Class Exercise #2 COSC600 Dennis Q. Weaver

Problem 1

$\Rightarrow 2^{50} \bmod 5$

$\Rightarrow 2^1 \bmod 5 = 2 \bmod 5 = 2$

$\Rightarrow 2^2 \bmod 5 = 4 \bmod 5 = 4$

$\Rightarrow 2^3 \bmod 5 = 8 \bmod 5 = 3$

$\Rightarrow 2^4 \bmod 5 = 16 \bmod 5 = 1$

$\Rightarrow 2^5 \bmod 5 = 32 \bmod 5 = 2$

$\vdots$

$\Rightarrow 2^{10} \bmod 5 = 1024 \bmod 5 = 4$

Then,

$\Rightarrow 2^{50} = (2^{10} \cdot 2^{10} \cdot 2^{10} \cdot 2^{10} \cdot 2^{10}) (\bmod 5)$

$= (4 \cdot 4 \cdot 4 \cdot 4 \cdot 4) \bmod 5$

$= 1024 \bmod 5 = 4$

$\therefore 2^{50} \bmod 5 = 4.$

$\Rightarrow 2^{100} \bmod 3$

$\Rightarrow 2^1 \bmod 3 = 2 \bmod 3 = 2$

$\Rightarrow 2^2 \bmod 3 = 4 \bmod 3 = 1$

$\Rightarrow 2^3 \bmod 3 = 8 \bmod 3 = 2.$

Observe even powers equal 2.

$\therefore 2^{100} \bmod 3 = 1.$

$\Rightarrow 3^{135} \bmod 2$

$\Rightarrow 3^1 \bmod 2 = 3 \bmod 2 = 1$

$\Rightarrow 3^2 \bmod 3 = 9 \bmod 2 = 1$

$\vdots$

$\therefore 3^{135} \bmod 2 = 1.$

2) Prove the following via induction.

(a) $\sum_{i=0}^{n} \frac{1}{(i+1)(i+2)} = \frac{n+1}{n+2}, \forall n \geq 0.$

Proof

Let $P(n) := \sum_{i=0}^{n} \frac{1}{(i+1)(i+2)} = \frac{n+1}{n+2}, \forall n \geq 0.$

Base Case: $n = 0$

$\Rightarrow P(0) = \sum_{i=0}^{0} \frac{1}{(i+1)(i+2)} = \frac{0+1}{0+2}$

$\Rightarrow \frac{1}{2} = \frac{1}{2}.$

Thus the base case holds. Let $P(K) := \sum_{i=0}^{K} \frac{1}{(i+1)(i+2)} = \frac{K+1}{K+2}.$ Assume $P(K)$ is true, then we'll show $P(K+1) = \sum_{i=0}^{K+1} \frac{1}{(i+1)(i+2)} = \frac{K+2}{K+3}$ is also true.

Then,

$$\Rightarrow P(K+1) = \sum_{i=0}^{K+1} \frac{1}{(i+1)(i+2)} = \sum_{i=0}^{K} \frac{1}{(i+1)(i+2)} + \frac{1}{(K+2)(K+3)}.$$

Next,

$$\Rightarrow \sum_{i=0}^{K} \frac{1}{(i+1)(i+2)} + \frac{1}{(K+2)(K+3)} = \frac{K+1}{K+2} + \frac{1}{(K+2)(K+3)}$$

$$= \frac{(K+3)(K+1)}{(K+2)(K+3)} + \frac{(K+2)}{(K+2)(K+3)}$$

$$= \frac{(K^2+4K+3)+(K+2)}{(K+2)(K+3)}$$

$$= \frac{(K+1)+1}{(K+3)}$$

$$= \frac{K+2}{K+3}.$$

$$\therefore P(n) := \sum_{i=0}^{n} \frac{1}{(i+1)(i+2)} = \frac{n+1}{n+2}, \forall n \geq 0.$$

2b) $\sum_{i=1}^{n}(2i-1) = n^2, \forall n \geq 1$

Proof

Let $P(n) := \sum_{i=1}^{n}(2i-1) = n^2, \forall n \geq 1.$

Base Case: $n=1$

$\Rightarrow P(1) = \sum_{i=1}^{1}(2i-1) = n^2$

$= (2(1)-1) = 1^2$

$= 1 = 1.$

Thus the base case holds. Let $P(K) := \sum_{i=1}^{K}(2i-1) = K^2, \forall K \geq 1.$ Assume $P(K)$ holds, then we'll show $P(K+1) = \sum_{i=1}^{K+1}(2i-1) = (K+1)^2, \forall K \geq 1.$ Then,

$\Rightarrow P(K+1) = \sum_{i=1}^{K+1}(2i-1) = \sum_{i=1}^{K}(2i-1) + 2(K+1)-1$

$= K^2 + 2K + 2 - 1$

$= K^2 + 2K + 1$

$= (K+1)(K+1)$

$= (K+1)^2.$

$\therefore$ By PMI, $P(n) := \sum_{i=1}^{n}(2i-1) = n^2, \forall n \geq 1.$

3) $\forall n \geq 1$, where $n \in \mathbb{Z}$, $(1 \cdot 3) + (2 \cdot 5) + (3 \cdot 7) + \dots + n(2n+1) = ?$

Rewrite in Σ notation & simplify.

$$\Rightarrow \sum_{i=1}^{n} n(2n+1)$$

$$\Rightarrow \sum_{i=1}^{n} 2n^2 + n$$

$$\Rightarrow \sum_{i=1}^{n} 2n^2 + \sum_{i=1}^{n} n$$

$$\Rightarrow 2\sum_{i=1}^{n} n^2 + \sum_{i=1}^{n} n$$

$$\Rightarrow \cancel{2}^{1}\left(\frac{n(n+1)(n+2)}{\cancel{6}_{3}}\right) + \frac{n(n+1)}{2}$$

$$\Rightarrow \frac{n(n+1)(n+2)}{3} + \frac{n(n+1)}{2}$$

$$\Rightarrow \frac{2n(n+1)(2n+1) + 3n(n+1)}{6}.$$

$$\therefore \sum_{i=1}^{n} n(2n+1) = \frac{2n(n+1)(2n+1) + 3n(n+1)}{6}, \ \forall n \geq 1.$$

4a) $8+12+16+\dots+104=?$

Recall, $a_n = a_1 + (n-1)d$. To find the number of terms,

$\Rightarrow 104 = 8 + (n-1)4$, where $d=4$, $a_1 = 8$,

$\Rightarrow 26 = 2 + n - 1$

$\Rightarrow 24 = n-1$

$\Rightarrow n = 25$.

Hence there's 25 terms, thus $n=25$. Then, use $\sum_{i=1}^{n} a_i = S_n = \frac{n(a_1+a_n)}{2}$.

$$\Rightarrow \sum_{i=1}^{n=25} a_i = S_{25} = \frac{25(8+104)}{2} = 1{,}400.$$

$$\therefore\ 8+12+16+\dots 104 = \sum_{i=1}^{8} 8+(i-1)4 = \boxed{1400}.$$

4b) $6+18+54+\dots+2(3^n)=?$

Recall, for a geometric series, $a_n = a_1 \cdot r^{n-1}$. Let $a_1 = 6$, $r=3$, then,

$a_n = 6 \cdot 3^{n-1}$. Then,

$$\Rightarrow \sum_{i=1}^{n} a_i = a_1 \cdot \frac{1-r^n}{1-r}, \text{ since } r \neq 1,$$

$$\therefore\ 6+18+54+\dots 2(3^n) = \sum_{i=1}^{n} 6 \cdot 3^{n-1} \text{ or } 6 \cdot \frac{1-3^n}{1-3}.$$

4c) $\lim_{n\to\infty}\left(\frac{1}{5}+\frac{1}{5^2}+\ldots\frac{1}{5^n}\right)=?$

Conjecture: $\lim_{n\to\infty}\left(\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\ldots\frac{1}{5^n}\right)=\frac{1}{4}.$

Proof

Rewrite the limit as, $\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\frac{1}{5^4}+\ldots\frac{1}{5^n},$

$\Rightarrow \frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\ldots$ as a geometric series, where $r=\frac{1}{5}.$

Observe $\frac{1}{5}<1$, then,

$$\Rightarrow S=\frac{\frac{1}{5}}{1-\frac{1}{5}}=\frac{1}{\cancel{5}}\cdot\frac{\cancel{5}}{4}=\frac{1}{4}.$$

$$\therefore \sum_{i=1}^{n}\frac{1}{5^i}=\frac{a_1}{1-r}=\frac{1}{4}.$$

4d) $\sum_{i=1}^{n}\left(i+\sum_{j=1}^{i}2n\right)$

$\Rightarrow \sum_{i=1}^{n} i + \sum_{i=1}^{n}\sum_{j=1}^{i} 2n$

$\Rightarrow \frac{n(n+1)}{2} + n^2(n+1)$

$\Rightarrow \frac{n^2+n+2n^3+2n^2}{2}$

$\Rightarrow \frac{2n^3+3n^2+n}{2}$

$\Rightarrow \frac{n(2n^2+3n+1)}{2}$

$\Rightarrow \frac{n(2n+1)(n+1)}{2}.$

$\therefore \sum_{i=1}^{n}\left(i+\sum_{j=1}^{i}2n\right) = \frac{n(2n+1)(n+1)}{2}.$

4e) $\sum_{i=1}^{n}\left(i + \sum_{j=1}^{i} 2j\right) = ?$

$$\Rightarrow \sum_{i=1}^{n} i + \sum_{i=1}^{n}\sum_{j=1}^{i} 2j$$

$$\Rightarrow \frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$\Rightarrow \frac{3n^2+3n}{6} + \frac{n(2n^2+n+2n+1)}{6} + \frac{3n^2+3n}{6}$$

$$\Rightarrow \frac{3n^2+3n}{6} + \frac{n(2n^2+3n+1)}{6} + \frac{3n^2+3n}{6}$$

$$\Rightarrow \frac{2n^3+9n^2+7n}{6}$$

$$\Rightarrow \frac{n(2n^2+9n+7)}{6}$$

$$\Rightarrow \frac{n(2n+7)(n+1)}{6}.$$

$$\therefore \sum_{i=1}^{n}\left(i + \sum_{j=1}^{i} 2j\right) = \frac{n(2n+7)(n+1)}{6}.$$

hanoi.cc

hanoi.cc > main()

```cpp
/* hanoi.cc - implementation of solution to Hanoi Tower Problem */
#include <iostream>

static int count{};

void towerOfHanoi(int n, char from_rod, char to_rod, char aux_rod)
{
    if (n == 1)
    {
        std::cout << "Move a disk from rod " << from_rod << " to rod " << to_rod << std::endl;
        count++;
        return;
    }
    towerOfHanoi(n - 1, from_rod, aux_rod, to_rod);
    std::cout << "Move a disk from rod " << from_rod << " to rod " << to_rod << std::endl;
    count++;
    towerOfHanoi(n - 1, aux_rod, to_rod, from_rod);
}

// Driver code
int main()
{
    int n{5};                        // Number of disks
    towerOfHanoi(n, 'A', 'C', 'B'); // A, B and C are names of rods return 0;
    std::cout << "\nTotal number of moves for " << n << " disks is : " << count << '\n';
    return 0;
}
```

```
~/Fall23/COSC600
❯ ./hanoi
Move a disk from rod A to rod C
Move a disk from rod A to rod B
Move a disk from rod C to rod B
Move a disk from rod A to rod C
Move a disk from rod B to rod A
Move a disk from rod B to rod C
Move a disk from rod A to rod C
Move a disk from rod A to rod B
Move a disk from rod C to rod B
Move a disk from rod C to rod A
Move a disk from rod B to rod A
Move a disk from rod C to rod B
Move a disk from rod A to rod C
Move a disk from rod A to rod B
Move a disk from rod C to rod B
Move a disk from rod A to rod C
Move a disk from rod B to rod A
Move a disk from rod B to rod C
Move a disk from rod A to rod C
Move a disk from rod B to rod A
Move a disk from rod C to rod B
Move a disk from rod C to rod A
Move a disk from rod B to rod A
Move a disk from rod B to rod C
Move a disk from rod A to rod C
Move a disk from rod A to rod B
Move a disk from rod C to rod B
Move a disk from rod A to rod C
Move a disk from rod B to rod A
Move a disk from rod B to rod C
Move a disk from rod A to rod C

Total number of moves for 5 disks is : 31
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