Unit : 4 Laplace Transforms.

Q:1. Define Laplace Transform.

→ Let $f(t)$ be a function defined for $t \geq 0$ then Laplace Transform of $f(x)$ can be denoted by $L\{f(t)\}$ or $\bar{f}(s)$ or $f(s)$ and is defined as.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) \cdot dt$$

Q:2. State the first shifting Property of L.T.

→ If $F(s)$ be the function then 1^{st}-shifting can be defined as,

$$F(s) = \int e^{-st} f(t) \cdot dt \qquad \therefore s > 0$$

$$F(s-a) = \int e^{-(s-a)t} F(t) \cdot dt \qquad \therefore s > 0$$

Q:3. State the linearity Property of L.T.

$$L\{f(t)\} = f(s) \; \cancel{\text{then } L\{e^{at} f(t)\}} =$$

and $L\{g(t)\} = G(s)$

$$L\{a f(t) \pm b(g(t))\} = a\, L\{f(t)\} \pm b\{g(t)\}.$$

$$\boxed{= a\, F(s) \pm b\, G(s)}$$

Q:4. Prove that If $L(f(t)) = F(s)$ then $L(f'(t)) = s\,L(f(t)) - f(0)$

→ By the def. of L.T. we have

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) \cdot dt$$

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t) \cdot dt$$

$$\int u \cdot v \cdot dx = u v_1 - u' v_2 + u'' v_3 + \ldots$$

$$L\{f'(t)\} = \left[e^{-st} f(t) \right]_0^\infty - \int_0^\infty -s e^{-st} f(t) \cdot dt$$

$$= \left[e^{\infty} f(\infty) - e^{0} f(0) \right] + s \underbrace{\int_0^\infty e^{-st} f(t) \cdot dt}_{L\{f(t)\}}$$

$$\boxed{L\{f'(t)\} = s\,L\{f(t)\} - f(0)}$$

Q:5. Write Values of followings.

1.) $L(t^n) = \frac{n!}{s^{n+1}} = \frac{\sqrt{n+1}}{s^{n+1}} = \frac{n\sqrt{n}}{s^{n+1}}$

(2.) $L(t^4) = \frac{4!}{s^5} = 24/s^5$

(3.) $L(t^3) = 3!/s^4 = 6/s^4$

(4.) $L(t^2) = 2!/s^3 = 2/s^3$

(5.) $L(\sin 3t) = 3/s^2+9$

(6.) $L(\cos 7t) = s/s^2+49$

(7.) $L(\cosh 3t) = s/s^2-9$

(8.) $L(\sinh 4t) = 4/s^2-16$

(9.) $L(\cosh 3t) = \frac{s}{s^2-9}$

(10.) $L(\cosh 8t) = \frac{s}{s^2-64}$

(11.) $L(\sinh 4t) = \frac{4}{s^2-16}$

(12.) $L(\cos 4t) = \frac{s}{s^2+16}$

(13.) $L(\sin 5t) = \frac{5}{s^2+25}$

(14.) $L(\cos 7t) = \frac{s}{s^2+49}$

(15.) $L(e^{2x}) = \frac{1}{s-2}$

(16.) $L(e^{-3x}) = \frac{1}{s+3}$

(17.) $L(e^{4x}) = \frac{1}{s-4}$

(18) $L(e^{-7x}) = \frac{1}{s+7}$

Q:6. Define Inverse L.T.

→ If $L\{f(t)\} = F(s)$ then $f(t)$ is called Inverse L.T of $F(s)$. It is denoted by $L^{-1}\{F(s)\} = F(t)$

Q:7. State change of Scale Property for Inverse L.T.

→ If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(as)\} = \frac{1}{a} f(\frac{t}{a})$

Q:8. State linearity Property for Inverse L.T.

$L^{-1}\{a f(s) \pm b g(s)\} = a L^{-1}\{f(s)\} \pm b L^{-1}\{g(s)\}$

$= a f(t) \pm b g(t)$

Q:9. Find the Values of following.

(1.) $L^{-1}\left(\frac{1}{s^n}\right) \Rightarrow \frac{t^{n-1}}{(n-1)!} = \frac{t^{n-1}}{\Gamma n}$

(2.) $L^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2!} = \frac{t^2}{2}$

(3.) $L^{-1}\left(\frac{1}{s^2}\right) = \frac{t^1}{1!} = t$

(4.) $L^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$

(5.) $L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2t$

(6.) $L^{-1}\left(\frac{1}{s^2+49}\right) = \frac{1}{7}\sin 7t$

Q:11 State Convolution theorem.

→ If $L^{-1}\{F(s)\} = F(t)$ and $L^{-1}\{G(s)\} = g(t)$

$$L^{-1}\{F(s)\cdot G(s)\} = \int_0^t f(u)\cdot g(t-u)\cdot du$$

$$= \int_0^t f(t-u)\, g(u)\cdot du$$

$$= f(t) * g(t)$$

Q:12. Define Convolution.

→ Convolution of a function $f(t)$ & $g(t)$ is denoted of $f(t) * g(t)$ and It is define as.

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\cdot du$$

$$= \int_0^t f(t-u)\, g(u)\cdot du$$

$$= g(t) * f(t)$$

Q:13. Find $1 * 1$ where $*$ denotes Convolution Product

$f(t) = 1$ $\quad g(t) = 1$

$f(u) = 1$ $\quad g(u) = 1$

$f(t-u) = 1.$

$$f(t) * g(t) = \int_0^t f(t-u)\cdot g(u)\cdot du$$

$$= \int_0^t 1\cdot 1\cdot du = [u]_0^t$$

$$\boxed{1 * 1 = t}$$

Q:14. State and Prove change of Scale Property for Inverse L.T.

→ If $L^{-1}\{F(s)\} = f(t)$ then

$$L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$$

Proof: By the Linearity Property of L.T. we have

$$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Taking Inverse L.T. on both sides we get,

$$L^{-1}\{f(at)\} = \frac{1}{a} L^{-1}\left\{F\left(\frac{s}{a}\right)\right\}$$

$$L^{-1}\left\{F\left(\frac{s}{a}\right)\right\} = a f(at)$$

$$L^{-1}\left\{F\left(\frac{a^2 s}{a}\right)\right\} = \frac{a}{a^2} F\left(\frac{at}{a^2}\right)$$

$$L^{-1}\{F(as)\} = \frac{1}{a} F\left(\frac{t}{a}\right).$$

Q:15. State and Prove Linearity Property for I.L.T.

→ If $L^{-1}\{F(s)\} = f(t)$ and $L^{-1}\{G(s)\} = g(t)$ then

$$L^{-1}\{a F(s) \pm b G(s)\} = a L^{-1}\{F(s)\} \pm b L^{-1}\{G(s)\}$$
$$= a f(t) \pm b g(t)$$

→ Proof.

→ From the L.P. of L.T. we have,

$$L\{a f(t) \pm b g(t)\} = a L\{f(t)\} \pm b L\{g(t)\}$$
$$= a F(s) \pm b G(s)$$

Taking Inverse L.T. on both the side

$$L^{-1}\{a f(t) \pm b g(t)\} = L^{-1}\{a F(s) \pm b G(s)\}$$

$$\boxed{a f(t) \pm b g(t) = L^{-1}\{a F(s) \pm b G(s)\}.}$$

Q:16 State & Prove second shifting Property for I.L.T.

→ $L\{f(t)\} = F(s)$ then $L\{f(t-a)\, u(t-a)\} = e^{-as}.F(s)$
$= e^{-as} L\{f(t)\}.$

→ If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{e^{-as} F(s)\} = L\{f(t-a)\, u(t-a)\}.$

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Q:17. Find the Values of followings.

(1) $L\{e^{at}t\sin at\}$

$\rightarrow$ By diff. by S property

$$L\{e^{at}t\sin at\} = (-1)^{1}\left[\frac{d}{ds} L\{e^{at}\sin at\}\right]$$

$$= -1\left[\frac{d}{ds}\frac{a}{(s+a)^2+a^2}\right]$$

$$= -a\frac{d}{ds}\frac{1}{(s-a)^2+a^2}$$

$$= -a\left[\frac{-2(s-a)}{((s-a)^2+a^2)^2}\right]$$

$$= \frac{+2a(s-a)}{(s^2-2as+a^2+a^2)^2}$$

$$\boxed{= \frac{+2a(s-a)}{(s^2-2as+2a^2)^2}}$$

(2.) $L(t^2\sinh at)$

$$= (-1)^2\left(\frac{d^2}{ds^2}L(\sinh at)\right)$$

$$= \frac{d}{ds}\frac{a}{s^2-a^2}$$

$$= a\frac{d^2}{ds^2}(s^2-a^2)^{-1}$$

$$\Rightarrow a\left[\frac{d}{ds}\frac{-1}{(s^2-a^2)^2}\cdot(2s)\right]$$

$$\boxed{= \frac{d}{ds}\frac{-2as}{(s^2-a^2)^2}}$$

$$= -2a\left[\frac{d}{ds}\frac{s}{(s^2-a^2)^2}\right]$$

$$= -2a\left[\frac{(s^2-a^2)^2 - 2(2s)(s^2-a^2)\cdot s}{(s^2-a^2)^4}\right]$$

$$= -2a\left[\frac{s^2-a^2-4s^2}{(s^2-a^2)^3}\right] \Rightarrow \frac{2a(3s^2+a^2)}{(s^2-a^2)^3}$$

(3) $L(\sin at + at\cos at)$

$= L(\sin at) + L(at\cos at)$

$= \frac{a}{s^2+a^2} + a(-1)^1\left[\frac{d}{ds}L(\cos at)\right]$

$= \frac{a}{s^2+a^2} - a\left[\frac{d}{ds}\frac{s}{s^2+a^2}\right]$

$\Rightarrow \frac{a}{s^2+a^2} - a\left[\frac{(s^2+a^2)-s(2s)}{(s^2+a^2)^2}\right]$

$= \frac{a}{s^2+a^2} - a\left[\frac{(a^2+s^2)}{(s^2+a^2)^{\cancel{2}}} - \frac{2s^2}{(s^2+a^2)^2}\right]$

$= \cancel{\frac{a}{s^2+a^2}} - \cancel{\frac{a}{s^2+a^2}} + \frac{2as^2}{(s^2+a^2)^2}$

$$\boxed{= \frac{2as^2}{(s^2+a^2)^2}}$$

(4) $L(t^2\sin^2 4t)$

$= (-1)^2\left[\frac{d^2}{ds^2}L(\sin^2 4t)\right]$

$= \frac{d^2}{ds^2}L\left[\frac{1-\cos 8t}{2}\right]$

$= \frac{d^2}{ds^2}\left[\frac{1}{2}L[1] - \frac{1}{2}L(\cos 8t)\right]$

$= \frac{1}{2}\left[\frac{d^2}{ds^2}\frac{1}{s} - \frac{d^2}{ds^2}\frac{s}{s^2+64}\right]$

$= \frac{1}{2}\left[\frac{d}{ds}\left[\frac{-1}{s^2} - \frac{(s^2+64)-s(2s)}{(s^2+64)^2}\right]\right]$

$= \frac{1}{2}\left(\frac{2}{s^3} - \frac{d}{ds}\frac{1}{s^2+64} + 2\frac{d}{ds}\frac{s^2}{(s^2+64)^2}\right)$

$\Rightarrow \frac{1}{2}\left[\frac{2}{s^3} - \frac{-1}{(s^2+64)^2} + 2\left[\frac{2s(s^2+64)^2 - s^2\,2(s^2+64)2s}{(s^2+64)^4}\right]\right]$

$$\boxed{= \frac{1}{s^3} + \frac{1}{2(s^2+64)^2} + \frac{2s}{(s^2+64)^2} - \frac{4s^3}{(s^2+64)^3}}$$

(5.) $L\left(\frac{e^{-at} - e^{-bt}}{t}\right)$

$= \frac{1}{s}\int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right] \cdot ds.$

$\Rightarrow \frac{1}{s}\left[\log(s+a) - \log(s+b)\right]_s^\infty$

$= \frac{1}{s}\left[\cancel{\log\infty} - \cancel{\log\infty} - \log(s+a) + \log(s+b)\right]$

$= \frac{1}{s}\log\left(\frac{s+b}{s+a}\right).$

(6) $L\left(\frac{\sin wt}{t}\right)$

$= \frac{1}{s}\int_s^\infty L(\sin wt) \cdot ds$

$= \frac{1}{s}\int_s^\infty \frac{w}{s^2+w^2} \cdot ds$

$= \frac{1}{s}\left[\tan^{-1}\left(\frac{s}{w}\right)\right]_s^\infty$

$\Rightarrow \frac{1}{s}\left[+\tan^{-1}\left(\frac{s}{w}\right)\right]_s^\infty$

$\Rightarrow \frac{1}{s}\{\tan^{-1}\infty - \tan^{-1}\frac{s}{w}\}$

$= \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1} s/w\right]$

or $= \frac{1}{s}\left[\cot^{-1} s/w\right]$

(7.) $L\left(\int_0^t (t^4 + \sin 3t)\right).$

$\Rightarrow L\left(\int_0^t t^4\right) + L\left(\int_0^t \sin 3t\right)$

$\rightarrow$ Dividing both side

$\Rightarrow \frac{1}{s}L\{t^4\} + \frac{1}{s}L\{\sin 3t\}$

$\Rightarrow \frac{1}{s}\frac{4!}{s^5} + \frac{1}{s}\frac{3}{s^2+9} \Rightarrow \frac{1}{s}\left[\frac{24}{s^5} + \frac{3}{s^2+9}\right]$

(8) $L(e^{3t}.t)$

$\Rightarrow (-1)^1 \frac{d}{ds} L(e^{3t})$

$\Rightarrow -1 \frac{d}{ds} \frac{1}{s-3}$

$\Rightarrow -1\left[\frac{-1}{(s-3)^2}\right]$

$\boxed{\Rightarrow \frac{1}{(s-3)^2}}$

9) $L(t\sin 3t)$

$\Rightarrow -1 \frac{d}{ds} L(\sin 3t)$

$= -1 \frac{d}{ds} \frac{3}{s^2+9}$

$\Rightarrow -3\left[\frac{(-2s)}{(s^2+9)^2}\right]$

$\boxed{\Rightarrow \frac{6s}{(s^2+9)^2}}$

(10) $L(t\cos 5t)$

$\Rightarrow (-1)\frac{d}{ds} L(\cos 5t)$

$\Rightarrow -\frac{d}{ds} \frac{s}{s^2+25}$

$\Rightarrow -5 \frac{(2s)}{(s^2+25)^2}$

$\boxed{\Rightarrow \frac{10s}{(s^2+25)^2}}$

11) $L\left(\frac{1-\cos 2t}{t}\right)$

$\Rightarrow \frac{1}{s}\int_s^\infty L(1-\cos 2t)\cdot ds$

$\cancel{\Rightarrow \frac{1}{s}\left[\log s\right]_s^\infty - \frac{1}{2}\frac{\infty}{s}}$

$\Rightarrow \frac{1}{s}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$\Rightarrow \frac{1}{s}\left[\log s\right]_s^\infty - \frac{1}{2s}\int_s^\infty \frac{2s}{s^2+4}\cdot ds.$

$\Rightarrow \frac{1}{s}(-\log s) - \frac{1}{2s}\left[\log(s^2+4)\right]_s^\infty$

$\Rightarrow \frac{-\log s}{s} + \frac{\log(s^2+4)}{2s}$

$\Rightarrow \frac{1}{s}\left[-\log s + \log(s^2+4)^{1/2}\right]$

$\boxed{\Rightarrow \frac{1}{s}\log\left(\frac{\sqrt{s^2+4}}{s}\right)}$

(12) $L\left[e^{-3t}(\cos 4t + 3\sin 4t + 2t^4)\right]$

$\Rightarrow \frac{s}{s^2+16} + \frac{12}{s^2+16} + 2\frac{4!}{s^5}$

$\Rightarrow \frac{(s+3)}{(s+3)^2+16} + \frac{12}{(s+3)^2+16} + \frac{48}{(s+3)^5}$

$\boxed{\Rightarrow \frac{s+15}{(s+3)^2+16} + \frac{48}{(s+3)^5}}$

(13.) $L(\sin^2 kt)$.

$\Rightarrow$

$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$

$f(t) = \sin^2 kt \qquad f(0) = 0$

$f'(t) = 2\sin kt.\cos kt = 2\sin 2kt \qquad f'(0) = 0$

$f''(t) = 2\cos kt$

$L\{2\cos kt\} = s^2 L\{\sin^2 kt\} - 0 - 0.$

$2\left(\frac{s}{s^2+k^2}\right) = s^2 L(\sin^2 kt)$

$$\boxed{L(\sin^2 kt) \Rightarrow \frac{2}{s(s^2+k^2)}}$$

(14.) $L(4e^{5t} + 6t^3 - 3\sin 4t + 2\cos 2t)$

$\Rightarrow 4\,L(e^{5t}) + 6\,L(t^3) - 3\,L(\sin 4t) + 2\,L(\cos 2t)$

$\Rightarrow 4\,\frac{1}{s-5} + 6\,\frac{3!}{s^4} - 3\,\frac{4}{s^2+16} + 2\,\frac{s}{s^2+4}$

$$\boxed{\Rightarrow \frac{4}{s-5} + \frac{36}{s^4} - \frac{12}{s^2+16} + \frac{2s}{s^2+4}}$$

(15.) $L(t\,u(t-a))$.

$\rightarrow$ By Corollary 1, we have

$L\{f(t).\,u(t-a)\} = e^{-as} L\{f(t+a)\}.$

$f(t) = t. \qquad f(t+a) = t+a$

$L\{f(t).\,u(t-a)\} = e^{-as} L(t+a)$

$= e^{-as}\left[\frac{1}{s^2} + \frac{a}{s}\right]$

$$\boxed{\Rightarrow e^{-as}\left[\frac{1}{s^2} + \frac{a}{s}\right]}$$

(16) $L\{e^{-3t} u(t-2)\}$

$L\{f(t)\cdot u(t-a)\} = e^{-as} L\{f(t+a)\}$

$f(t) = e^{-3t} \qquad a = +2$

$f(t+2) = e^{-3(t+2)}$

$f(t+2) = e^{-3t-6}$

$L\{f(t)\cdot u(t-a)\} = e^{-2s} L\{e^{-3t-6}\}$

$\Rightarrow e^{-2s} [L\{e^{-3t}\cdot e^{-6}\}]$

$= e^{-2s} \left[\frac{e^{-6}}{s+3}\right]$

$\boxed{\Rightarrow \frac{e^{-2s-6}}{s+3}}$

Q.18 Find the Values of Followings.

(1.) $L^{-1}\left(\frac{5}{(s+2)^5}\right)$

$\Rightarrow 5 L^{-1}\left(\frac{1}{(s+2)^5}\right)$

$\Rightarrow 5 e^{-2t} \frac{t^4}{4!}$

$\boxed{= \frac{5}{24} e^{-2t} t^4}$

(2.) $L^{-1}\left(\frac{s+1}{s^2-6s+25}\right)$

$= L^{-1}\left(\frac{s+1}{s^2-6s+9+16}\right)$

$\Rightarrow L^{-1}\left(\frac{s+1}{(s+3)^2+16}\right)$

$\Rightarrow L^{-1}\left(\frac{s-3}{(s-3)^2+16}\right) + 4 L^{-1}\left(\frac{4}{(s-3)^2+16}\right)$

$= e^{3t}\cos 4t + e^{3t}\sin 4t$

$\boxed{= e^{3t}(\cos 4t + \sin 4t)}$

(3.) $L^{-1}\left(\frac{2}{s^2-4}\right)$

$\boxed{\Rightarrow \sinh 2t}$

(4.) $L^{-1}\left(\log\left(\frac{s+a}{s+b}\right)\right)$

$\Rightarrow L^{-1}\left[\log(s+a) - \log(s+b)\right]$

$\Rightarrow \quad F(s) = \log(s+a) - \log(s+b)$

$L^{-1}\left(\frac{d}{ds}F(s)\right) = L^{-1}\left(\frac{1}{s+a} - \frac{1}{s+b}\right)$

$$\boxed{\Rightarrow e^{-at} - e^{-bt}}$$

$f(t) = -\frac{1}{t} L^{-1}\left(\frac{d}{ds}F(s)\right]$

$$\boxed{\Rightarrow \frac{e^{-bt} - e^{-at}}{t}}$$

(5.) $L^{-1}\left[\frac{1}{s(s+a)^3}\right]$

$\Rightarrow F(s) = \left[\frac{1}{s(s+a)^3}\right]$

$\int_0^t f(s(t))\cdot dt = \int_0^t L^{-1}\left(\frac{1}{(s+a)^3}\right)$

$= \int_0^t e^{-at}\cdot\frac{t^2}{2!}$

$\Rightarrow \frac{1}{2}\int_0^t e^{-at}\cdot t^2$

$= \frac{1}{2}\left[t^2\cdot\frac{e^{-at}}{-a} - \frac{2t\cdot e^{-at}}{a^2} + \frac{2e^{-at}}{-a^3}\right]_0^t$

$\Rightarrow \frac{1}{2}\left[-\frac{t^2e^{-at}}{a} - \frac{2te^{-at}}{a^2} - \frac{2e^{-at}}{a^3} + \frac{2}{a^3}\right]$

(6.) $L^{-1}\left(\log\left(1+\frac{w^2}{s^2}\right)\right)$

$= L^{-1}\left(\log\left(\frac{s^2+w^2}{s^2}\right)\right]$

$F(s) = \log\left(\frac{s^2+w^2}{s^2}\right)$

$f(t) = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\log\left(\frac{s^2+w^2}{s^2}\right)\right.$

$= -\frac{1}{t}\left(L^{-1}\left(\frac{2s}{s^2+w^2} - \frac{2s}{s^2}\right]\right.$

$\Rightarrow -\frac{1}{t}\left[2\cos wt - 2(1)\right]$

$$\boxed{\Rightarrow \frac{2}{t}(1-\cos wt)}$$

(7.) $L^{-1}\left(\frac{3s-12}{s^2+8}\right)$

$3\left[L^{-1}\left(\frac{s}{s^2+8}\right) - 4L^{-1}\left[\frac{1}{s^2+8}\right]\right]$

$= 3\cos 2\sqrt{2}t - \frac{12}{2\sqrt{2}}\sin 2\sqrt{2}t$

$$\boxed{\Rightarrow 3\left[\cos 2\sqrt{2}t - \sqrt{2}\sin 2\sqrt{2}t\right]}$$

(8) $L^{-1}\left(\frac{e^{-\frac{\pi s}{2}} + e^{-\frac{3\pi s}{2}}}{(s^2+1)}\right)$

Q:20 Using Method of Partial fraction to find followings :-

(I.) $L^{-1}\left(\frac{5s+3}{(s-1)(s^2+2s+5)}\right)$

$\Rightarrow$ Let $\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+2s+5)}$

$5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$

$5s+3 = As^2 + 2As + 5A + Bs^2 + Bs + Cs - C$

$5s+3 = s^2(A+B) + s(2A-B+C) + (5A-C)$

$A+B=0$ $\qquad$ $2A-B+C = 5$ $\qquad$ $5A - C = 3$

$B = -A$ $\qquad$ $2A+A+C = 5$ $\qquad$ $5A = 3+C$

$3A+C = 5$ $\qquad$ $5A-3 = C$

$3A+5A-3 = 5$

$8A-3 = 5$

$8A = 8$

$\boxed{A = 1}$ $\qquad$ $C = 5-3$

$\boxed{B = -1}$ $\qquad$ $\boxed{C = 2}$

$L^{-1}\left(\frac{5s+3}{(s-1)(s^2+2s+5)}\right) = L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{-s+2}{s^2+2s+5}\right]$

$= L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{s-2}{(s+1)^2+4}\right)$

$= L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{s+1}{(s+1)^2+4}\right) - \frac{3}{2} L^{-1}\left(\frac{2}{(s+1)^2+4}\right).$

$\Rightarrow e^{t} - e^{-t}\cos 2t - \frac{3}{2} e^{-t}\sin 2t$

$\boxed{\Rightarrow e^{t} - e^{-t}\left[\cos 2t + \frac{3}{2}\sin 2t\right]}$

(2.) $L^{-1}\left(\frac{s+3}{(s^2+6s+13)^2}\right)$

$\rightarrow$ let $\left(\frac{s+3}{(s^2+6s+13)^2}\right) = \frac{As+B}{(s^2+6s+13)} + \frac{Cs+D}{(s^2+6s+13)^2}$

$s+3 = (As+B)(s^2+6s+13) + (Cs+D)$

$s+3 = s^3(A) + s^2(6A+B) + s(13A+6B+C) + (13B+D)$

$\boxed{A=0}$ $\quad 6A+B=0$ $\quad 13A+6B+C=1$ $\quad 13B+D=3$

$\boxed{B=0}$ $\quad \boxed{C=1}$ $\quad \boxed{D=3}$

$L^{-1}\left\{\frac{s+3}{(s^2+6s+13)^2}\right\} = 0 + L^{-1}\left[\frac{s+3}{(s^2+6s+13)^2}\right]$

$= L^{-1}\left(\frac{s+3}{((s+3)^2+(2)^2)^2}\right) \Rightarrow e^{-3t} L^{-1}\left(\frac{s}{(s^2+4)^2}\right)$

$= \frac{1}{2(3)} t \cdot \sin 2t \cdot e^{-3t}$

$\boxed{= \frac{e^{-3t} t \cdot \sin 2t}{6}}$

(3.) $L^{-1}\left(\frac{s}{(s-1)(s^2+2s+2)}\right)$

let $\frac{s}{(s-1)(s^2+2s+2)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+2}$

$s = A(s^2+2s+2) + (Bs+C)(s-1)$

$s = s^2(A+B) + s(2A-B+C) + (2A-C)$

$A+B=0$ $\quad 2A-B+C=1$ $\quad 2A-C=0$

$B=-A$ $\quad 2A+A+2A=1$ $\quad 2A=C$

$\boxed{B=-1/5}$ $\quad 5A=1$ $\quad \boxed{C=2/5}$

$\boxed{A=1/5}$

$$L^{-1}(F(s)) = \frac{1}{5}L^{-1}\left(\frac{1}{s-1}\right) + \frac{1}{5}L^{-1}\left(\frac{-s+2}{s^2+2s+2}\right)$$

$$= \frac{1}{5}e^{t} - \frac{1}{5}L^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) + \frac{3}{5}L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$\Rightarrow \frac{1}{5}e^{t} - \frac{1}{5}e^{-t}\cos t + \frac{3}{5}\sin t$$

$$\boxed{= \frac{1}{5}e^{t} - \frac{1}{5}(\cos t - 3\sin t)}$$

(4.) $L^{-1}\left(\dfrac{4s+5}{(s-1)^2(s+2)}\right)$

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{(s-1)^2} + \frac{B}{(s-1)} + \frac{C}{(s+2)}$$

$$4s+5 = A(s+2) + B(s-1)(s+2) + C(s-1)^2$$

Let $s = 1$

$4(1) + 5 = A(3)$

$3A = 9$

$\boxed{A = 3}$

Let $s = -2$

$-8 + 5 = C(-3)^2$

$\boxed{C = -1/3}$

Let $s = 0$.

$5 = A(2) + B(-1)(2) + C(-1)^2$

$5 = 2A - 2B + C$

$5 = 2A - 2B + C$

$5 = 6 - 2B - \frac{1}{3}$

$2B = 1 - 1/3 = 2/3$

$\boxed{B = 1/3}$

$$L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\} = L^{-1}\left\{\frac{3}{(s-1)^2}\right\} + L^{-1}\left\{\frac{1/3}{s-1}\right\} + L^{-1}\left\{\frac{-1/3}{s+2}\right\}$$

$$\Rightarrow 3e^{t}\frac{t^{2-1}}{1!} + \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}$$

$$\boxed{= 3e^{t}t + \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}}$$

(5.) $L^{-1}\left(\frac{6}{(s+2)(s-4)}\right)$

$$\frac{6}{(s+2)(s-4)} = \frac{A}{(s+2)} + \frac{B}{s-4}$$

$$6 = A(s-4) + B(s+2)$$

let $s = 4$ let $s = -2$

$6 = A(-6)$

$6 = B(6)$ $\boxed{A = -1}$

$\boxed{B = 1}$

$$L^{-1}\left\{\frac{6}{(s+2)(s-4)}\right) = L^{-1}\left(\frac{-1}{s+2}\right) + L^{-1}\left(\frac{1}{s-4}\right)$$

$$\boxed{= -e^{-2t} + e^{4t}}$$

Q:2) State the Convolution theorem to solve following evaluate.

(I.) $L^{-1}\left[\frac{1}{s^2(s-1)}\right]$, $\cancel{f * g}$

~~f(t)=~~ $F(s) = \frac{1}{s^2}$ $G(s) = \frac{1}{s-1}$

$F(t) = L^{-1}(F(s))$ $g(t) = L^{-1}(G(s))$

$= L^{-1}\left(\frac{1}{s^2}\right)$ $= L^{-1}\left(\frac{1}{s-1}\right)$

$F(t) = t$ $g(t) = e^{t}$

$f(t-a) = t-u$ $g(a) = e^{a}$

$$f(t) * g(t) = \int_0^t f(t-u) \cdot g(u) \cdot du$$

$$\Rightarrow \int_0^t (t-u)(e^{u}) \, du$$

$$= \left[(t-u)e^{u} - (-1)(e^{u})\right]_0^t$$

$$\Rightarrow \cancel{(t-u)e}\,[(0)e^{u} + e^{t}] - [(t)e^{0} + e^{0}]$$

$$\Rightarrow e^{t} - te^{0} - 1$$

$$\Rightarrow \boxed{e^{t} - t - 1}$$

(2) $t * e^t$

$f(t) = t \qquad g(t) = e^t$

$f(t-u) = t-u \qquad g(u) = e^u$

$\rightarrow$ same (1)

$\boxed{= e^t - t - 1}$

(3) $1 * 1$

$f(t) = 1 \qquad g(t) = 1$

$f(t-1) = 1 \qquad g(u) = 1$

$$f(t) * g(t) = \int_0^t f(t-u) \cdot g(u) \cdot du$$

$$= \int_0^t 1 \cdot du$$

$$\Rightarrow [u]_0^t$$

$$\boxed{= t}$$

Q:22. Solve Initial Value Problems using L.T.

(1.) $y'' + 3y' + 2y = e^t$; $y(0) = 1$, $y'(0) = 0$.

$\rightarrow$ $y'' + 3y' + 2y = e^t$

Taking L.T both side

$$L\{y''\} + 3L\{y'\} + 2L\{y\} = L\{e^t\}$$

$$\{s^2 y(s) - s y(0) - y'(0)\} + 3\{s y(s) - y(0)\} + 2y(s) = \frac{1}{s-1}$$

$$s^2 y(s) - s + 3s\, y(s) - 3 + 2y(s) = \frac{1}{s-1}$$

$$y(s)\{s^2 + 3s + 2\} - (s+3) = \frac{1}{s-1}$$

$$y(s)\{s^2 + 3s + 2\} = \frac{1}{s-1} + (s+3)$$

$$y(s) = \frac{s^2 + 2s - 2}{(s-1)(s^2+3s+2)} = \frac{s^2+2s-3}{(s-1)(s+1)(s+2)}$$

→ let Partially function

$$\frac{s^2+2s-2}{(s-1)(s+1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s+1}$$

$$s^2+2s-2 = A(s+1)(s+2) + B(s-1)(s+1) + C(s-1)(s+2)$$

→ put $s=-2$

$4-4-2 = B(-3)(-1)$

$3B = -2$

$\boxed{B = -2/3}$

→ put $s=1$

$1+2-2 = A(2)(3)$

$6A = 1$

$\boxed{A = 1/6}$

→ put $s=-1$

$1-2-2 = C(-2)(1)$

$-2C = -3$

$\boxed{C = 3/2}$

$$y(s) = L^{-1}\{y(s)\} = \frac{1}{6}L^{-1}\left(\frac{1}{s-1}\right) - \frac{2}{3}L^{-1}\left(\frac{1}{s+2}\right) + \frac{3}{2}L^{-1}\left(\frac{1}{s+1}\right)$$

$$\boxed{= \frac{1}{6}e^{t} - \frac{2}{3}e^{-2t} + \frac{3}{2}e^{-t}}$$

(2) $y'' + a^2y = k\sin at$

$L(y') = sy(s) - y(0)$

$L(y'') = s^2y(s) - sy(0) - y'(0)$.

Take L.T. both side

$L(y'') + a^2L(y) = kL[\sin at]$

$s^2y(s) - sy(0) - y'(0) + y(s)a^2 = k\frac{a}{s^2+a^2}$

let $y(0) = A$, $y'(0) = B$

$s^2y(s) - sA - B + a^2 y(s) = \frac{ka}{s^2+a^2}$

$y(s)[s^2+a^2] - sA - B = \frac{ka}{s^2+a^2}$

$$y(s) = \frac{ka}{(s^2+a^2)^2} + \frac{sA+B}{(s^2+a^2)}$$

→ Take Inverse Laplace Transform

$$L^{-1}(y(s)) = L^{-1}\left\{\frac{ka}{(s^2+a^2)^2}\right\} + AL^{-1}\left\{\frac{s}{s^2+a^2}\right\} + BL^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$\Rightarrow \frac{ka}{2a^3}(\sin at - at\cos at) + A\cos at + B\sin at$$

$$\boxed{y = \cos at\left(A - \frac{kt}{2a}\right) + \sin at\left(B + \frac{k}{2a^2}\right)}$$

(3.) $\frac{d^2y}{dy^2} + 4y = 0$, $y(0) = 1$, $y'(0) = 6$.

$y'' + 4y = 0$

Taking L.T. both side

$L\{y''\} + 4\, L\{y\} = 0$

$s^2 y(s) - s\, y(0) - y'(0) + 4\, y(s) = 0$

~~$y(s)(s^2+4) - 6 - 1 = 0$~~

~~$y(s) = \frac{7}{s^2+4}$~~

$y(s)(s^2+4) - s - 6 = 0$

$\boxed{y(s) = \frac{s+6}{s^2+4}}$

By Partially function

$y(s) = \frac{s+6}{s^2+4}$

$y(s) = \frac{s}{s^2+4} + \frac{6}{s^2+4}$

$L^{-1}(y(s)) = L^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{6}{2} L^{-1}\left\{\frac{2}{s^2+4}\right\}$

$\boxed{y(t) = \cos 2t + 3 \sin 2t}$

(4.) $y'' - 2y' - 8y = 0$, $y(0) = 3$, $y'(0) = 6$

$[s^2 y(s) - s\,y(0) - y'(0)] - 2[s\,y(s) - y(0)] - 8[y(s)] = 0$

$y(s)(s^2 - 2s - 8) - 3s - 6 + 6 = 0$

$y(s)(s^2 - 2s - 8) = 3s$

$y(s) = \frac{3s}{s^2 - 2s - 8}$

$y(s) = \frac{3s}{(s-4)(s+2)}$

$\frac{3s}{(s-4)(s+2)} = \frac{A}{s-4} + \frac{B}{s+2}$

$3s = A(s+2) + B(s-4)$

let $s = -2$

$-6 = B(-6)$

$\boxed{B = 1}$

let $s = 4$

$3(4) = A(6)$

$6A = 12$

$\boxed{A = 2}$

$$L^{-1}\left\{\frac{3s}{s^2-2s-8}\right\} = 2L^{-1}\left\{\frac{1}{s-4}\right\} + 1L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$y(t) \boxed{= 2e^{4t} + e^{-2t}}$$

(4.) ~~$y'' - 2y' - 8y = 0$, $y(0) = 3$, $y'(0) = 6$.~~

~~Taking L.T on both side~~

~~$[s^2(y(s)) - sy(0) - y'(0)] - 2(sy(s) - y(0)) +$~~

(5.) $y'' + 2y' + y = 6te^{-t}$, $y(0) = y'(0) = 0$

→ Taking L.T on both side

$$[s^2y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}] + [2sy(s) + 2\underbrace{y(0)}_{0}] + y(s) = 6L^{-1}(te^{-t})$$

$$y(s)(s^2+2s+1) = 6\left[-\frac{d}{ds}F(s)\right]$$

$$y(s)(s^2+2s+1) = 6\left[-\frac{d}{ds}L[e^{-t}]\right]$$

$$= 6\left[-\frac{d}{ds}\frac{1}{s+1}\right]$$

$$\Rightarrow \frac{6}{(s+1)^2}$$

$$y(s) = \frac{6}{(s+1)^2(s+1)^2} - \frac{y'(s-2)}{(s+1)^2}$$

$$\Rightarrow \frac{6}{(s+1)^4} - \frac{y'(s-2)}{(s+1)^2}$$

Taking I.L.T. both side

$$L^{-1}(y(s)) = 6\, L^{-1}\left(\frac{1}{(s+1)^4}\right)$$

$$= \cancel{6}\, e^{-t}\, \frac{t^3}{\cancel{3!}}$$

$$\boxed{y(t) \Rightarrow e^{-t} t^3}$$