CSC 335 Special Session on Strong Induction February 21 2024

The only difference between

weak: Prove $Q(n)$ by assuming $Q(n-1)$

(equiv: $Q(n+1)$ by assuming $Q(n)$)

and

strong: Prove $Q(n)$ by assuming

$$\bigwedge_{i=0}^{n-1} Q(i)$$

(ie $Q(0) \wedge Q(1) \wedge \ldots \wedge Q(n-1)$)

(equiv: $Q(n+1)$ by assuming $\bigwedge_{i=0}^{n} Q(i)$)

induction is the IH

That is: for weak induction we use just $Q(n-1)$ to prove $Q(n)$ while for strong induction we may use any or all of $Q(0)$, $Q(1)$, ..., $Q(n-1)$ to prove $Q(n)$.

Example: we show by strong induction that for every integer $n >= 4$, n can be decomposed as a sum of 2's and 5's. That is: one need only have a stash of \$2 and \$5 bills to make change for any whole number of dollars ≥ 4.

For completeness, we shall also give a proof using weak induction.

So: let us consider the 2-5 problem, to be solve using strong induction. From above, we see right away that we should focus on the divide & conquer part first: how can we use the hypothesis

$$\bigwedge_{i=4}^{n-1} Q(i)$$

where $Q(i)$ is "change can be made for \$$i$ using just \$2 and \$5 bills"

to show that $Q(n)$ is true? Since we are told $n \geq 4$, we start by examining $n = 4, 5, \ldots$ — looking for a way to make use of the strong induction hypothesis. (SIH)

$n = 4 \longrightarrow 2 + 2$ (no use of SIH

$n = 5 \longrightarrow 5$ (no use of SIH)

$n=6 \longrightarrow 4+2$ use $Q(4)$

$n=7 \longrightarrow 5+2$ use $Q(5)$

$n=8 \longrightarrow 4+4$ use $Q(4)$

[OR]

$2+6$ use $Q(6)$

$n=9 \longrightarrow 4 + (9-4)$ use $Q(4)$, use $Q(5)$

[OR]

$2 + (9-2)$ use $Q(7)$

$n=10 \longrightarrow \overset{Q(4)}{4} + \overset{Q(6)}{(10-4)}$

OR

$2 + \overset{Q(8)}{(10-2)}$

OR

$\overset{Q(6)}{6} + \overset{Q(4)}{(10-6)}$

OR

$\overset{Q(8)}{8} + (10-8)$

Possibly use(d) decompositions

It appears that we can use any of

these decompositions – why not choose what appears to be simplest — ie — the decomposition of $n > 5$ into 2 and $n-2$? (Note that we could use 4 and $n-4$ for $n > 8$ — resulting in a 4-case basis step

$n = 4, 5, 6, 7$

But if we use 2 and $n-2$, we need only a 2-case basis step

$n = 4, 5$

As you can see, yet other - worse? - choices are possible)

What then does the argument look like?

Basis: show the 4 and 5 cases directly

SIH: . . .

SIS: Suppose $n \geq 6$. Then n can be decomposed as $2 + (n-2)$ where $n-2 \geq 4$. The '2' requires no additional analysis, and the $n-2$ case holds by the SIH.

Clearly, if change for $n-2$ can be made using \$2 and \$5, then change for n can similarly be made as

\$2 + [change for $n-2$ using just \$2 and \$5]

With this analysis in hand, you should go ahead and write the associated Scheme program to solve the 2-5 problem.

But before you do, let's ask whether our current solution idea yields a 'good' (as opposed to merely correct) solution.

Can you see (write a proof!) that the solution we have sketched always uses the max possible number of 2s?
Would a more balanced mix of 2's and 5's be preferable?

So there is another exercise for you: figure out how to do so, and prove that your solution has this balanced property (while still being correct).

Now: what about a solution by weak induction? It is interesting to note that anything provable by strong induction is also provable by weak induction, and conversely!

Here is one idea for a weak induction:

$n \geq 4$ is either even or odd. If it is even, then $n = 2m$ for some m — so change can be made using m \$2 bills.

If n is odd, then $n = 2m+1$ for some m. But $2m+1 = (2m-4)+5$ and $2m-4$ is even — make change using \$2 for $2m-4$, and throw on a fiver. Done!

Again, how about writing the program? And again, can you say anything about the number of \$2 and \$5 used?

OK — so what about an example where it seems that strong induction is much easier to use than weak induction?

② Most of the Fundamental Theorem of Arithmetic — Strong Induction Practice

We have the Fundamental Theorem of Arithmetic as follows:

Every integer $n \geq 2$ can be written ~~uniquely~~ as a product of powers of primes.

We'll ignore 'uniquely', even 'uniquely up to ordering of the factors'

$100 = 2 \times 2 \times 5 \times 5$

$= 5 \times 5 \times 2 \times 2$

$= 2 \times 5 \times 5 \times 2$

...

In fact, if we allow $p^0 = 1$, then we can say "every integer $n \geq 1$ can be written as a product of powers of primes"

How can we deploy strong induction to prove this?

Given $n > 1$. Either n is prime or n is not prime. If n is prime, then n itself is already a product of primes

$$n = \prod_{i=n}^{n} i$$

"a one-term product"

If n is not prime, then we know that n is composite – ie – n has factors p and q

$$n = p * q$$

By the SIH, both p and q

are products of primes. So

Then n is a product of primes

$$n = \overbrace{\underline{\text{prod of primes}}}^{p}\ \overbrace{\underline{\text{prod of primes}}}^{q}$$

Basis step: $n=2$, which is a prime.

We could start at $n=1$ — in this case, the basis step is

That — for any prime p —

$1 = p^0$, which indeed is a

product of powers of primes!

What about a weak induction for this problem ?

The issue is that $n \geq 2$ **Never** decomposes as

$$(n-1) * \text{somethingelse}$$

So how would the weak induction hypothesis help?

Another suggested hw: can you give a weak induction proof for this problem?