Question Bank

Section I

Unit 1: Ordinary linear differential equation of 2^{nd} order & higher order.

1) Define order and degree of Ordinary differential equation.

Ans: Order :-

- Order of a differential equation is the order of the higherst order derivative present in the equation.

degree :-

- The degree of differential equation is represented by the power of the highest order derivative in the given eq^n.

2) 1) Find y_c if $m = 2, 2$

$$y_c = \cancel{C_1 e^{2x}} \, (C_1 + C_2) e^{2x}$$

$$y_c = (C_1 + C_2 x) e^{2x}$$

2) Find CF for $m = 1, 2$

$$CF = C_1 e^{x} + C_2 e^{2x}$$

3) Find CF for $(D^2 - 2D + 1)y = \cos 3x$

For CF Auxillary eq^n is

$$D^2 - 2D + 1 = 0$$

putting $D = m$

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m - 1 = 0$$

$$\therefore m = 1$$

$$\therefore CF = C_1 e^{x}$$

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4) Find y_c if $m = 1, 2, 3$.

$$y_c = C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x}$$

3) 1) Find y_p if $(D^3 + 2D^2 - 10D + 4)y = 0$.

Here $f(x) = 0$.

$\therefore y_p = 0$.

2) For $(D^2 - 4)y = 0$ find PI

Here $f(x) = 0$.

$\therefore PI = 0$

3) Find PI For $(D^2 - 5D + 6)y = 0$.

Here $f(x) = 0$

$\therefore PI = 0$

4) Define Linearly Independent and Linearly Dependent function.

Ans: Linearly Independent function :-

- A set of function $y_1(x), y_2(x), ---, y_n(x)$ are said to be linearly Independent function if $k_1 y_1(x) + k_2 y_2(x) + \_\_\_ + k_n y_n(x) = 0$ where $k_1 = k_2 = k_3, --- = 0$.

→ Linearly dependent function :-

- A set of function $y_1(x), y_2(x), \_\_\_ y_n(x)$ are said to be linearly dependent function if $k_1 y_1(x) + k_2 y_2(x) + \_\_\_ + k_n y_n(x) = 0$ where k's are not zero.

5) Find order of the diff. eqⁿ :-

i) $\left(\frac{d^4x}{dt^4}\right)^6 + \left(\frac{d^5x}{dt^5}\right)^3 = m^4 x$

Order : 5

ii) $\frac{dy}{dx} + y = \sin x$

Order : 1

iii) $\left(\frac{d^2x}{dt^2}\right) + \left(\frac{dx}{dt}\right)^3 = \sin x$

order : 2

iv) $\frac{dy}{dx} + 2y = \cos x$

order : 1

6) Find degree of given diff. eqⁿ

$\left(\frac{d^3y}{dx^3}\right)^5 + \left(\frac{d^2y}{dx^2}\right)^4 + y = 0.$

Ans. degree :- 5

7) check whether the given sets are linearly Independent or linearly dependent

i) $\{1, e^x, e^{-x}\}$

$$W(1, e^x, e^{-x})(x) = \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix}$$

$= 1(e^x \cdot e^{-x} + e^{-x} e^x)$

$= e^0 + e^0$

$= 1 + 1 = 2 \neq 0.$

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$\therefore W \neq 0$

$\therefore$ The given function is L.I.

ii) $\{e^x, xe^x\}$

$$W(e^x, xe^x)(x) = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}$$

$$= xe^{2x} + e^{2x} - xe^{2x}$$

$$= e^{2x}$$

$$\neq 0$$

$W \neq 0$

$\therefore$ The given function is L.I.

iii) $\{e^x, e^{-x}, \cosh x\}$

$$W(e^x, e^{-x}, \cosh x)(x) = \begin{vmatrix} e^x & e^{-x} & \cosh x \\ e^x & -e^{-x} & \sinh x \\ e^x & e^{-x} & \cosh x \end{vmatrix}$$

$$= e^x(-e^{-x}\cosh x - e^x \sinh x)$$

$$-e^{-x}(e^x \cosh x - e^x \sinh x)$$

$$+ \cosh x(e^{\cancel{2x}} + e^0)$$

$$= -\cosh x - e^{2x}\sinh x$$

$$- \cosh x + \sinh x + 2\cosh x$$

$$\neq 0$$

$W \neq 0$

$\therefore$ The given function is L.I.

8) Solve $(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x$

Ans: For CF The Auxillary eqn. is

$$D^3 - 5D^2 + 7D - 3 = 0$$

	1	-5	7	-3
1	0	1	-4	3
	1	-4	3	0

$$(D-1)(D^2 - 4D + 3) = 0$$

$$(D-1)(D-3)(D-1)) = 0$$

$$D = 1, 1, 3$$

$$C.F = (C_1 + C_2 x)e^{x} + C_3 e^{3x}$$

→ For PI $= \frac{1}{F(D)} \cdot R(x)$

$$= \frac{1}{(D^3 - 5D^2 + 7D - 3)} e^{2x}\cosh x$$

$$= \frac{1}{D^3 - 5D^2 + 7D - 3} e^{2x}\left(\frac{e^{x} + e^{-x}}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{1}{D^3 - 5D^2 + 7D - 3} (e^{3x} + e^{x})$$

$$= \frac{1}{2}\left\{\frac{e^{3x}}{D^3 - 5D^2 + 7D - 3} + \frac{e^{x}}{D^3 - 5D^2 + 7D - 3}\right\}$$

$$= \frac{1}{2} \cancel{6} x \cdot \frac{e^{3x}}{3D^2 - 10D + 7} + \frac{1}{2} \cdot x \cdot \frac{e^{x}}{3D^2 - 10D + 7}$$

$D = a = 3$ $\qquad$ $D =$

$$= \frac{1}{2} x \cdot \frac{e^{3x}}{4} + \frac{1}{2} x^2 \cdot \frac{e^{x}}{6D - 10} \quad D = a = 1$$

$$= \frac{1}{8} x \cdot e^{3x} + \frac{1}{2} x^2 \cdot \frac{e^{x}}{-4}$$

$$PI = \frac{1}{8} x \cdot e^{3x} - \frac{1}{8} x^2 e^{x}$$

The general solution is

$$y = CF + PI$$

$$= (C_1 + C_2 x)e^{x} + C_3 e^{3x} + \frac{1}{8} x e^{3x} - \frac{1}{8} x^2 e^{x}$$

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9) solve $(D^2+4)y = e^x + \sin 2x$

Ans: For CF, The Auxillary eq^n is

$$D^2+4=0$$
$$D^2=-4$$
$$D=\pm 2i$$
$$CF = c_1\cos 2x + c_2 \sin 2x$$

→ For PI $= \frac{1}{F(D)}(e^x + \sin 2x)$

$$= \frac{1}{D^2+4} e^x + \frac{1}{D^2+4}\sin 2x$$

$D=a=1$ $\qquad$ $D^2=-a^2=-4$

$$= \frac{1}{1+4}e^x + x\cdot\frac{1}{2D}\sin 2x$$

$$= \frac{1}{5}e^x + \frac{x}{2}\left(\frac{1}{D}\times\frac{+D}{+D}\right)\sin 2x$$

$$= \frac{1}{5}e^x + \frac{x}{2}\left(\frac{+D}{+D^2}\right)\sin 2x$$

$D^2=-a^2=-4$

$$= \frac{1}{5}e^x + \frac{x}{2}\left(\frac{2\cos 2x}{-4}\right)$$

$$PI = \frac{1}{5}e^x - \frac{1}{4}x\cos 2x$$

∴ The General solution is

$$y = CF + PI$$
$$= c_1\cos 2x + c_2\sin 2x + \frac{1}{5}e^x - \frac{1}{4}x\cos 2x$$

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10) Solve $(D^2+16)y = x^4 + e^{3x} + \cos 3x$

Ans: For CF, The Auxillary eq^n

$$D^2 + 16 = 0$$
$$D^2 = -16$$
$$D = \pm 4i$$
$$CF = c_1 \cos 4x + c_2 \sin 4x$$

→ For PI $= \frac{1}{F(D)} (x^4 + e^{3x} + \cos 3x)$

$$= \frac{1}{D^2+16} x^4 + \frac{1}{D^2+16} e^{3x} + \frac{1}{D^2+16} \cos 3x$$

$a = 3$ $\quad D^2 = -a^2 = -9$

$$= \frac{1}{16(1 + D^2/16)} x^4 + \frac{1}{25} e^{3x} + \frac{1}{7} \cos 3x$$

$$= \frac{1}{16}\left(1 + \frac{D^2}{16}\right)^{-1} x^4 + \frac{1}{25} e^{3x} + \frac{1}{7} \cos 3x$$

$$= \frac{1}{16}\left[1 - \frac{D^2}{16} + \left(\frac{D^2}{16}\right)^2 - \left(\frac{D^2}{16}\right)^3\right] x^4 + \frac{1}{25} e^{3x} + \frac{1}{7} \cos 3x$$

$$= \frac{1}{16}\left[x^4 - \frac{12}{16} x^2 + \frac{(4 \times 3 \times 2 \times 1)}{256}\right] + \frac{1}{25} e^{3x} + \frac{1}{7} \cos 3x$$

$$PI = \frac{1}{16}\left(x^4 - \frac{3}{4} x^2 + \frac{3}{32}\right) + \frac{1}{25} e^{3x} + \frac{1}{7} \cos 3x$$

The General sol^n,

$$y = CF + PI$$

$$= c_1 \cos 4x + c_2 \sin 4x + \frac{1}{16}\left(x^4 - \frac{3}{4} x^2 + \frac{3}{32}\right) + \frac{1}{25} e^{3x} + \frac{1}{7} \cos 3x$$

11) Solve $(D^2+2)y = e^x \cos 2x$

Ans: For CF The Auxillary eq^n.

$$D^2+2=0$$
$$D^2=-2$$
$$D=\pm\sqrt{2}i$$
$$CF = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x$$

→ For PI $= \dfrac{1}{F(D)} e^x \cos 2x$

$$= \frac{1}{D^2+2} e^x \cos 2x \qquad a=1 \quad D+\cancel{0}\ D+a \to D+1$$

$$= \frac{1}{(D+1)^2+2} e^x \cos 2x$$

$$= e^x \frac{1}{D^2+2D+3} \cos 2x$$

$$D^2=-a^2=-4$$

$$= e^x \frac{1}{-4+2D+3} \cos 2x$$

$$= e^x \frac{1}{2D-1} \cos 2x$$

$$= e^x \left(\frac{1}{2D-1} \times \frac{2D+1}{2D+1}\right) \cos 2x$$

$$= e^x \left(\frac{2D+1}{4D^2-1}\right) \cos 2x$$

putting $D^2=-4^2=\cancel{4}-4$

$$= e^x \left(\frac{2D+1}{-17}\right) \cos 2x$$

$$= \cancel{0}\frac{e^x}{-17}\left(-4\sin 2x + \cos 2x\right)$$

$$= -\frac{e^x}{17}\left(-4\sin 2x + \cos 2x\right)$$

$$y = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x - \frac{e^x}{17}(\cos 2x - 4\sin 2x)$$

12) solve $(D^2-1)y = x\sin x$

Ans: For CF, The Auxillary eqn

$$D^2 - 1 = 0$$

$$D^2 = 1$$

$$D = \pm 1$$

$$CF = c_1 e^{x} + c_2 e^{-x}$$

$\rightarrow$ $PI = \frac{1}{F(D)} x\sin x$

$$= x \cdot \frac{1}{F(D)} \sin x - \frac{F'(D)}{[F(D)]^2} \sin x$$

$$= PI_1 - PI_2 \quad \text{①}$$

$\rightarrow$ $PI_1 = x \cdot \frac{1}{(D^2-1)} \sin x \qquad D^2 = -a^2 = -1$

$$= -\frac{x}{2} \sin x$$

$\rightarrow$ $PI_2 = \frac{F'(D)}{[F(D)]^2} \sin x = \frac{2D}{[(D^2-1)]^2} \sin x$

$$= \frac{2D}{D^4 - 2D^2 + 1} \sin x \qquad D^2 = -a^2 = -1$$

$$= \frac{2D}{(-1)(-1) - 2(-1) + 1} \sin x$$

$$= \frac{2D}{1+2+1} \sin x = \frac{2D\sin x}{4}$$

$$= \frac{1}{2} \cos x$$

$\therefore PI = -\frac{x}{2}\sin x - \frac{1}{2}\cos x$

$$y = c_1 e^{x} + c_2 e^{-x} - \frac{x}{2}\sin x - \frac{1}{2}\cos x$$

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13) Using method of undetermined coefficient solve $y'' - 2y' + 5y = 5x^3 - 6x^2 + 6x$

Ans:- For CF The Auxillary eq^n

$$D^2y - 2Dy + 5y = 0$$

$$(D^2 - 2D + 5)y = 0$$

$$D^2 - 2D + 5 = 0.$$

$$m^2 - 2m + 5 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$m = 1 \pm 2i$$

$$CF = (C_1 \cos 2x + C_2 \sin 2x)e^x + (C_3 \cos 2x - C_4 \sin 2x)e^x$$

$\rightarrow$ $$PI = 5 \cdot \frac{1}{D^2 - 2D + 5} x^3 - 6 \cdot \frac{1}{D^2 - 2D + 5} x^2 + 6 \cdot \frac{1}{D^2 - 2D + 5} x$$

$$= 5 \cdot \frac{1}{5\left(1 - \frac{2}{5}D + \frac{D^2}{5}\right)} x^3 - \frac{6 \cdot x^2}{5\left(1 - \frac{2}{5}D + \frac{D^2}{5}\right)} + \frac{6 \cdot x}{5\left(1 - \frac{2}{5}D + \frac{D^2}{5}\right)}$$

$$= \left(1 - \left(\frac{2}{5}D + \frac{D^2}{5}\right)\right)^{-1} x^3 - \frac{6}{5}\left(1 - \frac{2}{5}D + \frac{D^2}{5}\right)^{-1} x^2$$

$$+ \frac{6}{5}\left(1 - \frac{2}{5}D + \frac{D^2}{5}\right)^{-1} x$$

$$= \left[1 + \left(\frac{2}{5}D + \frac{D^2}{5}\right) + \left(\frac{2}{5}D + \frac{D^2}{5}\right)^2 + \left(\frac{2}{5}D + \frac{D^2}{5}\right)^3\right] x^3$$

$$\frac{4}{25}D^2 + \frac{4}{25}D^3 + \frac{D^4}{25} \qquad \frac{8}{125}D^3 +$$

$$-\frac{6}{5}\left[1+\left(\frac{2}{5}D+\frac{D^2}{5}\right)+\left(\frac{2}{5}D+\frac{D^2}{5}\right)^2\right]x^2$$

$$+\frac{6}{5}\left[1+\left(\frac{2}{5}D\right)\right]x$$

$$=\left[x^3+\frac{6}{5}x^2+\frac{6x}{5}+\frac{4}{25}\times 6x+\frac{4}{25}\times 6+\frac{8}{125}\times 6\right]$$

$$-\frac{6}{5}\left[x^2+\frac{4}{5}x+\frac{2}{5}+\frac{4}{25}\times 2\right]+\frac{6}{5}\left[x+\frac{2}{5}\right]$$

$$=x^3+\cancel{\frac{6}{5}x^2}+\frac{6}{5}x+\cancel{\frac{24}{25}x}+\frac{24}{25}+\cancel{\frac{48}{125}}$$

$$-\cancel{\frac{6}{5}x^2}-\cancel{\frac{24}{25}x}-\cancel{\frac{12}{25}}-\cancel{\frac{48}{125}}+\frac{6}{5}x+\cancel{\frac{12}{25}}$$

$$=x^3+\frac{12}{5}x+\frac{24}{25}$$

$$y=(c_1\cos 2x+c_2\sin 2x)e^{x}+(c_3\cos 2x-c_4\sin 2x)e^{x}$$

$$+x^3+\frac{12}{5}x+\frac{24}{25}$$

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13) Using method of undetermine co-efficient.

$y'' - 2y' + 5y = 5x^3 - 6x^2 + 6x$ — ①

Ans: Auxillary eq^n

$y(D^2 - 2D + 5)y = 0$

$m^2 - 2D + 5 = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4-20}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$= 1 \pm 2i$

$CF = (C_1 \cos 2x + C_2 \sin 2x) e^x$

→ Basis For $5x^3 = \langle x^3, x^2, x, 1 \rangle$

Basis For $6x^2 = \langle x^2, x, 1 \rangle$

Basis for $6x = \langle x, 1 \rangle$

Basis For $5x^3 - 6x^2 + 6x$ is =

$\langle x^3, x^2, x, 1 \rangle \cup \langle x^2, x, 1 \rangle \cup \langle x, 1 \rangle$

$= \langle x^3, x^2, x, 1 \rangle$

$y_p = PI = Ax^3 + Bx^2 + Cx + D$

$y'p = 3Ax^2 + 2Bx + C$

$y''p = 6Ax + 2B$

from eq^n ①

→ $y'' - 2y' + 5y = 5x^3 - 6x^2 + 6x$

$6Ax + 2B - 6Ax^2 - 4Bx - 2C + 5Ax^3 + 5Bx^2 + 5Cx + 5D$

$= 5x^3 - 6x^2 + 6x$

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$$\therefore\ 5Ax^3 + (5B - 6A)x^2 + (6A - 4B + 5C)x + (2B - 2C + 5D) = 5x^3 - 6x^2 + 6x$$

Comparing we get

$\rightarrow\ 5Ax^3 = 5x^3$

$\boxed{A = 1}$.

$\rightarrow\ (5B - 6A)x^2 = -6x^2$

$5B - 6 = -6$

$5B = 0$

$\boxed{B = 0}$

$\rightarrow\ (6A - 4B + 5C) = 6$

$6 + 5C = 6$

$5C = 0$

$\boxed{C = 0}$ $\cancel{C = -\frac{12}{5}}$

$\rightarrow\ 2B - 2C + 5D = 0$

$\cancel{-2 - \frac{12}{5} + 5D = 0}$

$\cancel{5D = \frac{24}{5}}$

$\cancel{D = \frac{24}{25}}$ $\boxed{D = 0}$

$\therefore\ y_P = x^3 + \cancel{\frac{12}{5}x} + \cancel{\frac{24}{25}}$

The General solⁿ

$y = (C_1 \cos 2x + C_2 \sin 2x)e^{x} + x^3 + \cancel{\frac{12}{5}x + \frac{24}{25}}$

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14) Using method of undetermined coefficients solve ~~$y''-9$~~ $y''-9y = x^3+e^{2x}-\sin x$ — (1)

Ans: Auxillary eq^n $(D^2-9)y = 0$

$\cancel{D(D-9)=0}$

$\cancel{D=}$ $D^2 = 9$

$D = \pm 3$

$CF = C_1e^{3x}+C_2e^{-3x}$

→ Basis for $x^3 = \langle x^3, x^2, x, 1\rangle$

Basis for $e^{2x} = \langle e^{2x}\rangle$

Basis for $\sin x = \langle \sin x, \cos x\rangle$

Basis for $x^3+e^{2x}-\sin x = \langle x^3, x^2, x, 1\rangle \cup \langle e^{2x}\rangle \cup \langle \sin x, \cos x\rangle$

$= \langle x^3, x^2, x, 1, e^{2x}, \sin x, \cos x\rangle$

$y_P = Ax^3+Bx^2+Cx+D+Ee^{2x}+F\sin x+G\cos x$

$y'_P = 3Ax^2+2Bx+C+2Ee^{2x}+F\cos x-G\sin x$

$y''_P = 6Ax+2B+4Ee^{2x}-F\sin x-G\cos x$

putting in eq^n (1)

$\therefore 6Ax+2B+4Ee^{2x}-F\sin x-G\cos x-9Ax^3-9Bx^2-9Cx-9D-9Ee^{2x}-9F\sin x-9G\cos x = x^3+e^{2x}-\sin x$

$\therefore -9Ax^3-9Bx^2+(4E-9E)e^{2x}-(F+9F)\sin x-(G+9G)\cos x+(6A-9C)x+(2B-9D) = x^3+e^{2x}-\sin x.$

$\therefore$ ~~$-9A=$~~ comparing both side

$-9A = 1$

$\boxed{A = -1/9}$

$-9B = 0$, $-5E = 1$, $10F = \sin x$!

$\boxed{B = 0}$ $\boxed{E = -1/5}$ $\boxed{F = \frac{1}{10}}$

$10G = 0$, $6A - 9C = 0$

$\boxed{G = 0}$

$-\frac{6}{9} - 9C = 0$

$9C = -\frac{\cancel{6}2}{\cancel{9}3}$

$\boxed{\cancel{C = \frac{-6}{81}}}$ $\boxed{C = \frac{-2}{27}}$

$2B - 9D = 0$

$-9D = 0$

$\boxed{D = 0}$

$\rightarrow y_P = -\frac{1}{9}x^3 - \frac{2}{27}x + \frac{-1}{5}e^{2x} + \frac{1}{10}\sin x$

The General sol^n is,

$y = c_1 e^{3x} + c_2 e^{-3x} - \frac{1}{9}x^3 - \frac{2}{27}x - \frac{1}{5}e^{2x} + \frac{1}{10}\sin x$

15) Solve $y'' + a^2y = \sec ax$ by the method of variation of parameters.

Ans: Auxillary eq^n is

$$D^2y + a^2y = 0$$

$$(D^2+a^2) = 0$$

$$D^2 = -a^2$$

$$D = \pm ai$$

$$CF = c_1 \cos ax + c_2 \sin ax$$

$PI \Rightarrow y_1 = \cos ax \quad , y_2 = \sin ax$

$$R(x) = \sec ax$$

$$W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a\cos^2 ax + a\sin^2 ax$$
$$= a$$

$$W_1 = \begin{vmatrix} 0 & \sin ax \\ 1 & a\cos ax \end{vmatrix} = -\sin ax$$

$$W_2 = \begin{vmatrix} \cos ax & 0 \\ -a\sin ax & 1 \end{vmatrix} = \cos ax$$

$$y_p = y_1 \int \frac{W_1}{W} R(x)\, dx + y_2 \int \frac{W_2}{W} R(x)\, dx$$

$$= \cos ax \int \frac{-\sin ax}{a} \cdot \sec ax \cdot dx + \cancel{\sin}$$
$$\sin ax \int \frac{\cos ax}{a} \sec ax \cdot dx$$

$$= \cos ax \int \frac{-\sin ax}{a} \cdot \frac{1}{\cos ax} \cdot dx + \sin ax \int \frac{\cos ax\, dx}{a\cos ax}$$

$$= \cos ax \left(-\frac{\log(\sec ax)}{a} \right) + \sin ax \cdot \frac{x}{a}$$

$$y_p = -\frac{1}{a} \log(\sec ax) \cos ax + \frac{x}{a} \sin ax$$

The General sol^n

$y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a} \cos ax \cdot \log(\sec ax)$

$+ \frac{x}{a} \sin ax$

16) using method of undetermined coefficients

$y'' + 2y' + 10y = 25x^2 + 3$ — ①

Ans: Auxillary eq^n is

$D^2y + 2Dy + 10y = 0$

$(D^2 + 2D + 10)y = 0$

$D^2 + 2D + 10 = 0 \qquad m^2 + 2m + 10 = 0$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-2 \pm \sqrt{4 - 40}}{2}$

$= \frac{-2 \pm \sqrt{-36}}{2}$

$= \frac{-2 \pm \cancel{\sqrt{6}}\, 6i}{2}$

$m = -1 \pm 3i$

$CF = (c_1 \cos 3x + c_2 \sin 3x) e^{-x}$

→ Basis for $25x^2 = \langle x^2, x, 1 \rangle$

Basis for $3 = \langle 1 \rangle$

Basis for $25x^2 + 3 = \langle x^2, x, 1 \rangle \cup \langle 1 \rangle = \langle x^2, x, 1 \rangle$

$y_p = Ax^2 + Bx + C$

$y'_p = 2Ax + B$

$y''_p = 2A$ From eq^n ①

$\therefore 2A + 4Ax + 2B + 10Ax^2 + 10Bx + 10C = 25x^2 +$

$\therefore \quad 10Ax^2 + (4A + 10B)x + (2A + 2B + 10C) = 25x^2 \ldots$

$\because$ Comparing both side

$$10A = 25$$

$$\boxed{A = \frac{5}{2}}$$

$$4A + 10B = 0$$

$$4\left(\frac{5}{2}\right) + 10B = 0$$

$$10 + 10B = 0$$

$$10B = -10$$

$$\boxed{B = -1}$$

$$2A + 2B + 10C = 3$$

$$5 - 2 + 10C = 3$$

$$10C = 0$$

$$\boxed{C = 0}$$

$$y_p = \frac{5}{2}x^2 - x$$

The General sol^n is

$$y = (C_1 \cos 3x + C_2 \sin 3x)e^{-x} + \frac{5}{2}x^2 - x$$

17) Solve the equation $\frac{d^2y}{dx^2} + 4y = 4e^{2x}$ by method of undetermined coefficients.

Ans: The Auxillary eq^n is

$$D^2y + 4y = 0$$
$$(D^2+4)y = 0$$
$$m^2 + 4 = 0$$
$$m^2 = -4$$
$$\boxed{m = \pm 2i}$$

$$CF = c_1\cos 2x + c_2 \sin 2x$$

→ Basis for $4e^{2x} = \{e^{2x}\}$

$$y_p = Ae^{2x}$$
$$y'_p = 2Ae^{2x}$$
$$y''_p = 4Ae^{2x}$$

$$4Ae^{2x} + 4Ae^{2x} = 4e^{2x}$$
$$8Ae^{2x} = 4e^{2x}$$
$$8A = 4$$
$$\boxed{A = \frac{1}{2}}$$

$$\therefore y_p = \frac{1}{2}e^{2x}$$

The General solution is

$$y = c_1\cos 2x + c_2\sin 2x + \frac{1}{2}e^{2x}$$

18) solve $(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x$

Ans: take $2x+3 = e^z$

$$2x \cancel{+} = e^z - 3$$

$$x = \frac{e^z - 3}{2}$$

for $2x+3 = e^z$

$$z = \log(2x+3)$$

Put $D = \frac{d}{dz}$

also $ax+b = 2x+3$

$$a = 2, b = 3$$

$$(2x+3)\frac{dy}{dx} = aDy = 2Dy$$

$$(2x+3)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)y = 4D(D-1)y$$

The given eq^n will be

$$4D(D-1)y - 4Dy - 12y = 6\left(\frac{e^z - 3}{2}\right)$$

$$\therefore (4D^2 - 4D - 4D - 12)y = 6\left(\frac{e^z - 3}{2}\right)$$

$$\therefore 4(D^2 - \cancel{4}2D - 3)y = 3e^z - 9$$

→ For CF ⇒ $\cancel{D^2} - 2\ m^2 - 2m - 3 = 0$

$$(m+1)(m-3) = 0$$

$$m = -1, 3$$

$$CF = C_1 e^{-z} + C_2 e^{3z}$$

$$CF = C_1 (2x+3)^3 + \frac{C_2}{(2x+3)}$$

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$$\rightarrow PI = \frac{1}{F(D)} \cdot 6\left(\frac{e^{z}-3}{2}\right)$$

$$= \frac{1}{4(D^2-2D-3)}(3e^{z}-9)$$

$$= \frac{3}{4}\frac{1}{D^2-2D-3}e^{z}\Big|_{a=1} - \frac{9}{4}\frac{1}{D^2-2D-3}\cdot e^{0}\Big|_{a=0}$$

$$= \frac{3}{4}\left[\frac{1}{1-2-3}e^{z} - \frac{3}{-3}\right]$$

$$= \frac{3}{4}\left[\frac{1}{-4}e^{z}+1\right]$$

$$PI = \frac{3}{-16}(e^{z}-4)$$

The general sol^n is,

$$y = c_1e^{-z} + c_2e^{3z} - \frac{3}{16}(e^{z}-4)$$

$$= \frac{c_1}{(2x+3)} + c_2(2x+3)^3 - \frac{3}{16}(2x+3-4)$$

$$= \frac{c_1}{(2x+3)} + c_2(2x+3)^3 - \frac{3}{16}(2x-1)$$

19) solve y'

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19) solve $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

<u>Ans:</u> For CF.

The Auxillary eq^n is,

$(D^2 - 6D + 9)y = 0$

$m^2 - 6m + 9 = 0$

$(m-3)(m-3) = 0$

~~CF = $C_1 e^{3}$~~ $m = 3, 3$

$CF = (C_1 + C_2 x)e^{3x}$

$CF = C_1 e^{3x} + C_2 x e^{3x}$

$y_1 = e^{3x}$

$y_2 = x e^{3x}$

$R(x) = \frac{e^{3x}}{x^2}$

$$W = \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & 3x e^{3x} + e^{3x} \end{vmatrix}$$

$= 3x e^{6x} + e^{6x} - 3x e^{6x}$

$= e^{6x}$

$$W_1 = \begin{vmatrix} 0 & x e^{3x} \\ 1 & 3x e^{3x} + e^{3x} \end{vmatrix}$$

$= -x e^{3x}$

$$W_2 = \begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & 1 \end{vmatrix}$$

$= e^{3x}$

$$y_p = y_1 \int \frac{W_1}{W} R(x)\, dx + y_2 \int \frac{W_2}{W} R(x)\, dx$$

$$= e^{3x} \int \frac{-x e^{3x}}{e^{6x}} \cdot \frac{e^{3x}}{x^2} \cdot dx + x e^{3x} \int \frac{e^{3x}}{e^{6x}} \cdot \frac{e^{3x}}{x^2} \cdot dx$$

$$= e^{3x}\int -\frac{1}{x}\cdot dx + xe^{3x}\int \frac{1}{x^2}\,dx$$

$$= -e^{3x}\log x + xe^{3x}\int [x^{-2}]\,dx$$

$$= -e^{3x}\log x + xe^{3x}\cdot\left(\frac{x^{-1}}{-1}\right)$$

$$= -e^{3x}\log x \cancel{+} - e^{3x}$$

$$y_P = -e^{3x}(\log x + 1)$$

The General sol^n is

$$y = CF + PI$$

$$= C_1 e^{3x} + (C_1 + C_2 x)e^{3x} - e^{3x}(\log x + 1)$$

20) Solve by Euler - Cauchy Method

$$x^3 y''' + 2x^2 y'' + 2y = 10\left(x + \frac{1}{x}\right)$$

Ans: $x = e^z$

$z = \log x$

$x \cdot \frac{dy}{dx} = Dy$ where $D = \frac{d}{dz}$

$x^2 \cdot \frac{d^2y}{dx^2} = D(D-1)y$

$x^3 \cdot \frac{d^3y}{dx^3} = D(D-1)(D-2)y$

$$\rightarrow (D(D-1)(D-2) + 2D(D-1) + 2)y = 10(e^z + e^{-z})$$

$$\therefore ((D^2-D)(D-2) + 2D^2 - 2D + 2)y = 10(e^z + e^{-z})$$

$$\therefore (D^3 - 2D^2 - D^2 + \cancel{2D} + \cancel{2D^2} - \cancel{2D} + 2)y = 10(e^z + e^{-z})$$

$$\therefore (D^3 - D^2 + 2)y = 10(e^z + e^{-z})$$

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→ For CF

$D^3 - D^2 + 2 = 0$

$m^3 - m^2 + 2 = 0$

$\cancel{(m^2}\ (m+1)(m^2-2m+2) = 0$

$$\begin{array}{r|rrrr} & 1 & -1 & 0 & 2 \\ -1 & 0 & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

$\therefore m = -1 \quad , m = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \dfrac{2 \pm \sqrt{4 - \cancel{16}\,8}}{2}$

$= \dfrac{2 \pm \sqrt{-4}}{2}$

$= \dfrac{2 \pm 2i}{2}$

$= 1 \pm i$

$m = -1 \, , 1+i \, , 1-i$

$CF = C_1 e^{-z} + (C_2 \cos z + C_3 \sin z) e^{z}$

→ $PI = \dfrac{1}{F(D)} \cancel{x}\, 10(e^z + e^{-z})$

$= 10 \cdot \dfrac{1}{D^3 - D^2 + 2} e^{z} + 10 \cdot \dfrac{1}{D^3 - D^2 + 2} e^{-z}$

$a = 1 \qquad \cancel{a=-1}$

$= 5e^z + 10 \cdot z \dfrac{1}{3D^2 - 2D} e^{-z}$

$a = -1$

$= 5e^z + 10 \cdot z \dfrac{1}{3+2} e^{-z}$

$PI = 5e^z + 2ze^{-z}$

The General solⁿ is

$y = C_1 e^{-z} + (C_2 \cos z + C_3 \sin z) e^{z} + 5e^{z} + 2ze^{-z}$

$= \cancel{\dfrac{C_1}{x}} + \cancel{(C_2 \cos x + C_3 \sin x)}$

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$$y = \frac{c_1}{x} + (c_2 \cos(\log x) + c_3 \sin(\log x))x + 5x + \frac{2}{x}\log x$$

27) Solve by Euler-cuchy method :

$$x^2 y'' - 2.5 x y' - 2.0 y = 0.$$

Ans: $x = e^z$

$z = \log x$

$$x \cdot \frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$\therefore [D(D-1) - 2.5 D - 2]y = 0$

$\therefore (D^2 - D - 2.5D - 2)y = 0$

$\therefore (D^2 - 3.5D - 2)y = 0$

→ For CF $m^2 - 3.5m - 2 = 0$

$(m-4)(m+\frac{1}{2}) = 0$

$\therefore m = 4, \frac{-1}{2}$

$\therefore CF = c_1 e^{4x} + c_2 e^{-1/2 x}$

Here $R(x) = 0$ $\therefore PI = 0$

$\therefore$ The General Sol^n is,

$$y = c_1 e^{4x} + c_2 e^{-1/2 x}$$

Unit 2 : Series solutions of ordinary differential equations.

1) Define :

→ Power series :-

- A power series in power of $(x - x_0)$ is an infinite series of the form

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k = a_0 + a_1 (x - x_0) + \ldots + a_n (x - x_0)^n + \ldots$$

where $a_0, a_1, a_2, \ldots, a_n$ are constant called co-efficient of power series.

→ Singular Point :-

- The point $x = x_0$ is called singular point of $y'' + P(x) y' + Q(x) y = 0$ if $P(x)$ & $Q(x)$ are not Analytic at $x = x_0$.
($\because$ $P(x)$ & $Q(x)$ are infinite)

→ Ordinary point :-

- The point $x = x_0$ is called ordinary point at $y'' + P(x) y' + Q(x) y = 0$ if $P(x)$ & $Q(x)$ are Analytic at $x = x_0$.
($\because$ $P(x)$ & $Q(x)$ are finite)

→ Regular Singular Point :-

- If $(x - x_0) P(x)$ & $(x - x_0)^2 Q(x)$ are Analytic at $x = x_0$ then the point x_0 is called Regular singular point. ($x = x_0$ be any ordinary point and $P(x)$ & $Q(x)$ are finite)

→ Irregular singular point :-

- If $x = x_0$ is not regular singular point then $x = x_0$ is called Irregular singular point.

2) Find ordinary point for

Ans: 1) $y'' + y = 0$ — ①

Here $y'' + P(x)y' + Q(x)y = 0$ compare with eqⁿ ①

$P(x) = 0$.

$Q(x) = 1$.

∴ Every points is an ordinary point

2) $y'' + e^{x} y' + (\sin x) y = 0$ — ①

$y'' + P(x) y' + Q(x) y = 0$

compare with eq^n ①

$P(x) = e^{x}$

$Q(x) = \sin x$

Here e^{x} & $\sin x$ are Analytic everywhere.

∴ All points are ordinary point.

4) $y'' - 2y = 0$ — ①

$y'' + P(x)y' + Q(x)y = 0$ compare with eq^n ①

$P(x) = 0$

$Q(x) = 2$.

∴ Every points is an ordinary point.

5) $y'' + e^{2x} y' + (\cos x) y = 0$ — ①

$y'' + P(x) y' + Q(x) y = 0$ compare with eq^n ①

$P(x) = e^{2x}$, $Q(x) = \cos x$

Here e^{2x} & $\cos x$ are Analytic everywhere

∴ All points are an ordinary point.

3) Find singular point for

Ans: 1) $y'' + \frac{1}{x-1} y' + \frac{1}{x-1} y = 0$

$$P(x) = \frac{1}{x-1}$$

$$Q(x) = \frac{1}{x-1}$$

$$x - 1 = 0$$

$$x = 1$$

$$P(x) = \infty$$

$$Q(x) = \infty$$

$\therefore x = x_0 = 1$ are singular point.

2) $(1-x^2)y'' - 6xy' - 4y = 0$

$$y'' - \frac{6x}{(1-x^2)} y' - \frac{4}{(1-x^2)} y = 0$$

$$P(x) = \frac{-6x}{(1-x^2)}$$

$$Q(x) = \frac{-4}{(1-x^2)}$$

$$1 - x^2 = 0$$

$$-6x = 0$$

$\rightarrow$ $x = \pm 1, 0$

$\rightarrow$ For $x = \pm 1$

$P(x) = \infty$ & $Q(x) = \infty$

$x = x_0 = \pm 1$ are singular points.

$\rightarrow$ $x = 0$

$P(x) = 0$ & $Q(x) = -4$

$x = x_0 = 0$ is an ordinary point.

3) $y'' + \frac{1}{x-2} y' + \frac{1}{x-2} y = 0.$

$x - 2 = 0$

$x = 2.$

$P(x) = \frac{1}{x-2}$, $Q(x) = \frac{1}{x-2}$

For $x = 2$

$P(x) = \infty$, $Q(x) = \infty$

$\therefore x = x_0 = 2$ are singular point.

4) $(4 - x^2) y'' - 6x y' - 4y = 0.$

$\therefore y'' - \frac{6x}{(4-x^2)} y' - \frac{4}{(4-x^2)} y = 0$

$P(x) = \frac{-6x}{4-x^2}$, $Q(x) = \frac{-4}{4-x^2}$

$4 - x^2 = 0$ $\quad -6x = 0$

$x^2 = 4$ $\quad x = 0.$

$x = \pm 2$

→ For $x = \pm 2.$ $\quad x = -2$

$P(x) = \frac{-12}{0} = \infty$ $\quad$, $P(x) = \frac{12}{0} = \infty$

$Q(x) = \frac{-4}{0} = \infty$ $\quad Q(x) = \frac{-4}{0} = \infty$

$\therefore f(x)\ x = x_0 = \pm 2.$ are singular points.

→ $x = 0.$

$P(x) = 0$; $Q(x) = -1$.

$\therefore x = x_0 = 0$ is an ordinary point.

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5) $y'' + \frac{2}{x+3} y' - \frac{5}{x+3} y = 0$

$P(x) = \frac{2}{x+3}$ $\quad Q(x) = \frac{-5}{x+3}$

$x+3 = 0$

$x = -3$

For $x = -3$

$P(x) = \frac{2}{0} = \infty$ $\quad Q(x) = \frac{-5}{0} = \infty$

$\therefore x = x_0 = -3$ is an ordinary point.

4) classify the singularities of

$(x^2+4)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = 0$

<u>Ans:</u> Here $x = 0$

$x = \pm 2i$

$P(x) = \frac{2x}{x^2+4}$ & $Q(x) = \frac{-12}{x^2+4}$

For $x = \pm 2i$ $\qquad x = -2i$

$P(x) = \frac{4i}{4i^2+4} = \frac{4i}{0} = \infty$ $\quad Q(x) = \frac{-4i}{4i^2+4} = \frac{-4i}{0} = \infty$

$Q(x) = \frac{-12}{0} = \infty$ $\qquad Q(x) = \frac{-12}{0} = \infty$

$\therefore x = x_0 = \pm 2i$ are an ordinary point

→ For $x = 0$

$P(x) = 0$, $Q(x) = \frac{-12}{4} = -3$

$x = x_0 = 0$ is a singular point

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$\rightarrow x_0 = +2i$

$$(x - x_0)\,P(x) = (x - 2i)\,\frac{2x}{x^2+4}$$

$$= (x-2i)\cdot\frac{2x}{(x^2-(2i)^2)}$$

$$= \cancel{(x-2i)} \times \frac{2x}{\cancel{(x-2i)}(x+2i)}$$

$$= \frac{2x}{x+2i}$$

$$= \frac{4i}{4i}$$

$$= 1.$$

$$(x - x_0)^2\,Q(x) = (x-2i)^2 \times \frac{-12}{x^2+4}$$

$$= (x-2i)^{\cancel{2}} \times \frac{(-12)}{\cancel{(x-2i)}(x+2i)}$$

$$= (2i-2i) \times \frac{(-12)}{2i+2i}$$

$$= 0$$

$x = x_0 = 2i$ is a Regular singular point.

$\rightarrow x_0 = -2i$

$$(x - x_0)\,P(x) = (x+2i)\,\frac{2x}{x^2+4}$$

$$= \cancel{(x+2i)}\,\frac{2x}{(x^2-2i)\cancel{(x+2i)}}$$

$$= \frac{2x}{x-2i}$$

$$= \frac{-4i}{-4i}$$

$$= 1$$

$$-(x-x_0)^2\, Q(x) = (x+2i)^2 \frac{2x}{(x-2i)(x+2i)}$$

$$= \frac{-4i \cdot x - 4i}{0} \quad 0$$

$$= \infty \quad 0$$

$x = x_0 = -2i$ is a ~~irregular~~ singular point.

5) Classify the singularities of

$(x^2+1)y'' - xy' - y = 0.$

Ans: $x^2+1=0 \qquad x=0$

$x^2 = -1$

$x = \pm i$

$\rightarrow$ ~~$P(x)$~~ $= y'' - \frac{x}{x^2+1} y' - \frac{1}{x^2+1} y = 0.$

$P(x) = \frac{-x}{x^2+1} \qquad Q(x) = \frac{-1}{x^2+1}$

$\rightarrow$ For $x = +i \qquad x = -i$

$P(x) = \frac{i}{0} = \infty \qquad P(x) = \frac{i}{0} = \infty$

$Q(x) = \frac{-1}{0} = \infty \qquad Q(x) = \frac{-1}{0} = \infty$

$\therefore x = x_0 = \pm i$ is a singular point.

$\rightarrow$ For $x=0$

$P(x) = 0 \qquad Q(x) = 1$

$\therefore x = x_0 = 0$ is an ordinary point.

→ For $x = i$

$$\cancel{f(x)}\ (x - x_0)\, P(x) = (x - i)\frac{-x}{(x^2 - i^2)}$$

$$= (x-i)\frac{-x}{x^2 - i^2}$$

$$= \frac{(x-i)(-x)}{(x-i)(x+i)}$$

$$= \frac{-x}{x+i}$$

$$= \frac{-i}{2i} = \frac{-1}{2}$$

$$(x-x_0)^2 Q(x) = (x-i)^2 \frac{(-1)}{(x-i)(x+i)}$$

$$= \frac{-(x-i)}{x+i}$$

$$= 0$$

$x = x_0 = i$ is a reguler singulur point.

→ For $x = -i$

$$(x - x_0) P(x) = \cancel{(x+i)}\frac{-x}{\cancel{(x+i)}(x-i)}$$

$$= \frac{i}{0-2i}$$

$$= \cancel{0}\ {-1/2}$$

$$(x - x_0)^2 Q(x) = (x+i)^2 \frac{(-1)}{\cancel{(x+i)}(x-i)}$$

$$= \cancel{-1}\ 0$$

$\therefore x = x_0 = -i$ is a Regular singular Point.

6) Solve the equation $\frac{d^2y}{dx^2} + y = 0$ by power series method.

Ans: Here $\frac{d^2y}{dx} + y = 0$ — (1)

$y'' + P(x)y' + Q(x) = 0$

$\therefore P(x) = 0 \quad Q(x) = 1$ are Analytic at x

$\therefore x = 0$ is an ordinary point.

Let $y = \sum_{k=0}^{\infty} a_k x^k$ be the solution of (i)

$y' = \sum_{k=1}^{\infty} k a_k x^{k-1}$

$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$

Put value of y & y'' in eq — (1)

$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^k = 0$

changing the index to get the same power in each term.

$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=0}^{\infty} a_k x^k = 0$

For $k \geq 0$

$(k+2)(k+1) a_{k+2} + \sum a_k = 0.$

$a_{k+2} = \frac{-a_k}{(k+1)(k+2)}$

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$K=0 \Rightarrow a_2 = \dfrac{-a_0}{1\cdot 2}$

$K=1 \Rightarrow a_3 = \dfrac{-a_1}{2\cdot 3}$

$K=2 \Rightarrow a_4 = \dfrac{-a_2}{3\cdot 4} = \dfrac{+a_0}{1\cdot 2\cdot 3\cdot 4}$

$K=3 \Rightarrow a_5 = \dfrac{-a_3}{4\cdot 5} = \dfrac{+a_1}{2\cdot 3\cdot 4\cdot 5}$

Now $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$= a_0 + a_1 x - \dfrac{a_0}{1\cdot 2} x^2 - \dfrac{a_1}{2\cdot 3} x^3$

$+ \dfrac{a_0}{1\cdot 2\cdot 3\cdot 4} x^4 + \dfrac{a_1}{2\cdot 3\cdot 4\cdot 5} x^5 - \ldots$

$= a_0\left(1 - \dfrac{x^2}{1\cdot 2} + \dfrac{1}{1\cdot 2\cdot 3\cdot 4} x^4 - \ldots\right)$

$+ a_1\left(x - \dfrac{x^3}{2\cdot 3} + \dfrac{x^5}{2\cdot 3\cdot 4\cdot 5} - \ldots\right)$

$= a_0\left(1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots\right) +$

$a_1\left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots\right)$

$y = a_0 \cos x + a_1 \sin x$ which is the required solution.

8) Find the series solution of $y'' = y'$

Ans: Here $y'' - y' = 0$ — ①

$P(x) = -1$, $Q(x) = 0$ are Analytic at $x = 0$

$\therefore x = 0$ is an Ordinary point

Let $y = \sum_{k=0}^{\infty} a_k x^k$ be the solution of (i)

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Put the value of y'' & y' in eq^n ①

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - \sum_{k=1}^{\infty} k a_k x^{k-1} = 0$$

- changing the index to get the same power in each term

$$\sum_{k=1}^{\infty} (k+1)k a_{k+1} x^{k-1} - \sum_{k=1}^{\infty} k a_k x^{k-1} = 0$$

→ For $k \geq 1$

$$(k+1)k a_{k+1} = k a_k$$

$$a_{k+1} = \frac{a_k}{k+1} \quad (k \neq 0)$$

$$k = 1 \Rightarrow a_2 = \frac{a_1}{2}$$

$$k = 2 \Rightarrow a_3 = \frac{a_2}{3} = \frac{a_1}{6}$$

$$k = 3 \Rightarrow a_4 = \frac{a_3}{4} = \frac{a_1}{24}$$

$$k = 4 \Rightarrow a_5 = \frac{a_4}{5} = \frac{a_1}{120}$$

Now $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$= a_0 + a_1 x + \frac{a_1}{2} x^2 + \frac{a_1}{6} x^3 + \frac{a_1}{24} x^4 + \frac{a_1}{120} x^5 + \ldots$

$= a_0 + a_1 \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots \right)$

$y = a_0 + a_1 (e^x - 1)$ is the required solution of (i)

9) Obtain the series solution of the differential equation $2x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (2x^2 - 3) y = 0$

Ans: Dividing by x^2

$$\frac{d^2y}{dx^2} + \frac{1}{2x} \frac{dy}{dx} + \left(\frac{2x^2 - 3}{2x^2} \right) y = 0 \quad \text{--- (1)}$$

Here $P(x) = \frac{1}{2x}$ & $Q(x) = \frac{2x^2 - 3}{2x^2}$ are not

Analytic at $x = 0$.

$\therefore x = 0$ is a singular point.

Let $y = \sum_{k=0}^{\infty} a_k x^{m+k}$ --- (2) be the required series sol^n of (1)

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} (m+k) a_k x^{m+k-1}$$

& $$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} (m+k)(m+k-1) x^{m+k-2}$$

Substi put the value of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in eq^n (1)

$$2x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (2x^2-3)y = 0$$

$$\therefore 2x^2 \sum_{k=0}^{\infty} (m+k)(m+k-1)a_k x^{m+k-2} + x\sum_{k=0}^{\infty}(m+k)a_k m^{m+k-1}$$

$$+ (2x^2-3)\sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\therefore \sum_{k=0}^{\infty} 2(m+k)(m+k-1)a_k x^{m+k} + \sum_{k=0}^{\infty}(m+k)a_k m x^{m+k}$$

$$+ \sum_{k=0}^{\infty} 2a_k x^{m+k+2} - \sum_{k=0}^{\infty} 3a_k x^{m+k} = 0.$$

→ changing the index to get same power in all terms.

$$\sum_{k=0}^{\infty} 2(m+k)(m+k-1)a_k x^{m+k} + \sum_{k=0}^{\infty}(m+k)a_k x^{m+k}$$

$$+ \sum_{k=2}^{\infty} 2a_{k-2} x^{m+k} - \sum_{k=0}^{\infty} 3a_k x^{m+k} = 0$$

→ $k=0 \Rightarrow 2m(m-1)a_0 + ma_0 - 3a_0 = 0$

$$(2m^2-2m+m-3)a_0 = 0$$

$$(2m^2-m-3)a_0 = 0$$

$$\therefore (2m^2+2m-3m-3)a_0 = 0$$

$$\therefore (2m(m+1)-3(m+1)]a_0 = 0$$

$$\therefore (m+1)(2m-3)a_0 = 0$$

$$m=-1,\ m=\frac{3}{2} \quad (\because a_0 \neq 0).$$

⇒ $k=1 \Rightarrow 2(m+1)(m)a_1 + (m+1)a_1 - 3a_1 = 0$

$$\therefore [2(m+1)(m) + m+1) - 3]a_1 = 0$$

$$\therefore (2m(m+1) + m - 2]a_1 = 0$$

$$\therefore a_1 = 0$$

$k \geq 2 \Rightarrow 2(m+k)(m+k-1)a_k + (m+k)a_k + 2a_{k-2} - 3a_k = 0$

Take $m+k = c$

$\therefore 2c(c-1)a_k + ca_k + 2a_{k-2} - 3a_k = 0.$

$\therefore [2c(c-1) + c - 3]a_k + 2a_{k-2} = 0$

$\therefore [2c^2 - 2c + c - 3]a_k = -2a_{k-2}$

$\therefore [2c^2 - c - 3]a_k = -2a_{k-2}$

$\therefore [2c^2 + 2c - 3c - 3]a_k = -2a_{k-2}$

$\therefore [2c(c+1) - 3(c+1)]a_k = -2a_{k-2}$

$\therefore (c+1)(2c-3)a_k = \cancel{2a_k} - 2a_{k-2}$

$\therefore a_k = \dfrac{-2a_{k-2}}{(c+1)(2c-3)}$

$= -\dfrac{2a_{k-2}}{(m+k+1)(2m+2k-3)}$

$\rightarrow$ For first sol^n take $m = -1$

$a_k = \dfrac{-2a_{k-2}}{k(2k-5)}$

$k = 2 \Rightarrow a_2 = -\dfrac{2a_0}{2(4-5)} = a_0$

$k = 3 \Rightarrow a_3 = -\dfrac{2a_1}{3(6-5)} = -\dfrac{2a_1}{3} = 0$

$k = 4 \Rightarrow a_4 = \dfrac{-2a_2}{4(8-5)} = \dfrac{-2a_2}{12} = -\dfrac{1}{6}\left(\dfrac{a_0}{2}\right) = \dfrac{-a_0}{12}$

$\rightarrow y_1 = x^m \sum_{k=0}^{\infty} a_k x^k$

$= x^m (a_0 + a_1 x + a_2 x^2 + \ldots)$

$= x^{-1}\left(a_0 + 0 + a_0 x^2 + 0 - \dfrac{a_0}{12} x^4 + \ldots\right)$

$= a_0 x^{-1}\left[1 + x^2 - \dfrac{x^4}{12} + \ldots\right]$

For 2^{nd} solution take $m = \frac{3}{2}$

$$a_k = -\frac{2a_{k-2}}{\left(\frac{3}{2}+k+1\right)(3+2k-3)}$$

$$= -\frac{2a_{k-2}}{\left(k+\frac{5}{2}\right)(2k)}$$

$$a_k = -\frac{2a_{k-2}}{(2k+5)(k)}$$

$$k=2 \Rightarrow a_2 = -\frac{2a_0}{2(9)} = -\frac{a_0}{9}$$

$$k=3 \Rightarrow a_3 = \frac{-2a_1}{2(11)} = -\frac{2a_1}{33} = 0$$

$$k=4 \Rightarrow a_4 = -\frac{2a_2}{4(13)} = -\frac{a_2}{26} = \frac{-1}{26}\left(\frac{-a_0}{9}\right) = \frac{a_0}{234}$$

$$y_2 = x^m \sum_{k=0}^{\infty} a_k x^k$$

$$= x^m(a_0 + a_1 x + a_2 x^2 + ---)$$

$$= x^{3/2}\left(a_0 + 0 + -\frac{a_0}{9}x^2 + 0 + \frac{a_0}{234}x^4 + ---\right)$$

$$= a_0 x^{3/2}\left(1 - \frac{x^2}{9} + \frac{x^4}{234} - 0 - 0\right)$$

→ The General solution is,

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 a_0 x^{-1}\left(1 + x^2 - \frac{x^4}{12} + --\right) +$$

$$c_2 a_0 x^{3/2}\left(1 - \frac{x^2}{9} + \frac{x^4}{234} - ---\right)$$

$$= Ax^{-1}\left(1 + x^2 - \frac{x^4}{12} + --\right) + Bx^{3/2}\left(1 - \frac{x^2}{9} + \frac{x^4}{234} - --\right)$$ where $A=$ $B=$

10) Find the series solution of the differential eqⁿ $xy'' + 2y' + xy = 0$

Ans:- Here, $xy'' + 2y' + xy = 0$ —— ①

$\therefore y'' + \frac{2}{x} y' + y = 0$

$P(x) = 2/x \quad Q(x) = 1$

Here, $P(x)$ is not analytic at $x = 0$

But $xP(x) = 2$ & $x^2 Q(x) = x^2$ are Analytic at $x = 0$

$\therefore x = 0$ is a regular singular point.

Let $y = \sum_{k=0}^{\infty} a_k x^{m+k}$ be the solⁿ of (1).

$$y' = \sum_{k=0}^{\infty} (m+k) a_k x^{m+k-1}$$

$$y'' = \sum_{k=0}^{\infty} (m+k)(m+k-1) a_k x^{m+k-2}$$

Substituting the expressions of y, y' & y'' in ①

$xy'' + 2y' + xy = 0$.

$$\therefore x \sum_{k=0}^{\infty} (m+k)(m+k-1) a_k x^{m+k-2} +$$

$$2 \sum_{k=0}^{\infty} (m+k) a_k x^{m+k-1} + x \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\therefore \sum_{k=0}^{\infty} (m+k)(m+k-1) a_k x^{m+k-1} + \sum_{k=0}^{\infty} 2(m+k) a_k x^{m+k-1}$$

$$+ \sum_{k=0}^{\infty} a_k x^{m+k+1} = 0.$$

changing the index to get the same power in each term.

$$\sum_{k=0}^{\infty} (m+k)(m+k-1) a_k x^{m+k-1} + \sum_{k=0}^{\infty} 2(m+k) a_k x^{m+k-1}$$

$$+ \sum_{k=2}^{\infty} a_{k-2} x^{m+k-1} = 0$$

$k=0 \Rightarrow m(m-1)a_0 + 2ma_0 = 0$

$\therefore (m^2 - m + 2m)a_0 = 0$

$\therefore (m^2+m)a_0 = 0$

$\therefore m(m+1)a_0 = 0$

$m=0, \ m=-1 \quad (\because a_0 \neq 0)$

$k=1 \Rightarrow (m+1)(m)a_1 + 2(m+1)a_1 = 0$

$\therefore (m+1)(m+2)a_1 = 0$

For the smaller root $m=-1$ the above eq^n becom

$0 \cdot a_1 = 0$

a_1 is undetermined

$\therefore a_i$ is an arbitarary constant for $m=-1$

$k \geq 2$

$\Rightarrow (m+k)(m+k-1)a_k + 2(m+k)a_k + a_{k-2} = 0$

$\therefore (m+k)(m+k+1)a_k = -a_{k-2}$

$$\therefore a_k = -\frac{a_{k-2}}{(m+k)(m+k+1)}$$

For the 1st sol^n take $m=-1$

$$\therefore a_k = -\frac{a_{k-2}}{(k-1)(k)}$$

$$k=2 \rightarrow a_2 = -\frac{a_0}{1 \cdot 2} = -\frac{a_0}{2}$$

$$k=3 \rightarrow a_3 = -\frac{a_1}{2 \cdot 3} = -\frac{a_1}{6}$$

$$k=4 \rightarrow a_4 = -\frac{a_2}{3 \cdot 4} = \frac{a_0}{24}$$

$$k=5 \rightarrow a_5 = -\frac{a_3}{4 \cdot 5} = \frac{a_1}{120}$$

Now $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$$= a_0 + a_1 x - \frac{a_0}{2}x^2 - \frac{a_1}{6}x^3 + \frac{a_0}{24}x^4 + \frac{a_1}{120}x^5 \cdots$$

$$= a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) + a_1\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$$

KRUSHI

Section - 2

Unit: 4 Laplace Transforms.

1) Define laplace transform.

Ans: Let f(t) be a given function of t define for all $t \geq 0$ then Laplace Transform of f(t) denoted by $L\{f(t)\}$ defined as

$$L\{f(t)\} = \bar{f}(s) = F(s) = \int_0^\infty e^{-st} f(t) \cdot dt.$$

2) state the first shifting property of laplace transform.

Ans: If $L\{f(t)\} = f(s)$
then $L\{e^{at} \cdot f(t)\} = f(s-a)$

3) state the linearity property of laplace transform.

Ans: If $L\{f(t)\} = F(s)$ and $L\{g(t)\} = G(s)$ then for any constants a and b

$$L[a \cdot f(t) + b \cdot g(t)] = a \cdot L\{f(t)\} + b \cdot L\{g(t)\}$$
$$= a \cdot F(s) + b \cdot G(s).$$

4) Prove that if $L(f(t)) = F(s)$ then
$L(f'(t)) = sL(f(t)) - f(0)$

Ans:
$$L[f'(t)] = \int_0^\infty e^{-st} f'(t) \cdot dt$$
$$= \left[e^{-st} \int f'(t) \cdot dt\right]_0^\infty - \int_0^\infty \left(e^{-st}(-s) \int_0^\infty f'(t)\, dt\right) dt$$

$$\left[\because \int uv = u \int v\, dx - \int \left(\frac{du}{dx} \int v \cdot dx\right) dx\right]$$

$$= \left[e^{-st} \cdot f(t)\right]_0^\infty + s \int_0^\infty e^{-st} \cdot f(t) \cdot dt$$
$$= [0 - f(0)] + s \cdot L[f(t)]$$
$$\therefore L(f'(t)) = s \cdot L(f(t)) - f(0)$$

5) Write the values of followings:

Ans: 1) $L(t^n) = \frac{\overline{|n+1}}{s^{n+1}}$ where $\overline{|n+1} = n!$

2) $L(t^4) = \frac{4!}{s^{4+1}} = \frac{4\times3\times2\times1}{s^5} = \frac{24}{s^5}$

3) $L(t^3) = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$

4) $L(t^2) = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

5) $L(\sin 3t) = \frac{3}{s^2+9}$

6) $L(\cos 7t) = \frac{s}{s^2+49}$

7) $L(\cosh 3t) = \frac{s}{s^2-9}$

8) $L(\sinh 4t) = \frac{4}{s^2-16}$

9) $L(\cosh 8t) = \frac{s}{s^2-64}$

10) $L(\sinh 4t) = \frac{4}{s^2-16}$

11) $L(\cos 4t) = \frac{s}{s^2+16}$

12) $L(\sinh 5t) = \frac{5}{s^2-25}$

13) $L(\cos 7t) = \frac{s}{s^2+49}$

14) $L(e^{2x}) = \frac{1}{s-2}$

15) $L(e^{-3x}) = \frac{1}{s+3}$

16) $L(e^{4x}) = \frac{1}{s-4}$

17) $L(e^{-7x}) = \frac{1}{s+7}$

6) Define Inverse laplace Transform.

Ans: $L^{-1}\{F(s)\} = F(t)$ to denote such a function $F(t)$, and it is called the inverse laplace ~~to~~ transform.

7) state change of scale property for Inverse laplace Transform.

Ans: If $L^{-1}\{f(s)\} = F(t)$ then

$$L^{-1}[F(as)] = \frac{1}{a} \cdot f\left(\frac{t}{a}\right)$$

8) state linearity property for Inverse L.T.

Ans: If $F(t) = L^{-1}\{F(s)\}$ and $g(t) = L^{-1}\{G(s)\}$ then

$$L^{-1}\{a \cdot F(s) \pm b \cdot G(s)\} = aL^{-1}\{F(s)\} \pm b\{G(s)\}$$

$$= a \cdot F(t) \pm b \cdot g(t)$$

9) Find the values of followings:

Ans: 1) $L^{-1}\left(\frac{1}{s^n}\right) = \frac{1}{(n-1)!} t^{n-1}$

2) $L^{-1}\left(\frac{1}{s^3}\right) = \frac{1}{(3-1)!} t^{3-1} = \frac{1}{2!} t^2 = \frac{1}{2} t^2$

3) $L^{-1}\left(\frac{1}{s^2}\right) = \frac{1}{(2-1)!} t^{2-1} = \cancel{t}\ t$

4) $L^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$

5) $L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2} \sin 2t$

6) $L^{-1}\left(\frac{1}{s^2+49}\right) = \frac{1}{7} \sin 7t$

10) Define unit step function.

Ans: The unit step function $u(t-u)$ is defined as

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \quad (a \geq 0) \\ 1 & \text{if } t \geq a. \end{cases}$$

11) State convolution theorem.

Ans: If $L^{-1}(F(s)) = F(t)$ and $L^{-1}(G(s)) = g(t)$ then

$$L^{-1}\{F(s)\cdot G(s)\} = \int_0^t F(u)\cdot g(t-u)\,du$$

$$= f * g$$

or

$$L^{-1}\{F(s)\cdot \overline{g(s)}\} = \int_0^t \ell g(u)\cdot F(t-u)\,du$$

$$= g * f$$

12) Define convolution.

Ans: Convolution of a function $f(t)$ & $g(t)$ denoted as $f(t) * g(t)$ is defined as,

$$f(t) * g(t) = \int_0^t f(u) \cdot g(t-u)\, du.$$

$$= \int_0^t f(t-u) \cdot g(u)\, du$$

$$\left[\because \int_0^a f(x) \cdot dx = \int_0^a f(a-x)\, dx\right]$$

$$= g(t) * f(t)$$

13) Find $1*1$ where $*$ ~~does not~~ denote convolution product.

Ans: $f(t) = 1 = g(t)$

$$f(t) * g(t) = \int_0^t f(u) \cdot g(t-u) \cdot du$$

$$= \int_0^t 1 \cdot 1 \cdot du$$

$$= [u]_0^t$$

$$= t$$

~~14) State and prove change of scale property for Inverse laplace transform:-~~

~~Ans: If~~

17) Find the values of followings:

Ans: 1) $L(e^{at} t \sin at)$

$= L(t \cdot e^{at} \sin at) = L\{t \cdot f(t)\}$

$= -\frac{d}{ds} L(f(t)\}$

$= -\frac{d}{ds} L\{e^{at} \sin at\}$

$= \frac{-d}{ds}\left(\frac{a}{(s-a)^2+a^2}\right)$

$= -\left(\frac{((s-a)^2+a^2)0 - 2(s-a)+0}{((s-a)^2+a^2)^2}\right)$

$= -\frac{(-2s+2a)}{((s-a)^2+a^2)^2}$

$= \frac{2s-2a}{((s-a)^2+a^2)^2}$

2) $L(t^2 \sinh at)$ $\left[\because L\{t^n f(t)\} = (-1)^n \cdot \frac{d^n}{ds^n} L\{f(t)\}\right]$

$= (-1)^2 \frac{d^2}{ds^2}[L(\sinh at)]$

$= \frac{d^2}{ds^2}\left[\frac{a}{s^2-a^2}\right]$

$= \frac{d}{ds}\left[\frac{2as}{(s^2-a^2)^2}\right]$

$= \left[\frac{(s^2-a^2)^2 2a - 2as(2(s^2-a^2)\cdot 2s)}{(s^2-a^2)^4}\right]$

$= \left[\frac{2a(s^2-a^2)^2 - 8as^2(s^2-a^2)}{(s^2-a^2)^4}\right]$

$= \left[\frac{2as^2-2a^3-8as^2}{(s^2-a^2)^3}\right]$

$= \left[\frac{-6as^2-2a^3}{(s^2-a^2)^3}\right]$

3) $L(\sin at + at\cos at)$

$= L\{\sin at\} + a\,L\{t \cdot \cos at\}$

$= \dfrac{a}{s^2+a^2} + a \cdot (-1)^1 \dfrac{d}{ds} L \cdot F(t)$

$= \dfrac{a}{s^2+a^2} - a \cdot \dfrac{d}{ds}\left(\dfrac{s}{s^2+a^2}\right)$

$= \dfrac{a}{s^2+a^2} - a\left(\cancel{\dfrac{2s^2}{(s^2+a^2)^2}}\right) a\left(\dfrac{(s^2+a^2)1 - s(2s)}{(s^2+a^2)^2}\right)$

$= \dfrac{a}{s^2+a^2} - a\left(\dfrac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right)$

$= \dfrac{a}{s^2+a^2} - a\left(\dfrac{a^2-s^2}{(s^2+a^2)^2}\right)$

4) $L(t^2 \sin^2 4t)$

$= (-1)^2 \dfrac{d^2}{ds^2}\left[L(\sin^2 4t)\right]$

$= \dfrac{d^2}{ds^2} L\left(1 - \left(\dfrac{\cos 8t + 1}{2}\right)\right)$

$= \dfrac{d^2}{ds^2}\left[L(1) - \dfrac{1}{2}L(\cos 8t) - \dfrac{1}{2}L(1)\right]$

$= \dfrac{d^2}{ds^2}\left[\dfrac{1}{s} - \dfrac{1}{2}\left(\dfrac{s}{s^2+64}\right) - \dfrac{1}{2}\times\dfrac{1}{s}\right]$

$= \dfrac{d^2}{ds^2}\left[\dfrac{1}{2s} - \dfrac{1}{2}\left(\dfrac{s}{s^2+64}\right)\right]$

$= \dfrac{d^2}{ds^2}\left[\dfrac{2s^2+128-2s^2}{2s^2+128}\right]$

$= \dfrac{d}{ds}\left[\dfrac{512s}{(2s^2+128)^2}\right]$

$= \left[\dfrac{(2s^2+128)^2\,512 - 512s\,(2(2s^2+128)4s)}{(2s^2+128)^4}\right]$

$= \left[\dfrac{(2s^2+128)\,512 - 4096s^2\,\cancel{(2s^2+128)}}{(2s^2+128)^3}\right]$

$$= \frac{2048s^2 + 65536 - 4096s^2}{(2s^2+128)^3}$$

$$= \frac{65536 - 2048s^2}{(2s^2+128)^3}$$

5) $L\left(\frac{e^{-at} - e^{-bt}}{t}\right)$

$$= L\left\{\frac{e^{-at}}{t}\right\} - L\left\{\frac{e^{-bt}}{t}\right\}$$

$$= \int_s^{\infty} \frac{1}{s+a} \cdot ds - \int_s^{\infty} \frac{1}{s+b} ds$$

$$= [\log(s+a)]_s^{\infty} - [\log(s+b)]_s^{\infty}$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^{\infty}$$

$$= \left[\log\left(\frac{1+a/s}{1+b/s}\right)\right]_s^{\infty}$$

$$= \left[0 - \log\left(\frac{s+a}{s+b}\right)\right]$$

$$= -\log\left(\frac{s+a}{s+b}\right)$$

6) $L\left(\frac{\sin \omega t}{t}\right)$

$$= \int_s^{\infty} \frac{\omega}{s^2+\omega^2} \cdot ds$$

$$= \omega \int_s^{\infty} \frac{1}{s^2+\omega^2} ds$$

$$= \omega\left[\frac{1}{\omega} \tan^{-1}\left(\frac{s}{\omega}\right)\right]_s^{\infty}$$

$$= \left[\tan^{-1}\left(\frac{s}{\omega}\right)\right]_s^{\infty}$$

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$= [\tan^{-1}(\infty) - \tan^{-1}(s/\omega)]$

$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{\omega}\right)$

7) $L\left(\int_0^t (t^4 + \sin 3t)\right)$

$= \frac{1}{s} L(t^4) + \frac{1}{s} L(\sin 3t)$

$= \frac{1}{s} \frac{\lceil 4+1}{s^{4+1}} + \frac{1}{s}\left[\frac{3}{s^2+9}\right]$

$= \frac{\lceil 5}{s^6} + \frac{1}{s}\left[\frac{3}{s^2+9}\right]$

8) $L(e^{3t} \cdot t)$

$= \frac{\lceil 1+1}{(s-3)^{1+1}}$

$= \frac{2}{(s-3)^2}$

9) $L(t \sin 3t)$

$= (-1) \frac{d}{ds} L(\sin 3t)$

$= (-1) \frac{d}{ds}\left[\frac{3}{s^2+9}\right]$

$= (-1)\left[\frac{(s^2+9)0 - 3(2s)}{(s^2+9)^2}\right]$

$= (-1)\left[\frac{-6s}{(s^2+9)^2}\right]$

$= \frac{6s}{(s^2+9)^2}$

10) $L(t \cos 5t)$

$= (-1) \frac{d}{ds} \left(\cos 5t \right)$

$= -\frac{d}{ds} \left(\frac{s}{s^2+25} \right)$

$= -\left(\frac{(s^2+25) - s(2s)}{(s^2+25)^2} \right)$

$= -\left(\frac{s^2+25-2s^2}{(s^2+25)^2} \right)$

$= -\left(\frac{25-s^2}{(s^2+25)^2} \right)$

$= \frac{s^2-25}{(s^2+25)^2}$

11) $L\left(\frac{1-\cos 2t}{t} \right)$

$= L\left(\frac{1}{t} \right) - L\left(\frac{\cos 2t}{t} \right)$

$= \int_s^\infty \frac{1}{s} ds - \int_s^\infty \frac{s}{s^2+4} ds$

$= [\log s]_s^\infty - \frac{1}{2} [\log (s^2+4)]_s^\infty$

$= [\log s]_s^\infty - [(\log (s^2+4))^{1/2}]_s^\infty$

$= \left[\log \left(\frac{s}{(s^2+4)^{1/2}} \right) \right]_s^\infty$

$\rightarrow L\left(1 - \frac{\cos at}{t} \right)$

same to 11

12) $L[e^{-3t}(\cos 4t + 3\sin 4t + 2t^4)]$

$= L(e^{-3t}\cos 4t) + 3L(e^{-3t}\sin 4t) + 2L(e^{-3t}t^4)$

$= \frac{s+3}{(s+3)^2+16} + \cancel{3}\ \frac{12}{(s+3)^2+16} + \frac{2\,\Gamma(4+1)}{(s+3)^{4+1}}$

$= \cancel{s+3}\ \frac{s+15}{(s+3)^2+16} + \frac{2\cdot 5!}{(s+3)^5}$

13) $L(\sin^2 kt)$

$= L\left[1 - \cos^2 kt\right]$

$= L\left[1 - \left(\frac{\cos 2kt + 1}{2}\right)\right]$

$= L(1) - \frac{1}{2}L(\cos 2kt) - \frac{1}{2}L(1)$

$= \frac{1}{s} - \frac{1}{2}\frac{s}{s^2+4k^2} - \frac{1}{2s}$

$= \cancel{\frac{7}{28}}\ \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\times\frac{s}{s^2+4k^2}$

$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4k^2}\right)$

$= \frac{1}{2}\left(\frac{s^2+4k^2-s^2}{s(s^2+4k^2)}\right)$

$= \frac{1}{2}\left(\frac{4k^2}{s(s^2+4k^2)}\right)$

14) $L(4e^{5t}+6t^3-3\sin 4t+2\cos 2t)$

$= 4L(e^{5t}) + 6L(t^3) - 3L(\sin 4t) + 2L(\cos 2t)$

$= 4\cdot\frac{1}{s-5} + 6\cdot\frac{\overline{|3+1}}{s^{3+1}} - 3\cdot\frac{4}{s^2+16} + 2\cdot\frac{2s}{s^2+4}$

$= \frac{4}{s-5} + \frac{144}{s^4} - \frac{12}{s^2+16} + \frac{2s}{s^2+4}$.

15) $L(t\,u(t-a))$

16) $L(e^{-3t}\,u(t-2))$ } pending

(18) Find the values of followings:

1) $L^{-1}\left(\frac{5}{(s+2)^5}\right)$

$= 5L^{-1}\left(\frac{1}{(s+2)^5}\right)$

$= 5\cdot\frac{e^{-2t}\cdot t^{5-1}}{(5-1)!}$

$= 5\cdot\frac{e^{-2t}\cdot t^4}{4!}$

$= \frac{5}{24}\cdot e^{-2t}\cdot t^4$

2) $L^{-1}\left(\frac{s+1}{s^2-6s+25}\right)$

$= L^{-1}\left(\frac{s+1}{s^2-6s+9+16}\right)$

$= L^{-1}\left(\frac{s+1}{(s-3)^2+16}\right)$

$= L^{-1}\left(\frac{(s-3)+4}{(s-3)^2+16}\right)$

$= L^{-1}\left[\frac{s-3}{(s-3)^2+16}\right] + 4 \cdot L^{-1}\left[\frac{1}{(s-3)^2+16}\right]$

$= e^{3t}\cos 4t + 4 \cdot \frac{1}{4} e^{3t}\sin 4t$

$= e^{3t}(\cos 4t + \sin 4t)$

3) $L^{-1}\left(\frac{2}{(s^2-4)}\right)$

$= 2\, L^{-1}\left(\frac{1}{(s^2-2^2)}\right)$

$= 2 \cdot \frac{1}{2}\sinh 2t$

$= \sinh 2t$

4) $L^{-1}\left(\log\left(\frac{s+a}{s+b}\right)\right)$

$F(s) = \log\left(\frac{s+a}{s+b}\right)$

$= \log(s+a) - \log(s+b)$

$\frac{d}{ds}F(s) = \frac{1}{s+a} - \frac{1}{s+b}$

$f(t) = -\frac{1}{t} L^{-1}\left(\frac{d}{ds}F(s)\right)$

$= \frac{-1}{t} L^{-1}\left(\frac{1}{s+a} - \frac{1}{s+b}\right)$

$= \frac{-1}{t}\left(e^{-at} - e^{-bt}\right)$

5) $L^{-1}\left[\frac{1}{s(s+a)^3}\right]$

$$F(s) = \frac{1}{(s+a)^3}$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{1}{(s+a)^3}\right) = e^{-at}\,\frac{t^2}{2!}$$

$$f(t) = \frac{1}{2}t^2 \cdot e^{-at}$$

By using property

$$L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(t)\,dt$$

$$= \frac{1}{2}\int_0^t t^2 \cdot e^{-at}\,dt$$

$$= \frac{1}{2}\left[t^2 \cdot \frac{e^{-at}}{-a} - 2t \cdot \frac{e^{-at}}{a^2} + \frac{2e^{-at}}{-a^3}\right]_0^t$$

$$= \frac{1}{2}\left[-t^2\frac{e^{-at}}{a} - \frac{2te^{-at}}{a^2} - \frac{2e^{-at}}{a^3}\right]_0^t$$

$$= \frac{1}{2}\left[-t^2\frac{e^{-at}}{a} - \frac{2te^{-at}}{a^2} - \frac{2e^{-at}}{a^3} - \left(\frac{-2}{a^3}\right)\right]$$

$$= \frac{-1}{2}\left[\frac{t^2e^{-at}}{a} + \frac{2te^{-at}}{a^2} + \frac{2e^{-at}}{a^3} - \frac{2}{a^3}\right]$$

6) $L^{-1}\left(\log\left(1+\frac{\omega^2}{s^2}\right)\right)$

$$= L^{-1}\left[\log\left(\frac{s^2+\omega^2}{s^2}\right)\right]$$

$$F(s) = \log\left(\frac{s^2+\omega^2}{s^2}\right)$$

$$= \log(s^2+\omega^2) - \log(s^2)$$

$$\frac{d}{ds}F(s) = \frac{1}{s^2+\omega^2}\cdot 2s - \frac{1}{s^2}\cdot 2s$$

$$= \frac{2s}{s^2+\omega^2} - \frac{2}{s}$$

$$f(t) = -\frac{1}{t} L^{-1}\left(\frac{2s}{s^2+\omega^2} - \frac{2}{s}\right)$$

$$= \frac{-1}{t}\left[2L^{-1}\left(\frac{s}{s^2+\omega^2}\right) - 2L^{-1}\left(\frac{1}{s}\right)\right]$$

$$= \frac{-1}{t}[2\cos\omega t - 2]$$

$$= \frac{-2}{t}[\cos\omega t - 1]$$

19) Using method of partial fraction to find followings:

1) $L^{-1}\left(\frac{s+3}{(s^2+6s+13)^2}\right)$

$$\frac{s+3}{(s^2+6s+13)^2} = \frac{As+B}{(s^2+6s+13)} + \frac{Cs+D}{(s^2+6s+13)^2}$$

Multiply both sides by $(s^2+6s+13)^2$

$$s+3 = (As+B)(s^2+6s+13) + Cs + D$$

$$= As^3 + 6As^2 + 13As + Bs^2 + 6Bs + 13B + Cs + D$$

$$= As^3 + (6A+B)s^2 + (13A+6B+C)s + (13B+D)$$

By comparison of coefficients

$A = 0$

$6A + B = 0$

$13A + 6B + C = 1$

$13B + D = 3$

Now, $A = 0$

$B = 0$

$C = 1$

$D = 3$

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$$\frac{s+3}{(s^2+6s+13)^2} = \frac{0}{(s^2+6s+13)} + \frac{s+3}{(s^2+6s+13)^2}$$

$$L^{-1}\left(\frac{s+3}{(s^2+6s+9+4)^2}\right) = L^{-1}\left(\frac{s+3}{((s+3)^2+2^2)^2}\right)$$

$$= e^{-3t} \times \frac{1}{2(2)} t\sin 2t$$

$$\left[\because L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{1}{2a} t\sin at\right]$$

$$= \frac{e^{-3t}}{4} t\sin 2t$$

2) $L^{-1}\left(\frac{4s+5}{(s-1)^2(s+2)}\right)$

Ans: There are 2 repeated linear factors in denomination by partial fraction.

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$$

multiplying both side by $(s-1)^2(s+2)$

$$4s+5 = A\cdot(s-1)(s+2) + B(s+2) + C\cdot(s-1)^2$$

For $s = 1, -2, \& 0$

For $s = 1$

$9 = 3B$

$\boxed{B = 3}$

For $s = -2$

$-3 = 9C$

$\boxed{C = \frac{-1}{3}}$

For $s = 0$

$5 = -2A + 2B + C$

$5 = -2A + 6 + (-1/3)$

$\boxed{A = \frac{1}{3}}$

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{1}{3(s-1)} + \frac{3}{(s-1)^2} - \frac{1}{3(s+2)}$$

$$L^{-1}\left[\frac{4s+5}{(s-1)^2(s+2)}\right] = L^{-1}\left[\frac{1}{3(s-1)} + \frac{3}{(s-1)^2} - \frac{1}{3(s+2)}\right]$$

$$= \frac{1}{3}L^{-1}\left\{\frac{1}{s-1}\right\} + 3L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{1}{3}e^{t} + 3\left(\frac{1}{1!}t^1e^{t}\right) - \frac{1}{3}e^{-2t}$$

$$= \frac{1}{3}e^{t} + 3te^{t} - \frac{1}{3}e^{-2t}$$

3) $L^{-1}\left(\frac{6}{(s+2)(s-4)}\right)$

$$\frac{6}{(s+2)(s-4)} = \frac{A}{s+2} + \frac{B}{s-4}$$

Multiply both side by $(s+2)(s-4)$

$$6 = A(s-4) + B(s+2)$$

putting $s = 4, -2$

For $s = 4$

$$6 = 6B$$

$$\boxed{B = 1}$$

For $s = -2$

$$6 = -6A$$

$$\boxed{A = -1}$$

$$\rightarrow \frac{6}{(s+2)(s-4)} = \frac{-1}{s+2} + \frac{1}{s-4}$$

$$L^{-1}\left[\frac{6}{(s+2)(s-4)}\right] = -L^{-1}\left[\frac{1}{s+2}\right] + L^{-1}\left[\frac{1}{s-4}\right]$$

$$= -e^{-2t} + e^{4t}$$

$$= e^{4t} - e^{-2t} \qquad \left[\because L^{-1}\left(\frac{1}{s-a}\right) = e^{at}\right]$$

4) $L^{-1}\left(\frac{5s+3}{(s-1)(s^2+2s+5)}\right)$

sol^n. Non repeated quadratic factor in denominator by partial fraction.

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{(s^2+2s+5)}$$

Multiplying both side by $(s-1)(s^2+2s+5)$

$$5s+3 = A(s^2+2s+5) + Bs+C(s-1)$$

$$= As^2 + 2As + 5A + Bs^2 - Bs + Cs - C$$

$$0s^2+5s+3 = (A+B)s^2 + (2A-B+C)s + (5A-C)$$

By Comparision of coefficients of s according to power.

$$A+B = 0 \quad \text{—} ①$$

$$2A-B+C = 5 \quad \text{—} ②$$

$$5A-C = 3 \quad \text{—} ③$$

putting $A=-B$ & ~~$C = 5A-3$~~ in eq^n ②

$$-2B-B+C = 5$$

$$-3B+C = 5$$

$$C = 5+3B$$

Now put $A=-B$ & $C=5+3B$ in eq^n ③

$$-5B-5-3B = 3$$

$$-8B = 8$$

$$\boxed{B=-1}$$

Put $B=-1$ in eq^n ①

$$\boxed{A=1}$$

→ Put value of A & B in eq^n ②

$$2+1+C = 5$$

$$C = 5-3$$

$$\boxed{C=2}$$

$$\therefore L^{-1}\left(\frac{5s+3}{(s-1)(s^2+2s+5)}\right) = L^{-1}\left[\frac{1}{s-1} - \frac{s}{(s^2+2s+5)} + \frac{2}{(s^2+2s+5)}\right]$$

$$= L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{(s+1)-1}{(s+1)^2+2^2}\right) + 2L^{-1}\left(\frac{1}{(s+1)^2+2^2}\right)$$

$$= e^{t} - L^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) + L^{-1}\left(\frac{1}{(s+1)^2+2^2}\right) + 2L^{-1}\left(\frac{1}{(s+1)^2+2^2}\right)$$

$$= e^{t} - L^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) + 3L^{-1}\left(\frac{1}{(s+1)^2+2^2}\right)$$

$$= e^{t} - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t$$

5) $L^{-1}\left(\dfrac{s}{(s-1)(s^2+2s+2)}\right)$

Sol^n: Non repeated quadratic factor in denomination by partial fraction.

$$\frac{s}{(s-1)(s^2+2s+2)} = \frac{A}{s-1} + \frac{Bs+C}{(s^2+2s+2)}$$

Multiplying both side by $(s-1)(s^2+2s+2)$

$$\cancel{s = A(s^2+2s+2) + B(s-1)}$$

$$\cancel{= As^2 + 2As + 2A + Bs - B}$$

$$\cancel{0s^2+s+0 = As^2 + (2A+B)s + (2A-B)}$$

~~By comparision of coefficients of s according to power~~

$$\cancel{A = 0}$$

$$\cancel{2A + 2A + B = 1}$$

$$\cancel{2A - B = 0}$$

$$\cancel{B = 1}$$

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$$s = A(s^2+2s+2) + Bs + c(s-1)$$
$$= As^2 + 2As + 2A + Bs^2 - Bs + Cs - C$$
$$0s^2 + s + 0 = (A+B)s^2 + (2A - B + C)s + (2A - C)$$

comparing both side

$A + B = 0$ — ①

$2A - B + C = 1$ — ②

$2A - C = 0$ — ③

putting $\underline{A = -B}$ in eq^n ②

$$-2B - B + C = 1$$
$$-3B + C = 1$$
$$\underline{C = 1 + 3B}$$

Now putting $A = -B$ & $C = 1 + 3B$ in eq^n ③

$$-2B - 1 - 3B = 0$$
$$-5B = 1$$
$$\boxed{B = -\frac{1}{5}}$$

$\therefore \boxed{A = \frac{1}{5}}$

$\therefore C = 1 - \frac{3}{5}$

$$\boxed{C = \frac{2}{5}}$$

$$L^{-1}\left(\frac{s}{(s-1)(s^2+2s+2)}\right) = L^{-1}\left[\left(\frac{1/5}{s-1}\right) + \left(\frac{-1/5\, s}{(s^2+2s+2)}\right) + \left(\frac{2/5}{(s^2+2s+2)}\right)\right]$$

$$= \frac{1}{5} L^{-1}\left(\frac{1}{s-1}\right) - \frac{1}{5} L^{-1}\left[\frac{s+1-1}{(s+1)^2+1^2}\right] + \frac{2}{5} L^{-1}\left[\frac{1}{(s+1)^2+1^2}\right]$$

$$= \frac{1}{5}e^{t} - \frac{1}{5}L^{-1}\left[\frac{s+1}{(s+1)^2+1^2}\right] + \frac{1}{5}L^{-1}\left(\frac{1}{(s+1)^2+1^2}\right) + \frac{2}{5}L^{-1}\left(\frac{1}{(s+1)^2+1^2}\right)$$

$$= \frac{1}{5}e^{t} - \frac{1}{5}e^{-t}\cos t + \frac{3}{5}\cdot\frac{1}{1}e^{-t}\sin t$$

$$= \frac{1}{5}e^{t} - \frac{1}{5}e^{-t}\cos t + \frac{3}{5}e^{-t}\sin t$$

20) state the convolution theorem to solve followings:
evaluate $L^{-1}\left[\frac{1}{s^2(s-1)}\right]$, $t * e^{t}$, $1 * 1$

Ans: 1) $L^{-1}\left[\frac{1}{s^2(s-1)}\right] = L^{-1}\left[\frac{1}{s^2} - \frac{1}{s-1}\right]$

$$= L^{-1}\langle F(s)\cdot G(s)\rangle$$

$$L^{-1}\langle F(s)\rangle = L^{-1}\left\langle\frac{1}{s^2}\right\rangle = \frac{1}{(2-1)!}t^{2-1}$$

$$= t = f(t)$$

$$L^{-1}\langle G(s)\rangle = L^{-1}\left\langle\frac{1}{s-1}\right\rangle = e^{t} = g(t)$$

$$L^{-1}\langle F(s) * G(s)\rangle = \int_0^t f(u)\cdot g(t-u)\cdot du$$

$$= \int_0^t u\cdot e^{t-u}\cdot du$$

$$= \left[u\cdot\frac{e^{t-u}}{-1} - \frac{e^{t-u}}{-(-1)}(1)\right]_0^t$$

$$= \left[-ue^{t-u} - e^{t-u}\right]_0^t$$

$$= (-t-1) - (0 - e^{t})$$

$$L^{-1}\langle F(s) + G(s)\rangle = e^{t} - t - 1$$

2) $1 * 1$

$f(t) = g(t) = 1$

$$f(t) * g(t) = \int_0^t (f(u) \cdot g(t-u)\, du$$

$$= \int_0^t 1 \cdot 1 \cdot du$$

$$= [u]_0^t$$

$$= t$$

3) $t * e^t$

$f(t) = t$

$g(t) = e^t$

$$f(t) * g(t) = \int_0^t f(u) \cdot g(t-u)\, du$$

$$= \int_0^t u \cdot e^{(t-u)} \cdot du$$

$$= \left[u \cdot \frac{e^{t-u}}{(-1)} - \frac{e^{t-u}}{-(-1)} (1) \right]_0^t$$

$$= [-u \cdot e^{t-u} - e^{t-u}]_0^t$$

$$= (-t-1) - (0 - e^t)$$

$$= e^t - t - 1$$

Que 18) $L^{-1}\left(\frac{3s-12}{s^2+8}\right)$

$L^{-1}\left(\frac{e^{-\frac{\pi s}{2}} + e^{-\frac{3\pi s}{2}}}{(s^2+1)}\right)$

pending.

Que(22)