Date: 18/6/18

MON

Chapter - 5

Fourier Series & Fourier Integral

- → Introduction
- → Euler's Formula
- → Fourier series for even & odd function
- → Fourier Series for a function having period 2L.
- → Half Range Fourier Series.
- → Fourier Integral Theorem.

- Periodic function A function $f(x)$ is called periodic if it is defined for all x and their exist some positive number P $f(x+P) = f(x), \forall x$. that the number P is called a Period of $f(x)$.

NOTES

1. $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic function with period 2π and $\tan x$, $\cot x$ are functions with period π.

2. The functions $\sin nx$ & $\cos nx$ are periodic function with period $2\pi/n$.

3. $x, x^2, x^3, e^x, \log x$ are non-periodic function.

- Some Important Results

1. $\int_{c}^{c+2\pi} \cos nx \, dx = 0 \quad ; n \neq 0$

Date.:
MON TUE WED THU FRI SAT

2. $\int_{c}^{c+2\pi} \sin nx \, dx = 0$; $n \neq 0$

3. $\int_{c}^{c+2\pi} \cos mx \sin nx \, dx = 0$; $\forall m, n$

4. $\int_{c}^{c+2\pi} \sin mx \cos nx \, dx = 0$; $\forall m, n \; [m = n]$

5. $\int_{c}^{c+2\pi} \cos^2 nx \, dx = \pi$

6. $\int_{c}^{c+2\pi} \sin mx \sin nx \, dx = 0$; $m \neq n$
$= \pi$; $m = n \neq 0$

7. $\int_{c}^{c+2\pi} \cos mx \cos nx \, dx = 0$; $m \neq n$
$= \pi$; $m = n \neq 0$

- Libnitz Rule: Libnitz Rule is defined as $\int uv \, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$
where $u', u'', u''', \ldots$ are derivative of u
$v_1, v_2, v_3, \ldots$ are Integration of v.

- Fourier Series: The Fourier Series for a function with the interval $c < x < c+2\pi$ is given by trignometric series as
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

Date.:
MON TUE WED THU FRI SAT

where a_0, a_n, b_n are real constant & are called the fourier Coefficient of the Series

- Euler's Formula We have define fourier Series as $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$ -----> (1)

$$\Rightarrow F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx] \quad \text{-----> (1)}$$

To Find a_0: integrating eq^n (1) with respect to x by takeing limit $c \to c+2\pi$ we get

$$\Rightarrow \int_c^{c+2\pi} f(x)dx = \int_c^{c+2\pi} \frac{a_0}{2}dx + \sum_{n=1}^{\infty}\left[\int_c^{c+2\pi}(a_n \cos nx + b_n \sin nx)dx\right]$$

$$\Rightarrow \int_c^{c+2\pi} \frac{a_0}{2}dx + \sum_{n=1}^{\infty}\left[a_n\int_c^{c+2\pi}\cos nx\,dx + b_n\int_c^{c+2\pi}\sin nx\,dx\right]$$

$$\Rightarrow \frac{a_0}{2}[x]_c^{c+2\pi} + 0$$

$$\Rightarrow \frac{a_0}{2}[c+2\pi-c]$$

$$\Rightarrow \frac{a_0}{2}[2\pi]$$

$$\Rightarrow a_0\pi = \int_c^{c+2\pi} f(x)dx$$

$$\Rightarrow \boxed{a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx}$$

Scanned by CamScanner

Date.: ______ MON TUE WED THU FRI SAT

To find a_n

Multiply on both side of eq^n (1) by $\cos mx$ & then integrating w.r.t x between the limit $c \to c+2\pi$ we get,

$$\Rightarrow \int_c^{c+2\pi} f(x)\cos mx\,dx = \int_c^{c+2\pi} \frac{a_0}{2}\cos mx\,dx + \sum_{n=1}^{\infty} \left[\int_c^{c+2\pi} a_n \cos mx \cos nx\,dx + \int_c^{c+2\pi} b_n \cos mx \sin nx\,dx\right]$$

$$\Rightarrow \frac{a_0}{2}\int_c^{c+2\pi} \cos mx\,dx + \sum_{n=1}^{\infty}\left[a_n \int_c^{c+2\pi} \cos mx \cos nx\,dx + b_n \int_c^{c+2\pi} \cos mx \sin nx\,dx\right]$$

$\Rightarrow$ $\boxed{\text{For } m=n}$,

$$\int_c^{c+2\pi} \cos^2 nx\,dx = \pi, \quad \int_c^{c+2\pi} \sin nx \cos mx\,dx = 0$$

$$\Rightarrow \int_c^{c+2\pi} f(x)\cos mx\,dx = 0 + a_n\pi + 0$$

$$\Rightarrow \boxed{a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx} \quad [m=n]$$

Date: MON TUE WED THU FRI SAT

To find b_n

Multiply on both side of eq^n(1) by $\sin mx$ & then integrating w.r.t x between the limit $c \to c+2\pi$ we get,

$$\Rightarrow \int_c^{c+2\pi} f(x)\sin mx\,dx = \int_0^{c+2\pi} \frac{a_0}{2}\sin mx\,dx + \sum_{n=1}^{\infty}$$

$$\left[\int_c^{c+2\pi} a_n \cos nx \sin mx\,dx + \int_c^{c+2\pi} b_n \sin nx \sin mx\,dx\right]$$

$$\Rightarrow \frac{a_0}{2}\int_c^{c+2\pi}\sin mx\,dx + \sum_{n=1}^{\infty}\left[a_n\int_c^{c+2\pi}\cos nx \sin mx\,dx\right.$$

$$\left.+\ b_n\int_c^{c+2\pi}\sin mx \sin nx\,dx\right]$$

$\Rightarrow$ For $m=n$

$$\int_c^{c+2\pi}\sin^2 x\,dx = \pi,\quad \int_c^{c+2\pi}\sin mx\cos nx\,dx = 0$$

$$\Rightarrow \int_c^{c+2\pi} f(x)\sin mx\,dx = 0 + 0 + b_n\pi$$

$$\Rightarrow \boxed{b_n = \frac{1}{\pi}\int_c^{c+2\pi}\sin nx\,dx}$$

Date.:
MON TUE WED THU FRI SAT

Ex: 1 Obtain the fourier Series of $f(x) = \frac{\pi - x}{2}$ in the interval $(0, 2\pi)$ & also deduce $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

$\Rightarrow$ The fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

we know that

$$a_0 = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} \left(\frac{\pi - x}{2}\right) dx$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} [x]_0^{2\pi} - \frac{1}{2} \left[\frac{x^2}{2}\right]_0^{2\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} [2\pi - 0] - \frac{1}{4} [(2\pi)^2 - 0] \right]$$

$$\Rightarrow \frac{1}{\pi} [\pi^2 - \pi^2]$$

$$\Rightarrow 0$$

$$\therefore \boxed{a_0 = 0}$$

we know that

$$a_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \cos nx\,dx$$

Date.:
MON TUE WED THU FRI SAT

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi}{2}\cos nx\,dx - \frac{x}{2}\cos nx\,dx\right)$$

$$= \frac{1}{2\pi}\left[\pi\int_0^{2\pi}\cos nx\,dx - \int_0^{2\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[\pi\left[\frac{\sin nx}{n}\right]_0^{2\pi} - \left[x\left[\frac{\sin nx}{x}\right]_0^{2\pi} - \int_0^{2\pi}\left[\frac{\sin nx}{n}\right]_0 dx\right]\right.$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n}\left[\sin 2\pi n - \sin 0n\right] - \left[\frac{x}{n}\left[\sin 2\pi n - \sin 0n\right] - \frac{1}{n}\int_0^{2\pi}\sin nx\,dx\right]\right.$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n}\left[0-0\right] - \left[\frac{x}{n}\left[0-0\right] - \frac{1}{n}\left[-\frac{\cos nx}{n}\right]_0^{2\pi}\right]\right.$$

$$= \frac{1}{2\pi}\left[\frac{1}{n^2}\left[\cos 2\pi n - \cos 0n\right]\right]$$

$$= \frac{1}{2\pi n^2}\left[1 - 1\right]$$

$$= \boxed{a_n = 0}$$

We know that

$$b_n = \frac{1}{\pi} \int_{c}^{c+2\pi} s \cdot f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} \left(\frac{\pi - x}{2} \right) \sin nx \, dx$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} (\pi \sin nx \, dx - x \sin nx \, dx)$$

$$= \frac{1}{2\pi} \left[\pi \int_{0}^{2\pi} \sin nx \, dx - \int_{0}^{2\pi} x \sin nx \, dx \right]$$

$$= \frac{1}{2\pi} \left[\pi \left[\frac{-\cos nx}{n} \right]_{0}^{2\pi} - \left[x \left[\frac{-\cos nx}{n} \right]_{0}^{2\pi} - \int_{0}^{2\pi} \left[\frac{-\cos nx}{n} \right] dx \right] \right.$$

$$= \frac{1}{2\pi} \left[-\frac{\pi}{n} \left[\cos 2\pi n - \cos 0 n \right] - \left[-\frac{x}{n} \left[\cos 2\pi n - \cos 0 n \right] + \frac{1}{n} \int_{0}^{2\pi} \cos nx \, dx \right] \right.$$

$$= \frac{1}{2\pi} \left[-\frac{\pi}{n} [1 - 1] - \left[\frac{-x}{n} [1 - 1] + \frac{1}{n} \left[\frac{\sin nx}{n} \right]_{0}^{2\pi} \right] \right.$$

$$= \frac{1}{2\pi} \left[0 - \left[0 + \frac{1}{n^2} \left[\sin \right. \right. \right.$$

Date.:

MON TUE WED THU FRI SAT

Date.:

MON TUE WED THU FRI SAT

We know that

$$b_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{2\pi} \int_{c}^{c+2\pi} (\pi - x) \sin nx \, dx$$

$$= \frac{1}{2\pi} \left[(\pi - x)\frac{(-\cos nx)}{n} - \int (-1)\left(\frac{-\cos nx}{n}\right) dx \right]$$

$$= \frac{1}{2\pi} \left[\frac{x \cos nx}{n} - \frac{\pi \cos nx}{n} - \frac{\sin nx}{n^2} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[\frac{2\pi}{n} - \frac{\pi}{n} + \frac{\pi}{n} \right]$$

$$b_n = \frac{1}{n}.$$

Put the Values of a_0, a_n & b_n in eq^n (1) then we get the fourier Series.

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

Put $n = 1, 2, 3.$

$$\therefore f(x) = \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{x} +$$

$\frac{\pi - x}{2}$. Put $x = \frac{\pi}{2}$

$$f(x) = \frac{\pi - \pi/2}{2} = \frac{\pi}{4}$$

Scanned by CamScanner

MON TUE WED THU FRI SAT

$$f(x) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Ex: 2 Obtain Fourier Series expansion of $f(x) = e^{ax}$ in the interval $(0, 2\pi)$.

$a_0 =$ The fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \rightarrow (1)$$

We know that

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} e^{ax}\,dx$$

$$= \frac{1}{\pi} \left[\frac{e^{ax}}{a} \right]_0^{2\pi}$$

$$a_0 = \frac{1}{a\pi} \left[e^{a 2\pi a} - 1 \right]$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} e^{ax} \cos nx\,dx$$

Date:
MON TUE WED THU FRI SAT

* $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$

$a_n = \frac{1}{(a^2+n^2)\times\pi}\left[e^{ax}(a\cos nx + n\sin nx)\right]_0^{2\pi}$

$= \frac{1}{(a^2+n^2)\pi}\left[e^{2\pi a}(a\times 1 + 0) - 1\times a\right]$

$= a_n = \frac{1\times a}{(a^2+n^2)\pi}\left[e^{2\pi a} - 1\right]$

* $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$= \frac{1}{\pi}\int_0^{2\pi} e^{ax}\sin nx\,dx$

$= \frac{1}{\pi(a^2+b^2)}\left[e^{ax}(a\sin nx - n\cos nx)\right]_0^{2\pi}$

$= \frac{1}{(a^2+b^2)\pi}\left[-ne^{2\pi a} + n\right]$

$b_n = \frac{n}{(a^2+b^2)\pi}\left[1 - e^{2\pi a}\right]$

Scanned by CamScanner

Date.: MON TUE WED THU FRI SAT

Ex: 3 Obtain a fourier Series for $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $(0, 2\pi)$. Also deduce $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$

• Fourier Series For Even And Odd functions

★ Some useful formula for determining fourier co-efficient,

1. $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$; $f(x)$ is even fu^n

$= 0$; $f(x)$ is odd fu^n

2. $\int e^{ax}\sin(bx+c)dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$

3. $\int e^{ax}\cos(bx+c)dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx+c) + b\sin(bx+c)\right]$

To Check the function is even or odd,

Put $x = -x$ in $f(x)$

1. If $f(-x) = f(x)$ then the function is even

For given fun^n $b_n = 0$

Only find a_0 & a_n for even fun^n.

Date.:
MON TUE WED THU FRI SAT

2. If $f(-x) = -f(x)$ odd fun^n
For this $a_0 = a_n = 0$

Only find the Value of b_n.

NOTE 1. $x^2, x^4, x^6, \ldots$ are the even fun^n.
2. $\sin x, \tan x, \operatorname{cosec} x$ are odd fun^n.
3. $\cos x, \sec x, \cot x$ are even fun^n.
4. $x, x^3, x^5, \ldots$ are odd fun^n.

Ex: 1 Find the fourier Series of the function
$f(x) = x^2, \quad -\pi < x < \pi$

$\Rightarrow$ Here given that $f(x) = x^2$
put $x = -x$
$f(-x) = (-x)^2$
$= x^2$.
$\therefore$ The given fun^n is even fun^n.
$\therefore \boxed{b_n = 0}$

Only we have to find a_0 & a_n of x^2
Now the fourier Series is given by
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad \text{------}\rightarrow (1)$$

$$a_0 = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

Scanned by CamScanner

Date.:
MON TUE WED THU FRI SAT

$$= \frac{1}{\pi} \times 2 \int_0^{\pi} x^2 dx$$

$$= \frac{2}{\pi} \left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^3}{3} - 0\right]$$

$$= \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx \, dx$$

$$= \frac{1}{\pi} \times 2 \int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi} \left[x^2 \left(\frac{\sin nx}{n}\right) - 2x \left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_0^{\pi}$$

Date.:
MON TUE WED THU FRI SAT

$$= \frac{2}{\pi}\left[\left[0 + \frac{2x}{n^2}(-1)^n - 0\right] - \left[0 + 2\times 0 - 0\right]\right]$$

$$\Rightarrow \frac{4}{n^2}(-1)^n$$

Now, we have

$$f(x) = \frac{1}{3}\times\pi^2 + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}(-1)^n \cos nx\right]$$

Ex: 2 Find the fourier Series of the function
$f(x) = x^3$ in $-\pi < x < \pi$

$\Rightarrow$ Here given that $f(x) = x^3$.
put $x = -x$
$f(-x) = (-x)^3$
$= -x^3$

$\therefore$ The given fu^n is odd.
$\therefore$ $a_0 = 0$, $a_n = 0$

Only we have to find b_n
Now the fourier Series is given by

$$f(x) = \cancel{b_n} \sum_{n=1}^{\infty}\left[b_n \sin nx\right] \quad \text{----}\rightarrow (1)$$

$$b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin nx\,dx$$

Date.: MON TUE WED THU FRI SAT

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[x^3\left(\frac{-\cos nx}{n}\right) - 3x^2\left(\frac{-\sin nx}{n^2}\right) + 6x\left(\frac{+\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left[\pi^3\left(-\frac{(-1)^n}{n}\right) - 3(\pi)^2\left(\frac{-0}{n^2}\right) + 6\pi\left(\frac{(-1)^n}{n^3}\right) - 6\times 0\right] - \left[\pi^3\left(\frac{+(-1)^n}{n}\right) - 0 + \frac{(-6\pi)(-1)^n}{n^3} - 0\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi^3}{n}(-1)^n + \frac{6\pi}{n^3}(-1)^n + \frac{\pi^3}{n}(-1)^n + \frac{6\pi}{n^3}(-1)^n\right]$$

$$= \frac{2}{n}(-1)^n\left[-\pi^2 + \frac{6}{n^2}\right]$$

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{2}{n}(-1)^n\left[-\pi^2 + \frac{6}{n^2}\right]\sin nx\right]$$

Scanned by CamScanner

Date.:
MON TUE WED THU FRI SAT

Ex: 3 Find the fourier Series of $f(x) = x + |x|$ in the interval $-\pi < x < \pi$

⇒ Fourier Series for $f(x) = x + |x|$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow ①$$

Now, we have to check $f(x)$ is even or odd.

$f(x) = x + |x|$

$f(-x) = -x + |-x|$

$f(-x) = -x + |x|$

∴ The given fu^n is niether even nor odd. So we have to find a_0, a_n & b_n.

$$a_0 = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + |x|)\,dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} + \int_{-\pi}^{\pi} x\,dx + \int_{-\pi}^{\pi} |x|\,dx \right]$$

$$= \frac{1}{\pi} \left[0 + 2 \int_{0}^{\pi} |x|\,dx \right]$$

(Here x is Odd fun^m ∴ $\int_{-\pi} x\,dx = 0$) and.

$f(x) = |x|$

$f(-x) = |-x| = |x|$

Date.:
MON TUE WED THU FRI SAT

$= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx \rightarrow$ even function

$a_0 = \frac{2}{\pi} \int_{0}^{\pi} |x| dx$

$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$

$= \frac{1}{\pi} [\pi^2 - 0]$

$\boxed{a_0 = \pi}$

$a_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \cos nx dx$

$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + |x|) \cos nx dx$

$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \cos nx dx + \int_{-\pi}^{\pi} |x| \cos nx dx \right]$

= Here $f(x) = x \cos nx$

$f(-x) = -x \cos n(-x)$

$= -x \cos nx$

$\therefore x \cos nx$ is an odd fu^n.

$\int_{-\pi}^{\pi} x \cos nx dx = 0$

For $f(x) = |x| \cos nx$

$f(-x) = |-x| \cos n(-x)$

$= |x| \cos nx$

$\therefore |x| \cos nx$ is an even fu^n.

Date.: ____
MON TUE WED THU FRI SAT

$$a_n = \frac{2}{\pi}\int_0^{\pi} |x| \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\frac{|x| \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x \sin \pi n}{n} + \frac{\cos n\pi}{n^2} \right]$$

$$= \frac{2}{\pi}\left[\left[0 + \frac{(-1)^n}{n^2} \right] - \left[0 + \frac{1}{n^2} \right] \right]$$

$$\boxed{a_n = +\frac{2}{\pi n^2}\left[(-1)^n - 1 \right]}$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + |x|) \sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{\pi}^{\pi} x \sin nx \, dx + \int_{-\pi}^{\pi} |x| \sin nx \, dx \right]$$

Here $f(x) = x \sin nx$

$f(-x) = (-x) \sin n(-x)$

$= x \sin nx$

$\therefore x \sin nx$ is an even fu^n

$f(x) = |x| \sin nx$

$f(-x) = |-x| \sin n(-x)$

$= -|x| \sin nx$

$\therefore |x| \sin nx$ is an odd fu^n

Date.:
MON TUE WED THU FRI SAT

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi(-1)^n}{n} + 0 + 0 - 0\right]$$

$$= \frac{-2(-1)^n}{n}$$

$$\boxed{b_n = \frac{-2(-1)^n}{n}} = \frac{2(-1)^{n+1}}{n}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{\pi n^2}[(-1)^n - 1]\cos nx + \frac{2(-1)^{n+1}\sin nx}{n}\right)$$

Fourier Series for Discontinuous Function

Ex:1 Find the fourier series of f(x) defined by $f(x) = \begin{cases} 0 ; & -1 < x < 0 \\ 1 ; & 0 < x < 1 \end{cases}$

= The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx] \dashrightarrow (1)$$

Scanned by CamScanner

Date.: ____

MON TUE WED THU FRI SAT

To find $a_0 = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\,dx$

$= \frac{1}{\pi}\left[\int_{-1}^{0} 0\,dx + \int_{0}^{1} 1\,dx\right]$

$= \frac{1}{\pi}\left[0 + [x]_0^1\right]$

$a_0 = \frac{1}{\pi}.$

To find $a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cos nx\,dx$

$= \frac{1}{\pi}\left[\int_{-1}^{0} 0\times\cos nx\,dx + \int_{0}^{1} 1\cos nx\,dx\right]$

$= \frac{1}{\pi}\left[0 + \left[\frac{\sin nx}{n}\right]_0^1\right]$

$a_n = \frac{1}{\pi}\left[\frac{\sin n(1)}{n}\right]$

$a_n = \frac{\sin n}{\pi n}$

To find $b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin nx\,dx$

$= \frac{1}{\pi}\left[\int_{-1}^{0} 0\times\sin nx\,dx + \int_{0}^{1} 1\sin nx\,dx\right]$

$= \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^1$

Date.: ______
MON TUE WED THU FRI SAT

$$b_n = \frac{-\cos n + 1}{\pi n}$$

$$f(x) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \left[\frac{\sin n}{\pi n} \cos nx + \frac{(-\cos n + 1)\sin nx}{\pi n} \right]$$

$\pi = 1$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{\sin n}{n} \cos nx + \frac{(1 - \cos n)\sin nx}{n} \right]$$

Find the fourier Series for f(x) defined by $f(x) = \begin{cases} -K ; & -\pi < x < 0 \\ K ; & 0 < x < \pi \end{cases}$

Hence deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

$$\boxed{b_n = \frac{-\cos n + 1}{\pi n}}$$

$$f(x) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \left[\frac{\sin n}{\pi n} \cos nx + \frac{(-\cos n + 1)\sin nx}{\pi n} \right]$$

$\pi = 1$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{\sin n}{n} \cos nx + \frac{(1-\cos n)\sin nx}{n} \right]$$

Find the fourier Series for $f(x)$ defined by $f(x) = \begin{cases} -k \; ; & -\pi < x < 0 \\ k \; ; & 0 < x < \pi \end{cases}$

Hence deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad \dots\dots \to (1)$$

To find a_0

$$a_0 = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x)\, dx$$

Date: MON TUE WED THU FRI SAT

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right]$$

$$= \frac{1}{\pi}\left[-\int_{-\pi}^{0} Kdx + \int_{0}^{\pi} Kdx\right]$$

$$= \frac{1}{\pi}\left[-K[x]_{-\pi}^{0} + K[x]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-K[0+\pi] + K[\pi - 0]\right]$$

$$= \frac{1}{\pi}\left[-K\pi + K\pi\right]$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[-K\int_{-\pi}^{0}\cos nx\,dx + K\int_{0}^{\pi}\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-K\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + K\left[\frac{\sin nx}{n}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{K}{n}[0] + \frac{K}{n}[\sin n\pi - \sin 0]\right]$$

$$= \frac{1}{\pi}[0]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin nx\,dx$$

Date: MON TUE WED THU FRI SAT

$$= \frac{1}{\pi}\left[-K\int_{-\pi}^{0}\sin nx\,dx + K\int_{0}^{\pi}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[K\left[\frac{\cos nx}{n}\right]_{-\pi}^{0} - K\left[\frac{\cos nx}{n}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[K\left[\frac{1}{n} - \frac{(-1)^n}{n}\right] - K\left[\frac{(-1)^n}{n} - \frac{1}{n}\right]\right]$$

$$= \frac{1}{\pi}K\left[\frac{1}{n} - \frac{(-1)^n}{n} - \frac{(-1)^n}{n} + \frac{1}{n}\right]$$

$$\boxed{b_n = \frac{2K}{n\pi}\left[1 - (-1)^n\right]}$$

$$b_n = \begin{cases} \frac{4K}{n\pi}, & n \text{ is odd } n = 1, 3, 5, \ldots \\ 0, & n \text{ is even } n = 2, 4, 6, 8, \ldots \end{cases}$$

Now from eq^n $\to$ (1)

We have,

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{4K}{n\pi}\sin nx\right]$$

Let $x = \pi/2$, $n = 1, 3, 5, \ldots$

$$f\left(\frac{\pi}{2}\right) = \frac{4K}{\pi}\left[\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \cdots\right]$$

$$0 < x < \pi \Rightarrow K = f(\pi/2)$$

$$\cancel{K} = \frac{4}{\pi}\cancel{K}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

$$\pi/4 = \left[1 - 1/3 + 1/5 - 1/7 + \cdots\right]$$

Change Of Interval

Fourier Series with period $P = 2l$

The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right].$$

where

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

where $n = 1, 2, 3, \dots$

Corollary:-1 put $C = 0$ in above eq^n. then we get the interval

$0 \le x \le 2l$.

$$a_0 = \frac{1}{l}\int_{0}^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{0}^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

$$b_n = \frac{1}{l}\int_{0}^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

where $n = 1, 2, 3, \dots$

Corollary:-2 Put $C = -l$ $\quad -l \le x \le l$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

where $n = 1, 2, 3, \dots$

Ex:1 Find a fourier series with period 3 to represent $f(x) = 2x - x^2$ in the range $0 < x < 3$

The fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right] \quad \text{------} \to (1)$$

Here range is $0 < x < 3$

$2l = 3$

$l = \frac{3}{2}$

$\therefore$ The fourier Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2n\pi x}{3}\right) + b_n \sin\left(\frac{2\pi n x}{3}\right)\right] \quad \text{------}\to (2)$$

To find

$$a_0 = \frac{1}{l} \int_0^3 (2x - x^2)\,dx$$

$$= \frac{2}{3} \int_0^3 (2x\,dx - x^2\,dx)$$

$$= \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3$$

$$= \frac{2}{3}\left[\left[9 - \frac{27}{3}\right] - \left[0 - 0\right]\right]$$

$$= \boxed{a_0 = 0}$$

To find a_n

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$a_n = \frac{2}{3}\int_0^3 (2x - x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[\int_0^3 2x\cos\left(\frac{2n\pi x}{3}\right)dx - \int_0^3 x^2\cos\left(\frac{2n\pi x}{3}\right)dx\right]$$

$$= \frac{2}{3}\left[(2x - x^2)\sin\left(\frac{2n\pi x}{3}\right)\left(\frac{3}{2n\pi}\right) - (2-2x)\left(-\cos\left(\frac{2n\pi x}{3}\right)\left(\frac{9}{4n^2\pi^2}\right)\right) + (0-2)\left(-\sin\left(\frac{2n\pi x}{3}\right)\left(\frac{27}{8n^3\pi^3}\right)\right)\right]_0^3$$

$$= -\frac{2}{3}\left[(2-6)\left(-\cos\left(\frac{6n\pi}{3}\right)\left(\frac{9}{4n^2\pi^2}\right)\right)\right]_0^3$$

$$= -\frac{2}{3}\left[\left[(2-6)\cdot -\frac{9}{4n^2\pi^2}\right] - \left[2 \times -\frac{9}{4n^2\pi^2}\right]\right]$$

$$= -\frac{2}{3}\left[+\frac{9}{n^2\pi^2} + \frac{9}{2n^2\pi^2}\right]$$

$$\Rightarrow -\frac{\cancel{2}}{\cancel{3}}\left[\frac{\cancel{3} \times 9}{\cancel{2}n^2\pi^2}\right]$$

$$= \frac{-9}{n^2\pi^2}$$

$$\boxed{a_n = \frac{-9}{\pi^2 n^2}}$$

To find b_n

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{2}{3}\int_0^3 (2x-x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(-\cos\left(\frac{2n\pi x}{3}\right)\left(\frac{3}{2n\pi x}\right)\right)\right.$$

$$-(2-2x)\left(-\sin\left(\frac{2n\pi x}{3}\right)\left(\frac{9}{4n^2\pi^2}\right)\right)$$

$$\left.+(0-2)\left(\cos\left(\frac{2n\pi x}{3}\right)\left(\frac{27}{8n^3\pi^3}\right)\right)\right]_0^3$$

$$= \frac{2}{3}\left[\left[(6-9)\left(-1\times\frac{3}{2n\pi}\right)+\right.\right.$$

$$\left.(-2)\left(1\times\frac{27}{8n^3\pi^3}\right)\right]-\left[(0)+(-2)\left(1\times\frac{27}{8n^3\pi^3}\right)\right]$$

$$= \frac{2}{3}\left[\frac{9}{2n\pi} - \frac{54}{8n^3\pi^3} + \frac{54}{8n^3\pi^3}\right]$$

$$\Rightarrow \boxed{b_n = \frac{3}{n\pi}}$$

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{-9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{9}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)\right]$$

Ex: 2 Find the fourier series for $f(x) = x^2 - 2$ in the interval $-2 \le x \le 2$

$\Rightarrow$ The fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right] \quad \text{---}\rightarrow (1).$$

Here $l = 2$.

Here $f(x) = x^2 - 2$

$f(-x) = (-x)^2 - 2$

$f(-x) = x^2 - 2$

$\therefore$ The given function is even.

Even fun^n $b_n = 0$

Only we have to find a_0 & a_n

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$= \frac{1}{2}\int_{-2}^{2}(x^2-2)dx$$

$$= \frac{1}{2}\times 2\int_0^2 (x^2dx - 2dx)$$

$\Rightarrow \left[\frac{x^3}{3} - 2x\right]_0^2$

$\Rightarrow \left[\frac{8}{3} - 4\right]$

$\Rightarrow \frac{8-12}{3}$

$\Rightarrow \boxed{a_0 = \frac{-4}{3}}$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{2}\int_{-2}^{2} (x^2-2)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow \int_0^2 \left(x^2-2\right)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow \left[(x^2-2)\left[\sin\left(\frac{n\pi x}{2}\right)\times\frac{2}{n\pi}\right] - (2x)\left[\left(-\cos\left(\frac{n\pi x}{2}\right)\times\frac{4}{n^2\pi^2}\right.\right.\right. + 2\left(-\sin\left(\frac{n\pi x}{2}\right)\times \left.\frac{8}{n^3\pi^3}\right]_0^2$$

$$\Rightarrow \left[+2\times 2\left[(-1)^n + 1\right]\times\frac{4}{n^2\pi^2}\right]$$

$$\Rightarrow \frac{16}{n^2\pi^2}\left[(-1)^n + 1\right]$$

$$\boxed{a_n = \frac{16}{n^2\pi^2}\left[(-1)^n + 1\right]}$$

$$f(x) = -\frac{2}{3} + \sum_{n=1}^{\infty}\left[\frac{16}{n^2\pi^2}\left[(-1)^n+1\right]\cos\left(\frac{n\pi x}{2}\right)\right]$$

Half Range Fourier Series Of Cosine Funⁿ.

For the cosine function find the value of a_0 & a_n

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

∴ The fourier series is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right)$$

Half Range Fourier Series of Sine Fun^n

For the sine function find the value of b_n

$$a_0 = a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$\therefore$ The fourier Series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

NOTE > Half range fourier series expansion of f(x) is valid only in the interval (0, l) not outside this interval.

Ex:1 Obtain Cosine Series for $f(x) = e^x$ in $(0, l)$

Here $f(x) = e^x$

we have to find cosine $\therefore$ a_0 & a_n.

$\therefore b_n = 0$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l e^x\,dx$$

$$= \frac{2}{l}\left[e^x\right]_0^l$$

$$\boxed{a_0 = \frac{2}{l}\left[e^l - 1\right]}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$a = 1,\quad b = \frac{n\pi}{l}$$

$$a_n = \frac{2}{l}\left[\frac{e^x}{1+\frac{n^2\pi^2}{l^2}}\left[\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right]\right]_0^l$$

$$= \frac{2l}{l^2+n^2\pi^2}\left[e^x\left[\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right]\right]_0^l$$

$$= \frac{2l}{l^2+n^2\pi^2}\left[e^l\left[(-1)^n\right] - e^0\left[1\right]\right]$$

$$\boxed{a_n = \frac{2l\left[e^l(-1)^n - 1\right]}{l^2+n^2\pi^2}}$$

$$f(x) = \frac{1}{l}\left[e^l - 1\right] + \sum_{n=1}^{\infty}\frac{2l\left[e^l(-1)^n - 1\right]}{l^2+n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)$$

Scanned by CamScanner

Ex:2 Obtain the Sine Series of $f(x) = 2x$ in $[-1 < x < 1$

Here $f(x) = 2x$

we have to find b_n.

$a_0 = a_n = 0$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$l = 1$.

$$b_n = 2\int_0^1 2x\sin(n\pi x)dx.$$

$$= 2\left[2x\left[-\cos(n\pi x)\times\frac{1}{n\pi}\right] - 2\left[-\sin(n\pi x)\times\frac{1}{n^2\pi^2}\right]_0^1\right.$$

$$= 2\left[-2\times\left[\frac{(-1)^n}{n\pi}\right] - 0\right]$$

$$\boxed{b_n = \frac{-4(-1)^n}{n\pi}}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{l}\right)$$

Ex:3 $f(x) = x^2 + x$ with the interval $-1 < x < 1$

Here $f(x) = x^2 + x$

$f(-x) = (-x)^2 + (-x)$

$= x^2 - x$

$\therefore$ $f(x)$ is neither even nor odd

To find a_0

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$= 2\int_0^1 (x^2+x)dx$$

$$= 2\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1$$

$$= 2\left[\frac{1}{3} + \frac{1}{2}\right]$$

$$= 2\left[\frac{2+3}{6}\right]$$

$$a_0 = \frac{5}{3}$$

To find a_n

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

$$a_n = 2\int_0^1 (x^2+x)\cos(n\pi x)\,dx$$

$$= 2\left[(x^2+x)\left(\frac{\sin(n\pi x)}{n\pi}\right) + (2x+1)\frac{\cos n\pi x}{(n\pi)^2} + 2\left(\frac{\sin(n\pi x)}{(n\pi)^3}\right)\right]_0^1$$

$$= 2\left[2\left(\frac{\sin n\pi}{n\pi}\right) + 3\left(\frac{\cos n\pi}{(n\pi)^2}\right) + 2\left(\frac{\sin n\pi}{(n\pi)^3}\right)\right] - \left[0 + \frac{\cos 0}{(n\pi)^2}\right]$$

$$= 2\left[\frac{3(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]$$

$$\therefore a_n = \frac{2}{n^2\pi^2}\left[3(-1)^n - 1\right]$$

- To find b_n,

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= 2\int_0^1 (x^2+x)\sin(n\pi x)\,dx$$

$$= 2\left[(x^2+x)\left(-\frac{\cos(n\pi x)}{n\pi}\right) + (2x+1)\left[\frac{\sin(n\pi x)}{(n\pi)^2}\right] - 2\left[\frac{\cos n\pi x}{(n\pi)^3}\right]\right]_0^1$$

$$= 2\left[-2\left(\frac{\cos n\pi}{n\pi}\right) + 3\left(\frac{\sin n\pi}{(n\pi)^2}\right) - 2\left(\frac{\cos n\pi}{n\pi}\right) - \left(0 - 2\left(\frac{\cos 0}{(n\pi)^3}\right)\right)\right]$$

$$= 2\left[\frac{-2(-1)^n}{n\pi} - \frac{2(-1)^n}{n\pi} + \frac{2}{(n\pi)^3}\right]$$

$$= 2\left[\frac{-4(-1)^n}{n\pi} + \frac{2}{(n\pi)^3}\right]$$

$$b_n = \frac{2}{n\pi}\left[\frac{2}{n^2\pi^2} - 4(-1)^n\right]$$

$$\therefore f(x) = \frac{5}{3} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}\left[3(-1)^n - 1\right]\cos n\pi x + \frac{2}{n\pi}\left[2/n^2\pi^2 - 4(-1)^n\right]\sin n\pi x\right]$$

Find the fourier Series for foll. function. $f(x) = \begin{cases} -\pi & ; -\pi < x < 0 \\ x & ; 0 < x < \pi \end{cases}$

Here $f(x) = \begin{cases} -\pi & ; -\pi < x < 0 \\ x & ; 0 < x < \pi \end{cases}$

The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad \text{---} \to (1)$$

To find a_0

$$a_0 = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \left[-\pi \int_{-\pi}^{0} dx + \int_{0}^{\pi} x\,dx \right]$$

$$= \frac{1}{\pi} \left[-\pi [x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[-\pi [0+\pi] + \left[\frac{\pi^2}{2} - 0\right] \right]$$

$$= \frac{1}{\pi} \left[-\pi^2 + \frac{\pi^2}{2} \right]$$

$$= \frac{1}{\cancel{\pi}} \times -\frac{\pi^{\cancel{2}}}{2}$$

$$a_0 = -\frac{\pi}{2}$$

To find a_n.

$$a_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi} \left[-\pi \int_{-\pi}^{0} \cos nx\,dx + \int_{0}^{\pi} x \cos nx\,dx \right]$$

$$= \frac{1}{\pi} \left[-\pi \left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[x \left[\frac{\sin nx}{n}\right] - \left[\frac{-\cos nx}{n^2}\right] \right]_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[0 + 0 + \frac{1}{n^2} [(-1)^n - 1] \right]$$

$$= \frac{1}{n^2 \pi} [(-1)^n - 1]$$

To find b_n

$$b_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \sin nx\,dx$$

$$= \frac{1}{\pi} \left[-\pi \int_{-\pi}^{0} \sin nx\,dx + \int_{0}^{\pi} x \sin nx\,dx \right]$$

$$= \frac{1}{\pi} \left[-\pi \left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \left[x \left[\frac{-\cos nx}{n}\right]_0^{\pi} - \left[\frac{-\sin nx}{n^2}\right]_0^{\pi} \right] \right]$$

$= \frac{1}{\pi}\left[\cdot \frac{\pi}{n}[1-(-1)^n] + \frac{(-\pi)}{n}[(-1)^n - 0]\right]$

$= \frac{1}{\pi}\left[\frac{\pi}{n}[(1-(-1)^n] + \frac{\pi}{n}[-(-1)^n]\right]$

$\Rightarrow \frac{1}{\pi}\left\{\frac{\pi}{n} - \frac{2(-1)^n \times \pi}{n}\right]$

$\Rightarrow \frac{1}{n}[1-2(-1)^n]$

$$b_n = \frac{1}{n}[1-2(-1)^n]$$

2. $f(x) = \begin{cases} -x^2 ; & -\pi \le x \le 0 \\ x^2 ; & 0 \le x \le \pi \end{cases}$

Here $f(x) = \begin{cases} -x^2 ; & -\pi \le x \le 0 \\ x^2 ; & 0 \le x \le \pi \end{cases}$

To find a_0

$a_0 = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\,dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x^2 dx + \int_{0}^{\pi} x^2 dx\right]$

$= \frac{1}{\pi}\left[-\left[\frac{x^3}{3}\right]_{-\pi}^{0} + \left[\frac{x^3}{3}\right]_{0}^{\pi}\right]$

$a_0 = \frac{1}{\pi}\left[-\left[0 + \frac{\pi^3}{3}\right] + \left[\frac{\pi^3}{3} - 0\right]\right]$

$= \frac{1}{\pi}\left[-\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$

$\boxed{a_0 = 0}$

To find a_n

$a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cos nx\,dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x^2\cos nx\,dx + \int_{0}^{\pi} x^2\cos nx\,dx\right]$

$= \frac{1}{\pi}\left[-\int_{0}^{-\pi} x^2\cos nx\,dx + \int_{0}^{\pi} x^2\cos nx\,dx\right]$

$= \frac{1}{\pi}\left[\left[-\left[x^2\left[\frac{\sin nx}{n}\right] - 2x\left[-\frac{\cos nx}{n^2}\right] + 2\left[-\frac{\sin nx}{n^3}\right]\right]_{0}^{-\pi}\right.\right.$

$\left.\left. + \left[x^2\left[\frac{\sin nx}{n}\right] - 2x\left[-\frac{\cos nx}{n^2}\right] + 2\left[-\frac{\sin nx}{n^3}\right]\right]_{0}^{\pi}\right]$

$= \frac{1}{\pi}\left[-\frac{2x}{n^2}[(-1)^n - 1] + \frac{2x}{n^2}[(-1)^n - 1]\right]$

$\boxed{a_n = 0}$

To find b_n

$$b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-\int_{-\pi}^{0} x^2\sin nx\,dx + \int_{0}^{\pi} x^2 \sin nx\,dx\right]$$

$$\Rightarrow \frac{1}{\pi}\left[-\left[x^2\left[-\frac{\cos nx}{n}\right] + 2x\left[\frac{\sin nx}{n^2}\right] + 2\left[\frac{\cos nx}{n^3}\right]\right]_{-\pi}^{0}\right.$$

$$\left. + \left[x^2\left[-\frac{\cos nx}{n}\right] + 2x\left[\frac{\sin nx}{n^2}\right] + 2\left[\frac{\cos nx}{n^3}\right]\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\left[0+0+2\left[\frac{+1}{n^3}\right] + \pi^2\left[+\frac{(-1)^n}{n}\right] + 2\pi\times 0\right.\right.$$

$$\left. + 2\frac{(-1)^n}{n^3}\right] + \left[-\pi^2\left(+\frac{(-1)^n}{n}\right) + 2\frac{(-1)^n}{n^3} - 0\right.$$

$$\left. + 0 + \frac{2}{n^3}\right]$$

$$= \frac{1}{\pi}\left[+\frac{2}{n^3} + \frac{\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} + \frac{\pi^2(-1)^n}{n}\right.$$
$$\left. + \frac{2(-1)^n}{n^3} + \frac{2}{n^3}\right]$$

$$= \quad b_n = \frac{1}{\pi}\left[\frac{-2\pi^2(-1)^n}{n} + \frac{4}{n^3}\left[(-1)^n - 1\right]\right]$$

$$b_n = \frac{2}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2}{n^3}\left[(-1)^n - 1\right]\right]$$

3. $f(x) = \begin{cases} 1 + \frac{4x}{3} & ; -\frac{3}{2} < x \le 0 \\ 1 - \frac{4x}{3} & ; 0 \le x < \frac{3}{2} \end{cases}$

$\Rightarrow$ Here $f(x) = \begin{cases} 1 + \frac{4x}{3} & ; -\frac{3}{2} < x \le 0 \\ 1 - \frac{4x}{3} & ; 0 \le x < \frac{3}{2} \end{cases}$

To find a_0

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{2}{3}\left[\int_{-3/2}^{0}(1 + 4x/3)\,dx + \int_{0}^{3/2}(1 - 4x/3)\,dx\right]$$

$$= \frac{2}{3}\left[\left[x + \frac{4}{3}\left[\frac{x^3}{3}\right]\right]_{-3/2}^{0} + \left[x - \frac{4}{3}\left[\frac{x^3}{3}\right]\right]_{0}^{3/2}\right]$$

$$= \frac{2}{3}\left[\left[0 - \frac{3}{2} + \frac{4}{9}\times\frac{+27}{8}\right] + \left[\frac{3}{2} - \frac{4}{9}\times\frac{27}{8}\right]\right]$$

$$= \frac{2}{3}\left[-\frac{3}{2} + \frac{3}{2} + \frac{3}{2} - \frac{3}{2}\right]$$

$$\boxed{a_0 = 0}$$

Date: MON TUE WED THU FRI SAT

To find a_n.

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos nx\,dx$$

$$= \frac{2}{3}\left[\int_{-\frac{3}{2}}^{0}\left(1+\frac{4x}{3}\right)\cos nx + \int_{0}^{3/2}\left(1+\frac{4x}{3}\right)\cos nx\right]$$

$$= \frac{2}{3}\left[\left[\left(1+\frac{4x}{3}\right)\left[\frac{\sin nx}{n}\right] + \left(x^{0}+\frac{4x^{2}}{3}\right)\frac{\cos nx}{n^2}\right]_{-\frac{3}{2}}^{0}\right.$$

$$\left. + \left[\left(1+\frac{4x}{3}\right)\left[\frac{\sin nx}{n}\right] + \left(0+\frac{4}{3}\right)\left[-\frac{\cos nx}{n^2}\right]\right]_{0}^{3/2}\right]$$

$$= \frac{2}{3}\left[0 + \frac{4}{3n^2} - (1-2)\times\frac{1}{n}\sin\left(-\frac{3n}{2}\right) + \frac{4}{3n^2}\cos\frac{3n}{2}\right.$$

$$\frac{(1+2)}{n}\sin\left(\frac{3n}{2}\right) - \frac{4}{3n^2}\left(\cos\frac{3n}{2}\right) +$$

$$\left. 0 + \frac{4}{3n^2}\right]$$

$$= \frac{2}{3}\left[\frac{8}{3n^2} - \frac{1}{n}\times(-1)^n + \frac{4}{3n^2}\times 0 + \frac{3}{n}(-1)^n\right]$$

Date: MON TUE WED THU FRI SAT

Date:
MON TUE WED THU FRI SAT

Fourier Integral

Fourier integral is defined as

$$f(x)=\int_{0}^{\infty}[A(\lambda)\cos\lambda x+B(\lambda)\sin\lambda x]\,d\lambda,$$

$$-\infty<x<\infty$$

where,

$$A(\lambda)=\frac{1}{\pi}\int_{-\infty}^{\infty}\cancel{A(x)}\,f(x)\cos\lambda x\,dx$$

$$B(\lambda)=\frac{1}{\pi}\int_{-\infty}^{\infty}\cancel{B(x)}\,f(x)\sin\lambda x\,dx$$

Fourier Cosine Integral

If $f(x)$ is even fun^n then
$B(\lambda)=0$, so the fourier cosine integral is defined as

$$f(x)=\int_{0}^{\infty}A(\lambda)\cos\lambda x\,d\lambda$$

where,

$$A(\lambda)=\frac{2}{\pi}\int_{0}^{\infty}f(x)\cos\lambda x\,dx$$

Fourier Sine Integral

If $f(x)$ is an odd fun^n then
$A(\lambda)=0$, so the fourier sine integral is defined as

$$f(x)=\int_{0}^{\infty}B(\lambda)\sin\lambda x\,dx$$

where,
$$B(\lambda)=\frac{2}{\pi}\int_{0}^{\infty}f(x)\sin\lambda x\,dx$$

Date:
MON TUE WED THU FRI SAT

Ex:1 Find fourier cosine & sine integral of $f(x)=e^{-kx}$ $(x>0, k>0)$

Hence, prove that $\int_{0}^{\infty}\frac{\cos\lambda x}{k^2+\lambda^2}\,d\lambda=\frac{\pi}{2k}e^{-kx}$

$$\int_{0}^{\infty}\frac{\lambda\sin\lambda x}{k^2+\lambda^2}\,d\lambda=\frac{\pi}{2}e^{-kx}$$

$$f(x)=\int_{0}^{\infty}A(\lambda)\cos\lambda x\,d\lambda \quad \rightarrow (1)$$

where,

$$A(\lambda)=\frac{2}{\pi}\int_{0}^{\infty}f(x)\cos\lambda x\,dx$$

$$=\frac{2}{\lambda}\int_{0}^{\infty}e^{-kx}\cos\lambda x\,dx$$

$$=\frac{e^{ax}}{[a^2+b^2]}[a\cos bx+b\sin bx]\quad\left[\begin{array}{l}\cos\infty=0\\ \because\sin\infty=0\\ e^{\infty}=0\end{array}\right]$$

$$a=-k,\ b=\lambda$$

$$A(\lambda)=\frac{2}{\pi}\left[\frac{e^{-kx}}{k^2+\lambda^2}(-k\cos\lambda x+\lambda\sin\lambda x)\right]_0^{\infty}$$

$$=\frac{2}{\pi}\left[\frac{e^{-kx}}{k^2+\lambda^2}(-k\cos\lambda x+\lambda\sin\lambda x)\right]_0^{\infty}$$

$$=\frac{2}{\pi}\left[0-\frac{1}{k^2+\lambda^2}(-k)\right]$$

$$A(\lambda)=\frac{2}{\lambda}\left[\frac{k}{k^2+\lambda^2}\right]$$

$$f(x)=\int_{0}^{\infty}\frac{2}{\pi}\left[\frac{k}{k^2+\lambda^2}\right]\cos\lambda x\,d\lambda$$

$$e^{-kx} = \int_0^\infty \frac{2}{\pi}\left[\frac{k}{k^2+\lambda^2}\right]\cos\lambda x\, d\lambda$$

$$\frac{\pi}{2k} e^{-kx} = \int_0^\infty \frac{\cos\lambda x}{k^2+\lambda^2}\, d\lambda.$$

$$f(x) = \int_0^\infty B(\lambda)\sin\lambda x\, d\lambda$$

Where,

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\sin\lambda x\, dx$$

$$= \frac{2}{\pi}\int_0^\infty e^{-kx}\sin\lambda x\, dx$$

$$= \frac{2}{\pi}\left[\frac{e^{-kx}}{k^2+\lambda^2}(-k\sin\lambda x-\lambda\cos\lambda x)\right]_0^\infty$$

$$= \frac{2}{\pi}\left[0 - \frac{1}{k^2+\lambda^2}(0-\lambda)\right]$$

$$= B(\lambda) = \frac{2}{\pi}\left[\frac{\lambda}{k^2+\lambda^2}\right].$$

$$f(x) = \int_0^\infty \frac{2}{\pi}\left[\frac{\lambda}{k^2+\lambda^2}\right]\sin\lambda x\, d\lambda$$

$$\frac{\pi}{2\lambda} e^{-kx} = \int_0^\infty \frac{\sin\lambda x}{k^2+\lambda^2}\, d\lambda$$

$$\frac{\pi}{2} e^{-kx} = \int_0^\infty \frac{\lambda\sin\lambda x}{k^2+\lambda^2}\, d\lambda$$

Find the fourier integral represented as $f(x) = \begin{cases} 0, & -\infty < x < -2 \\ 2, & -2 < x < 2 \\ 0, & 2 < x < \infty \end{cases}$

The fourier integral is

$$f(x) = \int_0^\infty [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\, d\lambda \quad \rightarrow ①$$

where,

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^\infty f(x)\cos\lambda x\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{-2} 0\cos\lambda x\, dx + \int_{-2}^{2} 2\cos\lambda x\, dx + \int_2^\infty 0\cos\lambda x\, dx\right]$$

$$= \frac{2}{\pi}\left[2\left[\frac{\sin\lambda x}{\lambda}\right]_0^2\right].$$

$$= \frac{2}{\pi}\left[\frac{2}{\lambda}[\sin 2\lambda \mp \sin 0\lambda]\right]$$

$$= \frac{2\times 2\sin 2\lambda}{\lambda\pi}$$

$$= \frac{4}{\pi\lambda}\sin 2\lambda$$

$$f(x) \neq$$

$$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{-2} 0\times\sin\lambda x\,dx + \int_{-2}^{2} 2\times\sin\lambda x\,dx + \int_{2}^{\infty} 0\times\sin\lambda x\,dx\right]$$

$$= \frac{2}{\pi}\left[2\int_{-2}^{2}\sin\lambda x\,dx\right]$$

$$= \frac{4}{\pi}\left[-\frac{\cos\lambda x}{\lambda}\right]_{-2}^{2}$$

$$= -\frac{4}{\pi\lambda}\left[\cos 2\lambda - \cos 2\lambda\right]$$

$$= \frac{4}{\pi\lambda}\left[1 - \cos 2\lambda\right] = 0$$

$$f(x) = \int_0^{\infty}\left[\frac{4}{\pi\lambda}\sin 2\lambda\cos\lambda x + \cancel{\frac{4}{\pi\lambda}(1-\cos 2\lambda)\sin\lambda x}\right]d\lambda$$

$$f(x) = \frac{4}{\pi\lambda}\int_0^{\infty}\sin 2\lambda\cos\lambda x\,d\lambda$$

Find fourier cosine integral of $f(x) = e^{-x}$ $x>0,\ k>0$

$$f(x) = \int_0^{\infty} A(\lambda)\cos\lambda x\,d\lambda \rightarrow ①$$

where,

$$A(\lambda) = \frac{2}{\pi}\int_0^{\infty} f(x)\cos\lambda x\,dx$$

$$= \frac{2}{\pi}\int_0^{\infty} e^{-x}\cos\lambda x\,dx$$

$$= \frac{2}{\pi}\left[\frac{e^{-x}}{1+\lambda^2}(-\cos\lambda x + \lambda\sin\lambda x)\right]_0^{\infty}$$

$$= A(\lambda) = \frac{2}{\pi}\left[0 - \frac{1}{1+\lambda^2}(-1)\right]$$

$$A(\lambda) = \frac{2}{\pi}\left[\frac{1}{1+\lambda^2}\right]$$

$$f(x) = \int_0^{\infty}\frac{2}{\pi}\left[\frac{1}{1+\lambda^2}\right]\cos\lambda x\,d\lambda$$

Scanned by CamScanner

Mid : 2

Date: 12-3-16

12-07 Chapter-4

Laplace Transform

Let $f(t)$ be a given fun^n that is defined for $t \geq 0$ then the laplace Transform of $f(t)$ is denoted by $L\{f(t)\}$ or $\bar{f}(s)$ or $F(s)$ and it is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

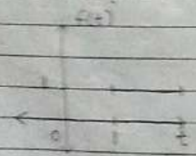
where,

L is a Laplace operator

Piecewise Continuous fu^n: A fun^n is said to be piecewise continuous on an infinite interval $[a, \infty)$ if and only if there exist a finite no. of sub intervals of $[a, \infty)$ such that $f(t)$ is continuous on each of this intervals & $f(t)$ is finite as t approaches to the end points of each interval from the interior.

Ex: $f(t) = 0 \quad ; 0 \leq t < 1$

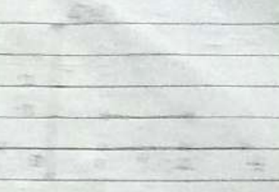
$\quad\quad\; = 1 \quad ; t \geq 1$



$f(t) = 0 \quad ; 0 \leq t < 1$

$\quad\;\; = 1 \quad ; 1 \leq t < 2$

$\quad\;\; = 2 \quad ; t \geq 2$



Ex: Find Laplace Transform of the foll. piecewise fun^n $f(t) = 4 \quad ; 0 \leq t < 1$

$\quad\quad = 3 \quad ; t \geq 1$

By the definition of laplace transform we have

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{f(t)\} = \int_0^1 e^{-st} \times 4 \times dt + \int_1^\infty e^{-st} \times 3 \times dt$$

$$= 4\int_0^1 e^{-st} dt + 3\int_1^\infty e^{-st} dt$$

$$= -\frac{4}{s}\left[e^{-st}\right]_0^1 - \frac{3}{s}\left[e^{-st}\right]_1^\infty$$

$$= -\frac{4}{s}\left[e^{-s} - 1\right] - \frac{3}{s}\left[0 - e^{-s}\right]$$

$$= \frac{-4e^{-s}}{s} + \frac{4}{s} + \frac{3}{s}e^{-s}$$

$$= \frac{4}{s} - \frac{e^{-s}}{s}$$

$$= \frac{1}{s}\left[4 - e^{-s}\right]$$

Ex-2

$$f(t) = \begin{cases} 0 & ; 0 \le t \le \pi \\ \sin t & ; t \geq \pi \end{cases}$$

By the definition of Laplace transform we have

$$L\{f(t)\} = \int_0^{\pi} 0 \times e^{-st} \times dt + \int_{\pi}^{\infty} e^{-st} \sin t \, dt.$$

$$= \left[\frac{e^{-st}}{s^2+1} [-s \sin t - \cos t] \right]_{\pi}^{\infty}$$

$$= \left[0 - \frac{e^{-\pi s}}{s^2+1} [0 - (-1)] \right]$$

$$= \frac{-e^{-\pi s}}{s^2+1}$$

Ex-3

$$f(t) = 0 \quad ; 0 \le t \le 1$$
$$= t \quad ; 1 \le t \le 4$$
$$= 0 \quad ; t \geq 4$$

$$L\{f(t)\} = \int_0^1 e^{-st} \times 0 \times dt + \int_1^4 e^{-st} t \, dt + \int_4^{\infty} e^{-st} \times 0 \times dt$$

$$= \int_1^4 e^{-st} \times t \times dt$$

$$= \left[t \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_1^4$$

$$= \frac{4e^{-4s}}{-s} - \frac{e^{-4s}}{s^2} + \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}$$

$$= -\frac{1}{s} [4e^{-4s} - e^{-s}] - \frac{1}{s^2} [e^{-4s} - e^{-s}]$$

$$= \frac{1}{s} [e^{-s} - 4e^{-4s}] + \frac{1}{s^2} [e^{-s} - e^{-4s}]$$

$$= e^{-s} \left[\frac{1}{s} + \frac{1}{s^2} \right] - e^{-4s} \left[\frac{4}{s} + \frac{1}{s^2} \right]$$

Find the Laplace transform of the foll.

(1) $L\{1\}$

By the definition of Laplace transform we have

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) \, dt$$

$$L\{1\} = \int_0^{\infty} e^{-st} \times 1 \times dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^{\infty}$$

$$= -\frac{1}{s} [e^{-\infty} - e^{0}]$$

$$= -\frac{1}{s} [0 - 1]$$

$$L\{1\} = \frac{1}{s} \quad ; s > 0$$

Scanned by CamScanner

(2) $L\{e^{at}\}$

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^\infty e^{-st} \times e^{at}\,dt$

$= \int_0^\infty e^{(-s+a)t}\,dt$

$= \left[\frac{e^{(-s+a)t}}{-s+a}\right]_0^\infty$

$= \frac{1}{-s+a}\left[e^{\infty} - e^{0}\right]$

$= \frac{1}{-s+a}\left[0-1\right]$

$L\{e^{at}\} = \frac{1}{s-a}$

Laplace transform doesn't exist if $(s \le a)$

NOTE 1. If $a=0 \Rightarrow L\{e^{at}\} = L\{e^{0}\}$
$= L\{1\}$
$= 1/s$

2. If $a=-a \Rightarrow L\{e^{at}\} = L\{e^{-at}\}$
$= \frac{1}{s+a}$

$\Gamma(n+1) = n\Gamma(n)$ } → real number
$\Gamma(n+1) = n!$ } → +ve number

3. $L\{t^n\}$

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$L\{t^n\} = \int_0^\infty e^{-st} \times t^n\,dt$

Suppose $st = u$

$t = \frac{u}{s}$

$dt = \frac{du}{s}$

$t \to 0 \Rightarrow u \to 0$

$t \to \infty \Rightarrow u \to \infty$

$L\{t^n\} = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^n \frac{du}{s}$

$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n\,du$

$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^{(n+1)-1}\,du$

$= \frac{1}{s^{n+1}} \Gamma(n+1)$ $\left[\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx\right]$

$L\{t^n\} = \frac{n!}{s^{n+1}} = \frac{n\Gamma(n)}{s^{n+1}}$

NOTE If $n=0$ $\because$ Laplace transform $\{t^n\}$

$L\{t^n\} = L\{t^0\} = L\{1\}$

$= 1/s$

If $n=-1/2$ then $L\{t^n\} = L\{t^{-1/2}\}$

$= \frac{\Gamma(-1/2+1)}{s^{-1/2+1}} = \frac{\Gamma(1/2)}{s^{1/2}}$

$= \sqrt{\pi/s}$

4. $L\{\cos at\}$ & $L\{\sin at\}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{iat} = \cos at + i\sin at$.

$L\{e^{iat}\} = L\{\cos at + i\sin at\}$

$L\{\cos at + i\sin at\} = \frac{1}{s-ia} \times \frac{s+ia}{s+ia}$

$= \frac{s+ia}{s^2 - i^2a^2}$

$= \frac{s+ia}{s^2+a^2}$ $(i^2=-1)$.

$L\{\cos at + i\sin at\} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$

equating real part & imaginary part at both side

$L\{\cos at\} = \frac{s}{s^2+a2}$

$L\{\sin at\} = \frac{a}{s^2+a^2}$

Find $L\{\cosh t\}$ & $L\{\sin aht\}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{iath} = \cosh at + i\sinh at$

$L\{e^{tinat}\} = L\{\cosh at + i\sinh at\}$

$L\{\cosh at + i\sinh at\} =$

$L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$= L\left\{\frac{e^{at}}{2} + \frac{e^{-at}}{2}\right\}$.

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$

$= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right]$

$= \frac{s}{s^2-a^2}$

$L\{\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\}$

$= L\left\{\frac{e^{at}}{2} - \frac{e^{-at}}{2}\right\}$

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$

$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$

$= \frac{a}{s^2-a^2}$

1. $L\{e^{3t}\} = L\{e^{3t}\} = \int_0^\infty e^{-st} \times e^{3t} \times dt$

$= \int_0^\infty e^{(-s+3)t}\,dt$

$= \left[\frac{e^{(-s+3)t}}{-s+3}\right]_0^\infty$

$= \frac{1}{-s+3}\left[e^{\infty} - e^0\right]$

Scanned by CamScanner

$= \frac{1}{s-3}$ $L\{e^{3t}\} = 1/s-3$

2. $L\{e^{7/2t}\}$

$\Rightarrow L\{e^{7/2t}\} = \int_0^\infty e^{-st} \times e^{7/2t}\,dt$

$= \int_0^\infty \{e^{(-s+7/2)t}\,dt$

$= \left[\frac{e^{(-s+7/2)}}{-s+\frac{7}{2}}\right]_0^\infty$

$= \frac{1}{-s+7/2}\left[e^{\infty} - e^{0}\right]$

$= L\{e^{7/2t}\} = \frac{1}{s-7/2}$

3. $L\{t^4\}$

$L\{t^4\} = \int_0^\infty e^{-st} \times t^4\,dt$

Let $st = u$

$t = \frac{u}{s}$

$dt = \frac{du}{s}$

$= \int_0^\infty e^{-u} \times \left(\frac{u}{s}\right)^4 \times \frac{du}{s}$

$= \frac{1}{s^5} \int_0^\infty e^{-u}\, u^{5-1}\,du$

$= \frac{1}{s^5} \times 4!$

$= L\{t^4\} = \frac{24}{s^5}$

4. $L\{t^{4/3}\}$

$\Rightarrow L\{t^{4/3}\} = \int_0^\infty e^{-st} \times t^{4/3}\,dt$

$=$ Let $st = u \Rightarrow t = \frac{u}{s} \Rightarrow dt = \frac{du}{s}$

$= \int_0^\infty e^{-u} \times \left(\frac{u}{s}\right)^{4/3} \frac{du}{s}$

$= \frac{1}{s^{7/3}} \int_0^\infty e^{-u}\, u^{7/3-1}\,du$

$= \frac{1}{s^{7/3}} \times \frac{4}{3}\Gamma\left(\frac{4}{3}\right)$

5. $L\{t^{7/2}\}$

$\Rightarrow L\{t^{7/2}\} = \int_0^\infty e^{-st} \times t^{7/2}\,dt$

Let $st = u \Rightarrow t = \frac{u}{s} \Rightarrow dt = \frac{du}{s}$

$\int_0^\infty e^{-u} \times \left(\frac{u}{s}\right)^{7/2} \frac{du}{s}$

$= \frac{1}{s^{9/2}} \int_0^\infty e^{-u}\, u^{9/2-1}\,du$

$= \frac{1}{s^{9/2}} \times \frac{105}{16}\sqrt{\pi}$

Scanned by CamScanner

6. $L\{\sin 3t\}$
$= \frac{3}{s^2+9}$

7. $L\{\cos 4t\}$
$= \frac{s}{s^2+16}$

8. $L\{\sinh t\}$
$= \frac{1}{s^2}$

9. $L\{\cosh 2t\}$
$= \frac{s}{s^2-4}$

10. $L\{\sinh 3t\}$
$= \frac{3}{s^2-9}$

Neariti Property for L.T

If $L\{f(t)\} = f(s)$ & $L\{g(t)\} = G(s)$

$L\{af(t) \pm bg(t)\} = aL[f(t)] \pm bL[g(t)]$

$= aF(s) \pm b'G(s)$

Property:

$L[af(t) \pm bg(t)]$

$= \int_0^\infty e^{-st}\{af(t) \pm \int_0^\infty e^{-st} bg(t)\,dt$

$= a\int_0^\infty e^{-st} f(t)\,dt \pm b\int_0^\infty e^{-st} g(t)\,dt$

$= aL\{f(t)\} \pm bL\{g(t)\}$

$= aF(s) \pm bG(s)$

where L is a Laplace linear Operator

Prove that Laplace Operator 'L' is a linear operator

Some important Formula

(1) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
(2) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
(3) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
(4) $2\sin A\sin B = -\cos(A+B) + \cos(A-B)$
(5) $\sin 2A = 2\sin A\cos A$
(6) $\cos 2A = \cos^2 A - \sin^2 A$
$= 2\cos^2 A - 1$
$= 1 - 2\sin^2 A$
(7) $\cos 3A = 4\cos^3 A - 3\cos A$
(8) $\sin 3A = 3\sin A - 4\sin^3 A$

Ex-1 Find the Laplace Transform of the foll.

(1) $L(\sin 2t\cos 2t)$

$= \frac{1}{2}L\{2\sin 2t\cos 2t\}$

$= \frac{1}{2}L\{\sin 4t\}$

$$= \frac{1}{2} \times \frac{4}{s^2+16}$$

$$= \frac{2}{s^2+16}$$

2. $L(\sin 2t \cos 3t) = \frac{1}{2} L\{2\sin 2t \cos 3t\}$

$$= \frac{1}{2} L\{\sin(2+3)t + \sin(2-3)t\}$$

$$= \frac{1}{2} L\{\sin 5t - \sin 1t\}$$

$$= \frac{1}{2} L\{\sin 5t\} - L\{\sin 1t\}$$

$$=$$

$$= \frac{1}{2}\left\{\frac{5}{s^2+25} - \frac{1}{s^2+1}\right\}$$

$$= \frac{1}{2}\left[\frac{5(s^2+1) - 1(s^2+25)}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{1}{2}\left[\frac{4s^2-20}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{2s^2-10}{(s^2+25)(s^2+1)}$$

3. $L(2t^3 + e^{-2t} + t^{4/3})$

$$= L\{2t^3\} + L\{e^{-2t}\} + L\{t^{4/3}\}$$

$$= \int_0^\infty e^{-st} \times 2t^3\,dt + \int_0^\infty e^{-st} \times e^{-2t} + \int_0^\infty e^{-st}\, t^{4/3}\,dt$$

Let $st = u \Rightarrow t = \frac{u}{s}$ $\quad dt = \frac{du}{s}$

$$= \frac{2 \cdot 3!}{s^4} + \frac{1}{s+2} + \frac{\frac{4}{3} \times \frac{1}{3}\Gamma(1/3)}{s^{7/3}}$$

$$= \frac{2 \cdot 3!}{s^4} + \frac{1}{s+2} + \frac{4/9\,\Gamma(1/3)}{s^{7/3}}$$

4. $L(\cos^2 \omega t)$

$$= \frac{1}{2} L(\cos 2\omega t + 1)$$

$$= \frac{1}{2}\left[L\{\cos 2\omega t\} + L(1)\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+4\omega^2} + \frac{1}{s}\right]$$

5. $L(\cosh^3 2t)$

$$= \frac{1}{4}\left[L\{\cos 3A + 3\cos A\}\right]$$

$$= \frac{1}{4}\left[L\{\cosh 6t\} + 3L(\cosh 2t)\right]$$

$$= \frac{1}{4}\left[L\{\cosh 6t\} + 3L\{\cosh 2t\}\right]$$

$$= \frac{1}{4}\left[\frac{s}{s^2-36} + 3 \times \frac{s}{s^2-4}\right]$$

$$= \frac{1}{4}\left[\frac{s(s^2-4) + 3s(s^2-36)}{(s^2-36)(s^2-4)}\right]$$

$$= \frac{s}{4}\left[\frac{s^2-4+3s^2-108}{(s^2-36)(s^2-4)}\right]$$

$$= \frac{s}{4}\left[\frac{4(s^2-28)}{(s^2-36)(s^2-4)}\right]$$

$$= \frac{s(s^2-28)}{(s^2-36)(s^2-4)}$$

* $L[\cos^2 4t]$
* $L[\cos 2A \cdot \cos A]$

* $\frac{1}{2} L[\cos 8t + 1]$

$= \frac{1}{2}\left[L\{\cos 8t\} + L(1)\right]$

$= \frac{1}{2}\left[\frac{s}{s^2+64} + \frac{1}{s}\right]$

$= \frac{1}{2}\left[\frac{2s^2+64}{s(s^2+64)}\right]$

$= \frac{s^2+32}{s(s^2+64)}$

* $\frac{1}{2} L[\cos(3A) + \cos A]$

$= \frac{1}{2}\left[L\{\cos 3A\} + L\{\cos A\}\right]$

$= \frac{1}{2}\left[\frac{s}{s^2+9} + \frac{s}{s^2+1}\right]$

$= \frac{s}{2}\left[\frac{s^2+1+s^2+9}{(s^2+9)(s^2+1)}\right]$

$= \frac{s(s^2+5)}{(s^2+9)(s^2+1)}$

First Shifting theorem:

If $L\{f(t)\} = F(s)$ then $L\{e^{at} f(t)\} = F(s-a)$
Determining fun^n Generating fun^n

Proof: Given that $L\{f(t)\} = F(s)$

By the definition of Laplace Transform

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} \times e^{at} f(t)\,dt$

$= \int_0^\infty e^{(-s+a)t} f(t)\,dt$

$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$

$= F(s-a)$

$L\{e^{at} f(t)\} = [L\{f(t)\}]_{s \to s-a}$

$L\{e^{-at} f(t)\} = [L\{f(t)\}]_{s \to s+a}$

First Shifting Theorem to Some Imp. Laplace Fun^n Formula

$L\{f(t)\}$	$L\{e^{at} f(t)\} \to$ 1st shifting
$L\{1\} = 1/s$	$L\{e^{at}(1)\} = \frac{1}{s-a}$
$L\{t^n\} = \frac{n!}{s^{n+1}}$	$L\{e^{at}(t^n)\} = \frac{n!}{(s-a)^{n+1}}$
$L\{\sin bt\} = \frac{b}{s^2+b^2}$	$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$

Scanned by CamScanner

$L\{\cosh bt\} = \frac{s}{s^2+b^2}$ | $L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$

$L\{\sinh bt\} = \frac{b}{s^2-b^2}$ | $L\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2}$

$L\{\cosh bt\} = \frac{s}{s^2-b^2}$ | $L\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$

Find the Laplace Transform for the foll.

1. $L\{e^{-2t}(4\cos 3t - 2\sin t)\}$

$\Rightarrow L\{4e^{-2t}\cos 3t\} - L\{2e^{-2t}\sin t\}$

$\Rightarrow 4L\{e^{-2t}\cos 3t\} - 2L\{e^{-2t}\sin t\}$

From the 1st shifting theorem.

$\Rightarrow 4\left[\frac{s+2}{(s+2)^2+9}\right] - 2\left[\frac{1}{(s+2)^2+1}\right]$

$= \frac{4s+8[(s+2)^2+1] - 2[(s+2)^2+9]}{((s+2)^2+9)[(s+2)^2+1]}$

$= \frac{4s+8}{s^2+4s+13} - \frac{2}{s^2+4s+5}$

2. $L\{5e^{2t}\sinh 2t\}$

$\Rightarrow 5 L\{e^{2t}\sinh 2t\}$

$\Rightarrow 5\times\left[\frac{2}{(s-2)^2-4}\right]$

$\Rightarrow \frac{10}{s^2-4s}$

3. $L\{e^{-3t}\sin^2 t\}$

$\Rightarrow L\{e^{-3t}\left(\frac{1-\cos 2t}{2}\right)\}$

$\Rightarrow \frac{1}{2} L\{e^{-3t} - e^{-3t}\cos 2t\}$

$\Rightarrow \frac{1}{2}L\{e^{-3t}\} - \frac{1}{2}L\{e^{-3t}\cos 2t\}$

$\Rightarrow \frac{1}{2}\left[\frac{1}{s+3}\right] - \frac{1}{2}\left[\frac{s+3}{(s+3)^2+4}\right]$

$\Rightarrow \frac{1}{2(s+3)} - \frac{s+3}{2[s^2+6s+13]}$

$\Rightarrow \frac{1}{2(s+3)} - \frac{s+3}{2s^2+12s+26}$

$\Rightarrow \frac{2}{(s+3)(s^2+6s+13)}$

4. $L\{e^{3t}t^{7/2}\}$

$\Rightarrow L\{t^{7/2}\} = \cancel{\frac{7/2\times 5/2\times 3/2\sqrt{\pi}}{s^{9/2}}}$

$= \cancel{105}$

$\Rightarrow \frac{\Gamma(7/2+1)}{s^{7/2+1}}$

$\Rightarrow \frac{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma\frac{1}{2}}{s^{9/2}}$

$\Rightarrow \frac{105\sqrt{\pi}}{16\, s^{9/2}}$

Scanned by CamScanner

Date: MON TUE WED THU FRI SAT

$\Rightarrow$ $L\{e^{3t} t^{7/2}\}$

$\Rightarrow$ $\frac{105}{16} \times \frac{\sqrt{\pi}}{(S-3)^{9/2}}$

* $L\{\cosh at \cos bt\}$

We know that $\cosh at = \frac{e^{at}+e^{-at}}{2}$

$L\{\cosh at \cos bt\}$

$= L\{\left(\frac{e^{at}+e^{-at}}{2}\right)\cos bt\}$

$= \frac{1}{2} L\{e^{at}\cos bt + e^{-at}\cos bt\}$

$= \frac{1}{2}\left[L\{e^{at}\cos bt\} + L\{e^{-at}\cos bt\}\right]$

$= \frac{1}{2}\left[\frac{S-a}{(S-a)^2+b^2} + \frac{S+a}{(S+a)^2+b^2}\right]$

* $L\{\sinh at \sin at\}$

$= L\{\left(\frac{e^{at}-e^{-at}}{2}\right)\sin at\}$

$= \frac{1}{2}\left[L\{e^{at}\sin at\} - L\{e^{-at}\sin at\}\right]$

$= \frac{1}{2}\left[\frac{a}{(S-a)^2+a^2} - \frac{a}{(S+a)^2+a^2}\right]$

$= \frac{a}{2}\left[\frac{(S+a)^2+a^2-(S-a)^2-a^2}{((S-a)^2+a^2)((S+a)^2+a^2)}\right]$

Date: MON TUE WED THU FRI SAT

$= \frac{a}{2}\left[\frac{4aS}{((S-a)^2+a^2)((S+a)^2+a^2)}\right]$

$= \frac{2a^2 S}{((S-a)^2+a^2)((S+a)^2+a^2)}$

$= \frac{2a^2 S}{S^4+4a^4}$

* $L\{e^{-2t}\cos bt\}$

$= \frac{(S+2)}{(S+2)^2+b^2}$

* $L\{e^t \cosh 3t\}$

$= \frac{S-1}{(S-1)^2-9} = \frac{S-1}{S^2-2S-8}$

* $L\{e^{2t}(3\sin 4t - 4\cos 4t)\}$

$= 3L\{e^{2t}\sin 4t\} - 4L\{e^{2t}\cos 4t\}$

$= 3\left[\frac{4}{(S-2)^2+16}\right] - 4\left[\frac{(S-2)}{(S-2)^2+16}\right]$

$= \frac{12-4S+8}{(S-2)^2+16}$

$= \frac{20-4S}{(S-2)^2+16}$

* Change of Scale Property

If $L\{f(t)\} = F(s)$ then
$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

Proof By the definition of Laplace Transform

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt \rightarrow ①$

Let $at = u$

$t = \frac{u}{a}$

$L\{f(at)\} = \int_0^\infty e^{-s(u/a)} f(u^2)\,\frac{du}{a}$

$= \frac{1}{a} \int_0^\infty e^{-(s/a)u} f(u)\,du$

$= \frac{1}{a} F\left(\frac{s}{a}\right)$ [By def^n of L.T]

$\boxed{\text{NOTE :}}$

(1) $L\{e^{at} f(t)\} = \frac{1}{b} F\left(\frac{s-a}{b}\right)$

$L\{f(t)\} = e^{-1/s}/s$ then

find ~~$L\{f(t)\}$~~ $L\{e^{-t} f(3t)\}$

Here given that $L\{f(t)\} = \frac{e^{-1/s}}{s}$

Now,

By the change of Scale property

$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

$L\{f(3t)\} = \frac{1}{3} F\left(\frac{s}{3}\right)$

$L\{e^{-t} f(3t)\} = L\{f(3t)\}_{s \to s+1}$

$= \left[\frac{1}{3} F\left(\frac{s}{3}\right)\right]_{s \to s+1}$

$= \frac{1}{3}\left[\frac{e^{-1/s/3}}{s/3}\right]_{s \to s+1}$

$= \left[\frac{e^{-3/s}}{s}\right]_{s \to s+1}$

$= \left[\frac{e^{-3/s}}{s}\right]_{s \to s+1}$

$= \frac{e^{-3/s+1}}{s+1}$

1. $L\{\sin t\} = \frac{1}{s^2+1}$

Find $L\{\sin 3t\, e^{2t}\}$

2. If $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right)$

Find $\left\{\frac{\sin at}{t}\right\}$

1) ⇒ Here given that $L\{\sin t\} = \frac{1}{s^2+1}$

By the change of Scale property

$L\{\sin 3t\} = \frac{1}{s^2+9}$

$L\{e^{2t} \sin 3t\} = \left[\frac{1}{s^2+9}\right]_{s \to s-2}$

$$= \left[\frac{1}{(s-2)^2+9}\right]$$
$$= \frac{1}{s^2-4s+5}$$

Laplace Transform of Derivative

Multiplication by 's' :- If $f(t)$ is continuous with a piecewise continuous derivative in every finite interval $0 \le t \le T$ and $f(t)$ is of order $e^{\alpha t}$ as t then for $s>0$ the laplace transform of $f'(t)$ exist & it is given by
$L\{f'(t)\} = s\,L\{f(t)\} - f(0)$.

Proof: By the definition of Laplace Transform we have $L\{f'(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$L\{f'(t)\} = \int_0^\infty \underset{u}{e^{-st}}\, \underset{v}{f'(t)}\,dt$$
$$= e^{-st}\int f'(t)dt - \int_\infty \frac{d}{dx}e^{-st}\left(\int f'(t)dt\right)dt$$
$$= \left[e^{-st}\left[f(t)\right]\right]_0^\infty - \int_0\left[(-s)e^{-st}f(t)dt\right]$$
$$= \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st}f(t)dt$$
$$= \left[0 - f(0)\right] + s\,L\{f(t)\}$$
$$L\{f'(t)\} = s\,L\{f(t)\} - f(0)$$

Similarly
$$L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f'(0)$$

$$L\{f^n(t)\} = s^n\,L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots$$

Find Laplace Transform of the foll. funⁿ by using transform of derivative.

(1) $L\{\sin^2 t\}$ = By the formula of derivative we have,

$\rightarrow L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f'(0) - 1$

$f(t) = \sin^2 t$
$f'(t) = 2\sin t\cos t$
$= \sin 2t$
$f''(t) = 2\cos 2t$
$f(0) = \sin 0 = 0$
$f'(0) = \sin 0 = 0$

$\rightarrow L\{f''(t)\} = L\{2\cos 2t\} = s^2\,L\{\sin^2 t\} - 0 - 0$

$$L\{\sin^2 t\} = \frac{2}{s^2}\,L\{\cos 2t\}$$
$$= \frac{2}{s^2}\left[\frac{s}{s^2+4}\right]$$
$$= \frac{2}{s^3+4s}$$

(2) $L\{t^2\}$

$f(t) = t^2$
$f'(t) = 2t$
$f''(t) = 2$
$f(0) = 0$
$f'(0) = 0$

$L\{2\} = s^2 L\{t^2\} - 0 - 0$

$L\{t^2\} = \frac{2}{s^2 \times s}$

$= \frac{2}{s^3}$

3) $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$ then Show that

$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$

Here, $f(t) = 2\sqrt{\frac{t}{\pi}}$

$f'(t) = \frac{2}{\sqrt{\pi}}\left(\frac{1}{2} t^{-1/2}\right)$

$f''(t) = -\frac{2}{3\sqrt{\pi}} t^{-3/2}$ $\quad f(0) = 0$

$L\{f'(t)\} = sL\{f(t)\} - f(0)$

$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = sL\left\{2\sqrt{\frac{t}{\pi}}\right\}$

$= \frac{2}{\sqrt{\pi}} \, s \, L\{t^{1/2}\}$

$= \frac{2}{\sqrt{\pi}} \, s \, \frac{\cancel{1/2}\,\Gamma\frac{1}{2}}{s^{1/2+1}}$

$= \frac{\cancel{2}}{\cancel{\sqrt{\pi}}} \, s \times \frac{1}{\cancel{2}} \, \frac{\cancel{\sqrt{\pi}}}{s^{3/2}}$

$= \frac{1}{\sqrt{s}}$

2) $f(t) = t\cos at$

$\cancel{f'(t) = -t\sin at + \cos at}$

$f''(t) = -t\cos at - \sin at - \sin at$

$=$

$f'(t) = -at\sin at + \cos at$

$f''(t) = -a^2 t\cos at - a\sin at - a\sin at$

$= -a^2 t\cos at - 2a\sin at$

$f(0) = 0$

$= L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$

$L\{-a^2 t\cos at - 2a\sin at\} = s^2 L\{t\cos at\}$

$= -a^2 L\{t\cos at\} - 2a L\{\sin at\} = s^2 L\{t\cos at\} - 0$

$= L\{t\cos at\} = -(s^2+a^2) \frac{2a}{(s^2+a^2)} L\{\sin at\}$

$= \frac{-2a \times a}{(s^2+a^2)(s^2+a^2)} + \frac{(s^2+a^2)}{(s^2+a^2)}$

$= \frac{-2a^2 + s^2 + a^2}{(s^2+a^2)^2}$

$L\{t\cos at\} = \frac{s^2 - a^2}{(s^2+a^2)^2}$

H.W $\quad L\{\cos at\}$

$L\{t^2 \sin at\}$

1) $L\{\cos at\}$

$f(t) = \cos at$

$f'(t) = -a\sin at$

$f''(t) = -a^2 \cos at$

$= L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$

$= L\{-a^2 \cos at\} = s^2 L\{\cos at\} - s$

$= L\{\cos at\}(s^2 + a^2) = s$

$L\{\cos at\} = \frac{s}{s^2+a^2}$

2) $L\{t^2 \sin at\}$

$f(t) = t^2 \sin at$

$f'(t) = 2t \sin at + t^2 a \cos at$

$f''(t) = 2 \sin at + 4at \cos at - t^2 a^2 \sin at$

$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$

$2L\{\sin at\} + 4a L\{t \cos at\} - a^2 L\{t^2 \sin at\}$

$= s^2 L\{t^2 \sin at\} - 0 - 0$

$(s^2+a^2) L\{t^2 \sin at\} = 2 \times \frac{a}{s^2+a^2} + \frac{4a(s^2-a^2)}{(s^2+a^2)^2}$

$L\{t^2 \sin at\} = \frac{2a}{(s^2+a^2)^2} + \frac{4a(s^2-a^2)}{(s^2+a^2)^3}$

Multiplication by t^n ~~write~~

Write differentiation of L.T

If $L\{f(t)\} = F(s)$. then

$L\{tf(t)\} = -\frac{d}{ds}F(s) = -\frac{d}{ds} L\{f(t)\}$

& $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

$= (-1)^n \frac{d^n}{ds^n} L\{f(t)\}$

Proof: By the definition of L.T for

$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt \rightarrow (1)$

Differentiating eq^n. no 1 w.r.t s.

$= \int_0^\infty \left(\frac{d}{ds} e^{-st}\right) f(t) dt$

$= \frac{d}{ds} F(s) = \int_0^\infty e^{-st} (-t) f(t) dt$

$= -\int_0^\infty e^{-st} (tf(t)) dt$

$= -L\{tf(t)\}$ By definition of L.T

$L\{tf(t)\} = -\frac{d}{ds}(F(s)) = -\frac{d}{ds} L\{f(t)\}$

Simillary L.T

$L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} F(s) = (-1)^2 \frac{d^2}{ds^2} L\{f(t)\}$

$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n \frac{d^n}{ds^n} L\{f(t)\}$

Ex: 1 Find $L\{t \sin 3t\}$.

$L\{\sin 3t\} = \frac{3}{s^2+9}$

$L\{t \sin 3t\} = -\frac{d}{ds} L\{\sin 3t\}$

$= -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$

$= -3 \frac{d}{ds} (s^2+9)^{-1}$

$= 3 \times \frac{2s}{(s^2+9)^2} = \frac{6s}{(s^2+9)^2}$

2) $L\{t\cos 5t\}$

$L\{\cos 5t\} = \frac{s}{s^2+25}$

$L\{t\cos 5t\} = -\frac{d}{ds} L\{\cos 5t\}$

$= -\frac{d}{ds}\left(\frac{s}{s^2+25}\right)$

$= \frac{-1\times(s^2+25)+2s^2}{(s^2+25)^2}$

$= \frac{s^2-25}{(s^2+25)^2}$

3) $L\{t^2 e^{at}\}$

$L\{e^{at}\} = \frac{1}{s-a}$

$L\{t^2e^{at}\} = (-1)^2 \frac{d^2}{ds^2} L\{e^{at}\}$

$= 1\times\frac{d^2}{ds^2}\left[\frac{1}{s-a}\right]$

$= \frac{d^2}{ds^2}(s-a)^{-1}$

$= \frac{2}{(s-a)^3}$

4) $L\{t^2\sin t\}$

$L\{\sin t\} = \frac{1}{s^2+1}$

$L\{t^2\sin t\} = (-1)^2\frac{d^2}{ds^2} L\{\sin t\}$

$= (-1)^2 \frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right]$

$= 1\times\frac{d}{ds}\frac{-1\times 2s}{(s^2+1)^2}$

$= -2\left[\frac{(s^2+1)^2(1) - 2(s^2+1)\times 2s\times s}{(s^2+1)^4}\right]$

$= -2\left[\frac{(s^2+1)^2 - 2(s^2+1)2s^2}{(s^2+1)^4}\right]$

$= \cancel{-2}\left[s^4+2s^2\cancel{+1} - 4s^4 - \cancel{4s^2}\right.$

$= (s^2\cancel{+1})\left[\frac{-2(s^2+1)+2\times 2s^2\times 2}{(s^2+1)^4}\right]$

$= \frac{6s^2-2}{(s^2+1)^3}$

5) $L\{t^3\cos t\}$

$L\{\cos t\} = \frac{s}{s^2+1}$

$L\{t^3\cos t\} = (-1)^3\frac{d^3}{ds^3}\left(\frac{s}{s^2+1}\right)$

$= -1\frac{d^2}{ds^2}\left[\frac{(s^2+1)\times 1 - s(2s)}{(s^2+1)^2}\right]$

$= -\frac{d^2}{ds^2}\left[\frac{1-s^2}{(s^2+1)^2}\right]$

$= -\frac{d}{ds}\left[\frac{(s^2+1)^2(-2s) - (2(s^2+1)(1-s^2)(2s))}{(s^2+1)^4}\right]$

$= -\frac{d}{ds}\left[\frac{(s^2+1)(-2s) - 4(s-s^3)}{(s^2+1)^3}\right]$

$= -\frac{d}{ds}\left[\frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}\right]$

Scanned by CamScanner

$$= -\frac{d}{ds}\left[\frac{2s^3-6s}{(s^2+1)^3}\right]$$

$$= -2\frac{d}{ds}\left[\frac{s^3-3s}{(s^2+1)^3}\right]$$

$$= -2\left[\frac{(s^2+1)^3(3s^2-3) - 3(s^3-3s)(s^2+1)^2 2s}{(s^2+1)^6}\right]$$

$$= -2\left[\frac{3(s^2+1)(s^2-1) - 6s^2(s^2-3)}{(s^2+1)^4}\right]$$

$$= -2\left[\frac{3(s^4-1) - 6s^4+18s^2}{(s^2+1)^4}\right]$$

$$= -2\left[\frac{3s^4-3-6s^4+18s^2}{(s^2+1)^4}\right]$$

$$\cancel{= -2\left[\frac{3-3s^4}{(s^2+1)^4}\right]}$$

$$\cancel{= \frac{6s^4-6}{(s^2+1)^4}}$$

$$= -2\left[\frac{-3-3s^4+18s^2}{(s^2+1)^4}\right]$$

$$= \frac{6+6s^4-36s^2}{(s^2+1)^4}$$

$$= \frac{6(s^4-6s^2+1)}{(s^2+1)^4}$$

Division by t property ($\int L\{f(t)\}$)

If $L\{f(t)\} = F(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

$$= \int_s^\infty L\{f(t)\}\,ds$$

Proof: By the definition of L.T

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Integrating w.r.t s from s to ∞ then we have

$$\int_s^\infty F(s)\,ds = \int_s^\infty\left(\int_0^\infty e^{-st} f(t)\,dt\right)ds$$

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right]dt$$

$$\int_s^\infty F(s)\,ds = \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty \left(\frac{-f(t)}{t}\right)(e^{-\infty} - e^{-st})\,dt$$

$$= \int_0^\infty t^{-1} f(t)\, e^{-st}\,dt$$

$$= \int_0^\infty t^{-1} e^{-st} f(t)\,dt.$$

$$= \int_0^\infty e^{-st}\left(\frac{f(t)}{t}\right)dt$$

$$= L\left\{\frac{f(t)}{t}\right\}$$

Ex: 1 Find $L\{f(t)\}$ of $L\left\{\frac{\sin \omega t}{t}\right\}$

By the divided by t property

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty L\{f(t)\}ds$$

$$= \int_s^\infty L\left\{\frac{\sin \omega t}{t}\right\} ds$$

$$= \int_s^\infty L\{\sin \omega t\} ds$$

$$= \int_s^\infty \left(\frac{\omega}{s^2+\omega^2}\right) ds$$

$$= \omega \times \frac{1}{\omega} \left[\tan^{-1}\left(\frac{s}{\omega}\right)\right]_s^\infty$$

$$= \left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{\omega}\right)\right]$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{\omega}\right)$$

$$= \cot^{-1}\left(\frac{s}{\omega}\right)$$

Ex: 2 $L\left(\frac{1-\cos 2t}{t}\right)$

By the divided by t property

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty L\{f(t)\} \times ds$$

$$= \int_s^\infty L\{1-\cos 2t\} ds$$

$$= \int_s^\infty L\{1\}ds - \int_s^\infty L\{\cos 2t\} ds$$

$$= \int_s^\infty \frac{1}{s} ds - \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+2^2}\right) ds$$

$$= [\log s]_s^\infty - \frac{1}{2}[\log(s^2+4)]_s^\infty$$

$$= -\log s + \frac{1}{2}\log(s^2+4).$$

Ex: 3 $L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\}$

$$= L\left\{\frac{e^{-at}}{t}\right\} - L\left\{\frac{e^{-bt}}{t}\right\}$$

$$= \int_s^\infty L\{e^{-at}\}ds - \int_s^\infty L\{e^{-bt}\}ds$$

$$= \int_s^\infty \frac{ds}{s+a} - \int_s^\infty \frac{ds}{s+b}$$

$$= [\log(s+a)]_s^\infty - [\log(s+b)]_s^\infty$$

$$= \cancel{-\log 2s + \log 2s} - \log(s+a) + \log(s$$

$$= \cancel{-2\log 2s} \text{ or } = \log\left(\frac{s+b}{s+a}\right)$$

Ex:4 Find $L\left\{e^t \frac{\sin t}{t}\right\}$

$$\Rightarrow L\left\{e^t \frac{\sin t}{t}\right\} = \int_s^\infty L\{e^t \sin t\}\, ds$$

$$= \int_s^\infty \frac{ds}{(s-1)^2+1}$$

$$= \int_s^\infty \frac{ds}{s^2-2s+2}$$

$$= \int_s^\infty \left[\frac{(2s-2)-(2s-2)}{s^2-2s+2}\right] ds$$

$$= \left[\log(s^2-2s+2) - \log(s^2-2s+2)\right]$$

$$\Rightarrow L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \left(\frac{1}{s^2+1}\right) ds$$

$$= \left[\tan^{-1}(s)\right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$= \frac{\pi}{2} - \tan^{-1}(s)$$

$$= \cot^{-1}(s)$$

$$L\left\{e^t \frac{\sin t}{t}\right\} = \cot^{-1}(s-1)$$

Inverse Laplace Transform

If $L\{f(t)\} = F(s)$ then $f(t)$ is called inverse laplace transform of $F(s)$. And it is denoted by $L^{-1}\{F(s)\} = f(t)$

Some functions and their inverse laplace transform $L^{-1}\{f(t)\}$.

$L\{f(t)\}$	$L^{-1}\{F(s)\}$
$L\{1\} = \frac{1}{s}$	$L^{-1}\left\{\frac{1}{s}\right\} = 1$
$L\{e^{at}\} = \frac{1}{s-a}$	$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
$L\{e^{-at}\} = \frac{1}{s+a}$	$L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$
$L\{\sin at\} = \frac{a}{s^2+a^2}$	$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$
$L\{\cos at\} = \frac{s}{s^2+a^2}$	$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$
$L\{\sinh at\} = \frac{a}{s^2-a^2}$	$L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{b}\sinh at$
$L\{\cosh at\} = \frac{s}{s^2-a^2}$	$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$
$L\{t^n\} = \frac{n!}{s^{n+1}} / \frac{\lceil n}{s^{n+1}}$	$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)\lceil n-1} = \frac{t^{n-1}}{(n-1)!}$
$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$	$L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b} e^{at}\sin bt$
$L\{e^{at}\cos bt\} = \frac{(s-a)}{(s-a)^2+b^2}$	$L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$

$L\{e^{at}\cosh bt\} = \dfrac{s-a}{(s-a)^2-b^2}$	$L^{-1}\left\{\dfrac{s-a}{(s-a)^2-b^2}\right\} = e^{at}\cosh bt$
$L\{e^{at}\sinh bt\} = \dfrac{b}{(s-a)^2-b^2}$	$L^{-1}\left\{\dfrac{1}{(s-a)^2-b^2}\right\} = \dfrac{1}{b}(e^{at}\sinh bt)$
$L\{e^{at}t^n\} = \dfrac{n!}{(s-a)^{n+1}}$	$L^{-1}\left\{\dfrac{1}{(s-a)^{n}}\right\} = \dfrac{e^{at}t^{n-1}}{(n-1)!}$

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}t\sin at$$

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}[\sin at - at\cos at]$$

$$L^{-1}\left\{\frac{1}{(s+a)^n}\right\} = \frac{e^{-at}t^{n-1}}{(n-1)!}\ ;\ n = 1, 2, 3, \ldots$$

Basic Properties of Inverse Laplace Transform

Linearity Property of $L^{-1}\{f(t)\}$

if $f(t) = L^{-1}\{F(s)\}$

$g(t) = L^{-1}\{G(s)\}$

$L^{-1}\{aF(s) \pm bG(s)\} = af(t) \pm bg(t)$

From the linearity property of L.T

$L\{af(t) \pm bg(t)\} = aF(s) \pm bG(s)$

Taking inverse L.T on both the side

$\therefore\ L^{-1}[L\{af(t) \pm bg(t)\}] = L^{-1}\{aF(s) \pm bG(s)\}$

$af(t) \pm bg(t) = L^{-1}\{aF(s) \pm bG(s)\}$

Ex : 1 Find $L^{-1}\left\{\dfrac{2s-1}{s^2-4}\right\}$

$\Rightarrow L^{-1}\left\{\dfrac{2s}{s^2-4}\right\} + L^{-1}\left\{\dfrac{1}{s^2-4}\right\}$

$\Rightarrow 2L^{-1}\left\{\dfrac{s}{s^2-2^2}\right\} + L^{-1}\left\{\dfrac{1}{s^2-2^2}\right\}$

$\Rightarrow 2\cosh 2t + \dfrac{1}{2}\sinh 2t$

Ex : 2 $L^{-1}\left\{\dfrac{3(s^2-2)^2}{2s^5}\right\}$

$\Rightarrow \dfrac{3}{2}L^{-1}\left\{\dfrac{s^4-4s^2+4}{s^5}\right\}$

$\Rightarrow \dfrac{3}{2}\left[L^{-1}\left\{\dfrac{1}{s}\right\} - 4L^{-1}\left\{\dfrac{1}{s^3}\right\} + 4L^{-1}\left\{\dfrac{1}{s^5}\right\}\right]$

$\Rightarrow \dfrac{3}{2}\left[1 - 4\dfrac{t^2}{2} + 4\dfrac{t^4}{4!}\right]$

$\Rightarrow \dfrac{3}{2}\left[1 - 2t^2 + \dfrac{t^4}{6}\right]$

$\Rightarrow \dfrac{3}{2} - 3t^2 + \dfrac{t^4}{4}$

Ex : 3 $L^{-1}\left\{\dfrac{(s+2)^3}{s^6}\right\}$

$\Rightarrow L^{-1}\left\{\dfrac{\cancel{8^3+8} + s^3 + 3s^2\times 2 + 3\times 4s + 8}{s^6}\right\}$

$\Rightarrow L^{-1}\left\{\dfrac{s^3+6s^2+12s+8}{s^6}\right\}$

$\Rightarrow L^{-1}\left\{\dfrac{1}{s^3}\right\} + 6L^{-1}\left\{\dfrac{1}{s^4}\right\} + 12L^{-1}\left\{\dfrac{1}{s^5}\right\} + 8L^{-1}\left\{\dfrac{1}{s^6}\right\}$

$\Rightarrow \frac{t^2}{2} + \frac{t^3}{\cancel{4}} + \frac{t^4}{2} + \frac{t^5}{15}$

Ex: 4 $L^{-1}\left\{\frac{2s^2+3s+4}{5s^3}\right\}$

$\Rightarrow L^{-1}\left\{\frac{2s^2}{5s^3} + \frac{3s}{5s^3} + \frac{4}{5s^3}\right\}$

$\Rightarrow L^{-1}\left\{\frac{2}{5s} + \frac{3}{5s^2} + \frac{4}{5s^3}\right\}$

$\Rightarrow \frac{2}{5} L^{-1}\left\{\frac{1}{s}\right\} + \frac{3}{5} L^{-1}\left\{\frac{1}{s^2}\right\} + \frac{4}{5} L^{-1}\left\{\frac{1}{s^3}\right\}$

$\Rightarrow \frac{2}{5} + \frac{3}{5}t + \frac{4}{5\times 2} t^2$

$\Rightarrow \frac{2}{5} + \frac{3}{5}t + \frac{2}{5}t^2$

Ex: 5 $L^{-1}\left\{\frac{6s-7}{s^2+5}\right\}$

$\Rightarrow 6L^{-1}\left\{\frac{s - 7/6}{s^2+(\sqrt{5})^2}\right\}$

$\Rightarrow 6L^{-1}\left\{\frac{s}{s^2+(\sqrt{5})^2}\right\} - \frac{7\times 6}{6} L^{-1}\left\{\frac{1}{s^2+(\sqrt{5})^2}\right\}$

$\Rightarrow 6\cos\sqrt{5}t - \frac{7\cancel{6}}{\sqrt{5}\times\cancel{6}} \sin\sqrt{5}t$

$\Rightarrow 6\cos\sqrt{5}t - \frac{7}{\sqrt{5}} \sin\sqrt{5}t$

First shifting theorem for Inverse Laplace Transform.

If $f(t) = L^{-1}\{F(s)\}$ then $L^{-1}\{F(s-a)\} = e^{at}f(t)$

Proof: As per first shifting property of Laplace transform applying L^{-1} on both side then it's give.

$LL^{-1} = 1$

Ex: 1 $L^{-1}\left\{\frac{3s+1}{(s+1)^4}\right\}$

$\Rightarrow L^{-1}\left\{\frac{3s}{(s+1)^4}\right\} + L^{-1}\left\{\frac{1}{(s+1)^4}\right\}$

$\Rightarrow 3L^{-1}\left\{\frac{s}{(s+1)^4}\right\} + L^{-1}\left\{\frac{1}{(s+1)^4}\right\}$

$\Rightarrow 3L^{-1}\left\{\frac{s+1-1}{(s+1)^4}\right\} + L^{-1}\left\{\frac{1}{(s+1)^4}\right\}$

$\Rightarrow 3L^{-1}\left\{\frac{s+1}{(s+1)^4}\right\} - 2L^{-1}\left\{\frac{1}{(s+1)^4}\right\}$

$\Rightarrow 3L^{-1}\left\{\frac{1}{(s+1)^3}\right\} - 2L^{-1}\left\{\frac{1}{(s+1)^4}\right\} = \left[3e^{-t}L^{-1}\left\{\frac{1}{s^3}\right\} + -2e^{-t}L^{-1}\{\right.$

$\Rightarrow 3e^{-t}\frac{t^2}{2} - 2\times e^{-t}\frac{t^3}{6}$

$\Rightarrow \frac{3}{2}e^{-t}t^2 - \frac{1}{3}e^{-t}t^3$

$\Rightarrow e^{-t}t^2\left[\frac{3}{2} - \frac{1}{3}t\right]$

Ex: 2 $L^{-1}\left\{\frac{10}{(s-2)^4}\right\}$

$\Rightarrow 10\, L^{-1}\left\{\frac{1}{(s-2)^4}\right\}$

$\Rightarrow 10\, e^{+2t}\, L^{-1}\left\{\frac{1}{s^4}\right\}$

$\Rightarrow 10 e^{2t} \times \frac{t^3}{6}$

$\Rightarrow \frac{5}{3} e^{2t} t^3$

Ex: 3 $L^{-1}\left\{\frac{5}{(s+2)^5}\right\}$

$\Rightarrow 5\, L^{-1}\left\{\frac{1}{(s+2)^5}\right\}$

$\Rightarrow 5 e^{-2t}\, L^{-1}\left\{\frac{1}{s^5}\right\}$

$\Rightarrow 5 e^{-2t} \frac{t^4}{24}$

$\Rightarrow \frac{5}{24} e^{-2t} t^4$

Laplace Transform of Integral of a fun^n -

→ Division by S property

If $L\{f(t)\} = F(s)$ then

$L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}F(s)$

Ex: 1 Find $L\left\{\int_0^t f(t)\,dt\right\} = L\left\{\int_0^t (t^4 + \sin 3t)\,dt\right\}$

$= L\left\{\int_0^t t^4\,dt + \int_0^t \sin 3t\,dt\right\}$

$= L\left\{\int_0^t t^4\,dt\right\} + L\left\{\int_0^t \sin 3t\,dt\right\}$

$= \frac{1}{s} L\{t^4\} + \frac{1}{s} L\{\sin 3t\}$

$= \frac{4!}{s \times s^5} + \frac{1}{s}\left[\frac{3}{s^2+9}\right]$

Ex: 2 $L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\}$

= By the divided by S property

$= \frac{1}{s} L\left\{\frac{\sin t}{t}\right\} \rightarrow (1)$

By divided by t property

$= L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty L\{f(t)\}\,ds$

$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}\,ds$

$= \int_s^\infty \left(\frac{1}{s^2+1}\right) ds$

$L\left\{\frac{\sin t}{t}\right\} = [\tan^{-1} s]_s^\infty$

$= \tan^{-1}\infty - \tan^{-1} s$

$= \cot^{-1} s$

$= \frac{1}{s}(\cot^{-1} s)$

Prove that $\int_0^\infty \frac{e^{-t}-e^{-4t}}{t}\,dt = 2\log 2$

Evaluation of an integral using L.T

$\Rightarrow \int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-4t}}{t}\right)dt$

$\Rightarrow L\left\{\frac{e^{-t}-e^{-4t}}{t}\right\}$

$\Rightarrow \cancel{\int_0^\infty L\left\{\frac{e^{-t}}{t}\right\}} - \cancel{\int_0^\infty L\left\{\frac{e^{-4t}}{t}\right\}}$

$= \int_s^\infty L\{e^{-t}-e^{-4t}\}\,ds$

$= \int_s^\infty \left(\frac{1}{s+1} - \frac{1}{s+4}\right)ds$

$= [\log(s+1) - \log(s+4)]_s^\infty$

$= \left[\log\left(\frac{s+1}{s+4}\right)\right]_s^\infty$

$= \left[\log(\infty) - \log\left(\frac{s+1}{s+4}\right)\right]$

$= -\log\left(\frac{s+1}{s+4}\right)$

$= \log\left(\frac{s+4}{s+1}\right)$

$= \int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-4t}}{t}\right)dt$

Let $s=0$ $\quad = \log\left(\frac{0+4}{0+1}\right)$

$= \log 4$

$= \log 2^2$

$= 2\log 2$

Ex: 4 Find L.T of the foll^n

i) $e^{2t}+4t^3-2\sin 3t+3\cos 3t$

ii) $e^{2t}+4t^3-\sin 2t * \cos 3t$

iii) $3t^2+e^{-t}+\sin^3 2t$

iv) $(t^2+a)^2$

v) if $L\{f(t)\} = \log\left(\frac{s+3}{s+1}\right)$ then find $L\{f(2t)\}$

vi) $(t+1)^2 * e^{at}$

vii) $e^{-3t}t^4$

viii) $e^{-4t}\cosh 2t$

ix) $t\sin at$

x) $t\cos^2 t$

xi) $L\{t^2\cos at\}$

xii) $\frac{1-e^{-t}}{t}$

xiii) $\frac{\cosh 2t \sin 2t}{t}$

xiv) $\frac{1-\cos t}{t}$

→ i) $e^{2t}+4t^3-2\sin 3t+3\cos 3t$

$L\{e^{2t}\}+4L\{t^3\}-2L\{\sin 3t\}+3L\{\cos 3t\}$

$= \frac{1}{s-2} + \frac{4\times 3!}{s^4} - 2\times\frac{3}{s^2+9} + 3\times\frac{s}{s^2+9}$

$= \frac{1}{s-2} + \frac{24}{s^4} - \frac{6}{s^2+9} - \frac{3s}{s^2+9}$

$\rightarrow$ ii) $e^{2t} + 4t^3 - \sin 2t \cos 3t$

$\Rightarrow L\{e^{2t}\} + 4L\{t^3\} - \frac{1}{2} L\{\sin 5t + (-\sin t)\}$

$\Rightarrow L\{e^{2t}\} + 4L\{t^3\} - \frac{1}{2} L\{\sin 5t\} + \frac{1}{2} L\{\sin t\}$

$= \frac{1}{s-2} + \frac{24}{s^4} - \frac{5}{2(s^2+25)} + \frac{1}{2(s^2+1)}$

$\rightarrow$ iii) $3t^2 + e^{-t} + \sin^3 2t$

$\Rightarrow 3L\{t^2\} + L\{e^{-t}\} + \frac{1}{4} L\{3\sin 2t - \sin 6t\}$

$\Rightarrow \frac{3 \times 2!}{s^3} + \frac{1}{s+1} + \frac{1}{4}\left[\frac{3 \times 2}{s^2+4} - \frac{6}{s^2+36}\right]$

$\Rightarrow \frac{6}{s^3} + \frac{1}{s+1} + \frac{6(s^2+36) - 6(s^2+4)}{(s^2+4)(s^2+36)} \times \frac{1}{4}$

$\Rightarrow \frac{6}{s^3} + \frac{1}{s+1} + \frac{102}{4(s^2+4)(s^2+36)}$

$\rightarrow$ iv) $(t^2 + a)^2$

$\Rightarrow t^4 + 2t^2 a + a^2$

$\Rightarrow L\{t^4\} + 2a\, L\{t^2\} + a^2 L\{1\}$

$\Rightarrow \frac{24}{s^5} + \frac{4 \cancel{a} 8}{s^3} + \frac{a^2\, 64}{s}$

$\Rightarrow \frac{24}{s^5} + \frac{32}{s^3} + \frac{64}{s}$

$\rightarrow$

$\rightarrow$ v) $L\{f(t)\} = \log\left(\frac{s+3}{s+1}\right)$

$= L\{f(2t)\} = \frac{1}{2} F\left(\frac{s}{2}\right)$

$= \frac{1}{2} \log\left(\frac{s+6}{s+2}\right)$.

$\rightarrow$ vi) $(t+1)^2 * e^{at}$.

Scanned by CamScanner

Derivative of Inverse Laplace Transform

* Multiplication by t Property.

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\left\{\frac{d}{ds} F(s)\right\} = -t f(t)$ or $f(t) = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds} F(s)\right\}$

Ex: 1 Find $L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\}$

$F(s) = \log\left(\frac{s+a}{s+b}\right)$.

$= \log(s+a) - \log(s+b)$

$\frac{d}{ds} F(s) = \frac{1}{s+a} - \frac{1}{s+b}$

By Multiplication by 't' pro.

$L^{-1}\left\{\frac{d}{ds} F(s)\right\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds} F(s)\right\}$

$= -\frac{1}{t} L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\}$

$= -\frac{1}{t}\left[L^{-1}\left\{\frac{1}{s+a}\right\} - L^{-1}\left\{\frac{1}{s+b}\right\}\right]$

$= -\frac{1}{t}\left\{e^{+at} - e^{-bt}\right\}$

$= \frac{1}{t}\left[e^{-bt} - e^{at}\right]$

Ex: 2 Find $L^{-1}\left\{\log\left(1+\frac{\omega^2}{s^2}\right)\right\}$

$= L^{-1}\left[\log\left(\frac{s^2+\omega^2}{s^2}\right)\right]$

$$F(s) = \log\left(\frac{s^2+\omega^2}{s^2}\right)$$

$$= \log(s^2+\omega^2) - 2\log s$$

$$\frac{d}{ds}F(s) = \frac{2s}{s^2+\omega^2} - \frac{2}{s} = \frac{s^2}{s^2+\omega^2}$$

By Multiplication by 't' pro.

$$f(t) = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}F(s)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{1}{s^2+\omega^2} - \frac{1}{s^2}\right\} = -\frac{1}{t} L^{-1}\left\{\frac{s^2}{s^2+\omega^2}\right\}$$

$$= -\frac{1}{t}\left[L^{-1}\left\{\frac{2s}{s^2+\omega^2}\right\} - L^{-1}\left\{\frac{2}{s}\right\}\right]$$

$$= -\frac{2}{t}\left[\cos\omega t - 1\right]$$

$$= \frac{2}{t}\left[1-\cos\omega t\right]$$

Ex: 3 Find $L^{-1}\left\{\log\left(\frac{1}{s}\right)\right\}$

$$F(s) = \log\left(\frac{1}{s}\right)$$

$$= \log(1) - \log s$$

$$= -\log s$$

$$\frac{d}{ds}F(s) = -\frac{1}{s}$$

By Multiplication by 't'

$$f(t) = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}F(s)\right\}$$

$$= +\frac{1}{t} L^{-1}\left\{\frac{1}{s}\right\}$$

$$= \frac{1}{t}$$

Multiplication by S Property

If $L^{-1}\{F(s)\} = f(t)$ at $f(0)=0$ then
$L^{-1}\{sF(s)\} = \frac{d}{dt}f(t) = f'(t)$

Ex: 1 Find $L^{-1}\left\{\frac{s}{s^2+1}\right\}$ by using diff^n of L.T

Sol^n: Here $F(s) = \frac{1}{s^2+1}$

$$\therefore L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \sin t$$

$$f(0) = \sin 0 = 0$$

$$L^{-1}\left\{\frac{s}{s^2+1}\right\} = \frac{d}{dt}\sin t = \cos t$$

Ex: 2 $L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}$

Here $F(s) = \frac{s}{(s^2+a^2)^2}$

$$\therefore L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

$$= \frac{1}{2a}\, t\sin at$$

Scanned by CamScanner

$f(0) = \sin 0 = 0$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \frac{d}{dt}\left(\frac{1}{2a} t \sin at\right)$$

$$= \frac{1}{2a}\sin at + \frac{a}{2a} t\cos at$$

$$= \left[\frac{1}{2} t\cos at + \frac{1}{2a}\sin at\right] //$$

Find $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$

Find $L^{-1}\left\{\frac{s}{s^2-a^2}\right\}$

1. ⇒ $F(s) = \frac{s}{(s^2+1)^2}$

$\frac{d}{ds}F(s) = \quad L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

$$= \frac{1}{2}[\sin t - t\cos t]$$

$f(0) = \sin 0 = 0$

$$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t\sin t \quad \frac{d}{dt}\left[\frac{1}{2}(\sin t - t\cos t\right]$$

$$= \frac{1}{2}\left[\cos t - \cos t + t\sin t\right]$$

$$= \frac{1}{2} t \sin t$$

2. ⇒ $F(s) = \frac{s}{s^2-a^2}$

$$L^{-1}\{F(s)\} = \frac{1}{a}L^{-1}\left\{\frac{a}{s^2-a^2}\right\}$$

$$= \sinh at / a \qquad \sinh 0 = 0$$

$$L^{-1}\left\{\frac{s}{(s^2-a^2)}\right\} = \frac{d}{dt}\left(\frac{\sinh at}{a}\right)$$

$$= \frac{a\cosh at}{a}$$

$$= \cosh at$$

Division By t property

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\left\{\int_s^\infty F(s)\,ds\right\} = \frac{f(t)}{t}$

Ex: 1 Find $L^{-1}\left\{\int_s^\infty \left(\frac{1}{s^2-1}\right) ds\right\}$

Here $F(s) = \frac{1}{s^2-1}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2-1}\right\}$$

$$= \sinh t$$

By Division by t property we can write

$$L^{-1}\left\{\int_s^\infty \left(\frac{1}{s^2-1}\right) ds\right\} = \frac{f(t)}{t} = \frac{\sinh t}{t}$$

Ex: 2 Find $L^{-1}\left\{\int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1}\right) ds\right\}$

Here $F(s) = \frac{1}{s} - \frac{1}{s+1}$

Scanned by CamScanner

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$
$$= 1 - e^{-t}$$

By division by t property

$$L^{-1}\left\{\int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1}\right) ds\right\} = \frac{f(t)}{t}$$
$$= \frac{1-e^{-t}}{t}$$

Ex:3 Find $L^{-1}\left\{\int_s^\infty \left(\frac{1}{s^2+4} + \frac{s}{s^2-4}\right) ds\right\}$

Here $F(s) = \frac{1}{s^2+4} + \frac{s}{s^2-4}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+4}\right\} + L^{-1}\left\{\frac{s}{s^2-4}\right\}$$
$$= \frac{1}{2}\sin 2t + \cosh 2t$$

By division by t property

$$L^{-1}\left\{\int_s^\infty \left(\frac{1}{s^2+4} + \frac{s}{s^2-4}\right) ds\right\} = \frac{f(t)}{t}$$
$$= \frac{\frac{1}{2}\sin 2t + \cosh 2t}{t}$$
$$= \frac{\sin 2t + 2\cosh 2t}{2t}$$

Division by s property

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt$

Ex:1 Find $L^{-1}\left\{\frac{1}{s(s+a)^3}\right\}$

$$L^{-1}\left\{\frac{1}{s} \times \frac{1}{(s+a)^3}\right\}$$

Here $F(s) = \frac{1}{(s+a)^3}$.

$$= \frac{e^{-at}\,t^2}{2!}$$
$$= \frac{e^{-at}t^2}{2}$$

By division by s property

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt$$
$$= \int_0^t \frac{e^{-at}t^2}{2}\,dt$$
$$= \frac{1}{2}\left[\frac{t^2e^{-at}}{-a} - \int \frac{2te^{-at}}{a^2}dt\right] + 2\frac{e^{-at}}{-a^3}\Bigg]_0^t$$
$$= \frac{1}{2}\left[\frac{t^2e^{-at}}{-a} - \frac{2te^{-at}}{a^2} - \frac{2e^{-at}}{a^3} + \frac{2e^{-at}}{a^3}\right]$$
$$= -\frac{t^2e^{-at}}{2a} - \frac{te^{-at}}{a^2} - \frac{e^{-at}}{a^3} + \frac{1}{a^3}$$
$$= \frac{e^{-at}}{2a}\left[-t^2 - \frac{2t}{a} - \frac{2}{a^2}\right] \frac{1}{a^3}$$

Ex:2 $L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\}$

$= L^{-1}\left\{\frac{1}{s} \times \frac{1}{s^2+a^2}\right\}$

$F(s) = \frac{1}{s^2+a^2}$

$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$

$= \frac{1}{a} \sin at$

$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt$

$= \frac{1}{a}\int_0^t \sin at \, dt$

$= -\frac{1}{a^2}[\cos at]_0^t$

$= -\frac{1}{a^2}[\cos at - 1]$

$= \frac{1}{a^2}[1 - \cos at]$

Ex:3 Find $L^{-1}\left\{\frac{4}{s^2+4s}\right\}$

$= L^{-1}\left\{\frac{1}{s} \times \frac{4}{s+4}\right\}$

$= F(s) = \frac{4}{s+4}$

$L\{t e^{-2t} \sin t\}$

1) $L(e^{3t}), e(e^{-2t}), L(e^{3/2 t}), L(e^{-5t})$

$= \frac{1}{3} = \frac{1}{-2}, = \frac{7}{3} = -\frac{1}{5}$

2) $\frac{1}{s+3} = \frac{1}{s-2} = \frac{1}{s+3/2} = \frac{1}{s-5}$

2) $\frac{5}{6}, \frac{-3}{5}, \frac{3}{25}, -\frac{5}{4s}$

3) $\frac{4}{s^2+16}, \frac{2}{s^2+4}, \frac{2/3}{s^2+4/9}$

4) $\frac{s}{s^2+49}, \frac{s}{s^2+9}, \frac{s}{s^2+(3/8)}$

5) $\frac{4}{s^2-16}, \frac{2}{s^2-4}, \frac{2/3}{s^2-4/9}$

6) $\frac{s}{s^2-49}, \frac{s}{}$

$= L^{-1}\{F(s)\} = 4L^{-1}\left\{\frac{1}{s+4}\right\}$

$= 4e^{-4t}$

By division by s property

$L^{-1}\left\{\frac{F(s)}{s}\right\} = 4\int_0^t e^{-4t}\,dt$

$= 4\left[\frac{e^{-4t}}{-4}\right]_0^t$

$= -1[e^{-4t} - 1]$

$= 1 - e^{-4t}$

Ex:4 $L^{-1}\left\{\frac{8}{s^2-4s}\right\}$

$= L^{-1}\left\{\frac{1}{s}\times\frac{8}{s-4}\right\}$

$F(s) = \frac{8}{s-4}$

$L^{-1}\{F(s)\} = 8L^{-1}\left\{\frac{1}{s-4}\right\}$

$= 8e^{4t}$

By division by s property

$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t 8e^{4t}\,dt$

$= 8\left[\frac{e^{4t}}{4}\right]_0^t = 2\left[e^{4t}\right]_0^t$

$= 2e^{4t} - 2$

Ex:5 $L^{-1}\left\{\frac{8}{s^3-4s}\right\}$

$= L^{-1}\left\{\frac{8}{s(s^2-4)}\right\}$

$F(s) = \frac{8}{s^2-4}$

$L^{-1}\{F(s)\} = 8L^{-1}\left\{\frac{1}{s^2-4}\right\}$

$= \frac{4\cancel{8}\,1}{2}\sinh 2t = 4\sinh 2t$

By division by s property

$L^{-1}\left\{\frac{F(s)}{s}\right\} = 4\int_0^t \sinh 2t\,dt$

$= \frac{\cancel{4}^{2}}{2}\left[\cosh 2t\right]_0^t = 2\left[\cosh 2t - 1\right]$

Date: MON TUE WED THU FRI SAT

Change of Scale Property

→ If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$

Method of Partial Fraction

There are four methods of partial fraction

i) Non-Repeated Linear Factor
ii) Repeated Linear Factor
iii) Non-Repeated Quadratic Factor
iv) Repeated Quadratic Factor

i) → Ex:1 Find $L^{-1}\left\{\frac{5s^2+3s-16}{(s-1)(s-2)(s-3)}\right\}$

Let $\frac{5s^2+3s-16}{(s-1)(s-2)(s-3)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$ —(*)

$\frac{5s^2+3s-16}{(s-1)(s-2)(s-3)} = \frac{A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2)}{(s-1)(s-2)(s-3)}$

$5s^2+3s-16 = A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2)$

i) $s=1$ ii) $s=2$

$A\times -1\times -2 = 5+3-16$ $20+6-16 = B\times 1\times -1$

$2A = -8 \Rightarrow \boxed{A=-4}$ $\boxed{B=-10}$

iii) $s=3$

$C\times 2\times 1 = 45+9-16$

$\boxed{C=19}$

$\frac{5s^2+3s-16}{(s-1)(s-2)(s-3)} = \frac{-4}{(s-1)} + \frac{(-10)}{(s-2)} + \frac{19}{(s-3)}$

$= L^{-1}\left\{\frac{5s^2+3s-16}{(s-1)(s-2)(s-3)}\right\} = -4L^{-1}\left\{\frac{1}{s-1}\right\} - 10L^{-1}\left\{\frac{1}{s-2}\right\} + 19L^{-1}\left\{\frac{1}{s-3}\right\}$

$= -4e^{t} - 10e^{2t} + 19e^{3t}$

Date: ______
MON TUE WED THU FRI SAT

Find $L^{-1}\left\{\frac{1}{(s-2)(s-3)}\right\}$

$= \text{Let } \frac{1}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$

$1 = A(s-3) + B(s-2)$

i) $s = 2$ ii) $s = 3$

$1 = A(-1)$ $1 = B(1)$

$A = -1$ $B = 1$

$= \frac{1}{(s-2)(s-3)} = -\frac{1}{s-2} + \frac{1}{s-3}$

$= L^{-1}\left\{\frac{1}{(s-2)(s-3)}\right\} = -L^{-1}\left\{\frac{1}{s-2}\right\} + L^{-1}\left\{\frac{1}{s-3}\right\}$

$= -e^{2t} + e^{3t}$

ii) $\Rightarrow$ Find $L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$

$= \text{Let } \frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s+2)}$

$= 4s+5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$

i) $s = 1$ ii) $s=-2$ iii) A $s = 0$

$4+5 = B(3)$ $-8+5 = C(-3)^2$ $5 = -2A+2B+$

$B = +\frac{9}{3}$ $C = \frac{-3}{9} = -1/3$ $A = +\frac{5-1}{2} = \frac{1}{3}$

$B = 3$

$\frac{4s+5}{(s-1)^2(s+2)} = +\frac{1}{3(s-1)} + \frac{3}{(s-1)^2} + \frac{-1/3}{(s+2)}$

$L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\} = \frac{1}{3}L^{-1}\left\{\frac{1}{s-1}\right\} + 3L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{s+2}\right\}$

$= \frac{1}{3}e^{t} + 3\, e^{t}t - \frac{1}{3}e^{2t}$

$a_n x^n$ $n+2$ $n+1$ $x^2=$

$n a_n x^{n-1}$ $(n+1)a_{n+1}x^n$ $a_{n+1}x^{n-1}$

$n(n-1)a_n x^{n-2}$ $n(n+1)$ $y'' - 4y = 0$

$(n+2)(n+1)a_{n+2}x^n$ $(D^2-4) =$

$-a_n + (n+1)a_{n+1} = 0$ $D^2 = 4$

$a_{n+1} = \frac{a_n}{n+1}$ // $D = \pm$

x

$1 = x$ $\cos 2x = \cos^2 x - \sin$

$\cos 2x = \cos^2 x -$

$= 1 - \sin^2 x$

$D^2 + D - 12 = 0$

$m^2 + 4m - 3m - 12 = 0$ $\left(\frac{D^2}{4}\right)^2 +$

$m(m+4) - 3(m+4)$

$64 - 80 + 28 - 3$ $a_3 =$

$64 - 92 - 83$

$8 - 20 + 7\times 23$ $20a_3$

$15 - 23$ $8 - 20$

$\frac{23}{15}$ $22-$

$(x^2 - x)^2$

$x^4 - 2x^3 + x^2$

$\cancel{s^2} + \cancel{a^2} + 2as + \cancel{a^2} -$ $-16 - 324$

$\cancel{s^2} - \cancel{a^2} + 2as - \cancel{a^2}$ -1.6

34

$xi - i^2 + i$

$xi - 1 + i$

$= \frac{x}{2}\cos x + \frac{\cos x}{2} - \frac{\sin x}{2}$

Scanned by CamScanner

iii) ⇒

1) Find $L^{-1}\left\{\frac{s}{(s^4+4a^4)}\right\}$ $\quad = (a+b)^2 - 2ab$

$\quad = (a-b)^2 + 2ab$

$s^4 + 4a^4 = (s^2)^2 + (2a^2)^2$

$= (s^2+2a^2)^2 - 2(s)^2(2a^2)$

$= (s^2+2a^2)^2 - 4a^2s^2$

$= (s^2+2a^2)^2 - (2as)^2$

$= (s^2+2a^2-2as)(s^2+2a^2+2as)$

$\frac{s}{(s^2+4a^4)} - \frac{As+B}{(s^2+2a^2+2as)} + \frac{Cs+D}{(s^2+2a^2-2as)}$

$s = A(s)+B(s^2+2a^2-2as) + Cs+D(s^2+2a^2+2as)$

$s = As^3 + 2Aa^2s - 2aAs^2 + Bs^2 + 2Ba^2 \mp 2aBs +$

$Cs^3 + 2Ca^2s + 2aCs^2 + Ds^2 + 2Da^2 + 2aDs$

$s = (A+C)s^3 + (2Aa^2 - 2aB + 2Ca^2 + 2aD)s$

$(-2aA + B + 2aC + D) + (2Ba^2 + 2a^2D)$

$A + C = 0 \qquad 2Ba^2 + 2a^2D = 0$

$A = -C \qquad D = -B$

$2Aa^2 - 2aB + 2Ca^2 + 2aD = 1$

$-2Ca^2 + 2aD + 2Ca^2 + 2aD = 1$

$4aD = 1$

$D = \frac{1}{4a}, \quad B = \frac{-1}{4a}, \quad A = 0, C = 0$

$\Rightarrow L^{-1}\left\{\frac{s}{s^2+4a^4}\right\} = -\frac{1}{4a} L^{-1}\left\{\frac{1}{s^2+2a^2+2as}\right\} +$

$\frac{1}{4a} L^{-1}\left\{\frac{1}{s^2-2as+2a^2}\right\}$

$= -\frac{1}{4a} L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\} + \frac{1}{4a} L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\}$

$= -\frac{1}{4a} \times \frac{e^{-at}\sin at}{-a} + \frac{1}{4a} \times e^{at}\frac{\sin at}{a}$

Scanned by CamScanner

$$= \frac{1}{2a^2} \sin at \left[\frac{e^{at} + e^{-at}}{2} \right]$$

$$= \frac{1}{2a^2} \sin at \cosh at$$

K) $\Rightarrow$ Find $L^{-1}\left\{ \frac{s+3}{(s^2+6s+13)^2} \right\}$

$$\Rightarrow \frac{s+3}{(s^2+6s+13)^2} = \frac{As+B}{(s^2+6s+13)} + \frac{Cs+D}{(s^2+6s+13)^2}$$

$$s+3 = As+B(s^2+6s+13) + Cs+D$$

$$s+3 = As^3 + 6As^2 + 13As + Bs^2 + 6sB + 13B + Cs + D$$

$$13A + 6B + C = 1 \qquad A = 0, \; B = 0$$

$$C = 1, \; D = 3$$

$$\Rightarrow L^{-1}\left\{ \frac{s+3}{(s^2+6s+13)^2} \right\} = L^{-1}\left\{ \frac{s}{(s^2+6s+13)^2} \right\} + 3L^{-1}\left\{ \frac{1}{(s^2+6s+13)^2} \right\}$$

$$= L^{-1}\left\{ \frac{s}{((s+3)^2+(2)^2)^2} \right\} + 3\left\{ \frac{1}{((s+3)^2+(2)^2)^2} \right\}$$

$$= \frac{e^{-3t}}{4} t \sin 2t + \frac{3}{16} e^{-3t} [\sin 2t - 2t \cos 2t]$$

$$= \frac{e^{-3t}}{4} \left[t \sin 2t + \frac{3}{2} \sin 2t \right] - \frac{3}{8} e^{-3t} t \cos 2t$$

$$L^{-1}\left\{ \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} \right\}$$

$$= \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{(s^2+2s+2)} + \frac{Cs+D}{(s^2+2s+5)}$$

$$= s^2+2s+3 = As+B(s^2+2s+5) + Cs+D(s^2+2s+2)$$

$$= s^2+2s+3 = As^3 + 2As^2 + 5As + Bs^2 + 2Bs + 5B + Cs^3 + 2Cs^2 + 2Cs + Ds^2 + 2sD + 2D$$

$$A + C = 0 \qquad 2A + B + 2C + D = 1$$

$$A = -C \qquad B = 1 - D$$

$$5A + 2B + 2C + 2D = 2$$

$$-3C = 0$$

$$C = 0 \Rightarrow A = 0$$

$$5B + 2D = 3$$

$$5 - 5D + 2D = 3$$

$$-3D = -2$$

$$D = \frac{2}{3}, \; B = \frac{1}{3}$$

$$\Rightarrow L^{-1}\left\{ \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} \right\} = \frac{1}{3} L^{-1}\left\{ \frac{1}{s^2+2s+2} \right\} + \frac{2}{3} L^{-1}\left\{ \frac{1}{s^2+2s+5} \right\}$$

$$\Rightarrow \frac{1}{3} L^{-1}\left\{ \frac{1}{(s+1)^2+1^2} \right\} + \frac{2}{3} L^{-1}\left\{ \frac{1}{(s+1)^2+(2)^2} \right\}$$

$$\Rightarrow \frac{1}{3} e^{-t} \sin t + \frac{2}{3} e^{-t} \sin 2t$$

$$\Rightarrow \frac{e^{-t}}{3} [e^{-t} \cdot \sin t + 2 \sin 2t]$$

Find $L^{-1}\left\{\frac{s}{(s^2-1)^2}\right\}$

$= \frac{s}{(s^2-1)^2} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s+1)} + \frac{D}{(s+1)^2}$

$s = A(s-1)(s+1)^2 + B(s+1)^2 + C(s-1)(s+1)^2 + D(s-1)^2$

i) $s=1$ ii) $s=-1$ iii) $s=0$

$1 = B(2)^2$ $D(4) = -1$ $A(-1) + B + C(-1) = 0$

$B = \frac{1}{4}$ $D = -\frac{1}{4}$ $A = -C$

iv) $s=2$

$2 = A(1)(9) + 9B + C(3) + D$

$C = \frac{4}{3}$ $A = -\frac{4}{3}$

$L^{-1}\left\{\frac{s}{(s^2-1)^2}\right\} = -\frac{4}{3}L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$
$+ \frac{4}{3}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{4}L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$

$= -\frac{4}{3}e^t + \frac{1}{4}e^t t + \frac{4}{3}e^{-t} - \frac{1}{4}e^{-t}t$

$\Rightarrow -\frac{8}{3}\left[\frac{e^t - e^{-t}}{2}\right] + \frac{t}{2}\left[\frac{e^t - e^{-t}}{2}\right]$

$= -\frac{8}{3}\sinh t + \frac{t}{2}\sinh t$

Find $L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\}$

$= \frac{s+2}{(s^2+4s+5)^2} = \frac{As+B}{(s^2+4s+5)} + \frac{Cs+D}{(s^2+4s+5)^2}$

$= s+2 = As+B(s^2+4s+5) + Cs+D$

$A=0$

$B=0$

$C=1$

$D=2$

$\Rightarrow L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\}$

$= L^{-1}\left\{\frac{s+2}{((s+2)^2+(1)^2)^2}\right\}$

$= \frac{t}{2}e^{-2t}\sin t$

Convolution

Convolution of a fun^n $f(t)$ & $g(t)$ is denoted as $f(t)*g(t)$ & it is defined as

$f(t)*g(t) = \int_0^t f(u)g(t-u)\,du$

$= \int_0^t f(t-u)g(u)\,du$

$= g(t) * f(t)$

This shows that Convolution of $f(t)$ & $g(t)$ is comutative

Convolution Theorem

If $L^{-1}\{F(s)\} = f(t)$ and $L^{-1}\{G(s)\} = g(t)$ then

$$L^{-1}\{F(s)\,G(s)\} = \int_0^t f(u)\,g(t-u)\,du$$

$$= f(t) * g(t).$$

Properties of Convolution

i) Commutative : $f*g = g*t$

ii) Associative : $f*(g*h) = (f*g)*h$

iii) Distributive : $f*(g+h) = (f*g)+(f*h)$

Ex: 1 Find the Value of $1*1$ where $*$ denote Convolution product

Let $f(t)=1$ & $g(t)=1$

By Convolution theorem

$$f(t)*g(t) = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t (1)(1)\,du$$

$$= [u]_0^t$$

$$= [t-0]$$

$$= t$$

Ex: 2 Find the Convolution of t & e^t

Evaluate $t * e^t$

Here $f(t) = t$ & $g(t) = e^t$

$\therefore f(u) = u$ & $g(t-u) = e^{t-u}$

By Convolution theorem

$$\therefore f(t)*g(t) = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t u \times e^{t-u}\,du$$

$$= \int_0^t u\,[e^t \times e^{-u}]\,du$$

$$= e^t \int_0^t u e^{-u}\,du$$

$$= e^t\left[-ue^{-u} - e^{-u}\right]_0^t$$

$$= e^t\left[-te^{-t} - e^{-t} - 0 + 1\right]$$

$$= -t - 1 + e^t$$

$$= -[t+1-e^t] = e^t - t - 1$$

Ex: 3 Evaluate $1 * \sin \omega t$

Here $f(t) = 1$ & $g(t) = \sin \omega t$

$f(t-u) = 1$ & $g(u) = \sin \omega u$

$$= f(t)*g(t) = \int_0^t 1 * \sin \omega u\,du$$

$$= \left[-\frac{\cos \omega u}{\omega}\right]_0^t$$

$$= -\frac{\cos \omega t}{\omega} + \frac{1}{\omega}$$

$$= \frac{1}{\omega} - \frac{\cos \omega t}{\omega}$$

Ex $L^{-1}\left\{\frac{1}{(s^2+\omega^2)^2}\right\}$

Here, $L^{-1}\left\{\frac{1}{(s^2+\omega^2)(s^2+\omega^2)}\right\}$

$F(s) = \frac{1}{s^2+\omega^2}$ and $G(s) = \frac{1}{s^2+\omega^2}$

$L^{-1}\{F(s)\} = f(t) = L^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{1}{\omega}\sin\omega t$

$L^{-1}\{G(s)\} = g(t) = L^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{1}{\omega}\sin\omega t.$ $\frac{1}{\omega}\sin\omega(t-u)$

By Convolution theorem

$L^{-1}\{F(s)\cdot G(s)\} = \int_0^t f(u)\, g(t-u)\, du$

$= \int_0^t \frac{1}{\omega}\sin\omega u \cdot \frac{1}{\omega}\sin\omega(t-u)\, du$

$= \frac{1}{2\omega^2}\int_0^t 2\sin\omega u\,\sin\omega(t-u)\, du$

$= \frac{1}{2\omega^2}\int_0^t\left[-\cos\omega t + \cos(2\omega u - \omega t)\right] du$

$= \frac{1}{2\omega^2}\left[-\int_0^t \cos\omega t\, du + \int_0^t \cos(2\omega u - \omega t)\, du\right]$

$= \frac{1}{2\omega^2}\left[\left[\frac{\sin(2\omega u-\omega t)}{2\omega}\right]_0^t - \cos\omega t\,[u]_0^t\right]$ $\cos\omega t\,[t-0]$

$= \frac{1}{2\omega^2}\left[\left[\frac{\sin\omega t}{2\omega} + \frac{\sin\omega t}{2\omega} - t\cos t\omega\right]\right.$

$= \frac{1}{2\omega^2}\left[\frac{\sin\omega t}{\omega} - t\cos\omega t\right]$

$= \frac{1}{2\omega^3}\left[\sin\omega t - \omega t\cos\omega t\right]$

Evaluate $L^{-1}\left\{\frac{1}{(s+1)(s^2-2s+2)}\right\}$

$F(s) = \frac{1}{s+1}$ and $G(s) = \frac{1}{s^2-2s+2}$

$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

$g(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s^2-2s+2}\right\} = e^{t}\sin t$

By Convolution theorem

$L^{-1}\{F(u)\cdot G(u)\} = \int_0^t f(t-u)\, g(u)\, du$

$= \int_0^t e^{-(t-u)}\cdot e^{u}\sin u\, du$

$= \int_0^t e^{-t}\cdot e^{u}\cdot e^{u}\sin u\, du$

$= e^{-t}\left[\int_0^t e^{2u}\sin u\, du\right]$

$= e^{-t}\left[\frac{e^{2u}}{5}\left[2\sin u - \cos u\right]\right]_0^t$

$= e^{-t}\left[\frac{e^{2t}}{5}\left[2\sin t - \cos t\right] - \frac{1}{5}[-1]\right]$

$= \frac{e^{t}}{5}(2\sin t - \cos t) + \frac{e^{-t}}{5}$

Using Convolution theorem to evaluate

$L^{-1}\left\{\frac{a}{s^2(s^2+a^2)}\right\}$

$F(s) = \frac{1}{s^2}$ and $G(s) = \frac{a}{s^2+a^2}$

$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t$

$g(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$

By Convolution theorem

$L^{-1}\{F(s)G(s)\} = \int_0^t f(t-u)\, g u\, du$

$= \int_0^t (t-u)\sin au\, du$

$= t\int_0^t \sin au\, du - \int_0^t u \sin au\, du$

$= t\left[-\frac{\cos au}{a}\right]_0^t - \left[u\left[-\frac{\cos au}{a}\right] + \frac{\sin au}{a^2}\right]_0^t$

$= t\left[-\frac{\cos at}{a} + \frac{1}{a}\right] - \left[t\left[-\frac{\cos at}{a}\right] + \frac{\sin at}{a^2} - 0\right]$

$= -\frac{t\cos at}{a} + \frac{t}{a} + \frac{t\cos at}{a} - \frac{\sin at}{a^2}$

$= \frac{t}{a} - \frac{\sin at}{a^2}$

$f(t) * g(t) = \int_0^t f(t)\cdot g(t-u)\, du$

Ex Apply convolution theorem to evaluate

$L^{-1}\left\{\frac{1}{s^2(s-1)}\right\}$

Let $F(s) = \frac{1}{s^2}$ and $G(s) = \frac{1}{s-1}$

$L^{-1}\{F(s)\} = f(t) = L^{-1}\left\{\frac{1}{s^2}\right\} = t$

$L^{-1}\{G(s)\} = g(t) = L^{-1}\left\{\frac{1}{s-1}\right\} = e^t$

= By Convolution theorem

$L^{-1}\{F(s)G(s)\} = \int_0^t f(u)\, g(t-u)\, du = \int_0^t f(t-u) g(u)\, du$

$= \int_0^t (t-u)e^u du$

$= t\int_0^t e^u du - \int_0^t u e^u du$

$= t\left[e^u\right]_0^t - \left[ue^u - e^u\right]_0^t$

$= te^t - t - \left[te^t - e^t - 0 + 1\right]$

$= te^t - t - te^t + e^t - 1$

$= e^t - t - 1$

Use Convolution theorem to evaluate

Unit-Step function

The unit-step fun^n $u(t-a)$ is defined as

$$u(t-a) = \begin{cases} 0 & \text{if } t<a \quad (a\geq 0) \\ 1 & \text{if } t\geq a \end{cases}$$

in particular if $a=0$ then we have $a=0$

$$u(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & t\geq 0 \end{cases}$$

Laplace Transform of unit-step function

By the definition of L.T

$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt$

$= \int_0^a e^{-st} u(t-a)\, dt + \int_a^\infty e^{-st} u(t-a)\, dt$

$= 0 + \int_a^\infty e^{-st}\, dt = \frac{1}{s}e^{-as}$

$L^{-1}\left\{ \frac{1}{s} e^{-as} \right\} = u(t-a)$

In Particular if $a=0$ then

$L^{-1}\left\{ \frac{1}{s} \right\} = u(t)$

Second Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{f(t-a) \cdot u(t-a)\}$
$= e^{-as}F(s)$

By definition of L.T $L\{f(t-a)u(t-a)\}$.

$\int_0^\infty e^{-st} f(t-a)u(t-a)dt$

$= \int_0^a e^{-st} f(t-a)u(t-a)dt + \int_a^\infty e^{-st} f(t-a) u(t-a)dt$

$= \int_a^\infty e^{-st} f(t-a)\,dt$

Let $t-a = r$
$t = a+r$
$dt = dr$
$t \to a$, $r \to 0$
$t \to \infty$, $r \to \infty$

$= \int_0^\infty e^{-s(a+r)} f(r)dr$

$\Rightarrow \int_0^\infty e^{-sa-sr} f(r)dr$

$\Rightarrow e^{-sa} \int_0^\infty e^{-sr} f(r)dr$

$\Rightarrow e^{-as} F(s)$

Corollary: 1 $L\{f(t)u(t-a)\} = e^{-as} L\{F(t+a)\}$

Corollary: 2 $L\{u(t-a)\} = L\{1 \cdot u(t-a)\}$
$= e^{-as} L\{1\} = \frac{e^{-as}}{s}$

Find the L.T of the foll.

1. $t \cdot u(t-a)$

By Corollary: 1 we have $L\{f(t)u(t-a)\} = e^{-as} L\{F(t+a)\}$

Let $f(t) = t$

$\therefore F(t+a) = t+a$

$\therefore L\{t \cdot u(t-a)\} = e^{-as} L\{t+a\}$
$= e^{-as}L\{t\} + e^{-as}L\{a\}$
$= e^{-as}\left\{ \frac{1}{s^2} + \frac{a}{s} \right\}$

$= e^{-as}\left\{ \frac{1+as}{s^2} \right\}$

2. $L\left\{ (t-\frac{1}{2})^2 u(t-\frac{1}{2}) \right\}$

By Corollary: 1 we can write

$L\{f(t)u(t-a)\} = e^{-as} L\{F(t+a)\}$

Let $f(t) = (t-\frac{1}{2})^2$

$\therefore F(t+a) = F(t+\frac{1}{2}) = \left[t+\frac{1}{2}-\frac{1}{2}\right]^2 = t^2$

$\therefore L\{(t-1/2)^2 \cdot u(t-1/2)\} = e^{-s/2} L\{t^2\}$
$= e^{-s/2} \times \frac{2}{s^3}$

$= \frac{2e^{-s/2}}{s^3}$

3. Find $L\{e^{-3t} u(t-2)\}$

$f(t) = e^{-3t}$

$F(t+a) = e^{-3t} + 2$

$\therefore L\{e^{-3t} u(t-2)\} = e^{-2s} L\{e^{-3t} + 2\}$

$= e^{-2s} \left[L\{e^{-3t}\} + L\{1\} \times 2 \right]$

$= e^{-2s} \left[\frac{1}{s+3} + \frac{2}{s} \right]$

$\Rightarrow e^{-2s} \left[\frac{s + 2(s+3)}{s(s+3)} \right]$

$\Rightarrow e^{-2s} \left[\frac{3s+3}{s^2+3s} \right]$

Find inverse laplace transform of the foll. by using Second shifting theorem of inverse laplace transform

1. $L^{-1}T$ of Second Shifting theorem is $L^{-1}\{e^{-as}F(s)\} = f(t-a)\, u(t-a)$

Find $L^{-1}\left\{ \frac{e^{-2s}}{s-5} \right\}$

= By Second shifting theorem we can write

$L^{-1}\{e^{-as}F(s)\} = f(t-a)\, u(t-a) \rightarrow (1)$

Here, $a = 2$

$F(s) = \frac{1}{s-5}$

$L^{-1}\{F(s)\} = L^{-1}\left\{ \frac{1}{s-5} \right\}$

$f(t) = e^{5t}$

$f(t-a) = e^{5(t-a)}$

$= e^{5(t-2)}$

$\therefore$ from eq^n (1)

$L^{-1}\left\{ \frac{e^{-2s}}{s-5} \right\} = e^{5(t-2)}\, u(t-2)$

Find $L^{-1}\left\{ \frac{se^{-3s}}{s^2+16} \right\}$

= By second shifting theorem is

$L^{-1}\{e^{-as}F(s)\} = f(t-a)\, u(t-a)$

Here, $a = 3$

$F(s) = \frac{s}{s^2+16}$

$L^{-1}\{F(s)\} = L^{-1}\left\{ \frac{s}{s^2+16} \right\}$

$f(t) = \cos 4t$

$f(t-a) = \cos 4(t-3)$

$\therefore$ from eq^n (1)

$L^{-1}\left\{ \frac{se^{-3s}}{s^2+16} \right\} = \cos 4(t-3)\, u(t-3)$

Application of Laplace Transform.

- The Laplace transform method is useful in solving linear ordinary differential eqn with Constant co-efficient.
- To solve the eqn by foll. steps

Steps-1 Linear diff. eqn

$\downarrow$ L.T.

Step-2 Algebraic eqn

$\downarrow$

Step-3 Soln of eqn

$\downarrow$ I.L.T

Step-4 Soln of diff. eqn

$y' = SF(s) - f(0)$

$y'' = S^2F(s) - SF(0) - F'(0)$

Ex:1 Solve $\frac{dy}{dt} + 2y = e^{-3t}$; $y(0) = 1$

Taking L.T on both the side

$L\{\frac{dy}{dt} + 2y\} = L\{e^{-3t}\}$

$L\{\frac{dy}{dt}\} + 2L\{y\} = \frac{1}{S+3}$

$SF(S) - F(0) + 2F(s) = \frac{1}{S+3}$

$(S+2)F(s) - 1 = \frac{1}{S+3}$

$F(s) = \frac{1 + S+3}{(S+3)(S+2)}$

$F(s) = \frac{S+4}{(S+3)(S+2)}$

$\frac{S+4}{(S+3)(S+2)} = \frac{A}{(S+3)} + \frac{B}{(S+2)}$

$S+4 = A(S+2) + B(S+3)$

$S+4 = (A+B)S + 2A + 3B$

$A + B = 1$ $\qquad$ $2A + 3B = 4$

$B = 1 - A$ $\qquad$ $2A + 3(1-A) = 4$

$B = 2$ $\qquad$ $2A + 3 - 3A = 4$

$-A = 1$

(i) $S = -2$ $\qquad$ $A = -1$

$B = 2$

$L^{-1}\{F(s)\} = 2L^{-1}\{\frac{1}{S+2}\} + (-1)L^{-1}\{\frac{1}{S+3}\}$

$= 2e^{-2t} - e^{-3t} = F(t)$

Ex:2 $y'' + 3y' + 2y = e^{t}$, $y(0) = 1$, $y'(0) = 0$

Taking L.T on both the side

$L\{y'' + 3y' + 2y\} = L\{e^{t}\}$

$L\{y''\} + 3L\{y'\} + 2L\{y\} = \frac{1}{S-1}$

$S^2F(s) - SF(0) - F'(0) + 3[SF(s) - F(0)] + 2F(s) = \frac{1}{S-1}$

$= (S^2F(s) + 3S + 2)F(s) - (S+3)F(0) - F'(0) = \frac{1}{S-1}$

$= (S^2 + 3S + 2)F(s) - (S+3) = \frac{1}{S-1}$

$= F(s) = \frac{1 + (S-1)(S+3)}{(S-1)(S^2+3S+2)}$

$= \frac{1 + S^2 + 2S - 3}{(S-1)(S^2+3S+2)}$

$$F(s) = \frac{s^2+2s-2}{(s-1)(s^2+3s+2)}$$

$$= \frac{s^2+2s-2}{(s-1)(s+1)(s+2)}$$

$$= \frac{s^2+2s-2}{(s-1)(s+1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$= s^2+2s-2 = A(s+1)(s+2) + B(s-1)(s+2)$$
$$C(s+1)(s-1)$$

(i) $s = -1$

$B(-2)(1) = 1-2-2$

$B(-2) = \frac{(-3)}{2} = \frac{-3}{-2}$

(ii) $s = 1$

$A(2)(3) = 1+2-2$

$A = \frac{1}{6}$

(iii) $s = -2$

$C(-1)(-3) = 4-4-2$

$C = -\frac{2}{3}$

$$L^{-1}\{F(s)\} = \frac{1}{6} L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{3}{2} L^{-1}\left\{\frac{1}{s+1}\right\}$$
$$- \frac{2}{3} L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{e^{t}}{6} + \frac{3e^{-t}}{2} - \frac{2e^{-2t}}{3}$$

Ex: 3 $y'' + y = t$, $y(0) = y'(0) = 1$

Taking L.T on both the side

$L\{y''+y\} = L\{t\}$

$L\{y''\} + L\{y\} = \frac{1}{s^2}$

$s^2F(s) - sF(0) - F'(0) + F(s) = \frac{1}{s^2}$

$s^2F(s) + F(s) - sF(0) - F'(0) = \frac{1}{s^2}$

$(s^2+1)F(s) - sF(0) - F'(0) = \frac{1}{s^2}$

$(s^2+1)F(s) - s - 1 = \frac{1}{s^2}$

$(s^2+1)F(s) = \frac{1}{s^2} + s + 1$

$(s^2+1)F(s) = \frac{1+s^2+s^3}{s^2}$

$F(s) = \frac{s^3+s^2+1}{s^2(s^2+1)}$

$$\frac{s^3+s^2+1}{s^2(s^2+1)} = \frac{s^3}{s^2(s^2+1)} + \frac{(s^2+1)}{(s^2+1)s^2}$$

$$= \frac{s}{(s^2+1)} + \frac{1}{s^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= \cos t + t$$

Ex:4 $y'' + a^2 y = K\sin at$

Taking L.T both the side

$L\{y'' + a^2 y\} = K\,L\{\sin at\}$

$L\{y''\} + a^2 L\{y\} = K\,L\{\sin at\}$

$s^2F(s) - sF(0) - F'(0) + a^2F(s) = K \cdot \frac{a}{s^2+a^2}$

Here the initial Condition is not given so we assume $F(0) = A$

$F'(0) = B$

$(s^2+a^2)F(s) - sA - B = \frac{Ka}{s^2+a^2}$

$(s^2+a^2)F(s) = \frac{Ka + (s^2+a^2)sA + (s^2+a^2)B}{(s^2+a^2)}$

$F(s) = \frac{As+B}{(s^2+a^2)} + \frac{Ka}{(s^2+a^2)^2}$

Taking inverse L.T on both the side

$L^{-1}\{F(s)\} = A L^{-1}\left\{\frac{s}{s^2+a^2}\right\} + B L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$

$+ Ka\, L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$

$= A\cos at + \frac{B}{a}\sin at + Ka\frac{1}{2a^3}[\sin at - at\cos at]$

$= A\cos at - \frac{B}{a}\sin at + \frac{K}{2a^2}[\sin at - at\cos at]$

Ex:5 $y'' - 2y' - 8y = 0$, $y(0) = 3$, $y'(0) = 6$

Taking L.T on both side

$L\{y'' - 2y' - 8y\} = L\{0\}$

$L\{y''\} - 2L\{y'\} - 8\{y\} = 0$

$s^2F(s) - sF(0) - F'(0) - 2sF(s) + 2F(0) - 8F(s) = 0$

$(s^2 - 2s - 8)F(s) - 3s - \cancel{6} + \cancel{6} = 0$

$F(s) = \frac{3s}{s^2 - 2s - 8}$

$L^{-1}\{F(s)\} = 3L^{-1}\left\{\frac{s}{s^2-2s-8}\right\}$

$\frac{s}{s^2-2s-8} = \frac{As+B}{s^2-2s-8}$

$s = As + B$

$As = s$

$A = 1$

$B = 0$

$L^{-1}\{F(s)\} = 3L^{-1}\left\{\frac{s}{(s^2-2s+1) - 9}\right\}$

$= 3L^{-1}\left\{\frac{s+1-1}{(s-1)^2-(3)^2}\right\}$

$= \frac{3}{3}e^t\cosh 3t + \frac{3}{3}e^t\sinh 3t$

$= 3e^t\cosh 3t + 3e^t\sinh 3t$

$= 3e^t(\cosh 3t + \frac{1}{3}\sinh 3t)$

Ex:6 $y'' + 2y' + y = 6te^{-t}$, $y(0) = y'(0) = 0$

$\Rightarrow$ Taking L.T on both side

$L\{y'' + 2y' + y\} = 6L\{te^{-t}\}$

$L\{y''\} + 2L\{y'\} + L\{y\} = 6\{te^{-t}\}$

$s^2F(s) - sF(0) - F'(0) + 2sF(s) - 2F(0) + F(s) = 6 \times \frac{1}{(s+1)^2}$

$= (s^2+2s+1)F(s) - (s+2)F(0) - F'(0) = \frac{6}{(s+1)^2}$

$F(s) = \frac{6}{(s+1)^2(s^2+2s+1)}$

$$\frac{6}{(s+1)^2(s^2+2s+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{(s^2+2s+1)}$$

$$6 = A(s+1)(s^2+2s+1) + B(s^2+2s+1) + Cs+D\,(s+1)^2$$

$s = -1$ $\qquad\qquad$ $s = 0$

$6 = B(1-2+1)$. $\qquad\qquad$ $s = 1$

$A + B + D = 6$ $\qquad\qquad$ $8A + 4B + 4C + 4D = 6$

$s = 2$ $\qquad\qquad$ $4A + 2B + 2C + 2D = 3$

$27A + 9B + 18C + 9D = 6$

$9A + 3B + 2C + 3D = 2$

$-\underline{4A - 2B - 2C - 2D = -3}$

$5A + B + D = -1$

$A + B + D = 6$

$\underline{-\quad -\quad -\qquad -}$

$4A = -7$

$A = -\frac{7}{4}$.

$$F(s) = \frac{6}{(s+1)^4}$$

$$L^{-1}\{F(s)\} = 6\, L^{-1}\left\{\frac{1}{(s+1)^4}\right\}$$

$$= \frac{6 \cdot e^{-t} \cdot t^3}{3}$$

$$= 2e^{-t} t^3$$

Ex: 79

Unit : 3

Probability And Probability Distribution

Sample Space: A Set of all possible Outcomes from an experiment is called a Sample Space.

Ex: Toss two Coins simultaneously the possible results are (H,H), (T,T), (T,H), (H,T) where (H,H), (T,T), (T,H), (H,T) are Sample point

The Collection of Sample points is called Sample Space & it is denoted by S.

$S = \{(H,H), (T,T), (T,H), (H,T)\}$

Event: The Subcollection of number of Sample points under a definite rule is called an event.

Ex: Let assume that dice has six faces i.e 1, 2, 3, 4, 5, 6 are Sample point. A be the event getting an even no. on the dice, so, $A = \{2, 4, 6\}$ which is the Subset of a Sample Space S.

B be the even getting an odd no. on the dice. So, $B = \{1, 3, 5\}$ which is the Subset of a Sample Space S.

Null Event: An Event having no Sample point is called Null even and it is denoted by ϕ.

Simple Event: An even having only one sample point is called simple event.
Ex: Let a dice be rolled once and A be the event having face 5. Then A is simple event.

Mutually exclusive event: If in an experiment the occurance of an event prevents or rules out the happening of all others events in the same experiment then this events are called mutually exclusive event.
Ex: In tossing a coin the events head or tail are mutually exclusive. If the outcome is head then possibility of getting a tail in the same trial is ruled out.

Equally likely event: Events are said to be equally likely then there is no reason to expect any one in ~~refer-~~ preference to other.
Ex: In ~~a~~ throwing the dice all the six faces are equally likely to occur.

Independent events And Dependent events
when the experiment are conducted in such a way that the occurance of an event in one trial doesn't have any effect on the occurance of other events at a subsequent experiment then the events are called independent events.
Two or more events are called independent. Events which are not independent is called Dependent.
Ex: If we draw a card from a pack of well shuffle cards and again draw a card from the rest of pack of cards containing 51 cards then the second draw is dependent on the first. but if on the other hand we draw a second card from the pack cards replacing the first card draw then the second draw is called independent of first.

In an experiment has n mutually exclusive, equally likely and exhaustive cases out of which m are favorable to happening of a event A that the Probability of happening of A is denoted by P(A) and defined as

$$P(A) = \frac{m}{n}$$

m = no. of favorable cases of A
n = total no. of cases.

NOTE > 1. Probability of an event which is certain to occur is 1.
Probability of an impossible is 0.

2. The Probability of occurance of any event lies between 0 & 1, both inclusive.

Ex:1 What is the probability of getting an even number in a single throw with a dice.

⇒ The total number of cases are 1, 2, 3, 4, 5, 6

But the no. of favorable cases i.e 2, 4, 6 getting an even no. in a single throw

∴ Probability of getting even no

$= \frac{m}{n} = \frac{3}{6} = \frac{1}{2}$

Ex:2 What is the probability of getting tail in a throw of a coin?

⇒ Total no. of cases are 2 {H, T}

⇒ But no. of favorable cases are 1

$P(A) = \frac{m}{n} = \frac{1}{2}$

What is the probability when a card is drawn at random from an ordinary pack of cards if it is

(i) a red card

(ii) club card

(iii) one of the court card

(i) Red card $= \frac{26}{52}$ Total no. of cards = 52. There are 26 red & 26 black

$= \frac{1}{2}$

(ii) Club card

No. of favorable cases = 13

∴ Probability of getting club

$= \frac{13}{52}$

$= \frac{1}{4}$

(iii) one of the court card (King, Queen, Jack)

There are $(4 \times 3) = 12$ court cards in the pack of card

∴ The no of favorable case $m = 12$

∴ Probability of getting a court card

$= \frac{12}{52} = \frac{3}{13}$

Ex:3 What is the probability of throwing a no greater than three with an ordinary dice?

⇒ Total no. of cases are 1, 2, 3, 4, 5, 6

⇒ But no. of favorable cases are 4, 5, 6

Probability of getting

$= \frac{m}{n} = \frac{3}{6} = \frac{1}{2}$

Ex:4 What is the probability of getting a total of more than 10 in a single throw with two dice

⇒ Total no. of cases are

{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)

(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)

(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)

(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)

(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)

$$P = \frac{m}{n} = \frac{3}{36} = \frac{1}{12}$$

Ex:5 A Card is thrown an ordinary pack of playing cards and a person bets that it is a Spade or ace that what is the probability of his winning this bets

⇒ Total number of cards = 52.

No. of favourable cases = 13 + 3

$$P(A) = \frac{13+3}{52} = \frac{16}{52}$$

$$= \frac{4}{13}$$

★ Addition theorem of Probability

If A and B are the events then the probability of happening of at list one of the event is defined as $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

NOTE ⟩ If A & B are mutually exclusive then probability of $A \cap B = 0$

∴ Probability of $(A \cup B) = P(A) + P(B)$

★ Multiplication theorem of Probability

Theorem of Compound Probability.

The Probability of the Simultaneous occurance of the two events A & B is equal to the probability of one of the events multiply by the Conditional Probability of other, given the occurance of the first

$P(AB) = P(A)$

$P(B/A) = P(B) \cdot P(A/B)$

Ex:1 A dice is rolled what is the probability that a number 1 or 6 may appear on the upper face?

⇒ The probability of appearing the no. '1' on the upper face $= \frac{1}{6}$

Similarly,

The probability of appearing the no '6' on the upper face $= \frac{1}{6}$

$$\therefore P(A) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

Ex:2 If the probability of the horse A winning the race is 1/5 & the Probability of the horse will win the Same race = 1/6 then what is the probability that one of the horse will win the race.

⇒ Probability of winning horse.

$A = \frac{1}{5}$

Probability of winning of horse B is $\frac{1}{6}$

Probability $= \frac{1}{5} + \frac{1}{6}$

$= \frac{6+5}{30}$

$= \frac{11}{30}$