Lecture 7a: Introduction to Matrices and Vectors

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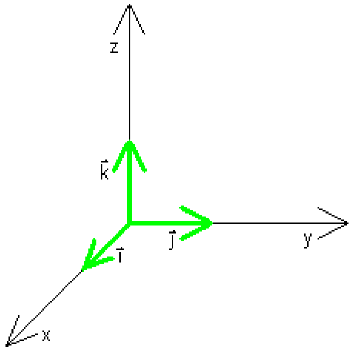
Overview

1. Introduction to Vectors
2. Introduction to Matrices
3. Change of Basis

Vectors

A vector is a quantity with both a magnitude and a direction, usually represented by an arrow.



[HMC, 2014]

Vector Operations

Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$.

Addition

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

$$\vec{u} - \vec{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3)$$

Dot Product

Also called **scalar product** or **inner product**.

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Properties of the Dot Product:

- Commutative: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- Distributive: $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$

Vector **norm**, the magnitude of the vector, is: $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$

[HMC, 2014]

Vector Operations

Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$.

Projection

The **projection** of $\vec{v}$ onto $\vec{u}$ is the portion of $\vec{v}$ that is parallel to $\vec{u}$.
The **dot product** gives the magnitude of the projection.

$$\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}||cos(\theta)$$

$$\text{projection of } \vec{v} \text{ onto } \vec{u} : \frac{\vec{v} \cdot \vec{u}}{||u||^2}\vec{u}$$

$$\text{perpendicular vector component of } \vec{v} \text{ from } \vec{u} : \vec{v} - \frac{\vec{v} \cdot \vec{u}}{||u||^2}\vec{u}$$

Cross Product

The **cross product** $\vec{u} \times \vec{v}$ yields a vector perpendicular to both $\vec{u}$ and $\vec{v}$.
Properties of the Cross Product:

- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
- $\vec{u} \times \vec{u} = \vec{0}$

[HMC, 2014]

Matrices

A **matrix** is a two-dimensional rectangular array of numbers. An $n \times k$ matrix has n rows and k columns.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} \end{pmatrix}$$

Terminology

- For an $n \times n$ **square** matrix A, the elements $a_{11}, a_{22}, \dots, a_{nn}$ for the **main diagonal** of the matrix.
- The **trace** of A is the sum of the main diagonal: $\sum_{i=1}^{n} a_{ii}$.
- The **transpose** of A is the matrix A^T formed by interchanging the rows and columns of A.
- If $A = A^T$, the matrix A is **symmetric**.

[HMC, 2014]

Let $A = [a_{ij}]$ and $B = [b_{ij}]$

Addition

If A and B are the same size, addition is defined. Each element of the new matrix is the element-wise sum of A and B.

$$A + B = [a_{ij} + b_{ij}]$$

Scalar Multiplication

$$cA = [ca_{ij}]$$

Matrix Multiplication

To multiply matrices A and B, the number of columns in A must be the same as the number of rows in B.

$$AB = [ab_{ij}] = [\sum_{k=1}^{n} a_{ik} b_{kj}]$$

[HMC, 2014]

Let $A = [a_{ij}]$ and $B = [b_{ij}]$

Matrix Inverse

The identity matrix I is the matrix with all main diagonal entries equal to 1, and all other elements equal to 0. This has the property:

$$AI = IA = A$$

The inverse of A is the matrix C such that:

$$AC = CA = I$$

and is denoted A^{-1}.

- Only square matrices can have inverses.
- Not every square matrix has an inverse. If an inverse exists, it is unique.
- If a matrix has an inverse, it is said to be **invertible**.

[HMC, 2014]

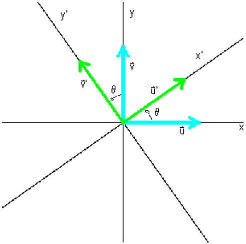
Matrices can form a map from one vector space to another.

Example: Rotation of Coordinates

The new basis $B' = \{\mathbf{u}', \mathbf{v}'\}$ of unit vectors along the $x'-$ and $y'-$ axies, respectively, has coordinates:

$$[\mathbf{u}']_{\mathbf{B}} = \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix}, [\mathbf{v}']_{\mathbf{B}} = \begin{bmatrix} -sin\theta \\ cos\theta \end{bmatrix}$$

in the original coordinate system B.



[HMC, 2014]

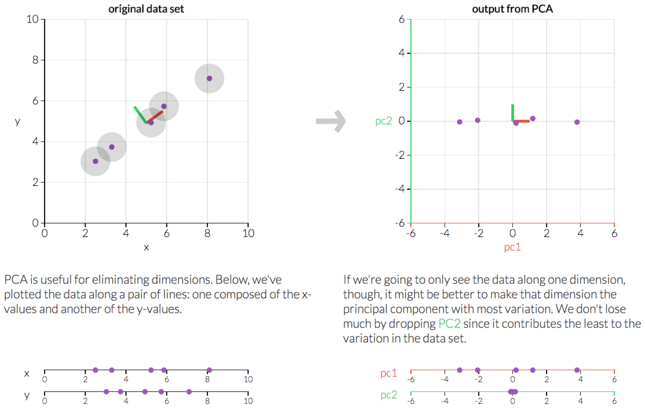
Change of Basis

The matrix $P = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ maps from the original coordinate system B to the rotated coordinate system B'.

A vector $\left[\begin{array}{c} x \\ y \end{array} \right]_B$ in the original coordinate system has coordinates $\left[\begin{array}{c} x' \\ y' \end{array} \right]_{B'}$ in the new system.

$$\left[\begin{array}{c} x' \\ y' \end{array} \right]_{B'} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \left[\begin{array}{c} x \\ y \end{array} \right]_B$$

Principal Component Analysis (PCA) is a matrix-based method that finds the dimensions in a set of data that hold the most variability.



[Explained Visually, 2015]

Principal Component Analysis

Covariance matrix: $C_X = \frac{1}{n-1}XX^T$ where X is an $m \times n$ data matrix where each row is a set of n observations.

- C_X is a square, symmetric $m \times m$ matrix.
- The diagonal terms of C_X are the measure variance.
- The off-diagonal terms are the covariance between measurements.

Symmetric matrices are diagonalizable, and can be rewritten:

$$XX^T = S \Lambda S^{-1}$$

The principal component basis $Y = S^T X$ gives a new basis set, such that most of the variability in the data lies along the first direction.

Exercise

Sparse Matrix Multiplication: A sparse matrix is one where most of the entries are zero. It is more efficient to store such a matrix in table format, with entries (i, j, a_{ij}), where i is the row number, j is the column number. Write a function that will perform matrix multiplication using this table input.

Principal Component Analysis:

- Visit Principal Component Analysis: Explained Visually to get an intuition about the analysis.
- Implement the algorithm on the data set gradientdescent.csv in the data folder.

References

HMC Mathematics Online Tutorial (2014)
Harvey Mudd College

Principal Component Analysis (2015)
Explained Visually

Recommended Reading

Data Science From Scratch, Chapter 4