Fixed-Point Iteration

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Definition. (Fixed-point)

The number p is a *fixed-point* for a given function g if $g(p) = p$.

Fixed-point Iteration

Given a root-finding problem, $f(x) = 0$, one can define multiple functions of g with a fixed-point in different ways,

$$g_1(x) = x - f(x), \;\; g_2(x) = x + 3f(x) \ldots$$

If g has a fixed-point at point p, then we can define f as

$$f(x) = x - g(x) = 0.$$

This means that f has a zero at p. In order to find roots using this method, we first need to learn to decide when a function has a fixed-point and then determine how the fixed-points can be approximated to within a specified accuracy.

Example: Determine any fixed-points of the function $g(x) = x^2 - 2$.

In order to do this, we can use the definition of a fixed-point above. Start by assuming p is a fixed-point of the function g. Then this means $g(p) = p$. Then,

$$\Rightarrow g(p) = p^2 - 2$$

$$p = p^2 - 2$$

$$0 = p^2 - p - 2$$

$$(p+1)(p-2) = 0.$$

Hence the roots of f are $p = -1, \; p = 2$. Therefore, the function g has two fixed points.

□

Graphically, a fixed point occurs when the graph of a function $y = g(x)$ intersects with the graph of $y = x$. At this point, the graphs is showing when the values of the input are equal to the output, in other words a fixed point of the function.

The following is a useful theorem that gives sufficient conditions for the existence and uniqueness of a fixed-point. Note, it gives *sufficient* conditions but not *necessary* conditions. In other words, the following theorem can be used to confirm if a unique fixed point exists in an interval for a given function, but a unique fixed-point may also exist even if it doesn't adhere to the hypotheses in the theorem.

Theorem (Theorem 2.3 in text). *For some function g,*

*(i) **(Existence)** if $g \in C[a,b]$ and $g(x) \in [a,b], \forall x \in [a,b]$, then g has at least one fixed-point in $[a,b]$.*

*(ii) **(Uniqueness)** if $g'(x)$ exists on (a,b), and there exists a positive constant $k < 1$ with*

$$|g'(x)| \leq k, \ \forall x \in (a,b),$$

then there is exactly one fixed point in $[a,b]$.

So, to show that a fixed point exists within a given interval, one can use methods developed in calculus to find the absolute min/max to ensure the function maps the interval onto itself.

A big question that concerns itself within the field of numerical analysis is, *"How can we find a fixed-point problem that produced a sequence that reliably and rapidly converges to s solution to a given root-finding problem?"*

To approximate the fixed-point of a function g, we choose an initial approximation p_0 to generate a sequence $\{p_n\}_{n=0}^{\infty}$ by letting $p_n = g(p_{n-1}),\ n \geq 1$. Then, if the sequence converges to p and g is continuous, then a solution to $x = g(x)$ can be obtained.

Theorem (Fixed-Point Iteration). *Let $g \in C[a,b]$ and $g(x) \in [a,b],\ \forall x \in [a,b]$. Suppose g' exists on (a,b) and there exists a constant k where $0 < k < 1$ with*

$$|g'(x)| \leq k, \ \forall x \in (a,b).$$

Then for any number p_0 in $[a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \ \forall n \geq 1,$$

converges to the unique fixed point p in $[a,b]$.

Corollary (Fixed-Point Iteration Corollary). *If g satisfies the Fixed-Point Iteration theorem, then the bounds for the error using p_n to approximate the root p are given by,*

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\},$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|, \ \forall n \geq 1.$$

Answer to previously posed question, *"Manipulate the root-finding problem into a fixed point problem that satisfies the conditions of the Fixed-Point theorem and has a derivative that is as small as possible near the fixed point."*