Question 1.

Given two functions $f(x)$ and $g(x)$, prove that

$$|\max_x f(x) - \max_x g(x)| \leq \max_x |f(x) - g(x)|. \tag{1}$$

Proof.
consider the following cases:

$$\begin{aligned} \max_x f(x) &\geq \max_x g(x) \\ \max_x f(x) &< \max_x g(x) \end{aligned} \tag{2}$$

For case 1, we have

$$\begin{aligned} |\max_x f(x) - \max_x g(x)| &= \max_x f(x) - \max_x g(x) \\ &\leq \max_x |f(x) - g(x)|. \end{aligned} \tag{3}$$

Because the difference between the max of two functions cannot exceed the max difference between the two functions.

For case 2, we have

$$\begin{aligned} |\max_x f(x) - \max_x g(x)| &= \max_x g(x) - \max_x f(x) \\ &\leq \max_x |g(x) - f(x)|. \\ &= \max_x |f(x) - g(x)|. \end{aligned} \tag{4}$$

Because, again, the difference between the max of two functions cannot exceed the max difference between the two functions.

Therefore, we have proved that

$$|\max_x f(x) - \max_x g(x)| \leq \max_x |f(x) - g(x)|. \tag{5}$$

Question 2.

Consider an infinite horizon discounted RL problem with finite state space and finite action space. Consider a model-based Q-learning algorithm with the following value iteration:

$$Q_{k+1}(i,u) = E[r(i,u)] + \alpha \sum_j p_{ij}(u) \max_v Q_k(j,v), \tag{6}$$

or written as

$$Q_{k+1} = T_Q(Q_k). \tag{7}$$

Prove that T_Q is a contraction mapping.

Proof.

$$\begin{aligned}
\|T(Q_k) - T(Q_{k+1})\|_\infty &= \max_{i,u} |T(Q_k(i,u)) - T(Q_{k+1}(i,u))| \\
&= \max_{i,u} |Q_{k-1}(i,u) - Q_k(i,u)| \\
&= \max_{i,u} \left| E[r(i,u)] + \alpha \sum_j p_{ij}(u) \max_v Q_k(j,v) - E[r(i,u)] - \alpha \sum_j p_{ij}(u) \max_v Q_{k+1}(j,v) \right| \\
&= \alpha \max_{i,u} \sum_j p_{ij}(u) \left| \max_v Q_k(j,v) - \max_v Q_{k+1}(j,v) \right| \\
&\geq \alpha \max_{i,u} \sum_j p_{ij}(u) \max_v |Q_k(j,v) - Q_{k+1}(j,v)| \\
&\geq \alpha \max_{j,v} |Q_k(j,v) - Q_{k+1}(j,v)| \\
&= \alpha \|Q_k - Q_{k+1}\|_\infty
\end{aligned} \tag{8}$$