Newton's Method and Its Extensions

Devere Anthony Weaver

Newton's Method (Newton-Raphson)

There are a few different ways in which one can approach Newton's method:

- graphically (often done in calculus)
- derive as a technique to obtain faster convergence than other types of functional iteration
- Taylor polynomials

This section of the notes introduces Newton's method based on Taylor polynomials. This type of derivation produces an approximation as well as a bound for the error of the approximation.

Suppose that $f \in C^2[a,b]$. Let $p_0 \in [a,b]$ be an approximation to p such that $f'(p_0) \neq 0$ and $|p - p_0|$ is "small". **Question:** What is considered "small" and how does one make this determination?

Using these hypotheses, consider the first Taylor polynomial for $f(x)$ expanded about p_0 and evaluated at $x = p$:

$$\Rightarrow f(p) = f(p_0) + f'(p_0)(p - p_0) + \frac{f''(\xi(p))}{2!}(p - p_0)^2$$

$$0 = f(p_0) + f'(p_0)(p - p_0) + \frac{f''(\xi(p))}{2!}(p - p_0)^2.$$

In the above equations, $\xi(p)$ lies between p and p_0. Recall we want to use iterative algorithms to find roots, in other words $f(p) = 0$. In our hypotheses, we assume $|p - p_0|$ to be small, this implies $(p - p_0)^2$ is even smaller. Thus,

$$0 \approx f(p_0) + f'(p_0)(p - p_0).$$

Then solving for p gives

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1.$$

This leads to the generation of a sequence that starts off with the initial approximation, p_0. The sequence $\{p_n\}_{n=0}^{\infty}$ is given by,

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, n \geq 1.$$

In other words, the n^{th} approximation of the sequence is a function of the previous approximation.

When looking at Newton's method graphically, the approximations are obtained using successive tangents. Start with the initial approximation. The next approximation in the sequence is the x-intercept of the tangent line of the function evaluated at the previous approximation. Each iteration, the successive approximation **should** be approaching the actual root of the function over the given interval.

Convergence Using Newton's Method

As can be seen from examples in the text, Newton's Method often gives provides very accurate approximations with very few iterations. The reason for this lies in the Taylor series derivation of Newton's method. The important point for this derivation is the need for an accurate initial approximation. The term involving $(p - p_0)^2$ should be so small that it can be deleted when compared with $|p-p_0|$. This will be false unless p_0 is a good approximation to p.

The following convergence theorem for Newton's Method illustrates the theoretical importance of the choice of p_0.

Theorem. *Let $f \in C^2[a, b]$. If $p \in (a, b)$ such that $f(p) = 0$ **and** $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$.*

In other words, if the first two hypotheses hold, then any initial guess of p_0 that is within the $\delta - nbd$ of p will converge to p.

TODO: Review the proof for this theorem after submitting the assignment and double read the same material concerning this in the other recommended textbook to get a better grasp on it.