Section : 1

UNIT: 1 ORDinary L.D.E of Second Order & Higher Order

1. Define linearly Independent function. &

→ Define linearly dependent function

→ Two function $y_1(x)$ and $y_2(x)$ are said to be linearly Independent If,

$$k_1 y_1(x) + k_2 y_2(x) = 0. \quad \text{—— (1)}$$

implies k_1 & k_2 are both zero.

i.e. when y_1 or y_2 can not be expressed as Proportional to the other.

→ otherwise, y_1 and y_2 are linearly dependet if (1) holds for some constants k_1 and k_2 ~~noth~~ not both zero.

→ Define Wronskian

A Mathematical determinant whose first row Consists of n functions of x and whose fall. rows consist of the successive derivatives of these same fun.n with respect to x.

Q:2. Check whether the given sets are linearly independent or linearly dependent.

(1.) $\{1, e^x, e^{-x}\}$

$$= \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix}$$

$$W(x) = 1\,[e^x \cdot e^{-x} + e^x \cdot e^{-x}]$$

$$= 1\,[1+1]$$

$$w(x) = 2 \neq 0$$

→ Here $w(1, e^x, e^{-x}) = 2$

The set $\{1, e^x, e^{-x}\}$ is L.I.

(2) $\{e^x, e^{-x}, \cosh x\}$.

$$w(x) = \begin{vmatrix} e^x & e^{-x} & \cosh x \\ e^x & -e^{-x} & \sinh x \\ e^x & e^{-x} & \cosh x \end{vmatrix}$$

$= 0$

$\rightarrow$ The set $\{e^x, e^{-x}, \cosh x\}$ is L.D.

(3) $\{1, e^x\}$.

$$w(x) = \begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}$$

$w(x) = e^x$

$\rightarrow$ The set $\{1, e^x\}$ is L. I.

Q: 3. Using method of of Undermined coefficient solve

(i) $y'' - 2y' + 5y = 5x^3 - 6x^2 + 6x$.

$\rightarrow$ $(D^2 - 2D + 5)y = R(x)$ which is non-Homogeneous fun.

auxillary eq^n

$m^2 - 2m + 5 = 0$

$m = \dfrac{2 \pm 4i}{2} = 1 \pm 2i$

$y_c = e^x [c_1 \cos 2x + c_2 \sin 2x]$

$\rightarrow$ By Method of undetermined Coefficient.

To Find y_p.

$R(x) = 5x^3 - 6x^2 + 6x$

$y_p = k_0 + k_1 x + k_2 x^2 + k_3 x^3$

$y'_p = 0 + k_1 + 2k_2 x + 3k_3 x^2$

$y'_p = k_1 + 2k_2 x + 3k_3 x^2$

$y''_p = 2k_2 + 6k_3 x$

Replace y_p Inpluce in given eq.^{n} y in D.E.

~~$2k_2 - 2(k_1 + 2k_2 x) + 5(k_0 + k_1 x + k_2 x^2) = 5x^3 - 6x^2 + 6x$~~

~~$2k_2 - 2k_1 + 4k_2 x + 5k_0 + 5k_1 x + 5k_2 x^2 = 5x^3 - 6x^2 + 6x$~~

$$(2k_2 + 6k_3 x) - 2(k_1 + 2k_2 x + 3k_3 x^2) + 5(k_0 + k_1 x + k_2 x^2 + k_3 x^3) = 5x^3 - 6x^2 + 6x$$

$$5k_3 x^3 + x^2(-6k_3 + 5k_2) + x(6k_3 - 4k_2 + 5k_1) + (2k_2 - 2k_1 + 5k_0) = 5x^3 - 6x^2 + 6x$$

→ Compare the both side

$5k_3 = 5$ $-6k_3 + 5k_2 = -6$ $6k_3 - 4k_2 + 5k_1 = 6$

$\boxed{k_3 = 1}$ $-6(1) + 5k_2 = -6$ $6(1) - 0 + 5k_1 = 6$

$\boxed{k_2 = 0}$ $\boxed{k_1 = 0}$

$2k_2 - 2k_1 + 5k_0 = 0$

$\boxed{k_0 = 0}$

$\boxed{y_p = x^3}$

→ The General solution

$$y = y_c + y_p = e^x[c_1 \cos 2x + c_2 \sin 2x] + x^3$$

→ (ii) Solve The eq.^{n} $\frac{d^2y}{dx^2} + 4 = 4e^{2x}$ by Method of Undetemined Coefficient

→ which is non-Homogeneous eq.^{n}

→ $(D^2 + 4)y = R(x)$

auxillary eq.^{n}

$m^2 + 4 = 0$

$\boxed{m = \pm 2i}$

$\boxed{y_c = c_1 \cos 2x + c_2 \sin 2x}$

→ To Find y_p :-

$$R(x) = 4e^{2x}$$

$$y_p = k e^{2x}$$

$$y_p' = 2k e^{2x}$$

$$y_p'' = 4k e^{2x}$$

$$4k e^{2x} + 4k e^{2x} = 4e^{2x}$$

$$8k e^{2x} = 4e^{2x}$$

$$\boxed{k = 1/2}$$

$$\boxed{y_p = e^{2x}/2}$$

→ (iii) Solve Using Method of Undetermined co-efficients solve the following $y'' + 2y' + 10y = 25x^2 + 3$

→ which is non-Homogeneous. D.E.

→ To Find y_c :-

$$(D^2 + 2D + 10)y = R(x)$$

Auxiliary eq.^{n}

$$m^2 + 2m + 10 = 0$$

$$a = 1, \quad b = 2, \quad c = 10$$

$$m = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm \sqrt{-36}}{2}$$

$$\boxed{m = -1 \pm 3i}$$

$$y_c = e^{-x}[c_1 \cos 3x + c_2 \sin 3x]$$

→ To Find y_p :-

$$y_p = k_0 + k_1 x + k_2 x^2$$

$$y_p' = k_1 + 2k_2 x$$

$$y_p'' = 2k_2$$

→ Replace y_p inplace of given eq.^{n} y in D.E.

$$2k_2 + 2(k_1 + 2k_2 x) + 10(k_0 + k_1 x + k_2 x^2) = 25x^2 + 3$$

$$10k_2 x^2 + x(4k_2 + 10k_1) + (2k_2 + 2k_1 + 10k_0) = 25x^2 + 3$$

Compare the both side.

$10k_2 = 25$, $\boxed{k_2 = 5/2}$

$4k_2 + 10k_1 = 0$

$4\left(\frac{5}{2}\right) = -10k_2$

$\boxed{k_2 = -1}$

$2k_2 + 2k_1 + 10k_0 = 3$

$2\left(\frac{5}{2}\right) + 2(-1) + 10k_0 = 3$

$5 - 2 + 10k_0 = 3$

$\boxed{k_0 = 0}$

$$\boxed{y_p = 5/2\, x^2 - x}$$

→ The general solⁿ

$$y = e^{-x}(c_1\cos 3x + c_2 \sin 3x) + 5/2\, x^2 - x.$$

→ (iv) Solve the eqⁿ $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 6\sin 3x$ by method of undetermined coefficients.

$$(D^2 + 4D + 4)y = 6\sin 3x = R(x)$$

Auxillary Eqⁿ

$$m^2 + 4m + 4 = 0.$$

$$(m+2)^2 = 0$$

$$\boxed{m = -2, -2}$$

$y_c = (c_1 + c_2 x)e^{-2x}$

→ To Find y_p.

$$y_p = k_1 \sin 3x + k_2 \cos 3x$$

$$y_p' = 3k_1 \cos 3x - 3k_2 \sin 3x$$

$$y_p'' = -9k_1 \sin 3x - 9k_2 \cos 3x.$$

→ Replace.

$$(-9k_1 \sin 3x - 9k_2\cos 3x) + 4(3k_1\cos 3x - 3k_2 \cancel{\cos 3x}\sin 3x) + 4(k_1 \sin 3x + k_2 \cos 3x) = 6\sin 3x.$$

$$\sin 3x(-9k_1 - 12k_2 + 4k_1) + \cos 3x(-9k_2 + 12k_1 + 4k_2) = 6\sin 3x$$

$-9k_1 - 12k_2 + 4k_1 = 6$ $\qquad$ $-9k_2 + 12k_1 + 4k_2 = 0$

$-5k_1 - 12k_2 = 6$ $\qquad$ $-5k_2 + 12k_1 = 0$

$5k_2 = 12k_1$

$$-5k_1 + 12k_2 = 6$$

$$-5\left(\frac{5k_2}{12}\right) + 12k_2 = 6$$

$$k_1 = \frac{5k_2}{12}$$

$$\frac{-25k_2 + 144k_2}{12} = 6$$

$$k_1 = \frac{5}{12}\left(-\frac{72}{169}\right)$$

$$-169k_2 = 72$$

$$\boxed{k_2 = -72/169}$$

$$\boxed{k_1 = -30/169}$$

$$\boxed{y_p = -\frac{30}{169}\sin 3x - \frac{72}{169}\cos 3x}$$

Q: 4. Solve $y'' + a^2y = \sec ax$ by the method of Variation of Parameters.

$$y'' + a^2y = \sec ax$$

→ Auxillary eq.

$$m^2 + a^2 = 0$$

$$m^2 = -a^2$$

$$\boxed{m = \pm ai}$$

$$\boxed{y_c = c_1\cos ax + c_2\sin ax}$$

$$y_1 = \cos ax \qquad y_2 = \sin ax$$

$$w(x) = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$$

$$= a\cos^2 ax + a\sin^2 ax$$

$$\boxed{w(x) = a}$$

$$y_p = -y_1(x)\int \frac{y_2(x)\cdot R(x)}{w(x)}\,dx + y_2(x)\int \frac{y_1(x)\cdot R(x)}{w(x)}\,dx$$

$$= -\cos ax\int \frac{\sin ax\cdot\sec ax}{a}\,dx + \sin ax\int \frac{\cos ax\cdot\sec ax}{a}\,dx$$

$$= -\frac{\cos ax}{a}\int \tan ax\,dx + \frac{\sin ax}{a}\int 1\,dx$$

$$\boxed{y_p = -\frac{\cos ax}{a^2}\log(\sec ax) + \frac{\sin ax}{a}\cdot x}$$

$\ast \longrightarrow$ Solve $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$ by MVP.

$\rightarrow$ Auxillary Eq^n

$$m^2 - 6m + 9 = 0$$

$$(m-3)^2 = 0$$

$$m = 3, 3$$

$$y_c = (C_1 + C_2 x) e^{3x}$$

$$y_1 = e^{3x} \qquad y_2 = e^{3x} \cdot x.$$

$$w(x) = \begin{vmatrix} e^{3x} & e^{3x} x \\ 3e^{3x} & (1+3x)e^{3x} \end{vmatrix}$$

$$w(x) = (e^{3x})^2 (1+3x) - 3x (e^{3x})^2$$

$$= (e^{3x})^2 (1 + 3x - 3x)$$

$$w(x) = (e^{3x})^2$$

$$y_p = -y_1(x) \int \frac{y_2(x) \cdot R(x)}{w(x)} \cdot dx + y_2(x) \int \frac{y_1(x) \cdot R(x)}{w(x)} \cdot dx$$

$$= -e^{3x} \int \frac{\cancel{e^{3x}} \cdot \cancel{x} \left(\frac{\cancel{e^{3x}}}{\cancel{x^2}\, x} \right)}{\cancel{(e^{3x})^2}} + e^{3x} \cdot x \int \frac{\cancel{e^{3x}} \cdot \left(\cancel{e^{3x}} / x^2 \right)}{\cancel{(e^{3x})^2}}$$

$$= -e^{3x} \int \frac{1}{x} \cdot dx + e^{3x} \cdot x \int \frac{1}{x^2} \cdot dx$$

$$y_p = -e^{3x} \log x + e^{3x} \cdot x \left(-\frac{1}{x} \right)$$

$$y_p = -e^{3x} \log x - e^{3x}$$

$$\boxed{y_p = -e^{3x} (\log x + 1)}$$

$\ast \longrightarrow$ Solve $y'' + 9y = \sec 3x$ by MVP.

$\rightarrow$ Auxillary Eq.^{m}

$$m^2 + 9 = 0$$

$$\boxed{m = \pm 3i} \qquad y_c = c_1 \cos 3x + c_2 \sin 3x$$

$y_1 = \cos 3x \qquad y_2 = \sin 3x$

$$w(x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

Direct $\quad w(x) = 3$

$$y_p = \frac{-\cos 3x}{9} \log(\sec 3x) + \frac{\sin 3x}{3} \cdot x.$$

→ Solve $y'' + 16y = \sec 4x$ by MVP.

→ Same (1) & (3).

Q:6. Solve Legendre's Differential equation

→ $(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x$

→ which is non-Homogeneous D.E.

→ To Find y_c.

Let $(2x+3) = e^z$

$x = \frac{e^z - 3}{2} \qquad z = \log(2x+3)$

Let $(2x+3)\, y' = aDy = 2Dy$

$(2x+3)\, y'' = a^2 D(D-1)y = 4D(D-1)y$

$4D(D-1)y - 2(2Dy) - 12y = 6\left(\frac{1}{2}(e^z - 3)\right)$

$(4D^2 - 4D - 4D - 12)y = 3(e^z - 3)$

$(4D^2 - 8D - 12)y = 3e^z - 9$

$f(D)\, y = R(D)$

→ A.E. is. $\quad 4m^2 - 8m - 12 = 0$

$m^2 - 2m - 3 = 0$

$(m-3)(m+2) = 0$

$\boxed{m = -2, 3}$

$y_c = c_1 e^{-2z} + c_2 e^{3z} \Rightarrow y_c = c_1 (2x+3)^{-1} + c_2 (2x+3)^3$

→ To Find y_p

$$y_p = \frac{1}{f(D)} . R(z)$$

$$= \frac{1}{4D^2-8D-12}\left[3e^{z} - 9e^{0}\right]$$

$$= \frac{3e^{z}}{4D^2-8D-12} - \frac{9e^{0}}{4D^2-8D-12}$$

$$= \frac{3e^{z}}{4-8-12} - \frac{9e^{0}}{0-0-12}$$

$$= \frac{3e^{z}}{-16} - \frac{9}{-12}$$

$$= \frac{3}{4} - \frac{3e^{z}}{16}$$

$$= 3/4\left(1 - e^{z}/4\right)$$

$$\boxed{y_p = \frac{3}{4}\left(1 - \frac{(2x+3)}{4}\right)}$$

The General Solⁿ

$$y = c_1(2x+3)^{-1} + c_2(2x+3)^{3} + \frac{3}{4}\left(1 - \frac{(2x+3)}{4}\right)$$

→ (ii) $(2x+3)^2\frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 0$

→ y_c Same. which are Homogeneous fun.

$y_p = 0.$

→ (iii) $(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 4\cos(\log(1+x))$

→ which is non-Homogeneous D.E.

→ let $a=1,\ b=1,$ where $(ax+b) = (x+1)$

let $z = \log(x+1)$

$e^{z} = x+1.$

let $(x+1)y' = aDy = 1Dy$

$(x+1)^2y'' = a^2D(D-1).y = D(D-1)y$

$$D(D-1)y + Dy + y = 4\cos z$$
$$(D^2 - D + D + 1)y = 4\cos z$$
$$(D^2+1)y = 4\cos z$$
$$f(D)y = R(z)$$

A.E. is. $m^2+1 = 0$

$$\boxed{m = \pm i}$$

$$y_c = c_1 \cos z + c_2 \sin z$$
$$y_c = c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1))$$

→ To Find y_p :

$$y_p = \frac{1}{f(D)} R(z)$$
$$= \frac{1}{D^2+1} 4\cos z$$
$$= \frac{4\cos z}{2D}$$
$$= 2z \int \cos z \, dz$$
$$= 2z(\sin z)$$
$$y_p = 2z \sin z$$
$$y_p = 2\log(x+1) \sin(\log(x+1))$$

→ The G.S.

$$y = c_1 \cos(\log(1+x)) + c_2 \sin(\log(1+x)) + 2\log(x+1)\sin(\log(x+1))$$

Q: 8. Solve by Euler - Cauchy Method

(I) $x^3 y''' + 2x^2 y'' + 2y = 10(x + \frac{1}{x})$

→ which is non-Homogeneous Euler Cauchy D.E.

Let $x^3 y''' = D(D-1)(D-2)y$

$$x^2 y'' = D(D-1)y$$
$$xy' = Dy$$

$$D(D-1)(D-2)y + 2D(D-1)y + 2y = 10\left[e^z + \frac{1}{e^z}\right]$$
$$(D^2 - D)(D-2)y + 2D^2 y - 2Dy + 2y = 10\left[e^z + \frac{1}{e^z}\right]$$
$$(D^3 - D^2 - \cancel{2D^2} + \cancel{2D} + \cancel{2D^2} - \cancel{2D} + 2)y = 10\left[e^z + \frac{1}{e^z}\right]$$
$$(D^3 - D^2 + 2)y = 10\left[e^z + \frac{1}{e^z}\right]$$
$$f(D)y = R(x).$$

A.E. eq.^{n}

$$m^3 - m^2 + 2 = 0$$

$$(m+1)(m^2 - 2m + 2) = 0$$

$$a = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = \frac{2 \pm \sqrt{4-8}}{2}$$

$$\boxed{a = 1 \pm 1i}$$

$$\boxed{m = -1,\ 1 \pm i}$$

$$y_c = c_1 e^{-z} + e^{z}[c_2 \cos z + c_3 \sin z]$$

$$y_c = c_1 e^{-\log x} + e^{\log x}[c_2 \cos(\log x) + c_3 \sin(\log x)]$$

$$\boxed{y_c = c_1 x^{-1} + x[c_2 \cos(\log x) + c_3 \sin(\log x)]}$$

→ To Find y_p

$$y_p = \frac{1}{f(D)} \cdot R(z)$$

$$= 10 \frac{1}{D^3 - D^2 + 2} e^z + 10 \frac{1}{D^3 - D^2 + 2} \cdot \frac{1}{e^z}$$

$$= \cancel{10}^{5} \frac{e^z}{\cancel{1-1+2}} + 10 \frac{z e^{-z}}{3D^2 - 2D}$$

$$= 5e^z + \cancel{10}^{2} \frac{z e^{-z}}{\cancel{3+2}}$$

$$\boxed{y_p \Rightarrow 5e^z + 2ze^{-z}}$$

$$y_p = 5x + 2\log x \cdot \frac{1}{x}$$

$$\boxed{y_p = 5x + \frac{2\log x}{x}}$$

→ $x^3 y''' + 2x^2 y'' + 2y = 0.$

same as y_c

$$y_p = 0.$$

→ (iii) $x^2y'' - 3xy + 4y = x^2$

→ which is Non-Homogeneous Euler's Cauchy D.E.

let $z = \log x$

$e^z = x$

$e^{2z} = x^2$

let $xy' = Dy$

$x^2y'' = D(D-1)y.$

$D(D-1)y - 3Dy + 4y = e^{2z}$

$(D^2 - D - 3D + 4)y = e^{2z}$

[$f(D)y = R(z)$]

→ Auxillary eq.^{m}

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$\boxed{m = 2, 2}$

$y_c = [c_1 + c_2 z] e^{2z}$

$\boxed{y_c = [c_1 + c_2 \log x] x^2}$

→ To Find y_p

$y_p = \frac{1}{f(z)} \cdot R(z)$

$= \frac{1}{D^2 - 4D + 4} e^{2z}$

$= z \times \frac{1}{2D - 4} e^{2z}$

$= \frac{z^2 e^{2z}}{2}$

$\boxed{y_p = \frac{(\log x)^2 x^2}{2}}$

→ $x^2y'' - 2.5xy' - 2.0y = 0.$

same

$y_p = 0$

The G.S.

$\boxed{y = [c_1 + c_2 \log x] x^2 + \frac{(\log x)^2 x^2}{2}.}$