Page No.: 1

Unit : 3. Partial D.E. And It's Applications

Q:1. Define Order and degree for Partial D.E.

→ Order : The order of a Partial D.E is the ordered of the highest Partial Derivatives appeared in the equation

→ Degree: The Degree of the highest Partial Derivatives in eq^n is the Degree of the Partial D.E.

Q:2. Find the Order and Degree of the following PDE.

(1.) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$O = 2 \quad D = 1$

(2.) $\left(\frac{\partial^3 u}{\partial x^3}\right)^2 + \frac{\partial^2 u}{\partial x^2} = 0$

$O = 3 \quad D = 2$

(4.) $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 4z$

$O = 1 \quad D = 2$

(5.) $\left(\frac{\partial u}{\partial x}\right)^4 = y\left(\frac{\partial u}{\partial y}\right)$

$O = 1 \quad D = 4$

(5.) $\left(\frac{\partial u}{\partial x}\right)^4 + \left(\frac{\partial^2 y}{\partial y^2}\right)^5$

$O = 2 \quad D = 5$

(6.) $\left(\frac{\partial^3 y}{\partial x^3}\right)^2 + \left(\frac{\partial^2 y}{\partial y^2}\right)^5 = 10z$

$O = 3 \quad D = 2$

Q:3. Write the following.

1. One Dimensional Heat eq^n $\Rightarrow \partial u/\partial t = c^2 \, \partial^2 u/\partial x^2$
2. One Dimensional wave eq^n $\Rightarrow \partial^2 u/\partial t^2 = c^2 \left[\partial^2 u/\partial x^2\right]$
3. Two " Laplace eq^n $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
4. Two " wave eq^n $\Rightarrow \partial^2 u/\partial t^2 = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$
5. Three " Laplace eq^n $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Q: 4. classify the following PDE.

(1.) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ => Parabolic

(2.) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ => Elliptic

(3.) $\frac{\partial^2 u}{\partial x^2} + 3 \frac{\partial^2 u}{\partial x . \partial t} + \frac{\partial^2 u}{\partial t^2} = 0$ => Hyperbolic

(4) $\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 v}{\partial t^2}$ => Hyperbolic

(5) $\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 v}{\partial x . \partial y} + \frac{\partial^2 v}{\partial y^2} = 0$ => Parabolic.

Q: 5. Form a P.D.E from the following Relation (By eliminating arbitrary functions).

Derive P.D.E by eliminating arbitary fun. from the relation

(1.) $f(xy + z^2, x+y+z) = 0$

→ Let $u = xy + z^2$ $v = x+y+z$.

$f(u, v) = 0$. — ①

Let $xy + z^2 = f(x+y+z)$. — ②

Diff. w.r.t x & y

$y(1) + 2z \frac{\partial z}{\partial x} = F'(x+y+z)(1 + 0 + \frac{\partial z}{\partial x})$

$y + 2zp = F'(x+y+z)(1+p)$ — ③

$x(1) + 2z \frac{\partial z}{\partial y} = F'(x+y+z)(0 + 1 + \frac{\partial z}{\partial y})$

$x + 2zq = F'(x+y+z)(1+q)$ — ④

Dividing ③ & ④

$$\frac{y + 2zp}{x + 2zq} = \frac{F'\cancel{(x+y+z)}\,(1+p)}{F'\cancel{(x+y+z)}\,(1+q)}$$

$$(y+2zp)(1+q) = (x+2zq)(1+p)$$

$$y + yq + 2zp + \cancel{2zpq} = x + xp + 2zq + \cancel{2zpq}$$

$$yq - 2zq + 2zp - xp = x-y$$

$$q(y-2z) + (2z-x)p = x-y$$

$$y-x = -q(y-2z) - (2z-x)p$$

$$y-x = q(2z-y) + (x-2z)p$$

(2.) $f(x+y+z,\ x^2+y^2+z^2) = 0$

$$x+y+z = f(x^2+y^2+z^2)$$

$\rightarrow$ let $u = x+y+z$ $\quad v = x^2+y^2+z^2$

$$f(u,v) = 0.$$

Diff. Partially w.r.t x & y.

$$\rightarrow \quad 1 + 0 + \frac{\partial z}{\partial x} = f'(x^2+y^2+z^2)\left(2x + 2z\cdot\frac{\partial z}{\partial x}\right)$$

$$1+p = f'(x^2+y^2+z^2)(2x+2zp) \quad \text{—} (1)$$

$$\rightarrow \quad 0 + 1 + \frac{\partial z}{\partial y} = f'(x^2+y^2+z^2)\left(2y + 2z\cdot\frac{\partial z}{\partial y}\right)$$

$$1+q = f'(x^2+y^2+z^2)(2y+2z\cdot q) \quad \text{—} (2)$$

Dividing (1) & (2)

$$\frac{1+p}{1+q} = \frac{x+zp}{y+zq}$$

$$y + zq + yp + \cancel{zpq} = x + zp + xq + \cancel{zpq}$$

$$y + zq + yp = x + zp + xq$$

$$zq - xq + yp - zp = x-y$$

$$q(z-x) + p(y-z) = x-y$$

(3) $z = f\left(\frac{x}{y}\right)$

$\rightarrow$ Diff w.r.t x & y

$\frac{\partial z}{\partial x} = f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)$ $\qquad \frac{\partial z}{\partial y} = f'\left(\frac{x}{y}\right) x\left(-\frac{1}{y^2}\right)$

~~Again~~ Dividing both

$$\frac{p}{q} = \frac{\cancel{f'\left(\frac{x}{y}\right)}\left(\frac{1}{y}\right)}{\cancel{f'\left(\frac{x}{y}\right)}\left(-\frac{x}{y^2}\right)}$$

$$\frac{p}{q} = -\frac{y}{x}$$

$$xp = -yq$$

$$\boxed{xp + yq = 0}$$

(4) $z = y + f(x + \ln y)$

Diff w.r.t x & y.

$\frac{\partial z}{\partial x} = 0 + f'(x+\ln y)(1+0)$ $\qquad \frac{\partial z}{\partial y} = 1 + f'(x+\ln y)\left(0 + \frac{1}{y}\right)$

$p = f'(x+\ln y)(1)$ — (1) $\qquad q = 1 + f'(x+\ln y)\frac{1}{y}$ — (2).

$\rightarrow$ Again Diff. w.r.t x & y.

Put $p = f'(x+\ln y)$ in eq. (2)

$$q = 1 + \frac{p}{y}$$

$$qy = y + p$$

$$-p + qy = y$$

$$y = qy - p$$

(5.) $z = f(x+6y) + g(x-6y)$

Partial Diff. w.r.t x & y.

$$\frac{\partial z}{\partial x} = f'(x+6y)(1+0) + g'(x-6y)(1)$$

$$p = f'(x+6y) + g'(x-6y) \quad \text{— (1)}$$

$$\frac{\partial z}{\partial y} = f'(x+6y)(6) + g'(x-6y)(-6)$$

$$q = 6[f'(x+6y) - g'(x-6y)] \quad \text{--- (2)}$$

Again Diff. w.r.t x & y.

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x+6y) + g''(x-6y) \quad \text{--- (3)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = 36[f''(x+6y) + g''(x-6y)] \quad \text{--- (4)}$$

$$\frac{\partial^2 z}{\partial x^2} = 36\,\frac{\partial^2 z}{\partial y^2}$$

Q: 6. From P.D.E from the following relation (By eliminating arbitary constant.)
Derive P.D.E by eliminating arbitary constant from the relation

(I.) $z = (x-a)^2 + (y-b)^2$

$\rightarrow$ D.P. w.r.t x & y.

$$\frac{\partial z}{\partial x} = 2(x-a) \qquad \frac{\partial z}{\partial y} = 2(y-b)$$

$$p = 2(x-a) \qquad q = 2(y-b)$$

$$(x-a) = p/2 \qquad (y-b) = q/2.$$

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$4z = p^2 + q^2$$ $\rightarrow$ which is Required P.D.E.

(2.) $z = (x-a)(y-b)$

$\rightarrow$ D.P. w.r.t x & y

$$\frac{\partial z}{\partial x} = (1-0)(y-b) \qquad \frac{\partial z}{\partial y} = (x-a)(1-0)$$

$$p = y-b \qquad q = x-a$$

$$z = pq$$

(3) $z = ax + by$

P.D. w.r.t x & y

$$\frac{\partial z}{\partial x} = a \qquad \frac{\partial z}{\partial y} = b$$

$$p = a \qquad q = b$$

$$\boxed{z = px + qy}$$

(4.) $z = ax + by + c.$

$$p = a \qquad q = b$$

$$\boxed{z = px + qy + c}$$

Q: 8. Solve the following P.D.E by Direct Integration method

$$\frac{\partial^2 z}{\partial x . \partial y} = 2x - 3y^2$$

→ Taking Integration both side w.r.t x

$$\int \frac{\partial^2 z}{\partial x . \partial y} . dx = \int 2x . dx + 3y^2 \int 1 . dx$$

$$\frac{\partial z}{\partial y} = x^2 + 3xy^2 + f(y)$$

→ Taking Integration both side w.r.t y.

$$\int \frac{\partial z}{\partial y} . dy = x^2 \int 1 . dy - 3x \int y^2 . dy$$

$$z = x^2 y - \frac{\cancel{3} xy^3}{\cancel{3}} + f(y) + g(x)$$

$$\boxed{z = x^2 y - xy^3} + f(y) + g(x)$$ — constant.

→ (2) $\frac{\partial^2 z}{\partial x.\partial y} = \sin x. \sin y$

→ Taking Integration w.r.t x.

$$\int \frac{\partial^2 z}{\partial x.\partial y} = \int \sin x \sin y . dx$$

$$\frac{\partial z}{\partial y} = \sin y(-\cos x) + f(y)$$

→ Taking Integration w.r.t y

$$\int \frac{\partial z}{\partial y} = -\cos x \int \sin y . dy + f(y)$$

$$\boxed{z = \cos x. \cos y + f(y) + g(x).}$$

(3) $\frac{\partial^2 y}{\partial x.\partial y} = \cos x. \cos y.$

→ w.r.t x

$$\frac{\partial z}{\partial y} = \cos y \sin x + f(x)$$

→ w.r.t y

$$\boxed{z = \cos y. \cos x + f(y) + g(x)}$$

Q : 9. Solve the P.D.E $p + q^2 = 1$.

→ compare with $P_p + Q_q = R$.

$P = 1 \quad Q = q^2 \quad R = 1$

→ Now. Auxillary eq.ⁿ

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{1} = \frac{dy}{q} = \frac{dz}{1}$$

→ Now,

$$dx = \frac{dy}{q}$$

$$q. dx = dy$$

Taking Integration

$$q \int 1. dx = \int 1. dy$$

$$\boxed{qx = y} \qquad \boxed{y - qx = 0}$$

→ Now,

$$\frac{dy}{q} = dz.$$

$$dy = q \cdot dz.$$

→ Taking Integration

$\int 1 . dy = q \int 1 . dz$

$\boxed{y = qz}$ $\qquad y - qz = 0$

→ The eq.^{n} is.

$F(u, v) = f(y - qx, y - qz) = 0$

(2.) Solve the P.D.E $p^2 + q^2 = npq.$

$P = p \qquad Q = q \qquad R = npq$

$A.E = \dfrac{dx}{p} = \dfrac{dy}{q} = \dfrac{dz}{npq}$

$p . dx = q . dy. \qquad np . dy = dz \qquad nq . dx = dz$

$\int p . dx = \int q . dy \qquad \int np . dy = \int . dz \qquad \int nq . dx = \int 1 . dz$

$\qquad np y = z \qquad nq x = z$

$px = qy$

$f(u, v) = f(px - qy, npy - z) = 0$

(3.) $p^3 + q^3 = 0.$

$P = p^2 \quad Q = -q^2 \quad R = 0.$

$\dfrac{dx}{p^2} = \dfrac{dy}{q^2} = 0$

$q^2 . dx = p^2 . dy$

$q^2 x + p^2 y = 0$

$f'(u, v) = 0$

$f(q^2 x - p^2 y, 0) = 0$