MAD4401 - Numerical Analysis

Homework 1

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Problem 1

Use bisection method to find the third approximation p_3 for the solution of $f(x) = \sqrt{x} - \cos x = 0$ on $[0, 1]$.

First iteration

Let $a_1 = 0$ and $b_1 = 1$. Then,

$$p_1 = \frac{a_1 + b_1}{2} = \frac{1+0}{2} = \frac{1}{2}.$$

Evaluate $f(p_1)$ and $f(a_1)$. Then,

$$f\left(\frac{1}{2}\right) =\approx -0.170475781,$$

and

$$f(0) = -1.$$

Hence, $f(p_1)f(a_1) > 0$. Thus, the root lies to the right of p_1.

Second Iteration

To shift the interval to the right, let $a_2 = p_1 = \frac{1}{2}$ and let $b_2 = b_1 = 1$. Then,

$$p_2 = \frac{a_2 + b_2}{2} = \frac{\frac{1}{2} + 1}{2} = \frac{3}{4}.$$

Evaluate $f(p_2)$ and $f(a_2)$. Then,

$$f(p_2) = f\left(\frac{3}{4}\right) \approx 0.134336545,$$

and

$$f(a_2) = f\left(\frac{1}{2}\right) \approx -0.170475781.$$

Hence, $f(p_2)f(a_2) < 0$. Therefore, the root lies to the left of p_2.

<u>Third Iteration</u>

To shift the interval to the left, let $a_3 = a_2 = \frac{1}{2}$ and let $b_3 = p_2 = \frac{3}{4}$. Then,

$$f(p_3) = \frac{\frac{1}{2} + \frac{3}{4}}{2} = \frac{5}{8} \approx 0.625.$$

Therefore, the third approximation for the solution of $f(x) = \sqrt{x} - \cos x = 0$ on $[0, 1]$ is given by $p_3 = 0.625$.

$\square$

Problem 2

One can obtain an approximation of $\sqrt{3}$ by using the bisection method, since $\sqrt{3}$ is a root of the function $f(x) = x^2 - 3$.

(i) Use Theorem 2.1 to find an upper bound on the number of iterations of bisection method needed to achieve an approximation with accuracy 10^{-3} to $\sqrt{3}$ starting with the initial interval $[1, 2]$.

To find an upper bound on the number of iterations of the bisection method, use

$$|p_n - p| \leq \frac{b-a}{2^n}$$

where p_n is the n^{th} approximation for the value p. To find the number of iterations, n, to achieve approximation 10^{-3},

$$\Rightarrow |p_n - p| \leq \frac{b-a}{2^n} = 2^{-n}(a-b) < 10^{-3}$$

$$= 2^{-n}(2-1) < 10^{-3}$$

$$= 2^{-n}(1) < 10^{-3}$$

$$= -n \log 2 < \log 10^{-3}$$

$$= -n \log 2 < -3$$

$$= -n < \frac{-3}{\log 2}$$

$$= n > \frac{-3}{\log 2} \approx 9.965784285.$$

Hence, n would take approximately 10 iterations to achieve an approximation with an accuracy of 10^{-3}.

(ii) Carry out 4 iterations of bisection method and find the error associated to each approximation p_n for $n = 1, 2, 3, 4$.

The approximate errors for p_n for $n = 1, 2, 3, 4$ are,

$$p_1 = 1.5, \text{ then } |1.5 - \sqrt{3}| \approx 0.2321,$$

$$p_2 = 1.75, \text{ then } |1.75 - \sqrt{3}| \approx 0.0179,$$

$$p_3 = 1.625, \text{ then } |1.625 - \sqrt{3}| \approx 0.1071,$$

$$p_4 = 1.6875, \text{ then } |1.6875 - \sqrt{3}| \approx 0.0446.$$

□

Problem 3

Use Theorem 2.1 to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-3} to the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$.

To find a bound for the number of iterations,

$$\begin{aligned}
\Rightarrow |p_n - p| \leq \frac{b-a}{2^n} &= 2^{-n}(b-a) \\
&= 2^{-n}(3) < 10^{-3} \\
&= 2^{-n} < \frac{10^{-3}}{3} \\
&= -n \log 2 < \log \frac{10^{-3}}{3} \\
&= n > -\frac{\log\left(\frac{10^{-3}}{3}\right)}{\log(2)} \approx 11.55074679.
\end{aligned}$$

Hence, the number of iterations for the desired accuracy is approximately 12.

□

Problem 4

Let $f(x) = (x-1)^{10}$, $p = 1$, and $p_n = 1 + 1/n$. Show that $|f(p_n)| < 10^3$ whenever $n > 1$ but that $|p - p_n| < 10^3$ requires $n > 1000$.

(i) To show $|f(p_n)| < 10^{-3}$,

$$\Rightarrow |f(p_n)| < 10^{-3} = \left|(1 + \frac{1}{n} - 1)^{10}\right| < 10^{-3}$$

$$= \left|\left(\frac{1}{n}\right)^{10}\right| < 10^{-3}$$

$$= \left|(n^{-1})^{10}\right| < 10^{-3}$$

$$= \left|n^{-10}\right| < 10^{-3}$$

$$= \left|\frac{1}{n^{10}}\right| < 10^{-3}.$$

So when $n = 1$, $|\frac{1}{n^{10}} < 10^{-3}$ doesn't hold. But for $n > 1$ it does. As n gets larger, $\frac{1}{n^{10}}$ gets smaller.

(ii) To show $|p - p_n| < 10^{-3}$ requires $n < 1000$,

$$\Rightarrow |p - (p_n)| < 10^{-3} = \left|1 - \left(\frac{1}{n} + 1\right)\right| < 10^{-3}$$

$$= \left|-\frac{1}{n}\right| < 10^{-3}$$

$$= \frac{1}{n} < 10^{-3}$$

$$= n > \frac{1}{10^{-3}} = 1000.$$

Therefore, for $|p - p_n| < 10^{-3}$, $n > 1000$.

□