MAD4401 - Numerical Analysis

Homework 2

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Problem 1

Use algebraic manipulation to show that each of the following functions has a fixed-point p at precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

a) $g_2(x) = \left(\frac{x+3-x^4}{2}\right)^{\frac{1}{2}}$

A fixed point for $f(p) = 0$ exists if $f(p) = p - g(p) = 0$. To show this, show that $p = g(p)$.

$$\Rightarrow p - g(p) = 0$$

$$g(p) = p$$

$$\left(\frac{p+3-p^4}{2}\right)^{\frac{1}{2}} = p$$

$$\frac{p+3-p^4}{2} = p^2$$

$$p + 3 - p^4 = 2p^2$$

$$p^4 + 2p^2 - p - 3 = 0.$$

Hence, $p - g(p) = 0$, therefore f has a fixed point at p.

b) $g_3(x) = \left(\frac{x+3}{x^2+2}\right)^{\frac{1}{2}}$

$$\Rightarrow p - g(p) = 0$$

$$\left(\frac{p+3}{p^2+2}\right)^{\frac{1}{2}} = p$$

$$\frac{p+3}{p^2+2} = p^2$$

$$p+3 = p^2(p^2+2)$$

$$p+3 = p^4+2p^2$$

$$p^4+2p^2-p+3 = 0$$

Hence, $p - g(p) = 0$, therefore f has a fixed point at p.

□

Problem 2

a) Perform four iterations of fixed-point iteration, if possible, on each of the functions defined in Exercise 1 with starting guess $p_0 = 1$.

- $g_2(x) = \left(\frac{x+3-x^4}{2}\right)^{\frac{1}{2}}$

```
n        P
___________________________________
01       1.224744871391589
02       0.9936661590774817
03       1.228568645274987
04       0.9875064291508866

The approximated value of P after 4 iterations is: 0.9875064291508866
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- $g_3(x) = \left(\frac{x+3}{x^2+2}\right)^{\frac{1}{2}}$

```
n        P
___________________________________
01       1.1547005383792515
02       1.116427409872122
03       1.1260522330022757
04       1.1236388847132548

The approximated value of P after 4 iterations is: 1.1236388847132548
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b) Which function do you think gives the best approximation to the solution? Why?

Of the two functions above, I believe g_3 gives the better approximation to the solution. Using the bisection method to an accuracy of 10^{-8}, one can find the approximated root is 1.124123029410839. g_3 is closer than the fourth approximation for g_2. g_3 also converges faster than g_2, if g_2 converges at all.

□

Problem 3

a) Use Theorem 2.3 (fixed-point theorem) to show that

$$g(x) = \frac{1}{10}\left(\frac{5}{x^2} + 2x + 9\right)$$

has a unique fixed point on $[1, 3]$.

<u>Existence</u>

To show existence of a unique point in the interval, let $x \in [1, 3]$. Then,

$$\Rightarrow x^2 = (x)(x) > 0.$$

Thus, there are no values of $x \in [1, 3]$ such that $g(x)$ does not exist. Therefore $g(x)$ is continuous on $[1, 3]$. To show that $g(x) \in [a, b], \forall x \in [a, b]$, we need to find the absolute minimum and absolute maximum. This occurs either at the endpoints of the interval or when $g'(x) = 0$. Evaluating $g(x)$ at the endpoints,

$$\Rightarrow g(1) = \frac{1}{10}\left(\frac{5}{1^2} + 2(1) + 9\right)$$

$$= \frac{16}{10} = 1.6$$

and

$$\Rightarrow g(3) = \frac{1}{10}\left(\frac{5}{3^2} + 2(3) + 9\right)$$

$$= \frac{140}{90} \approx 1.555.$$

Then, $g'(x) = \frac{1}{5} - \frac{1}{x^3} = 0$.

$$\Rightarrow \frac{1}{5} - \frac{1}{x^3} = 0$$

$$-\frac{1}{x^3} = -\frac{1}{5}$$

$$-x^3 = -5$$

$$x^3 = 5$$

$$x = 5^{\frac{1}{3}} \approx 1.7099.$$

So the absolute minimum and maximum values lie within $[1, 3]$. Thus $g(x) \in [a, b], \forall x \in [a, b]$. Therefore, there exists at least one fixed point for $g(x)$ on $[1, 3]$.

Uniqueness

To show uniqueness of a fixed point on [1,3], we must show that $g'(x)$ exists on $(1, 3)$ and $|g'(x)| \leq k, \forall x \in (a, b)$ and $0 < k < 1$. Since $g'(x) = \frac{1}{5} - \frac{1}{x^3}$, observe the only value of x that would cause a domain error would be 0 and $0 \notin (1, 3)$. Thus, $g'(x)$ is exists on $(1, 3)$.

To show that $|g'(x)| \leq k, \forall x \in (a, b)$ and $0 < k < 1$, observe that for all values of $x \in (1, 3)$, $g'(x)$ should lie between $g'(1)$ and $g'(3)$ since $g'(x)$ is strictly increasing over $(1, 3)$. Then evaluating $g'(1)$ and $g'(3)$,

$$\Rightarrow |g'(1)| = \left|\frac{1}{5} - 1\right|$$

$$= \left|-\frac{4}{5}\right|$$

$$= 0.8$$

and

$$\Rightarrow |g'(3)| = \left|\frac{1}{5} - \frac{1}{3^3}\right|$$

$$= \left|\frac{22}{135}\right|$$

$$\approx 0.162962963.$$

Since $g'(x)$ is strictly increasing over $(1, 3)$, every value $g'(x)$ in that interval should be between $g'(1)$ and $g'(3)$. Therefore, $\forall x \in (1, 3)$, $|g'(x)| \leq k$, $0 < k < 1$.

Thus there exists at least one fixed point on $[1, 3]$ and that fixed point is unique. Therefore by the Fixed-Point Theorem, for any $p_0 \in [1, 3]$, the sequence $p_n = g(p_{n-1}$ will converge to this unique fixed-point.

b) Use Corollary 2.5 to estimate the number of iterations required to find an approximation to the fixed point accurate to within 10^-5 using fixed-point iteration with any starting guess p_0 in the interval $[1, 3]$.

From previous results, the maximum value for $|g'(x)| = 0.8 \leq k$. If we set the value of $k = 0.8$ we can use

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

to estimate the number of iterations to achieve an accuracy of 10^{-5}. Let $\max\{p_0 - a, b - p_0\} = \frac{3-1}{2} = 1$. Then,

$$\Rightarrow k^n \max\{p_0 - a, b - p_0\} < 10^{-5}$$

$$0.8^n(1) < 10^{-5}$$

$$0.8^n < \frac{10^{-5}}{1}$$

$$n \log 0.8 < \log 10^{-5}$$

$$n < \frac{\log 10^{-5}}{\log 0.8}$$

$$\approx 51.59425579.$$

Therefore about 52 iterations are needed. Notice k is fairly close to 1 indicating the sequence may converge slowly.

c) Use fixed-point iteration starting with $p_0 = 2$ to find an approximation to the fixed point accurate within 10^{-5}.

```
01      1.425
02      1.4312296091104955
03      1.43033670439075
04      1.4304629717248538
05      1.430445081365641

The approximated value of P after 5 iterations to 0.000010 accuracy is: 1.4304476154908503
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It appears the fixed-point iteration achieved and accuracy of 10^{-5} after 5 iterations.

$\square$