Stat 471: Lecture 3
Accept-Reject Method

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1. Generate $X \sim Unif(0, \frac{1}{2})$ using uniform distribution $U \sim Unif(0,1)$.
   Step 1. generate U.
   Step 2. if $U < 1/2$, $X = U$, else go to step 1.
   The method seems intuitively correct. This is the simplest version of the accept-reject method. But how do you actually show this is in fact correct? Let's study the general setting.

2. *Accept-reject method.* Given known random number generators $U \sim Unif(0,1)$ and $X \sim g$, we can generate $Y \sim f$ by the following algorithm. Let c be a constant such that $f(x) \leq cg(x)$ for all x.
   Step 1. Generate $X \sim g$, $U \sim Unif(0,1)$.
   Step 2. Accept $Y = X$ if $U \leq \frac{f(X)}{cg(X)}$ otherwise go to Step 1.
   proof. We show that the conditional distribution $P(X < y | U \leq \frac{f(X)}{cg(X)}) = P(Y \leq y)$. We see that the conditional distribution is
   $$\frac{P(X \leq y, U \leq \frac{f(X)}{cg(X)})}{P(U \leq \frac{f(X)}{cg(X)})} = \frac{\int_{-\infty}^{y} \int_0^{f(x)/cg(x)} g(x)\, du\, dx}{\int_{-\infty}^{\infty} \int_0^{f(x)/cg(x)} g(x)\, du\, dx}.$$
   Simplifying this, it can be shown to be $\int_{-\infty}^{y} f(x)\, dx$.

3. From $U \sim Unif(0,1)$, generate 1000 random numbers that follow probability density $f(y) = 2y, 0 < y < 1$. *solution.* Let g follows $Uinf(0,1)$. $g = 1$. Then $f(y)/g(y) = f(y) = 2y < 2$ for $y \in (0,1)$. So we choose $c = 2$. Usually choose the smallest c that satisfies the inequality.

```
i=1;
while i <= 1000
  U = rand;X = rand;
  if U < X
    Y(i)=X;
    i=i+1;
  end;
end;
```

4. A simple version of the accept-reject method. Choose $X \sim Unif(0,1)$.
   Step 1. Generate $X, U \sim Unif(0,1)$.
   Step 2. Accept $Y = X$ if $U \leq \frac{1}{c} f(X)$ otherwise go to Step 1.
   But this may not be an efficient algorithm. The following example illustrate this point.

5. Generate 1000 exponential random numbers with parameter $\lambda = 2$ from $U \sim Unif(0,1)$. *solution.* Let $f = e^{-2x}/2, 0 \leq x$ and $g \sim Unif(0,1)$. Since $f(y) \leq 1/2$ for all $0 \leq y$, choose $c = 1/2$.

```
>> f=inline('exp(-2*x)/2')
f =
     Inline function:
     f(x) = exp(-2*x)/2
i=1;
while i <= 1000
   U = rand; X = rand;
   if (U < 2*f(X))
      Y(i)=X;
      i=i+1;
   end;
end;
```

6. In lecture 2, we studied the integral transform based algorithm for generating exponential random variables, i.e. `>> U=rand(1000,1); X=-log(U)/2`. Which one performs better? It is easy to check by computing the sample mean which should converge to the population mean as the number of samples increases.

```
>> mean(X)
ans = 0.4857
>> mean(Y)
ans = 0.3444
```

To use the accept-reject method, the distributions f and g should be somewhat similar to have a sufficiently good algorithm.