4.1 Solve the following system of equations using the Gauss elimination method
$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 9 \\ 2x_1 + 3x_2 + x_3 &= 23 \\ 3x_1 + 2x_2 - 4x_3 &= 11 \end{aligned}.$$

Solution

Step 1: Write the system of equations in matrix form:

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & 1 \\ 3 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 23 \\ 11 \end{bmatrix}$$

Using the first element of the matrix as a pivot, $a_{11} = 1$ and $m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2$.

Step 2: Multiply the first row by $m_{21} = 2$, subtract the result from the second row, and replace the second row with the final result:

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 5 \\ 3 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 11 \end{bmatrix}$$

Step 3: Next, multiply the first row by $m_{31} = \frac{a'_{31}}{a'_{11}} = \frac{3}{1} = 3$, subtract the result from the third row, and replace the third row with the final result:

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 5 \\ 0 & -4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ -16 \end{bmatrix}$$

Step 4: Use the element a'_{22} as the pivot. Multiply the second row by $m_{32} = \frac{a'_{32}}{a'_{22}} = \frac{-4}{-1} = 4$, subtract the result from the third row, and replace the third row with the final result:

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 5 \\ 0 & 0 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ -36 \end{bmatrix}$$

Step 4: The matrix is in upper triangular form, apply back-substitution:

$$x_3 = \frac{-36}{-18} = 2$$

$$-x_2 + 5(2) = 5 \quad \Rightarrow \quad x_2 = 10 - 5 = 5$$

$$x_1 + 2(5) - 2(2) = 9 \quad \Rightarrow \quad x_1 = 9 + 4 - 10 = 3$$

Thus, the solution to the given set of simultaneous equations is $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 2 \end{bmatrix}$.

4.2 Given the system of equations $[a][x] = [b]$, where $a = \begin{bmatrix} 2 & -4 & 1 \\ 6 & 2 & -1 \\ -2 & 6 & -2 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $b = \begin{bmatrix} 4 \\ 10 \\ -6 \end{bmatrix}$, determine the solution using the Gauss elimination method.

Solution

Following the same procedure as in Problem 4.1, the row elimination operations proceed as follows:

Step 1: Multiply the first row by $m_{21} = \frac{a_{21}}{a_{11}} = \frac{6}{2} = 3$, subtract the result from the second row, and replace the second row with the final result:

$$\begin{bmatrix} 2 & -4 & 1 \\ 0 & 14 & -4 \\ -2 & 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$$

Step 2: Multiply the first row by $m_{31} = \frac{a'_{31}}{a'_{11}} = \frac{-2}{2} = -1$, subtract the result from the third row, and replace the third row with the final result:

$$\begin{bmatrix} 2 & -4 & 1 \\ 0 & 14 & -4 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -2 \end{bmatrix}$$

Step 3: Use the element a'_{22} as the pivot. Multiply the second row by $m_{32} = \frac{a'_{32}}{a'_{22}} = \frac{2}{14} = \frac{1}{7}$, subtract the result from the third row, and replace the third row with the final result:

$$\begin{bmatrix} 2 & -4 & 1 \\ 0 & 14 & -4 \\ 0 & 0 & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -\frac{12}{7} \end{bmatrix}$$

Step 4: Solving by back-substitution, $x_3 = \frac{12}{3} = 4$, $14x_2 - 4(4) = -2$ or $x_2 = 1$, and $2x_1 - 4(1) + 4 = 4$ or $x_1 = 2$. Thus, the solution is $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \end{bmatrix}$.

4.9 Determine the LU decomposition of the matrix $a = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 2 \end{bmatrix}$ using the Gauss elimination procedure.

Solution

LU decomposition using Gausian elimination transforms the above matrix into a lower triangular matrix $[L]$ multiplied by an upper triangular matrix $[U]$. $[U]$ is the upper triangular matrix that would normally result after applying Gaussian elimination to the given matrix. $[L]$ consists of the multipliers that are used in the Gaussian elimination procedure and 1s along the diagonal. Thus, with $m_{21} = 4$ and $m_{31} = 3$, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -4 & -7 \end{bmatrix}$. Next, with $m_{32} = 4/3$, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -4 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix}$. Thus, $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix}$, and $L = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 4/3 & 1 \end{bmatrix}$. Therefore, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 2 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 4/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix}$