MAD4401 - Numerical Analysis
Homework 3

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Problem 1

Consider the function $f(x) = \cos x$.

a) Find the 3rd Taylor polynomial approximation $P_3(x)$ of $f(x)$ at $x_0 = \frac{\pi}{3}$.

Let $f(x) = \cos x$. To find the 3rd Taylor Polynomial, start by computing the necessary derivatives of $f(x)$,

$$\begin{aligned}
&\Rightarrow f(x) = \cos x \\
&\Rightarrow f'(x) = -\sin x \\
&\Rightarrow f''(x) = -\cos x \\
&\Rightarrow f'''(x) = \sin x.
\end{aligned}$$

Then evaluate the function and each derivative at $x_0 = \frac{\pi}{3}$,

$$\Rightarrow f(x_0) = f\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\Rightarrow f'(x_0) = f'\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow f''(x_0) = f''\left(\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

$$\Rightarrow f'''(x_0) = f'''\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Then the 3rd Taylor polynomial is given by,

$$\Rightarrow P_3(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{\frac{1}{2}}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{\frac{\sqrt{3}}{2}}{3!}\left(x - \frac{\pi}{3}\right)^3$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3.$$

b) Show that the error associated to approximating $f(\frac{\pi}{3}+h)$ by $P_3(\frac{\pi}{3}+h)$ is of order $O(h^4)$.

To compute the estimate error of approximating $f(x)$ using $P_3(x)$, use the remainder given by,

$$\Rightarrow R_3(x) = \frac{f^{n+1}(\xi)}{n+1!}(x-x_0)^{n+1}$$

$$= \frac{f^{(4)}(\xi)}{4!}\left(x-\frac{\pi}{3}\right)^4$$

$$= \frac{f^{(4)}(\xi)}{24}\left(x-\frac{\pi}{3}\right)^4. \tag{1}$$

By Taylor's Theorem,

$$\Rightarrow f(x) = P_3(x) + R_3(x) \to |R_3(x)| = |f(x) - P_3(x)|, \text{ for } x = \left(\frac{\pi}{3}+h\right). \tag{2}$$

Using (1) and (2), we can compute the error by,

$$\Rightarrow \left|f\left(\frac{\pi}{3}+h\right) - P_3\left(\frac{\pi}{3}+h\right)\right| = \left|R_3\left(\frac{\pi}{3}\right)\right|$$

$$= \left|\frac{f^{(4)}(\xi)}{24}\left(x-\frac{\pi}{3}\right)^4\right|$$

$$= \left|\frac{\cos(\xi)}{24}\left(\frac{\pi}{3}+h-\frac{\pi}{3}\right)^4\right|$$

$$= \left|\frac{\cos(\xi)}{24}h^4\right|$$

$$= \frac{|\cos(\xi)|}{24}\cdot h^4$$

$$\leq \frac{1}{24}\cdot h^4$$

$$= \frac{h^4}{24}.$$

The error is less than the constant h^4. Therefore it is of order $O(h^4)$.

$\square$

Problem 2

Consider the function $f(x) = e^{5x-3}$.

a) Find the 4th Taylor polynomial approximation $P_4(x)$ of $f(x)$ at $x_0 = 0$.

Let $f(x) = e^{5x-3}$. To find the 4rd Taylor Polynomial, start by computing the necessary derivatives of $f(x)$,

$$\begin{aligned}
&\Rightarrow f(x) = e^{5x-3} \\
&\Rightarrow f'(x) = 5e^{5x-3} \\
&\Rightarrow f''(x) = 25e^{5x-3} \\
&\Rightarrow f'''(x) = 125e^{5x-3} \\
&\Rightarrow f''''(x) = 625e^{5x-3}
\end{aligned}$$

Then evaluate the function and each derivative at $x_0 = 0$,

$$\Rightarrow f(x_0) = f(0) = e^{5(0)-3} = e^{-3}$$

$$\Rightarrow f'(x_0) = f(0) = 5e^{5(0)-3} = 5e^{-3}$$

$$\Rightarrow f''(x_0) = f(0) = 25e^{5(0)-3} = 25e^{-3}$$

$$\Rightarrow f'''(x_0) = f(0) = 125e^{5(0)-3} = 125e^{-3}$$

$$\Rightarrow f''''(x_0) = f(0) = 625e^{5(0)-3} = 625e^{-3}$$

Then the 4rd Taylor polynomial is given by,

$$\Rightarrow P_3(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \frac{f''''(x_0)}{4!}(x - x_0)^4$$

$$= e^{-3} + 5e^{-3}x + \frac{25e^{-3}}{2}x^2 + \frac{125e^{-3}}{3!}x^3 + \frac{625e^{-3}}{4!}x^4$$

$$= e^{-3} + 5e^{-3}x + \frac{25e^{-3}}{2}x^2 + \frac{125e^{-3}}{6}x^3 + \frac{625e^{-3}}{24}x^4.$$

b) Show that if $0 \leq h \leq 1$ the error associated to approximating $f(h)$ by $P_4(h)$ is of order $O(h^5)$.

To compute the estimate error of approximating $f(x)$ using $P_4(x)$, use the remainder given by,

$$\Rightarrow R_4(x) = \frac{f^{n+1}(\xi)}{n+1!}(x-x_0)^{n+1}$$

$$= \frac{f^{(5)}(\xi)}{5!}(x-0)^5$$

$$= \frac{f^{(5)}(\xi)}{120}(x-0)^5. \tag{3}$$

By Taylor's Theorem,

$$\Rightarrow f(x) = P_3(x) + R_3(x) \rightarrow |R_3(x)| = |f(x) - P_3(x)|, \text{ for } x = (0+h). \tag{4}$$

Using (3) and (4), we can compute the error by,

$$\Rightarrow |f(0+h) - P_3(0+h)| = |R_3(0+h)|$$

$$= \left| \frac{f^{(5)}(\xi)}{120}(x-0)^5 \right|$$

$$= \left| \frac{e^{5(\xi)-3}}{120}(0+h-0)^5 \right|$$

$$= \left| \frac{e^{5(\xi)-3}}{120}(h)^5 \right|$$

$$= \left| \frac{e^{5(\xi)-3}}{120} \right| \cdot h^5$$

$$0 \leq \frac{\left|e^{5(\xi)-3}\right|}{120} \cdot h^5 < 1, \text{ for } 0 \leq h \leq 1.$$

Therefore the error approximation is of order $0(h^5)$.

$\square$

Problem 3

Use Newton's method to find solutions accurate to within 10^{-4} for the following problem,

$$f(x) = x^3 - 2x^2 - 5 = 0, \; x \in [1, 4].$$

In order to use Newton's method, $f \in C^2[a, b]$, $f(p) = 0$, and $f'(p) \neq 0$. If these hypotheses hold, then we can find some $\delta > 0$ such that Newton's method generates a sequence that converges to p for any initial guess p_0 that is within the δ-neighborhood of p. In other words, the initial guess p_0 should be "close" to the actual root p.

To compute an approximation for p_n, iteratively use

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}. \tag{5}$$

The first derivative of $f(x)$ is $f'(x) = 3x^2 - 4x$. Using graphical methods, $f(x)$ has a unique root on $[1, 4]$ that appears to be between 2 and 3. So, for the initial attempt, let $p_0 = 2$, then using (5)

$$\Rightarrow p_1 = 2 - \frac{f(2)}{f'(2)} = 3.25$$

$$p_2 = 3.25 - \frac{f(3.25)}{f'(3.25)} \approx 2.81103$$

$$p_3 = 2.81103 \frac{f(2.81103)}{f'(2.81103)} \approx 2.69798$$

$$p_4 = 2.69798 - \frac{f(2.69798)}{f'(2.69798)} \approx 2.69067$$

$$p_5 = 2.69067 - \frac{f(2.69067)}{f'(2.69067)} \approx 2.69065.$$

Using $p_0 = 2$, the sequence converged to within 10^{-4} error around 5 iterations.

Using a computer, different values of p_0 were selected to see how they compared. When $p_0 = 2.5$, the sequence converged quicker.

```
N                Pn
--------------------------------------------
1                2.7142857142857144
2                2.6909515167228415
3                2.6906474992568943

After 3 iterations and error tolerance 0.0001
the approximated root is 2.6906474992568943.
```

When $p_0 = 3$ the sequence converged the same as $p_0 = 2.5$.

```
N               Pn
---------------------------------------------
1               2.7333333333333334
2               2.6916247257710673
3               2.6906479769300162

After 3 iterations and error tolerance 0.0001
the approximated root is 2.6906479769300162.
```

However, to the point that p_0 should be "close" to the root, when $p_0 = 1$, the sequence took longer to converge with the given accuracy; however, it still ended up with a good approximation.

```
N               Pn
---------------------------------------------
1               -5.0
2               -3.105263157894737
3               -1.7937858983897073
4               -0.7712643594642501
5               0.594038366674229
6               -3.577575730649256
7               -2.1282993172828815
8               -1.0560162494472352
9               0.054743278689767916
10              -23.784511787562405
11              -15.643125422170678
12              -10.217705491762906
13              -6.6011112023606735
14              -4.184040522642873
15              -2.5486433237950648
16              -1.3847433572227172
17              -0.3671392188646079
18              2.4728324557293426
19              2.7222804675706396
20              2.691188664218553
21              2.6906476102946812

After 21 iterations and error tolerance 0.0001
the approximated root is 2.6906476102946812.
```

□

Problem 4

Consider the equation $4x^2 - e^x - e^{-x} = 0$. Use Newton's method to approximate the solution to within 10^{-5} with the given values of p_0.

a) $p_0 = 3$

To solve the solution by hand, the same method can be used as in Problem 3, take the first derivative of the given function and compute the n^{th} term of the sequence using,

$$f'(x) = 8x + e^{-x} - e^x,$$

and

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}.$$

So, for $p_0 = 3$,

```
N               Pn
----------------------------------------------
1               -1.0019361613022864
2               -0.8385205483091717
3               -0.8246057692142937
4               -0.8244985916713463

After 4 iterations and error tolerance 1e-05
the approximated root is -0.8244985916713463.
```

The approximation for p is ≈ -0.82 when $p_0 = 3$.

b) $p_0 = 1$

The same method is used to compute an approximation for p when $p_0 = 1$,

```
N               Pn
----------------------------------------------
1               0.8382471119158565
2               0.8246016670341486
3               0.8244985911923535

After 3 iterations and error tolerance 1e-05
the approximated root is 0.8244985911923535.
```

Here, the approximation is ≈ 0.82. Looking the graph of $f(x)$, this would make sense as to why different values of p_0 would result in one of the possible roots for this function.

$\square$