Linear Algebra and Calculus

Unit-I: Matrices

- Introduction of Matrix
- Rank of a Matrix
- Canonical Form of a Matrix (Echelon Form)
- Normal Form of a Matrix
- **System of Linear Equations**
- Orthogonal Transformations
- Eigen Values and Eigen Vectors
- Diagonalization of Matrices
- Cayley Hamilton Theorem
- Applications to problems in Engineering

- **Example 4.** *Show that the system of equations* $3x + 4y + 5z = a;\ 4x + 5y + 6z = b;\ 5x + 6y + 7z = c$ *is consistent only when* a, b, c *are in arithmetic progression.*
- **Solution:** We have to prove that the following system of simultaneous equations is

$$3x + 4y + 5z = a;\ 4x + 5y + 6z = b;\ 5x + 6y + 7z = c \qquad \text{- - - - - - -} \qquad (1)$$

is consistent only when a, b, c are in arithmetic progression i. e. $a + c = 2b$.

The system of simultaneous equations (1) can be written as

$$AX = B$$

$$\Rightarrow \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Where $A = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}$ = Coefficient Matrix, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ = Variable matrix, $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ = Constant matrix.

Let us consider the Augmented matrix $[A \colon B] = \begin{bmatrix} 3 & 4 & 5 & : & a \\ 4 & 5 & 6 & : & b \\ 5 & 6 & 7 & : & c \end{bmatrix}$.

$$[A:B]=\begin{bmatrix}3 & 4 & 5 & : & a\\ 4 & 5 & 6 & : & b\\ 5 & 6 & 7 & : & c\end{bmatrix}$$

Apply $R_1 \rightarrow R_1 - R_2$

$$\sim\begin{bmatrix}-1 & -1 & -1 & : & a-b\\ 4 & 5 & 6 & : & b\\ 5 & 6 & 7 & : & c\end{bmatrix}$$

Apply $R_2 \rightarrow R_2 + 4R_1$ and $R_3 \rightarrow R_3 + 5R_1$

$$\sim\begin{bmatrix}-1 & -1 & -1 & : & a-b\\ 0 & 1 & 2 & : & b+4a-4b\\ 0 & 1 & 2 & : & c+5a-5b\end{bmatrix}$$

Apply $R_3 \rightarrow R_3 - R_2$

$$\sim\begin{bmatrix}-1 & -1 & -1 & : & a-b\\ 0 & 1 & 2 & : & 4a-3b\\ 0 & 0 & 0 & : & c+5a-5b-4a+3b\end{bmatrix}$$

This is the Echelon form of the given matrix $[A:B]$. It is clear that the rank of A is 2 i.e. $\rho(A)=2$.

- **Example 5.** *Investigate for what values of* λ *and* μ *the system of simultaneous equations* $x+y+z=6;\ x+2y+3z=10;\ x+2y+\lambda z=\mu$ *have (i) no solution (ii) an infinite number of solution and (iii) a unique solution.*

- **Solution:** We have to investigate for what values of λ and μ the system of simultaneous equations

$x+y+z=6;\ x+2y+3z=10;\ x+2y+\lambda z=\mu$ - - - - - - - - - (1)

have (i) no solution (ii) an infinite number of solution and (iii) a unique solution.

The system of simultaneous equations (1) can be written as

$$AX=B$$

$$\Rightarrow \begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ 10\\ \mu\end{bmatrix}$$

Where $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 2 & \lambda\end{bmatrix}$ =Coefficient Matrix, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ =Variable matrix, $B=\begin{bmatrix}6\\ 10\\ \mu\end{bmatrix}$ =Constant matrix.

Let us consider the Augmented matrix $[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 6\\ 1 & 2 & 3 & : & 10\\ 1 & 2 & \lambda & : & \mu\end{bmatrix}$.

$$[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 6\\ 1 & 2 & 3 & : & 10\\ 1 & 2 & \lambda & : & \mu\end{bmatrix}$$

Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$[A:B]\sim\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 1 & 2 & : & 4\\ 0 & 1 & \lambda-1 & : & \mu-6\end{bmatrix}$$

Apply $R_3 \to R_3 - R_2$

$$[A:B]\sim\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 1 & 2 & : & 4\\ 0 & 0 & \lambda-3 & : & \mu-10\end{bmatrix} \text{- - - - - - - - -} \qquad (2)$$

This is the Echelon form of the given matrix $[A:B]$.

We have to find the values of λ and μ for which the system has

(i) No Solution: If $\rho(A) \neq \rho(A:B)$, then the system has no solution.

Clearly if $\lambda - 3 = 0 \Rightarrow \lambda = 3$ and $\mu - 10 \neq 0 \Rightarrow \mu \neq 10$, then $\rho(A) = 2$ and $(A:B) = 3$.

Hence the system of equation have no solution, if $\boxed{\lambda = 3, \mu \neq 10}$

From (1), we have

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda - 3 & : & \mu - 10 \end{bmatrix}$$

(ii) An infinite number of solution: If $\rho(A) = \rho(A:B) \leq$ n = the number of variables, then the system has infinitely many solutions.

Clearly if $\lambda - 3 = 0 \Rightarrow \lambda = 3$ and $\mu - 10 = 0 \Rightarrow \mu = 10$, then $\rho(A) = 2 = (A:B)$. *Hence the system of equation have infinitely many solutions, if* $\boxed{\lambda = 3,\ \mu = 10}$

(iii) Unique solution: If $\rho(A) = \rho(A:B) =$ n = the number of variables, then the system has unique solution.

Clearly if $\lambda - 3 \neq 0 \Rightarrow \lambda \neq 3$ and μ can have any value, then $\rho(A) = 3 = (A:B)$. *Hence the system of equation has unique solution, if* $\boxed{\lambda \neq 3,\ \mu = any\ value}$

Home Assignment

Solve the following problems of system of linear equations.

1. Use matrix method to determine the values of λ for which the equations $x + 2y + z = 3;\ x + y + z = \lambda;\ 3x + y + 3z = \lambda^2$ are consistent and solve them for these values of λ. $[Ans: \lambda = 2, 3; for\ \lambda = 2,\ \ x = 1 - t,\ \ y = 1,\ z = t; for\ \lambda = 3,\ \ x = 3 - t,\ \ y = 0,\ z = t]$
2. *Investigate for what values of* λ *and* μ *the system of equations* $2x + 3y + 5z = 9;\ 7x + 3y - 2z = 8;\ 2x + 3y + \lambda z = \mu$ *have (i) no solution (ii) an infinite number of solution and (iii) a unique solution.* $[Ans: (i)\ \lambda = 5, \mu \neq 9, (ii)\ \lambda = 5, \mu = 9, (iii)\ \lambda \neq 5, \mu = any\ values]$
3. *Investigate for what values of* a *and* b *the system of equations* $2x - y + 3z = 2;\ x + y + 2z = 2;\ 5x - y + az = b$ *have (i) no solution (ii) an infinite number of solution and (iii) a unique solution.* $[Ans: (i)\ a = 8, b \neq 6, (ii)\ a = 8, b = 6, (iii)\ a \neq 8, b = any\ values]$
4. *Show that the system of equations* $3x + 4y + 5z = \alpha;\ 4x + 5y + 6z = \beta;\ 5x + 6y + 7z = \gamma$ *is consistent only when* $\alpha + \gamma = 2\beta$.
5. *Investigate for what values of* k *the equations* $x + y + z = 1;\ 2x + y + 4z = k;\ 4x + y + 10z = k^2$ *have an infinite number of solution? Hence find the solutions.* $[\ k = 1,\ \ 2]$

Questions?

Thanks

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System of Homogeneous Linear Equations

Consider the system of linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = 0$$

....

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = 0$$

These system of equations can be written as in the matrix form

$AX = O$ - - - - - - - - - - *(1)*

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Here A = Coefficient Matrix

X = Variable Matrix

O = Zero Matrix

Augmented Matrix :

If $AX = O$ be a system of m linear equations with n unknowns, then **Augmented Matrix** is denoted by $[A : 0]$ and is defined by

$$[A : 0] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & 0 \\ a_{31} & a_{32} & \cdot\cdot & {}_{3} & : & 0 \\ \vdots & \vdots & \vdots & \vdots & : & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & : & 0 \end{bmatrix} = A$$

Here

(

$$\Rightarrow \rho\ (A) = \rho\ (A\!:O)$$

$\Rightarrow$ System of homogeneous linear equations is always consistent.

Condition for consistency of homogeneous Equations

Let $AX = O$ be a system of m homogeneous linear equations with n unknowns, then

m = Total number of equations and n = Total number of unknowns

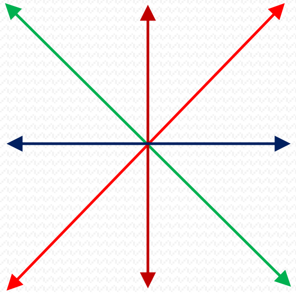
$\rho\,(A)$ = Rank of Coefficient Matrix $= \rho\,(A{:}\,B) =$ O Rank of Augmented matrix

Here it is clear that $\boldsymbol{\rho\,(A) = \rho\,(A{:}\;O)}$, *hence the system of homogeneous equations are always consistent. So, the given system of equations have solutions.*

(i) ***Trivial Solution:*** *If* $\boldsymbol{\rho\,(A) = \rho\,(A{:}\;O) = n =}$ ***Number of variables****, then the system have unique solution.*

(ii) ***Non-trivial Solution:****If* $\boldsymbol{\rho\,(A) = \rho\,(A{:}\,B\;O < n =}$ ***Number of variables****, then the system of equations have infinitely many solutions.*

Condition for consistency of Homogeneous Equations



Lines are intersecting at origin

Trivial Solution or only one solution

Lines are overlapping to each others

Non Trivial Solutions

Condition for consistency of homogeneous Equations

Homogeneous System AX = B, n = unknowns

Always consistent
$\rho\,(A) = \rho\,(A{:}\,B) = r$

Trivial Solution if
$n = r$ or $|A| \neq 0$

Lines are intersecting at origin only

Non trivial solutions if
$r < n$ or $|A| = 0$

Lines are overlapping to each others

- **Example 1.** *Examine for nontrivial solutions of the following system of linear equations and hence find solutions:* $x+2y+3z=0;\ 2x+3y+z=0;\ 4x+5y+4z=0.$
- **Solution:** We have to examine for nontrivial solution of the following system of equations

$x+2y+3z=0;\ 2x+3y+z=0;\ 4x+5y+4z=0$ - - - - - (1)

The system of simultaneous equations (1) can be written as

$$AX=O$$

$$\Rightarrow \begin{bmatrix}1&2&3\\2&3&1\\4&5&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{- - - - - - - - - - - - -} \quad (2)$$

$$\text{Consider}|A|=\begin{vmatrix}1&2&3\\2&3&1\\4&5&4\end{vmatrix}=1(3\times 4-5\times 1)-2(2\times 4-4\times 1)+3(2\times 5-3\times 4)$$

$$=1(12-5)-2(8-4)+3(10-12)$$

$$=1(7)-2(4)+3(-2)=7-8-6=-7\neq 0$$

$$\because |A|\neq 0$$

$\therefore$ The system has trivial solution. i.e. $\boxed{x=0,y=0,z=0}$

- **Example 2.** *Examine for nontrivial solutions of the following system of linear equations and hence find solutions:* $2x - y + 3z = 0;\ 3x + 2y + z = 0;\ x - 4y + 5z = 0.$
- **Solution:** We have to examine for nontrivial solution of the following system of equations

$2x - y + 3z = 0;\ 3x + 2y + z = 0;\ x - 4y + 5z = 0$ - - - - - - (1)

The system of simultaneous equations (1) can be written as

$$AX = O$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{- - - - - - - - - - - - - -} \quad (2)$$

Consider $|A| = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{vmatrix}$

$$= 2 \times [2 \times 5 - (-4) \times 1] - (-1) \times [3 \times 5 - 1 \times 1] + 3 \times [3 \times (-4) - 1 \times 2]$$

$$= 1(10 + 4) + 2(15 - 1) + 3(-12 - 2)$$

$$= 1(14) + 2(14) + 3(-14) = 14 + 28 - 42 = 0$$

$$\because |A| = 0$$

$\therefore$ The system has non trivial solution.

From (2), we have

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply R_{13}

$$\begin{bmatrix} 1 & -4 & 5 \\ 3 & 2 & 1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & -4 & 5 \\ 0 & 14 & -14 \\ 0 & 7 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_2\left(\frac{1}{14}\right)$

$$\begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & -1 \\ 0 & 7 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_3 \to R_3 - 7R_2$

$$\begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x - 4y + 5z = 0$ - - - (3)

$y - z = 0$ - - - (4)

We have three variables and two equation. So, one variable will be independent.

Put $z = t$ in (4), then we have

$$y - t = 0$$

$$\Rightarrow y = t$$

Putting the values of y and z in (3), we have

$$\Rightarrow x - 4t + 5t = 0$$

$$\Rightarrow x + t = 0$$

$$\Rightarrow x = -t$$

Hence the nontrivial solutions is

$$\boxed{x = -t, y = t, z = t}$$

- **Example 3.** *Show that the system of equations:* $ax + by + cz = 0;\ bx + cy + az = 0;\ cx + ay + bz = 0$ has non-trivial solution only if $a + b + c = 0$ or $a = b = c$.
- **Solution:** We have to show that the following system of equations

$ax + by + cz = 0;\ bx + cy + az = 0;\ cx + ay + bz = 0$ - - - - - (1)

has a nontrivial solutions only if $a + b + c = 0$ or $a = b = c$.

The system of simultaneous equations (1) can be written as

$$AX = O$$

$$\Rightarrow \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{- - - - - - - - - - - -} \quad (2)$$

$\because$ The system has nontrivial solution, then

$$\therefore |A| = 0$$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

Apply $R_1 \to R_1 + R_2 + R_3$, we have

$$\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

Apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have

$$\Rightarrow (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c)[1 \times \{(c-b)(b-c)-(a-c)(a-b)\}+0+0] = 0$$

$$\Rightarrow (a+b+c)[1 \times \{bc-c^2-b^2+bc-(a^2-ab-ac+bc)\}] = 0$$

$$\Rightarrow (a+b+c)[bc-c^2-b^2+bc-a^2+ab+ac-bc] = 0$$

$$\Rightarrow (a+b+c)[-(a^2+b^2+c^2-ab-ac-bc)] = 0$$

$\Rightarrow a+b+c=0$ or $a^2+b^2+c^2-ab-ac-bc=0$

$\Rightarrow a + b + c = 0 \quad$ or $\; a^2 + b^2 + c^2 - ab - ac - bc = 0$

$\Rightarrow a + b + c = 0 \quad$ or $\; 2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc = 0$

$\Rightarrow a + b + c = 0$ or $\; (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ac + a^2) = 0$

$\Rightarrow a + b + c = 0 \quad$ or $\; (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

$\Rightarrow a + b + c = 0 \quad$ or $\; a - b = 0$ and $b - c = 0$ and $c - a = 0$

$\Rightarrow a + b + c = 0 \quad$ or $\; a = b = c$

$\therefore$ The given system has non trivial solution if

$$\boxed{\boldsymbol{a + b + c = 0 \text{ or } a = b = c}}$$

(Hence Proved)

Home Assignment

Check whether the following system of linear equations have non trivial solutions or not. If yes, then solve them.

1. $x + 2y + 3z = 0;\ \ 2x + 3y + z = 0;\ \ 4x + 5y + 4z = 0;\ \ x + y - 2z = 0.$

   $[Ans\colon x = 0,\quad y = 0,\quad z = 0]$

2. $5x + 2y - 3z = 0;\ \ 3x + y + z = 0;\ \ 2x + y + 6z = 0.$ $[Ans\colon x = 0,\ y = 0,\ z = 0]$
3. $x + 3y + z = 0;\ \ 2x - 2y - 6z = 0;\ \ 3x + y - 5z = 0.$ $[Ans\colon x = 2t,\ y = -t,\ z = t]$
4. $4x_1 - x_2 + 2x_3 + x_4 = 0; 2x_1 + 3x_2 - x_3 - 2x_4 = 0; 7x_2 - 4x_3 - 5x_4 = 0; 2x_1 - 11x_2 + 7x_3 + 8x_4 = 0.$ $\left[Ans\colon x = \frac{-a-5b}{14}, y = \frac{5a+4b}{7},\ z = b,\ \ t = a\right]$
5. For different values of k, discuss the nature of solution of the following equations:

$x + 2y - z = 0;\ \ 3x + (k + 7)y - 3z = 0;\ \ 2x + 4y + (k - 3)z = 0.$

$[Ans\colon For\ k = 1, x = t,\quad y = 0,\quad z = t, for\ k = -1, x = -2t,\quad y = t,\quad z = 0]$

6. Show that the system of equations $x_1 + 2x_2 + 3x_3 = \lambda x_1;\ 3x_1 + x_2 + 2x_3 = \lambda x_2; 2x_1 + 3x_2 + x_3 = \lambda x_3$ can possess a non-trivial solution only $\lambda = 6$. Obtain the general solution for real values of λ . $[Ans\colon For\ \lambda = 6, x = t,\ y = t,\ z = t]$

Questions?

Thanks

Cayley Hamilton Theorem

Caley Hamilton Theorem states that

“Every square matrix satisfies its own characteristic equation”.

Q.1. *Verify Cayley Hamilton Theorem for the matrix* $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ *and find its inverse. Also express* $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ *as linear polynomial in A.*

Solution:

Cayley Hamilton Theorem: *Every square matrix satisfies its own characteristics equation.*

We have to verify the Cayley Hamilton Theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

Characteristics equation of A is given by

$|A - \lambda I| = 0 \; or \lambda^2 - S_1\lambda + |A| = 0$ - - - - - - - - - (1)

where $S_1 =$ Sum of diagonal elements $= 1 + 3 = 4$

$$|A| = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 1 \times 3 - 4 \times 2 = 3 - 8 = -5$$

So (1) becomes

$\lambda^2 - 4\lambda - 5 = 0$ - - - - - - - - - (2)

This is characteristic equation of the given matrix A.

Hence by the Cayley Hamilton Theorem

$A^2 - 4A - 5I = 0$ - - - - - - - - - (3)

Now, we have to verify the Cayley Hamilton Theorem for the given matrix A.

Let us compute

$$A^2 = A \times A$$

$$= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 1 + 4 \times 2 & 1 \times 4 + 4 \times 3 \\ 2 \times 1 + 3 \times 2 & 2 \times 4 + 3 \times 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

Now consider

$$L.H.S. = A^2 - 4A - 5I$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4 \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 15 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 - 4 - 5 & 16 - 16 - 0 \\ 8 - 8 - 0 & 17 - 12 - 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 = R.\,H.\,S.$$

$$\therefore \boxed{A^2 - 4A - 5I = 0}$$

Hence Cayley Hamilton Theorem is verified.

We have to calculate A^{-1}.

Since $|A| = -5 \neq 0$, so A^{-1}exists. Now operating A^{-1} on both the sides of (3), we have

$$A^{-1}\,(A^2 - 4A - 5I) = A^{-1}(0)$$

$$\Rightarrow (A^{-1}A)A - 4(A^{-1}A) - 5A^{-1} = 0[\because A^{-1}A = I]$$

$$\Rightarrow IA - 4I - 5A^{-1} = 0$$

$$\Rightarrow -5A^{-1} = -A + 4I \left[\because A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\right]$$

$$\Rightarrow A^{-1} = \frac{1}{5}[A - 4I]$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} 1-4 & 4-0 \\ 2-0 & 3-4 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

$$\therefore \boxed{A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}}$$

Now, we have to express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as linear polynomial in A.

Consider

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I \qquad [\because A^2 - 4A - 5I = 0]$$

$$= A^5 - 4A^4 - (5A^3 + 2A^3) + (8A^2 + 3A^2) + (10A - 11A) - 10I$$

$$= A^5 - 4A^4 - 5A^3 - 2A^3 + 8A^2 + 3A^2 + 10A - 11A - 10I$$

$$= A^5 - 4A^4 - 5A^3 - 2A^3 + 8A^2 + 10A + 3A^2 - 12A - 15I + A + 5I$$

$$= A^3(A^2 - 4A - 5I) - 2A(A^2 - 4A - 5I) + 3(A^2 - 4A - 5I) + A + 5I$$

$$= A^3 \times 0 - 2A \times 0 + 3 \times 0 + A + 5I$$

$$= A + 5I$$

$$\therefore \qquad \boxed{A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5I}$$

This is linear polynomial in A

- **Q.2.** *Verify Cayley Hamilton Theorem for the matrix* $A=\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}$. *Apply it to find* A^{-1}.

Solution:

- **Cayley Hamilton Theorem:** *Every square matrix satisfies its own characteristics equation.*

- We have to verify the Cayley Hamilton Theorem for the matrix $A=\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}$

Characteristics equation of A is given by

$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$ - - - - - - - - - - (1)

where $S_1 =$ Sum of diagonal elements $= 0+2+1=3$

$S_2 =$ Sum of minors of diagonal elements

$$= \begin{vmatrix}2&3\\1&1\end{vmatrix} + \begin{vmatrix}0&2\\3&1\end{vmatrix} + \begin{vmatrix}0&1\\1&2\end{vmatrix}$$

$$= 2\times 1 - 3\times 1 + 0\times 1 - 3\times 2 + 0\times 2 - 1\times 1$$

$$= 2-3+0-6+0-1 = -8$$

And $|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$

$$= 0 \times [2 \times 1 - 1 \times 3] - 1 \times [1 \times 1 - 3 \times 3] + 2 \times [1 \times 1 - 2 \times 3]$$

$$= 0 - 1 \times (-8) + 2 \times (-5) = 8 - 10 = -2$$

Putting the values of S_1, S_2 and $|A|$ in (1), we have

$$\lambda^3 - 3\lambda^2 - 8\lambda - (-2) = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - 8\lambda + 2 = 0 \quad \text{---------} \quad (2)$$

According to Cayley- Hamilton Theorem

$$A^3 - 3A^2 - 8A + 2I = 0 \quad \text{---------} \quad (3)$$

Consider

$$A^2 = A \times A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0+1+6 & 0+2+2 & 0+3+2 \\ 0+2+9 & 1+4+3 & 2+6+3 \\ 0+1+3 & 3+2+1 & 6+3+1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix}$$

Similarly

$$A^3 = A^2 \times A = \begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0+4+15 & 7+8+5 & 14+12+5 \\ 0+8+33 & 11+16+11 & 22+24+11 \\ 0+6+30 & 4+12+10 & 8+18+10 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 20 & 31 \\ 41 & 38 & 57 \\ 36 & 26 & 36 \end{bmatrix}$$

Now taking L.H.S. $= A^3 - 3A^2 - 8A + 2I$

$$= \begin{bmatrix} 19 & 20 & 31 \\ 41 & 38 & 57 \\ 36 & 26 & 36 \end{bmatrix} - 3\begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix} - 8\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 20 & 31 \\ 41 & 38 & 57 \\ 36 & 26 & 36 \end{bmatrix} - \begin{bmatrix} 21 & 12 & 15 \\ 33 & 24 & 33 \\ 12 & 18 & 30 \end{bmatrix} - \begin{bmatrix} 0 & 8 & 16 \\ 8 & 16 & 24 \\ 24 & 8 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 19-21-0+2 & 20-12-8+0 & 31-15-16+0 \\ 41-33-8+0 & 38-24-16+2 & 57-33-24+0 \\ 36-12-24+0 & 26-18-8+0 & 36-30-8+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R.H.S.$$

$$\Rightarrow \boxed{A^3 - 3A^2 - 8A + 2I\ = 0}$$

Hence Cayley- Hamilton Theorem is verified.

Now, we have to find A^{-1} using Cayley Hamilton Theorem.

Since $|A| = -2 \neq 0 \Rightarrow A^{-1}$ exists. Operating (3) by A^{-1}, we have

$$A^{-1}\,[A^3 - 3A^2 - 8A + 2I]\ = A^{-1}0$$

$$\Rightarrow A^{-1}A^3 - 3A^{-1}A^2 - 8A^{-1}A + 2A^{-1}I = 0$$

$$\Rightarrow (A^{-1}A)A^2 - 3(A^{-1}A)A - 8(A^{-1}A) + 2A^{-1} = 0$$

$$\Rightarrow \mathrm{I}A^2 - 3IA - 8I + 2A^{-1} = 0$$

$$\Rightarrow A^2 - 3A - 8 + 2A^{-1} = 0$$

$$\Rightarrow 2A^{-1} = -A^2 + 3A + 8I$$

$$\Rightarrow A^{-1} = \frac{1}{2}\{-A^2 + 3A + 8I\} = \frac{1}{2}\left\{-\begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} -7+0+8 & -4+3+0 & -5+6+0 \\ -11+3+0 & -8+6+8 & -11+9+0 \\ -4+9+0 & -6+3+0 & -10+3+8 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

Home Assignment

Q.1. *Verify Cayley Hamilton Theorem for the matrix* $A=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$ *and use it find* A^4 *and* A^{-1}.

Answer: Characteristic Equation is $\lambda^3-5\lambda^2+9\lambda-1=0$,

$A^4=\begin{bmatrix}-55 & 104 & 24\\ -20 & -15 & 32\\ 32 & -40 & -23\end{bmatrix}$ $A^{-1}=A^2-5A+9I=\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}$.

Q.2. *Verify Cayley Hamilton Theorem for the matrix* $A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$ *and use it find the matrix* $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$.

Answer: Characteristic Equation is $\lambda^3-5\lambda^2+7\lambda-3=0$, A^2+A+I

Q.3. *Verify Cayley Hamilton Theorem for the matrix* $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ *and use it find* A^{-1}.

Answer: Characteristic Equation is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0,\quad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

Q.4. *Verify Cayley Hamilton Theorem for the matrix* $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ *and use it find* A^{-1}.

Answer: Characteristic Equation is $\lambda^3 - 3\lambda^2 - 8\lambda + 2 = 0$, $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$.

Q.5. *Verify Cayley Hamilton Theorem for the matrix* $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$. *Find* A^{-1}, A^{-2}, A^{-3}.

Answer: Characteristic Equation is $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$, $A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}$,

$$A^{-2} = \frac{1}{16}\begin{bmatrix} 1 & -18 & -18 \\ -5 & 10 & -6 \\ 5 & 6 & 22 \end{bmatrix}, A^{-3} = \frac{1}{64}\begin{bmatrix} 1 & 78 & 78 \\ -21 & 90 & 26 \\ 21 & -154 & -90 \end{bmatrix}.$$

Eigen Values and Eigen Vectors

Let us consider the linear transformation

$Y = AX$ - - - - - - - - *(1)*

Now replace the matrix A by any number say λ, then (1) becomes

$Y = \lambda X$ - - - - - - - - *(2)*

From (1) and (2), we have

$$AX = \lambda X$$

$$\Rightarrow AX - \lambda X = 0$$

$$\Rightarrow AX - \lambda IX = 0 [\because IX = X]$$

$\Rightarrow (A - \lambda I)X = 0$- - - - - - - - *(3)*

This is a matrix equation on X.

1. Characteristic Matrix: The matrix $(A - \lambda I)$ is called Eigen Matrix or Characteristic Matrix or LatentMatrix.
2. Characteristic Determinant: The determinant $|A - \lambda I|$ is called Eigen Determinant or Characteristic Determinant or LatentDeterminant.

4. Characteristic Polynomial: The polynomial $|A-\lambda I|$ in λ is called Eigen Polynomial or Characteristic Polynomial or LatentPolynomial.

5. Characteristic Equations: The equation $|A-\lambda I|=0$ in λ is called Eigen equation or Characteristic equation or Latentequation. The degree of the characteristic equation of a matrix is equal to the order of that matrix.

6. Characteristic Value or Root: The values of λ which is obtained from the polynomial equation $|A-\lambda I|=0$ are called Eigen values or Characteristic values or latent values or Proper values.

7. Spectrum of a Matrix: The set of all characteristic roots of a matrix is called the spectrum of the matrix.

8. Eigen Vector: The values of X are called Eigen vectors or characteristic vectors or latent vectors corresponding to eigen value λ.

   Any nonzero vector X is said to be a characteristic vector of a matrix A, if there exits a number λ such that $AX=\lambda X$.

Properties of Eigen Values

1. Trace of a Matrix A: The sum of the entries on the main diagonal of a matrix is called the Trace of A.

$$\text{Trace of A} = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$$

2. The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal.

$$\text{Trace of A} = a_{11} + a_{22} + a_{33} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n$$

3. The eigen values of an upper or lower or diagonal matrix are the elements on its main diagonal.
4. The product of the eigen values of a matrix is equal to the determinant of the matrix.

$$\lambda_1 \times \lambda_2 \times \lambda_3 \times \cdots \times \lambda_n = |A|$$

5. If $\lambda_1,\ \lambda_2,\ \lambda_3, \dots, \lambda_n$ are eigen values of A, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \dots, \frac{1}{\lambda_n}$ are eigen values of A^{-1}.
6. If $\lambda_1,\ \lambda_2,\ \lambda_3, \dots, \lambda_n$ are eigen values of A, then k$\lambda_1,\ k\lambda_2,\ k\lambda_3, \dots, k\lambda_n$are eigen values of the matrix kA.
7. The eigen values of A and its transpose A^T are same.

Properties of Eigen Values

8. If $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are eigen values of A, then $\lambda_1^m, \lambda_2^m, \lambda_3^m, \dots, \lambda_n^m$ are eigen values of A^m.
9. If $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are eigen values of A, then $\lambda_1 - k, \lambda_2 - k, \lambda_3 - k, \dots, \lambda_n - k$ are eigen values of $A - kI$.
10. The eigen values of a symmetric matrix are real.

Short Cut Method to Find Characteristic Equation

1. Characteristic Equation of 2×2 is given by

   $\boldsymbol{\lambda^2 - S_1\lambda - |A| = 0}$, +

   Where $S_1 =$ Sum of diagonal elements, $|A| =$ Determinant of A.

2. Characteristic Equation of 3×3 is given by

   $\boldsymbol{\lambda^3 - S_1\lambda^2 - S_1\lambda + |A| = 0}$,

   Where $S_1 =$ Sum of diagonal elements,

   $S_2 =$ Sum of minors of diagonal elements

   $|A| =$ Determinant of A.

Properties of Eigen Vectors

1. Eigen vector of a matrix corresponding to an eigen value λ is not unique.
2. Let $\lambda_1,\ \lambda_2,\ \lambda_3, \dots, \lambda_n$ be distinct eigen values of A, then corresponding eigen vectors X_1, $X_2,\ X_3, \dots, X_n$ are linearly independent.
3. It may or may not be possible to get linearly independent eigen vectors corresponding to the repeated roots.
4. An $n \times n$ matrix has at most n linearly independent eigen vectors.
5. Eigen vector of a square matrix can not correspond to two distinct eigen values.
6. Orthogonal Eigen Vectors: Two eigen vectors X_1 and X_2 are said to be orthogonal if $X_1 X_2^T = 0$.
7. Eigen vectors of a symmetric matrix corresponding to different eigen values are orthogonal.
8. Corresponding to zero eigen value, eigen vector will not be zero.

- **Example 1.** *Find the Eigen values and Eigen vectors of the matrix* $A = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}$.
- **Solution:** We have to find the Eigen values and Eigen vectors of the matrix.

$$A = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}$$

Characteristics equation of A is given by

$\lambda^2 - S_1\lambda + |A| = 0$ - - - - - - - - - (1)

where $S_1 =$ Sum of diagonal elements $= 14 + (-1) = 13$

And $|A| = \begin{vmatrix} 14 & -10 \\ 5 & -1 \end{vmatrix} = 14 \times (-1) - 5 \times (-10) = -14 + 50 = 36$

Hence (1) becomes

$\lambda^2 - 13\lambda + 36 = 0$ - - - - - - - - - (2)

$$\Rightarrow \lambda^2 - 9\lambda - 4\lambda + 36 = 0$$
$$\Rightarrow \lambda(\lambda - 9) - 4(\lambda - 9) = 0$$
$$\Rightarrow (\lambda - 9)(\lambda - 4) = 0$$

$\Rightarrow \lambda - 9 = 0, \lambda - 4 = 0$

$$\Rightarrow \boxed{\lambda = 4, \quad 9}$$

- Now we have to find the Eigen vectors corresponding to eigen values $\lambda = 4,\ 9$.
- (i) Eigen vector corresponding to eigen value $\lambda = 4$:

Let $X_1 = [x_1 \quad x_2]^T$ be Eigen vector corresponding to the eigen value $\lambda = 4$. Then the characteristics equation of A is given by

$[A - \lambda I]X = 0$ - - - - - - - - - (3)

$$\Rightarrow \left\{\begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left\{\begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 14 - \lambda & -10 \\ 5 & -1 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ - - - - - - - - - (4)

Putting $\lambda = 4$ in (4), we have

$$\begin{bmatrix} 14 - 4 & -10 \\ 5 & -1 - 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 10 & -10 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 10x_1 - 10x_2 = 0,\ 5x_1 - 5x_2 = 0$

$\Rightarrow 10x_1 - 10x_2 = 0, 5x_1 - 5x_2 = 0$

$$\Rightarrow x_1 - x_2 = 0$$

Putting $x_2 = t$, then $x_1 = t$.

∴Eigen vector corresponding to $\lambda = 4$ is $X_1 = [t \quad t]^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(i) Eigen vector corresponding to eigen value $\lambda = 9$: Let $X_2 = [x_1 \quad x_2]^T$ be Eigen vector corresponding to the eigen value $\lambda = 9$. Putting $\lambda = 9$ in (4), we have

$$\begin{bmatrix} 14 - 9 & -10 \\ 5 & -1 - 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & -10 \\ 5 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 5x_1 - 10x_2 = 0, 5x_1 - 10x_2 = 0$

$$\Rightarrow x_1 - 2x_2 = 0$$

Putting $x_2 = t$, then $x_1 = 2t$.

∴Eigen vector corresponding to $\lambda = 9$ is $X_1 = [2t \quad t]^T = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Hence the Eigen values and Eigen vectors of the given matrix are given by

$$\boxed{\lambda_1 = 4, \quad X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_2 = 9, \quad X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

- **Example 2.** *Find the Eigen values and Eigen vectors of the matrix* $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$.
- **Solution:** We have to find the Eigen values and Eigen vectors of the matrix.

$$A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

Characteristics equation of A is given by

$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$ - - - - - - - - - (1)

where $S_1 =$ Sum of diagonal elements $= 8 + (-3) + 1 = 6$

$S_2 =$ Sum of minors of diagonal elements

$$= \begin{vmatrix} -3 & -2 \\ -4 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -8 \\ 4 & -3 \end{vmatrix}$$

$$= (-3) \times 1 - (-4) \times (-2) + 8 \times 1 - 3 \times (-2)$$

$$+8 \times (-3) - (4) \times (-8)$$

$$= -3 - 8 + 8 + 6 - 24 + 32 = 11$$

And $|A| = \begin{vmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{vmatrix} = 8 \times [(-3) \times 1 - (-4) \times (-2)]$

$$-(-8) \times [4 \times 1 - 3 \times (-2)] - 2 \times [4 \times (-4) - 3 \times (-3)]$$

And $|A| = 8\times[(-3)\times 1-(-4)\times(-2)]-(-8)\times[4\times 1-3\times(-2)]$
$$-2\times[4\times(-4)-3\times(-3)]$$
$$= 8\times[-3-8]-(-8)\times[4+6]-2\times[-16+9]$$
$$= 8\times(-11)+8\times 10-2\times(-7)$$
$$= -88+80+14 = 6$$

Putting above values in (1), we have

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$
$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 [\because S_1 = 6, \quad S_2 = 11, \quad |A| = 6]$$
$$\Rightarrow \lambda^3 - \lambda^2 - 5\lambda^2 + 5\lambda + 6\lambda - 6 = 0$$
$$\Rightarrow (\lambda^3 - \lambda^2) - (5\lambda^2 + 5\lambda) + (6\lambda - 6) = 0$$
$$\Rightarrow \lambda^2(\lambda - 1) - 5\lambda(\lambda - 1) + 6(\lambda - 1) = 0$$
$$\Rightarrow (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$$
$$\Rightarrow (\lambda - 1)(\lambda^2 - 3\lambda - 2\lambda + 6) = 0$$
$$\Rightarrow (\lambda - 1)[(\lambda^2 - 3\lambda) - (2\lambda - 6)] = 0$$
$$\Rightarrow (\lambda - 1)[\lambda(\lambda - 3) - 2(\lambda - 3)] = 0$$
$$\Rightarrow (\lambda - 1)(\lambda - 3)(\lambda - 2) = 0$$
$$\Rightarrow \lambda = 1, \quad 2, \quad 3$$

These are the eigen values of the given matrix A.

- Now we have to find the Eigen vectors corresponding to eigen values $\lambda = 1,\ 2,\ 3.$
- (i) Eigen vector corresponding to eigen value $\lambda = 1$:

Let $X_1 = [x_1 \quad x_2 \quad x_3]^T$ be Eigen vector corresponding to the eigen value $\lambda = 1$. Then the characteristics equation of A is given by

$[A - \lambda I]X = 0$ - - - - - - - - - (2)

$$\Rightarrow \left\{\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left\{\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{- - - - - - - - -} \quad (3)$$

Putting $\lambda = 1$ in (3), we have

$$\begin{bmatrix} 8-1 & -8 & -2 \\ 4 & -3-1 & -2 \\ 3 & -4 & 1-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It can written as

$7x_1 - 8x_2 - 2x_3 = 0$ - - - - - - - - - (4)

$4x_1 - 4x_2 - 2x_3 = 0$ - - - - - - - - - (5)

$3x_1 - 4x_2 + 0.x_3 = 0$ - - - - - - - - - (6)

Solving (4) and (5) using Cremer's Rule, we have

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -4 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 7 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 7 & -8 \\ 4 & -4 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{(-8)\times(-2)-(-4)\times(-2)} = \frac{-x_2}{(7)\times(-2)-(4)\times(-2)} = \frac{x_3}{(7)\times(-4)-(4)\times(-8)}$$

$$\Rightarrow \frac{x_1}{16-8} = \frac{-x_2}{-14+8} = \frac{x_3}{-28+32}$$

$$\Rightarrow \frac{x_1}{16-8} = \frac{-x_2}{-14+8} = \frac{x_3}{-28+32}$$

$$\Rightarrow \frac{x_1}{8} = \frac{-x_2}{-6} = \frac{x_3}{4}$$

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{3} = \frac{x_3}{2} = k(says)$$

$$\Rightarrow x_1 = 4k,\ x_2 = 3k, \qquad x_3 = 2k$$

$\therefore$ The Eigen vector corresponding to the eigen value $\lambda = 1$ is given by

$$\boxed{X_1 = [4k \quad 3k \quad 2k]^T \Rightarrow X_1 = [4 \quad 3 \quad 2]^T}$$

- (ii) Eigen vector corresponding to eigen value $\lambda = 2$:

Let $X_2 = [y_1 \quad y_2 \quad y_3]^T$ be Eigen vector corresponding to the eigen value $\lambda = 2$.

Putting $\lambda = 2$ in (3), we have

$$\begin{bmatrix} 8-2 & -8 & -2 \\ 4 & -3-2 & -2 \\ 3 & -4 & 1-2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8-2 & -8 & -2 \\ 4 & -3-2 & -2 \\ 3 & -4 & 1-2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It can written as

$6y_1 - 8y_2 - 2y_3 = 0$ - - - - - - - - - - (7)

$4y_1 - 5y_2 - 2y_3 = 0$ - - - - - - - - - - (8)

$3y_1 - 4y_2 - y_3 = 0$ - - - - - - - - - - (9)

Solving (7) and (8) using Cremer's Rule, we have

$$\frac{y_1}{\begin{vmatrix} -8 & -2 \\ -5 & -2 \end{vmatrix}} = \frac{-y_2}{\begin{vmatrix} 6 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{y_3}{\begin{vmatrix} 6 & -8 \\ 4 & -5 \end{vmatrix}}$$

$$\Rightarrow \frac{y_1}{(-8)\times(-2)-(-5)\times(-2)} = \frac{-y_2}{(6)\times(-2)-(4)\times(-2)} = \frac{y_3}{(6)\times(-5)-(4)\times(-8)}$$

$$\Rightarrow \frac{y_1}{16-10} = \frac{-y_2}{-12+8} = \frac{y_3}{-30+32}$$

$$\Rightarrow \frac{y_1}{16-10} = \frac{-y_2}{-12+8} = \frac{y_3}{-30+32}$$

$$\Rightarrow \frac{y_1}{6} = \frac{-y_2}{-4} = \frac{y_3}{2}$$

$$\Rightarrow \frac{y_1}{3} = \frac{y_2}{2} = \frac{y_3}{1} = k(says)$$

$$\Rightarrow y_1 = 3k,\ y_2 = 2k, \qquad y_3 = k$$

$\therefore$ The Eigen vector corresponding to the eigen value $\lambda = 2$ is given by

$$\boxed{X_2 = [3k \quad 2k \quad k]^T \Rightarrow X_1 = [3 \quad 2 \quad 1]^T}$$

- (iii) Eigen vector corresponding to eigen value $\lambda = 3$:

Let $X_3 = [z_1 \quad z_2 \quad z_3]^T$ be Eigen vector corresponding to the eigen value $\lambda = 3$.

Putting $\lambda = 3$ in (3), we have

$$\begin{bmatrix} 8-3 & -8 & -2 \\ 4 & -3-3 & -2 \\ 3 & -4 & 1-3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8-3 & -8 & -2 \\ 4 & -3-3 & -2 \\ 3 & -4 & 1-3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It can written as

$5z_1 - 8z_2 - 2z_3 = 0$ - - - - - - - - - (10)

$4z_1 - 6z_2 - 2z_3 = 0$ - - - - - - - - - (11)

$3z_1 - 4z_2 - 2z_3 = 0$ - - - - - - - - - (12)

Solving (10) and (11) using Cremer's Rule, we have

$$\frac{z_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = \frac{-z_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{z_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$$\Rightarrow \frac{z_1}{(-8)\times(-2)-(-6)\times(-2)} = \frac{-z_2}{(5)\times(-2)-(4)\times(-2)} = \frac{z_3}{(5)\times(-6)-(4)\times(-8)}$$

$$\Rightarrow \frac{z_1}{16-12} = \frac{-z_2}{-10+8} = \frac{z_3}{-30+32}$$

$$\Rightarrow \frac{z_1}{16-12} = \frac{-z_2}{-10+8} = \frac{z_3}{-30+32}$$

$$\Rightarrow \frac{z_1}{4} = \frac{-z_2}{-2} = \frac{z_3}{2}$$

$$\Rightarrow \frac{z_1}{2} = \frac{z_2}{1} = \frac{z_3}{1} = k(says)$$

$$\Rightarrow z_1 = 2k,\ z_2 = k, \qquad y_3 = k$$

$\therefore$ The Eigen vector corresponding to the eigen value $\lambda = 2$ is given by

$$\boxed{X_2 = [2k \quad k \quad k]^T \Rightarrow X_1 = [2 \quad 1 \quad 1]^T}$$

Hence the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ *are given by*

$$\boxed{\lambda_1 = 1, \qquad X_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}, \lambda_2 = 2, \qquad X_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ and } \lambda_3 = 3, \qquad X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}$$

Home Assignment

Find the Eigen values and Eigen vectors of the following matrices.

1. $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \left\{ \because \lambda_1 = 1, X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \lambda_2 = 2, X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \lambda_3 = 3, X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \right\}$

2. $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} \left\{ \because \lambda_1 = -1, X_1 = \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix}, \lambda_2 = 1, X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \lambda_3 = 4, X_3 = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \right\}$

3. $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \left\{ \because \lambda_1 = -2, X_1 = \begin{bmatrix} 11 \\ 1 \\ -14 \end{bmatrix}, \lambda_2 = 1, X_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \lambda_3 = 3, X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

4. $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} \left\{ \because \lambda_1 = 1, X_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \lambda_2 = 2, X_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \lambda_3 = 5, X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

5. $\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \left\{ \because \lambda_1 = -1, X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \lambda_2 = -2, X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \lambda_3 = -3, X_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \right\}$

6. $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\left\{\because\ \lambda_1 = -1, X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \lambda_2 = 1, X_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \lambda_3 = 2, X_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right\}$

7. $\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}\left\{\because\ \lambda_1 = 0, X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \lambda_2 = 1, X_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \lambda_3 = 2, X_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right\}$

8. $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\left\{\because\ \lambda_1 = 1, X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \lambda_2 = 2, X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \lambda_3 = 3, X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

9. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}\left\{\because\ \lambda_1 = 1, X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \lambda_2 = 2, X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \lambda_3 = 4, X_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$

10. $\begin{bmatrix} 0 & 2 & 0 \\ 3 & -2 & 3 \\ 0 & 3 & 0 \end{bmatrix}\left\{\because\ \lambda_1 = 0, X_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \lambda_2 = -5, X_2 = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}, \lambda_3 = 3, X_3 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}\right\}$

Questions?

Thanks

Linear Algebra and Calculus

Unit-I: Matrices

- Introduction of Matrix
- Rank of a Matrix
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Orthogonal Transformation

The linear transformation $Y = AX$ i.e.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

is said to be orthogonal, if it transforms $x_1^2 + x_2^2 + \cdots + x_n^2$ into $y_1^2 + y_2^2 + \cdots + y_n^2$.

Orthogonal Matrix

A matrix 'A' is said to be orthogonal, if $AA^T = A^T A = I$.

Properties of Orthogonal Transformation

1. If A is orthogonal then $A^{-1} = A^T$.
2. If A is orthogonal then $|A| = \pm 1$.
3. If A is orthogonal then A^{-1} and A^T is also orthogonal.

- **Example 1.** *Show that* $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ *is orthogonal.*

- **Solution:** We know that a matrix A is orthogonal, then

$AA^T = A^T A = I$ - - - - - - - - (1)

Now consider

$$AA^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3 \times 3}\begin{bmatrix} 1+4+4 & 2+2-4 & 2-4+2 \\ 2+2-4 & 4+1+4 & 4-2-2 \\ 2-4+2 & 4-2-2 & 4+4+1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Similarly, we can prove that $A^T A = I$.

$\therefore$ *The given A matrix is orthogonal.*

- **Example 2.** *Show that the transformation* $y_1 = \frac{2}{3}x_1 + \frac{1}{3}x_2 + \frac{2}{3}x_3,\ y_2 = -\frac{2}{3}x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3$, $y_3 = \frac{1}{3}x_1 + \frac{2}{3}x_2 - \frac{2}{3}x_3$ *is orthogonal.*
- **Solution:** Given transformation will be orthogonal if the coefficient matrix is orthogonal. i.e.

$$AA^T = A^T A = I \qquad \text{- - - - - - - -} \qquad (1)$$

Here $A = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{bmatrix}$

Now consider $AA^T = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$

$$= \frac{1}{3 \times 3}\begin{bmatrix} 4+1+4 & -4+2+2 & 2+2-4 \\ -4+2+2 & 4+4+1 & -2+4-2 \\ 2=2-4 & -2+4-2 & 1+4+4 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Similarly, we can prove that $A^T A = I$. *So, the given transformation is orthogonal.*

- **Example 3.** *Apply the orthogonal property of the orthogonal matrix* $\frac{1}{13}\begin{bmatrix} -12 & -5 \\ a & -12 \end{bmatrix}$ *to find the value of* a.

- **Solution:** We have to find the value of a, if the following matrix is orthogonal.
- $A = \frac{1}{13}\begin{bmatrix} -12 & -5 \\ a & -12 \end{bmatrix}$

Orthogonal Matrix: A matrix 'A' is called orthogonal, iff $AA^T = A^T A = I$.

Since the given matrix is orthogonal.

$$\therefore AA^T = I$$

$$\Rightarrow \frac{1}{13}\begin{bmatrix} -12 & -5 \\ a & -12 \end{bmatrix} \times \frac{1}{13}\begin{bmatrix} -12 & a \\ -5 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{13 \times 13}\begin{bmatrix} (-12)\times(-12)+(-5)\times(-5) & (-12)\times a+(-5)\times(-12) \\ a\times(-12)+(-12)\times(-5) & a\times a+(-12)\times(-12) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{169}\begin{bmatrix} 144+25 & -12a+60 \\ -12a+60 & a^2+144 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 169 & -12a+60 \\ -12a+60 & a^2+144 \end{bmatrix} = 169 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 169 & -12a+60 \\ -12a+60 & a^2+144 \end{bmatrix} = \begin{bmatrix} 169 & 0 \\ 0 & 169 \end{bmatrix}$$

Equality of two matrices, we have

$-12a + 60 = 0 \quad$ *and* $\quad a^2 + 144 = 169$

$\Rightarrow -12a = -60 \quad$ *and* $\quad a^2 = 169 - 144$

$\Rightarrow a = \frac{60}{12} \quad$ *and* $\quad a^2 = 25$

$\Rightarrow a = \frac{12\times 5}{12} \quad$ *and* $\quad a = \sqrt{25}$

$\Rightarrow a = 5 \quad$ *and* $\quad a = \pm 5$

$\therefore \quad$ *The value of* $\boxed{a = \pm 5}$

- **Example 4.** *Determine the values of a, b and c when* $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ *is orthogonal.*

- **Solution:** We know that a matrix A is orthogonal, then

$$AA^T = A^TA = I \quad \text{- - - - - - - -} \quad (1)$$

Now consider $AA^T = I$

$$\Rightarrow \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \times \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0+4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equality of two matrices, we have

$$4b^2+c^2 = 1, \quad 2b^2-c^2 = 0, a^2-b^2-c^2 = 0, \quad a^2 \overset{+}{-} b^2 \overset{+}{-} c^2 = 1$$

From $2b^2 - c^2 = 0$, we have

$$c^2 = 2b^2 \text{ - - - - - - - -} \quad (2)$$

Putting the value of $c^2 = 2b^2$ in $4b^2 + c^2 = 1$, we have

$$4b^2 + 2b^2 = 1$$

$$\Rightarrow 6b^2 = 1 \Rightarrow b^2 = 1/6$$

$$\Rightarrow \boxed{b = \pm 1/\sqrt{6}}$$

Putting the value of $b = \pm 1/\sqrt{6}$ in $c^2 = 2b^2$, we have

$$c^2 = 2 \times \frac{1}{6} = \frac{1}{3}$$

$$\Rightarrow \boxed{c = \pm 1/\sqrt{3}}$$

Putting the values of $b = \pm 1/\sqrt{6}$ and $c = \pm 1/\sqrt{3}$ in $a^2 - b^2 - c^2 = 0$, we have

$$a^2 - \frac{1}{6} - \frac{1}{3} = 0$$

$$\Rightarrow a^2 = \frac{1}{6} + \frac{1}{3} = \frac{1+2}{6} = \frac{3}{6} = \frac{1}{2}$$

Home Assignment

1. *Verify whether the following matrix is orthogonal or not, if so write A^{-1}.*

(i) $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $[Ans: Yes, A^T]$

(ii) $A = \frac{1}{13}\begin{bmatrix} -12 & -5 \\ 5 & -12 \end{bmatrix}$ $[Ans: Yes, A^T]$

(iii) $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $[Ans: Yes, A^T]$

(iv) $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ $[Ans: Yes, A^T]$

(v) $A = \begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$ $[Ans: Yes, A^T]$

(vi) $A = \begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{5}} & \frac{3}{\sqrt{70}} \\ \frac{2}{\sqrt{14}} & -\frac{1}{\sqrt{5}} & \frac{6}{\sqrt{70}} \\ \frac{3}{\sqrt{14}} & 0 & \frac{-5}{\sqrt{70}} \end{bmatrix}$ $[Ans: Yes, A^T]$

2. *Find the values of a, b, c if the following matrix is orthogonal.*

$$A = \begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix}$$

$$\left[Ans: a = \pm\frac{2}{3}, b = \pm\frac{2}{3}, c = \pm\frac{1}{3},\right]$$

Questions?

Thanks

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Linear Transformation

Let us consider

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n$$

$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n$$

$$y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n$$

- -

$$y_n = a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n$$

It can be written as

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow Y = AX$$

$\Rightarrow y_1,\ y_2,\ \ y_3,\ \ , \dots ,\ y_n$ can be expressed in terms of $x_1,\ x_2,\ \ x_3,\ \ , \dots ,\ x_n$.

$\Rightarrow y_1,\ y_2,\ \ y_3,\ \ , \dots ,\ y_n$ is linear transformation of $x_1,\ x_2,\ \ x_3,\ \ , \dots ,\ x_n$.

Properties of Linear Transformation

Suppose $Y = AX$ is a linear transformation from X to Y.

1. A is called Linear Operator.
2. $|A|$ is called Modulus of Transformation.
3. If $|A| \neq 0$, then the transformation is called Regular or Non-singular.
4. If $|A| = 0$, then the transformation is called Non-Regular or Singular.
5. If the transformation is Regular, then $X = A^{-1}Y$ is called Inverse Regular Transformation.
6. If the A is regular, then A^{-1} is also Regular.
7. If $Y = AX$ is linear transformation from X to Y and $Z = BY$ is a linear transformation from Y to Z, then $Z = BY = B(AX) = (BA)X$ is called Composite Transformation of from X to Z.
8. A regular linear transformation carries linearly independent vectors into linearly independent vectors.
9. A regular linear transformation carries linearly dependent vectors into linearly dependent vectors.

- **Example 1.** *Show that the transformation* $y_1 = 2x_1 - 2x_2 - x_3;\quad y_2 = -4x_1 + 5x_2 + 3x_3;$ $y_3 = x_1 - x_2 - x_3$ *is regular.*
- **Solution:** We have to show that the transformation $y_1 = 2x_1 - 2x_2 - x_3;\ y_2 = -4x_1 + 5x_2 + 3x_3;\ y_3 = x_1 - x_2 - x_3$ is regular. i.e.

$$|A| \neq 0$$

Now consider

$$|A| = \begin{vmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{vmatrix}$$

$$= 2 \times [5 \times (-1) - (-1) \times 3] - (-2) \times [(-4) \times (-1) - 1 \times 3]$$

$$+(-1) \times [(-4) \times (-1) - 1 \times 5]$$

$$= 2 \times [-5 - (-3)] + 2 \times [4 - 3] - 1 \times [4 - 5]$$

$$= 2 \times [-5 + 3] + 2 \times 1 - 1 \times (-1)$$

$$= 2 \times (-2) + 2 + 1 = -4 + 3 = -1$$

$\therefore$ *The given transformation is regular since* $\boxed{|A| = -1 \neq 0}$

- **Example 2.** *Given the transformation* $Y = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. *Find the coordinate* $(x_1, \quad x_2, \quad x_3)$ *in X corresponding to* $(1, \quad 2, \quad -1)$ *in Y.*

- **Solution:** Given transformation is $Y = AX$

$$Y = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Now consider

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix}$$

$$= 2 \times [1 \times (-2) - (0) \times 2] - 1 \times [1 \times (-2) - 0 \times 2] \quad 1$$

$$+ 1 \times [1 \times 1 - 1 \times 0] \quad 0 \quad 1$$

$$= 2 \times [-2 - 0] - 1 \times [-2 - 0] \; 2 \; 1 \times [1 - 0 \quad 1$$

$$= 2 \times (-2) - 1 \times \; 4 \text{-}2) + 1 \times (1)$$

$$= - \; 4 \; \text{-} \; 2 \quad + 3 = -1$$

$$\because |A| = -1 \neq 0$$

$\therefore$ A^{-1}exists which is given by

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

Here $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = -2 - 0 = -2$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = -(-2 - 2) = 4$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = -(-2 - 0) = 2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -4 - 1 = -5$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -(0 - 1) = 1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -(4 - 1) = -3$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1$$

$$\therefore A^{-1} = \frac{1}{-1}\begin{bmatrix} -2 & 4 & -1 \\ 2 & -5 & 1 \\ 1 & -3 & 1 \end{bmatrix}^T$$

$$\Rightarrow A^{-1} = -\begin{bmatrix} -2 & 2 & 1 \\ 4 & -5 & -3 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Hence we have

$$Y = AX$$

$$\Rightarrow X = A^{-1}Y$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \times 1 + (-2) \times 2 + (-1) \times (-1) \\ (-4) \times 1 + 5 \times 2 + 3 \times (-1) \\ 1 \times 1 + (-1) \times 2 + (-1) \times (-1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - 4 + 1 \\ -4 + 10 - 3 \\ 1 - 2 + 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

$\therefore$ Hence the coordinate in X is $(-1, \quad 3, \quad 0)$ corresponding to $(1, \quad 2, \quad -1)$ in Y.

- **Example 3.** *Express each of the transformation* $x_1 = 3y_1 + 2y_2,\ x_2 = -y_1 + 4y_2$ *and*
- $y_1 = z_1 + 2z_2,\ y_2 = 3z_1$ *in the matrix form and find the composite transformation which expresses* x_1, x_2 *in terms of* z_1, z_2.
- **Solution:** The transformation $x_1 = 3y_1 + 2y_2,\ x_2 = -y_1 + 4y_2$ can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow X = AY$$

- Similarly, the transformation $y_1 = z_1 + 2z_2,\ y_2 = 3z_1$ can be written as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \Rightarrow Y = BZ$$

$\therefore$ The composite transformation is given by

$$X = AY = A(BZ) = (AB)Z$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 3 \times 1 + 2 \times 3 & 3 \times 2 + 2 \times 0 \\ -1 \times 1 + 4 \times 3 & -1 \times 2 + 4 \times 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\Rightarrow \boxed{x_1 = 9z_1 + 6z_2,\ x_2 = 11z_1 - 2z_2}$$

Home Assignment

1. *Given the transformation* $Y = \begin{bmatrix} 4 & -5 & 1 \\ 3 & 1 & -2 \\ 1 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. *Find the coordinate* $(x_1, \quad x_2, \quad x_3)$ *in* X *corresponding to* $(2, \quad 9, \quad 5)$ *in* Y. $[Ans: (2, \quad 1, \quad -1)]$
2. *Given the transformation* $Y = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. *Find the coordinate* $(x_1, \quad x_2, \quad x_3)$ *in* X *corresponding to* $(3, \quad 0, \quad 8)$ *in* Y. $[Ans: (8/5, \quad 5, \quad 9/5)]$
3. *Given the transformation* $Y = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. *Find the coordinate* $(x_1, \quad x_2, \quad x_3)$ *in* X *corresponding to* $(2, \quad 3, \quad 0)$ *in* Y. $[Ans: (21/19, \quad -16/19, \quad -5/19)]$
4. *Express each of the transformation* $x_1 = 3y_1 + 5y_2$, $x_2 = -y_1 + 7y_2$ *and* $y_1 = z_1 + 3z_2$, $y_2 = 4z_1$ *in the matrix form and find the composite transformation which expresses* x_1, x_2 *in terms of* z_1, z_2. $[Ans: x_1 = 23z_1 + 9z_2, x_2 = 27z_1 - 3z_2]$
5. *Show that the transformation* $y_1 = x_1 + x_2 + x_3$; $y_2 = 2x_1 + 3x_2 + 4x_3$; $y_3 = x_1 - x_2 + x_3$ *is regular. Also find the coordinate* $(x_1, \quad x_2, \quad x_3)$ *in* X *corresponding to* $(6, \quad 20, \quad 2)$ *in* Y. $[Ans: (1, \quad 2, \quad 3)]$

Questions?

Thanks

Linear Algebra and Calculus

Unit-I: Matrices

- Introduction of Matrix
- Rank of a Matrix
- Canonical Form of a Matrix (Echelon Form)
- Normal Form of a Matrix
- **System of Linear Equations**
- Orthogonal Transformations
- Eigen Values and Eigen Vectors
- Diagonalization of Matrices
- Cayley Hamilton Theorem
- Applications to problems in Engineering

System of Linear Equtions

Consider the system of linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$

....

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

These system of equations can be written as in the matrix form

$AX = B$ - - - - - - - - - (1)

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

Here A = Coefficient Matrix
X = Variable Matrix
B = Constant Matrix

Augmented Matrix :

If $AX = B$ be a system of m linear equations with n unknowns, then **Augmented Matrix** is denoted by $[A : B]$ and is defined by

$$[A : B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & b_2 \\ a_{31} & a_{32} & \cdots & a_{33} & : & b_3 \\ \vdots & \vdots & \vdots & \vdots & : & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & : & b_m \end{bmatrix}$$

System of Nonhomogeneous Linear Equations :

A system of m linear equations with n unknowns $AX = B$ is called system of nonhomogeneous linear equations if $\boldsymbol{B \neq 0}$.

System of homogeneous Linear Equations :

A system of m linear equations with n unknowns $AX = B$ is called system of homogeneous linear equations if $\boldsymbol{B = 0}$.

Condition for consistency of Nonhomogeneous Equations

Let $AX = B$ be a system of m nonhomogeneous linear equations with n unknowns, then

m = Total number of equations and n = Total number of unknowns

$\rho\,(A)$ = Rank of Coefficient Matrix, $\rho\,(A\!:B)$ = Rank of Augmented matrix

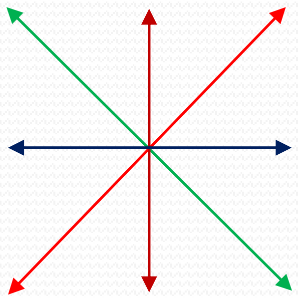
1. *If* $\boldsymbol{\rho\,(A) = \rho\,(A\!:B)}$*, then the system of equations is* ***consistent*** *and hence the given system of equations have solutions.*

   (i) ***Unique Solution:*** *If* $\boldsymbol{\rho\,(A) = \rho\,(A\!:B) = n =}$ ***Number of variables****, then the system have unique solution.*

   (ii) ***Infinitely Many Solutions:*** *If* $\boldsymbol{\rho\,(A) = \rho\,(A\!:B) < n =}$ ***Number of variables****, then the system of equations have infinitely many solutions.*

2. *If* $\rho\,(A) \neq \rho\,(A\!:B)$*, then the system of equations is not consistent and hence the given system of equations have* ***no solution****.*

Condition for consistency of Nonhomogeneous Equations

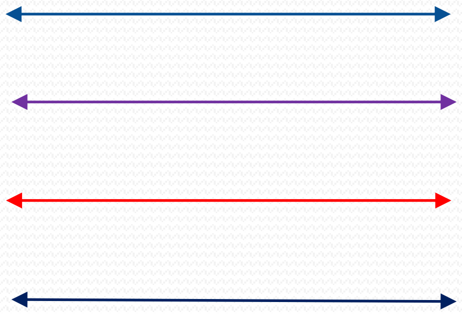


Lines are intersecting at one point

Unique Solution or only one solution

Lines are overlapping to each others

Infinitely many Solutions



Lines are parallel to each others

No Solution

Condition for consistency of Nonhomogeneous Equations

Non-Homogeneous System AX = B, n = unknowns

Consistent if, $\rho\ (A) = \rho\ (A{:}\, B) = r$

Inconsistent if, $\rho\ (A) \neq \rho\ (A{:}\, B)$

Unique Solution if $n = r$

Lines are intersecting at one point

Infinitely many solutions if $r < n$

Lines are overlapping to each others

The System has no solution

Lines are parallel to each others

- **Example 1.** *Examine for consistency and if consistent, then solve the following system of equations:* $x+y+z=6;\ x-y+2z=5;\ 3x+y+z=8;\ 2x-2y+3z=7$
- **Solution:** We have to examine the consistency of the following system of equations

$x+y+z=6;\ x-y+2z=5;\ 3x+y+z=8;\ 2x-2y+3z=7$ - - - - - - (1)

The system of simultaneous equations (1) can be written as

$$AX=B$$

$$\Rightarrow \begin{bmatrix}1 & 1 & 1\\1 & -1 & 2\\3 & 1 & 1\\2 & -2 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\5\\8\\7\end{bmatrix} \quad \text{- - - - - - - - - - - - -} \quad (2)$$

Where $A=\begin{bmatrix}1 & 1 & 1\\1 & -1 & 2\\3 & 1 & 1\\2 & -2 & 3\end{bmatrix}$ = Coefficient Matrix,

$X=\begin{bmatrix}x\\y\\ \end{bmatrix}$ = Variable matrix, $B=\begin{bmatrix}6\\5\\8\end{bmatrix}$ = Constant matrix.

Let us consider the Augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & -1 & 2 & : & 5 \\ 3 & 1 & 1 & : & 8 \\ 2 & -2 & 3 & : & 7 \end{bmatrix}$$

Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$
and $R_4 \to R_4 - 2R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & -2 & -2 & : & -10 \\ 0 & -4 & 1 & : & -5 \end{bmatrix}$$

Apply $R_3 \to R_3 - R_2$ and $R_4 \to R_4 - 2R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & 0 & -3 & : & -9 \\ 0 & 0 & -1 & : & -3 \end{bmatrix}$$

Apply R_{13}, we have

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & 0 & -1 & : & -3 \\ 0 & 0 & -3 & : & -9 \end{bmatrix}$$

Apply $R_4 \to R_4 - 3R_3$, we have

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & 0 & -1 & : & -3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \text{---} \quad (3)$$

This is Echelon Form of the given matrix.
Here Rank of $[A:B]$ = Number of nonzero rows i.e. $\rho\ (A:B) = 3$. Also, $\rho\ (A) = 3$.
Since $\rho\ (A:B) = \rho\ (A) = 3 =$ Number of variables
$\therefore$ The given system of equations is consistent and have unique solution.
Now have to find the solution of (1).

From (2) and (3), we have

$$\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 2\\ 3 & 1 & 1\\ 2 & -2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ 5\\ 8\\ 7\end{bmatrix} \qquad ---- \qquad (2)$$

$$[A:B]\sim\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & -2 & 1 & : & -1\\ 0 & 0 & -1 & : & -3\\ 0 & 0 & 0 & : & 0\end{bmatrix} -- \qquad (3)$$

It can be written as

$$\begin{bmatrix}1 & 1 & 1\\ 0 & -2 & 1\\ 0 & 0 & -1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}6\\ -1\\ -3\\ 0\end{bmatrix}$$

$x+y+z=6 \qquad ---- \qquad (4)$

$-2y+z=-1 \qquad ---- \qquad (5)$

$-z=-3 \qquad ---- \qquad (6)$

- **Example 2.** *Examine for consistency and if consistent, then solve the following system of equations:* $2x - y - z = 2;\ x + 2y + z = 2;\ 4x - 7y - 5z = 2$

- **Solution:** We have to examine the consistency of the following system of equations

$2x - y - z = 2;\ x + 2y + z = 2;\ 4x - 7y - 5z = 2$ - - - - - (1)

The system of simultaneous equations (1) can be written as

$$AX = B$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \text{- - - - - - - - - - - - -} \quad (2)$$

Where $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix}$ = Coefficient Matrix,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ = Variable matrix, $B = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ = Constant matrix.

Let us consider the Augmented matrix

$$[A:B] = \begin{bmatrix} 2 & -1 & -1 & : & 2 \\ 1 & 2 & 1 & : & 2 \\ 4 & -7 & -5 & : & 2 \end{bmatrix}$$

Apply R_{12}

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 2 & -1 & -1 & : & 2 \\ 4 & -7 & -5 & : & 2 \end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 4R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & -5 & -3 & : & -2 \\ 0 & -15 & -9 & : & -6 \end{bmatrix}$$

Apply $R_3 \to R_3 - 3R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & -5 & -3 & : & -2 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \quad \text{- - - (3)}$$

This is an Echelon Form of the given matrix.

From (2) and (3), we have

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad ---- \quad (2)$$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & -5 & -3 & : & -2 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} -- \quad (3)$$

It can be written as

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & -5 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

$$2x - y - z = 2 \quad ---- \quad (4)$$

$$-5y - 3z = -2 \quad ---- \quad (5)$$

Here the number of variables is 3 and number of equations is 2. So, out of three variable one variable is independent.

Let $z = t$, then putting the value of z in (5), we have

$$-5y - 3t = -2$$

$$\Rightarrow -5y = -2 + 3t$$

$$\Rightarrow 5y = 2 - 3t$$

$$\Rightarrow y = \frac{2 - 3t}{5}$$

Putting the values of y and z in (4), we have

$$2x - \left(\frac{2 - 3t}{5}\right) - t = 2$$

$$\Rightarrow 10x - (2 - 3t) - 5t = 10$$

$$\Rightarrow 10x - 2 + 3t - 5t = 10$$

$$\Rightarrow 10x - 2t = 10 + 2$$

$$\Rightarrow 10x = 2t + 12 = 2(t + 6)$$

$$\Rightarrow x = \frac{2(t + 6)}{10}$$

$$\Rightarrow x = \frac{t + 6}{5}$$

Hence the solution of the given system of linear equations is given by

- **Example 3.** *Examine for consistency and if consistent, then solve the following system of equations:* $2x - 3y + 7z = 5;\ 3x + y - 3z = 13;\ 2x + 19y - 47z = 32$
- **Solution:** We have to examine the consistency of the following system of equations

$2x - 3y + 7z = 5;\ 3x + y - 3z = 13;\ 2x + 19y - 47z = 32$ - - - - - (1)

The system of simultaneous equations (1) can be written as

$$AX = B$$

$$\Rightarrow \begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix} \quad \text{- - - - - - - - - - - - -} \quad (2)$$

Where $A = \begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}$ = Coefficient Matrix,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ = Variable matrix, $B = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$ = Constant matrix.

Let us consider the Augmented matrix

$$[A:B] = \begin{bmatrix} 2 & -3 & 7 & : & 5 \\ 3 & 1 & -3 & : & 13 \\ 2 & 19 & -47 & : & 32 \end{bmatrix}$$

Apply $R_1 \to R_1 - R_2$

$$[A:B] \sim \begin{bmatrix} -1 & -4 & 10 & : & -8 \\ 3 & 1 & -3 & : & 13 \\ 2 & 19 & -47 & : & 32 \end{bmatrix}$$

Apply $R_2 \to R_2 + 3R_1$ and $R_3 \to R_3 + 2R_1$

$$[A:B] \sim \begin{bmatrix} -1 & -4 & 10 & : & -8 \\ 0 & -11 & 27 & : & -11 \\ 0 & 11 & -27 & : & 16 \end{bmatrix}$$

Apply $R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} -1 & -4 & 10 & : & -8 \\ 0 & -11 & 27 & : & -11 \\ 0 & 0 & 0 & : & 5 \end{bmatrix} \qquad \text{- - - (3)}$$

This is an Echelon Form of the given matrix.

Home Assignment

Check whether the following system of linear equations are consistent or not. If yes, then solve them.

1. $3x + y + 2z = 3;\ \ 2x - 3y - z = -3;\ \ x + 2y + z = 4\ \ [Ans: x = 1,\ y = 2,\ z = -1]$
2. $x + 2y + z = -1;\ \ 6x + y + z = -4;\ \ 2x - 3y - z = 0;\ -x - 7y - 2z = 7; x - y = 1.\ [Ans: x = -1,\ y = -2,\ z = 4]$
3. $2x + 3y + 4z = 11;\ \ x + 5y + 7z = 15;\ \ 3x + 11y + 13z = 25\ \ [Ans: x = 2,\ y = -3,\ z = 4]$
4. $4x - 2y + 6z = 8;\ \ x + y - 3z = -1;\ \ 15x - 3y + 9z = 21\ [Ans: x = 1,\ y = 3t - 2,\ z = t]$
5. $2x_1 - 3x_2 + 5x_3 = 1; 3x_1 + x_2 - x_3 = 2; x_1 + 4x_2 - 6x_3 = 1\left[Ans: x_1 = \frac{7-2t}{11}, x_2 = \frac{1+17t}{11},\ x_3 = t\right]$
6. $2x + z = 4;\ \ x - 2y + 2z = 7;\ \ 3x + 2y = 1\ \left[Ans: x = \frac{4-t}{2}, y = \frac{3t-10}{4},\ z = t\right]$
7. $x + y + z = 3;\ \ 2x - y + 3z = 1;\ \ 4x + y + 5z = 2;\ \ 3x - 2y + z = 4[Ans: No\ ]$
8. $x_1 + x_2 + 2x_3 + x_4 = 5; 2x_1 + 3x_2 - x_3 - 2x_4 = 2; 4x_1 + 5x_2 + 3x_3 = 7[Ans: No]$

Home Assignment

Check whether the following system of linear equations are consistent or not. If yes, then solve them.

9. $3x + 3y + 2z = 1;\ \ x + 2y = 4;\ \ 10y + 3z = -2; 2x - 3y - z = 5$ $[Ans: x = 2,\ y = 1,\ z = -4]$

10. $x + 2y + 2z = 1;\ \ 2x + 2y + 3z = 3;\ \ x - y + 3z = 5$ $[Ans: x = 1,\ y = -1,\ z = 1]$

11. $x + y + z = 6;\ \ 2x + y + 3z = 13;\ \ 5x + 2y + z = 12;\ \ 2x - 3y - 2z = -10$ $[Ans: x = 1,\ y = 2,\ z = 3]$

12. $2x_1 + x_2 - x_3 + 3x_4 = 8; x_1 + x_2 + x_3 - x_4 + 2 = 0; 3x_1 + 2x_2 - x_3 = 6; 4x_2 + 3x_3 + 2x_4 + 8 = 0$ $[Ans: x_1 = 2,\ x_2 = -1,\ x_3 = -2, x_4 = 1]$

13. $2x_1 + x_2 - x_3 + 3x_4 = 11; x_1 - 2x_2 + x_3 + x_4 = 8; 4x_1 + 7x_2 + 2x_3 - x_4 = 0; 3x_1 + 5x_2 + 4x_3 + 4x_4 = 17$ $[Ans: x_1 = 2,\ x_2 = -1,\ x_3 = 1, x_4 = 3]$

Questions?

Thanks

Linear Algebra and Calculus

Unit-I: Matrices

- Introduction of Matrix
- Rank of a Matrix
- Canonical Form of a Matrix (Echelon Form)
- Normal Form of a Matrix
- **System of Linear Equations**
- Orthogonal Transformations
- Eigen Values and Eigen Vectors
- Diagonalization of Matrices
- Cayley Hamilton Theorem
- Applications to problems in Engineering

System of Homogeneous Linear Equations

Consider the system of linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = 0$$

....

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = 0$$

These system of equations can be written as in the matrix form

$AX = O$ - - - - - - - - - *(1)*

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Here A = Coefficient Matrix

X = Variable Matrix

O = Zero Matrix

Augmented Matrix :

If $AX = O$ be a system of m linear equations with n unknowns, then **Augmented Matrix** is denoted by $[A : 0]$ and is defined by

$$[A : 0] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & 0 \\ a_{31} & a_{32} & \cdots & {}_{3} & : & 0 \\ \vdots & \vdots & \vdots & \vdots & : & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & : & 0 \end{bmatrix} = A$$

Here

$$\Rightarrow \rho\,(A) = \rho\,(A: O)$$

$\Rightarrow$ System of homogeneous linear equations is always consistent.

Condition for consistency of homogeneous Equations

Let $AX = O$ be a system of m homogeneous linear equations with n unknowns, then

m = Total number of equations and n = Total number of unknowns

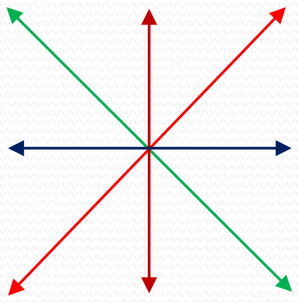
$\rho\,(A)$ = Rank of Coefficient Matrix $= \rho\,(A{:}\,B) =$ o Rank of Augmented matrix

Here it is clear that $\boldsymbol{\rho\,(A) = \rho\,(A{:}}$ **o**), *hence the system of homogeneous equations are always consistent. So, the given system of equations have solutions.*

(i) ***Trivial Solution:*** *If* $\boldsymbol{\rho\,(A) = \rho\,(A{:}}$ **o**$\boldsymbol{) = n =}$ ***Number of variables****, then the system have unique solution.*

(ii) ***Non-trivial Solution:****If* $\boldsymbol{\rho\,(A) = \rho\,(A{:}\,B}$ **o**$\boldsymbol{< n =}$ ***Number of variables****, then the system of equations have infinitely many solutions.*

Condition for consistency of Homogeneous Equations



Lines are intersecting at origin

Trivial Solution or only one solution

Lines are overlapping to each others

Non Trivial Solutions

Condition for consistency of homogeneous Equations

Homogeneous System AX = B, n = unknowns

↓

Always consistent
$\rho\,(A) = \rho\,(A{:}\,B) = r$

↓

Trivial Solution if
$n = r$ or $|A| \neq 0$

Lines are intersecting at origin only

Non trivial solutions if
$r < n$ or $|A| = 0$

Lines are overlapping to each others

- **Example 1.** *Examine for nontrivial solutions of the following system of linear equations and hence find solutions:* $x + 2y + 3z = 0;\ 2x + 3y + z = 0;\ 4x + 5y + 4z = 0.$
- **Solution:** We have to examine for nontrivial solution of the following system of equations

$x + 2y + 3z = 0;\ 2x + 3y + z = 0;\ 4x + 5y + 4z = 0$ - - - - - (1)

The system of simultaneous equations (1) can be written as

$$AX = O$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{- - - - - - - - - - - - -} \quad (2)$$

$$\text{Consider}|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 5 & 4 \end{vmatrix} = 1(3 \times 4 - 5 \times 1) - 2(2 \times 4 - 4 \times 1) + 3(2 \times 5 - 3 \times 4)$$

$$= 1(12 - 5) - 2(8 - 4) + 3(10 - 12)$$

$$= 1(7) - 2(4) + 3(-2) = 7 - 8 - 6 = -7 \neq 0$$

$$\because |A| \neq 0$$

$\therefore$ The system has trivial solution. i.e. $\boxed{x = 0, y = 0, z = 0}$

- **Example 2.** *Examine for nontrivial solutions of the following system of linear equations and hence find solutions:* $2x - y + 3z = 0;\ 3x + 2y + z = 0;\ x - 4y + 5z = 0$.
- **Solution:** We have to examine for nontrivial solution of the following system of equations

$$2x - y + 3z = 0;\ 3x + 2y + z = 0;\ x - 4y + 5z = 0 \quad \text{- - - - - -} \quad (1)$$

The system of simultaneous equations (1) can be written as

$$AX = O$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{- - - - - - - - - - - - - -} \quad (2)$$

$$\text{Consider} |A| = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{vmatrix}$$

$$= 2 \times [2 \times 5 - (-4) \times 1] - (-1) \times [3 \times 5 - 1 \times 1] + 3 \times [3 \times (-4) - 1 \times 2]$$

$$= 1(10 + 4) + 2(15 - 1) + 3(-12 - 2)$$

$$= 1(14) + 2(14) + 3(-14) = 14 + 28 - 42 = 0$$

$$\because |A| = 0$$

$\therefore$ The system has non trivial solution.

From (2), we have

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply R_{13}

$$\begin{bmatrix} 1 & -4 & 5 \\ 3 & 2 & 1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_2 \rightarrow R_2 - 3R_1$ and $R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & -4 & 5 \\ 0 & 14 & -14 \\ 0 & -7 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_2\left(\frac{1}{14}\right)$

$$\begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & -1 \\ 0 & 7 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_3 \rightarrow R_3 - 7R_2$

$$\begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x - 4y + 5z = 0$ - - - (3)

$y - z = 0$ - - - (4)

We have three variables and two equation. So, one variable will be independent.

Put $z = t$ in (4), then we have

$$y - t = 0$$

$$\Rightarrow y = t$$

Putting the values of y and z in (3), we have

$$\Rightarrow x - 4t + 5t = 0$$

$$\Rightarrow x + t = 0$$

$$\Rightarrow x = -t$$

Hence the nontrivial solutions is

$$\boxed{x = -t, y = t, z = t}$$

- **Example 3.** *Show that the system of equations:* $ax + by + cz = 0; bx + cy + az = 0; cx + ay + bz = 0$ has non-trivial solution only if $a + b + c = 0$ or $a = b = c$.
- **Solution:** We have to show that the following system of equations

$ax + by + cz = 0; bx + cy + az = 0; cx + ay + bz = 0$ - - - - - (1)

has a nontrivial solutions only if $a + b + c = 0$ or $a = b = c$.

The system of simultaneous equations (1) can be written as

$$AX = O$$

$$\Rightarrow \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{- - - - - - - - - - - -} \quad (2)$$

$\because$ The system has nontrivial solution, then

$$\therefore |A| = 0$$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

Apply $R_1 \to R_1 + R_2 + R_3$, we have

$$\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

Apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have

$$\Rightarrow (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c)[1 \times \{(c-b)(b-c)-(a-c)(a-b)\}+0+0] = 0$$

$$\Rightarrow (a+b+c)[1 \times \{bc-c^2-b^2+bc-(a^2-ab-ac+bc)\}] = 0$$

$$\Rightarrow (a+b+c)[bc-c^2-b^2+bc-a^2+ab+ac-bc] = 0$$

$$\Rightarrow (a+b+c)[-(a^2+b^2+c^2-ab-ac-bc)] = 0$$

$\Rightarrow a+b+c=0$ or $a^2+b^2+c^2-ab-ac-bc=0$

$$\Rightarrow a+b+c=0 \quad \text{or} \quad a^2+b^2+c^2-ab-ac-bc=0$$

$$\Rightarrow a+b+c=0 \quad \text{or} \quad 2a^2+2b^2+2c^2-2ab-2ac-2bc=0$$

$$\Rightarrow a+b+c=0 \quad \text{or} \quad (a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ac+a^2)=0$$

$$\Rightarrow a+b+c=0 \quad \text{or} \quad (a-b)^2+(b-c)^2+(c-a)^2=0$$

$$\Rightarrow a+b+c=0 \quad \text{or} \quad a-b=0 \text{ and } b-c=0 \text{ and } c-a=0$$

$$\Rightarrow a+b+c=0 \quad \text{or} \quad a=b=c$$

$\therefore$ The given system has non trivial solution if

$$\boxed{a+b+c=0 \text{ or } a=b=c}$$

(Hence Proved)

Home Assignment

Check whether the following system of linear equations have non trivial solutions or not. If yes, then solve them.

1. $x + 2y + 3z = 0;\ \ 2x + 3y + z = 0;\ \ 4x + 5y + 4z = 0;\ \ x + y - 2z = 0.$

   $[Ans: x = 0,\quad y = 0,\quad z = 0]$

2. $5x + 2y - 3z = 0;\ \ 3x + y + z = 0;\ \ 2x + y + 6z = 0.$ $[Ans: x = 0,\ y = 0,\ z = 0]$
3. $x + 3y + z = 0;\ \ 2x - 2y - 6z = 0;\ \ 3x + y - 5z = 0.$ $[Ans: x = 2t,\ y = -t,\ z = t]$
4. $4x_1 - x_2 + 2x_3 + x_4 = 0; 2x_1 + 3x_2 - x_3 - 2x_4 = 0; 7x_2 - 4x_3 - 5x_4 = 0; 2x_1 - 11x_2 + 7x_3 + 8x_4 = 0.$ $\left[Ans: x = \frac{-a-5b}{14}, y = \frac{5a+4b}{7},\ z = b,\ \ t = a\right]$
5. For different values of k, discuss the nature of solution of the following equations:

$x + 2y - z = 0;\ \ 3x + (k+7)y - 3z = 0;\ \ 2x + 4y + (k-3)z = 0.$

$[Ans: For\ k = 1, x = t,\quad y = 0,\quad z = t, for\ k = -1, x = -2t,\quad y = t,\quad z = 0]$

6. Show that the system of equations $x_1 + 2x_2 + 3x_3 = \lambda x_1;\ 3x_1 + x_2 + 2x_3 = \lambda x_2; 2x_1 + 3x_2 + x_3 = \lambda x_3$ can possess a non-trivial solution only $\lambda = 6$. Obtain the general solution for real values of λ . $[Ans: For\ \lambda = 6, x = t,\ y = t,\ z = t]$

Questions?

Thanks

Linear Algebra and Calculus

Unit-I: Matrices

- Introduction of Matrix
- **Rank of a Matrix**
- **Canonical Form of a Matrix (Echelon Form)**
- Normal Form of a Matrix
- System of Linear Equations
- Orthogonal Transformations
- Eigen Values and Eigen Vectors
- Diagonalization of Matrices
- Cayley Hamilton Theorem
- Applications to problems in Engineering

Rank of a Matrix

A matrix 'A' is said to be of **rank r,** *if there is*

(i) at least one non zero minor of the order 'r' and

(ii) every minor of order $(r+1)$ *is equal to zero.*

i.e. The rank of a matrix 'A' is the maximum order of its non-vanishing minor. Rank of a matrix A is denoted by $\rho(A) = r$.

Properties of a rank of matrix:

1. *If a matrix A has a non-zero minor of order r, then* $\rho(A) \geq r$.
2. *If a matrix A has all the minors of order* $(r+1)$ *is equal to zero, then* $\rho(A) \leq r$.
3. *If a matrix A of order* $m \times n$*, then the* $\rho(A) \leq$ *minimum of m and n.*
4. *Elementary transformations of a matrix do not alter the rank of a matrix.*
5. $\rho(A) = \rho(A') = \rho(A^{-1})$.

Example1: *Find the rank of the following matrices:*

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ *(ii)* $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ *(iii)* $C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 2 \\ 5 & 8 & 9 \end{bmatrix}$

Solution: (i) We have to find the rank of the matrix A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

Rank of the matrix is the order of highest order non-vanishing minor of it.

Here the order of the given matrix is 2×3. So, the rank of the matrix will not be more than 2. It may be 2 if at least one minor of order 2×2 is not equal to zero.

Now consider all the minor of order 2×2 of A, we have

$$|A_1| = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 2 - 2 = 0; \qquad |A_2| = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 3 - 3 = 0$$

$$|A_3| = \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 6 - 6 = 0$$

Since all the minors of order 2×2 of the matrix A are zero. So the rank of the matrix will be less than 2. Again 1, 2, 3 are nonzero elements of A.

Hence the rank of the given matrix is 1. i.e. $\boxed{\rho(A) = 1}$.

Solution: (ii) We have to find the rank of the matrix A.

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Rank of the matrix is the order of highest order non-vanishing minor of it.

Here the order of the given matrix is 3×3. So, the rank of the matrix may be 3, if the determinant of B is not equal to zero i.e. $|B| \neq 0$.

Now consider

$$|B| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1(5 \times 9 - 8 \times 6) - 2(4 \times 9 - 7 \times 6) + 3(4 \times 8 - 7 \times 5)$$

$$= 1(45 - 48) - 2(36 - 42) + 3(32 - 35)$$

$$= 1 \times (-3) - 2 \times (-6) + 3 \times (-3)$$

$$= -3 + 12 - 9 = 0$$

Since $|B| = 0$. So, the rank of the matrix is less than 3.

Now consider $|B_1| = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$

Hence the rank of the given matrix is 2. i.e. $\boxed{\rho(B) = 2}$.

Solution: (iii) We have to find the rank of the matrix A.

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 2 \\ 5 & 8 & 9 \end{bmatrix}$$

Rank of the matrix is the order of highest order non-vanishing minor of it.

Here the order of the given matrix is 3×3. So, the rank of the matrix may be 3, if the determinant of C is not equal to zero i.e. $|C| \neq 0$.

Now consider

$$|C| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 2 \\ 5 & 8 & 9 \end{vmatrix} = 1(5 \times 9 - 8 \times 2) - 2(4 \times 9 - 5 \times 2) + 3(4 \times 8 - 5 \times 5)$$

$$= 1(45 - 16) - 2(36 - 10) + 3(32 - 25)$$

$$= 1 \times (29) - 2 \times (16) + 3 \times (7)$$

$$= 29 - 32 + 21 = 18 \neq 0$$

Since $|C| = 18 \neq 0$. So, the rank of the matrix is 3.

$$\boxed{\rho(C) = 3}.$$

Elementary transformations of mAtrix

1. The interchange of i^{th} and j^{th} rows denoted by R_{ij}
2. The multiplication of each element of i^{th} row by a non zero scalar k is denoted by kR_i
3. Multiplication of every element of j^{th} row by scalar k and adding to the corresponding element of i^{th} row is denoted by $R_i + kR_j$

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - R_2$

Canonical or Echelon Form of a Matrix

A matrix A is said to be in echelon form if

(i) All zero rows, if any, are at the bottom of the matrix.

(ii) Each leading non-zero entry in a row is to the right of the leading non-zero entry in the preceding row.

Note: The rank of a matrix in echelon form is equal to the number of non zero rows of the matrix.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad Rank(A) = 3$$

Row Echelon Form : *An Echelon form of a matrix A is called* ***Row Echelon Form*** *if all the principal diagonal elements are one (1).*

Example: *Following matrices are in the Row Echelon form or canonical form*

$$(i)\ \ A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \qquad (ii)\quad B = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (iii)\quad C = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Row Reduced Echelon Form : *An Echelon form of a matrix A is called* ***Row Reduced Echelon Form*** *if all the principal diagonal elements are one (1) and all other elements are zero.*

Example: *Following matrices are in the Row Reduced Echelon form*

$$(i)\ \ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (ii)\quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (iii)\quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Properties of an Echelon or Canonical Form

(i) If a matrix is reduced to its Echelon form, then the rank of a matrix is equal to the number of nonzero rows present in its Echelon form.

(ii) If a matrix is square, then the Echelon form of the matrix is same as Upper Triangular matrix.

(iii) Echelon form of a matrix is called Gauss Elimination Method which is used to solve the system of linear equations.

(iv) Row Echelon form is used to check whether the given vectors are linearly independent or linearly dependent.

(v) Row reduced Echelon form is used to find the rank and nullity of a matrix.

Example2: *Reduce the following matrices to its Echelon Form and hence find the rank.*

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ *(iii)* $B = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Solution: (i) We have to reduce the following matrix A to its Echelon form and also the rank.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$, we have

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Apply $R_3 \to R_3 - R_2$, we have

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the Echelon Form of the given matrix.

Hence the rank of the given matrix is equal to nonzero rows. i.e. $\boxed{\rho(A) = 2}$.

Solution: (ii)

We have to reduce the following matrix A to its Echelon form and also the rank.

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Apply R_{12}, we have

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Apply $R_3 \to R_3 - 3R_1, R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

Apply $R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the Echelon Form of the given matrix.

Hence the rank of the given matrix is equal to nonzero rows. i.e. $\boxed{\rho(A) = 2}$.

Home Assignment

1. Find the rank of the following matrices:

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ $[Ans: \rho(A) = 1]$

(b) $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$ $[Ans: \rho(A) = 2]$

(c) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ $[Ans: \rho(A) = 3]$

(d) $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$ $[Ans: \rho(A) = 2]$

(e) $\begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$ $[Ans: \rho(A) = 3]$

2. Reduce the following matrices to its Echelon form and hence find the rank.

(a) $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$ $[Ans: \rho(A) = 3]$

(b) $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$ $[Ans: \rho(A) = 2]$

3. *Determine the values of p such that the rank of* $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ p & 2 & 2 & 2 \\ 9 & 9 & p & 3 \end{bmatrix}$ is 3.

$[Ans: p = -6]$

Questions?

Thanks

Linear Algebra and Calculus

Unit-I: Matrices

- Introduction of Matrix
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- **Normal Form of a Matrix**
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NORMAL FORM OF A MATRIX

The following forms are called the normal forms of the matrix.

$$[I_r], \qquad [I_r \quad O], \qquad \begin{bmatrix} I_r \\ O \end{bmatrix}, \qquad \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$$

where I_r is the identity matrix of order 'r' and O is null matrix of suitable order. If any matrix is reduced to its normal form, then the rank of matrix will be equal to the order of $\mathrm{I_r}$.

Example: *Following matrices are in the Normal form*

$$(i)\ \ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (ii) \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} (iii)\ \ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Here the rank of the matrices are $\rho(A) = \rho(B) = \rho(C) = 3$.

Procedure to convert Normal Form of a Matrix

1. *We have to first check that* a_{11} *is one(1) or not?*

2. *If it is not 1, then by using elementary transformation we should make it 1.*

Example: *Following matrices are in the Normal form*

$$(i)\ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (ii) \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (iii) \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Here the rank of the matrices are $\rho(A) = \rho(B) = \rho(C) = 3$.

- ***Example 1:*** *Reduce the matrix* $A = \begin{bmatrix} 3 & 2 & 5 & 2 & 7 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$ *to its normal form and hence find its rank.*
- **Solution:** Let us define the following terms first.
- Normal Form: *The following forms are called the normal forms of the matrix.*

$$[I_r], \qquad [I_r \quad O], \qquad \begin{bmatrix} I_r \\ O \end{bmatrix}, \qquad \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$$

where I_r is the identity matrix of order 'r' and O is null matrix of suitable order.

Rank of a Matrix: *Rank of a matrix A is the order of highest order non-vanishing (nonzero) minor of it. If the matrix is reduced to its normal form, then the rank of matrix will be equal to the order of* I_r.

- We have to reduce the following matrix to its normal form and also find the rank.

$$A = \begin{bmatrix} 3 & 2 & 5 & 2 & 7 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix} \quad \text{- - - - - - - - -} \qquad (1)$$

We have

- $A = \begin{bmatrix} 3 & 2 & 5 & 2 & 7 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$

Apply R_{12}

- $A \sim \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 3 & 2 & 5 & 2 & 7 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$

Apply $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 3R_1$

- $A \sim \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 0 & -1 & -1 & -7 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Apply $C_2 \to C_2 - C_1$, $C_3 \to C_3 - 2C_1$,

$C_4 \to C_4 - 3C_1$ and $C_5 \to C_5 - 5C_1$, we have

- $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -7 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

- ***Example 2:*** *Reduce the matrix* $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ *to its normal form and hence find its rank.*

- **Solution:** Let us define the following terms first.

- Normal Form: *The following forms are called the normal forms of the matrix.*

$$[I_r], \qquad [I_r \quad O], \qquad \begin{bmatrix} I_r \\ O \end{bmatrix}, \qquad \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$$

where I_r is the identity matrix of order 'r' and O is null matrix of suitable order.

Rank of a Matrix: *Rank of a matrix A is the order of highest order non-vanishing (nonzero) minor of it. If the matrix is reduced to its normal form, then the rank of matrix will be equal to the order of* I_r.

- We have to reduce the following matrix to its normal form and also find the rank.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \end{bmatrix} \text{ - - - - - - - - - - } \qquad (1)$$

We have

- $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Apply R_{12}

- $A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Apply $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ and $R_4 \to R_4 - 6R_1$

- $A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$

Apply $C_2 \to C_2 + C_1$, $C_3 \to C_3 + 2C_1$, $C_4 \to C_4 + 4C_1$, we have

- $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$

Apply $R_2 \to R_2 - R_3$

- $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$

- Apply $R_3 \to R_3 - 4R_2$, $R_4 \to R_4 - 9R_2$

- $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$

- $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$
- Apply $C_3 \to C_3 + 6C_2,\ C_4 \to C_4 + 3C_2$
- $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$
- Apply $C_3 \to \frac{1}{33} C_3$
- $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 2 & 44 \end{bmatrix}$
- Apply $R_4 \to R_4 - 2R_3$

$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$

Home Assignment

Reduce the following matrices to its normal form and hence find the rank.

1. $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ $[Ans: \rho(A) = 3]$

2. $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ $[Ans: \rho(A) = 3]$

3. $\begin{bmatrix} 1 & 2 & -1 & 2 \\ -2 & -5 & 3 & 0 \\ 1 & 0 & 1 & 10 \end{bmatrix}$ $[Ans: \rho(A) = 2]$

4. $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ $[Ans: \rho(A) = 2]$

5. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$ $[Ans: \rho(A) = 3]$

6. $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$ $[Ans: \rho(A) = 4]$

7. $\begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 7 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ $[Ans: \rho(A) = 4]$

8. $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -3 \end{bmatrix}$ $[Ans: \rho(A) = 4]$

9. $\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$ $[Ans: \rho(A) = 4]$

- ***Example 3:*** *Find non singular matrices P and Q such that PAQ in a normal form. Also find the rank and inverse of the matrix if exits for* $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$.

- **Solution:** We have to find non-singular matrices P and Q such that PAQ is in normal form, where

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \qquad \text{---------} \qquad (1)$$

Now consider

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $R_1 \to R_1 - R_2$, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1$, we have

$$\begin{bmatrix}1 & 0 & 0\\ 0 & -3 & 4\\ 0 & -1 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\ -2 & 3 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Apply $C_2 \to C_2 + C_3$, we have

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 4\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\ -2 & 3 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$$

Apply $C_3 \to C_3 - 4C_2$, we have

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\ -2 & 3 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -4\\ 0 & 1 & -3\end{bmatrix}$$

$$\Rightarrow \qquad I \quad = \quad P \quad A \quad Q$$

Here $P = \begin{bmatrix}1 & -1 & 0\\ -2 & 3 & 0\\ 0 & 0 & 1\end{bmatrix}$ and $Q = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -4\\ 0 & 1 & -3\end{bmatrix}$ we have

Here $P = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$

Now we have to find the A^{-1}.

Since P and Q are nonsingular matrices, so P^{-1} and Q^{-1} exist.

Consider

$$PAQ = I$$

Operating P^{-1} and Q^{-1} on both sides, we have

$$P^{-1}(PAQ)Q^{-1} = P^{-1}(I)Q^{-1}$$

$$\Rightarrow (P^{-1}P)A(QQ^{-1}) = P^{-1}Q^{-1}$$

$$\Rightarrow (I)A(I) = P^{-1}Q^{-1} [\because AA^{-1} = A^{-1}A = I, \qquad IA = A]$$

$$\Rightarrow A = P^{-1}Q^{-1}$$

Taking inverse on both the sides

$$A^{-1} = (P^{-1}Q^{-1})^{-1}$$

$$= (Q^{-1})^{-1} (P^{-1})^{-1} [\because (AB)^{-1} = B^{-1}A^{-1}]$$

$$\therefore \boxed{A^{-1} = QP} [\because (A^{-1})^{-1} = A]$$

Here $P = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$

Putting the values of P and Q , we have

$$A^{-1} = QP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & -1+0+0 & 0+0+0 \\ 0-2+0 & 0+3+0 & 0+0-4 \\ 0-2+0 & 0+3+0 & 0+0-3 \end{bmatrix}$$

$$\therefore \boxed{A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}}$$

Again Consider $AA^{-1} = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- ***Example 4:*** *Find non singular matrices P and Q such that PAQ in a normal form. Also find the rank and inverse of the matrix if exits for* $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

- **Solution:** We have to find non-singular matrices P and Q such that PAQ is in normal form, where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix} \qquad \text{- - - - - - - - -} \qquad (1)$$

Now consider

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$, we have

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 3 & 4\\ 0 & -3 & -2 & -5\\ 0 & -6 & -4 & -22\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Apply $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$, $C_4 \to C_4 - 4C_1$, we have

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -3 & -2 & -5\\ 0 & -6 & -4 & -22\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & -2 & -3 & -4\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Apply $R_2 \to (-1)R_2$, we have

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 3 & 2 & 5\\ 0 & -6 & -4 & -22\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 2 & -1 & 0\\ -3 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & -2 & -3 & -4\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Apply $C_2 \to C_2 - C_3$, we have

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 2 & 5\\ 0 & -2 & -4 & -22\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 2 & -1 & 0\\ -3 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 1 & -3 & -4\\ 0 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply $R_3 \to R_3 + 2R_2$, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply $C_3 \to C_3 - 2C_2$, $C_4 \to C_4 - 5C_2$, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -5 & -9 \\ 0 & 1 & -2 & -5 \\ 0 & -1 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply C_{34}, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -12 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -9 & -5 \\ 0 & 1 & -5 & -2 \\ 0 & -1 & 5 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & -12 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\2 & -1 & 0\\1 & -2 & 1\end{bmatrix} A \begin{bmatrix}1 & 1 & -9 & -5\\0 & 1 & -5 & -2\\0 & -1 & 5 & 3\\0 & 0 & 1 & 0\end{bmatrix}$$

Apply $R_3(-1/12)$, we have

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\2 & -1 & 0\\-1/12 & 1/6 & -1/12\end{bmatrix} A \begin{bmatrix}1 & 1 & -9 & -5\\0 & 1 & -5 & -2\\0 & -1 & 5 & 3\\0 & 0 & 1 & 0\end{bmatrix}$$

$\Rightarrow \quad I \quad = \quad P \quad A \quad Q$

Here $\mathrm{P} = \begin{bmatrix}1 & 0 & 0\\2 & -1 & 0\\-1/12 & 1/6 & -1/12\end{bmatrix}$ and $\mathrm{Q} = \begin{bmatrix}1 & 1 & -9 & -5\\0 & 1 & -5 & -2\\0 & -1 & 5 & 3\\0 & 0 & 1 & 0\end{bmatrix}$

Rank of the given matrix is 3.

Since the matrix is not a square matrix, so A^{-1} does not exist.

Home Assignment

Find non-singular matrices P and Q such that PAQ is in normal form. Hence find the rank of A and also find A^{-1}.

1. $\begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \left\{ Ans: \rho(A) = 3,\ A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 4 & -1 \\ 5 & -5 & 5 \\ 7 & -6 & 4 \end{bmatrix} \right\}$

2. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \left\{ Ans: \rho(A) = 3, \quad A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix} \right\}$

3. $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \left\{ Ans: \rho(A) = 3,\ A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \right\}$

4. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \left\{ Ans: \rho(A) = 3,\ A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right\}$

5. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix} \{ Ans: \rho(A) = 3, \quad A^{-1} does\ not\ exist \}$

Questions?

Thanks