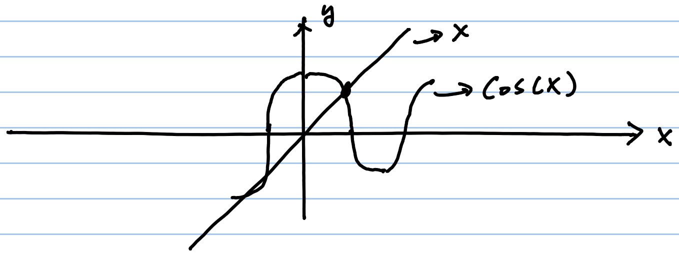
Example:

Carry out 3 iterations of the bisection method for the function $f(x) = x - \cos(x)$ on $[0, \frac{\pi}{2}]$



1. $f(0) = 0 - (1) = \underline{-1}$ < 0 , $f(\frac{\pi}{2}) = \frac{\pi}{2} - \cos(\frac{\pi}{2}) = \frac{\pi}{2} - 0 = \underline{\frac{\pi}{2}}$ > 0

$\Rightarrow f(0) \cdot f(\frac{\pi}{2}) = (-1) \cdot (\frac{\pi}{2}) = -\frac{\pi}{2} \underline{< 0}$

Based on IVT, there exist at least one root. on $[0, \frac{\pi}{2}]$

2. Let $a_1 = 0$, $b_1 = \frac{\pi}{2}$, and

$P_1 = \frac{a_1 + b_1}{2}$ (Midpoint)

$= \frac{\pi}{4}$

$\Rightarrow f(P_1) = f(\frac{\pi}{4}) = \frac{\pi}{4} - \cos(\frac{\pi}{4})$

$= \frac{\pi}{4} - \frac{\sqrt{2}}{2} \approx 0.0782913$

$\Rightarrow f(a_1) \cdot f(P_1) = \overset{-}{(-1)} \cdot \overset{+}{(0.0782913)} = ? < 0$

IVT ensures at least a zero on $[a_1, P_1]$

3. Let $a_2 = 0$, $b_2 = P_1 = \frac{\pi}{4}$,

$P_2 = \frac{a_2 + b_2}{2} = \frac{\pi}{8}$

$\Rightarrow f(P_2) = f(\frac{\pi}{8}) = \frac{\pi}{8} - \cos(\frac{\pi}{8}) = -0.531180 < 0$

$f(a_2) \cdot f(P_2) > 0$

$f(b_2) \cdot f(P_2) < 0$ (v), IVT ensures at leasts one root on $[P_2, b_2]$

$\Rightarrow [\frac{\pi}{8}, \frac{\pi}{4}]$

4. Let $a_3 = p_2 = \frac{\pi}{8}$, $b_3 = p_1 = \frac{\pi}{4}$,

$$p_3 = \frac{a_3 + b_3}{2} = \frac{\frac{\pi}{8} + \frac{\pi}{4}}{2} = \frac{3\pi}{16}$$

3rd iterate

$\square$

✱ Can we predict the number of iterations needed to ensure the error of approximation is small?

i.e.,
$$|p_n - p| < \varepsilon$$

Theorem 2.1.

Suppose that $f \in C[a,b]$, and $f(a)\cdot f(b) < 0$ the Bisection method generates a sequence

$$\{p_n\} = \{p_1, p_2, p_3, \dots\}$$

such that $p_n \to p$ where $f(p) = 0$ and

$$|p_n - p| \le \frac{b-a}{2^n}, \quad \text{when } n \geq 1$$

✱ The THM states that the size of interval $[a,b]$ is halved at each step.

Proof: For each $n \geq 1$, we have

$$b_n - a_n = \frac{b-a}{2^{n-1}}, \quad \text{and } p \in (a_n, b_n)$$

since $p_n = \frac{b_n - a_n}{2}$, then

$$|p_n - p| \le \frac{b_n - a_n}{2} = \frac{b-a}{2^n}$$

$\square$

Ex.

How many iterations of Bisection are needed to approximate the root/zero of $f(x) = x - \cos(x)$ on $[0, \frac{\pi}{2}]$ to within 10^{-6} accuracy?

\# • Based on THM 2.1, we know

$$|P_n - P| \leq \frac{b-a}{2^n} = \frac{\frac{\pi}{2} - 0}{2^n} = \frac{\pi}{2^{n+1}}$$

• we want the accuracy ε

$$|P_n - P| \leq \varepsilon = 10^{-6}$$

• Find n, such that

$$\frac{\pi}{2^{n+1}} \leq 10^{-6}$$

$\Leftrightarrow$ Error bound is less than tolerance

$$\Rightarrow \frac{\pi}{10^{-6}} \leq 2^{n+1}$$

$$\Rightarrow 2^{n+1} \geq 10^6 \cdot \pi$$

$$\Rightarrow \ln(2^{n+1}) \geq \ln(10^6 \cdot \pi) = \ln(10^6) + \ln(\pi)$$

$$\Rightarrow (n+1) \cdot \ln(2) \geq 6 \cdot \ln(10) + \ln(\pi)$$

$$\Rightarrow n+1 \geq \frac{6 \cdot \ln(10) + \ln(\pi)}{\ln(2)}$$

$$\Rightarrow n > \frac{6 \cdot \ln(10) + \ln(\pi)}{\ln(2)} - 1 \approx 20.58\ldots$$

\# We need $n \geq 21$ to guarantee the accuracy of the root of the approximation, P_n, smaller than 10^{-6}.

Ex.

Consider $g(x) = \frac{1}{2}(10 - x^2)^{1/2}$ on $[1, 2]$

a) Show $g(x)$ has a unique fixed point on $[1,2]$

b) How many iteration are needed to ensure fixed point iteration converges to the fixed point, P, for any $P_0 \in [1, 2]$ with accuracy 10^{-6}

a) we need to show

① $g(x)$ is continuous on $[a, b]$, and

$$1 \leq g(x) \leq 2$$

② $|g'(x)| \leq K$ for $0 < K < 1$ on $(1, 2)$

- by inspection $g(x) = \frac{1}{2}(10 - x^2)^{1/2}$ is continuous on $[1, 2]$.

- To show $1 \leq g(x) \leq 2$ on $[1, 2]$, Let's show the max/min of $f(x)$ on $[1, 2]$ are between 1 and 2.

Critical Points

$g'(x) = \frac{-x}{2\sqrt{10 - x^2}}$ ① $\Rightarrow g'(x) = 0 \Rightarrow x = 0$

not in the interval

② $10 - x^2 = 0 \Rightarrow x = \pm\sqrt{10}$

Plug in critical points and end points

$g(1) = 1.5$ $\uparrow$ max

$g(2) \approx 1.224745$ $\uparrow$ min

*. the max and min value of $g(x)$ are between 1 and 2.

② To show $|g'(x)| \leq K < 1$

We will show the magnitude of max and min of $g'(x)$ is less than 1.

* critical points of $g'(x)$

$$g''(x) = -\frac{5}{(10-x^2)^{3/2}}$$

$\Rightarrow g''(x) = 0 \Rightarrow$ Nowhere

$10 - x^2 = 0 \Rightarrow x = \pm\sqrt{10}$ (not in the interval)

Plug in the critical points and end points

$g'(1) = -0.16666\cdots7.$

$g'(2) = -0.408248$ ↳ max magnitude

Thus $|g'(x)| \leq K$ where $K = 0.408248$.

* by THM 2.3 (2.4), there exists a unique fixed point. on $[1,2]$

b) Error estimate from Corollary 2.5

$$|P_n - P| \le K^n \max\{P_0 - a, b - P_0\}$$

$k = 0.408248$

for any $P_0 \in [a,b]$ this will $\le \frac{b-a}{2}$

Thus $|P_n - P| \le (0.408248)^n \cdot \frac{1}{2}(2-1)$

we need some n such that

0.4082482

$$\frac{1}{2}(0.4808248)^n \le 10^{-6}$$

$$\Rightarrow \quad 0.4808248^n \le 2\times10^{-6}$$

$$\Rightarrow \quad n \cdot \ln(0.4808248) \le \ln(2\times10^{-6})$$

$$\Rightarrow \quad n \le \frac{\ln(2\times10^{-6})}{\ln(0.4808248)} \approx 17.920549.$$

In other words, for function $g(x)$. fixed-point method need 18 iterations to ensure convergence with accuracy 10^{-6}.

Newton's Method

- To approximate p such that $f(p)=0$ (root)

→ Generate a series of approximations by

$P_0 \leftarrow$ given

$$P_1 = P_0 - \frac{f(P_0)}{f'(P_0)}$$

$$P_2 = P_1 - \frac{f(P_1)}{f'(P_1)}$$

$$P_3 = P_2 - \frac{f(P_2)}{f'(P_2)}$$

$$\vdots$$

$$P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}$$

Ex. Carry out 3 steps of Newton's method for

$$f(x) = x - \cos(x)$$

starting with $P_0 = 0$.

step 1. $f'(x) = ?$

$$f'(x) = 1 + \sin(x)$$

step 2 Newton Method.

$P_0 = 0$

$$P_1 = P_0 - \frac{f(P_0)}{f'(P_0)} = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{(-1)}{1} = 1$$

$$P_2 = P_1 - \frac{f(P_1)}{f'(P_1)} = 1 - \frac{f(1)}{f'(P_1)} = 1 - \frac{1-\cos(1)}{1+\sin(1)} \approx 0.750364$$

$$P_3 = P_2 - \frac{f(P_2)}{f'(P_2)} = 0.750364 - \frac{1-\cos(0.750364)}{1+\sin(0.750364)}$$

$$= 0.739113$$

THM 1.14. Taylor THM (P8)

Suppse $f \in C[a,b]$, $f^{n+1}(x)$ exists on $[a,b]$, and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x)$ between x_0 and x with

$$f(x) = P_n(x) + R_n(x)$$

n th Taylor polynomial.

$$\Rightarrow P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x-x_0)^n$$

$$= \sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x-x_0)^k$$

Remainder Term (truncation error)

$$\Rightarrow R_n(x) = \frac{f^{n+1}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}$$

Ex. Find the 3rd Taylor polynomial approximation of $\sin(x)$ at $x = \frac{\pi}{6}$. Use remainder term to estimate produced when using this to approximate $\sin(\frac{\pi}{6}+h)$, where h is small.

① $f(x) = \sin(x)$, $x_0 = \frac{\pi}{6}$.

$$P_3(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$

$$= f(\tfrac{\pi}{6}) + f'(\tfrac{\pi}{6})(x-\tfrac{\pi}{6}) + \frac{f''(\frac{\pi}{6})}{2!}(x-\tfrac{\pi}{6})^2 + \frac{f'''(x_0)}{3!}(x-\tfrac{\pi}{6})^3$$

$$P_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x-\tfrac{\pi}{6}) - \frac{1}{4}(x-\tfrac{\pi}{6})^2 - \frac{\sqrt{3}}{12}(x-\tfrac{\pi}{6})^3$$

Remainder:

$$R_3(x) = \frac{f^{(4)}(\xi)}{4!}\left(x-\frac{\pi}{6}\right)^4$$, where ξ is some unknown number between x to $\frac{\pi}{6}$.

$$= \frac{\sin(\xi)}{24}\left(x-\frac{\pi}{6}\right)^4$$

* what about at $x = \frac{\pi}{6} + h$

by Taylor's THM

$$f(x) = P_3(x) + R_3(x)$$

$$\Rightarrow |f(x) - P_3(x)| = |R_3(x)|$$ ← Error.

when $x = \frac{\pi}{6} + h$

$$\left|\sin\left(\frac{\pi}{6}+h\right) - P_3\left(\frac{\pi}{6}+h\right)\right|$$

$$= \left|\frac{\sin(\xi)}{24}\left(\frac{\pi}{6}+h-\frac{\pi}{6}\right)^4\right|$$

$$= \frac{|\sin(\xi)|}{24}(h)^4$$

$$= \frac{h^4}{24}|\sin(\xi)| \le \frac{h^4}{24}$$

- The error is less than $(\text{Constant}) \cdot h^4$.

- We say the error is of the order $O(h^4)$