MAD4401 - Numerical Analysis

Homework 4

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Problem 1

Consider the function $f(x) = \sin e^2 - 2$. Approximate $f(0.9)$ using each of the following:

a) Lagrange interpolating polynomial of degree one using $x_0 = 0.8$ and $x_1 = 1.0$. The Lagrange interpolating polynomial of degree one is of the form,

$$\Rightarrow P(x) = f(x_0)L_{1,0}(x) + f(x_1)L_{1,1}(x).$$

Compute $L_{1,0}(x)$, and $L_{1,1}(x)$

$$\Rightarrow L_{1,0}(x) = \frac{(x - x_1)}{(x_0 - x_1)} = \frac{(x - 1.0)}{(0.8 - 1.0)} = -5x + 5.0,$$

$$\Rightarrow L_{1,1}(x) = \frac{(x - x_0)}{(x_1 - x_0)} = \frac{(x - 0.8)}{(1.0 - 0.8)} = 5x - 4.$$

Then,

$$\Rightarrow P(x) = f(0.8)(-5x + 5.0) + f(1.0)(5x - 4)$$

$$= \sin(e^{0.8} - 2)(-5x + 5.0) + \sin(e^{1.0} - 2)(5x - 4).$$

We can now approximate the function $f(0.9)$ using the above polynomial,

$$\Rightarrow P(0.9) \approx 0.440862796.$$

b) Lagrange interpolating polynomial of degree two using $x_0 = 0.7$, $x_1 = 0.8$, and $x_2 = 1.0$. The Lagrange interpolating polynomial of degree two is of the form,

$$\Rightarrow P(x) = f(x_o)L_{2,0}(x) + f(x_1)L_{2,1}(x) + f(x_2)L_{2,2}(x).$$

Compute $L_{2,0}(x)$, $L_{2,1}(x)$, and $L_{2,2}(x)$,

$$\Rightarrow L_{2,0}(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 0.9)(x - 1.0)}{(0.7 - 0.8)(0.7 - 1.0)} = \frac{x^2 - 1.8x + 0.8}{0.03},$$

$$\Rightarrow L_{2,1}(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - 0.7)(x - 1.0)}{(0.8 - 0.7)(0.8 - 1.0)} = \frac{x^2 - 1.7x + 0.7}{-0.02},$$

$$\Rightarrow L_{2,2}(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - 0.7)(x - 0.8)}{(1.0 - 0.7)(1.0 - 0.8)} = \frac{x^2 - 1.5x + 0.56}{0.06}.$$

Then,

$$\Rightarrow P(x) = f(0.7)\left(\frac{x^2-1.8x+0.8}{0.03}\right) + f(0.8)\left(\frac{x^2-1.7x+0.7}{-0.02}\right) + f(1.0)\left(\frac{x^2-1.5x+0.56}{0.06}\right)$$

$$= \sin\left(e^{0.7}-2\right)\left(\frac{x^2-1.8x+0.8}{0.03}\right) + \sin\left(e^{0.8}-2\right)\left(\frac{x^2-1.7x+0.7}{-0.02}\right) + \sin\left(e^{1.0}-2\right)\left(\frac{x^2-1.5x+0.56}{0.06}\right).$$

We can now approximate the function $f(0.9)$ using the above polynomial,

$$\Rightarrow P(0.9) \approx 0.438413527.$$

c) Lagrange interpolating polynomial of degree three using $x_0 = 0.6$, $x_1 = 0.7$, $x_2 = 0.8$, and $x_3 = 1.0$. The Lagrange interpolating polynomial of degree three is of the form,

$$\Rightarrow P(x) = f(x_o)L_{3,0}(x) + f(x_1)L_{3,1}(x) + f(x_2)L_{3,2}(x) + f(x_3)L_{3,3}(x).$$

Compute $L_{3,0}(x)$, $L_{3,1}(x)$, $L_{3,2}(x)$ $L_{3,3}(x)$,

$$\Rightarrow L_{3,0}(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-0.7)(x-0.8)(x-1.0)}{(0.6-0.7)(0.6-0.8)(0.6-1.0)} = \frac{x^3-2.5x^2+2.06x-0.56}{-0.008},$$

$$\Rightarrow L_{3,1}(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0.7)(x-0.8)(x-1.0)}{(0.7-0.6)(0.7-0.8)(0.7-1.0)} = \frac{x^3-2.4x^2+1.88x-0.48}{0.003},$$

$$\Rightarrow L_{3,2}(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-0.6)(x-0.7)(x-1.0)}{(0.8-0.6)(0.8-0.7)(0.8-1.0)} = \frac{x^3-2.3x^2+1.72x-0.42}{-0.004},$$

$$\Rightarrow L_{3,3}(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-0.6)(x-0.7)(x-0.8)}{(1.0-0.6)(1.0-0.7)(1.0-0.8)} = \frac{x^3-2.1x^2+1.46x-0.336}{0.024}.$$

Then,

$$\Rightarrow P(x) = f(0.6)\left(\frac{x^3-2.5x^2+2.06x-0.56}{-0.008}\right) + f(0.7)\left(\frac{x^3-2.4x^2+1.88x-0.48}{0.003}\right) + f(0.8)\left(\frac{x^2-1.5x+0.56}{0.06}\right)$$
$$+ f(1.0)\left(\frac{x^3-2.1x^2+1.46x-0.336}{0.024}-0.004\right)$$

$$= \sin\left(e^{0.6}-2\right)\left(\frac{x^3-2.5x^2+2.06x-0.56}{-0.008}\right) + \sin\left(e^{0.7}-2\right)\left(\frac{x^3-2.4x^2+1.88x-0.48}{0.003}\right)$$
$$+ \sin\left(e^{0.8}-2\right)\left(\frac{x^2-1.5x+0.56}{0.06}\right) + \sin\left(e^{1.0}-2\right)\left(\frac{x^3-2.1x^2+1.46x-0.336}{0.024}\right).$$

We can now approximate the function $f(0.9)$ using the above polynomial,

$$\Rightarrow P(0.9) \approx 0.441985002.$$

$\square$

Problem 2

Use the error formula to find and upper bound for the error in problem 1 parts a) and b) and compared the bound to the actual error.

a) For the initial step, we'll find the derivatives for the function f,

$$\Rightarrow f(x) = \sin(e^x - 2),$$

$$\Rightarrow f'(x) = e^x \cos(e^x - 2),$$

$$\Rightarrow f''(x) = e^x \cos(e^x - 2) - e^{2x} \sin(e^x - 2).$$

With the derivatives, we an now find the error term for the Lagrange interpolating polynomial of degree one.

$$\Rightarrow \frac{f''(\xi(x))}{2!} = \frac{e^{(\xi(x))} \cos(e^{(\xi(x))} - 2) e^{2(\xi(x))} \sin(e^{(\xi(x))} - 2)}{2}(x - x_0)(x - x_1).$$

$$= \frac{e^{(\xi(x))} \cos(e^{(\xi(x))} - 2) e^{2(\xi(x))} \sin(e^{(\xi(x))} - 2)}{2}|x^2 - 1.8x + .8|.$$

To maximize the first part of the term, we can evaluate it at $x_1 = 1.0$ which gives us its max over the interval,

$$\Rightarrow \left| \frac{e^1 \cos(e^1 - 2) e^2 \sin(e^1 - 2)}{2} \right| \approx 1.407990999.$$

Then to maximize the second term, we can evaluate it at the critical point .9,

$$\Rightarrow |(.9)^2 - 1.8(.9) + .8| = | - 0.01|.$$

Then the error bound can be estimated using,

$$\Rightarrow 1.407990999| - 0.01| = 0.01407991.$$

For comparison, we can find the actual error using,

$$\Rightarrow |f(0.9) - P_1(0.9)| \approx 0.002729643.$$

b) For the initial step, we'll find the third derivative for the function f,

$$\Rightarrow f'''(x) = -e^{3x}\cos(e^x - 2) - 2e^{2x}\sin(e^x - 2) - e^{2x}\sin(e^x - 2) + e^x\cos(e^x - 2).$$

With the derivatives, we an now find the error term for the Lagrange interpolating polynomial of degree one. To find the estimated error bound, we need to maximize $\frac{f'''}{3!}$ by again evaluating at $x_3 = 1.0$. The logic is the same as in part a) and won't be repeated for the sake of brevity,

$$\Rightarrow \sup_{0.6<x<1.0} \left| \frac{f'''(\xi(x))}{3!} \right| = 4.61.$$

Then to maximize the $(x - 0.7)(x - 0.8)(x - 1.0)$ we can evaluate it at the critical point .9,

$$\begin{aligned} \Rightarrow |(x - 0.7)(x - 0.8)(x - 1.0)| &= |x^3 - 2.5x^2 + 2.06x - 0.56| \\ &= |(0.9)^3 - 2.5(0.9)^2 + 2.06 - 0.56| \\ &= | - 0.002|. \end{aligned}$$

Then the error bound can be estimated using,

$$\Rightarrow 4.61| - 0.002| = 0.00922.$$

For comparison, we can find the actual error using,

$$\Rightarrow |f(0.9) - P_2(0.9)| \approx 0.005178938.$$

$\square$

Problem 3

Use algebra to find the equation of a linear function which passes through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$. Show that this function is the first Lagrange interpolating polynomial of $f(x)$ at x_0 and x_1.

To show algebraically, we can use the point-slope form of a line as a starting point and begin to work backwards using substitution.

$$\Rightarrow y - y_1 = m(x - x_1) \tag{1}$$

From (1), we can substitute y_1 and x_1 with one of our points,

$$\begin{aligned} \Rightarrow y - f(x_0) &= m(x - x_0) \\ y &= m(x - x_0) + f(x_0) \\ f(x) &= m(x - x_0) + f(x_0) \quad \text{(we let y=f(x)).} \end{aligned}$$

Next, we can solve for our slope, m, in order to substitute it into the point-slope form equation.

$$\Rightarrow m = \frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Then,

$$\begin{aligned} \Rightarrow f(x) &= m(x - x_0) + f(x_0) \\ f(x) &= \frac{f(x_1) - f(x_0)}{x_1 - x_0}\,(x - x_0) + f(x_0) \\ f(x) &= \frac{f(x_1)(x - x_0)}{x_1 - x_0} - \frac{f(x_0)(x - x_0)}{x_1 - x_0} + f(x_0) \\ f(x) &= \frac{f(x_1)(x - x_0)}{x_1 - x_0} - \frac{f(x_0)(x - x_0)}{x_1 - x_0} + \frac{f(x_0)(x_1 - x_0)}{x_1 - x_0} \\ f(x) &= \frac{f(x_1)(x - x_0)}{x_1 - x_0} + \left[\frac{(-x + x_0)}{x_1 - x_0} + \frac{(x_1 - x_0)}{x_1 - x_0}\right] f(x_0) \\ f(x) &= \frac{f(x_1)(x - x_0)}{x_1 - x_0} + \left[\frac{(-x + x_1)}{x_1 - x_0}\right] f(x_0) \\ f(x) &= \frac{f(x_1)(x - x_0)}{x_1 - x_0} + \left[\frac{(x - x_1)}{x_0 - x_1}\right] f(x_0) \\ f(x) &= P_1(x) = f(x_1)\frac{(x - x_0)}{x_1 - x_0} + f(x_0)\frac{(x - x_1)}{x_0 - x_1}. \end{aligned} \tag{2}$$

Observe (2) is equivalent to the Lagrange interpolating polynomial of degree one.

$\square$