Taylor's Theorem

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Theorem (Taylor's Theorem)**.** *Let f be n times continuously differentiable on $[a, b]$ and suppose $f^{(n+1)}$ derivative exists on $[a, b]$. Let $x_0 \in [a, b]$. Then for $x \in [a, b]$, $f(x)$ can be expressed as Taylor expansion $f(x) = P_n(x) + R_n(x)$ where*

$$\begin{aligned} \Rightarrow P_n(x) &= \sum_{j=0}^{n} \frac{f^j x_0}{j!}(x - x_0)^j \\ &= f(x_0) + f'(x_0)(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \end{aligned}$$

and

$$\Rightarrow R_n(x) = \int_{x_0}^{x} \frac{f^{(n+1)}(s)}{n!}(x - s)^n ds$$

$$= \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}$$

where $\xi(x)$ is between x_0 and x.

In the above theorem,

- $P_n(x)$ is called the *Taylor Polynomial* of f with center x_0
- $R_n(x)$ is called the *Taylor Remainder* of $P_n(x)$ (truncation error of $P_n(x)$

Also note, when the center $x_0 = 0$, the Taylor Polynomial is known as the nth *Maclaurin Polynomial.*