\Gamma_{(x)} = \frac{Z_L - Z_0}{Z_L + Z_0}e^{-j2\beta x}$

(d) $Z(x) = Z_0\left(\frac{1+\Gamma_{(x)}}{1-\Gamma_{(x)}}\right)$

(e) $\Gamma_{(x)} = \frac{Z(x) - Z_0}{Z(x) + Z_0}$

(f) $Z(x)\ Z_0 = \left(\frac{Z_L + jZ_0 \tan(\beta x)}{Z_0 + jZ_L \tan(\beta x)}\right)$(For lossless Tx-line)

(g) $2(x) = Z_0\left(\frac{Z_L + Z_O \tan n(\gamma x)}{Z_o + Z_L \tan n(\gamma x)}\right)$(For lossy Tx-line)

6. Impedance:

(a) $Z_{OC} = -jZ_o \cot \beta l$

(b) $Z_{SC} = jZ_o \tan \beta l$

(c) $Z_{OC}\ .\ Z_{SC} = (Z_o)^2$

(d) For $Z_L = Z_0$, $Z_{in} = Z_0$

Length	Z_{SC}	Z_{OC}
$O < l < \frac{\lambda}{4}$	Inductive	Capacitive
$\frac{\lambda}{4} < l < \frac{\lambda}{2}$	Capacitive	Inductive
$\frac{\lambda}{2} < l < \frac{3\lambda}{4}$	Inductive	Capacitive
$\frac{3\lambda}{4} < l < \lambda$	Capacitive	Inductive

(e)

Series Resonance		Parallel Resonance	
Z_{SC}	Z_{OC}	Z_{SC}	Z_{OC}
$l = \frac{n\lambda}{2}$	$l = \frac{(2n+1)\lambda}{4}$	$l = \frac{(2n+1)\lambda}{4}$	$l = \frac{n\lambda}{2}$
$\lambda = \frac{2l}{n}$	$\lambda = \frac{4l}{(2n+1)}$	$\lambda = \frac{4l}{(2n+1)}$	$\lambda = \frac{2l}{n}$
$f_o = \frac{nV_p}{2l}$	$f_o = \frac{(2n+1)V_p}{4l}$	$f_o = \frac{(2n+1)V_p}{4l}$	$f_o = \frac{nV_p}{2l}$

(f) $l = \frac{(2n+1)\lambda}{4} \Rightarrow Z_{in} = \frac{Z_o^2}{Z_L} \Rightarrow \bar{Z}_{in} = \frac{1}{\bar{Z}_L}$ impedance in version

(g) Normalised Impedance

Input impedance $\Rightarrow \bar{Z}_{in} = \frac{Z_{in}}{Z_o}$

Normalised load impedance $\Rightarrow \bar{Z}_L = \frac{Z_L}{Z_o}$

(h) $l = \frac{n\lambda}{2} \Rightarrow Z_{in} = Z_L$

(i) $l = \frac{\lambda}{8}, \frac{5\lambda}{8}, \frac{9\lambda}{8}, \ldots$

$$Z_{in} = Z_o\left(\frac{Z_L + jZ_o}{Z_o + jZ_L}\right)$$

(j) $l = \frac{3\lambda}{8}, \frac{7\lambda}{8}, \frac{11\lambda}{8}, \ldots$

$$Z_{in} = Z_o\left(\frac{Z_L - jZ_o}{Z_o - jZ_L}\right)$$

5.2. VSWR

SWR : Standing wave ratio

$$V_{\text{SWR}} = \frac{V_{\max}}{V_{\min}}, \quad I_{\text{SWR}} = \frac{I_{\max}}{I_{\min}}, \quad E_{\text{SWR}} = \frac{E_{\max}}{E_{\min}}, \quad H_{\text{SWR}} = \frac{H_{\max}}{H_{\min}} \rightarrow \rho = \frac{1+|\Gamma|}{1-|\Gamma|}$$

$Z_L < Z_0$ $\qquad$ $Z_L > Z_0$

$Z_L = 0$ $\qquad$ $Z_L = Z_0$ $\qquad$ $Z_L = \infty$

Case I: $Z_L = 0 \Rightarrow$ Short Circuit

1. $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{0 - Z_0}{0 + Z_0} = -1 = 1e^{j\pi} \Rightarrow |\Gamma_L| = 1, \theta_\Gamma = \pi$

2. $V_{\max} = V_f(1 + \Gamma_L) = 2V_f$

   $V_{\min} = V_f(1 - \Gamma_L) = 0$

3. $V_{SWR} = \dfrac{1+|\Gamma_L|}{1-|\Gamma_L|} = \infty$

4. $Z_{\max} = \dfrac{(2n\pi + \pi)\lambda}{4\pi} = \dfrac{(2n+1)\lambda}{4}$

   $Z_{\max} = \dfrac{\lambda}{4}, \dfrac{3\lambda}{4}, \dfrac{5\lambda}{4}, \dfrac{7\lambda}{4}, \dfrac{9\lambda}{4}, \ldots$

   $\uparrow$

   1st Maxima

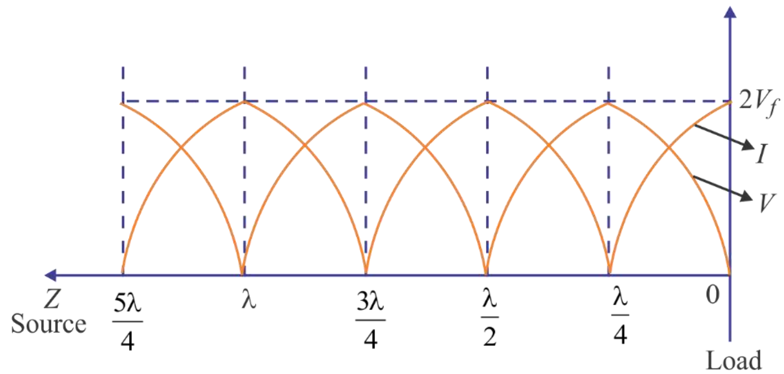
5. $Z_{\min} = \dfrac{((2n+1)\pi + \pi)\lambda}{4\pi} = \dfrac{(n+1)\lambda}{2}$

   $0, \ \lambda/2, \ \dfrac{3\lambda}{2}, \ \ldots$

   $\downarrow \quad \downarrow$

   $Z_{\min}$ = 1st Minima

6.



Case II: $Z_L < Z_0$

1. $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0} = -K = Ke^{j\pi} \Rightarrow \begin{array}{c} |\Gamma_L| = K \\ \underline{\theta^{\Gamma} = \pi} \end{array} \quad 0 < K < 1$

2. $V_{max} = V_f(1 + K) < 2V_f$

   $V_{min} = V_f(1 - K) > 0$

3. $V_{SWR} = \dfrac{1+K}{1-K}$

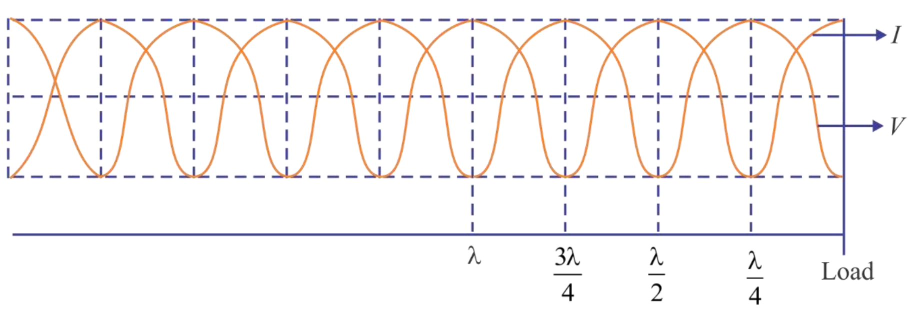
4. $Z_{max} = \dfrac{(2n\pi + \pi)\lambda}{4\pi} = \dfrac{(2n+1)\lambda}{4}$

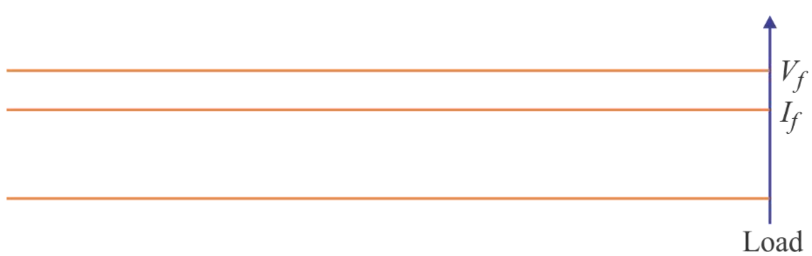
   $Z_{max} = \dfrac{\lambda}{4}, \dfrac{3\lambda}{4}, \dfrac{5\lambda}{4}, \ldots$

5. $Z_{min} = \dfrac{((2n+1)\pi + \pi)\lambda}{4\pi} = \dfrac{n\lambda}{2}$

   $Z_{min} = 0, \dfrac{\lambda}{2}, \lambda, \dfrac{3\lambda}{2}, 2\lambda, \ldots$

6.





Case III: $Z_L > Z_0$

1. $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0} = K \Rightarrow \begin{array}{l} |\Gamma_L| = K \\ \theta_\Gamma = 0 \end{array}, \quad 0 < K < 1$

2. $V_{max} = V_f(1 + K) < 2V_f = V_f(1 - K) > 0$

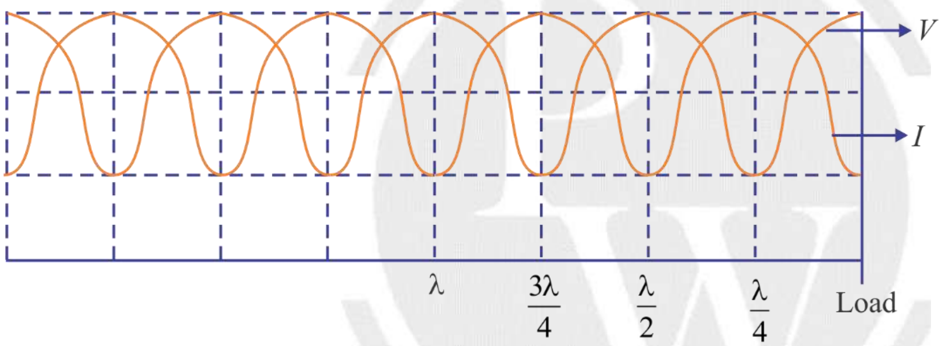
3. $V_{SWR} = \dfrac{1+K}{1-K}$

4. $Z_{max} = \dfrac{(2n\pi + 0)\lambda}{4\pi} = \dfrac{n\lambda}{2}$

   $Z_{max} = 0, \dfrac{\lambda}{2}, \lambda, \dfrac{3\lambda}{2}, 2\lambda, \ldots$

5. $Z_{min} = \dfrac{(2n\pi + \pi)\lambda}{4} = \dfrac{(2n+1)\lambda}{4}$

   $Z_{min} = \dfrac{\lambda}{4}, \dfrac{3\lambda}{4}, \dfrac{5\lambda}{4}, \ldots$

6. 

Case IV: $\Gamma_L = \infty \Rightarrow$ Open circuit

1. $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{1 - Z_0/Z_L}{1 + Z_0/Z_L} = 1 \Rightarrow |\Gamma_L| = 1, \ \theta_\Gamma = 0$

2. $V_{max} = V_f(1 + |\Gamma_L|) = 2V_f$

   $V_{min} = V_f(1 - |\Gamma_L|) = 0$

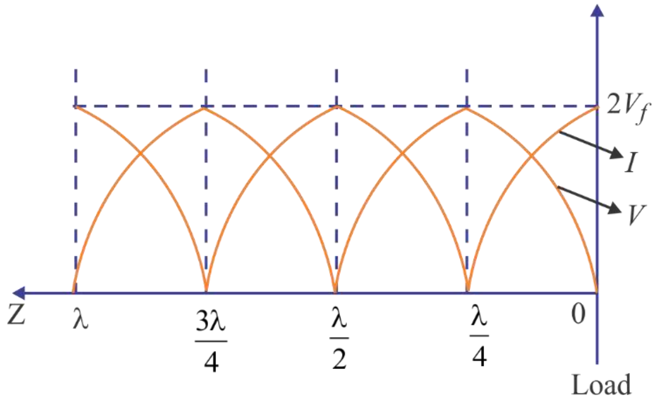
3. $V_{SWR} = \dfrac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \infty$

4. $Z_{max} = \dfrac{(2n\pi + 0)\lambda}{4\pi} = \dfrac{n\lambda}{2}$

   $Z_{max} = 0, \dfrac{\lambda}{2}, \lambda, \dfrac{3\lambda}{2}, 2\lambda, \ldots$

5. $Z_{min} = \dfrac{((2n+1)\pi + 0)\lambda}{4\pi} = \dfrac{(2n+1)\lambda}{4}$

   $Z_{min} = \dfrac{\lambda}{4}, \dfrac{3\lambda}{4}, \dfrac{5\lambda}{4}, \ldots$

6. 

1. Distance between two successive maxima = $\lambda/2$

   Distance between two successive minima = $\lambda/2$

   Distance between two successive maxima & minima = $\lambda/4$

2.

V	I	Z	Z
Maximum	Minimum	Maximum	$Z_{max} = \frac{(2n\pi + \theta_\Gamma)\lambda}{4\pi}$
Minimum	Maximum	Minimum	$Z_{min} = \frac{((2n+1)\pi + \theta_\Gamma)\lambda}{4\pi}$

3. (a) Complete matched or matched

$$\Gamma_L = 0,\ VSWR = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = 1$$

   (b) Complete mismatched

$$|\Gamma_L| = 1,\ VSWR = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \infty$$

   For example: S/C, O/C, pure inductive, pure resistive.

   (c) Mismatched:

$$-1 < \Gamma_L < 1 \quad \Rightarrow 1 < \text{VSWR} < \infty$$

4. VSWR $\in [1, \infty)$

$$\Gamma \in [-1, 1]$$

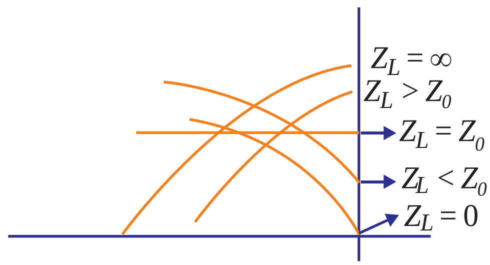
5. $\Gamma(z) = \Gamma_L\, e^{-j2fz} = |\Gamma_L| e^{+j\theta_\Gamma} e^{-j\theta Z} = |\Gamma_L| e^{j(\theta_\Gamma - 2fz)}$

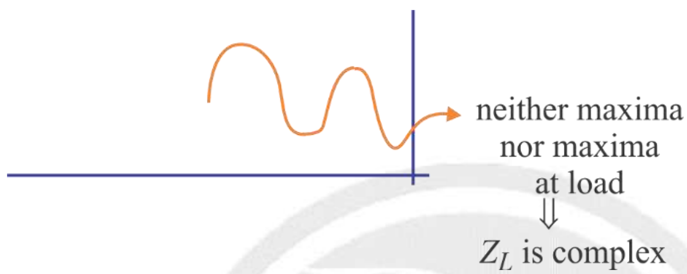
   Magnitude of reflection coefficient does not depend on distance. But phase of reflection coefficient depends upon distance.

$$VSWR = \frac{1+|\Gamma(Z)|}{1-|\Gamma(Z)|} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|}$$

VSWR does not depends upon distance.

6.





7.

8. (a) In case of maxima at load, $Z_L > Z_0$.

$$\Gamma_L = \frac{Z_L - Z_O}{Z_L + Z_O} \Rightarrow |\Gamma_L| = \frac{(Z_L - Z_O)}{(Z_L + Z_O)}$$

$$VSWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \frac{Z_L}{Z_O}$$

(b) In case of minima at load ($Z_L < Z_O$)

$$\Gamma_L = \frac{Z_L - Z_O}{Z_L + Z_O} < 0$$

$$|\Gamma_L| = -\Gamma_L = \frac{Z_O - Z_L}{Z_O + Z_L}$$

$$VSWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \frac{Z_O}{Z_L}$$

$$VSWR = \rho = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \Rightarrow |\Gamma_L| = \frac{\rho - 1}{\rho + 1}$$

$VSWR = \infty$
- Maxima at load $\Rightarrow Z_L = \infty,\ \Gamma_L = 1$
- Minima at load $\Rightarrow Z_L = 0,\ \Gamma_L = -1$

$VSWR = \text{finite} = \rho$
- Maxima at load $\Rightarrow Z_L > Z_O,\ VSWR = \dfrac{Z_L}{Z_O},\ |\Gamma_L| = \Gamma_L$
- Minima at load $\Rightarrow Z_L < Z_O,\ VSWR = \dfrac{Z_O}{Z_L},\ |\Gamma_L| - \Gamma_L$

12. In ideal case, matched $\Rightarrow |\Gamma_L| = 0$, VSWR = 1

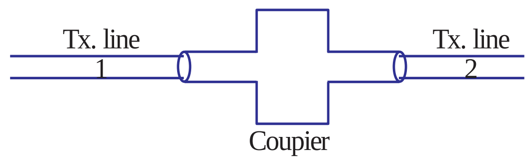
In practical case, matched $\Rightarrow 1 \le \text{VSWR} \le 2$.

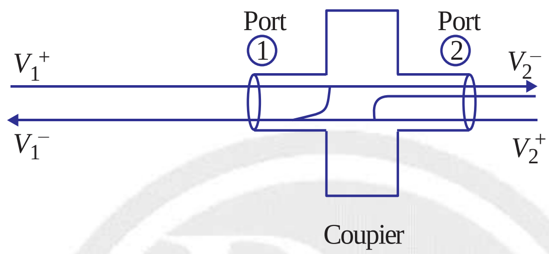
Scattering Matrix:

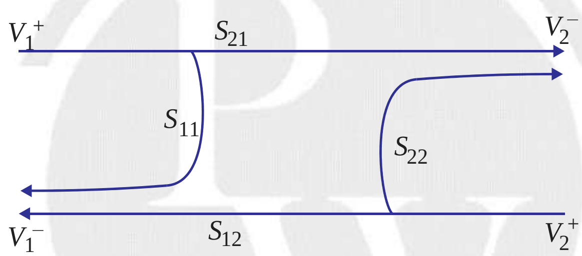
In Y, Z, h, g parameter, open circuit and short circuit conditions are required. But in transmission line, short and open circuit cannot be achieved throughout the transmission line.

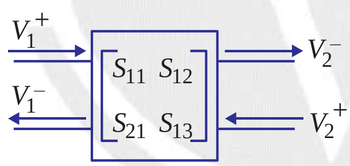
So, S-parameter concept is introduced in microwave devices.

⇒ S-parameter for coupler.









$$V_1^- = S_{11}V_1^+ + S_{12}V_2^+$$
$$V_2^- = S_{21}V_1^+ + S_{22}V_2^+$$

(a) When $V_1^+ = 0 \Rightarrow$ Port (1) matched

$$S_{12} = \left.\frac{V_1^-}{V_2^+}\right|_{V_1^+=0}$$

S_{12} = Transmission coefficient at port (2) due to port (2) when port (1) matched.

= Reverse voltage gain

$$S_{22} = \left.\frac{V_2^-}{V_2^+}\right|_{V_1^+=0}$$

S_{22} = Reflection coefficient at port (2) due to port (1) when port (1) matched

= Output reflection coefficient

(b) When $V_2^+ = 0 \Rightarrow$ Port (2) matched

$$S_{11} = \left.\frac{V_1^-}{V_1^+}\right|_{V_2^+=0}$$

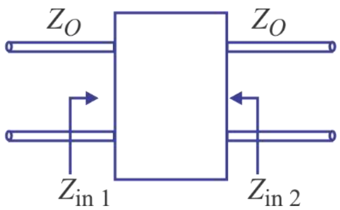
S_{11} = Reflection coefficient at port (1) due to port (1) when port (2) is matched

= Input reflection coefficient

$$S_{21} = \left.\frac{V_2^-}{V_2^+}\right|_{V_2^+=0}$$

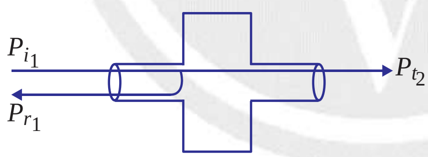
S_{21} = Transmission coefficient at port (2) due to port (1) when port (2) is matched

= Forward voltage gain

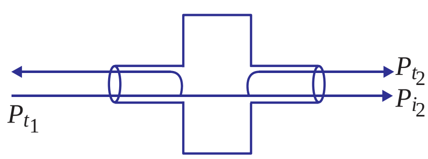


1. $S_{11} = \Gamma_1 = \dfrac{Z_{in_1} - Z_O}{Z_{in_1} + Z_O}$ 2. $S_{21} = \tau_1 = 1 + \Gamma_1$ 3. $S_{22} = \Gamma_1 = \dfrac{Z_{in_2} - Z_O}{Z_{in_2} + Z_O}$ 4. S12 = $\tau_2 = 1 + \Gamma_2$	(a) For Symmetry $Z_{in_1} = Z_{in_2}$ $\Rightarrow \boxed{S_{11} = S_{22}}$ (b) For reciprocal Forward gain = Reverse gain $\boxed{S_{12} = S_{21}}$

For Lossless:



$$S_{11} = \sqrt{\frac{P_{r_1}}{P_{i_1}}},\ S_{21} = \sqrt{\frac{P_{t_1}}{P_{i_1}}}$$



$$S_{22} = \sqrt{\frac{P_{t_2}}{P_{i_2}}},\ S_{12} = \sqrt{\frac{P_{t_1}}{P_{i_2}}}$$

For lossless device

$$P_{i_1} = P_{r_1} + P_{t_2} \Rightarrow\ 1 = \frac{P_{r_1}}{P_{i_1}} + \frac{P_{t_2}}{P_{i_2}} \Rightarrow 1 = |S_{11}|^2 + |S_{21}|^2$$

$$P_{i_2} = P_{r_2} + P_{t_2} \Rightarrow\ 1 = |S_{22}|^2 + |S_{12}|^2$$

For lossy device:

$$P_{i_1} > P_{r_1} + P_{t_2} \Rightarrow |S_{11}|^2 + |S_{21}|^2 < 1$$

$$P_{i_2} > P_{r_2} + P_{t_2} \Rightarrow |S_{22}|^2 + |S_{12}|^2 < 1$$

For Amplifier

$$P_{i_1} < P_{r_1} + P_{t_2} \Rightarrow |S_{11}|^2 + |S_{21}|^2 > 1$$

$$P_{i_2} < P_{r_2} + P_{t_1} \Rightarrow |S_{11}|^2 + |S_{12}|^2 > 1$$

For Lossless

S*S = I

$$\begin{bmatrix} S_{11}^* & S_{21}^* \\ S_{12}^* & S_{22}^* \end{bmatrix}\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left.\begin{matrix} |S_{11}|^2 + |S_{21}|^2 = 1 \\ |S_{12}|^2 + |S_{22}| = 1 \end{matrix}\right] \rightarrow \text{Unitar property} \left|\begin{matrix} S_{11}^* S_{12} + S_{21}^* S_{22} = 0 \\ S_{12}^* S_{11} + S_{22}^* S_{21} = 0 \end{matrix}\right] \rightarrow \text{zero property}$$

1. Return loss at port (1) = – 20 $\log_{10}|S_{11}|$

2. Return loss at port (2) = – 20 $\log_{10}|S_{22}|$

3. Gain at port (2) = 20 $\log_{10}|S_{21}|$

4. Gain at port (1) = 20 $\log_{10}|S_{12}|$

5. Intertion loss at port (1) = $-20\log_{10}\left(\frac{(S_{11})^2}{1-|S_{11}|^2}\right)$

   Intertion loss at port (2) = $-20\log_{10}\left(\frac{(S_{22})^2}{1-|S_{22}|^2}\right)$

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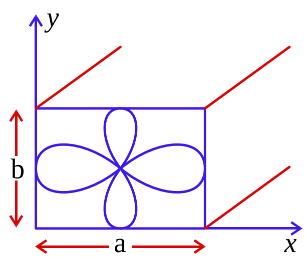
6 WAVEGUIDE

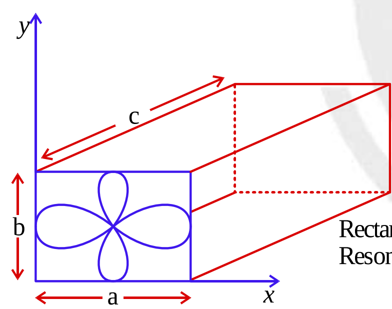
6.1. Introduction

- Waveguide directs the wave in one direction



Rectangular waveguide.

		E	H
X direction	Standing wave	sin (min)	cos (max)
Y direction	Standing wave	sin (min)	cos (max)
Z direction	Propagating wave	$e^{-j\beta_z z}$	$e^{-j\beta_z z}$

- 

Rectangular Resonator.

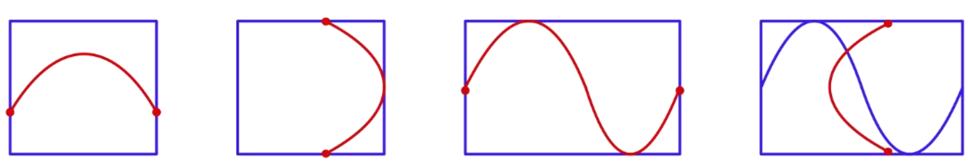
		E	H
X direction	Standing wave	Min.	Max.
Y direction	Standing wave	Min.	Max.
Z direction	Standing wave	Max.	Min.

(1) Analysis of propagation constants.

(a) $f < f_c \Rightarrow \beta_z = -j\sqrt{\beta_c^2 - \beta^2}$

(b) $f = f_c \Rightarrow \beta_z = 0$

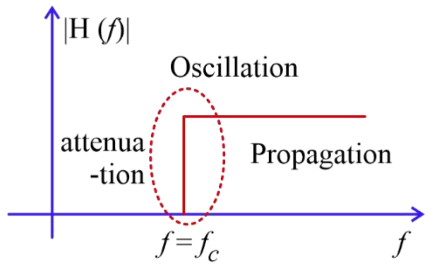
No Propagation only Oscillation



m = 1, n = 0 | m = 0, n = 1 | m = 2, n = 0 | m = 2, n = 1

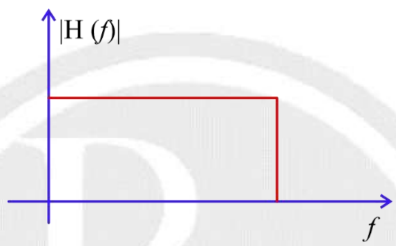
(c) $f > f_c \Rightarrow \beta_z = \sqrt{\beta^2 - \beta_c^2} \neq 0$ Only Propagation occurs.

2. (a) $f < f_c$ & $\lambda > \lambda_c$ Attenuation, Evancent mode.
   (b) $f = f_c$, & $\lambda = \lambda_c$ Only Attenuation
   (c) $f > f_c$ & $\lambda < \lambda_c$ Propagation



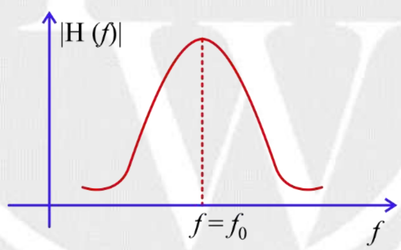
Rectangular
Waveguide

(3) Transmission Line:



It will act as a low pass filter. It will behave like an all pass filter within Frequency range.

(4) Resonator:



It will act as a band pass filter.

(5) Cut-off frequency and Cut-off wavelength.

$$f_c = \left(\frac{C_0}{2\sqrt{\mu_r \in_r}}\right)\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\lambda_c = \frac{2\sqrt{\mu_r \in_r}}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{p}\right)^2}}$$

(6) λ_x = Wavelength in X-direction.

λ_y = Wavelength in Y-direction.

λ_z = Wavelength in Z-direction

λ_g = Guided wavelength

λ = Operating wavelength.

$$\boxed{\lambda_g = \lambda_z}$$

- $\lambda_x = \frac{2a}{m}, \qquad \lambda_y = \frac{2b}{n}$
- $\lambda_z = \lambda_g = \frac{\lambda}{\cos\theta}$
- $\frac{1}{\lambda^2} = \frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2}$
- θ = tilt angle
- $\sin\theta = \frac{f_c}{f} = \frac{\lambda}{\lambda_c} = \frac{\omega_c}{\omega} = \frac{\beta_c}{\beta}$
- $\beta_x = \frac{m\pi}{a}, \beta_y = \frac{n\pi}{b}, \beta_z = \beta\cos\theta$
- $\beta_c = \sqrt{\beta_x^2 + \beta_y^2} = \beta\sin\theta$.

(7) Phase velocity and group velocity.

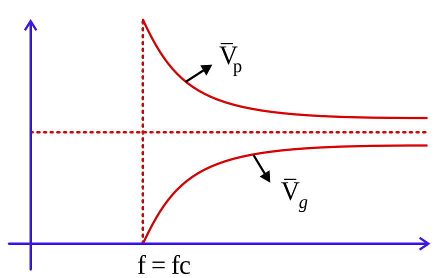
$\bar{V}_p$ = phase velocity inside waveguide

$\bar{V}_g$ = group velocity inside waveguide

V_p = phase velocity in free space

V_g = group velocity in free space

- $\bar{V}_p = \frac{V_p}{\cos\theta}, \bar{V}_g = V_p\cos\theta$ for $f > f_c$
- $V_p = \frac{C_0}{\sqrt{\mu_r \in_r}}$
- $$\bar{V}_p = \begin{cases} 0 & f < f_c \\ \infty & f = f_c \\ \dfrac{V_P}{\sqrt{1-\left(\dfrac{f_c}{f}\right)^2}} & f > f_c \end{cases}$$



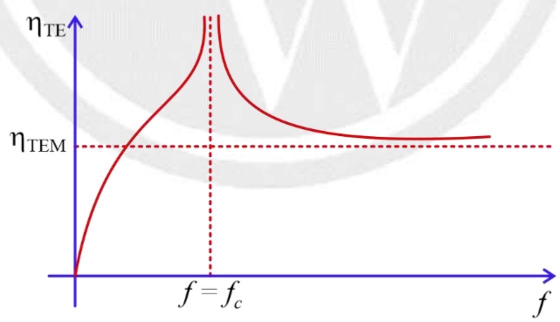
- $\bar{V}_P.\bar{V}_g = \frac{C_0^2}{\mu_r \in_r}$

- Inside waveguide, $\bar{V}_P > V_P$
- Inside waveguide, $\bar{V}_g < V_P$
- At very high frequency, $\bar{V}_P = \bar{V}_g = V_P$

(8) Intrinsic Impedance:

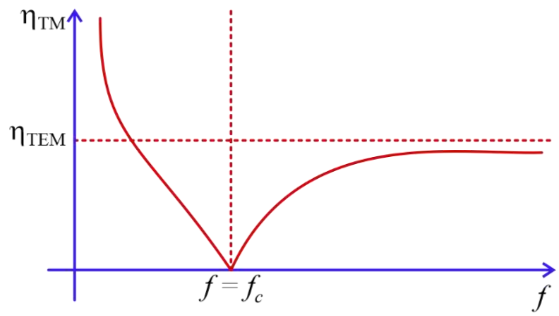
- η_{TEM} = Wave Impedance in free space or TEM wave.
- η_{TE} = Wave Impedance in TE mode
- η_{TM} = Wave Impedance in TM mode
- $\eta_{TEM} = \eta_0 \sqrt{\dfrac{\mu_r}{\in_r}}$
-

$$\eta_{TE} \rightarrow \begin{cases} -jx & \text{(Capacitive)} \\ \infty & \text{Open circuit (Parallel resonance)} \\ \dfrac{\eta_{TEM}}{\sqrt{1-\left(\dfrac{f_c}{f}\right)^2}} & f > f_c \end{cases}$$



- $\eta_{TE} > \eta_{TEM}$
- At very high frequency, $\eta_{TE} = \eta_{TEM}$

$$\eta_{TM} \rightarrow \begin{cases} jx & f < f_c & \text{Inductive} \\ \text{O} & f = f_c & \text{Short circuit (series resonance)} \\ \eta_{TEM}\sqrt{1-\left(\dfrac{f_c}{f}\right)^2} & f > f_c & \end{cases}$$



- $\eta_{TM} < \eta_{TEM}$
- At very high frequency, $\eta_{TM} = \eta_{TEM}$
- $\eta_{TE} \cdot \eta_{TM} = \eta_{TEM}^2 = \eta_0^2 \left(\dfrac{\mu_r}{\in_r}\right)$
- $\eta_{TE} = \dfrac{\eta_{TEM}}{\cos\theta}$ $\qquad f > f_c$
- $\eta_{TM} = (\eta_{TEM})\cos\theta$ $\qquad f > f_c$

(9) Some Important Terms

(i) Dominant mode: The mode which has lowest cut-off frequency and highest cut-off wavelength, is known as dominant mode.

- For a > b → TE_{10} dominant mode → $f_c = \dfrac{C_0}{2a}$
- For a < b → TE_{01} dominant mode → $f_c = \dfrac{C_0}{2b}$
- For c > a > b → TE_{101} dominant mode → $f_r = \dfrac{C_0}{2\sqrt{\mu_r \in_r}}\sqrt{\left(\dfrac{1}{a}\right)^2 + \left(\dfrac{1}{c}\right)^2}$
- For c > b > a → TE_{011} dominant mode → $f_r = \dfrac{C_0}{2\sqrt{\mu_r \in_r}}\sqrt{\left(\dfrac{1}{b}\right)^2 + \left(\dfrac{1}{c}\right)^2}$

(ii) Single mode frequency operation

(a) For $a > b \rightarrow \dfrac{C_0}{2a} < f < \dfrac{C_0}{2b} \rightarrow 2b < \lambda < 2a$

(b) For $a < b \rightarrow \dfrac{C_0}{2b} < f < \dfrac{C_0}{2a} \rightarrow 2a < \lambda < 2b$

(iii) Non-Existence Mode:

The mode which does not exist.

(a) Rectangular waveguide:

TEoo, TMoo, TMon, TMmo

(b) Rectangular resonator

TEooo, TEmoo, TEono, TEmno, TEoop, TMooo, TMoop, TMmop, TMmoo, TMono, TMonp

(iv) Existence Mode:

The mode which exist

(a) Rectangular waveguide

TEon, TEmo, TEmn, TMmn

(b) Rectangular resonator:

TEonp, TEmop, TEmnp, TMmno, TMmnp

(v) Evanescent Mode:

The mode which exist but operating frequency is less than cutt-off frequency.

(vi) Degenerate Mode:

- Two modes have same cutt-off frequency
- These modes are existance mode
- These modes exist in a square waveguide (a = b) and cubic resonator (a = b = c).

e.g. TEmn ≡ TEnm, TEon ≡ TEmo

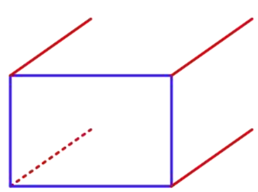
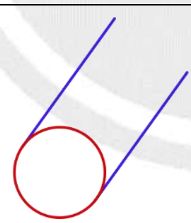
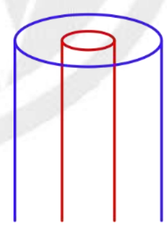
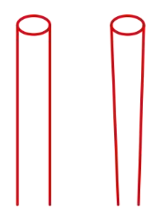
TMmn ≡ TMnm,

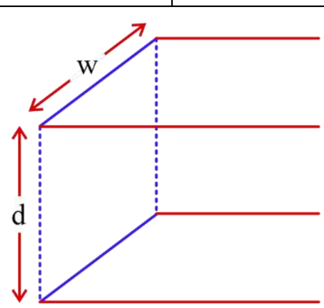
TEmop ≡ TEonp ≡ TEpom ≡ TEopm.

(vii) TEM mode:

- The mode which has no cutt-off frequency is known as TEM mode.

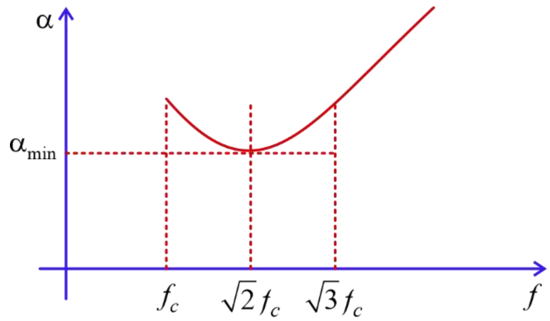
- TEoo, TMoo, TEooo, TMooo are TEM mode

				
•	Rectangular	Circular	Co-axial	Twin-wire
•	Single conductor	Single conductor	Double conductor	Double conductor
•	$f_c \neq o$	$f_c \neq o$	$f_c = o$	$f_c = o$
•	TEM does not exist	TEM does not exist	TEM exist	TEM exist

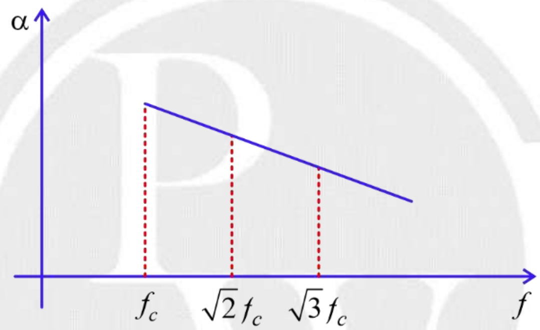


- Semi-infinite parallel plate waveguide
- $f_c = 0$
- TEM exist

(viii) Attenuation Constant in Rectangular Waveguide and Circular Waveguide.



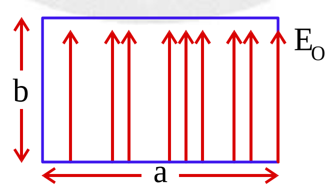
This graph is valid for all modes of rectangular and circular waveguide except TE_{01} mode in circular waveguide.



TE_{01} mode of circular waveguide has lowest cutt-off frequency

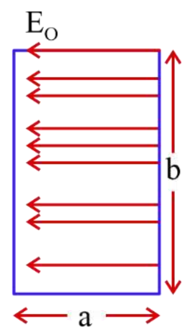
(10) Open circuit voltage:

- For a > b → TE_{10} mode is dominant



$V_{OC} = E_0 b$

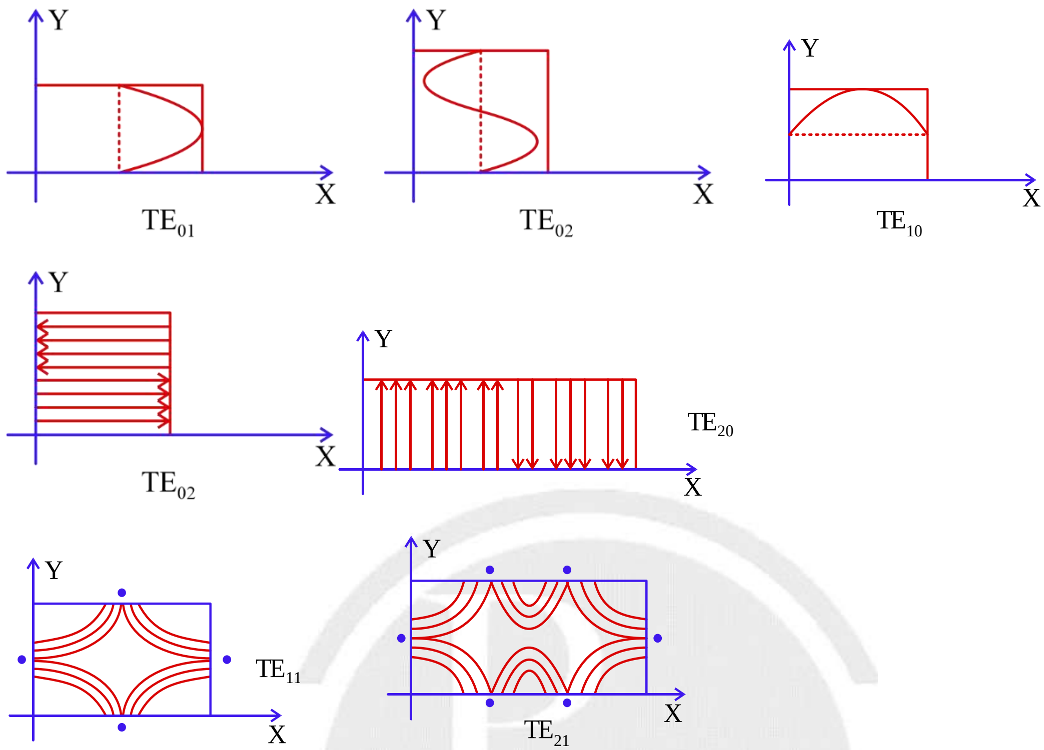
- For a < b → TE_{01} mode is dominant



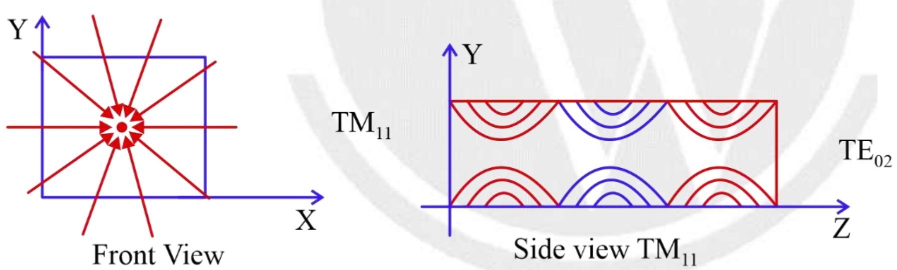
$V_{OC} = E_0 a$

(11) Electric Field Pattern:

(a)



(b) TM mode :



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7 ANTENNA

6.1. Antenna and its Radiation Pattern.

- **Antenna:** It is a device which converts electrical signal into EM wave & EM wave into electrical signal.
- Some Important terms related to antenna.
  - W_o**:** Total Radiated Power (Watt)
  - P_{rad} **or** P_{av}**:** Time average power density or time average poynting vector or Radiated power density (Watt/m^2)
  - $U(\theta, \phi)$**:** Radiated power Intensity or Radiation Pattern (Watt/steradian)
  - Ω_{eff}**:** effective solid angle (Steradian)
  - θ_E**:** Half power beam width or elevation angle. (radian)
  - ϕ_H **or** θ_H**:** Half power beam width or azimuthal angle. (radian)
  - D **or** G_D**:** Directive Gain
  - D_o = Maximum directive gain or directivity.
  - G_P**:** Power gain
  - G_{Po}**:** Maximum Power gain.
  - e_c**:** efficiency due to conductor loss.
  - e_d**:** efficiency due to dielectric loss.
  - e_{cd}**:** efficiency due to conductor and dielectric loss.
  - e_r**:** efficiency due to reflection
  - R_r**:** Radiation Resistance (Ω)
  - R_l**:** Loss Resistance (Ω)
  - P_{in}**:** Direct of transmitting Antenna.
  - $\hat{\rho}_t$ **:** Direct on transmitting Antenna.
  - $\hat{\rho}_r$ **:** Direction of receiving Antenna.
  - **PLF:** Polarization Loss Factor
  - A_{eff}**:** Effective Aperture Area (m^2)
  - L_{phy}**:** Physical length (m)

- **l_{eff} or l_{av}:** Effective length or average length (m)
- **V_{OC}:** Open circuit Voltage (Volt)

1. Solid Angle

(a) $d\Omega = \sin\theta d\theta d\phi$

(b) $\Omega_{sphere} = 4\pi$

(c) $\Omega_{eff} = \theta_E \times \theta_H \rightarrow$ Non-uniform cone

$= \theta^2_E \rightarrow$ Uniform cone

(d)1 Steradian = 1 radian × 1 radian

$$= \frac{(180°)^2}{\pi^2} = 3282.8$$

2. Radiated Power Density:

$$\vec{P}_{av} = \vec{P}_{rad} = \frac{E_{rms}^2}{\eta}\hat{a}_p = E_{rms} \cdot H_{rms}\hat{a}_p$$

$$= \frac{E_o H_o}{2}\hat{a}_p$$

The strength of EM wave transmitted by the antenna which depends upon distance (r) and direction (θ, ϕ) is radiated power density.

3. Radiated Power Intensity:

$$U(\theta,\phi) = \frac{dW}{d\Omega}\left(\frac{\text{Watt}}{\text{Steradian}}\right)$$

- $P_{rad} = \dfrac{dW_0}{dS}$
- $W_o = \int\int P_{rad}\, r^2 \sin\theta d\theta d\phi$
- $U(\theta,\phi) = \dfrac{dW_o}{d\Omega}$
- $W_o = \int\int U(\theta,\phi)\sin\theta d\theta d\phi$

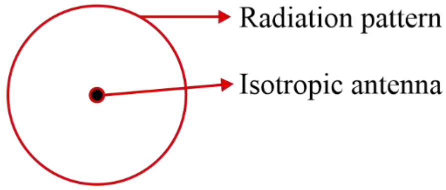
4. Isotropic Antenna:

(a) It is point size antenna.

(b) It is ideal Antenna.

(c) It is reference Antenna.

(d) It is reference Antenna

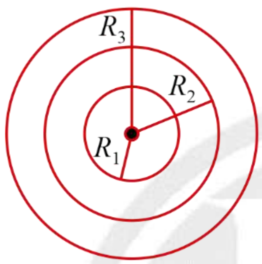


(e) W_o = Total power radiated by isotropic

(f) $\Omega_{sphere} = 4\pi$ Steradian

(g) $U_o(\theta, \phi) = \dfrac{W_o}{4\pi}$

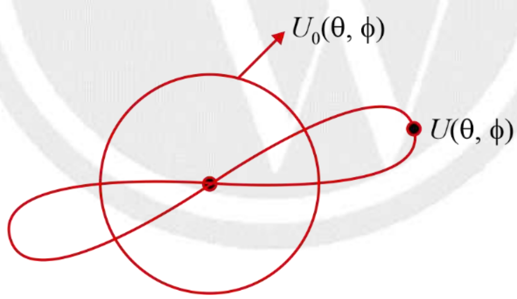
(h)



$$\left.\begin{aligned} P_1 &= \frac{W_o}{4\pi R_1^2} \\ P_2 &= \frac{W_o}{4\pi R_2^2} \\ P_3 &= \frac{W_o}{4\pi R_3^2} \end{aligned}\right\} \text{Radiated Power Density}$$

(i) $D_o = 1,\ G_{P_0} = 1,\ e_{cd} = 1,\ e_r = 1,\ e_t = 1$

5. Directive Gain And Directivity



- $D = \dfrac{\text{Radiation Pattern of general antenna}}{\text{Radiation Pattern of isotropic antenna}}$
- $D = \dfrac{U(\theta, \phi)}{U_o(\theta, \phi)} = \dfrac{4\pi U(\theta, \phi)}{W_o} = \dfrac{4\pi U(\theta, \phi)}{\iint U(\theta, \phi) \sin\theta\, d\theta\, d\phi}$
- For General Antenna → $U(\theta, \phi) = U_o(\theta, \phi)$
- $D_o = D_{max} = \dfrac{4\pi U(\theta, \phi)|_{max}}{U_o(\theta, \phi)}$
- $0 \le D \le D_o,\ D_o \ge 1$
  - $D_0 > 1 \rightarrow$ General Antenna
  - $D_0 = 1 \rightarrow$ Isotropic Antenna

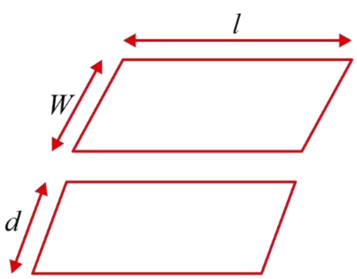
- $D_o = \dfrac{4\pi}{\Omega_{eff}} = \dfrac{4\pi}{\theta_E \cdot \theta_H} \rightarrow$ Non-uniform conical beam

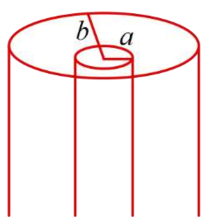
  $= \dfrac{4\pi}{\theta_E^2} \rightarrow$ Uniform conical beam

- $D = f_1(\theta, \varphi) \rightarrow$ directional antenna.
- $D = f_1(\theta)$ or $f_1(\varphi) \rightarrow$ Omnidirectional Antenna.
- $D = K$ = constant → All directional antenna.

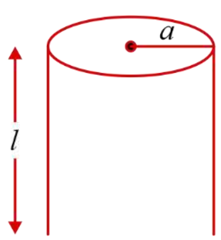
6. R_L = Loss Resistance

$$R_S = \sqrt{\frac{wu}{2\sigma_c}}, R = \frac{2R_S}{W}, R_L = \left(\frac{2R_S}{W}\right)l$$



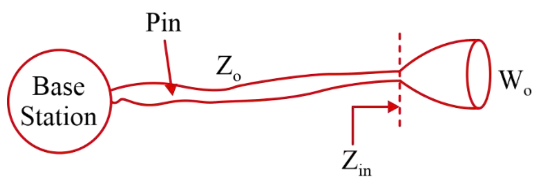


$$R_L = \left(\frac{R_S}{2\pi a} + \frac{R_S}{2\pi b}\right)l$$



$$R_L = \left(\frac{R_S}{2\pi a}\right)l$$

7. Efficiency:

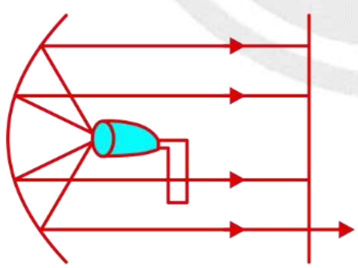


- $e_c = 1$ (Perfect conductor)
- $e_c = \dfrac{R_r}{R_r + R_L} \rightarrow$ (Good or Poor Conductor)

- $e_d = 1 \rightarrow$ Perfect dielectric
- $0 < e_c < 1 \rightarrow$ Conductor loss
- $0 < e_d < 1 \rightarrow$ Dielectric loss
- $e_r = (^2)$
- $e_r = \left(1-|\Gamma|^2\right)$
- $e_r = 1 \rightarrow$ Perfectly matched
- $0 < e_r < 1 \rightarrow$ Mismatched
- $e_r = 0 \rightarrow$ Perfectly mismatched
- $e_{cd} = e_c \cdot e_d = \dfrac{R_r}{R_r + R_L}$
- $e_t = e_{cd} \cdot e_r = \dfrac{R_r}{R_r + R_L}\left(1-|\Gamma|^2\right)$
- $W_0 = e_t P_{\text{in}}$
- $G_{P\text{o}} = e_t D_\text{o}$
- $D_o|_{dB} = \log_{10} D_\text{o}$
- $G_{P_o}\Big|_{dB} = 10\log_{10} G_{P_o}$
- $\text{PLF}|_{dB} = 10\log_{10} \text{PLF}$

8. Parabolic Antenna

- $A_{eff} = \dfrac{\lambda^2}{4\pi} D_o$
- $A_{eff} = \dfrac{\lambda^2}{4\pi} D_o(\text{PLF})$



- Area efficiency = $\dfrac{A_{eff}}{A_{phy}}$
- $G_{P_o} = e_t D_o = e_A e_t \pi^2 \left(\dfrac{d}{\lambda}\right)^2$
- $G_{P_o} = 6.5\left(\dfrac{d}{\lambda}\right)^2 \rightarrow$ Gain of parabolic antenna
- First Null Beam width = FNBW = $\left(\dfrac{70\lambda}{d}\right)^0$

- Half Power Beam width = HPBW = $\left(\dfrac{58\lambda}{d}\right)^{0}$

Cassegrain Antenna

(a) A Horn antenna is used as feed antenna.

(b) A secondary reflector which is hyperbolic in Shape.

(c) A primary reflector which is paraboloid

9. Polarization Loss Factor (PLF):

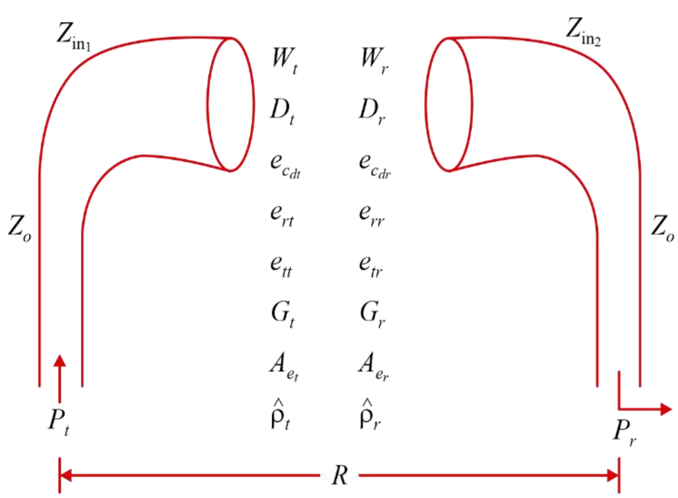
$\hat{\rho}_t$ = Direction of transmitting Antenna

$\hat{\rho}_r$ = Direction of receiving Antenna

$$\text{PLF} = |\hat{\rho}_t \cdot \hat{\rho}_r|^2$$

O-Polarization Received = 100%, Loss = 0%		X-Polarization Received = 0%, Loss = 100%	
T_x	R_x	T_x	R_x
H	H	H	V
V	V	V	H
θ	θ	θ	90 – θ
RCP	LCP	RCP	RCP
LCP	RCP	LCP	LCP
LEP	REP	REP	REP
REP	LEP	REP	REP

10. Friis Formulae:



- $f\lambda = V_P$
- $e_{tt} = \dfrac{W_t}{P_t}$
- $e_{tr} = \dfrac{P_r}{W_r}$
- $G_t = e_{tt}\, D_t$
- $G_r = e_{tr}\, D_r$
- $e_{rt} = 1 - |\Gamma_t|^2$
- $e_{rr} = 1 - |\Gamma_r|^2$
- $\Gamma_t = \dfrac{Z_{in_1} - Z_o}{Z_{in_2} + Z_o}$
- $\Gamma_r = \dfrac{Z_{in_2} - Z_o}{Z_{in_2} + Z_o}$
- $e_{c_{dt}} = \dfrac{R_{r_t}}{R_{r_t} + R_{L_t}}$
- $e_{c_{dr}} = \dfrac{R_{rr}}{R_{rr} + R_{Lr}}$
- $e_{tt} = e_{cdt}\, e_{rt}$
- $e_{tr} = e_{cdr}\, e_{rr}$
- $\text{PLF} = |\hat{\rho}_t \cdot \hat{\rho}_r|^2$
- $A_{et} = \dfrac{\lambda^2}{4\pi} D_t$
- $A_{er} = \dfrac{\lambda^2}{4\pi} D_r$
- Power density at the receiving antenna due to transmitting Antenna when transmitting antenna is isotropic = $\dfrac{W_t}{4\pi R^2}$
- Power density at the receiving antenna due to transmitting antenna when transmitting antenna is not isotropic = $\dfrac{W_t\, D_t}{4\pi R^2}$
- Power density of the receiving antenna in terms of Electric field = $\dfrac{E_{rms}^2}{\eta}$

- where $D_o = 1$ (Isotropic)
- $D_o = 1.5$ (dipole antenna)
- $D_o = 1.63$ ($l = \lambda/2 \rightarrow$ half wave dipole).
- Power Received

$$W_r = \frac{W_t D_t A_{er}}{4\pi R^2} = \frac{W_t A_{et} A_{er}}{R^2 \lambda^2}$$

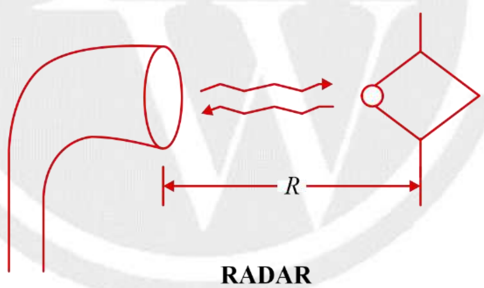
- $P_r = \frac{P_t D_t D_r}{(4\pi R)^2}\left(\frac{R_{rt}}{R_{rt}+R_{Lt}}\right)(1-|\Gamma_t|^2)\left(\frac{R_{rr}}{R_{rr}+R_{Lr}}\right)(1-|\Gamma_r|^2)\text{PLF}$
- $P_r = \frac{P_t G_t F_r}{\left(\frac{4\pi R}{\lambda}\right)^2}$

Link Formulae

$$P_r|_{dB} = P_t|_{dB} + G_t|_{dB} + G_r|_{dB} - 20\log_{10}\left(\frac{4\pi R}{\lambda}\right)$$

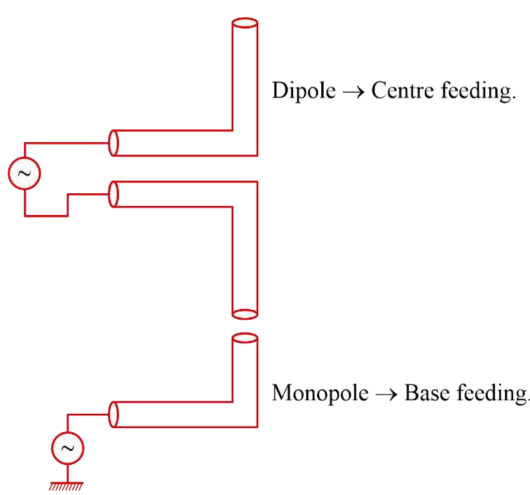
- Path Loss = $20\log_{10}\left(\frac{4\pi R}{\lambda}\right)$
- $P_r = \frac{W_t D_t^2 \sigma^2 + \lambda^2}{64\pi^3 R^4}$

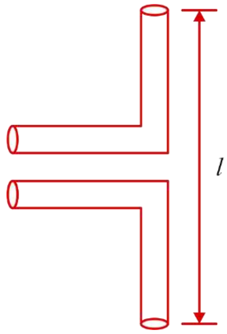
σ^2 = Cross-Sectional area of object.



RADAR

11. Dipole Antenna





$l = \frac{\lambda}{2} \rightarrow$ Half wave dipole antenna

$l = \frac{\lambda}{4} \rightarrow$ Quater wave dipole antenna.

12. Types Of Dipole Antenna

(a) Infinitesimal dipole antenna/Hertzian dipole.

$$dl < \frac{\lambda}{50}$$

(b) Small dipole antenna/short dipole antenna

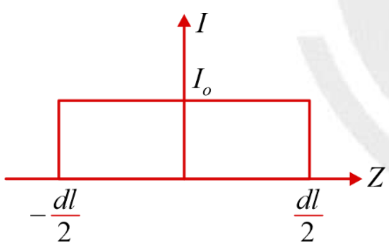
$$\frac{\lambda}{50} < l < \frac{\lambda}{10}$$

(c) Large dipole antenna

$$l > \frac{\lambda}{10}$$

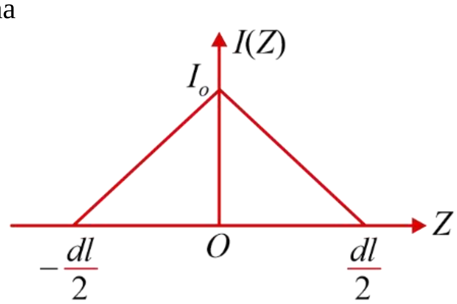
13. Current Distribution In Dipole Antenna

(a)



$\rightarrow$ Hertzian dipole antenna.

(b) Small/Short dipole Antenna

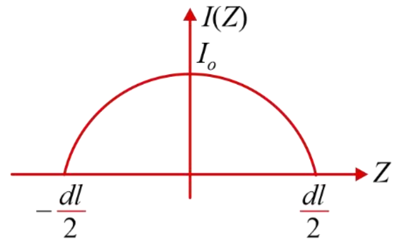


$$I_{av} = \frac{I_o}{2}$$

$$I(Z) = I_o\left(1 + \frac{2Z}{l}\right) \quad -\frac{l}{2} \le Z \le 0$$

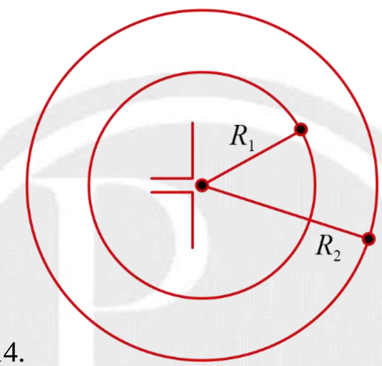
$$= I_o\left(1 - \frac{2Z}{l}\right) \quad 0 \le Z \le \frac{l}{2}$$

(c) Large dipole Antenna:



$$I(Z) = I_o \sin\left(K\left(1+\frac{2Z}{l}\right)\right), -\frac{l}{2} \le Z \le 0$$

$$= I_o \sin\left(K\left(1-\frac{2Z}{l}\right)\right) \quad 0 \le Z \le \frac{l}{2}$$



14.

(a) r > R_1 → Near field

(b) R_1 < r < R_2 → Fresnel's region.

(c) r > R_2 → Far field.

(d) $R_1 = 0.63\sqrt{\frac{l_3}{\lambda}}, R_2 = \frac{2l^2}{\lambda}$

(e) The distance at which radiation field and inductive field.

$$\beta r = 1 \rightarrow \boxed{r = \frac{\lambda}{2\pi}}$$

15. Far Field of Hertizian Dipole Antenna

$$\vec{E} = \frac{I_o dl}{4\pi r}(\sin\theta)(\eta)(j\beta)e^{-j\beta r}\hat{a}_\theta$$

$$\vec{H} = \frac{I_o dl}{4\pi r}(\sin\theta)(j\beta)e^{-j\beta r}\hat{a}_\phi$$

$$\eta = \frac{E_\theta}{H_\phi}$$

(a) $U(\theta, \phi) = K\sin^2\theta$ → Omnidirectional antenna.

(b) $R_r = 80\pi^2\left(\frac{dl}{\lambda}\right)^2 \rightarrow$ Radiation Resistance.

(c) $D = \frac{3}{2}\sin^2\theta \rightarrow$ Directive Gain.

(d) $D_o = \frac{3}{2} \rightarrow$ Directivity

16. Radiation Resistance

(a) Hertzian dipole Antenna

$$R_r = 80\pi^2\left(\frac{dl}{\lambda}\right)^2$$

(b) Small dipole antenna.

$$R_r = 20\pi^2\left(\frac{l}{\lambda}\right)^2$$

(c) Half wave dipole antenna.

$R_r = 73\ \Omega$
$Z_{in} = (73 + j42.5)\ \Omega$
$D_o = 1.63$

(d) Quarter water dipole Antenna.

$Z_{in} = (36.5 + j21.25)\ \Omega$
$D_o = 3.26$

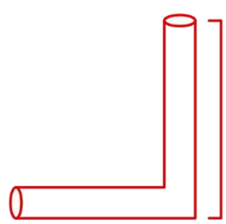
(e) $l = \frac{\lambda}{2} \rightarrow R_r = 73\,\Omega$

$$l = \frac{\lambda}{4} \rightarrow R_r = 36.5\,\Omega$$

$$l = \frac{\lambda}{8} \rightarrow R_r = 18.25\,\Omega$$

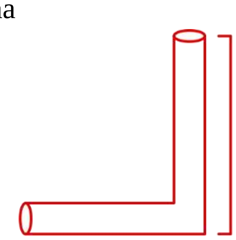
$$l = \frac{\lambda}{16} \rightarrow R_r = 20\pi^2\left(\frac{\frac{\lambda}{16}}{\lambda}\right)^2 = \frac{5\pi^2}{64}$$

(f) Hertzian monopole antenna.



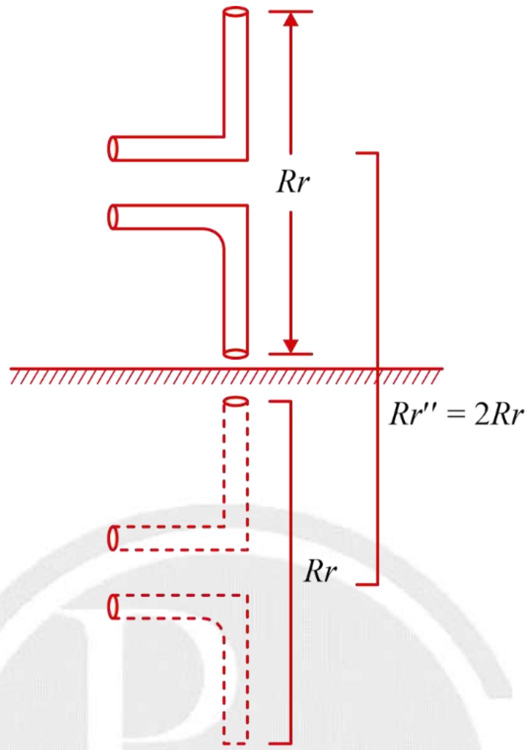
$$R_r' = \frac{R_r}{2} = 40\pi^2\left(\frac{dl}{\lambda}\right)^2$$

(g) Small/Short Monopole Antenna



$$R_r' = 10\pi^2\left(\frac{l}{\lambda}\right)^2$$

(g) Grounded Antenna



(i) Hertzian Grounded dipole antenna

$$R_r'' = 2R_r' = 160\pi^2\left(\frac{dl}{\lambda}\right)^2$$

(j) Hertzian Grounded monopole antenna

$$R_r'' = 2R_r' = 80\pi^2\left(\frac{dl}{\lambda}\right)^2$$

(k) Small/short Grounded dipole Antenna

$$R_r'' = 2R_r = 40\pi^2\left(\frac{l}{\lambda}\right)^2$$

(l) Small/Short Grounded monopole antenna.

$$R_r'' = 2R_r' = 20\pi^2\left(\frac{l}{\lambda}\right)^2$$

(m) Folded Antenna (n-fold)

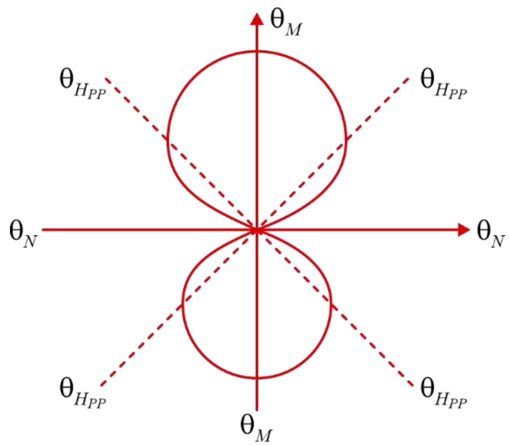
$$R_r' = n^2 R_r$$

R_r' = radiation resistance due to n fold

R_r = radiation resistance due to one fold

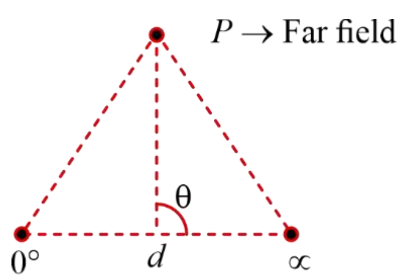
17. Radiation Pattern:

(a) Hertzian Dipole/small dipole



- $\theta_M = \frac{\pi}{2}, \frac{3\pi}{2}$
- $\theta_N = 0, \pi$
- $\theta_{HPP} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
- $\text{HPBW} \equiv \frac{\pi}{2}$, ie. $\theta_H = \frac{\pi}{2}, \theta_E = \frac{\pi}{2}$
- $\Omega_{eff} = \frac{\pi^2}{4}$
- $A_{eff} = \frac{\lambda^2}{\Omega_{eff}} = \frac{4\lambda^2}{\pi^2}$
- $F_{NBW} = \pi$ i.e. $F_{NBW} = 2(\text{HPBW})$

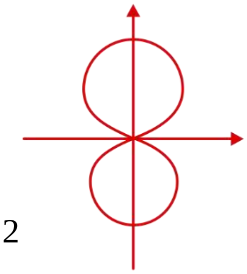
(b) Two element array of antenna



$$|E_P| = 2E_o\left|\cos\left(\frac{\psi}{2}\right)\right|$$

$$\psi = Bd\cos\theta + \propto$$

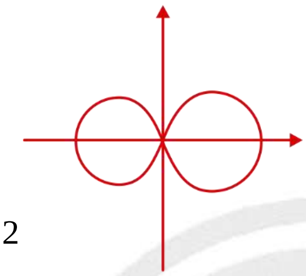
- $d = \frac{\lambda}{2}, \alpha = 0$



Board side array

Number of Lobes = $\dfrac{4d}{\lambda} = 2$

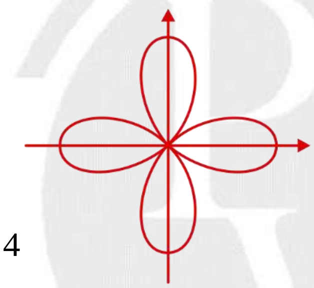
- $d = \dfrac{\lambda}{2}, \alpha = \pi$



End fire array

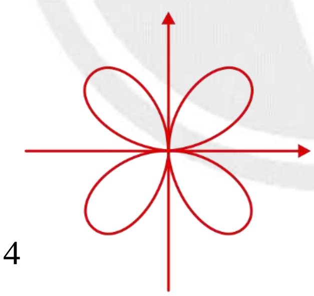
Number of lobes = $\dfrac{4d}{\lambda} = 2$

- $d = \lambda, \propto = 0$



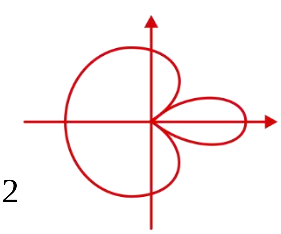
Number of lobes = $\dfrac{4d}{\lambda} = 4$

- $d = \lambda, \alpha = \pi$



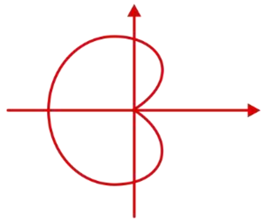
Number of lobes = $\dfrac{4d}{\lambda} = 4$

- $d = \dfrac{\lambda}{2}, \alpha = \dfrac{\pi}{2}$



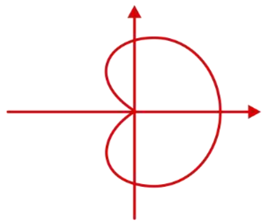
Number of lobes = $\dfrac{4d}{\lambda} = 2$

- $d = \dfrac{\lambda}{4}, \alpha = \dfrac{\pi}{2}$



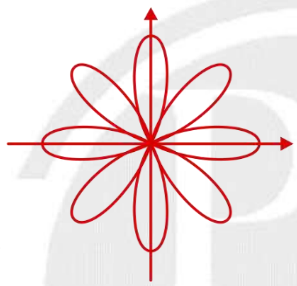
Number of lobes = $\frac{4d}{\lambda} = 1$

- $d = \frac{\lambda}{4}, \alpha = -\frac{\pi}{2}$



Number of lobes = $\frac{4d}{\lambda} = 1$

- $d = 2\lambda, \alpha = 0$



Number of lobes = $\frac{4d}{\lambda} = 8$

Note:-Number of lobes = $\frac{4d}{\lambda}$

(c) N-element Linear array

$$|\vec{E}_P| = E_o \left| \frac{\sin\left(\frac{N\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)} \right|$$

(d) Vertical Grounded dipole antenna.

Number of Lobes = $\frac{2h}{\lambda} + 1$

h = distance of dipole antenna from Ground.

(e) Horizontal Grounded dipole antenna

Number of Lobes = $\frac{2h}{\lambda}$

h = distance of dipole antenna from ground.

❒❒❒

Computer Organization

COMPUTER ORGANIZATION

INDEX

1 INTRODUCTION

1.1. Introduction of Computer Architecture

Computer architecture refers to those attributes of a system visible to a programmer or, put another way, those attributes that have a direct impact on the logical execution of a program.

Computer organization refers to the operational units and their interconnections that realize the architectural specifications.

Examples of architectural attributes include the instruction set, the number of bits used to represent various data types (e.g., numbers, characters), I/O mechanisms, and techniques for addressing memory. Organizational attributes include those hardware details transparent to the programmer, such as control signals; interfaces between the computer and peripherals; and the memory technology used.

For example, it is an architectural design issue whether a computer will have a multiply instruction. It is an organizational issue whether that instruction will be implemented by a special multiply unit or by a mechanism that makes repeated use of the add unit of the system. The organizational decision may be based on the anticipated frequency of use of the multiply instruction, the relative speed of the two approaches, and the cost and physical size of a special multiply unit.

1.1.1. Computer Architecture

Deals with Conceptual design and fundamental operational structure mainly it involves. The following

(1) CPU design

(2) Instruction

(3) Addressing modes

(4) Data format

1.1.2. Computer Organization

Deals with physical devices and their interconnections with a perspective of improving the performance.

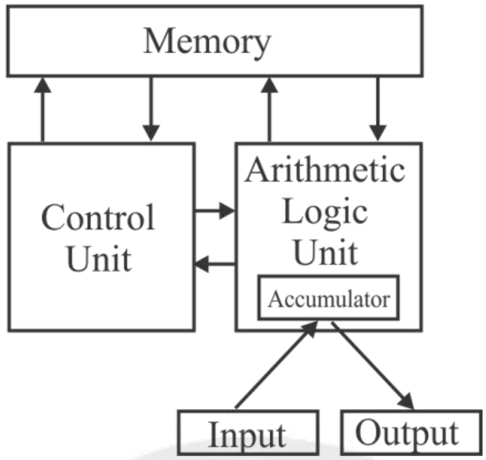
(1) IO organization

(2) Memory organization

(3) Parallel processing → pipelining

Von Neumann's Architecture :

Also known as **'Stored Program Architecture'**



Floating point Numbers

- **Motive:** wanted to represent a larger range of numbers using less number of bits.
- The numbers are represented in this format

S	E	M

S = Sign: 0 +ve; 1 –ve

E = Exponent

M = Mantissa

- The numbers are represented in this format

S	E	M

- Mantissa is signed normalized (implicit/explicit) fraction number
- Exponent is stored in stored in biased form.

Biased-Exponent

S	E	M

E = 4-bits $\Rightarrow$ Range = –8 to +7 transform Range 0 to 15

Original Exponent	Stored Exponent	Excess-8 code (Bias)	
–8	0		
–7	1	1	1
–6	2	7	15
0	8		

If E is represented by k-bits

$$\boxed{\text{bias} = 2^{k-1}}$$

Mamtissa : [Number after point]

101.01 → Explicit Normalization → 0.10101×2^3 → $M = 1$
always non zero

101.01 → Implicit Normalization → 1.0101 → $M = 0101$
always one

2 Components of Computer

2.1. Introduction

- CPU
- Memory
- I/O Devices

CPU :

- Control Unit
- Arithmetic Logic Unit (ALU)

Memory:

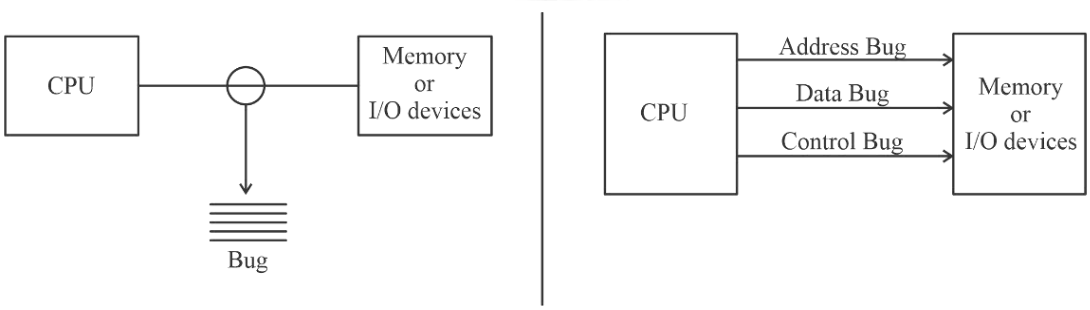
- Primary Memory
- Secondary Memory

I/O Deices:

Other Component

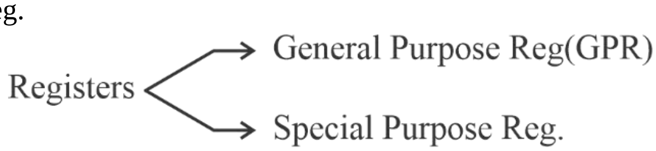
- System Buses
- CPU Registers

System Buses



1.2. CPU Registers

Small memories inside CPU $\Rightarrow$ Reg.

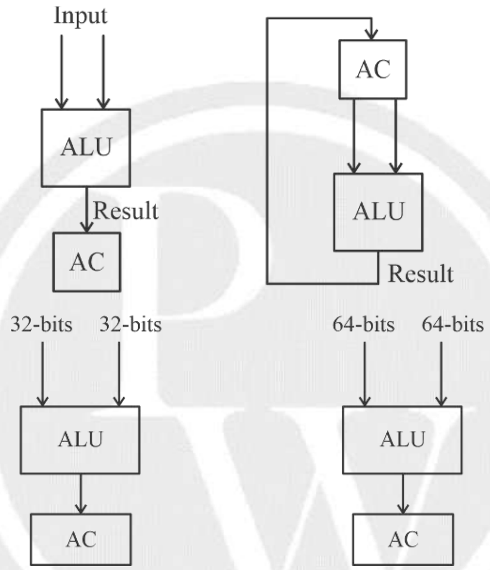


Special Purpose Reg:-

1. Accumulator (AC)
2. Status Reg./Flag Reg./Prog. Status word (PSW)
3. Instructions Reg.(IR)
4. Program Counter (PC)
5. Stack Pointer (SP)
6. Adress Reg. (AR)/Memory AR(MAR)

AC:

Used t store result of ALU and sometimes one of the input to ALU.

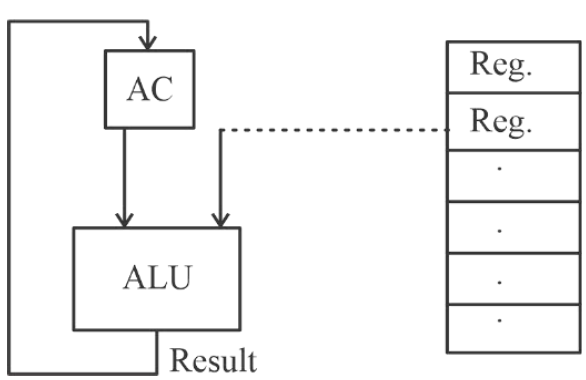


32-bit Architecture | 64-bit Architecture
Word = 32-bits | word = 64-bits

1.2.1. Type of Architecture:- (based on ALU's Input)

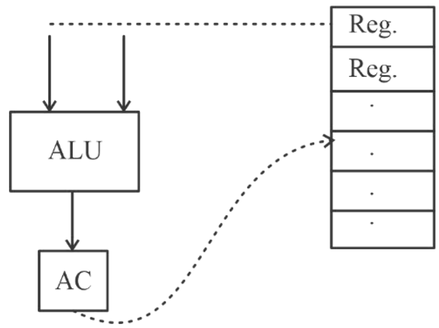
AC Based Architecture:

One of the inputs to ALU should be taken from AC.



1. Reg.-Based Architecture:

Both inputs of ALU are taken from Reg.



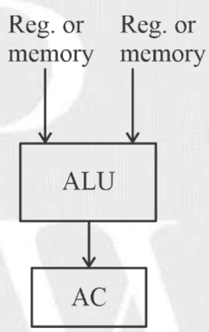
Add R_1, R_2

$\downarrow$

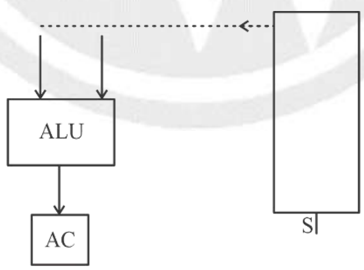
$R_1 \leftarrow R_1 + R_2$

2. Complex System:

Both input taken either from reg. or memory.

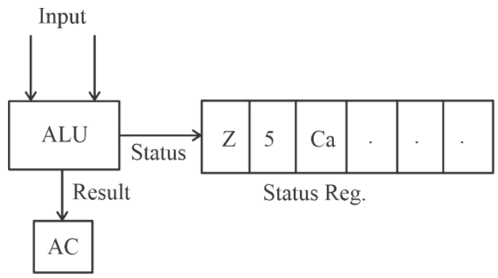


3. Stack-Based:



PSW:

Store status of result of ALU.



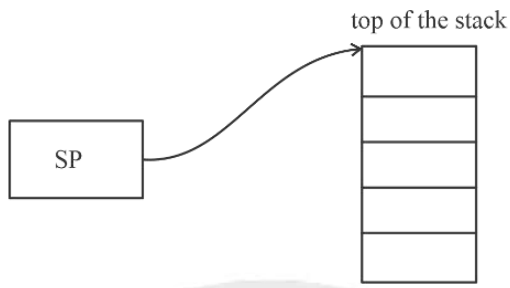
PC:

It store the address of next instruction to be calculate.

Instruction Register:

It is used to store the current instruction which is CPU executing.

Stack Pointer:



Address Reg:

It is used to send address to memory.

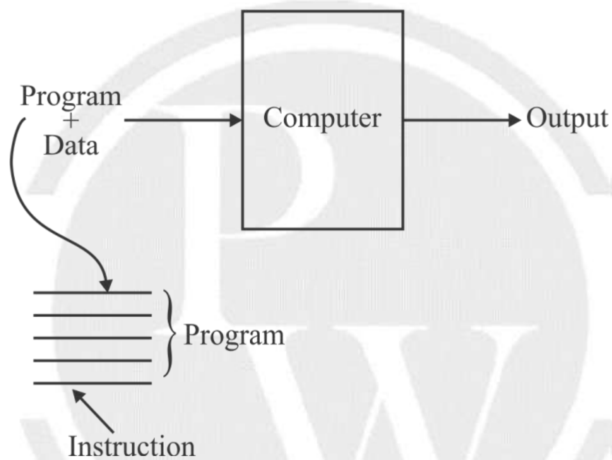
Data Reg:

It is used to send data to memory (memory white) and to receive data from memory (memory read)

3 INSTRUCTIONS

3.1. Digital Computer

A computer which takes input in binary form and provides output in binary form.



Instruction:

High Level Language		**Low Level/Bite Code/Machine Code**
#include<stdio.h>		1 0 1 1 1 0 0 0
void main()		1 0 0 0 0 0 0 1
{		1 1 1 1 0 0 1 0
int a. b. c:	**Language Translation** →	0 1 0 1 0 1 0 1
printf("Enter 2 values: ");		1 1 1 1 0 1 1 0
scanf('%d %d", &a. &b):		0 1 0 1 0 1 0 1
c = a + b;		1 0 0 0 1 1 1 1
printf("Sum = %d", c);		1 0 1 0 0 0 1 1
}		0 0 1 1 1 1 0 1

- A group of bits which instructs computer to perform some operation.

Opcode ←

Operation	operands information

Example Assume 8-bits instruction

← 8 →

Op Code	operands information
3	5

⇓

max opcode $= 2^3 = 8$ →

$000 \rightarrow$ ADD
$001 \rightarrow$ Comp.
$010 \rightarrow$ Inc.
⋮
$111 \rightarrow$ ADD

\+ max. number of instruction supported by this CPU = 8

max. number of operation supported by this CPU = 8

Types of Instruction :

Based on operation

- Data Transfer : MOV, LDI, LDA
- Arithmetic & Logic : ADD, SUB, AND, OR
- Machine Control : El, Dl, PUSH, POP
- Iterative : LOOP, LOOPE, LOOPZ
- Branch : JMP, CALL, RET, JZ, JNZ

Based on operand information

- 3-Address Instruction :
- 2-Address Instruction :
- 1-Address Instruction :
- O-Address Instruction :

3-Address Instruction :

Within an instruction max 3-address can be specified for operands.

OpCode	add 1	add 2	add 3
10101	01	11	10
↓	↓	↓	↓
ADD	R_1	R_3	R_2

$R_1 \leftarrow R_3 + R_2$

Operation ⇒

$R_1 \leftarrow R_2 + R_3 + R_4$

↓

$R_5 \leftarrow R_2 + R_3$

$R_1 \leftarrow R_5 + R_4$

2-Address Instruction :

max 2-address within an instruction.

OpCode	add 1	add 2
10101	01	11
↓	↓	↓
ADD	R_1	R_3

$$R_1 \leftarrow R_1 + R_3$$

One of the operand is used as source and destination both.

1-Address Instruction :

max 1-address in an instruction.

OpCode	add 1
10101	01
↓	↓
ADD	R_1

$$A_C \leftarrow A_C + R_1$$

It supports AC based architechure.

0-Address Instruction :

No any address in instruction.

OpCode
10101
↓
ADD

Both operand are taken from stack.

It supports **0-Address** stack based architecture.

4 ADDRESSING MODES

4.1. Introduction

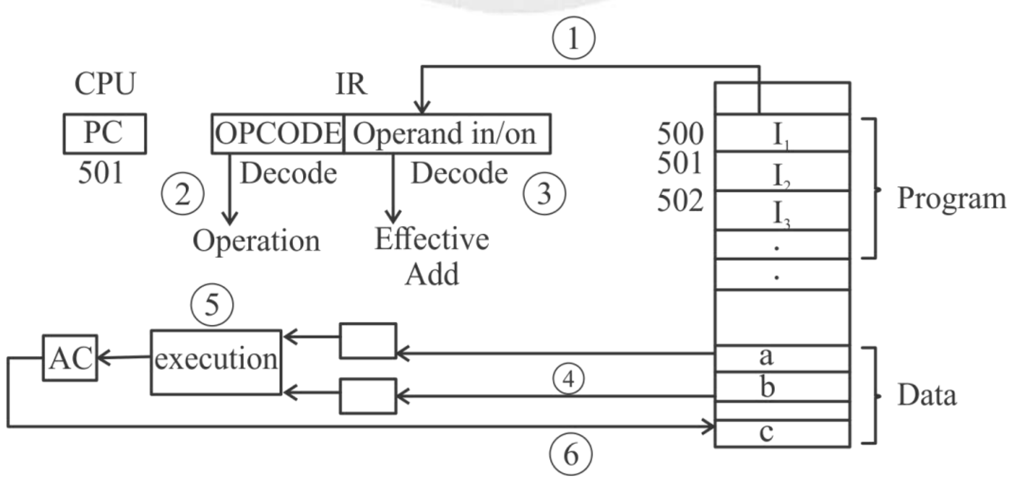
Classification of Modes

- Computable (Computation required for obtaining effective add)
  - Autoincrement/Autodecrement
  - Index Register Mode
  - Base Register Mode
  - PC-Relative Mode
- Non-Computable (No computation)
  - Implied
  - Immediate
  - Direct
  - Indirect
  - Register Direct
  - Register Indirect

Effective Address:

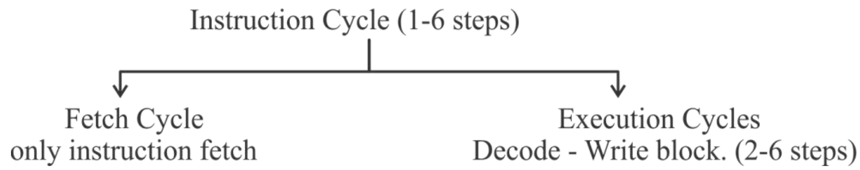
Address of operand in a computation-type instruction or the target address in a branch-type instruction.

Instruction Cycle:



1. Instruction Fetch
2. Instruction Decode
3. Effective Add Calculation

4. Operand Fetch
5. Execution
6. Write Back Result

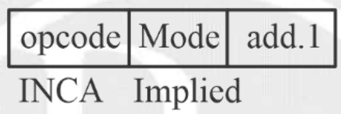


Addressing Modes:

Mode specifies that how & from whose the operand is obtained using the add of instruction.

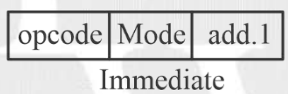
Implied Mode:

When opcode itself specifies the operand.



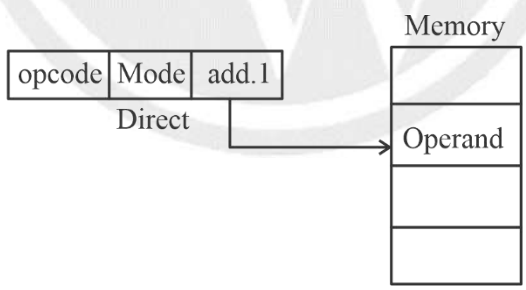
Immediate Mode:

Add. field value specifies the operand.



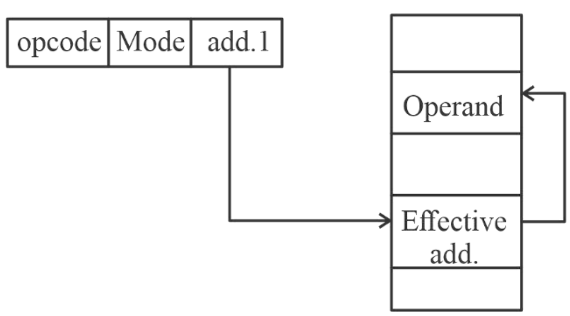
Use: To initialize registers with constant value.

Direct Mode:



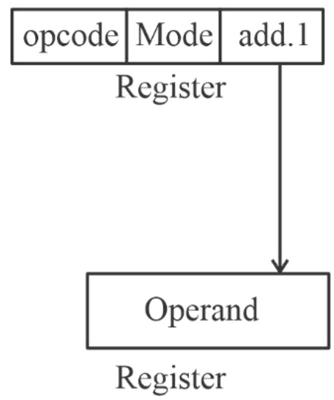
Add field specifies effective address.

Indirect Mode:



Add. field of instruction specifies address of effective address.

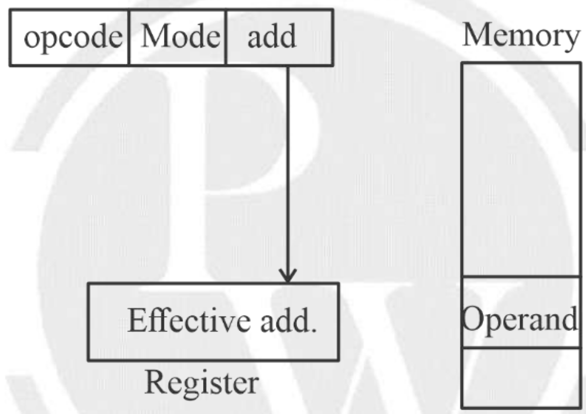
Register Mode:



Address field of instruction specifies a register and the register holds operand.

Register Indirect Mode:

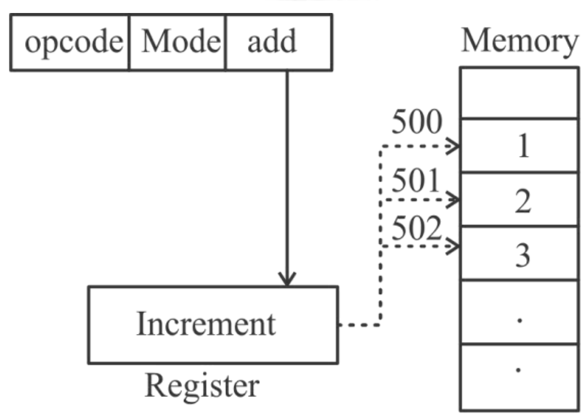
It used to shorten the instruction length.



add dield of instruction specifies a register which holds effective add.

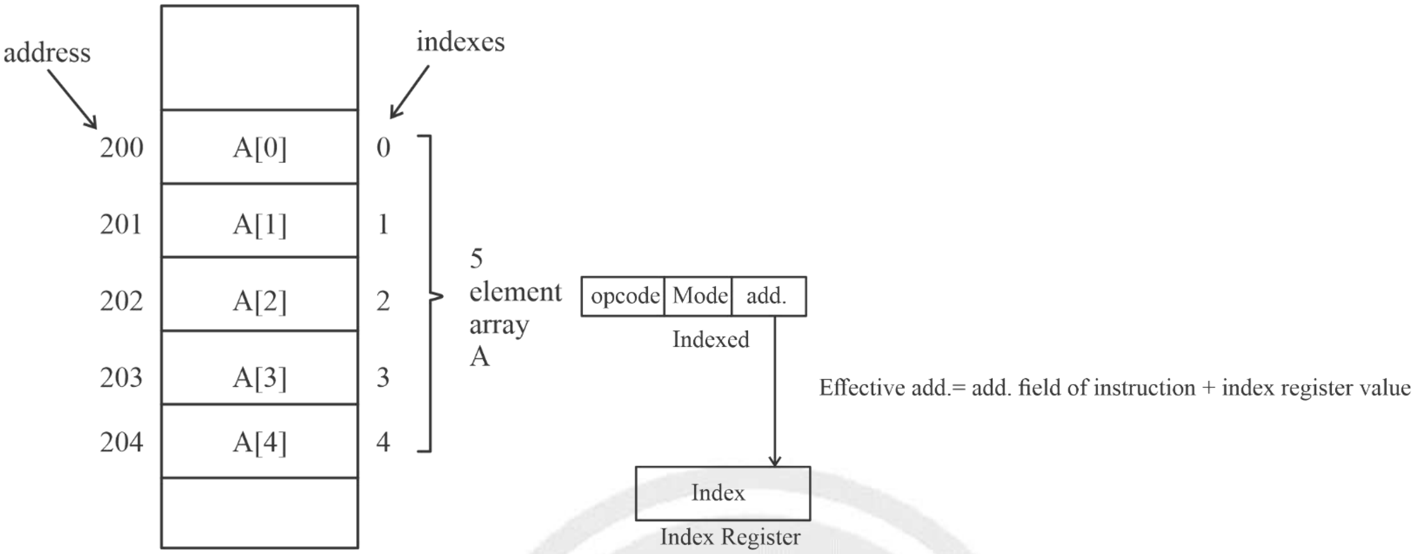
Autoincrement/Autodecrement Mode:

Variant of register indirect mode:- but value of register (effective add.) is automatically incremented/decremented.



Indexed Mode/Index-Register Mode:

Used to access array element.



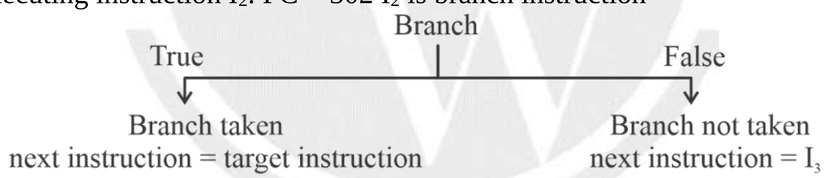
Disadvantage of Indexed Mode:

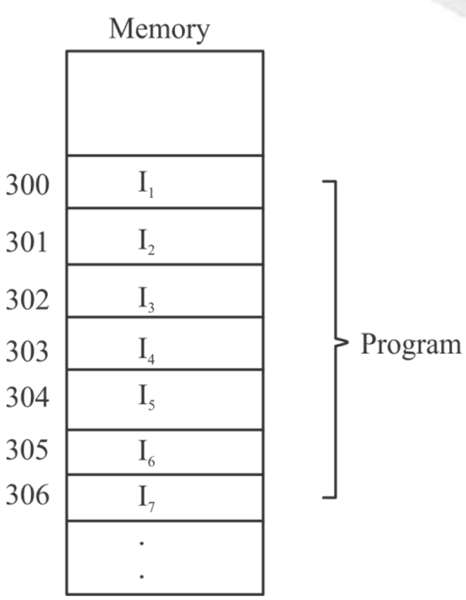
- If array is relocated then the new base address should be updated in the instruction.
- Updating of instruction is a costly (time) operation.

PC-Relative Mode:

Used for branching:

- Assume, CPU is executing instruction I_2. PC = 302 I_2 is branch instruction





(assume I_6)

Target add. = 305 (302 + 3) = 305

Relative location to jump

Example:

	Memory	
200	Load to AC	Mode
201	Address = 500	
202	Next Instruction	
399	450	
400	700	
500	800	
600	900	
702	325	
800	300	

PC = 200

R1 = 400

XR = 100

AC

Example:

Mode	Effective Address	Operand
1. Immediate Mode	201	500
2. Direct Mode	500	800
3. Indirect Mode	800	300
4. Register Mode	-	400
5. Register Indirect Mode	400	700
6. Autodecrement	399	450
7. Indexed Mode	500+100 = 600	900
8. PC-Relative Mode	202+500 = 702	-

CPU – Specification :

Master device of computer.

CPI(Cycle Per Instruction):

No. of CPU cycles required to execute 1 instruction.

CPU Cycle Time:

The time in which |C P U| can perform 1 operation.

Execution Time:

1 instruction execution time = CPI_{avg} * 1 cycle time

Program execution time = n*CPI_{avg}*1cycle time

$$\text{Program execution time} = \frac{n * CPI_{avg}}{\text{clock rate(freq.)}}$$

MIPS (Million Instruction per second) :

To give the performance of CPU.

It does not provide correct result always.

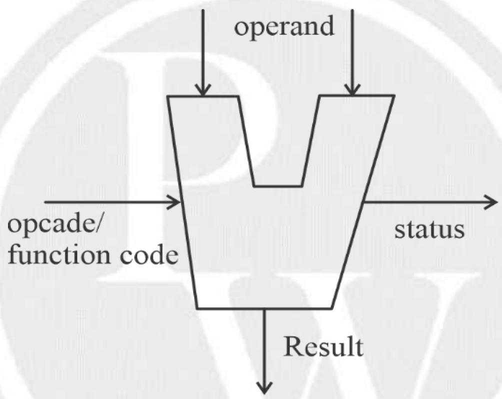
$$\text{MIPS} = \frac{\text{no.of executed instuctions}}{\text{execution time}}$$

$$\text{MIPS} = \frac{\text{n*clock rate}}{\text{n*CPI}_{\text{avg}}}$$

$$\text{MIPS} = \frac{\text{clock rate}}{\text{CPI}_{\text{avg}}}$$

ALU :

Part of CPU, which performs logical & arithmetic operation



Control Unit:

Part of CPO, which generates control signals and Sends those to diff -diff. Components. All components operate accordingly.

Parallel Processing:

Parallel Processing: Simultaneous Data Processing

Types :

1. Vector Processing
2. Array Processing
3. Pipeline Processing (Pipelining)

Flynn's Classification of Computers:

1. **SISD:** Single Instruction stream, Single Data stream ex:- Von neumann
2. **SIMD:** Single Instruction stream, Multiple Data stream ex:- Pipeline processor
3. **MISD:** Multiple Instruction stream, Single Data stream ex: only hypothetical
4. **MIMD:** Multiple Instruction stream, Multiple Data stream

5 PIPELINING

5.1. Introduction

- It is the technique to decompose a sequential process into sub-operatons
- Sub-operations are performed in segments (stage)
- **Task:** One operation performed in all segments
- Each segment can perform it's respective suboperation over different input in parallel to other segments.

Example:

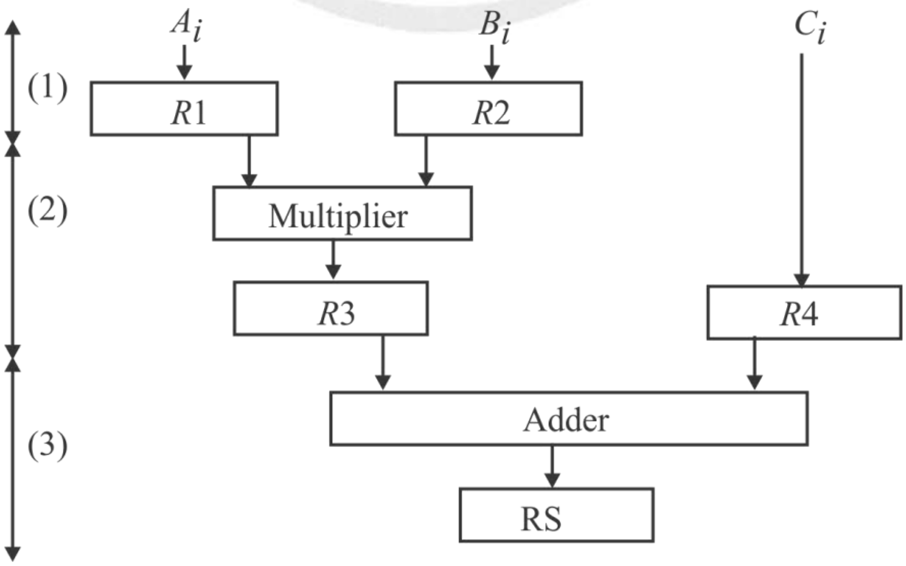
$A_i * B_i + C_i$ for $i = 1, 2, \ldots, 5$ all operands are initially memory

5.2. Sub-operations

Seg 1: $R_1 \leftarrow A_i, R_2 \leftarrow B_i$

Seg 2: $R_3 \leftarrow R_1 * R_2, R_4 \leftarrow C_i$

Seg 3: $R_5 \leftarrow R_3 + R_4$



Clock pulse	Segment 1		Segment 2		Segment 3
	R1	R2	R3	R4	R5
1	A_1	B_1			
2	A_2	B_2	$A_1 * B_1$	C_1	
3	A_3	B_3	$A_2 * B_2$	C_2	$A_1 * B_1 + C_1$
4	A_4	B_4	$A_3 * B_3$	C_3	$A_2 * B_2 + C_2$
5	A_5	B_5	$A_4 * B_4$	C_4	$A_3 * B_3 + C_3$
6	—	—	$A_5 * B_5$	C_5	$A_4 * B_4 + C_4$
7					$A_5 * B_5 + C_5$
8					

Sequential		Pipeline
$n = 5$	$3 * 5 = 15$	7
$n = 6$	18	8
$n = 7$	21	9

5.2.1. General Consideration About Pipeline

Consider a k segment pipeline

With clock cycle time = t_p

To perform n tasks

Time required to perform 1st task = $k * t_p$

Time required to perform remaining $(n-1)$ tasks = $(n-1)t_p$

Time required for all n tasks = $\underbrace{(k + n - 1)}_{\text{total cycles}}t_p$

5.2.2. General Consideration About Pipeline

Performance of a pipeline is given by **Speed up ratio**.

$$\text{Speed up ratio} = \frac{\text{non-pipeline time}}{\text{pipeline time}}$$

$$\boxed{S = \frac{n * t_n}{(k + n - 1)t_p}}$$

as the number of tasks increases, $n >> k$ (ignore $k - 1$)

$$\boxed{S_{\text{ideal}} = \frac{t_n}{t_p}}$$

[Ignoring $k - 1$ cycles to fill the pipe]

Or $\Downarrow$

S_{max} for 1 input $\Rightarrow$ 1 cycle

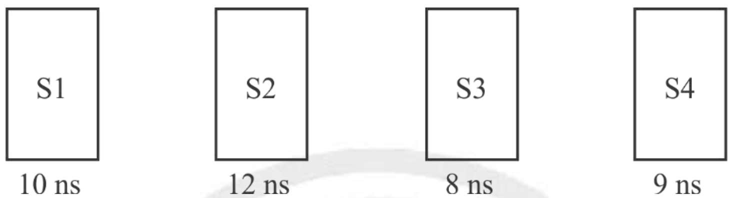
+

Special case: If to perform 1 task pipeline and non-pipeline system take equal time.

$t_n = k * t_p$

$\boxed{S_{ideal} = k}$

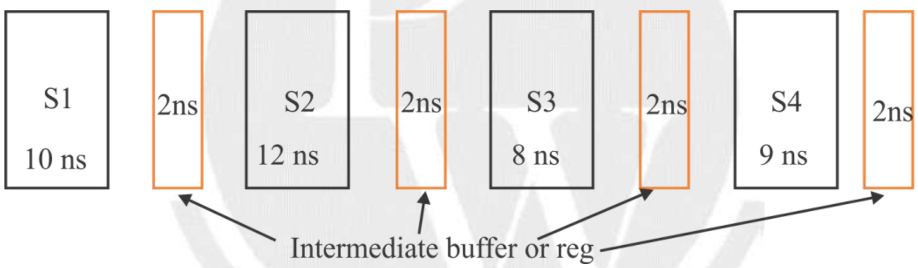
5.3. Synchronous Pipeline



t_p = max (seg delay)

= max (10, 12, 8, 9)

= 12 ns



t_p = max of seg delays + seg delay = max (10, 12, 8, 9) + 2 = 14 ns	t_n = sum of all reg delay = 10 + 12 + 8 + 9 = 39 n sec

5.4. Cycle Time in Synchronous Pipeline

t_p = maximum of seg delays + reg delay

t_n = sum of all seg delay

Speed up

f value *n* is given

$$S = \frac{n * t_n}{(k + n - 1)t_p}$$

if *n* is not given

$$S_{max} = \frac{t_n}{t_p}$$

Latency & Throughput

Time after which machine takes next input
Latency in non-pipeline = t_n
Latency in pipeline = t_p

No. of inputs processed per unit of time

$$= \frac{n}{(k+n-1)t_p}$$

In $(k + n - 1)t_p$ time, number of inputs processed = n

In 1 time, number of inputs processed = $\left[\frac{n}{(k+n-1)t_p}\right]$

Ideal Case

$$\boxed{\text{throughput} = \frac{1}{t_p}}$$

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GENERAL APTITUDE

GENERAL APTITUDE

INDEX

1 PERCENTAGES

1.1. Understanding Percentages

The word percent can be understood as follows:

Per cent $\Rightarrow$ for every 100.

So, when percentage is calculated for any value, it means that that you calculate the value for every 100 of the reference value.

Why Percentage?

Percentage is a concept evolved so that there can be a uniform platform for comparison of various things. (Since each value is taken to a common platform of 100.)

Example:

- To compare three different students depending on the marks they scored we cannot directly compare their marks until we know the maximum marks for which they took the test. But by calculating percentages they can directly be compared with one another.
- Before going deeper into the concept of percentage, let u have a look at some basics and tips for faster calculations:

1.2. Calculation of Percentage

$$\text{Percentage} = \left(\frac{\text{Value}}{\text{Total value}}\right) \times 100$$

Example: 50 is what % of 200?

Solution: $\text{Percentage} = \left(\frac{50}{200}\right) \times 100 = 25\%$.

1.2.1. Calculation of Value:

$$\text{Value} = \left(\frac{\text{Percentage}}{100}\right) \times \text{ total value}$$

Example: What is 20% of 200?

Solution: $\text{Value} = \left(\frac{20}{100}\right) \times 200$

Note: Percentage is denoted by "%", which means "/100".

Example: What is the decimal notation for 35%?

Solution: $35\% = \frac{35}{100} = 0.35$.

For faster calculations we can convert the percentages or decimal equivalents into their respective fraction notations.

1.3. Percentages – Fractions Conversions:

The following is a table showing the conversions of percentages and decimals into fractions:

Percentage	Decimal	Fraction
10%	0.1	$\frac{1}{10}$
12.5%	0.125	$\frac{1}{8}$
16.66%	0.1666	$\frac{1}{6}$
20%	0.2	$\frac{1}{5}$
25%	0.25	$\frac{1}{4}$
30%	0.3	$\frac{3}{10}$
33.33%	0.3333	$\frac{1}{3}$
40%	0.4	$\frac{2}{5}$
50%	0.5	$\frac{1}{2}$
60%	0.6	$\frac{3}{5}$
62.5%	0.625	$\frac{5}{8}$
66.66%	0.6666	$\frac{2}{3}$

Similarly we can go for converting decimals more than 1 from the knowledge of the above cited conversions as follows:

We know that $12.5\% = 0.125 = \frac{1}{8}$

Then, $1.125 = \frac{[8(1)+1]}{8} = \frac{9}{8}$ (i.e., the denominator will add to numerator once, denominator remaining the same.

Also, $2.125 = \frac{[8(2)+1]}{8} = \frac{17}{8}$ (here the denominator is added to numerator twice)

$3.125 = \frac{[8(3)+1]}{8} = \frac{25}{8}$ and so on.

Thus we can derive the fractions for decimals more than 1 by using those les than 1.

We will see how use of fractions will reduce the time for calculations:

Example: What is 62.5% of 320?

Solution: Value $= \left(\frac{5}{8}\right) \times 320 \left(\text{since } 62.5\% = \frac{5}{8}\right) = 200$.

1.4. Percentage Change

A change can be of two types – an increase or a decrease.

When a value is changed from initial value to a final value,

% change = (Difference between initial and final value/initial value) × 100

Example: If 20 changes to 40, what is the % increase?

Solution: $\%\ increase = \frac{(40-20)}{20} \times 100 = 100\%$.

Note:

1. If a value is doubled the percentage increase is 100.
2. If a value is tripled, the percentage change is 200 and so on.

1.5. Percentage Difference

% Difference = (Difference between values/value compared with) × 100.

Example: By what percent is 40 more than 30?

Solution: % difference = $\frac{(40-30)}{30} \times 100 = 33.33\%$

(Here 40 is compared with 30. So 30 is taken as denominator)

Example: By what % is 60 more than 30?

Solution: % difference = $\frac{(60-30)}{30} \times 100 = 100\%$.

(Here is 60 is compared with 30.)

Hint: To calculate percentage difference the value that occurs after the word "than" in the question can directly be used as the denominator in the formula.

1.6. Important Points to Note

1. When any value increases by

(a) 10%, it becomes 1.1 times of itself. (since 100+10 = 110% = 1.1)

(b) 20%, it becomes 1.2 times of itself.

(c) 36%, it becomes 1.36 times of itself.

(d) 4%, it becomes 1.04 times of itself.

Thus we can see the effects on the values due to various percentage increases.

2. When any value decreases by

(a) 10%, it becomes 0.9 times of itself. (Since 100 – 10 = 90% = 0.9)

(b) 20%, it becomes 0.8 times of itself

(c) 36%, it becomes 0.64 times of itself

(d) 4%, it becomes 0.96 times of itself.

Thus we can see the effects on a value due to various percentage decreases.

Note:
1. When a value is multiplied by a decimal more than 1 it will be increased and when multiplied by less than 1 it will be decreased.
2. The percentage increase or decrease depends on the decimal multiplied.

Example: 0.7 $\Rightarrow$ 30% decrease, 0.67 $\Rightarrow$ 33% decrease, 0. 956 $\Rightarrow$ 4.4% decrease and so on.

Example: When the actual value is x, find the value when it is 30% decreased.

Solution: 30% decrease $\Rightarrow$ 0.7 x.

Example: A value after an increase of 20% became 600. What is the value?

Solution: 1.2x = 600 (since 20% increase)

$\Rightarrow$ x = 500.

Example: If 600 is decrease by 20%, what is the new value?

Solution: New value = 0.8 × 600 = 480. (Since 20% decrease)

Thus depending on the decimal we can decide the % change and vice versa.

Example: When a value is increased by 20%, by what percent should it be reduced to get the actual value?

Solution: (It is equivalent to 1.2 reduced to 1 and we can use % decrease formula)

$$\% \text{ decrease} = \frac{(1.2 - 1)}{1.2} \times 100 = 16.66\%.$$

3. When a value is subjected multiple changes, the overall effect of all the changes can be obtained by multiplying all the individual factors of the changes.

Example: The population of a town increased by 10%, 20% and then decreased by 30%. The new population is what % of the original?

Solution: The overall effect = 1.1 × 1.2 × 0.7 (Since 10%, 20% increase and 30% decrease)

= 0.924 = 92.4%.

Example: Two successive discounts of 10% and 20% are equal to a single discount of ___

Solution: Discount is same as decrease of price.

So, decrease = 0.9 × 0.8 = 0.72 ⇒ 28% decrease (Since only 72% is remaining).

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2 AVERAGES & AGES

2.1. What is Average?

The concept of average is equal distribution of the overall value among all the things or persons present there. So the formula for finding the average is as follows:

Average, $$A = \frac{\text{Total of all things, } T}{\text{Number of things, } N}$$

Therefore, Total, $T = AN$

If any person joins a group with more value than the average of the group then the overall average increases. This is because the value in excess than the average will also be distributed equally among all the members.

Similarly when any value less than the average joins the group the overall group decreases as the deficit is divided equally among all the people present there.

Example:

Consider three people A, B and C with total of Rs. 30/-. Their average becomes Rs. 10/- for each. If another person D joins them with Rs. 50/- then he has Rs. 40/- more than actual average of Rs. 10/-.

So this Rs. 40/- will get distributed among those four and each gets Rs. 10/-. Thus the average becomes Rs. 20/- each.

Example:

The average age of a class of 30 students is 12. If the teacher is also included the average becomes 13 years. Find the teacher's age.

Solution:

- When the teacher is included there are totally 31 members in the class and the average is increased by 1 year. This means that everyone got 1 extra year after distributing the extra years of the teacher.
- So extra years of the teacher are as follow: $31 \times 1 = 31$ years.
- Age of the teacher = actual avg + extra years = 12 + 31 = 43 years.

3 PROFIT AND LOSS

3.1. What is Profit?

When a person does a business transaction and gets more than what he had invested, then he is said to have profit. The profit he gets will be equal to the additional money he gets other than his investment.

So profit can be understood as the extra money one gets other than what he had invested.

Example: A person bought an article for Rs. 100 and sold it for Rs. 120. Then he got Rs. 20 extra and so his profit is Rs. 20.

3.2. What is Loss?

When a person gets an amount less than what he had invested, then he is said to have a loss. The loss will be equal to the deficit he got than the investment.

Example: A person bought an article at Rs. 100 and sold it for Rs. 90. Then he got a deficit of Rs. 10 and so his loss is Rs. 10.

3.3. Cost Price (CP)

- The money that the trader puts in his business is called Cost Price. The price at which the articles are bought is called Cost Price.
- In other words, Cost Price is nothing but the investment in the business.

3.4 Selling Price (SP)

- The price at which the articles are sold is called the Selling Price. The money that the trader gets from the business is called Selling Price.
- In other words, Selling Price is nothing but the returns from a business.

3.5. Marked/Market/List Price (MP):

- The price that a trader marks or lists his articles to is called the Marked Price.
- This is the only price known to the customer.

3.6. Discount

The waiver of cost from the Marked Price that the trader allows a customer is called Discount.

Note:

1. Profit or loss percentage is to be applied always to the Cost Price only.
2. Discount percentage is to be applied always to the Marked Price only.

3.7. Relationship Among CP, SP and MP:

A trader adds his profit to the investment and sells it at that increased price.
Also he allows a discount on Marked Price and sells at the discounted price.
So, we can say that,

- SP = CP + Profit. (CP applied with profit is SP)
- SP = MP – Discount. (MP applied with discount is SP)

3.8. Understanding Profit and Loss:

So, by now we came to know that if CP is increased and sold it would result in profit and vice versa.

Also whatever increase is applied to CP, that increase itself is the profit.

For Rs. 10 profit, CP is to be increased by RS. 10 and the increased price becomes SP.

For 10% profit, CP is to be increased by 10% and it is the SP.

(From previous chapter we know that any value increased by 10% becomes 1.1 times.)

So, for 10% profit, CP increased by 10% $\Rightarrow$ 1.1CP = SP.

- $SP = 1.1CP \Rightarrow \dfrac{SP}{CP} = 1.1 \Rightarrow$ 10% profit
- $SP = 1.07CP \Rightarrow \dfrac{SP}{CP} = 1.07 \Rightarrow$ 7% profit
- $SP = 1.545CP \Rightarrow \dfrac{SP}{CP} = 1.545 \Rightarrow$ 54.5% profit and so on.

Similarly,

- $SP = 0.9CP \Rightarrow \dfrac{SP}{CP} = 0.9 \Rightarrow$ 10% loss (Since 10% decrease)
- $SP = 0.7CP \Rightarrow \dfrac{SP}{CP} = 0.76 \Rightarrow$ 24% loss and so on.

So, to calculate profit % or loss %, it is enough for us to find the ratio of SP to CP.

Note:

1. If SP/CP > 1, it indicates profit.
2. If SP/CP < 1, it indicates loss.

3.9. Multiple Profits or Losses

A trader may sometimes have multiple profits or losses simultaneously. This is equivalent to having multiple changes and so all individual changes are to be multiplied to get the overall effect.

Examples: A trader uses a 800gm weight instead of 1 kg. Find his profit %.

Solution: (He is buying 800 gm but selling 1000 gm.

So, CP is for 800 gm and SP is for 1000 gm.)

$$\frac{SP}{CP}=\frac{1000}{800} = 1.25 \Rightarrow 25\% \text{ profit.}$$

Examples: A trader uses 1 kg weight for 800 gm and increases the price by 20%. Find his profit/loss %.

Solution: 1 kg weight for 800 gm ⇒ loss (decrease) ⇒ 800/1000 = 0.8

20% increase in price ⇒ profit (increase) ⇒ 1.2

So, net effect = (0.8) × (1.2) = 0.96 ⇒ 4% loss.

Examples: A milk vendor mixes water to milk such that he gains 25%. Find the percentage of water in the mixture.

Solution: To gain 25%, the volume has to be increased by 25%.

So, for 1 lt of milk, 0.25 lt of water is added ⇒ total volume = 1.25 lt

$$\% \text{ of water} = \frac{0.25}{1.25}\times 100 = 20\%.$$

Examples: A trader bought an item for Rs. 200. If he wants a profit of 22%, at what price must he sell it?

Solution: CP=200, Profit = 22%.

So, SP = 1.22CP = 1.22 × 200 = 244/-.

Examples: A person buys an item at Rs. 120 and sells to another at a profit of 25%. If the second person sells the I tem to another at Rs. 180, what is the profit % of the second person?

Solution: SP of 1^{st} person = CP of 2^{nd} person = 1.25 × 120 = 150.

SP of 2^{nd} person = 180.

$$\text{Profit \%} = \frac{SP}{CP} = \frac{180}{150} = 1.2 \Rightarrow 20\%.$$

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4 RATIOS AND PROPORTIONS

4.1. What is a Ratio?

A ratio is a representation of distribution of a value present among the persons present and is shown as follows:

If a total is divided among A, B and C such that A got 4 parts, B got 5 parts and C got 6 parts then it is represented in ratio as A:B:C = 4:5:6.

So, 4:5:6 means that the total value is divided into 4+5+6 = 15 equal parts and then distributed as per the ratio.

Example 1: Divide Rs. 580 between A and B in the ratio of 14:15.

Solution: A:B = 14:15 $\Rightarrow$ 580 is divided into 29 equal parts $\Rightarrow$ each part = Rs. 20.

So A's share = 14 parts = 14 × 20 = Rs. 280

B's share = 15 parts = Rs. 300.

Example 2: If A:B = 2:3 and B:C = 4:5 then find A:B:C.

Solution: To combine two ratios the proportions common for them shall be in equal parts. Here the common proportion is B for the given ratios.

Making B equal in both ratios they become 8:12 and 12:15 $\Rightarrow$ A:B:C = 8:12:15.

Example 3: Three numbers are in the ratio of 3: 4 : 8 and the sum of these numbers is 975. Find the three numbers.

Solution: Let the numbers be $3x$, $4x$ and $8x$. Then their sum = $3x + 4x + 8x = 15x = 975 \Rightarrow x = 65$.

So the numbers are $3x = 195$, $4x = 260$ and $8x = 520$.

Example 4: Two numbers are in the ratio of 4 : 5. If the difference between these numbers is 24, then find the numbers.

Solution: Let the numbers be $4x$ and $5x$. Their difference = $5x - 4x = x = 24$ (given).

So the numbers are $4x = 96$ and $5x = 120$.

Example 5: Given two numbers are in the ratio of 3 : 4. If 8 is added to each of them, their ratio is changed to 5 : 6. Find two numbers.

Solution: Let the numbers be a and b.

$A:B = 3:4 \Rightarrow \frac{A}{B} = \frac{3}{4}$. Also, $\frac{(A+8)}{(B+8)} = \frac{5}{6}$.

Solving we get, A=12 and B = 16.

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5 TIME AND DISTANCE

5.1. Speed

We have the relation between speed, time and distance as follows:

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

So the distance covered in unit time is called speed.

This forms the basis for Time and Distance. It can be re-written as Distance = Speed X Time or

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}.$$

5.1.1. Units of Speed

The units of speed are kmph (km per hour) or m / s.

$$1 \text{ kmph} = \frac{5}{18} \text{ m/s}$$

$$1 \text{ m/s} = \frac{18}{5} \text{ kmph}$$

5.1.2. Average Speed

When the travel comprises of various speeds then the concept of average speed is to be applied.

$$\text{a Average Speed} = \frac{\text{Total distance covered}}{\text{Total time of travel}}$$

Note: In the total time above, the time of rest is not considered.

Example 1: If a car travels along four sides of a square at 100 kmph, 200 kmph, 300 kmph and 400 kmph find its average speed.

Solution: $\text{Average Speed} = \frac{\text{Total distance}}{\text{Total time}}$.

Let each side of square be x km. Then the total distance = $4x$ km.

The total time is sum of individual times taken to cover each side.

To cover x km at 100 kmph, time $= \frac{x}{100}$.

For the second side time $= \frac{x}{200}$.

Using this we can write average speed = $\frac{4x}{\left(\frac{x}{100}+\frac{x}{200}+\frac{x}{300}+\frac{x}{400}\right)} = 192$ kmph.

Example 2: A man if travels at $\frac{5}{6}$ th of his actual speed takes 10 min more to travel a distance. Find his usual time.

Solution: Let s be the actual speed and t be the actual time of the man.

Now the speed is $\left(\frac{5}{6}\right)$s and time is (t +10) min. But the distance remains the same.

So distance 1 = distance 2 $\Rightarrow s \times t = \left(\frac{5}{6}\right)$s × (t+10) $\Rightarrow$ t = 50 min.

Example 3: If a person walks at 30 kmph he is 10 min late to his office. If he travels at 40 kmph then he reaches to his office 5 min early. Find the distance to his office.

Solution: Let the distance to his office be d. The difference between the two timings is given as 15 min = $\frac{1}{4}$ hr.

Now if d km are covered at 30 kmph then time = d/30. Similarly second time = d/40.

So, $\frac{d}{30} - \frac{d}{40} = \frac{1}{4} \Rightarrow d = 30$ km.

Note: When two objects move with speeds s_1 and s_2

- In opposite directions their combined speed = $s_1 + s_2$
- In same direction their combined speed = $s_1 \sim s_2$.

Example 4: Two people start moving from the same point at the same time at 30 kmph and 40 kmph in opposite directions. Find the distance between them after 3 hrs.

Solution: Speed = 30 + 40 = 70 kmph (since in opposite directions)

Time = 3 hrs

So distance = speed × time = 70 × 3 = 210 km.

Example 5: A starts from X to Y at 6 am at 40 kmph and at the same time B starts from Y to X at 50 kmph. When will they meet if X and Y are 360 km apart?

Solution: Distance = 360 km, Speed = 40 + 50 = 90 kmph.

Time $= \frac{\text{distance}}{\text{speed}} = \frac{360}{90} = 4$hrs from 6 am $\Rightarrow$ 10 am.

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6 TIME AND WORK

6.1. Introduction

If a person can complete a work in 'n' days then he can do $\frac{1}{n}$ part of the work in one day.

The amount of work done be a person in 1 day is called his efficiency.

Example: A can do a work in 10 days. Then the efficiency of A is given by $A = \frac{1}{10}$.

Note: Number of days required to do a work = work to be done/work per day.

Example 1: If A can do a work in 10 days, B can do it in 20 days and C in 30 days in how many days will the three together do it?

Solution: The efficiencies are $A = \frac{1}{10}, B = \frac{1}{20}$ *and* $C = \frac{1}{30}$

So work done per day by the three = $\frac{1}{10} + \frac{1}{20} + \frac{1}{30} = \frac{11}{60} \Rightarrow No\ of\ days = \frac{60}{11} = 5.45\ days$.

Example 2: If A and B can do a work in 10 days, B and C can do it in 20 days and C and A can do it in 40 days in what time all the three can do it?

Solution:

$$A + B = \frac{1}{10}$$

$$B + C = \frac{1}{20}$$

$$C + A = \frac{1}{40}$$

Adding all the three we get $2(A+B+C) = \frac{7}{40} \Rightarrow A+B+C = \frac{7}{80} \Rightarrow No\ of\ days = \frac{80}{7}\ days$.

Note: If all the people do not work for all the time then the principle below can be used:

$$mA + nB + oC = 1. \quad \text{(1 is the total work)}$$

Here, m = no of days A worked

n = no of days B worked

o = no of days C worked

A, B, C = efficiencies

Example 3: If A can do a work in 12 days, B can do it in 18 days and C in 24 days. All the three started the work. A left after two days and C left three days before the completion of the work. How many days are required to complete the work?

Solution: Let the total no of days be x.

A worked only for 2 days, B worked for x days and C worked for $x - 3$ days.

So, mA + nB + oC = 1

$$\Rightarrow 2\left(\frac{1}{12}\right) + x\left(\frac{1}{18}\right) + (x-3)\left(\frac{1}{24}\right) = 1$$

$\Rightarrow$ 12 + 4x + 3(x-3) = 72

$$\Rightarrow x = \frac{69}{7}\ days.$$

Note: The ratio of dividing wages = ratio of efficiencies = ratio of parts of work done

Example 4: A can do a work in 10 days and B can do it in 30 days and C in 60 days. If the total wages for the work is Rs. 1800 what is the share of A?

Solution: Ratio of wages = $\frac{1}{10} : \frac{1}{30} : \frac{1}{60} = 6 : 2 : 1$ (Multiplying each term by LCM 60)

So total 9 equal parts in Rs. 1800 $\Rightarrow$ each part = Rs. 200 $\Rightarrow$ share of A = 6 parts = Rs. 1200.

Note: When pipes are used filling the tank they are treated similar to the men working but some outlet pipes emptying the tank are present whose work will be considered negative.

Example 5: A pipe can fill a tank in 5 hrs but because of a leak a the bottom it takes 1 hr extra. In what time can the leak alone empty the tank?

Solution: Let the filling pipe be A.

$$A = \frac{1}{5}$$

But with the leak L, $A - L = \frac{1}{6}$ (A-L because leak is outlet)

So, $\frac{1}{L} = \frac{1}{5} - \frac{1}{6} = \frac{1}{30} \Rightarrow$ Leak can empty the tank in 30 hrs.

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7 CLOCKS

7.1. Introduction

In a clock the most important hands are the minutes hand and the hours hand. Whatever may be the shape of the dial they move in a circular track.

The total angle of 360 degrees in a watch is divided into 1 sectors, one for each hour.

So one hour sector = $\frac{360}{12} = 30$ degrees.

For every one hour (60 min),

- The minutes hand moves through 360 deg.
- The hours hand moves through 30 deg.

So for every minute,

- The minutes hand moves through 6 deg
- The hours hand moves through 0.5 deg.

They move in same direction. So their relative displacement for every minute is 5.5 deg.

This 5.5 deg movement constitutes the movements of both the hands.

So for every minute both the hands give a displacement of 5.5 deg.

Note:

1. Between every two hours i.e., between 1 and 2, 2 and 3 and so on the hands of the clock coincide with each other for one time except between 11, 12 and 12, 1.
   In a day they coincide for 22 times.
2. Between every two hours they are perpendicular to each other two times except between 2, 3 and 3, 4 and 8, 9 and 9, 10.
   In a day they will be perpendicular for 44 times.
3. Between every two hours they will be opposite to each other one time except between 5, 6 and 6, 7.
   In a day they will be opposite for 22 times.

Examples: At what time between 5 and 6 will the hands of the clock coincide?

Solution: At 5 the angle between the hands is 150 deg.

To coincide, they collectively have to travel this distance. Every minute they travel 5.5 deg.

So no. of minutes required to coincide = $\frac{150}{5.5} = \frac{300}{11} = 27\frac{3}{11}$ *min*.

Examples: At what time between 6 and 7 will the hands be perpendicular?

Solution: At 6 the angle between the hands is 180 deg.

To form 90 deg they have to cover 90 deg (out of 180 if 90 is covered 90 will remain)

So no. of minutes required = $\frac{90}{5.5} = \frac{180}{11} = \frac{164}{11}$ *min*.

But they will be perpendicular for two times. The second one will happen after the minutes hand crosses the hours hand and then for 90 deg.

So it has to travel 180 + 90 = 270 deg.

So time = $\frac{270}{5.5} = \frac{540}{11} = 49\frac{1}{11}$ *min*.

Examples: What is the angle between the hands of the clock at 3.45?

Solution: At 3, the angle between the hands = A = 90 deg.

In 45 min the hands will move angle of B = 45 × 5.5 deg (since 5.5 deg for 1 min)

B = 247.5 deg.

Required angle = A ~ B = 157.5 deg.

Examples: What is the angle between the hands at 4.40?

Solution: At 4 the angle between the hands, A = 120 deg.

In 40 min, B = 40 × 5.5 = 220 deg.

The required angle = A ~ B = 100 deg.

Examples: A clock loses 5 min for every hour and another gains 5 min for every hour. If they are set correct at 10 am on Monday then when will they be 12 hrs apart?

Solution: For every hour watch A loses 5 min and watch B gains 5 min.

So for every hour they will differ by 10 min.

For 12 hrs (720 min) difference between them the time required = $\frac{720}{10} = 72$ *hrs*

So they will be 12 hrs apart after 3 days i.e., at 10 am on Thursday.

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8 CALENDARS

8.1. Calendars

Here you mainly deal in finding the day of the week on a particular given date.

The process of finding this depends on the number of odd days.

Odd days are quite different from the odd numbers.

- **Odd Days:** The days more than the complete number of weeks in a given period are called odd days.
- **Ordinary Year:** An year that has 365 days is called Ordinary Year.
- **Leap Year:** The year which is exactly divisible by 4 (except century) is called a leap year.

Example: 1968, 1972, 1984, 1988 and so on are the examples of Leap Years.

1986, 1990, 1994, 1998, and so on are the examples of non leap years.

Note: The Centuries divisible by 400 are leap years.

Important Points:

- An ordinary year has 365 days = 52 weeks and 1 odd day.
- A leap year has 366 days = 52 weeks and 2 odd days.
- Century = 76 Ordinary years + 24 Leap years.
- Century contain 5 odd days.
- 200 years contain 3 odd days.
- 300 years contain 1 odd day.
- 400 years contain 0 odd days.
- Last day of a century cannot be Tuesday, Thursday or Saturday.
- First day of a century must be Monday, Tuesday, Thursday or Saturday.

Explanation:

100 years = 76 ordinary years + 24 leap years

= 76 odd days + 24 × 2 odd days

= 124 odd days = 17 weeks + 5 days

∴ 100 years contain 5 odd days.

No. of odd days in first century = 5

∴ Last day of first century is Friday.

No. of odd days in two centuries = 3

∴ Wednesday is the last day.

No. of odd days in three centuries = 1

∴ Monday is the last day.

No. of odd days in four centuries = 0

∴ Sunday is the last day.

Since the order is continually kept in successive cycles, the last day of a century cannot be Tuesday, Thursday or Saturday.

So, the last day of a century should be Sunday, Monday, Wednesday or Friday.

Therefore, the first day of a century must be Monday, Tuesday, Thursday or Saturday.

8.2. Working Rules

Working rule to find the day of the week on a particular date when reference day is given:

Step 1: Find the net number of odd days for the period between the reference date and the given date (exclude the reference day but count the given date for counting the number of net odd days).

Step 2: The day of the week on the particular date is equal to the number of net odd days ahead of the reference day (if the reference day was before this date) but behind the reference day (if this date was behind the reference day).

Working rule to find the day of the week on a particular date when no reference day is given

Step 1: Count the net number of odd days on the given date

Step 2: Write:

For 0 odd days – Sunday

For 1 odd day – Monday

For 2 odd days – Tuesday

⋮ ⋮ ⋮

For 6 odd days - Saturday

Examples: If 11^{th} January 1997 was a Sunday then what day of the week was on 10^{th} January 2000?

Solution: Total number of days between 11^{th} January 1997 and 10^{th} January 2000

= (365 – 11) in 1997 + 365 in 1998 + 365 in 1999 + 10 days in 2000

= (50 weeks + 4 odd days) + (52 weeks + 1 odd day) + (52 weeks + 1 odd day) + (1 week + 3 odd days)

Total number of odd days = 4 + 1 + 1 + 3 = 9 days = 1 week + 2 days

Hence, 10^{th} January, 2000 would be 2 days ahead of Sunday i.e. it was on Tuesday.

Examples: What day of the week was on 10^{th} June 2008?

Solution: 10^{th} June 2008 = 2007 years + First 5 months up to May 2008 + 10 days of June

2000 years have 0 odd days.

Remaining 7 years has 1 leap year and 6 ordinary years $\Rightarrow$ 2 + 6 = 8 odd days

So, 2007 years have 8 odd days.

No. of odd days from 1^{st} January 2008 to 31^{st} May 2008 = 3+1+3+2+3 = 12

10 days of June has 3 odd days.

Total number of odd days = 8+12+3 = 23

23 odd days = 3 weeks + 2 odd days.

Hence, 10^{th} June, 2008 was Tuesday.

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9 BLOOD RELATIONS

9.1. Introduction

The standard definitions of relations are given below

A/An ⇓	is related to a PERSON as ⇓
Grandfather	The father of his/her mother or father
Grandmother	The mother of his/her mother or father
Grandson	The son of his/her daughter/son
Granddaughter	The daughter of his/her daughters/son
Uncle	The brother of his/her mother or father
Aunt	The sister of his/her mother or father
Nephew	The son of his/her brother or sister
Cousin	The son or daughter of his/her aunt of uncle
Niece	The daughter of his/her brother or sister
Spouse	as her husband or his wife
Father-in-law	the father of his/her spouse
Mother-in-law	the mother of his/her spouse
Sister-in-law	the sister of his/her spouse
Brother-in-law	the brother of his/her spouse
Son-in-law	the spouse of his/her daughter
Daughter-in-law	the spouse of his/her son

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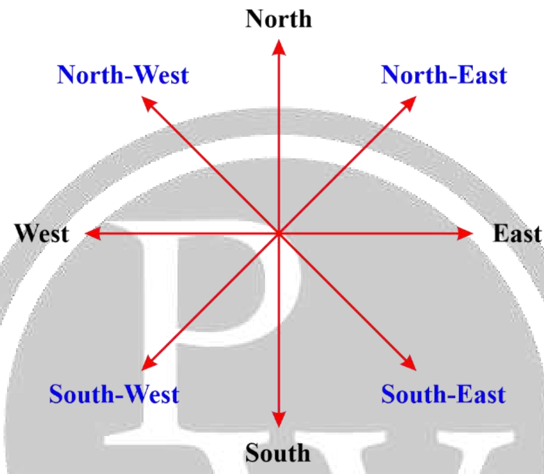
10 DIRECTIONS

10.1. Introduction

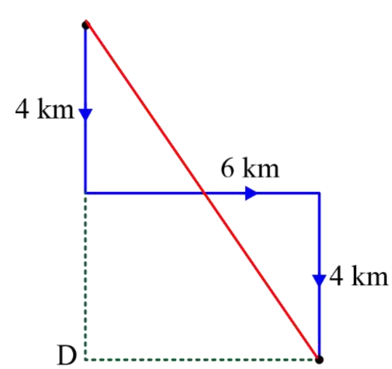


Examples:

Example 1: Ravi traveled 4 km straight towards south. He turned left and traveled 6 km straight, then turned right and traveled 4 km straight. How far is he from the starting point?

(a) 8 km (b) 10 km (c) 18 km (d) 12 km

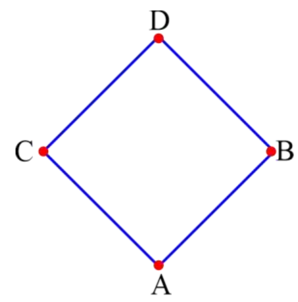
Solution. 10 km. B is the finishing point and A is starting point. The distance of A from B is



Example 2. A is to the South-East of C, B is to the East of C and North-East of A. If D is to the North of A and North-West of B. In which direction of C is D located?

(a) North-West (b) South-West (c) North-East (d) South-East

Solution. North-East D is located to the North-East of C.

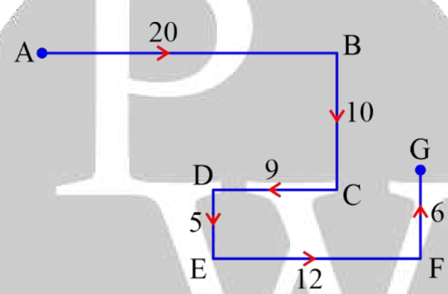


Example 3. A rat runs 20' towards East and turns to right, runs 10' and turns to right, runs 9' and again turns to left, runs 5' and then runs to left, runs 12' and finally turns to left and runs 6'. Now, which direction is the rat facing?

(a) East (b) West

(c) North (d) South

Solution. North. The movements of the rat from A to G are as shown in the fig. It is clear, rat is walking in one direction FG, *i.e.*, North.

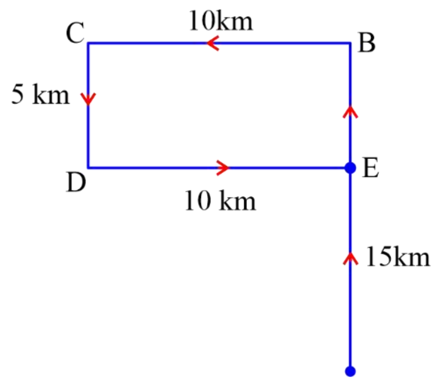


Example 4. From his house, Lokesh went 15 km to the North. Then he turned west and covered 10 km, then he turned South and covered 5 km. Finally, turning to East, he covered 10 km. In which direction is he from his house?

(a) East (b) South

(c) North (d) West

Solution. North. Starting point is A and ending point is E. E is to the north of his house at A.



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11 DATA INTERPRETATION

11.1. Data Interpretation

- It deals with careful reading, understanding, organizing and interpreting the data provided so as to derive meaningful conclusions.
- Mostly used tools for interpretation of a data are
  - Ratio
  - Percentage
  - Rate
  - Average

11.2. Types of Data Interpretation:

The numerical data pertaining to any event can be presented by any one or more of the following methods.

1. Tables
2. Line Graphs
3. Bar Graphs or Bar Charts
4. Pie Charts or Circle Graphs

11.2.1. Tables

It is the systematic presentation of data in tabular form to understand the given information and to make clear the problem in a certain field of study. It has six elements namely:

- **Title:** It is the heading of the table.
- **Stule:** It is the section of the table containing row headings
- **Column Captions:** It is the heading of each column
- **Body:** It consists the numerical figures
- **Footnotes:** It is for further explanation of the table
- **Source:** It is the authority of the data

Example: Study the following table and answer the questions given below it.

Annual Income of Five Schools

Figures in ’00 rupees

Sources of Income	School A	School B	School C	School D	School E
Tuition Fee	120	60	210	90	120
Term Fee	24	12	45	24	30
Donations	54	21	60	51	60
Funds	60	54	120	42	55
Miscellaneous	12	3	15	3	15
Total	270	150	450	210	280

Example: The income by way of donation to school D is what per-cent of its miscellaneous?

Solution: Required percentage = $\frac{5100}{300} = 27\%$

11.2.2. Line Graph

A line graph indicates the variation of a quantity w.r.t two parameters calibrated on X and Y-axis respectively.

Note:
1. Any part of the line graph parallel to X-axis represents no change in the value of Y parameter w.r.t the value of X parameter.
2. The steepest or maximum part of the line graph indicates maximum percentage change of the value during the two consecutive period in which the related part lies.
3. If the steepest part is a rise slope, then it is the highest percentage growth.
4. If the steepest part is a decline slope, it will represent a maximum percentage fall of the value calibrated in the other axis.

11.2.3. Bar Graph

Bar graphs are diagrammatic representation of a discrete data.

Types of Bar Graphs:

- **Simple Bar Graphs:** A simple bar graph relates to only one variable. The values of the variables may relate to different years or different terms.
- **Sub-divided Bar Graph:** It is used to represent various parts of sub-classes of total magnitude of the given variable.
- **Multiple Bar Graphs:** In this type, two or more bars are constructed adjoining each other, to represent either different components of a total or to show multiple variables.

11.2.4. Pie Chart

In this method of representation, the total quantity is distributed over a total angle of 360° which is one complete circle or pie.

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12 DATA SUFFICIENCY

12.1. Introduction

Data sufficiency questions are designed to measure your ability to analyze a quantities problem, recognize which given information is relevant, and determine at what point there is sufficient information to solve a problem. In these questions, you are to classify each problem according to the five or four fixed answer choice, rather than find a solution to the problem.

Each Data sufficiency question consists of a question, often accompanied by some initial information, and two statements, labeled (1) and (2), which contain additional information. You must decide whether the information in each statement is sufficient to answer the question or- if neither statement provides enough information –whether the information in the two statements together is sufficient. It is also possible that the statements in combination do not give enough information answer the question.

Begin by reading the initial information and the question carefully. Next, consider the first statement. Does the information provided by the first statement is sufficient to answer the question? Go on the statement. Try to ignore the information given in the first statement when you consider the second statement. Now you should be able to say, for each statement, whether it is sufficient to determine the answer.

Next, consider the two statements in tandem. Do they, together, enable you to answer the question?

Give our answers as per the following statements

A Statement (1) alone is sufficient but
Statement (2) alone is not sufficient

B Statement (2) alone is sufficient but
Statement (1) alone is not sufficient

C Both statements together are sufficient
but neither statement alone is sufficient

D Each statement alone is sufficient

E Both statement together are still not sufficient.

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THANK YOU