Unit 2: Series Solution Of Ordinary D. E.

Q: 1. Define

(i) **Power Series** :-

A Power Series in Power of $(x - x_0)$ is an Infinite Series of the form,

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k = a_0 (x - x_0)^0 + a_1 (x - x_0)^1 + \dots + (a_n)(x-x_0)^n + \dots$$

(ii) **Singular Point** :-

→ Consider the Linear D.E is,

$$y'' + P(x) y' + Q(x) y = 0.$$

The Point $x = x_0$ is called a singular Point of eq. If $P(x)$ & $Q(x)$ are not analytic at x_0.

(iii) **Ordinary Point** :-

→ Consider the L.D.E. is

$$y'' + P(x) y' + Q(x) y = 0$$

The Point $x = x_0$ is called ordinary point of eq.ⁿ If $P(x)$ & $Q(x)$ are analytic at $x = x_0$.

(iv) **Regular Singular Point**

→ If both $(x - x_0) P(x)$ & $(x - x_0)^2 Q(x)$ are Analytic at $x = x_0$ is called RSP.

(v) **Irregular Singular Point**

If both $(x - x_0) P(x)$ & $(x - x_0)^2 Q(x)$ are not Analytic that $x = x_0$ is called ISP.

Q:2. Find Ordinary Point for

(1.) $y'' + y = 0$

→ Compare with

$y'' + P(x)\, y' + Q(x)\, y = 0$

$P(x) = 0 \quad Q(x) = 1.$

→ Both $P(x)$ & $Q(x)$ are finite. And It's $P(x)$ & $Q(x)$ are analytic.

→ ∴ Therefore every Point are Ordinary Point

2.) $y'' + e^{x} y' + (\sin x)\, y = 0.$

$P(x) = e^{x} \quad Q(x) = \sin x$

⇒ $x = 0$

$P(x) = e^{0} \quad Q(x) = \sin 0$

$P(x) = 1 \quad Q(x) = 0.$

→ e^{x} and $\sin x$ are analytic everywhere

∴ All Points are OP.

(3.) $y'' + e^{x} y' + (\cos x) y = 0.$

$P(x) = e^{x} \quad Q(x) = \cos x.$

→ $x = 0$

$P(x) = 1 \quad Q(x) = 1.$

→ e^{x} and $\cos x$ are Analytic

→ All Point are OP.

(4.) $y'' - 2y = 0.$

→ All Point Are OP.

(5.) $y'' + e^{2x} y' + x^{2} y = 0.$

→ ∴ All Points are OP.

Q:3. Find Singular Point for.

(1.) $y'' + \dfrac{1}{x-1} y' + \dfrac{1}{x-1} y = 0$

$P(x) = \dfrac{1}{x-1} \quad Q(x) = \dfrac{1}{x-1}$

$\boxed{x = 1}$

$P(x) = \infty \quad Q(x) = \infty$

$P(x)$ & $Q(x)$ are not Analytic at Point $x = 1.$

$P(x)$ & $Q(x)$ are Infinite.

∴ $x = 1$ is a singular point.

(2.) $(1-x^2)\,y'' - 6xy' - 4y = 0$

$$P(x) = \frac{-6x}{(1-x^2)} \qquad Q(x) = \frac{-4}{(1-x^2)}$$

$\rightarrow$ $x = \pm 1$.

$\cancel{P(x) = \infty}$, $Q(x) = \infty$

$P(x)$ & $Q(x)$ Are not Analytic at $x = \pm 1$.

$\therefore$ $x = \pm 1$ is SP.

(3.) $y'' + \dfrac{1}{x-2}\,y' + \dfrac{1}{x-2}\,y = 0$

$\therefore$ $x = 2$ is S.P.

(4.) $y'' + \dfrac{2}{x+3}\,y' + \dfrac{5}{x-3}\,y = 0$

$\therefore$ $x = -3$ is S.P.

Q:4. Classify the Singularities OR. Find ordinary Point, SP, RSP, ISP for the following eq.^{n}

(1.) $(x^2+4)\dfrac{d^2y}{dx^2} + 2x\left(\dfrac{dy}{dx}\right) - 12y = 0$

$\rightarrow$ Divided both side by (x^2+4)

$$y'' + \frac{2x}{x^2+4}\,y' - \frac{12}{x^2+4}\,y = 0.$$

$\rightarrow$ Compare with

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

$$P(x) = \frac{2x}{x^2+4} \qquad Q(x) = \frac{-12}{x^2+4}$$

$\boxed{x = \pm 2i}$

$\rightarrow$ For $x = \pm 2i$, $P(x) = \infty$, $Q(x) = -\infty$

∴ Here $P(x)$ & $Q(x)$ are Infinite. therefor is $P(x)$ & $Q(x)$ are not Analytic. $x = \pm 2i$.

∴ $x = \pm 2i$ is singuler Point.

→ For $x_0 = \pm 2i$

$$(x-x_0)P(x) = (x-2i)\left(\frac{2x}{x^2+4}\right) \quad \text{or} \quad (x+2i)\left(\frac{2x}{x^2+4}\right)$$

$$= \cancel{(x-2i)}\left(\frac{2x}{\cancel{(x-2i)}(x+2i)}\right) \quad \text{or} \quad \cancel{(x+2i)}\left(\frac{2x}{(x-2i)\cancel{(x+2i)}}\right)$$

$$= \frac{2x}{x+2i} \quad \text{or} \quad \frac{2x}{x-2i}$$

– finite. – finite.

$$(x-x_0)^2 Q(x) = (x+2i)^2\left(\frac{12}{(x-2i)\cancel{(x+2i)}}\right) \quad \text{or} \quad (x-2i)^2\left(\frac{12}{\cancel{(x-2i)}(x+2i)}\right)$$

$$= \frac{12(x+2i)}{(x-2i)} \quad \text{or} \quad 12\left(\frac{x-2i}{x+2i}\right).$$

– finite – finite

→ Here, both $(x-x_0)P(x)$ & $(x-x_0)^2 Q(x)$ both are finite. There for $x = \pm 2i$ is a RSP.

(2) $(x+1)y'' - xy' - y = 0$

$$y'' - \frac{x}{x+1}y' - \frac{1}{x+1}y = 0.$$

$x = -1$. $P(x) = \infty$ $Q = \infty$.

→ both Infinite, $x = -1$ is SP.

→ For $x = -1$,

$(x-x_0)P(x) = \cancel{(x+1)}\left(\frac{-x}{\cancel{x+1}}\right) \Rightarrow -x \Rightarrow$ finite

$(x-x_0)^2 Q(x) = (x+1)^2\left(\frac{-1}{\cancel{x+1}}\right) \Rightarrow -(x+1) \Rightarrow$ finite.

→ Here, both are finite

there for $x = -1$ is RSP.

Q:6. Solve the equation $\frac{d^2y}{dx^2} + y = 0$ by Power series Method.

(i) $y'' + y = 0.$

→ Let $y = \sum_{k=0}^{\infty} c_k x^k$

$y' = \sum_{k=1}^{\infty} k c_k x^{k-1}$

$y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}$

→ Now, substitute all values in given eq.ⁿ

$$\sum_{k=2}^{\infty} c_k k(k-1) x^{k-2} + \sum_{k=0}^{\infty} c_k x^k = 0.$$

$$[c_2\, 2(1)x^0 + c_3(3)(2)x + c_4(4)(3)x^2 + c_5(5)(4)x^3 + \dots] + [c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots] = 0$$

$$[2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \dots] + [c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots] = 0$$

$$(c_0 + 2c_2) + x(6c_3 + c_1) + x^2(12c_4 + c_2) + x^3(20c_5 + c_3) + x^4(30c_6 + c_4) + \dots = 0$$

→ Comparing the co-efficient of power x both side.

$c_0 + 2c_2 = 0 \Rightarrow c_2 = -c_0/2$

$6c_3 + c_1 = 0 \Rightarrow c_3 = -c_1/6$

$12c_4 + c_2 = 0 \Rightarrow c_4 = -c_2/12 \Rightarrow c_0/24$

$20c_5 + c_3 = 0 \Rightarrow c_5 = -c_3/20 \Rightarrow c_1/120$

$$y = \sum_{k=0}^{\infty} c_k x^k$$

$$= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots$$

$$= c_0 + c_1 x - \frac{c_0}{2} x^2 - \frac{c_1}{6} x^3 + \frac{c_0}{24} x^4 + \frac{c_1}{120} x^5 - \dots$$

$$y = c_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right] + c_1\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right]$$

(ii) $y' = 2xy$

→ Clearly Here $x=0$ is an ordinary Point.

$$y = \sum_{k=0}^{\infty} c_k x^k$$

$$y' = \sum_{k=1}^{\infty} c_k k x^{k-1}$$

$$y'' = \sum_{k=2}^{\infty} c_k k(k-1) x^{k-2}$$

→ Sub. all the values in given eqⁿ

$$y' - 2xy = 0$$

$$\sum_{k=1}^{\infty} c_k k x^{k-1} - 2x\left[\sum_{k=0}^{\infty} c_k x^k\right] = 0$$

$$\Rightarrow [c_1 x^0 + 2c_2 x^1 + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \dots]$$
$$- 2x[c_0 x^0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots] = 0.$$

$$\Rightarrow [c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \dots]$$
$$- 2x[c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots] = 0.$$

$$\Rightarrow c_1 + [2c_2 - 2c_0]x + [3c_3 - 2c_1]x^2 + [4c_4 - 2c_2]x^3$$
$$+ [5c_5 - 2c_3]x^4 + \dots = 0$$

→ Compare

$c_1 = 0 \quad 2c_2 - 2c_0 = 0 \Rightarrow c_2 = c_0$

$3c_3 - 2c_1 = 0 \Rightarrow c_3 = 0$

$4c_4 - 2c_2 = 0 \Rightarrow 2c_4 = c_2 \Rightarrow c_4 = c_0/2$

$5c_5 - 2c_3 = 0 \Rightarrow c_5 = 0$

$6c_6 - 2c_4 = 0 \Rightarrow 6c_6 = 2c_0 \Rightarrow c_6 = c_0/3$

$$y = \sum_{k=0}^{\infty} c_k x^k$$

$$= c_0 + c_1 x + c_2 x + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots$$

$$= c_0 + c_0 x^2 + \frac{c_0 x^4}{2} + \frac{c_0 x^6}{3} + \dots$$

$$\boxed{y = c_0\left[1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3} + \dots\right]}$$

(iii) $y'' = y'$

$y'' - y' = 0.$

Let $y = \sum_{k=0}^{\infty} c_k x^k$

$y' = \sum_{k=1}^{\infty} k c_k x^{k-1}$

$y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}$

→ Now, sub. All the Value in given eq^n.

$$\sum_{k=2}^{\infty} a_k k(k-1) x^{k-2} - \sum_{k=1}^{\infty} a_k k x^{k-1} = 0.$$

$\Rightarrow [c_2 2(1) x^0 + c_3 (3)(2) x^1 + c_4 (4)(3) x^2 + c_5 (5)(4) x^3 + \dots]$
$- [c_1 x^0 + c_2 (2) x + c_3 (3) x^2 + c_4 (4) x^3 + \dots] = 0$

$\Rightarrow [2c_2 + 6c_3 x + 12 c_4 x^2 + 20 c_5 x^3 + 30 c_6 x^4 + \dots]$
$- [c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots] = 0$

$\Rightarrow [2c_2 - c_1] + [6c_3 - 2c_2] x + [12 c_4 - 3c_3] x^2 + [20c_5 - 4c_4] x^3$
$+ [30 c_6 - 5c_5] x^4 + \dots = 0$

→ Compare both side

$2c_2 - c_1 = 0 \Rightarrow c_2 = c_1/2$

$6c_3 - 2c_2 = 0 \Rightarrow c_3 = c_2/3 \Rightarrow c_1/6$

$12c_4 - 3c_3 = 0 \Rightarrow c_4 = c_3/4 \Rightarrow c_1/24$

$20c_5 - 4c_4 = 0 \Rightarrow c_5 = c_4/5 \Rightarrow c_1/120$

$30c_6 - 5c_5 = 0 \Rightarrow c_6 = c_5/6 \Rightarrow c_1/720$

→ Now,

$y = \sum_{k=0}^{\infty} c_k x^k$

$= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$

$= c_0 + c_1 x + \frac{c_1 x^2}{2} + \frac{c_1 x^3}{6} + \frac{c_1 x^4}{24} + \frac{c_1 x^5}{120} + \dots$

$= c_0 + c_1 x \left[1 + \frac{1}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \dots\right]$

(iv) $y'' + xy' = 0$

→ let $y = \sum_{k=0}^{\infty} c_k x^k$

$y' = \sum_{k=1}^{\infty} k c_k x^{k-1}$

$y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}$

→ Now, sub all value

$$\sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} + x\left[\sum_{k=1}^{\infty} k c_k x^{k-1}\right] = 0$$

$$\Rightarrow [2(1)c_2 + 3(2)c_3 x + (4)(3)c_4 x^2 + (5)(4)c_5 x^3 + (6)(5)c_6 x^4 + \dots]$$
$$+ x[1c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots] = 0.$$

$$\Rightarrow [2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + \dots]$$
$$+ [c_1 x + 2c_2 x^2 + 3c_3 x^3 + 4c_4 x^4 + \dots] = 0.$$

$$\Rightarrow [2c_2] + x[6c_3 + c_1] + x^2[12c_4 + 2c_2] + x^3[20c_5 + 3c_3]$$
$$+ x^4[30c_6 + 4c_4] + \dots = 0$$

→ Compare

$c_2 = 0$

$6c_3 + c_1 = 0 \Rightarrow c_3 = -c_1/6$

$12c_4 + 2c_2 = 0 \Rightarrow c_4 = 0$

$20c_5 + 3c_3 = 0 \Rightarrow c_5 = \frac{-3c_3}{20} \Rightarrow c_5 = c_1/40$

$30c_6 + 4c_4 = 0 \Rightarrow c_6 = 0.$

$$y = \sum_{k=0}^{\infty} c_k x^k$$
$$= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$
$$= c_0 + c_1 x + 0 + \frac{-c_1}{6} x^3 + 0 + \frac{c_1}{40} x^5 + 0 \dots$$

$$y = c_0 + c_1\left[x - \frac{x^3}{6} + \frac{x^5}{40} - \dots\right]$$

(IV) $y'' - y = 0$

$$\rightarrow \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} - \sum_{k=0}^{\infty} c_k x^k = 0.$$

$$[2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + \dots]$$
$$- [c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots] = 0.$$

$$[2c_2 - c_0] + x[6c_3 - c_1] + x^2[12c_4 + c_2] + x^3[20c_5 + c_3] + \dots = 0$$

$2c_2 - c_0 = 0 \Rightarrow c_2 = c_0/2$

$6c_3 - c_1 = 0 \Rightarrow c_3 = c_1/6$

$12c_4 \cancel{+} c_2 = 0 \Rightarrow c_4 = +c_2/12 \Rightarrow c_0/24$

$20c_5 - c_3 = 0 \Rightarrow c_5 = c_3/20 \Rightarrow c_1/120.$

$$y = c_0 + c_1 x + \frac{c_0}{2} x^2 + \frac{c_1}{6} x^3 + \frac{c_0}{24} x^4 + \frac{c_1}{120} x^5 + \dots$$

$$y = c_0 \left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots\right] + c_1 \left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right]$$

(VI) $y'' + 2xy = 0.$

$$\Rightarrow [2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \dots]$$
$$- 2x[c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots] = 0$$

$$[2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \dots]$$
$$- [2c_0 x + 2c_1 x^2 + 2c_2 x^3 + 2c_3 x^4 + \dots] = 0.$$

$$2c_2 + x(6c_3 - 2c_0) + x^2(12c_4 - 2c_1) + x^3(20c_5 - 2c_2)$$
$$+ x^4(30c_6 - 2c_3) + \dots = 0.$$

$c_2 = 0.$

$6c_3 - 2c_0 = 0 \Rightarrow c_3 = c_0/3$

$20c_5 - 2c_2 = 0 \Rightarrow c_5 = c_2/10 \Rightarrow c_5 = 0$

$30c_6 - 2c_3 = 0 \Rightarrow c_6 = c_3/15 \Rightarrow c_6 = c_0/45$

$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \dots$

$y = c_0 + c_1 x + 0 + \frac{c_0}{3} x^3 + \frac{c_1}{6} x^4 + 0 + \frac{c_0}{45} x^6 + \dots$

$y = c_0 + c_1 x + \frac{c_0}{3} x^3 + \frac{c_1}{6} x^4 + \frac{c_0}{45} x^6 + \dots$

(vi) $y' + 2xy = 0.$

$[c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots]$
$+ 2x[c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots] = 0$

$c_1 + [2c_2 + 2c_0]x + [3c_3 + 2c_1]x^2 + [4c_4 + 2c_2]x^3$
$+ [5c_5 + 2c_3]x^4 + \dots = 0.$

$c_1 = 0.$

$2c_2 + 2c_0 = 0 \Rightarrow c_2 = -c_0$

$3c_3 + 2c_1 = 0 \Rightarrow c_3 = 0$

$4c_4 + 2c_2 = 0 \Rightarrow c_4 = -c_2/2 \Rightarrow +c_0/2$

$5c_5 + 2c_3 = 0 \Rightarrow 0.$

$6c_6 + 2c_4 = 0 \Rightarrow c_6 = -c_0/2 \cdot \frac{1}{3} \Rightarrow -c_0/6$

$y = c_0 + 0 + (-c_0)x^2 + 0 + \frac{c_0}{2} x^4 + 0 + \dots$

$y = c_0 \left[1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \dots\right]$