BCA

MATHEMATICS

NEW SYLLABUS

For G.G.S.I.P. University

FIRST SEMESTER

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Contents

CHAPTER - 1

[Matrices & Determinants]

1.1 [Introduction To Matrices]

Definition :

An array of real numbers (or imaginary) arranged in m rows and n columns is called $m \times n$ matrix. It is represented by

$$A = [a_{ij}]_{m \times n} \qquad \text{Where } A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots a_{ij} \ldots\ldots a_{in} \\ a_{21} & a_{22} \ldots\ldots a_{2j} \ldots\ldots a_{2n} \\ \vdots & \vdots \quad\quad \vdots \quad\quad \vdots \\ a_{i1} & a_{i2} \ldots\ldots a_{ij} \ldots\ldots a_{in} \\ \vdots & \vdots \quad\quad \vdots \quad\quad \vdots \\ a_{m1} & a_{m2} \ldots\ldots a_{mj} \ldots\ldots a_{mn} \end{bmatrix}$$

a_{ij}'s are called elements of the matrix, $i = 1, 2 \cdots\cdots m, j = 1, 2 \cdots\cdots n$
(a_{ij} is the element in the i^{th} row and j^{th} column)

Example: $A = \begin{bmatrix} 2 & 3 \\ 7 & 0 \end{bmatrix}$ is a 2×2 matrix where $a_{11} = 2, a_{12} = 3\ a_{21} = 7, a_{22} = 0$.

1.2 [Types Of Matrices]

1. **Rectangular matrix :**

$m \times n$ ($m \neq n$) matrix is called Rectangular matrix. **Example :** $\begin{bmatrix} 1 & 2 \\ -3 & 1 \\ -7 & 0 \end{bmatrix}$ is 3×2 rectangular matrix

2. **Row matrix :**

Matrix with only one row is called row matrix **Example :** Matrix [1 2] is row matrix

3. **Column matrix :**

Matrix with only one column is called column matrix **Example :** $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is column matrix

4. **Square Matrix :**

Matrix in which number of rows is equal to the number of column (*i.e.* $m = n$) is called Square Matrix

Example : $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 1 & 2 \end{bmatrix}$ is 3×3 square matrix

5. **Diagonal Matrix :**

Diagonal matrix is a square matrix in which all the elements except the main diagonal elements are zero.

i.e. $a_{ij} = 0\ \forall\ i \neq j$ (a_{ij} wher $i = j$ are called diagonal elements)

Example : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is a diagonal matrix

6. Scaler Matrix :

Scaler Matrix is a square matrix in which all the diagonal elements are equal, all other elements being zeros.

Example : $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scaler matrix

7. Identity (Unit Matrix) :

Identity matrix is a square matrix $(a_{ij})_{n \times n}$ in which

1. $a_{ij} = 0 \ \forall \ i \neq j$
2. $a_{ij} = 1 \ \forall \ i$

i.e. Indentity matrix is a square matrix in which all the diagonal elements are unity and non-diagonal elements are zero.

Example : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3×3 identity matrix

8. Null Matrix :

A matrix in which all elements are equal to zero is called a null matrix

Example : $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a null matrix.

9. Upper Triangular Matrix :

A square matrix $A = [a_{ij}]$ is called an upper triangular matrix if $a_{ij} = 0 \ \forall \ i > j$

i.e. all elements below the main diagonals are zero.

Example : $A = \begin{bmatrix} 1 & 5 & 6 & 7 \\ 0 & 9 & 2 & 1 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is an upper triangular matrix.

10. Lower Triangular Matrix :

A square matrix $A = [a_{ij}]$ is called a lower triangular matrix if $a_{ij} = 0 \ \forall \ i < j$

i.e. all elements above the main diagonals are zero.

Example : $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 2 & 5 & 3 \end{bmatrix}$ is a lower triangular matrix.

11. Sub Matrix :

A matrix obtained by deleting the rows or columns (or both) of a matrix is called sub matrix.

Example : $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ is the sub matrix of

matrix $B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ by deleting 2nd row and 2nd column.

Examples

1. Construct a 2×2 matrix $B = [b_{ij}]$ where elements are given by : (i) $b_{ij} = \dfrac{(i-2j)^2}{2}$ (ii) $b_{ij} = \dfrac{1}{2}|-3i+j|$

Solution :

(i) $b_{11} = \dfrac{(1-2)^2}{2} = \dfrac{1}{2}$; $b_{12} = \dfrac{(1-4)^2}{2} = \dfrac{9}{2}$; $b_{21} = \dfrac{(2-2)^2}{2} = 0$; $b_{22} = \dfrac{(2-2\times 2)^2}{2} = \dfrac{4}{2} = 2.$ $\quad \therefore B = \begin{bmatrix} \frac{1}{2} & \frac{9}{2} \\ 0 & 2 \end{bmatrix}$

(ii) $b_{11} = \frac{1}{2}|-3+1| = 1$ $b_{12} = \frac{1}{2}|-3+2| = \frac{1}{2}$ $b_{21} = \frac{1}{2}|-6+1| = \frac{5}{2}$

$b_{22} = \frac{1}{2}|-3 \times 2 + 2| = 2$ $\therefore\ B = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{5}{2} & 2 \end{bmatrix}$

2. Construct a 2×3 matrix A where elements are given by $a_{ij} = \frac{3i - j}{2}$

Solution :

As $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

Where

$a_{11} = \frac{3-1}{2} = 1$ $a_{12} = \frac{3-2}{2} = \frac{1}{2}$ $a_{13} = \frac{3-3}{2} = 0$

$a_{21} = \frac{6-1}{2} = \frac{5}{2}$ $a_{22} = \frac{6-2}{2} = 2$ $a_{23} = \frac{6-3}{2} = \frac{3}{2}$ $\therefore\ A = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{5}{2} & 2 & \frac{3}{2} \end{bmatrix}$

1.3 [Equality Of Matrices]

Definition : Order of a matrix -

If A is a $m \times n$ matrix, then $m \times n$ is called the order of the matrix

Definition : Equality of Matrices -

Two matrices A and B are equal if and only if both matrices are of same order and each element of one matrix is equal to the corresponding element of other matrix.

Example : $\begin{bmatrix} 3 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 6-3 & -1+2 \\ 3 & 9-9 \end{bmatrix}$

Examples

3. Find x, y and z so that $A = B$ where $A = \begin{bmatrix} x-2 & 3 & 2z \\ 18z & y+2 & 6z \end{bmatrix}$, $B = \begin{bmatrix} y & z & 6 \\ 6y & x & 2y \end{bmatrix}$.

Solution:

As $\begin{bmatrix} x-2 & 3 & 2z \\ 18z & y+2 & 6z \end{bmatrix} = \begin{bmatrix} y & z & 6 \\ 6y & x & 2y \end{bmatrix}$ *i.e.* $a_{12} = b_{12} \Rightarrow 3 = z$

Also $a_{11} = b_{11} \Rightarrow x - 2 = y$ and $a_{21} = b_{21} \Rightarrow 18z = 6y \Rightarrow 54 = 6y \Rightarrow y = 9$.

Now as $x - 2 = y$ $\therefore\ x = y + 2$ *i.e.* $x = 9 + 2 = 11$ *i.e.* $x = 11, y = 9$ and $z = 3$

4. If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ 4x+6 & a-1 & 0 \\ b-3 & 3b & z+2c \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ 2x & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$. Obtain the values of a, b, c, x, y and z

Solution :

Let $A = \begin{bmatrix} x+3 & z+4 & 2y-7 \\ 4x+6 & a-1 & 0 \\ b-3 & 3b & z+2c \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 6 & 3y-2 \\ 2x & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$

As $A = B \therefore a_{11} = b_{11} \Rightarrow x + 3 = 0 \Rightarrow x = -3$

Also $a_{12} = b_{12} \Rightarrow z + 4 = 6 \Rightarrow z = 2$

$a_{22} = b_{22} \Rightarrow a - 1 = -3 \Rightarrow a = -2$

$a_{23} = b_{23} \Rightarrow 0 = 2c + 2 \Rightarrow c = -1$

$a_{31} = b_{31} \Rightarrow b - 3 = 2b + 4 \Rightarrow b = -7$

$a_{13} = b_{13} \Rightarrow 2y - 7 = 3y - 2 \Rightarrow y = -5$

Hence $a = -2, b = -7, c = -1, x = -3, y = -5$ and $z = 2$.

1·4 [Operation On Matrices]

(1) Addition of Matrices :

Two matrices A and B can be added only iff they have same number of rows and columns. Sum is denoted by $A + B$. It is done by adding corresponding elements of A and B.

Example : $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 6 & 9 \\ 1 & 0 & 0 \end{bmatrix}$ then $A + B = \begin{bmatrix} 8 & 8 & 12 \\ 5 & 5 & 6 \end{bmatrix}$

In general :

$A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ then $A + B = [a_{ij} + b_{ij}]_{m \times n}$

(2) Subtraction of Matrices :

Two matrices A and B can be subtracted only iff they have same number of rows and columns. Subtraction is denoted by $A - B$. It is done by subtracting elements of B from the corresponding elements of A.

Example : $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 0 & 1 \\ 0 & 3 & -1 \end{bmatrix}$ then $A - B = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix}$.

(3) Scaler Multiplication :

Let $A = [a_{ij}]$ be a $m \times n$ matrix and k be the scaler. Scaler multiplications is denoted by kA. It is done by multiplying every element of A with k.

Example : $A = \begin{bmatrix} 5 & 4 \\ 2 & 1 \end{bmatrix}$ then $2A = \begin{bmatrix} 10 & 8 \\ 4 & 2 \end{bmatrix}$.

(4) Multiplication of Matrices :

Two matrices A and B are multiplied for product AB iff the number of columns in A is same as the number of rows in B.

Remark :

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ be two matrices, then product AB is of the order $m \times p$.

Examples

5. If $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 \\ 9 & -1 \end{bmatrix}$ find : (i) AB (ii) BC.

Solution :

(i) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}$, $AB = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}$, $AB = \begin{bmatrix} 1\times1+0\times0 & 1\times5+0\times6 \\ 2\times1+1\times0 & 2\times5+1\times6 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 2 & 16 \end{bmatrix}$.

(ii) $BC = \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 9 & -1 \end{bmatrix}$, $BC = \begin{bmatrix} 1\times2+5\times9 & 1\times1-5\times1 \\ 0\times2+6\times9 & 0\times1+6\times(-1) \end{bmatrix} = \begin{bmatrix} 47 & -4 \\ 54 & -6 \end{bmatrix}$

6. If $A=\begin{bmatrix}1&2&-1\\2&0&3\\0&1&2\end{bmatrix}$ and $B=\begin{bmatrix}3&-1&1\\0&0&2\\4&-3&2\end{bmatrix}$ verify the result $(A+B)^2=A^2+AB+BA+B^2$.

Solution :

$$A=\begin{bmatrix}1&2&-1\\2&0&3\\0&1&2\end{bmatrix} \text{ and } B=\begin{bmatrix}3&-1&1\\0&0&2\\4&-3&2\end{bmatrix} \quad \therefore \quad A+B=\begin{bmatrix}4&1&0\\2&0&5\\4&-2&4\end{bmatrix}$$

$$\text{Now } (A+B)^2=(A+B)(A+B)=\begin{bmatrix}4&1&0\\2&0&5\\4&-2&4\end{bmatrix}\begin{bmatrix}4&1&0\\2&0&5\\4&-2&4\end{bmatrix}$$

$$=\begin{bmatrix}4\times4+1\times2+0\times4 & 4\times1+1\times0+0\times-2 & 4\times0+1\times5+0\times4\\ 2\times4+0\times2+5\times4 & 2\times1+0\times0+5\times-2 & 2\times0+0\times5+5\times4\\ 4\times4-2\times2+4\times4 & 4\times1-2\times0+4\times-2 & 4\times0-2\times5+4\times4\end{bmatrix}=\begin{bmatrix}18&4&5\\28&-8&20\\28&-4&6\end{bmatrix}$$

$$A^2=A\cdot A=\begin{bmatrix}1&2&-1\\2&0&3\\0&1&2\end{bmatrix}\begin{bmatrix}1&2&-1\\2&0&3\\0&1&2\end{bmatrix}$$

$$=\begin{bmatrix}1\times1+2\times2-1\times0 & 1\times2+2\times0-1\times1 & 1\times-1+2\times3-1\times2\\ 2\times1+0\times2+3\times0 & 2\times2+0\times0+3\times1 & 2\times-1+0\times3+3\times2\\ 0\times1+1\times2+2\times0 & 0\times2+1\times0+2\times1 & 0\times-1+1\times3+2\times2\end{bmatrix}=\begin{bmatrix}5&1&3\\2&7&4\\2&2&7\end{bmatrix}$$

$$AB=\begin{bmatrix}1&2&-1\\2&0&3\\0&1&2\end{bmatrix}\begin{bmatrix}3&-1&1\\0&0&2\\4&-3&2\end{bmatrix}=\begin{bmatrix}3+0-4&-1+0+3&1+4-2\\6+0+12&-2+0-9&2+0+6\\0+0+8&0+0-6&0+2+4\end{bmatrix}=\begin{bmatrix}-1&2&3\\18&-11&8\\8&-6&6\end{bmatrix}$$

$$BA=\begin{bmatrix}3&-1&1\\0&0&2\\4&-3&2\end{bmatrix}\begin{bmatrix}1&2&-1\\2&0&3\\0&1&2\end{bmatrix}=\begin{bmatrix}3-2+0&6+0+1&-3-3+2\\0+0+0&0+0+2&0+0+4\\4-6+0&8+0+2&-4-9+4\end{bmatrix}=\begin{bmatrix}1&7&-4\\0&2&4\\-2&10&-9\end{bmatrix}$$

$$B^2=\begin{bmatrix}3&-1&1\\0&0&2\\4&-3&2\end{bmatrix}\begin{bmatrix}3&-1&1\\0&0&2\\4&-3&2\end{bmatrix}=\begin{bmatrix}9+0+4&-3+0-3&3-2+2\\0+0+8&0+0-6&0+0+4\\12-0+8&-4+0-6&4-6+4\end{bmatrix}=\begin{bmatrix}13&-6&3\\8&-6&4\\20&-10&2\end{bmatrix}$$

$A^2+AB+BA+B^2$

$$=\begin{bmatrix}5&1&3\\2&7&4\\2&2&7\end{bmatrix}+\begin{bmatrix}-1&2&3\\18&-11&8\\8&-6&6\end{bmatrix}+\begin{bmatrix}1&7&-4\\0&2&4\\-2&10&-9\end{bmatrix}+\begin{bmatrix}13&-6&3\\8&-6&4\\20&-10&2\end{bmatrix}$$

$$=\begin{bmatrix}5-1+1+13&1+2+7-6&3+3-4+3\\2+18+0+8&7-11+2-6&4+8+4+4\\2+8-2+20&2-6+10-10&7+6-9+2\end{bmatrix}=\begin{bmatrix}18&4&5\\28&-8&20\\28&-4&6\end{bmatrix}, \ (A+B)^2=A^2+AB+BA+B^2$$

7. If $A=\begin{bmatrix}0&3\\-7&5\end{bmatrix}$ find k so that $kA^2=5A-21I$

Solution :

$$A^2=A\cdot A\begin{bmatrix}0&3\\-7&5\end{bmatrix}\begin{bmatrix}0&3\\-7&5\end{bmatrix},\begin{bmatrix}0\times0+3\times-7&0\times3+3\times5\\-7\times0+5\times(-7)&-7\times3+5\times5\end{bmatrix}=\begin{bmatrix}-21&15\\-35&4\end{bmatrix}$$

$$kA^2 = \begin{bmatrix} -21k & 15k \\ -35k & 4k \end{bmatrix},\ 5A = \begin{bmatrix} 0 & 15 \\ -35 & 25 \end{bmatrix}$$

Now $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore\ 21\,I = \begin{bmatrix} 21 & 0 \\ 0 & 21 \end{bmatrix}$, $5A - 21\,I = \begin{bmatrix} 0 & 15 \\ -35 & 25 \end{bmatrix} - \begin{bmatrix} 21 & 0 \\ 0 & 21 \end{bmatrix} = \begin{bmatrix} -21 & 15 \\ -35 & 4 \end{bmatrix}$

Now $kA^2 = 3A - 21\,I$ *i.e.* $\begin{bmatrix} -21k & 15k \\ -35k & 4k \end{bmatrix} = \begin{bmatrix} -21 & 15 \\ -35 & 4 \end{bmatrix}$

Now by equality of matrices $-21\,k = -21$ $\therefore\ k = 1$.

8. Show that matrix $A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$ satisfies the equation $A^3 - 7A^2 - 5A + 13\,I = 0$

Solution :

$$A^2 = A.A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 3 & 1 \\ 2 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 5\times5+3\times2+1\times4 & 5\times3+3\times(-1)+1\times1 & 5\times1+3\times2+1\times3 \\ 2\times5-1\times2+2\times4 & 2\times3-1\times-1+2\times1 & 2\times1-1\times2+2\times3 \\ 4\times5+1\times2+3\times4 & 4\times3+1\times(-1)+3\times1 & 4\times1+1\times2+3\times3 \end{bmatrix} = \begin{bmatrix} 35 & 13 & 14 \\ 16 & 9 & 6 \\ 34 & 14 & 15 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 35 & 13 & 14 \\ 16 & 9 & 6 \\ 34 & 14 & 15 \end{bmatrix}\begin{bmatrix} 5 & 3 & 1 \\ 2 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 35\times5+13\times2+14\times4 & 35\times3+13\times(-1)+14\times1 & 35\times1+13\times2+14\times3 \\ 16\times5+9\times2+6\times4 & 16\times3+9\times(-1)+6\times1 & 16\times1+9\times2+6\times3 \\ 34\times5+14\times2+15\times4 & 34\times3+14\times(-1)+15\times1 & 34\times1+14\times2+15\times3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 257 & 106 & 103 \\ 122 & 45 & 52 \\ 258 & 103 & 107 \end{bmatrix};\ 7A^2 = \begin{bmatrix} 245 & 91 & 98 \\ 112 & 63 & 42 \\ 238 & 98 & 105 \end{bmatrix};\ 5A = \begin{bmatrix} 25 & 15 & 5 \\ 10 & -5 & 10 \\ 20 & 5 & 15 \end{bmatrix};\ 13\,I = \begin{bmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{bmatrix}$$

$\therefore\ A^3 - 7A^2 - 5A + 13\,I$

$$= \begin{bmatrix} 257 & 106 & 103 \\ 122 & 45 & 52 \\ 258 & 103 & 107 \end{bmatrix} - \begin{bmatrix} 245 & 91 & 98 \\ 112 & 63 & 42 \\ 238 & 98 & 105 \end{bmatrix} - \begin{bmatrix} 25 & 15 & 5 \\ 10 & -5 & 10 \\ 20 & 5 & 15 \end{bmatrix} + \begin{bmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

9. Solve $[1\ x\ 1]\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = 0.$

Solution :

Clearly $[1\ x\ 1]$ is a matrix of order 1×3 $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}$ is a matrix of order 3×3

$\therefore$ $[1\ x\ 1]\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}$ is a matrix of order 1×3 which is equal to $= [1 + 2x + 15\quad 3 + 5x + 3\quad 2 + x + 2]_{1\times3}$

Now $\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix}$ is a matrix of order 3×1

$\therefore$ $[1 + 2x + 15\quad 3 + 5x + 3\quad 2 + x + 2]_{1\times3}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix}_{3\times1}$ is a matrix of order

1×1 which is equal to $= [(16 + 2x) \times 1 + (6 + 5x) \times 2 + (4 + x) \times x]$

$= [16 + 2x + 12 + 10x + 4x + x^2] = [28 + 16x + x^2]$

Now $[28 + 16x + x^2] = O$ *i.e.* $28 + 16x + x^2 = 0$; $x^2 + 14x + 2x + 28 = 0$

$\Rightarrow\ x(x + 14) + 2(x + 14) = 0\ \Rightarrow\ x + 14 = 0$ or $x + 2 = 0\ \Rightarrow\ x = -14$ or $x = -2$

1.5 [Transpose of a Matrix]

Let A be a $m \times n$ matrix then, the matrix $n \times m$, obtained by interchanging the rows and columns of A, is called the transpose of A and is denoted by A^T of A'. In symbols,

If $A = [a_{ij}]_{m \times n}$ then $A^T = [a_{ij}]_{n \times m}$. **Example :** If $\begin{bmatrix} 2 & 5 & 7 \\ 2 & -2 & 5 \end{bmatrix}$, then $A^T = \begin{bmatrix} 2 & 2 \\ 5 & -3 \\ 7 & 5 \end{bmatrix}$

Properties: (i) $(A^T)^T = A$ (ii) $(A+B)^T = A^T + B^T$

(iii) $(KA)^T = KA^T$ (iv) $(AB)^T = B^T A^T$

1.6 [Symmetric and Skew Symmetric Matrices]

(*i*) A square matrix $A = [a_{ij}]$ is said to be a **symmetric matrix** if $a_{ij} = a_{ji}$ for all (i, j) i.e., $(i, j)^{th}$ element of A is the same as $(j, i)^{th}$ element of A, i.e., $A^T = A$.

Example : $\begin{bmatrix} 4 & 7 \\ 7 & 6 \end{bmatrix}$, $\begin{bmatrix} 5 & -7 & 1 \\ -7 & 2 & 9 \\ 1 & 9 & 8 \end{bmatrix}$ and $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ are all symmetric matrices.

(*ii*) A square matrix A [aij] is said to be a **skew-symmetric matrix** if $a_{ij} = -a_{ij}$ for i, j, *i.e.*, (i, j) in element of A $= -(j, i)^{th}$ element of A, *i.e.*, $A^T = -A$.

Example : $A = \begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & 6 \\ 4 & -6 & 0 \end{bmatrix}$ is skew symmetric, $B = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$ is skew symmetric.

NOTE : For a skew symmetric matric A. $a_{ij} = -a_{ji}$ for all i, j

$\therefore$ Putting $j = i$, we get $a_{ii} = -a_{ii}$, *i.e.*, $2a_{ii} = 0$, *i.e.*, $a_{ii} = 0$, i.e., every diagonal elements of A is zero.

Thus, every diagonal entry of a skew symmetric matrix is zero.

i.e., **elements on the main diagonal of a skew symmetric matrix are zero.**

1.7 [Trace of a Square Matrix]

The sum of the main diagonal elements of a square matrix A is called the trace of A and is denoted by $tr(A)$, *i.e.*,

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

Properties:

(i) If A and B are of the same order then tr $(A+B) = tr(A) + tr(B)$

(ii) If A and B are of order $n \times m$ and $m \times n$ respectively then $tr(AB) = tr(BA)$.

(iii) $tr(A) = tr(A^T)$.

1.8 [Some Results on Symmetric and Skew Symmetric Matrices]

Theorem 1:

Prove that

(i) the sum of two symmetric matrices is symmetric. i.e., $A + B$ is symmetric if A, B are symmetric

(ii) the sum of two skew symmetric matrices is skew symmetric. i.e., $A + B$ is skew symmetric if A, B are skew symmetric.

Proof:

(i) Let A and B be two symmetric matrices of the same order. Then $A^T = A$ and $B^T = B$.

$\therefore \quad (A+B)^T = A^T + B^T = A + B \qquad [\because A^T = A, B^T = B] \qquad \therefore A + B$ is symmetric.

(ii) Let A, B be two skew symmetric matrices of the same other. Then $A^T = -A$ and $B^T = -B$.

$\therefore (A+B)^T = A^T + B^T = -A - B = -(A+B)$ $\therefore A + B$ is skew symmetric.

Theorem 2:

Let A be a square matrix and k be a scalar. Prove that

(i) if A is symmetric, then kA is symmetric

(ii) if A is skew symmetric, then KA is skew symmetric.

Proof:

(i) Let A be symmetric, $\therefore A^T = A$ $\therefore$ $(kA)^T = kA^T = kA$ $[\because A^T = A]$ $\therefore kA$ is symmetric.

(ii) Let A be skew symmetric $\therefore A^T = -A$ $\therefore$ $(kA)^T = kA^T = k(-A) = -kA$ $[\therefore A^T = -A]$

$\therefore kA$ is skew symmetric.

Theorem 3

If A is any square matrix, then prove that

(i) $A + A^T$ is symmetric (ii) $A - A^T$ is skew symmetric. (iii) AA^T, A^TA are symmetric.

Proof:

(i) $(A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$ $[\because (A^T)^T = A]$ $\therefore A + A^T$ is symmetric.

(ii) $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$ $[\because (A^T)^T = A]$

$\therefore$ A – AT is skew symmetric. $[\because (AB)^T = B^TA$ and $(A^T)^T = A]$

(iii) $(AA^T)^T = (A^T)^TA^T = AA^T$ $[\because (AB)^T = B^TA$ and $(A^T)^T = A]$ $\therefore AA^T$ is symmteric.

Again $(A^TA)^T = A^T(A^T)^T = A^TA$ $[\because (A^T)^T = A^T]$ $\therefore A^TA$ is symmetric.

Theorem 4:

Prove that every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew-symmetric matrix.

Proof:

Let A be any square matrix. Then we can write A as

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = P + Q \text{ (say)}, \quad \text{where } P = \frac{1}{2}(A + A^T) \text{ and } Q = \frac{1}{2}(A - A^T).$$

Now, $$P^T = \left\{\frac{1}{2}(A + A^T)\right\}^T = \frac{1}{2}(A + A^T)^T \quad [\therefore (kA)^T = kA^T]$$

$$= \frac{1}{2}(A^T + (A^T)^T) = \frac{1}{2}(A^T + A) \quad [\because (A^T)^T = A]$$

$$P^T = \frac{1}{2}(A + A^T) = P \quad [\because (A^T + A) = (A + A^T)]$$

$\therefore P$ is symmetric.

Again $$Q^T = \left\{\frac{1}{2}(A - A^T)\right\}^T = \frac{1}{2}(A - A^T)^T \quad [\because (kA)^T = kA^T]$$

$$= \frac{1}{2}(A^T - (A^T)^T) = \frac{1}{2}(A^T - A) \quad [\because (kA)^T = kA^T]$$

$$Q^T = -\frac{1}{2}(A - A^T) = -Q$$ $\therefore Q$ is skew-symmetric.

Thus, $A = P + Q$, where P is symmetric and Q is skew-symmetric, which shows that every square matrix is expressible as the sum of a symmetric matrix and a skew-symmetric matrix.

To prove that the representation is unique, let $A = R + S$, where R is symmetric and S is skew-symmetric.

Now, $A = R + S \Rightarrow A^T = (R+S)^T = R^T + S^T = R - S$ $[\because R^T = R$ and $S^T = -S]$

Thus, $A = R + S$ and $A^T = R - S$ $\therefore A + A^T = 2R$ and $A - A^T = 2S$.

So, $R = \frac{1}{2}(A + A^T) = P$ and $S = \frac{1}{2}(A - A^T) = Q$ Hence, the representation is unique.

Theorem 5:

Prove that every diagonal element of a skew-symmetric matrix is zero.

Proof:

Let $A = [a_{ij}]_{n \times n}$ be skew-symmetric. Then by definition, we have $A^T = -A$ and therefore, $a_{ji} = -a_{ij}$.

Taking $j = i$, we get $a_{ii} = -a_{ii}$ or $2a_{ii} = 0$ or $a_{ii} = 0$ for each i.

Hence, **every diagonal element of a skew-symmetric matrix is zero.**

Examples

10. If $A = \begin{bmatrix} 3 & 2 & 1 \\ -5 & 0 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & -5 & -2 \\ 3 & 1 & 8 \end{bmatrix}$, verify that $(A+B)^T = A^T + B^T$.

Solution:

We have $A^T = \begin{bmatrix} 3 & -5 \\ 2 & 0 \\ 1 & -6 \end{bmatrix}$ and $B^T = \begin{bmatrix} -4 & 3 \\ -5 & 1 \\ -2 & 8 \end{bmatrix}$

$\therefore \quad A^T + B^T = \begin{bmatrix} 3 & -5 \\ 2 & 0 \\ 1 & -6 \end{bmatrix} + \begin{bmatrix} -4 & 3 \\ -5 & 1 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -3 & 1 \\ -1 & 2 \end{bmatrix}$.

Also, $A + B = \begin{bmatrix} 3 & 2 & 1 \\ -5 & 0 & -6 \end{bmatrix} + \begin{bmatrix} -4 & -5 & -2 \\ 3 & 1 & 8 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 1 \\ -2 & 1 & 2 \end{bmatrix}$.

$\therefore \quad (A+B)^T = \begin{bmatrix} -1 & -2 \\ -3 & 1 \\ -1 & 2 \end{bmatrix}$. Hence, $(A+B)^T = A^T + B^T$.

11. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, show that $A - A^T$ is skew symmetric, where A^T is transpose of matrix A.

Solution:

We have $A^T = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$ $\quad \therefore \quad A - A^T = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$

which is skew symmetric matrix because $a_{ij} = -a_{ji}$ for all i and j.

12. Show that the matrix $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ is skew-symmetric.

Solution:

We have $A^T = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = (-1) \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = -A$. Hence, A is skew-symmetric.

13. If $A = \begin{bmatrix} 5 & 7 & 3 \\ 1 & 2 & 6 \\ 1 & 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 5 \\ 0 & 3 & 2 \end{bmatrix}$ then verify $(AB)^T = B^T A^T$.

Solution:

$AB = \begin{bmatrix} 5 & 7 & 3 \\ 1 & 2 & 6 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 5 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 33 & 19 & 41 \\ 9 & 20 & 22 \\ 17 & 11 & 26 \end{bmatrix}$ ...(1) $\quad \therefore \quad (AB)^T = \begin{bmatrix} 33 & 9 & 17 \\ 19 & 20 & 11 \\ 41 & 22 & 26 \end{bmatrix}$ ...(2)

Also $$A^T = \begin{bmatrix} 5 & 1 & 1 \\ 7 & 2 & 4 \\ 3 & 6 & 3 \end{bmatrix} \text{ and } B^T = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 0 & 3 \\ 0 & 5 & 2 \end{bmatrix}$$

$$\therefore \quad B^T A^T = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 0 & 3 \\ 0 & 5 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 & 1 \\ 7 & 2 & 4 \\ 3 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 33 & 9 & 17 \\ 19 & 20 & 11 \\ 41 & -22 & 26 \end{bmatrix} \quad \text{...(3)}$$

From Eqs. (2) and (3) we observe that $(AB)^T = B^T A^T$.

14. If A and B are symmetric matrices, prove that $AB - BA$ is a skew symmetric matrix, and $AB + BA$ is symmetric.

Solution:

Since A, B are symmetric

$\therefore \quad A^T = A, B^T = B \quad$ ...(1)

Again $(AB - BA)^T = (AB)^T - (BA)^T \quad [\because (A - B)^T = A^T - B^T]$

Again $(AB - BA)^T = (AB)^T - (BA)^T \quad [\because (A - B)^T = A^T - B^T]$

$= B^T A^T - A^T B^T \quad [\because (AB)^T = B^T A^T]$

$= BA - AB \quad$ [By (1)]

$(AB - BA)^T = -(AB - BA) \quad \therefore AB - BA$ is skew symmetric.

Again $(AB + BA)^T = (AB)^T + (BA)^T = B^T A^T + A^T B^T = BA + AB$

$(AB + BA)^T = AB + BA \quad \therefore AB + BA$ is symmetric.

15. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ verify that $AA^T = A^T A = I_2$.

Solution:

Let $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\cos\alpha & \cos\alpha \end{bmatrix}$ then $A^T = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ $\therefore AA^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha = \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \quad [\because \cos^2\alpha + \sin^2\alpha = 1]$$

Similarly $A^T A = I_2$, $\therefore AA^T = A^T A = I_2$.

16. Express $\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$ as a sum of symmetric and skew symmetric matrices.

Solution:

Let $A = \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$. Then $A^T = \begin{bmatrix} 1 & -6 & -4 \\ 3 & 8 & 6 \\ 5 & 3 & 5 \end{bmatrix}$

$$\therefore \quad A + A^T = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix} \text{ and } \frac{A + A^T}{2} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -\frac{3}{2} & 8 & \frac{9}{2} \\ \frac{1}{2} & \frac{9}{2} & 5 \end{bmatrix}$$

Again $A - A^T = \begin{bmatrix} 0 & 9 & 9 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix}$ and $\dfrac{A - A^T}{2} = \begin{bmatrix} 0 & \frac{9}{2} & \frac{9}{2} \\ -\frac{9}{2} & 0 & -\frac{3}{2} \\ -\frac{9}{2} & \frac{3}{2} & 0 \end{bmatrix}$

Since $\dfrac{A + A^T}{2}$ is symmetric and $\dfrac{A - A^T}{2}$ is skew-symmetric and $A = \dfrac{A + A^T}{2} + \dfrac{A - A^T}{2}$.

$$\therefore \quad A = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -\frac{3}{2} & 8 & \frac{9}{2} \\ \frac{1}{2} & \frac{9}{2} & 5 \end{bmatrix} + \begin{bmatrix} 0 & \frac{9}{2} & \frac{9}{2} \\ -\frac{9}{2} & 0 & -\frac{3}{2} \\ -\frac{9}{2} & \frac{3}{2} & 0 \end{bmatrix}$$

1.9 [Conjugate of a Matrix]

Matrix obtained after replacing each element of given matrix by its complex conjugate, is called conjugate of matrix. It is denoted by $\bar{A}$, *i.e.*,

$$\bar{A} = [\bar{a}_{ij}]_{m \times n} \ ; \ \bar{a}_{ij} = \text{complex conjugate of } a_{ij}$$

Example : Let $A = \begin{bmatrix} 3+5i & 2-7i & 11i \\ 3 & 9+2i & -8i \end{bmatrix}$. Then $\bar{A} = \begin{bmatrix} 3-5i & 2+7i & -11i \\ 3 & 9-2i & 8i \end{bmatrix}$

1.10 [Conjugate Transpose of a Matrix]

The conjugate of transpose of a matrix A is called conjugate of A and is denoted by A^*. Thus

$$A^* = (\overline{A^T}) = (\bar{A})^T$$

Example : Let $A = \begin{bmatrix} 2+5i & 6-i & 5+2i \\ 3 & 2 & -1+5i \\ 0 & 7-3i & -5+6i \end{bmatrix}$

then $A^* = \begin{bmatrix} 2-5i & 3 & 0 \\ 6+i & 2 & 7+3i \\ 5-2i & -1-5i & -5-6i \end{bmatrix}$

$$(A^*)^* = A, (A+B)^* = A^* + B^*, (KA)^* = \bar{K}A^*, (AB)^* = B^*A^*.$$

These results are similar to results of transpose of a matrix.

1.11 [Hermitian and Skew-Hermitian Matrices]

(i) A square matrix A = $[a_{ij}]$ is said to be **Hermitian** iff $a_{ij} = \overline{a_{ji}}$ i.e., (i, j) the element is the conjugate of the $(j, i)^{\text{th}}$ element.

From above def., it is clear that a matrix A is Hermitian

iff $A^* = A$.

Again as $a_{ij} = \overline{a_{ji}}$ $\therefore$ $a_{ij} = \overline{a_{ji}}$, *i.e.*, the conjugate of any diagonal element is the same element. Therefore, **every diagonal element must be real.**

Example : $\begin{bmatrix} 2 & 5-6i & 3-4i \\ 5+6i & 0 & 1-2i \\ 5+4i & 1+2i & 7 \end{bmatrix}, \begin{bmatrix} 0 & a+ib & c+id \\ a-ib & 1 & m+in \\ c-id & m-in & 2 \end{bmatrix}$ are Hermitian Matrices.

(ii) A square matrix $A = [a_{ij}]$ is said to be **Skew-Hermitian** if $a_{ij} = -\overline{a_{ji}}$ i.e., $(i, j)^{th}$ element is the negative conjugate of $(j, i)^{th}$ element,

Again as $a_{ij} = -a_{ji}$ $\therefore a_{ii} = -\overline{a_{ii}}$ i.e., $a_{ii} + a_{ii} = 0$, i.e., $Re\,(a_{ij}) = 0$.

$\therefore$ **every diagonal element must be either zero or a purely imaginary number.**

Example : $\begin{bmatrix} 2i & 6-3i & 5+3i \\ -6-3i & 0 & 3+6i \\ -5+3i & -3+6i & -7i \end{bmatrix}, \begin{bmatrix} 2i & 5-2i \\ -5-2i & 4i \end{bmatrix}$ are Skew Hermitian Matrices.

Examples

17. If A is a square matrix, prove that (i) $A + A^*$ is Hermitian (ii) $A = A^*$ is skew-Hermition.

Solution:

(i) $(A + A^*)^* = A^* + (A^*)^* = A^* + A = A + A^*$ $[\because (A^*)^* = A]$ $\therefore$ $A + A^*$ is Hermitian

(ii) $(A - A^*)^* = A^* - (A^*)^* = A^* - A$ $[\because (A^*)^* = A]$ $\therefore$ $A - A^*$ is Skew Hermitian.

18. Every square matrix can be expressed in one and only one way is $P + i_Q$ where P and Q are hermitian.

Solution:

We have $A = \frac{1}{2}(A + A^*) + i \cdot \frac{1}{2i}(A - A^*)$, where A is any square matrix.

$\therefore$ $A = P + iQ$, where $P = \frac{1}{2}(A + A^*)$, $Q = \frac{1}{2i}(A - A^*)$

Now $P^* = \frac{1}{2}(A + A^*)^* = \frac{1}{2}[A^* + (A^*)^*] = \frac{1}{2}[A^* + A] = \frac{1}{2}[A^* + A] = P$

$\therefore$ $P^* = P$ $\therefore$ is Hermitian.

Again $Q^* = \left(\frac{1}{2i}(A - A^*)\right)^* = -\frac{1}{2i}(A - A^*)^* = \frac{1}{2i}(A^* - (A^*)^*) = -\frac{1}{2i}(A^* - A)$

$Q^* = \frac{1}{2i}(A - A^*) = Q$ $\therefore$ Q is hermitian. $\therefore$ We have $A = P + iQ$...(1)

Now we want to prove that representation (1) is unique.

For this, if possible, let $A = R + iS$

be another representation, where R, S are Hermitian. ...(2) $\therefore$ $A^* = (R + iS)^* = R^* - iS^*$

i.e., $A^* = R - iS$ $[\because R^* = R, S^* = S$ as R, S are Hermitian$]$ ...(3)

From Eqs. (2) and (3), we have $R = \frac{A + A^*}{2} = P$ and $S = \frac{A - A^*}{2i} = Q$

Therefore, representation Eq. (2) is the same as Eq. (1). Hence representation Eq. (1) is unique.

19. Show that every Hermitian matrix A can be uniquely expressed as $P + iQ$ where P, Q are real symmetric and real skew-symmetric and also show that $A^* A$ is real iff $PQ = -QP$.

Solution:

We have $A = \frac{1}{2}(A + \bar{A}) + i\frac{1}{2i}(A - \bar{A}) = P + iQ$ where $P = \frac{1}{2}(A + \bar{A}), Q = \frac{1}{2i}(A - \bar{A})$

Since $\bar{P} = \frac{1}{2}\overline{(A + \bar{A})} = \frac{1}{2}(\bar{A} + \overline{(\bar{A})}) = \frac{1}{2}(\bar{A} + A) = \frac{1}{2}(A + \bar{A}) = P$

$\therefore$ P is real.

Again $\quad \bar{Q} = \frac{-1}{2i}\overline{(A-\bar{A})} = -\frac{1}{2i}(\bar{A}-\overline{(\bar{A})}) = -\frac{1}{2i}(\bar{A}-A) = \frac{1}{2i}(A-\bar{A}) = Q \qquad \therefore Q$ is real.

Next we shall prove that P is symmetric and Q is skew-symmetric.

Now $\quad P^T = \frac{1}{2}(A+\bar{A})^T = \frac{1}{2}(A^T + (\bar{A})^T) = \frac{1}{2}(A^T + A^*)$

$$= \frac{1}{2}[(A^*)^T + A] \qquad [A = \text{Hermetian, } i.e., A^* = A]$$

$$P^T = \frac{1}{2}[((\bar{A})^T)^T + A] = \frac{1}{2}[\bar{A}+A] = \frac{1}{2}[A+\bar{A}] = P$$

$\therefore P$ is symmetric.

Again $\quad Q^T = \left(\frac{1}{2i}(A-\bar{A})\right) = \frac{1}{2i}(A^T - (\bar{A})^T) = \frac{1}{2i}(A^T - A^*) = \frac{1}{2i}(A^T - A)$

$$= \frac{1}{2i}((A^*)^T - A) \qquad [\because A^* = A]$$

$$= \frac{1}{2}(((\bar{A})^T)^T - A) = \frac{1}{2i}(\bar{A}-A) = -\frac{1}{2i}(A-\bar{A}) = -Q \qquad \therefore Q \text{ is skew-symmetric.}$$

Thus $A = P + iQ$ where P is real symmetric and Q real skew-symmetric.

For uniqueness, let $A = R + iS$ be another representation where R is real symmetric and S is real skew symmetric.

$\therefore \quad \bar{A} = \overline{(R+iS)} = \bar{R} - i\bar{S} = R - iS \qquad [\because \bar{R} = R, \bar{S} = S$ as R, S are real$]$

$\therefore \quad A + \bar{A} = 2R \Rightarrow \frac{A+\bar{A}}{2} = R \Rightarrow P = R$

$A - \bar{A} - 2iS \Rightarrow \frac{A-\bar{A}}{2i} = S \Rightarrow Q = S$

Hence the representation of A is unique.

Again $\quad A = P + iQ$

$A^* = (P+iQ)^* = P^* - iQ^* = (\bar{P})^T - i(\bar{Q})^T = P^T - iQ^T = P + iQ \quad [\because P, Q$ are real and $P^T = P, Q^T = -Q]$

$\therefore \quad A^*A = (P+iQ)(P+iQ) = P^2 + i(PQ+QP) - Q^2$

$A^*A = P^2 - Q^2 + i(PQ+QP) \qquad$ which is real iff $PQ + QP = 0$, $i.e.$, $PQ = -OP$.

1·12 [Determinants]

Introduction :

A determinant is a uniquely defined scaler associated with only square matrix.

Let $A = [a_{ij}]_{n\times n}$ be square matrix then determinant of A is denoted by $|A|$ or det A

$$|A| = \begin{vmatrix} a_{11} & a_{12} \cdots\cdots a_{ij} \cdots\cdots a_{in} \\ a_{21} & a_{22} \cdots\cdots a_{2j} \cdots\cdots a_{2n} \\ \vdots & \vdots \qquad \vdots \qquad \vdots \\ a_{i1} & a_{i2} \cdots\cdots a_{ij} \cdots\cdots a_{in} \\ \vdots & \vdots \qquad \vdots \qquad \vdots \\ a_{n1} & a_{n2} \cdots\cdots a_{nj} \cdots\cdots a_{nn} \end{vmatrix}$$

1·13 [Minors]

Definition :

A minor of an element a_{ij} in a matrix A is a subdeterminant of $|A|$ which is obtained by deleting its i^{th} row and j^{th} column

It is denoted by M_{ij}

Example : $M_{12} = \begin{vmatrix} a_{21} & a_{23} & \cdots\cdots & a_{2n} \\ a_{31} & a_{33} & \cdots\cdots & a_{3n} \\ a_{n1} & a_{n3} & \cdots\cdots & a_{nn} \end{vmatrix}$

[subdeterminant obtained from $|A|$ by deleting its 1^{st} row and 2^{nd} column]

1·14 [Cofactors]

Definition :

A cofactor of an element a_{ij} in a matrix A is defined as $C_{ij} = (-1)^{i+j} M_{ij}$

Example : $C_{11} = (-1)^{1+1} M_{11} = M_{11}$; $C_{12} = (-1)^{1+2} M_{12} = -M_{12}$

1·15 [Evaluation Of Determinants]

1. For Square Matrix of Order 1 :

Let $A = [a_{11}]$ be the square matrix of order 1 then $|A| = a_{11}$

2. For Square Matrix of Order 2 :

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be the square matrix of order 2, then $|A| = a_{11}a_{22} - a_{12}a_{21}$

Example : If $A = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}$ then $|A| = 1 \times 6 - 5 \times 2 \quad 6 - 10 = -4$

3. For Square Matrix of Order 3 :

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be the square matrix of order 3

then $|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Example : Let $A = \begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix}$ then det A or $|A| = 2\begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ -2 & -3 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}$

$= 2\,[2 \times (-3) - (1 \times 3)] - 3\,[1 \times (-3) - (-2 \times 3)] - 2\,[1 \times 1 - (-2 \times 2)]$

$= 2\,[-6 - 3] - 3\,[-3 + 6] - 2\,[1 + 4] = 2\,[-9] - 3\,(3) - 2\,(5) - 18 - 9 - 10 = -37$

Examples

20. Write the minors and co-factors of each element of matrix $A = \begin{bmatrix} 0 & 2 & 6 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{bmatrix}$ and hence evaluate the determinant.

Solution :

$A = \begin{bmatrix} 0 & 2 & 6 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{bmatrix}$; $M_{11} = \begin{vmatrix} 5 & 0 \\ 7 & 1 \end{vmatrix} = 5 \times 1 - 0 \times 7 = 5$; $C_{11} = (-1)^{1+1} M_{11} = 5$

$$M_{12}=\begin{vmatrix}1&0\\3&1\end{vmatrix}=1\times1-3\times0=1 \quad ; \quad C_{12}=(-1)^{1+2}M_{12}=-M_{12}=-1$$

$$M_{13}=\begin{vmatrix}1&5\\3&7\end{vmatrix}=1\times7-3\times5=-8 \quad ; \quad C_{13}=(-1)^{1+3}M_{13}=-8$$

$$M_{21}=\begin{vmatrix}2&6\\7&1\end{vmatrix}=2\times1-7\times6=-40 \quad ; \quad C_{21}=(-1)^{2+1}M_{21}=-(-40)=40$$

$$M_{22}=\begin{vmatrix}0&6\\3&1\end{vmatrix}=0\times1-6\times3=-18 \quad ; \quad C_{22}=(-1)^{2+2}M_{22}=-18$$

$$M_{23}=\begin{vmatrix}0&2\\3&7\end{vmatrix}=0\times7-3\times2=-6 \quad ; \quad C_{23}=(-1)^{2+3}M_{23}=-(-6)=6$$

$$M_{31}=\begin{vmatrix}2&6\\5&0\end{vmatrix}=2\times0-5\times6=-30 \quad ; \quad C_{31}=(-1)^{3+1}M_{31}=M_{31}=-30$$

$$M_{32}=\begin{vmatrix}0&6\\1&0\end{vmatrix}=0\times0-1\times6=-6 \quad ; \quad C_{32}=(-1)^{3+2}M_{32}=-M_{32}=-(-6)=6$$

$$M_{33}=\begin{vmatrix}0&2\\1&5\end{vmatrix}=0\times5-2\times1=-2 \quad ; \quad C_{33}=(-1)^{3+3}M_{33}=M_{33}=-2$$

also $|A|=0\begin{vmatrix}5&0\\7&1\end{vmatrix}-2\begin{vmatrix}1&0\\3&1\end{vmatrix}+6\begin{vmatrix}1&5\\3&7\end{vmatrix}-2(1)+6(7-15)-2+6(-8)=-50$

21. For what value of x, the Matrix $A=\begin{bmatrix}x-1&1&1\\1&x-1&1\\1&1&x-1\end{bmatrix}$ is singular.

Recall : A square matrix is called singular matrix if its determinant is zero.

Solution :

Now $|A|=0$ *i.e.* $\begin{vmatrix}x-1&1&1\\1&x-1&1\\1&1&x-1\end{vmatrix}=0$; $(x-1)\begin{vmatrix}x-1&1\\1&x-1\end{vmatrix}-1\begin{vmatrix}1&1\\1&x-1\end{vmatrix}+1\begin{vmatrix}1&x-1\\1&1\end{vmatrix}=0$

$(x-1)[(x-1)^2-1]-1[x-1-1]+1[1-(x-1)]=0$

$(x-1)(x^2-2x)-x+2+2-x=0$

$x^3-2x^2-x^2+2x-x+2+2-x=0$

$x^3-3x^2+4=0$; $(x+1)(x^2-4x+4)=0$

$(x+1)(x-2)^2=0 \Rightarrow x=-1, 2, 2.$

22. If $A=\begin{bmatrix}2&5\\2&1\end{bmatrix}$ and $B=\begin{bmatrix}4&-3\\2&5\end{bmatrix}$. The verify that $|AB|=|A||B|$.

Solution :

$$AB=A.B=\begin{bmatrix}2&5\\2&1\end{bmatrix}\cdot\begin{bmatrix}4&-3\\2&5\end{bmatrix}=\begin{bmatrix}2\times4+5\times2&2\times(-3)+5\times5\\2\times4+1\times2&2\times(-3)+1\times5\end{bmatrix} \; ; \; AB=\begin{bmatrix}18&19\\10&-1\end{bmatrix}$$

$|AB|=18(-1)-10\times19-18-190=-208$ —(i)

Also $|A|=2\times1-2\times1=-8$

$|B|=4\times5-2\times(-3)=26$

$|A||B|=-8(26)=-208$ —(ii)

from (i) and (ii) result follows.

1·16 [Properties Of Determinants]

1. If the rows of a determinant are changed into columns and vice versa determinant gives the same value *i.e.* determinant of a matrix is equal to the transpose of the same matrix

i.e. $\det A = \det A^T$

Example : Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$; $|A| = 5 - 6 = -1$; $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$; $|A^T| = 5 - 6 = -1$ $\therefore$ $|A| = |A^T|$.

2. The interchange of any two rows or (column) of a matrix alters the value of determinant by a multiple of -1.

Example : Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 5 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ then $|A| = 1 \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix} + 0 \begin{vmatrix} 5 & 2 \\ 2 & 2 \end{vmatrix}$ $4 - 2(15 - 2) + 0$

$4 - 2(13) = -22$

Let $A' = \begin{bmatrix} 5 & 2 & 1 \\ 1 & 2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$ [interchanging first two rows] ; $|A'| = 5 \begin{vmatrix} 2 & 0 \\ 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}$

$5(6) - 2(3) - 2$

$30 - 6 - 2 = 22$ *i.e.* $|A| = -|A'|$

3. If all the elements of a row (or column) are multiplied by a scaler k, then the new determinant of the new matrix is k times the original matrix

Example : Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ then $|A| = -6$

$A' = \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix}$ *i.e.* multiply all the element of first row by $k = 2$

$|A'| = 2 \times 0 - 12 = -12$ *i.e.* $|A'| = 2|A|$

4. The addition (or subtraction) of a multiple of any row (or column) to another row (or column) does not effect the value of determinant

Example : Let $A = \begin{bmatrix} 1 & 5 \\ 7 & 0 \end{bmatrix}$; $|A| = -35$.

Let the operation be $R_2 \to R_2 + 2R_1$

$A' = \begin{bmatrix} 1 & 5 \\ 9 & 10 \end{bmatrix}$; $|A| = 10 - 45 = -35$ $\therefore A = A'$ also consider $C = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}$ $|C| = 10 - 6 = 4$

Let the operation be $C_2 \to C_2 - C_1$ then $C' = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$ $|C'| = 5 - 1 = 4$ $\therefore |C| = |C'|$

5. If any two rows or columns of a square matrix are identical, then the determinant is zero.

Example : Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}$. Now Row R_1 = Row R_2

$|A| = 1(14 - 18) - 2(7 - 15) + 3(6 - 10) - 4 - 2(-8) + 3(-4)$ $-4 + 16 - 12$; $12 - 12 = 0$.

6. When every element in a row or coulumn of a square matrix is zero, then its determinant is also zero.

Example : Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 5 & 0 \end{bmatrix}$ then $|A| = 1(0 - 0) + 2(0 - 0) + 0(5 - 3) = 0$.

Examples

23. Show that $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$

Solution :

Let $\Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$. Perform $C_2 \to C_2 + C_3$ $\Delta = \begin{vmatrix} 1 & a+b+c & b+c \\ 1 & b+c+a & c+a \\ 1 & c+a+b & a+b \end{vmatrix}$

$\Delta = (a+b+c) \begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}$ [Take $a+b+c$ common from C_2]

$\Delta = (a+b+c)\,0 = 0$ [as $C_1 = C_2$].

24. Find $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Solution :

Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$. Perform $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

then $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$ *i.e.* $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix}$; $\Delta = (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$

[By taking $b-a$ and $c-a$ common from R_2 and R_3 respectively]. Now perform $R_3 \to R_3 - R_2$

$\Delta = (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$; $\Delta = (b-a)(c-a)(c-b)$

$= (b-a)(c-a)(c-b) = (a-b)(b-c)(c-a)$

25. Show that $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

Solution :

Let $\Delta = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$. Perform $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$\Delta = \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & b^3-a^3 \\ 0 & c-a & c^3-a^3 \end{vmatrix}$; $\Delta = \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & (b-a)(b^2+a^2+ab) \\ 0 & c-a & (c-a)(c^2+a^2+ca) \end{vmatrix}$

$$\Delta = (b-a)(c-a)\begin{vmatrix} 1 & a & a^3 \\ 0 & 1 & b^2+a^2+ab \\ 0 & 1 & c^2+a^2+ca \end{vmatrix}$$

(By taking $(b-a)$ and $(c-a)$ common from R_2 and R_3 respectively). Now perform $R_2 \to R_2 - R_3$

$$\Delta = (b-a)(c-a)\begin{vmatrix} 1 & a & a^3 \\ 0 & 0 & (b^2-c^2)+(ab-ac) \\ 0 & 1 & c^2+a^2+ca \end{vmatrix}$$

$$\Delta = (b-a)(c-a)\begin{vmatrix} 1 & a & a^3 \\ 0 & 0 & (b-c)(b+c)+a(b-c) \\ 0 & 1 & c^2+a^2+ca \end{vmatrix}$$

$$\Delta = (b-a)(c-a)(b-c)\begin{vmatrix} 1 & a & a^3 \\ 0 & 0 & a+b+c \\ 0 & 1 & c^2+a^2+ca \end{vmatrix}$$

$$\Delta = (b-a)(c-a)(b-c)[0-(a+b+c)+0] = -(b-a)(c-a)(b-c)(a+b+c)$$
$$= (a-b)(b-c)(c-a)(a+b+c)$$

26. Without expanding prove that : $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0.$

Solution :

Let $\Delta = \begin{vmatrix} x+y & y+z & z+y \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$. Performe $R_2 \to R_2 + R_1$

$$\Delta = \begin{vmatrix} x+y+z & y+z+x & z+x+y \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

$\Delta = (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$ (By taking $x+y+z$ common from R_1)

$= (x+y+z) . 0$ (as $R_1 = R_2$)

27. Show that $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$

Solution :

Let $\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$

$\Delta = xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$ (by taking x, y, z common from C_1, C_2, C_3 respectively)

Perform $C_2 \to C_2 - C_1$
$C_3 \to C_3 - C_1$

$$\Delta = xyz\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix} \; ; \; \Delta = xyz\begin{vmatrix} 1 & 0 & 0 \\ x & (y-x) & (z-x) \\ x^2 & (y-x)(y+x) & (z-x)(z+x) \end{vmatrix}$$

$$\Delta = (xyz)(y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y+x & z+x \end{vmatrix}$$ (by taking $(y-x)$ and $(z-x)$ common from C_2 and C_3)

$$\Delta = (xyz)(y-x)(z-x)[1 \cdot (z+x)-(y+x)]$$

$$\Delta = (xyz)(y-x)(z-x)[z-y] = xyz\,(x-y)(y-z)(z-x)$$

28. Prove that $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$

Solution :

Let $\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$. Perform $R_3 \to R_1 + R_3$

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha+\beta+\gamma & \beta+\gamma+\alpha & \gamma+\alpha+\beta \end{vmatrix} = (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix}$$

Perform $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$(\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$(\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^2 & (\beta-\alpha)(\beta+\alpha) & (\gamma-\alpha)(\gamma+\alpha) \\ 1 & 0 & 0 \end{vmatrix}$$

$$(\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \alpha & 1 & 1 \\ \alpha^2 & \beta+\alpha & \gamma+\alpha \\ 1 & 0 & 0 \end{vmatrix}$$

(By taking $(\beta-\alpha)$, $(\gamma-\alpha)$ common from C_2 and C_3 respectively)

$$(\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)[1(\gamma+\alpha)+1-(\beta+\alpha)]$$

$$(\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma-\beta) = (\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$$

29. Prove that $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

Solution :

Let $\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$

$$\Delta = abc\begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$$ (by taking a, b and c common from C_1, C_2 and C_3)

$$\Delta = a^2 b^2 c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \text{(by taking } a, b \text{ and } c \text{ common from } R_1, R_2 \text{ and } R_3)$$

Perform $C_2 \to C_2 + C_1$, $C_3 \to C_3 + C_1$

$$\Delta = a^2 b^2 c^2 \begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} \; ; \; \Delta = [-1(-4)]\, a^2 b^2 c^2 = 4\, a^2 b^2 c^2$$

30. Prove that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2\,abc\,(a+b+c)^3$

Solution :

Let $\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

Perform $C_1 \to C_1 - C_3$, and $C_2 \to C_2 - C_3$

$$\Delta = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} (b+c-a)(b+c+a) & 0 & a^2 \\ 0 & (c+a-b)(c+a+b) & b^2 \\ (c+a+b)(c-a-b) & (c-a-b)(c+a+b) & (a+b)^2 \end{vmatrix}$$

$$\Delta = (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

[By taking $a+b+c$ common from C_1 and C_2 perform $R_3 \to R_3 - (R_1 + R_2)$

$$\Delta = (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} = \frac{(a+b+c)^2}{ab} \begin{vmatrix} ab+ac-a^2 & 0 & a^2 \\ 0 & cb+ab-b^2 & b^2 \\ -2ab & -2ab & 2ab \end{vmatrix}$$

[By applying $C_1 \to C_1 a$, $C_2 \to C_2 b$

Now perform $C_1 \to C_1 + C_3$, $C_2 \to C_2 + C_3$

$$\Delta = \frac{(a+b+c)^2}{ab} \begin{vmatrix} ab+ac & a^2 & a^2 \\ b^2 & bc+ba & b^2 \\ 0 & 0 & 2ab \end{vmatrix} \; ; \; \frac{(a+b+c)^2}{ab} \cdot a \cdot b \cdot 2ab \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ 0 & 0 & 1 \end{vmatrix}$$

[By taking a, b and $2ab$ common from R_1, R_2 and R_3 respectively]

$= (a+b+c)^2\, 2ab\, [(b+c)(c+a) - a(b) + a(0)]$

$= (a+b+c)^2\, 2ab\, [(b+c)(c+a) - ab]$

$= 2ab\,(a+b+c)^2\, [bc + ba + c^2 + ca - ab]$

$= 2ab\,(a+b+c)^2\, [bc + c^2 + ca] = 2abc\,(a+b+c)(b+c+a) = 2abc\,(a+b+c)^3$

31. Show that : $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1$

Solution :

Let $\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$ Perform $C_2 \to C_2 - pC_1$
$C_3 \to C_3 - qC_1$

$\Delta = \begin{vmatrix} 1 & 1 & 1+p \\ 2 & 3 & 1+3p \\ 3 & 6 & 1+6p \end{vmatrix}$ Perform $C_3 \to C_3 - pC_2$

$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 6 & 1 \end{vmatrix}$ Perform $C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$

$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 5 & 1 \end{vmatrix} \Rightarrow 5 - 4 = 1$

32. Show that $\begin{vmatrix} (b+c)^2 & ba & ca \\ ab & (c+a)^2 & cb \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$

Solution :

Let $\Delta = \begin{vmatrix} (b+c)^2 & ba & ca \\ ab & (c+a)^2 & cb \\ ac & bc & (a+b)^2 \end{vmatrix}$. Multiply $R_1\, R_2\, R_3$ by a, b, c respectively

$$\Delta = \frac{1}{abc}\begin{vmatrix} a(b+c)^2 & ba^2 & ca^2 \\ ab^2 & (c+a)^2 b & cb^2 \\ ac^2 & bc^2 & (a+b)^2 c \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

By taking a, b, c common from C_1, C_2 and C_3

$= \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$ Now see example no. 22.

33. If $x \neq y \neq z$ and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ then prove that $xyz = -1$

Solution :

Let $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}; \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

[by taking x, y, z common from C_1, C_2, C_3 respectively from second determinant]

$$= \begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} ; \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

[by interchanging C_1 and C_2 in first determinant)

$$= (1 + xyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \qquad \text{Perform } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1$$

$$= (1 + xyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = (1 + xyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & (y-x)(y+x) \\ 0 & z-x & (z-x)(z+x) \end{vmatrix}$$

$$= (1 + xyz)(y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix} = (1+xyz)(y-x)(z-x)[(z+x)-(y+x)]$$

$$= (1 + xyz)(y-x)(z-x)(z-y) ; (1 + xyz)(x-y)(y-z)(z-x).$$

1·17 [Adjoint of A Matrix]

Definition :

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Transpose of the matrix of cofactors of elements of A is called adjoint of A. It is denoted by $adj\ A$

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then } adj\ A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

where C_{ij} denotes the cofactors of a_{ij} in A

Remark : Adjoint of matrix $A = [a_{ij}] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Examples

34. Compute the adjoint of the Matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

Solution :

Let C_{ij} be cofactors of a_{ij} in A. Then, cofactors of elements of A are given by

$$C_{11} = \begin{vmatrix} 5 & 0 \\ 4 & 3 \end{vmatrix} = 15 \times 3 - (4 \times 0) = 15 \qquad C_{12} = -\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} = -[0 \times 3 - (2 \times 0)] = 0$$

$$C_{13} = \begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix} = (0 \times 4 - 2 \times 5) = -10 \qquad C_{21} = -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = -(2 \times 3 - 4 \times 3) = 6$$

$$C_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = (3 \times 1 - 2 \times 3) = -3 \qquad C_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = -(1 \times 4 - 2 \times 2) = 0$$

$$C_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} = 2 \times 0 - 5 \times 3 = -15 \qquad C_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = -(1 \times 0 - 3 \times 0) = 0$$

$$C_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} = (1 \times 5 - 2 \times 0) = 5 \qquad adj\ A = \begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}^T = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$$

35. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ Show that $adj\, A = 3\, A^T$

Solution :

Let C_{ij} be cofactors of a_{ij} in A. Then cofactors elements of A are given by

$$C_{11} = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = 1 - 4 = -3 \qquad C_{12} = -\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = -(2+4) = -6$$

$$C_{13} = \begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} = -4 - 2 = -6 \qquad C_{21} = -\begin{vmatrix} -2 & -2 \\ -2 & 1 \end{vmatrix} - (-2-4) = 6$$

$$C_{22} = \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = -1 + 4 = 3 \qquad C_{23} = -\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} = -(2+4) = -6$$

$$C_{31} = \begin{vmatrix} -2 & -2 \\ 1 & -2 \end{vmatrix} = 4 + 2 = 6 \qquad C_{32} = -\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} = -(2+4) = -6$$

$$C_{33} = \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = -1 - (-4) = 3$$

$$adj\, A = \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}^T = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix} \qquad \text{—(i)}$$

Now $A^T = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$

$$3A^T = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix} \qquad \text{—(ii)}$$

from (i) to (ii) $\qquad adj\, A = 3A^T$.

1·18 [Inverse of A Matrix]

Definition : Singular Matrix

A square matrix is a singular matrix if its determinant is zero. Otherwise it is called non-singular.

Definition : Inverse of a Matrix

If A and B are two square matrices such that $AB = BA = I$, then B is called inverse of a matrix A. It is denoted as A^{-1} [A is called inverse of B, denoted as $B^{-1} = A$]

Remark :

1. A square matrix A is invertible (*i.e.* A^{-1} exist) iff it is non singular.
2. If $A = [aij]_{n \times n}$ is any square matrix then Inverse of A is given by

$$A^{-1} = \frac{1}{|A|}\, adj\, A$$

3. $(AB)^{-1} = B^{-1} A^{-1}$

Examples

36. Compute Inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

Solution :

Let $|A| = 1(3\times 2-1) - 2(2\times 2-3) + 3(2\times 1 - 3\times 3)$; $(6-1) - 2(1) + 3(-7)$; $5-2-21$

$5-23 = -18 \neq 0$ $\quad$ A is non singular matrix $\quad \therefore$ A^{-1} exists

Let C_{ij} be the cofactors of a_{ij} in A, Then cofactors of elements of A are given by

$$C_{11} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = (6-1) = 5 \qquad C_{12} = -\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = -1$$

$$C_{13} = \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = -7 \qquad C_{21} = -\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -(4-3) = -1$$

$$C_{22} = \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = (2-9) = -7 \qquad C_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -(1-6) = 5$$

$$C_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 2-9 = -7 \qquad C_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -(1-6) = 5$$

$$C_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3-4 = -1$$

$$adj\,A = \begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix}^T = \begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix} \; ; \; A^{-1} = \frac{1}{|A|}\,adj\,A = \frac{1}{-18}\begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix}$$

37. Find Inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ and verify that $A^{-1}A = I_3$

Solution :

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \; ; \; |A| = [(16-9) - 3(4-3) + 3(3-4)] \; ; \; 7 - 3(1) + 3(-1) \; ; \; 7-3-3 = 1 \neq 0$$

$\therefore$ A is non singular matrix $\qquad \therefore$ A^{-1} exists

Let C_{ij} be cofactors of a_{ij} in A Then cofactors of elements of A is given by

$$C_{11} = \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 16-9 = 7 \qquad C_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -[4-3] = -1$$

$$C_{13} = \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 3-4 = -1 \qquad C_{21} = -\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = -(12-9) = -3$$

$$C_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 4-3 = 1 \qquad C_{23} = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = -(3-3) = 0$$

$$C_{31} = \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = 9-12 = -3 \qquad C_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = -(3-3) = 0$$

$$C_{33} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 4-3 = 1$$

$$adj\,A = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{Now } A^{-1} = \frac{1}{|A|}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{where } |A| = 1 \quad = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Verification :

$$A^{-1}\cdot A=\begin{bmatrix}7&-3&-3\\-1&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&3&3\\1&4&3\\1&3&4\end{bmatrix}$$

$$=\begin{bmatrix}7\times1-3\times1-3\times1 & 7\times3-3\times4-3\times3 & 7\times3-3\times3-3\times4\\-1\times1+1\times1+0\times1 & -1\times3+1\times4+0\times3 & -1\times3+1\times3+0\times4\\-1\times1+0\times1+1\times1 & -1\times3+0\times4+1\times3 & -1\times3+0\times3+1\times4\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I_3$$

38. Show that $A=\begin{bmatrix}5&3\\-1&-2\end{bmatrix}$ satifies the equation : $x^2-3x-7=0$. Hence find A^{-1}.

Solution :

In order to show that $A=\begin{bmatrix}5&3\\-1&-2\end{bmatrix}$ satifies the equation : $x^2-3x-7=0$, we must show that $A^2-3A-7I=O$

$$\text{Now } A^2=A\,.\,A=\begin{bmatrix}5&3\\-1&-2\end{bmatrix}\begin{bmatrix}5&3\\-1&-2\end{bmatrix}=\begin{bmatrix}5\times5+3\times(-1) & 5\times3+3\times(-2)\\-1\times5-2\times(-1) & -1\times3-2\times(-2)\end{bmatrix}=\begin{bmatrix}22&9\\-3&1\end{bmatrix};\ 3A=\begin{bmatrix}15&9\\-3&-6\end{bmatrix}$$

$$A^2-3A-7I=\begin{bmatrix}22&9\\-3&1\end{bmatrix}-\begin{bmatrix}15&9\\-3&-6\end{bmatrix}-7\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}22-15-7 & 9-9-0\\-3+3-0 & 1+6-7\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$A^2-3A-7I=0\ ;\ A^2-3A=7I$$

Pre multiply by A^{-1}

$$A^{-1}(A^2-3A)=7A^{-1}I_2 \quad ;\quad A^{-1}A^2-3A^{-1}A=7A^{-1}\quad ;\quad A-3I=7A^{-1};$$

$$A^{-1}=\frac{A-3I}{7}=\frac{1}{7}\left[\begin{bmatrix}5&3\\-1&-2\end{bmatrix}-\begin{bmatrix}3&0\\0&3\end{bmatrix}\right]\ ;\ \frac{1}{7}\begin{bmatrix}2&3\\-1&-5\end{bmatrix}=\begin{bmatrix}2/7&3/7\\-1/7&-5/7\end{bmatrix}$$

1·19 [Cramer's Rule]

Consider the system of simultaneous linear equations $a_1x+b_1y=c_1;\ a_2x+b_2y=c_2$.

Let $D=\begin{vmatrix}a_1&b_1\\a_2&b_2\end{vmatrix}$ $\quad D_1=\begin{vmatrix}c_1&b_1\\c_2&b_2\end{vmatrix}$ $\quad D_2=\begin{vmatrix}a_1&c_1\\a_2&c_2\end{vmatrix}$

1. If $D\neq0$, then the system of equations is consistent and has unique solution. Solutions is given by

$$x=\frac{D_1}{D}\qquad y=\frac{D_2}{D}$$

2. If $D=0$, $D_1=0$ and $D_2=0$, then system of equations is consistent and has infinitely many solutions.
3. If $D=0$ and one of D_1 and D_2 is non zero, then the system of equations is inconsistent and has no solution.

Consider the system of simultaneous linear equations

$$a_1x+b_1y+c_1z=d_1\ ;\ a_2x+b_2y+c_2z=d_2\ ;\ a_3x+b_3y+c_3z=d_3$$

Let $D=\begin{vmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix}$, $D_1=\begin{vmatrix}d_1&b_1&c_1\\d_2&b_2&c_2\\d_3&b_3&c_3\end{vmatrix}$, $D_2=\begin{vmatrix}a_1&d_1&c_1\\a_2&d_2&c_2\\a_3&d_3&c_3\end{vmatrix}$, $D_3=\begin{vmatrix}a_1&b_1&d_1\\a_2&b_2&d_2\\a_3&b_3&d_3\end{vmatrix}$.

4. If $D\neq0$, then the system of equations is consistent and has unique solution. Solution is given by

$$x=\frac{D_1}{D}\ ;\ y=\frac{D_2}{D}\ ;\ z=\frac{D_3}{D}$$

5. If $D=0$ and $D_1=D_2=D_3=0$ then the system of equations in consistent and has infinitely many solutions.
6. If $D=0$ and if atleast one of the determinants (D_1, D_2 or D_3) is non-zero, then system of equations is inconsistent

and has no solutions.

Examples

39. Solve using Cramer's rule : $x - 2y = 4$; $-3x + 5y = -7$.

Solution :

$$D = \begin{vmatrix} 1 & -2 \\ -3 & 5 \end{vmatrix} = 5 - 6 = -1 \neq 0$$

$\therefore$ System has unique solution given by $x = \dfrac{D_1}{D}$ and $y = \dfrac{D_2}{D}$

$$\text{where } D_1 = \begin{vmatrix} 4 & -2 \\ -7 & 5 \end{vmatrix} = 20 - 14 = 6 \ ; \quad D_2 = \begin{vmatrix} 1 & 4 \\ -3 & -7 \end{vmatrix} = -7 - (-12) = 5$$

$$\therefore \quad x = \frac{6}{-1} = -6 \quad ; \quad y = \frac{5}{-1} = -5$$

40. Solve using Cramer's rule : $x + 2y = 5$; $3x + 6y = 15$.

Solution :

$$D = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6 - 6 = 0$$

$\therefore$ System either has infinitely many solutions or no solution.

$$D_1 = \begin{vmatrix} 5 & 2 \\ 15 & 6 \end{vmatrix} 30 - 30 = 0 \ ; \quad D_2 = \begin{vmatrix} 1 & 5 \\ 3 & 15 \end{vmatrix} = 15 - 15 = 0 \quad \text{as} \quad D = D_1 = D_2 = 0$$

$\therefore$ System has infinitely many solutions. Let $y = k$ (where k is any real number)

$\therefore$ $x = 5 - 2k$ $\Rightarrow$ $x = 5 - 2k,\ \ y = k$

41. Show that the system of equations $3x - y + 2z = 3$; $2x + y + 3z = 5$; $x - 2y - z = 1$ is consistent and has no solution.

Solution :

$$D = \begin{vmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 1 & -2 & -1 \end{vmatrix} \quad 3(-1+6) + 1(-2-3) + 2(-4-1) \ ; \ 15 - 5 - 10 \ ; \ 10 - 10 = 0$$

$$D_1 = \begin{vmatrix} 3 & -1 & 2 \\ 5 & 1 & 3 \\ 1 & -2 & -1 \end{vmatrix} = 3(-1+6) + 1(-5-3) + 2(-10-1) \ ; \ 15 - 8 - 22 \ ; \ -15 \neq 0$$

$$D_2 = \begin{vmatrix} 3 & 3 & 2 \\ 2 & 5 & 3 \\ 1 & 1 & -1 \end{vmatrix} = 3(-5-3) - 3(-2-3) + 2(2-5) - 24 + 15 - 6 - 30 + 15 = -15 \neq 0$$

$$D_3 = \begin{vmatrix} 3 & -1 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 1 \end{vmatrix} = 3(1+10) + 1(2-5) + 3(-4-1) \ ; \ 33 - 3 - 15 = 15 \neq 0$$

Even if one of the D_1, D_2 or D_3 is zero when $D = 0$, system has no solution.
Clearly the system is inconsistent and has no solution.

42. Solve using determinants $x - y + 3z = 6$; $x + 3y - 3z = -4$; $5x + 3y + 3z = 10$.

Solution :

$$D = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 3 & -3 \\ 5 & 3 & 3 \end{vmatrix} = 1(9+9) + 1(3+15) + 3(3-15) \ ; \ 18 + 18 - 36 = 0$$

$$D_1 = \begin{vmatrix} 6 & -1 & 3 \\ -4 & 3 & -3 \\ 10 & 3 & 3 \end{vmatrix} = 6(9+9) + 1(-12+30) + 3(-12-30) \ ; \ 6 \times 18 + 18 - 126 \ ; \ 108 - 108 = 0$$

$$D_2 = \begin{vmatrix} 1 & 6 & 3 \\ 1 & -4 & -3 \\ 5 & 10 & 3 \end{vmatrix} = (-12+30) - 6(3+15) + 3(10+20) \quad ; \quad 18 - 108 + 90 = 0$$

$$D_3 = \begin{vmatrix} 1 & -1 & 6 \\ 1 & 3 & -4 \\ 5 & 3 & 10 \end{vmatrix} = 1(30+12) + 1(10+20) + 6(3-15)$$

$42 + 30 + 6(-12)$; $72 - 72 = 0$; $D = D_1 = D_2 = D_3 = 0$

$\therefore$ Clearly system has infinitely many solutions

Put $z = k$ in any of two equations say Ist & IIIrd (k is any real number), we get $x - y = 6 - 3k$; $+3y = 10 - 3k$

Solving these two by cramer's rule, we have

$$D = \begin{vmatrix} 1 & -1 \\ 5 & 3 \end{vmatrix} = 3 + 5 = 8 \neq 0 ; D_1 = \begin{vmatrix} 6-3k & -1 \\ 10-3k & 3 \end{vmatrix} = 18 - 9k - (-10 + 3k) - 18 - 9k + 10 - 3k - 28 - 12k$$

$$D_2 = \begin{vmatrix} 1 & 6-3k \\ 5 & 10-3k \end{vmatrix} = 10 - 3k - (30 - 15k) = 10 - 3k - 30 + 15k - 20 + 12k$$

$$x = \frac{D_1}{D} = \frac{28-12k}{8} = \frac{7-3k}{2} \quad ; \quad y = \frac{-20+12k}{8} = \frac{-5+3k}{2}$$

$$\therefore \; x = \frac{7-3k}{2}, \; y = \frac{3k-5}{2}, \; z = k \; (k \text{ is any real number})$$

43. Using Cramer's rule solve the following system of equations:

$2x - 3y + z = 7$; $2x + y - z = 1$; $4x + 3x = -11$ using Cramer's rule.

Solution:

We have $$|A| = \begin{vmatrix} 2 & -3 & 1 \\ 2 & 1 & -1 \\ 4 & 0 & 3 \end{vmatrix} = 1(3-0) + 3(6+4) + 1(0-4) = 6 + 30 - 4 = 32$$

$$|A_1| = \begin{vmatrix} 7 & -3 & 1 \\ 1 & 1 & -1 \\ -11 & 0 & 3 \end{vmatrix} = 7(3-0) + 3(3-11) + 1(0+11) = 21 - 24 + 11 = 8$$

$$|A_2| = \begin{vmatrix} 2 & 7 & 1 \\ 2 & 1 & -1 \\ 4 & -11 & 3 \end{vmatrix} = 2(3-11) - 7(6+4) + 1(-22-4) = -16 - 70 - 26 = -112$$

$$|A_3| = \begin{vmatrix} 2 & -3 & 7 \\ 2 & 1 & 1 \\ 4 & 0 & -11 \end{vmatrix} = 2(-11-0) + 3(-22-4) + 7(0\;\; 4) = -22 - 78 - 28 = -128$$

Hence, $$x = \frac{|A_1|}{|A|} = \frac{8}{32} = \frac{1}{4} ; y = \frac{|A_2|}{|A|} = \frac{-112}{32} = -\frac{7}{2} ; z = \frac{|A_3|}{|A|} = \frac{-128}{32} = -4$$

1·20 [Matrix Method For The Solutions of A Non-Homogeneous System of Simultaneous Equations]

If A is a non-singular matrix, then the system of equations given by $AX = B$ has a unique solution given by $X = A^{-1}B$

Remark :

1. If $|A| = 0$, then the given system of equations is either inconsistent. (*i.e.* it has no solution) or it has infinitely many solutions.

2. If $(adj\ A) \cdot B \neq 0$, then the given system of equations is inconsistent.
3. If $(adj\ A) \cdot B = 0$ then the given system of equations is consistent having infinitely many solutions

Examples

44. Solve the following system of equations by matrix method : $\begin{matrix} 5x + 7y + 2 = 0 \\ 4x + 6y + 3 = 0 \end{matrix}$

Solution :

The given system of equation can be $\begin{matrix} 5x + 7y = -2 \\ 4x + 6y = -3 \end{matrix}$ or $\begin{bmatrix} 5 & 7 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

or $AX = B$ where $A = \begin{bmatrix} 5 & 7 \\ 4 & 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

Consider $|A| = 30 - 28 = 2 \neq 0$ $\quad \therefore$ given system will have unique solution given by $X = A^{-1}B$

clearly $adj\ A = \begin{bmatrix} 6 & -7 \\ -4 & 5 \end{bmatrix}$ $\quad \therefore\ A^{-1} = \dfrac{adj\ A}{|A|} = \dfrac{1}{2}\begin{bmatrix} 6 & -7 \\ -4 & 5 \end{bmatrix}$ $\quad \therefore\ A^{-1} = \begin{bmatrix} 3 & -7/2 \\ -2 & 5/2 \end{bmatrix}$

$$\therefore\ X = \begin{bmatrix} 3 & -7/2 \\ -2 & 5/2 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} -6 - \frac{7}{2}(-3) \\ 4 + \frac{5}{2}(-3) \end{bmatrix} = \begin{bmatrix} -6 + \frac{21}{2} \\ 4 - \frac{15}{2} \end{bmatrix} = \begin{bmatrix} \frac{9}{2} \\ -\frac{7}{2} \end{bmatrix} \quad \therefore\ x = \frac{9}{2},\ y = -\frac{7}{2}$$

45. Show that the following system of equations is consistent:

$$2x - y + 3z = 5\ ; \quad 3x + 2y - z = 7\ ; \quad 4x + 5y - 5z = 9$$

Also, find the solution.

Solution :

The given system of equations can be re-written as $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$

where $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$|A| = 2\begin{vmatrix} 2 & -1 \\ 5 & -5 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -1 \\ 4 & -5 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 2(-10 + 5) + 1(-15 + 4) + 3(15 - 8)$$

$$= 2(-5) + (-11) + 3(7) = -10 - 11 + 21 = 0$$

i.e. A is a singular matrix.

Now we shall check whether $(adj\ A) \cdot B = O$ or $(adj\ A) \cdot B \neq O$

So we will first calculate $adj\ A$

Let C_{ij} be the cofactors of a_{ij} in A. Then cofactors of elements of A are given by

$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -1 \\ 5 & -5 \end{vmatrix} = -5$; $\quad C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -1 \\ 4 & -5 \end{vmatrix} = 11$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 7$; $\quad C_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 3 \\ 5 & -5 \end{vmatrix} = 10$

$C_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix} = -22$; $\quad C_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -1 \\ 4 & 5 \end{vmatrix} = -14$

$C_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} = -5$; $\quad C_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = 11$

$C_{33} = (-1)^{3+3}\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 7$

$$adj\ A = \begin{bmatrix} -5 & 11 & 7 \\ 10 & -22 & -14 \\ -5 & 11 & 7 \end{bmatrix}^T = \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & 11 \\ 7 & -14 & 7 \end{bmatrix}$$

$$\text{Now } (adj\ A) \cdot B = \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & 11 \\ 7 & -14 & 7 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} -25+70-45 \\ 55-154+99 \\ 35-98+63 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now as $(adj\ A) \cdot B = 0$ given system of equations has infinitely many solutions

Put $z = k$ in first two equations, we get $2x - y = 5 - 3k$

$3x + 2y = 7 + k$

i.e. $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5-3k \\ 7+k \end{bmatrix}$ *i.e.* $AX = B$ where $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 5-3k \\ 7+k \end{bmatrix}$

Now $|A| = 7 \neq 0$ and $adj\ A = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$ $\therefore$ $A^{-1} = \dfrac{adj\ A}{|A|} = \begin{bmatrix} 2/7 & 1/7 \\ -3/7 & 2/7 \end{bmatrix}$

Now $X = A^{-1}B$ (A is non singular)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2/7 & 1/7 \\ -3/7 & 2/7 \end{bmatrix} \begin{bmatrix} 5-3k \\ 7+k \end{bmatrix} = \begin{bmatrix} \frac{2}{7}(5-3k)+\frac{1}{7}(7+k) \\ -\frac{3}{7}(5-3k)+\frac{2}{7}(7+k) \end{bmatrix} ; \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{17-5k}{7} \\ \frac{11k-1}{7} \end{bmatrix}$$

$\therefore$ $x = \dfrac{17-5k}{7}, y = \dfrac{11k-1}{7}, z = k$ (k is any real number)

1·21 [Rank of A Matrix]

Definition : Order of Minor

If any r rows and any r columns from an $m \times n$ matrix A are retained and remaining $(m - n)$ rows and $(n - r)$ columns are removed, then the determinant of the remaining $r \times r$ submatrix of A is called a minor of A of order r

Example : Consider the matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(i) elements a_{11}, a_{21} etc are minors of order 1

(ii) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$ are minors of order 2.

Definition : Rank of a Matrix

The rank of a matrix A is called r if

(a) It has atleast one non-zero minor of order r

(b) Every minor of A of order higher than r is zero. It is denoted by $\rho(A)$.

Remark :

1. There exists a non-zero minor of order $r \Rightarrow$ the rank is $\geq r$.
2. All minors of order $(r + 1)$ are zero $\Rightarrow$ the rank is $\leq r$.

Examples

46. Find the rank of the following matrices

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 9 & 7 & 1 \\ 2 & 4 & 0 \end{bmatrix}$

Solution :

Now as
$|A| = 1(0-4) - 2(0-2) + 3(36-14) - 4 + 4 + 66 = 66 \neq 0$ $\therefore \rho(A) = 3$
'3' rowed minor in A is non-zero and there is no other higher order minor

2. $A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore \rho(A) \neq 3$

Consider $|A| = -1(0-0) - 1(0-0) + 2(0-0) = 0$

Now consider a '2' rowed minor in A say $\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix}$ now $\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1 \neq 0$ $\therefore \rho(A) = 2$

3. $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}_{3\times 4}$; $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}$. The highest order minor is 3 $\therefore \rho(A) \leq 3$.

Now consider minor of order 3 say $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{vmatrix}$; $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} = 2 \cdot 0 = 0$

Similarly all thes minors of order '3' vanishes $\therefore \rho(A) < 3$

Now consider a minor of order '2' say $\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix}$ where $\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} = 6 - 3 = 3 \neq 0$ $\therefore \rho(A) = 2$

4. $A = \begin{bmatrix} 3 & 4 & 12 \\ 9 & 12 & 15 \\ -6 & -8 & -10 \end{bmatrix}$; $A = \begin{bmatrix} 3 & 4 & 12 \\ 9 & 12 & 15 \\ -6 & -8 & -10 \end{bmatrix}$

Consider $|A| = 3(-120+120) - 4(-90+90) + 12(-72+72) = 0$. Clearly $\rho(A) \leq 2$

Now consider a minor of order '2' say $\begin{vmatrix} 4 & 12 \\ 12 & 15 \end{vmatrix}$. Now $\begin{vmatrix} 4 & 12 \\ 12 & 15 \end{vmatrix} = -84 \neq 0$ $\therefore \rho(A) = 2$

5. $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 3 \end{bmatrix}$. Now $\rho(A) \leq 3$. Consider any 3 rowed minor say $\begin{vmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ 1 & 3 & 4 \end{vmatrix}$

Now $3\begin{vmatrix} 1 & 3 & 4 \\ 1 & 3 & 4 \\ 1 & 3 & 4 \end{vmatrix} = 0$ (Taking 3 common from row 2)

Similarly we find that all other '3' rowed minor are 0 $\therefore \rho(A) \leq 2$

Consider any '2' rowed minor say $\begin{vmatrix} 1 & 4 \\ 3 & 12 \end{vmatrix}$. Where $\begin{vmatrix} 1 & 4 \\ 3 & 12 \end{vmatrix} = 0$

Similarly all other 2-rowed minor are zero $\therefore \rho(A) \neq 2$ Hence $\rho(A) = 1$

1·22 [Elementary Transformation]

1. Elementary transformation on a matrix can be done by :

   Interchanging any two rows (or colum no.) represented as R_{ij} or C_{ij} (if i^{th} rows (or column) and j^{th} row (or column) are interchanged).

2. Multiplication of the elements of any row (or column) by a non-zero scaler quantity say k, represented as $k\,R_j$ or $(k\,C_j)$ [If j^{th} rows or j^{th} columns is multiplied by k]

3. The addition (or subtraction) of k times of j^{th} row (or column) to i^{th} row (or column) to be denoted as $R_i \rightarrow R_i \pm k\,R_j\,(C_i = C_i \pm k\,C_j)$

Remark :

1. Two matrices are said to be equivalent if one is obtained from the other by elementary transformations. Equivalence is indicated by '~'.
2. Elementary transformations does not alter the rank of a matrix.

1·23 [Echelon form of a Matrix]

Rank = Number of non-zero rows in upper triangular matrix

[Upper triangular matrix is a square matrix in which all the elements below the principal diagonal are zero.]

Steps to Reduce matrix in the upper triangular form.

Step 1

Use row or columns transformations to get a non-zero element (1) in the first row and first column position (a_{11}). [Divide the first row by the element in the position of a_{11} if it is not 1].

Step 2

Subtract suitable multiples of the first row from the other rows to get 'zeros' in the reminder of first column.

Step 3

Subtract suitable multiples of the one column from the other column to get zeros in remainder of first row.

Step 4

Repeat the steps from (i) to (iii) until all elements below the main diagonal becomes zero.

Examples

47. Reduce the matrix given below into upper triangular matrix $A = \begin{bmatrix} 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -2 \\ 3 & -1 & -1 & 7 \end{bmatrix}$.

Solution :

As the element at position 'a_{11}' is 1. $\therefore$ Perform the following operations to get zeros at position a_{21} and a_{31}.

$$A = \begin{bmatrix} 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -2 \\ 3 & -1 & -1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 3 & -3 & -12 \\ 0 & 2 & -4 & -8 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 1 & -1 & -4 \\ 0 & 2 & -4 & -8 \end{bmatrix} R_2 \rightarrow \frac{R_2}{3}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & -2 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - 2R_2$$

48. Find the rank of the following matrices using echelon form.

Solution :

$$\text{(a)} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

Number of non zero rows = 2 $\quad \therefore \rho(A) = 2$

(b) $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 & 2/3 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix} R_1 \to \frac{R_1}{3} \sim \begin{bmatrix} 1 & -1/3 & 2/3 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 6R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & -1/3 & 2/3 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{bmatrix} R_2 \to R_2/2 \sim \begin{bmatrix} 1 & -1/3 & 2/3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2$

Number of non-zero rows = 2 $\rho(A) = 2$

(c) $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -\frac{6}{7} & \frac{11}{7} \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

$R_2 \to -\frac{R_2}{7} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -\frac{6}{7} & \frac{11}{7} \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} R_3 \to R_3 + 7R_2 \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -\frac{6}{7} & \frac{11}{7} \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 \to R_4 + \frac{1}{2} R_3$

Clearly $\rho(A) = 3$ [as number of non-zero rows = 3]

(d) $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 2 & 3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & -1 & -2 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -2 & 1 \end{bmatrix} R_2 \to -R_2$

$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & 4 \end{bmatrix} R_3 \to R_3 + R_2 \qquad \rho(A) = 3 \qquad$ (as number of non-zero rows = 3)

1·24 [Normal Form]

With the help of elementary transformations, a non-zero matrix say 'A' can be reduced to any one of the following form.

(a) I_r (b) $[I_r \ 0]$ (c) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

(Normal Forms)

Here I_r is an identity (Normal Forms) matrix of order r.

'r' obtained from elementary transformation is called as rank of a matrix A *i.e.* $\rho(A) = r$

Remark :

Both row and column transformations could be used.

Steps do reduce Matrix in the Normal form

Step 1

Interchange rows (or columns) to get a non-zero element (1) in the first row and first column position (a_{11}) [divide the first row by the element in the position of a_{11} if it is not 1].

Step 2

Subtract the suitable multiples of the first row from the other rows to get 'zeros' in the remainder of the first column.

Step 3

Subtract the suitable multiples of the first column from the other column to get the zeros in the remainder of the first row.

Step 4

Repeat the steps from (1) to (3) starting with the element in the second row and second column position (a_{22}).

Step 5

Continue this process until the matrix is obtained of any one of the form (a, b, c or d)

Examples

49. Reduce the matrix given below into normal form $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$.

Solution :

Divide R_1 by 3 to get the element '1' at the position 'a_{11}'

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & -1 & \frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} \quad R_2 \to R_2 - 2R_1 \text{ (to get } a_{21} = 0)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & \frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} \quad \begin{array}{ll} C_2 \to C_2 + C_1 & (\text{to get } a_{21} = 0) \\ C_3 \to C_3 - \frac{4}{3}C_1 & (\text{to get } a_{23} = 0) \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} \quad R_2 \to -R_2 \text{ (to get '}a_{21}\text{' = 1)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \quad C_3 \to C_3 + \frac{4}{3}C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C_3 \to C_3(-3)$$

Hence Normal Form is obtained.

50. Find the rank of the following matrices using nornal form.

Solution :

(a) $$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -2 & 1 & 5 \\ 0 & 7 & 2 & 3 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 5 \\ 0 & 7 & 2 & 3 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2 - 2C_1 \\ C_4 \to C_4 + C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & 7 & 2 & 3 \end{bmatrix} \quad R_2 \to -\frac{R_2}{2}$$

(to get '1' at the position 'a_{22}')

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{5}{2} \\ 0 & 0 & \frac{11}{2} & \frac{41}{2} \end{bmatrix} \quad R_3 \to R_3 - 7R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{11}{2} & -\frac{41}{2} \end{bmatrix} \quad \begin{matrix} C_3 \to C_3 + \frac{1}{2}C_2 \\ C_4 = C_4 + \frac{5}{2}C_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{41}{11} \end{bmatrix} \quad R_3 \to R_3/\frac{11}{2}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \quad C_4 \to C_4 - \frac{41}{11}C_3$$

$$= [I_3\ 0] \quad \therefore \rho(A) = 3$$

(b) $$A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -4 & -4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -4 & -4 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2 - 3C_1 \\ C_3 \to C_3 - 4C_1 \\ C_4 \to C_4 - 5C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -2 \\ 0 & 2 & -4 & -4 \end{bmatrix} \quad R_2 \to -R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} C_3 \to C_3 + 2C_2 \\ C_4 \to C_4 + 2C_2 \end{matrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \quad \rho(A) = 2$$

(c) $$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & -5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & -5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - 2C_1 \\ C_4 \to C_4 - 3C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & -5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \to \frac{R_2}{2}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -8 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -8 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_3 \to C_3 + C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{6}{8} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to -\frac{R_3}{8}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_4 \to C_4 - \frac{6}{8}C_3$$

$$\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \quad \therefore \rho(A) = 3$$

1.25 [Consistency of Linear Simultaneous Equations]

Consider a system of m simultaneous linear equations in n unknowns $x_1, x_2, ..., x_n$ given by

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2$$
$$...$$
$$...$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = b_m$$

The system of equations can be written in matrix form as $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ or $AX = B$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ & $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times n}$

The matrix A is called the coefficient matrix and the matrix

$$C = [A : B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & \vdots & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} & \vdots & b_m \end{bmatrix}$$

is called the augmented matrix of the given system of equations.

if $b_1 = b_2 = b_3 ... = b_m = 0$, then $B = 0$ and the matrix equation $AX = B$ reduce to $AX = 0$.

Such a system of equation is called a system of homogeneous linear equation.

If at least one of $b_1, b_2, b_3, ..., b_m$ is non-zero then $B \neq 0$. Such a system of equation is called a system of non-homogeneous linear equation.

Working Rule for Finding the Solution of the equation $AX = B$

Suppose the coefficient matrix A is of the type $m \times n$, *i.e.*, we have m equations in n unknowns. Write the augmented matrix C and reduce it to a Echelon form by applying only E-row transformations on it. This echelon form will enable us to know the ranks of the augmented matrix C and the coefficient matrix A. Then the following different cases arise:

Case I : Rank of $A \neq$ Rank of C

In this case the equation $AX = B$ are inconsistent *i.e.* they have no solution.

Case II : Rank of A = Rank of C = number of unknowns

In this case the equation $AX = B$ has a unique solution.

Case III: Rank of A = Rank of C < Number of unknowns

In this case the equation $AX = B$ has an infinite number of solutions.

Working Rule for Finding the Solution of the Equation $AX = 0$

In case of a homogenous system of linear equations the rank of the augmented matrix C is always same as that of the coefficient matrix. So a homogeneous system of linear equations is always consistent.

Case I : Rank of A = Number of unknowns.

In this case, the equation $AX = 0$ has only the trivial solution.

Case II: Rank of A < Number of unknowns.

In this case the equation $AX = 0$ has an infinite number of non-trivial solutions.

51. Test the consistency and hence solve the following set of equations

$$3x + 3y + 2z = 1 \ ; \ x + 2y = 4 \ ; \ 10y + 3z = -2 \ ; \ 2x - 3y - z = 5$$

Solution. We have $\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$. Then the augmented matrix is given by

$$C = [A : B] = \begin{bmatrix} 3 & 3 & 2 & \vdots & 1 \\ 1 & 2 & 0 & \vdots & 4 \\ 0 & 10 & 3 & \vdots & -2 \\ 2 & -3 & -1 & \vdots & 5 \end{bmatrix}$$

Operate $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & \vdots & 4 \\ 3 & 3 & 2 & \vdots & 1 \\ 0 & 10 & 3 & \vdots & -2 \\ 2 & -3 & -1 & \vdots & 5 \end{bmatrix}$$

Operate $R_2 \rightarrow R_2 - 3R_1$, $R_4 \rightarrow R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 & \vdots & 4 \\ 0 & -3 & 2 & \vdots & -11 \\ 0 & 10 & 3 & \vdots & -2 \\ 0 & -7 & -1 & \vdots & -3 \end{bmatrix}$$

Operate $R_3 \rightarrow R_3 + 3R_2$, $R_4 \rightarrow R_4 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & \vdots & 4 \\ 0 & -3 & 2 & \vdots & -11 \\ 0 & 1 & 9 & \vdots & -35 \\ 0 & -1 & -5 & \vdots & 19 \end{bmatrix}$$

Operate $R_1 \rightarrow R_1 - 2R_3$, $R_2 \rightarrow R_2 + 3R_3$ and $R_4 \rightarrow R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 0 & -18 & \vdots & 74 \\ 0 & 0 & 29 & \vdots & -116 \\ 0 & 1 & 9 & \vdots & -35 \\ 0 & 0 & 4 & \vdots & -16 \end{bmatrix}$$

Operate $R_2 \leftrightarrow R_3$, $R_4 \rightarrow \frac{1}{4} R_4$

$$\sim \begin{bmatrix} 1 & 0 & -18 & \vdots & 74 \\ 0 & 1 & 9 & \vdots & -35 \\ 0 & 0 & 29 & \vdots & -116 \\ 0 & 0 & 1 & \vdots & -4 \end{bmatrix}$$

Operate $R_1 \rightarrow R_1 + 18R_4$, $R_2 \rightarrow R_2 - 9R_4$, $R_3 \rightarrow R_3 - 29R_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & -4 \end{bmatrix}$$

Operate $R_3 \leftrightarrow R_4 \sim \begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & -4 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$

$\Rightarrow$ Rank of $C = 3$, Rank of $A = 3$.

$\Rightarrow$ Rank of A = Rank of C = number of unknowns, hence the given system is consistent and possesses a unique solution.

In matrix notation the above matrix reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4 \\ 0 \end{bmatrix} \Rightarrow x = 2,\ y = 1,\ z = -4.$

52. Test for consistency and solve the system.

$$5x + 3y + 7z = 4\ ;\ 3x + 26y + 2z = 9\ ;\ 7x + 2y + 10z = 5.$$

Solution. We have $A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *and* $B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$

Then the augumented matrix is given by $C = [A : B] = \begin{bmatrix} 5 & 3 & 7 & \vdots & 4 \\ 3 & 26 & 2 & \vdots & 9 \\ 7 & 2 & 10 & \vdots & 5 \end{bmatrix}$

Operate $R_1 \to \frac{1}{5} R_1 \sim \begin{bmatrix} 1 & 3/5 & 7/5 & \vdots & 4/5 \\ 3 & 26 & 2 & \vdots & 9 \\ 7 & 2 & 10 & \vdots & 5 \end{bmatrix}$

Operate $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 7R_1 \sim \begin{bmatrix} 1 & 3/5 & 7/5 & \vdots & 4/5 \\ 0 & 121/5 & -11/5 & \vdots & 33/5 \\ 0 & -11/5 & 1/5 & \vdots & -3/5 \end{bmatrix}$

Operate $R_3 \to R_3 + \frac{1}{11} R_2 \sim \begin{bmatrix} 1 & 3/5 & 7/5 & \vdots & 4/5 \\ 0 & 121/5 & -11/5 & \vdots & 33/5 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \Rightarrow$ Rank of $C = 2$, Rank of $A = 2$.

$\Rightarrow$ Rank of A = Rank of C < number of unknowns, hence the given system is consistent and possess infinite number of solutions.

The matrix form of the system is given by

$$\begin{bmatrix} 1 & 3/5 & 7/5 \\ 0 & 121/5 & -11/5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4/5 \\ 33/5 \\ 0 \end{bmatrix} \Rightarrow x + \frac{3y}{5} + \frac{7}{5}z = \frac{4}{5};\ \ \frac{121}{5}y - \frac{11}{5}z = \frac{33}{5}$$

Let $z = k$; then we have $11y - k = 3 \Rightarrow y = \frac{3}{11} + \frac{k}{11}$. Also $x = -\frac{16}{11}k + \frac{7}{11}$

Here k can take infinite number of valves so x, y, z also take infinite number of values. Thus there exists infinite number of solutions.

53. Solve the equation

$$x + 2y + 3z = 0\ ;\ 3x + 4y + 4z = 0\ ;\ 7x + 10y + 12z = 0.$$

Solution. We have

$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and hence $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now $$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$$

Operate $R_3 \to R_3 - 3R_1 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 7 & 10 & 12 \end{bmatrix}$

Operate $R_3 \to R_3 - 7R_1 - 2R_2 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ Rank of $A = 3$ = number of unknown, hence the given equations have only a trivial solution *i.e.* $x = y = z = 0$.

54. For what value of η the equations

$$x + y + z = 1 \;;\; x + 2y + 4z = \eta \;;\; x + 4y + 10z = \eta^2$$

have a solution and solve them completely in each case.

Solution. The matrix form of the given system is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \eta \\ \eta^2 \end{bmatrix}$; $C = [A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 4 & : & \eta \\ 1 & 4 & 10 & : & \eta^2 \end{bmatrix}$

Operate $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1 \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \eta - 1 \\ 0 & 3 & 9 & : & \eta^2 - 1 \end{bmatrix}$

Operate $R_3 \to R_3 - 3R_2 \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \eta - 1 \\ 0 & 0 & 0 & : & \eta^2 - 3\eta + 2 \end{bmatrix}$ ... (*i*)

Now the given equation will be consistent if and only if

$\eta^2 - 3\eta + 2 = 0$ *i.e.*, iff $(\eta - 2)(\eta - 1) = 0$ *i.e.*, iff $\eta = 2$ or $\eta = 1$ ... (*i*)

Case I. If $\eta = 2$, the equation (*i*) becomes $y + 3z = 1, x + y + z = 1$ $\therefore$ $y = 1 - 3z$, $x = 2z$

Thus, $x = 2k$, $y = 1 - 3k, z = k$ constitute the general solution, where k is any arbitrary constant.

Case II. If $\eta = 1$, the equation (*i*) becomes $y + 3z = 0, x + y + z = 1, y = -3z$, $x = 1 + 2z$

Thus $x = 1 + 2c$, $y = -3c, z = c$ constitute the general solution, where c is any arbitrary constant.

55. Find what values of a and b do the equations $x + 2y + 3z = 6$; $x + 3y + 5z = 9$; $2x + 5y + az = b$ have (*i*) no solution (*ii*) a unique solution (*iii*) more than one solution.

Solution. The given system of equations is $AX = B$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & a \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 9 \\ b \end{bmatrix}$.

The augmented matrix $C = \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 1 & 3 & 5 & : & 9 \\ 2 & 5 & a & : & b \end{bmatrix}$; $C \sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 2 & : & 3 \\ 0 & 1 & a - b & : & b - 12 \end{bmatrix}$

(On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$)

$\sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 2 & : & 3 \\ 0 & 0 & a - 8 & : & b - 15 \end{bmatrix}$ (On applying $R_3 \to R_3 - R_2$)

If $a = 8,\ b \neq 15$ we have $\rho(A) = 2,\ \rho(C) = 3$, so $\rho(A) \neq \rho(C)$ hence system has no solution.
Similarly if $a \neq 8$ and $b \neq 15$ we have $\rho(A) = 3,\ \rho(C) = 3$

$\therefore\ \rho(A) = \rho(C) =$ number of unknowns hence the system will have unique solution.
Further if $a = 8$ and $b = 15$ we have $\rho(A) = 2,\ \rho(C) = 2$

$\therefore\ \rho(A) = \rho(C) < 3$ hence more than one solution.

56. Find the values of a and b such that the rank of matrix $A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & a & b \end{bmatrix}$ is 2.

Solution. $A \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & -5 & a+3 & b-6 \end{bmatrix}$ On applying $R_3 \to R_3 - 3R_2$ and $R_2 \to R_2 - 2R_1$

$A \sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & 0 & a-4 & b-6 \end{bmatrix}$ On applying $R_3 \to R_3 + R_2$

Since rank of A is given to be 2 so we have $a - 4 = 0, b - 6 = 0 \Rightarrow a = 4, b = 6$.

57. For what value of μ the rank of matrix $\begin{bmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ -1 & 0 & \mu \end{bmatrix}$ is 2?

Solution. Any row or column operations on a matrix do not change its rank.

Let $A = \begin{bmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ -1 & 0 & \mu \end{bmatrix}$. Operating $R_1 \leftrightarrow R_3$ thus $A \sim \begin{bmatrix} -1 & 0 & \mu \\ -1 & \mu & -1 \\ \mu & -1 & 0 \end{bmatrix}$

Then applying $R_3 \to R_3 + \mu R_1$ given $A \sim \begin{bmatrix} -1 & 0 & \mu \\ 0 & \mu & -1 \\ 0 & -1 & \mu^2 \end{bmatrix}$

Applying $R_3 \to \mu_3 + \frac{1}{\mu} R_2$ we get $A \sim \begin{bmatrix} -1 & 0 & \mu \\ 0 & \mu & -1 \\ 0 & 0 & \mu^2 - \frac{1}{\mu} \end{bmatrix}$

Since rank of A is 2 we must have $\mu^2 - \frac{1}{\mu} = 0$ or $\mu^3 = 1\ \therefore\ \mu = 1$.

Exercise

1. Contstruct a 2x3 matrix whose elements a_{ij} are given by

(1) $i \cdot j$ **Ans :** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ (2) $\frac{(2i+j)^2}{2}$ **Ans :** $\begin{bmatrix} \frac{9}{2} & 8 \\ \frac{25}{2} & 18 \end{bmatrix}$

2. If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$ find x, y, z, w **Ans.** $x = 1; y = 2; z = 4, w = 5$

3. Find x, y, a and b if $\begin{bmatrix} 2x-3y & a-b & 3 \\ 1 & x+4y & 3a+4b \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 6 & 29 \end{bmatrix}$ **Ans :** $x = 2, y = 1, a = 3, b = 5$

4. If $A = [a_{ij}] = \begin{bmatrix} 1 & 3 & 2 \\ 9 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ and $B = [b_{ij}] = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 9 & 1 \end{bmatrix}$ find $a_{11}b_{11} + a_{22}b_{22}$ **Ans: 2**

5. Find matrices X and Y if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$ **Ans :** $X = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}, Y = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

6. If $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$, find the matrix C such that A+B+C is zero matrix **Ans :** $C = \begin{bmatrix} -3 & 4 & -1 \\ -3 & 0 & -1 \end{bmatrix}$

7. Find x ,y, z and t, if $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 15 & 14 \end{bmatrix}$ **Ans :** $x = 2, y = 9$

8. Find the value of λ, a non-zero scaler if $\lambda\begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 10 \\ 4 & 2 & 14 \end{bmatrix}$ **Ans :** $\lambda = 2$

9. If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, find the value of α for which $A^2 = B$ **Ans : There is no value of α**

10. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 5 & 2 \\ -2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 2 \\ -3 & 5 \\ 5 & 0 \end{bmatrix}$ verify that AB=AC though $B \neq C$, $A \neq 0$.

11. If $A = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & 5 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$ find $A^2 - B^2$ **Ans :** $\begin{bmatrix} -2 & -9 & -1 \\ 3 & 26 & 3 \\ 35 & 15 & 34 \end{bmatrix}$

12. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ then prove that $A^2 - 4A - 5I = 0$

13. Solve the matrix equation $[x \quad -5 \quad -10]\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = 0$ **Ans :** $x = -3, 5$

14. If $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 & -4 \\ -2 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 5 & 2 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. Verify that $A(B - C) = AB - AC$

15. If $A = \begin{bmatrix} 2 & 4 & -1 \\ -1 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$ Find $(AB)^T$ **Ans:** $\begin{bmatrix} 0 & 1 \\ 15 & -2 \end{bmatrix}$

16. If $A = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$ and $B = [1 \quad 0 \quad 4]$, verify that $(AB)^T = B^T A^T$

17. If $A = \begin{bmatrix} 2 & -3 \\ -7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & -4 \end{bmatrix}$, verify that $(A - B)^T = AT - B^T$

18. If $A = \begin{bmatrix} \cos a & -\sin a & 0 \\ \sin a & \cos a & 0 \\ 0 & 0 & 1 \end{bmatrix}$, find adj A and verify that $A(\text{adj}A) = (\text{adj})A = |A| I_3$

Ans: adj $A = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

19. If $A=\begin{bmatrix}-1 & -2 & -2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$, show that adj $A = 3A^T$

20. Find the inverse of the Matrix $A=\begin{bmatrix}a & b\\ c & \frac{1+bc}{a}\end{bmatrix}$ and show that $aA^{-1} = (a^2 + bc + 1)\,I - aA$

Ans. $A^{-1}=\begin{bmatrix}\frac{1+bc}{a} & -b\\ -c & a\end{bmatrix}$

21. Show that $\begin{bmatrix}1 & -\tan\phi/2\\ \tan\phi/2 & 1\end{bmatrix}\begin{bmatrix}1 & \tan\phi/2\\ -\tan\phi/2 & 1\end{bmatrix}^{-1} = \begin{bmatrix}Cos\phi & -Sin\phi\\ Sin\phi & Cos\phi\end{bmatrix}$

22. For the matrix $A=\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}$ and $B=\begin{bmatrix}4 & 5\\ 3 & 4\end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$

23. If $A=\begin{bmatrix}5 & 0 & 4\\ 2 & 3 & 2\\ 1 & 2 & 1\end{bmatrix}$ and $B^{-1}=\begin{bmatrix}1 & 2 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$ find $(AB)^{-1}$ Hint : $(AB)^{-1} = B^{-1}A^{-1}$ **Ans :** $\begin{bmatrix}-2 & 20 & -29\\ -2 & 18 & -25\\ -3 & 29 & -42\end{bmatrix}$

24. Find the inverse of the matrix $\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}$ Also verify your answer (Verify $AA^{-1} = A^{-1}A = I$).

Ans : $A^{-1}=\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$

25. Show that $A=\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}$ satisfies the equation $x^2 - 6x + 17 = 0$. Hence find A^{-1}.

26. Show that $A=\begin{bmatrix}6 & 5\\ 7 & 6\end{bmatrix}$ satisfies the equation $x^2 - 12x + 1 = 0$. Hence find A^{-1}. **Ans :** $A^{-1}=\begin{bmatrix}6 & -5\\ -7 & 6\end{bmatrix}$

27. Show that matrix $A=\begin{bmatrix}1 & 0 & -2\\ -2 & -1 & 2\\ 3 & 4 & 1\end{bmatrix}$ satisfies the equation $A^3 - A^2 - 3A - I_3 = 0$. Hence find A^{-1}.

Ans : $A^{-1}=\begin{bmatrix}-9 & -8 & -2\\ 8 & 7 & 2\\ -5 & -4 & -1\end{bmatrix}$

28. Solve by Cramer's rule.

(1) $2x - y = 17$; $3x + 5y = 6$ **Ans : x = 1, y = – 1, z = –1**

(2) $3x + ay = 4$; $2x + ay = 2, a \neq 0$ **Ans :** $x = 2, y = \frac{-2}{a}$

(3) $2x + 3y = 10$; $x + 6y = 4$ **Ans :** $x = \frac{16}{3}, y = \frac{-2}{9}$

(4) $\frac{2}{x}+\frac{3}{y}+\frac{10}{z} = 4$; $\frac{4}{x}-\frac{6}{y}+\frac{5}{z} = 1$; $\frac{6}{x}+\frac{9}{y}-\frac{20}{z} = 2$ **Ans : x = 2, y = 3, z = 5**

(5) $5x - 7y + z = 11$; $6x - 8y - z = 15$; $3x + 2y - 6z = 7$ **Ans : x = 1, y = – 1, z = – 1**

(6) $2x + y - 2z = 4$; $x - 2y + 2 = -2$; $5x - 5y + z = -2$ **Ans :** $x = \frac{6+3k}{5},\ y = \frac{8+4}{5}, z = k$

(7) $x + 2y = 3$; $4x + 8y = 12$ **Ans : x = 3 – 2k, y = k**

29. Find the rank of the following matrices :

(i) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & 5 \\ 4 & 6 & 10 \\ -8 & -12 & -20 \end{bmatrix}$ **Ans : (i) 3, (ii) 1**

30. Use elementary transformation to find rank of matrix $A=\begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 3 & 5 & 4 \end{bmatrix}$ **Ans : 2**

31. Reduce to normal form of the matrix. And hence find the rank:

$$A=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$$ **Ans : 3**

32. Evaluate $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ **Ans : $abc + 2fgh - af^2 - bg^2 - ch^2$**

33. Evaluate $\begin{vmatrix} 1 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix}$ **Ans : – 20**

34. Evaluate $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ **Ans : $(a-b)(b-c)(c-a)$**

35. Evaluate $\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$ **Ans : $-(a+b+c)(a-b)(b-c)(c-a)$**

36. Using properties of determinants, show that $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

37. Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

38. Prove that $\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$

39. Prove that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

40. Prove that $\begin{vmatrix} a & b-c & c-b \\ a-c & b & c-a \\ a-b & b-a & c \end{vmatrix} = (a+b-c)(b+c-a)(c+a)$.

41. Prove that $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2 - ac)(ax^2 + 2bxy + cy^2)$.

42. Prove that $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2)$

43. Prove that $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ca)^3$

44. Prove that $\begin{vmatrix} a^2 & 2ab & b^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix} = (a^3+b^3)^2$

45. Prove that $\begin{vmatrix} z & x & y \\ z^2 & x^2 & y^2 \\ z^4 & x^4 & y^4 \end{vmatrix} = xyz(x-y)(y-z)(z-x)(x+y)$

CHAPTER - 2

[Eigen Vectors And Cayley Hamilton Theorem]

2·1 [Linear Dependence and Independence of Vectors]

Definition :

A single row or column matrix is called a **vector**.

Example : $X = (1, 2, 3)$, $Y = (3, 1, 2, 4)$ etc.

Definition :

Vectors (Matrices) $X_1, X_2, X_3 \ldots\ldots X_n$ are said to be linearly dependent if

1. All the vectors (row or column matrices) are of the same order
2. $\exists$ scalers $(\lambda_1\ \lambda_2\ \ldots\ldots - \lambda_n)$ (n in number, not all zero)

Such that $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 \ldots\ldots \lambda_n X_n = 0$

Definition :

Vectors (Matrices) $X_1, X_2, X_3 \ldots\ldots X_n$ are said to be linearly independent iff

$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_n \ldots\ldots \lambda_n X_n = 0$ (λ_i 's are scalers) $\Rightarrow \lambda_i = 0\ \forall\ i = 1$ to n

Remark :

Vectors which are not linearly dependent are called linearly independent

Examples

1. Examine the following system of vectors for linearly dependence. If dependent, find the relation between them

(a) $X_1 = (1, -1, 1)$, $X_2 = (2, 1, 1)$, $X_3 = (3, 0, 2)$

Solution :

(a) Consider the matrix equation

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0 \qquad \text{—(A)}$$

i.e. $\lambda_1 (1, -1, 1) + \lambda_2 (2, 1, 1) + \lambda_3 (3, 0, 2) = 0$

$$\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0 \qquad \text{—(1)}$$

$$-\lambda_1 + \lambda_2 + 0\lambda_3 = 0 \qquad \text{—(2)}$$

$$\lambda_1 + \lambda_2 + 2\lambda_3 = 0 \qquad \text{—(3)}$$

This is a homogeneous system of equations

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Perform $R_2 \to R_2 + R_1$
$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_3 \to R_3 + \frac{1}{3} R_2 \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad \lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$ —(4)

$\lambda_2 + \lambda_3 = 0$ —(5)

Let $\lambda_3 = t$

$\lambda_2 = -t$ [From (5)]

$\therefore \quad \lambda_1 + 2(-t) + 3t = 0$ [Substitute $\lambda_2 = -t$ in (4)]

$\therefore \quad \lambda_1 - 2t + 3t = 0 \qquad \therefore \quad \lambda_1 + t = 0 \qquad \therefore \quad \lambda_1 = -t$

Substitute these values in (A) we get

i.e. $-tX_1 - tX_2 + tX_3 = 0$ *i.e.* $t(X_1 + X_2 - X_3) = 0$

$\therefore$ X_1, X_2, X_3 are linearly dependent and relation is $X_1 + X_2 - X_3 = 0$

(b) $X_1 = (1, 2, 3)$, $X_2 = (2, -2, 6)$

Consider the matrix equation $\lambda_1 X_1 + \lambda_2 X_2 = 0$

i.e. $\lambda_1(1, 2, 3) + \lambda_2(2, -2, 6) = 0$

i.e. $\lambda_1 + 2\lambda_2 = 0$ —(1)

$2\lambda_1 - 2\lambda_2 = 0$ —(2)

$3\lambda_1 - 6\lambda_2 = 0$ —(3)

From first two equations we get

$\lambda_1 = \lambda_2$ and $3\lambda_2 = 0 \quad \Rightarrow \quad \lambda_2 = 0$ and $\lambda_1 = 0$

$\therefore$ X_1 and X_2 are linearly independent

(c) $X_1 = (1, -1, 2, 0)$, $X_2 = (2, 1, 1, 1)$, $X_3 = (3, -1, 2, -1)$ $X_4 = (3, 0, 3, 1)$

Consider the matrix equation

$\lambda_1(X_1) + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 = 0$ —(A)

$\lambda_1(1, -1, 2, 0) + \lambda_2(2, 1, 1, 1) + \lambda_3(3, -1, 2, -1) + \lambda_4(3, 0, 3, 1) = 0$

$\lambda_1 + 2\lambda_2 + 3\lambda_3 + 3\lambda_1 = 0 - \lambda_1 + \lambda_2 - \lambda_3 + 0\lambda_4 = 0$

$2\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_4 = 0 \; ; \; 0\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 = 0$

This is a homogeneous system $\begin{bmatrix} 1 & 2 & 3 & 3 \\ -1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 3 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

i.e. $\lambda_1 + 2\lambda_2 + 3\lambda_3 + 3\lambda_4 = 0$ —(1)

$-\lambda_1 + \lambda_2 - \lambda_3 = 0$ —(2)

$2\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_4 = 0$ —(3)

$\lambda_2 - \lambda_3 + \lambda_4 = 0$ —(4)

Now $\lambda_1 = -2\lambda_2 - 3\lambda_3 - 3\lambda_4$ [From (1)]

$-(-2\lambda_2 - 3\lambda_3 - 3\lambda_4) + \lambda_2 - \lambda_3 = 0$ [Substitute in (2)]

$\Rightarrow \quad 3\lambda_2 + 2\lambda_3 + 3\lambda_4 = 0$ —(A)

also $\lambda_2 - \lambda_3 + \lambda_4 = 0$ [4th equation]

i.e. $3\lambda_2 - 3\lambda_3 + 3\lambda_4 = 0$ [Multiplying by (3)] —(B)

Subtracting (A) and (B), we get : *i.e.* $5\lambda_3 = 0$; $\lambda_3 = 0$

i.e. Now $\lambda_2 - 0 + \lambda_4 = 0$ [Substituting $\lambda_3 = 0$ in (4)] *i.e.* $\lambda_2 = -\lambda_4$

Substituting $\lambda_3 = 0$ and $\lambda_2 = -\lambda_4$ in (1) & (3), we get : $\lambda_1 - 2\lambda_4 + 3 \cdot 0 + 3\lambda_4 = 0$

and $2\lambda_1 - \lambda_4 + 0 + 3\lambda_4 = 0$ *i.e.* $\lambda_1 = -\lambda_4$ so we get $\lambda_1 = \lambda_2 = -\lambda_4$ and $\lambda_3 = 0$

Let $\lambda_4 = t$ $\therefore \lambda_1 = \lambda_2 = -t$

Substituting in (A), we get : $-t(X_1) + (-t)\lambda_2 + 0 \cdot X_3 + tX_4 = 0$

$$-tX_1 - tX_2 + tX_4 = 0$$

$$-t(X_1 + X_2 - X_4) = 0$$

$\therefore$ X_1, X_2, X_3 are linearly dependent in the relation is $X_1 + X_2 - X_4 = 0$

2·2 [Eigen Values and Eigen Vectors]

Definition:

(1) Matrix Polynomial :

An expression of the form $A_0 x^m + A_1 x^{m-1} + A_2 x^{m-2} + \ldots\ldots A_{m-1} x + A_m$ where $A_0, A_1, A_2 \ldots\ldots A_m$ are all square matrices of same order n and m is a positive integer, is called n rowed matrix polynomial of degree m.

(2) Eigen Values and Eigen Vectors of Matrix :

Consider square matrix A of order n. A scaler λ is called Eigen Value of A iff there exist a non-zero $n \times 1$ column

matrix $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ such that $AX = \lambda X$.

X is called eigen vector of the matrix A corresponding to eigen value λ of A.

Note : λ is an eigen value of n-rowed square matrix A iff $|A - \lambda I| = 0$

(3) Characteristic Equation of a Matrix :

Consider a square matrix A over a field F and Let λ be an indeterminate, then the matrix $A - \lambda I$ is called the characteristic matrix of A.

(4) Characteristic Polynomial of a Matrix :

The determinant $|A - \lambda I|$ which is an algebraic polynomial in λ of degree n is called characteristic polynomial of A.

(5) Characteristic Equation of a Matrix :

The equation $|A - \lambda I|$ is called characteristic equation of A.

Note : An eigen value λ of a matrix A is always a root of its characteristic equation. Converse is also true *i.e.* every root of the characteristic equation of A is an eigen value of A.

$\therefore$ If we have to find the eigen values of A, we should find the roots of characteristic equation of A.

Result :

If vectors $X_1, X_2 \ldots\ldots X_n$ are arranged as columns of the matrix (say) A and if $\rho(A) = r$, then A will have $n - r$ linearly independent solutions.

Examples

2. Find the eigen values of A where

(i) $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

Solution :

(i) $A=\begin{bmatrix}1&1&-2\\-1&2&1\\0&1&-1\end{bmatrix}$ and $I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$; $\lambda I=\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}$ $\therefore A-\lambda I=\begin{bmatrix}1-\lambda&1&-2\\-1&2-\lambda&1\\0&1&-1-\lambda\end{bmatrix}$

Characteristic equation of A is $|A-\lambda I|=0$ $\begin{vmatrix}1-\lambda&1&-2\\-1&2-\lambda&1\\0&1&-1-\lambda\end{vmatrix}=0$

$(1-\lambda)[(2-\lambda)(-1-\lambda)-1]-1[1(1+\lambda)-0]-2[-1]=0$

$(1-\lambda)[-2-2\lambda+\lambda+\lambda^2-1]-1(1+\lambda)+2=0$

$(1-\lambda)[-3-\lambda+\lambda^2]-1-\lambda+2=0$; $-3-\lambda+\lambda^2+3\lambda+\lambda^2-\lambda^3-1-\lambda+2=0$

$-\lambda^3+2\lambda^2+\lambda-2=0$; $\lambda^3-2\lambda^2-\lambda+2=0$

$(\lambda-1)(\lambda^2-\lambda-2)=0$; $(\lambda-1)(\lambda+1)(\lambda-2)=0$ $\therefore$ $\lambda=1, \lambda=-1, \lambda=2$

(ii) $A=\begin{bmatrix}2&-3&1\\3&1&3\\-5&2&-4\end{bmatrix}$. Let $I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$; $\lambda I=\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}$; $A-\lambda I=\begin{bmatrix}2-\lambda&-3&1\\3&1-\lambda&3\\-5&2&-4-\lambda\end{bmatrix}$

Characteristics equation of A is $|A-\lambda|=0$ *i.e.* $\begin{vmatrix}2-\lambda&-3&1\\3&1-\lambda&3\\-5&2&-4-\lambda\end{vmatrix}=0$

$(2-\lambda)[(1-\lambda)(-4-\lambda)-6]+3[3(-4-\lambda)+15]+1[6+5(1-\lambda)]=0$

$(2-\lambda)[(1-\lambda)(-4-\lambda)-6]+3[-12-3\lambda+15]+[6+5-5\lambda]=0$

$(2-\lambda)[-4-\lambda+4\lambda+\lambda^2-6]+3[3-3\lambda]+11-5\lambda=0$

$(2-\lambda)[\lambda^2+3\lambda-10]+9-9\lambda+11-5\lambda=0$

$2\lambda^2+6\lambda-20-\lambda^3-3\lambda^2+10\lambda+9-9\lambda+11-5\lambda=0$

$-\lambda^3-\lambda^2+2\lambda=0$; $-\lambda[\lambda^2+\lambda-2]=0$; $-\lambda[(\lambda+2)(\lambda-1)]=0$

$\lambda=0, \lambda=-2, \lambda=1$

3. Find the characteristic roots of the matrix $A=\begin{bmatrix}1&0&2\\0&2&0\\0&0&3\end{bmatrix}$.

Solution :

$A=\begin{bmatrix}1&0&2\\0&2&0\\0&0&3\end{bmatrix}$; $I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $\lambda I=\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}$ $\therefore A-\lambda I=\begin{bmatrix}1-\lambda&0&2\\0&-2-\lambda&0\\0&0&3-\lambda\end{bmatrix}$

Characteristic equation of A is $|A-\lambda I|=0$

i.e. $\begin{vmatrix}1-\lambda&0&2\\0&2-\lambda&0\\0&0&3-\lambda\end{vmatrix}=0$ *i.e.* $(1-\lambda)[(2-\lambda)(3-\lambda)]=0$

$-(1-\lambda)(\lambda-2)(3-\lambda)=0$; $(\lambda-2)[-(1-\lambda)(3-\lambda)]=0$; $(\lambda-2)[-3+\lambda+3\lambda-\lambda^2]=0$

$(\lambda-2)(-\lambda^2+4\lambda-3)=0$; $-(\lambda-2)(\lambda^2-4\lambda+3)=0$; $-(\lambda-2)[\lambda^2-3\lambda-\lambda+3]=0$

$-(\lambda-2)[\lambda(\lambda-3)-1(\lambda-3)]=0$; $-(\lambda-2)[(\lambda-1)(\lambda-3)]=0$; $\lambda=2, \lambda=1, \lambda=3$

4. Find the characteristic roots and their corresponding characteristic vectors for the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -7 & 5 & 1 \end{bmatrix}$

Solution :

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -7 & 5 & 1 \end{bmatrix}$. Let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$; $A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 3 & 1-\lambda & 0 \\ -7 & 5 & 1-\lambda \end{bmatrix}$

characteristic equation of A is $|A - \lambda I| = 0$ or $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 3-\lambda & 1-\lambda & 0 \\ -7 & 5 & 1-\lambda \end{vmatrix} = 0$

or $(+1-\lambda)[(1-\lambda)^2] = 0 + (1-\lambda)(1-\lambda)^2 = 0$ *i.e.* $\lambda = 1, \lambda = 1, \lambda = 1$

Now characteristic vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq 0$ corresponding to $\lambda = 1$ is given by $AX = \lambda X$

or $(A - \lambda I)X = 0$ or $(A - I)X = 0$ [Putting $\lambda = 1$] or $\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ -7 & 5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Clearly the coefficient matrix of these equation is of rank 2 Therefore these equation will $3 - 2 = 1$ linearly independent solution

$\therefore$ $3x = 0$; $-7x + 5y = 0$ Let $z = 1$ $\therefore$ $x = 0, y = 0, z = 1$ $\therefore$ $X = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

5. Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

Solution :

$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$; $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\therefore$ $\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$ $A - \lambda I = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$ *i.e.* $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

$(-2-\lambda)[(1-\lambda)(-\lambda)-(12)] - 2[-2\lambda - 6] - 3[-4 + (1-\lambda)] = 0$

$(-2-\lambda)[-\lambda + \lambda^2 - 12] + 4\lambda + 12 + 12 - 3 + 3\lambda = 0$

$= 2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 7\lambda + 21 = 0$

$= -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$ —(1)

Clearly $\lambda = 5$ is root of equation no (1)

$\therefore$ $\lambda^3 + \lambda^2 - 21\lambda - 45 = (\lambda - 5)(\lambda^2 + 6\lambda + 9) = (\lambda - 5)(\lambda + 3)^2$

$\therefore$ $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0 \Rightarrow (\lambda - 5)(\lambda + 3)^2 = 0$ *i.e.* $\lambda = 5, \lambda = -3, \lambda = -3$

The characteristic vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq 0$ corresponding to characteristic root $\lambda = 5$ is given by :

$AX = \lambda X \Rightarrow (A - \lambda I)X = 0$ [Putting $\lambda = 5$] $= (A - 5I)X = 0$

$$= \begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_1 \Leftrightarrow R_3$$

$$= \begin{bmatrix} 1 & 2 & 5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_1 \to \frac{R_1}{-1}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} R_2 \Rightarrow R_2 - 2R_1 \\ R_3 \Rightarrow R_3 + 7R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_3 \Rightarrow R_3 + 2R_2$$

Now coefficient matrix of these equation has rank = 2

$\therefore$ Equations will have $3 - 2 = 1$ linearly independent solution $\quad \therefore \; x + 2y + 5z = 0$

$-8y - 16z = 0 \Rightarrow y = -2z$ and $\quad \therefore \; x + 2y + 5z \Rightarrow x = -z$

Take $z = 1$ $\quad x = -1, y = -2 \quad \therefore \; X = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ is characteristic vector of A

Now the characteristic vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq 0$ corresponding to characteristic root $\lambda = -3$ is

$AX = \lambda X \Rightarrow (A - \lambda I)X = 0$; $(A + 3I)X = 0$ *i.e.* $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e. $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$

Coefficient matrix of these equation has rank = 1

$\therefore$ The equation will have $3 - 1 = 2$ linearly independent solution $\quad x + 2y - 3z = 0$

Take $y = 1, z = 0$ then $x + 2(1) - 3(0) = 0 \Rightarrow x = -2$

Take $y = 0, z = 1$, then $x + 2(0) - 3(1) = 0 \Rightarrow x = 3$

$\therefore$ Characteristic vectors of A are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

6. Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

Solution. The characteristic equation of the given matrix is $|A - \lambda I| = 0$

or $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$ or $(-2-\lambda)[-\lambda(1-\lambda)-12]-2[-2\lambda-6]-3[-4+1(1-\lambda)]=0$

or $\lambda^3+\lambda^2-21\lambda-45=0$ By trial, $\lambda=-3$ satisfies it $\therefore (\lambda+3)(\lambda^2-2\lambda-15)=0$

$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5)=0 \Rightarrow \lambda=-3,-3,5.$ Thus the eigen values of A are $-3, -3, 5$.

(*i*) Corresponding to $\lambda=-3$, the eigen vector is given by

$$\begin{bmatrix}-2-(-3) & 2 & -3\\ 2 & 1-(-3) & -6\\ -1 & -2 & -(-3)\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \text{ or } \begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

which gives one independent equation $x_1+2x_2-3x_3=0$ $\therefore$ Choosing $x_2=0$, we have $x_1-3x_3=0$

$\therefore \quad \frac{x_1}{3}=\frac{x_2}{0}=\frac{x_3}{1}$ giving the eigen vector $(3, 0, 1)$.

Choosing $x_3=0$, we have $x_1+2x_2=0$

$\therefore \quad \frac{x_2}{2}=\frac{x_2}{-1}=\frac{x_3}{0}$ giving the eigen vector $(2, -1, 0)$.

Any other eigen vector corresponding to $\lambda=-3$ will be linear combination of there two.

(*ii*) Corresponding to $\lambda=5$, the eigen vector is given by $\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$-7x_1+2x_2-3x_3=0 \Rightarrow 2x_1-4x_2-6x_3=0\ ;\ -x_1-2x_2-5x_3=0$$

Since only two of them are independent, we can omit one of them from first two equations, we have

$$\frac{x_1}{-12-12}=\frac{x_2}{-6-42}=\frac{x_3}{28-4} \text{ or } \frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{-1} \text{ giving the eigen vector } (1, 2, -1).$$

7. Determine the characteristic roots and the corresponding characteristic vectors of the matrix $A=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$.

Solution. The characteristic equation of the matrix A is $|A-\lambda I|=0$. or $\begin{vmatrix}8-\lambda & -6 & 2\\ -6 & 7-\lambda & -4\\ 2 & -4 & 3-\lambda\end{vmatrix}=0$

or $\lambda^3-18\lambda^2+45\lambda=0 \Rightarrow \lambda(\lambda-3)(\lambda-15)=0 \Rightarrow \lambda=0, 3, 15$ are the characteristic roots of A.

(*i*) Corresponding to $\lambda=0$, the eigen vector $X_1=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ of A are given by the non-zero solution of the equation

$$(A-0I)X=0 \text{ or } \begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Operate $R_1 \leftrightarrow R_3$ $\begin{bmatrix}2 & -4 & 3\\ -6 & 7 & -4\\ 8 & -6 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

Operate $R_2 \rightarrow R_2+3R_1,\ R_3 \rightarrow R_3-4R_1$ $\begin{bmatrix}2 & -4 & 3\\ 0 & -5 & 5\\ 0 & 10 & -10\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

Operate $R_3 \rightarrow R_3+2R_2$ $\begin{bmatrix}2 & -4 & 3\\ 0 & -5 & 5\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ $2x_1-4x_2+3x_3=0$ or $-5x_2+5x_3=0$

or $x_2 = x_3$ and $2x_1 - 4x_2 + 3x_3$. Let us take $x_2 = 1, x_3 = 1$. Then $x_1 = \frac{1}{2}$. Therefore

$$X_1 = \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$ is an eigen vector of A corresponding to the eigen value $\lambda_1 = 0$.

(*ii*) Corresponding to $\lambda = 3$, the eigen vector $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A are given by the non-zero solution of the equations

$$[A - 3I]X = 0 \quad \text{or} \quad \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_1 \to R_1 + R_2$
$$\begin{bmatrix} -1 & -2 & -2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_2 \to R_2 + 6R_1, R_3 \to R_3 + 2R_1$
$$\begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & -8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_3 \to R_3 + \frac{1}{2}R_2$
$$\begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $-x_1 - 2x_2 - 2x_3 = 0$; $16x_2 + 8x_3 = 0$ or $x_3 = -2x_2$ and $-x_1 - 2x_2 - 2x_3$.
Let us take $x_2 = 1, x_3 = -2$, then $x_1 = 2$. Therefore

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$ is an eigen vector of A corresponding to the eigen value 3.

(*iii*) Corresponding to $\lambda = 15$, the eigen vector $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A are given by the non-zero solution of the equation

$$[A - 15I]X = 0 \quad \text{or} \quad \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_1 \to R_1 - R_2$
$$\begin{bmatrix} -1 & 2 & 6 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_2 \to R_2 - 6R_1, R_3 \to R_3 + 2R_1$
$$\begin{bmatrix} -1 & 2 & 6 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $-x_1 - 2x_2 - 6x_3 = 0$; $-20x_2 - 40x_3 = 0$

or $x_2 = -2x_3$ and $-x_1 - 2x_2 - 6x_3 = 0$. Let us take $x_3 = 1$, then $x_2 = -2$ and $x_1 = 2$. Therefore

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$ is an eigen vector of A corresponding to the eigen value $\lambda = 15$.

8. Find the characteristic roots and vectors of the matrix $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.

Solution:

The characteristic equation of A is $|A - \lambda| = \begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$

i.e., $(3 - l)(2 - l)(5 - l) = 0$ i.e., $l = 2, 3, 5$ ...(1)

The characteristic vector of A corresponding to the characteristic root is non-zero solution $(A - \lambda I)X = 0$.

For $\lambda = 2$, $\begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x + y + 4z = 0, 6z = 0, 3z = 0 \Rightarrow x = -y, z = 0$

Hence $X = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. For $\lambda = 3$, $\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow y + 4z = 0, -y + 6z = 0, 2z = 0 \Rightarrow y = 0 = z$

Hence $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. For $\lambda = 5$, $\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow 2x + y + 4z = 0, -3y + 6z = 0 \Rightarrow y = 2z, x = 3z$

Hence $X = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

9. Find the characteristic roots and corresponding characteristic vectors of $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$.

Solution:

The characteristic equation of matrix A is $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda 3 - 13\lambda + 12 = 0$ i.e., $\lambda = 1, 3, -4$

The characteristic vector of A corresponding to characteristic roots is non-zero solution of $(A - \lambda I)X = 0$.

For $\lambda = 1$, $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ -7 & 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_2 \to 2R_1, R_3 \to R_3 + 7R_1$, we get $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 1 \\ 0 & 16 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Applying $R_3 \to R_3 + + 4R_2$, we get $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. $\Rightarrow x + 2y = 0, -4y + z = 0 \Rightarrow z = 4y, x = -2y$

Hence $X = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$. For $\lambda = 3$, $\begin{bmatrix} -1 & 2 & 0 \\ 2 & -2 & 1 \\ -7 & 2 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_2 \to R_2 + 2R_1, R_3 \to R_3 - 7R_1$, we get $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & -12 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 \to R_3 + 6R_2$, we get $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow -x + 2y = 0, 2y + z = 0 \Rightarrow x = 2y, z = -2y$

Hence $X = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. For $\lambda = -4$, $\begin{bmatrix} 6 & 2 & 0 \\ 2 & 5 & 1 \\ -7 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_2 \to R_2 - \frac{R_1}{3}, R_3 \to R_3 + \frac{7}{6}R_1$, we get $\begin{bmatrix} 6 & 2 & 0 \\ 0 & \frac{13}{3} & 1 \\ 0 & \frac{13}{3} & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Applying $R_3 \to R_3 - R_2$, we get $\begin{bmatrix} 6 & 2 & 0 \\ 0 & \frac{13}{3} & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 6x + 2y = 0, \frac{13}{3}y + z = 0 \Rightarrow x = -\frac{y}{3}, z = -\frac{13y}{3}$. Hence $X = \begin{bmatrix} 1 \\ -3 \\ 13 \end{bmatrix}$

2·3 [Cayley Hamilton Theorem]

Cayley Hamilton Theorem :

Every square matrix satisfies its characteristic equation.

Remark :

Cayley Hamilton Theorem helps us to find the inverse of the square matrix.

Examples

10. Verify Cayley Hamilton theorem for the matrix $A = \begin{pmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{pmatrix}$.

Solution :

$A = \begin{pmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\therefore \lambda I = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ $\therefore A - \lambda I = \begin{pmatrix} 3-\lambda & 2 & 4 \\ 4 & 3-\lambda & 2 \\ 2 & 4 & 3-\lambda \end{pmatrix}$.

The characteristic equation of A is $|A - \lambda I| = 0$ $\therefore \begin{vmatrix} 3-\lambda & 2 & 4 \\ 4 & 3-\lambda & 2 \\ 2 & 4 & 3-\lambda \end{vmatrix} = 0$

$\therefore (3-\lambda)[(3-\lambda)^2 - 8] - 2[4(3-\lambda) - 4] + 4[16 - 2(3-\lambda)] = 0$

$(3-\lambda)(9 + \lambda^2 - 6\lambda - 8) - 2(12 - 4\lambda - 4) + 4(16 - 6 + 2\lambda) = 0$

$(3-\lambda)\left[\lambda^2 - 6\lambda + 1) - 2(-4\lambda + 8) + 4(10 + 2\lambda) = 0\right]$

$$3(\lambda^2-6\lambda+1)-\lambda(\lambda^2-6\lambda+1)+8\lambda-16+40+8\lambda=0$$

$$3\lambda^2-18\lambda+3-\lambda^3+6\lambda^2-\lambda+8\lambda-16+40+8\lambda=0$$

$$-\lambda^3+9\lambda^2-3\lambda+27=0$$

Now we shall show show $A^3-9A^2+3A-27I=0$ $A=\begin{pmatrix}3&2&4\\4&3&2\\2&4&3\end{pmatrix}$; $A^2=AA=\begin{pmatrix}3&2&4\\4&3&2\\2&4&3\end{pmatrix}\begin{pmatrix}3&2&4\\4&3&2\\2&4&3\end{pmatrix}$

$$=\begin{bmatrix}9+8+8&6+6+16&12+4+12\\12+12+4&8+9+8&16+6+6\\6+16+6&4+12+12&8+8+9\end{bmatrix};\begin{pmatrix}25&28&28\\28&25&28\\28&28&25\end{pmatrix};\ A^3=A^2\cdot A=\begin{pmatrix}25&28&28\\28&25&28\\28&28&25\end{pmatrix}\begin{pmatrix}3&2&4\\4&3&2\\2&4&3\end{pmatrix}$$

$$=\begin{pmatrix}25\times3+28\times4+28\times2&25\times2+28\times3+28\times4&25\times4+28\times2+28\times3\\28\times3+25\times4+28\times2&28\times2+25\times3+28\times4&28\times4+25\times2+28\times3\\28\times3+28\times4+25\times2&28\times2+28\times3+25\times4&28\times4+28\times2+25\times2\end{pmatrix}=\begin{pmatrix}243&246&240\\240&243&246\\246&240&243\end{pmatrix}$$

Now $A^3-9A^2+3A-27I$

$$=\begin{pmatrix}243&246&240\\240&243&246\\246&240&243\end{pmatrix}-9\begin{pmatrix}25&28&28\\28&25&28\\28&28&25\end{pmatrix}+3\begin{pmatrix}3&2&4\\4&3&2\\2&4&3\end{pmatrix}-27\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}243-225+9-27&246-252+6&240-252+12\\240-252+12+0&243-225+9-27&246-252+6\\246-252+6&240-252+12&243-225+9-27\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

Hence Cayley - Hamilton Theorem is verified.

11. Find the characteristic equation of the matrix. And hence find the inverse $A=\begin{pmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{pmatrix}$.

Solution :

$$A=\begin{pmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{pmatrix};\ I=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad\therefore \lambda I=\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}\quad\therefore A-\lambda I=\begin{pmatrix}2-\lambda&-1&1\\-1&2-\lambda&-1\\1&-1&2-\lambda\end{pmatrix}$$

Characteristic equation of A is $|A-\lambda I|=0$ $\begin{vmatrix}2-\lambda&-1&1\\-1&2-\lambda&-1\\1&-1&2-\lambda\end{vmatrix}=0$

$$(2-\lambda)((2-\lambda)2-1)+1(-(2-\lambda)+1)+1(1-(2-\lambda))=0$$

$$(2-\lambda)\left(4+\lambda^2-4\lambda-1\right)+1(\lambda-2+1)+(1-2+\lambda)=0$$

$$(2-\lambda)((\lambda^2-4\lambda+3)+(\lambda-1)+(\lambda-1)=0$$

$$2\lambda^2-8\lambda+6-\lambda^3+4\lambda^2-3\lambda+2\lambda-2=0;\ -\lambda^3+6\lambda^2-9\lambda+4=0;\ \lambda^3-6\lambda^2+9\lambda-4=0$$

Now to show $A^3-6A^2+9A-4I=0$

$$A=\begin{pmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{pmatrix};\ A^2=\begin{pmatrix}2&-1&-1\\-1&2&-1\\1&-1&2\end{pmatrix}\begin{pmatrix}2&-1&-1\\-1&2&-1\\1&-1&2\end{pmatrix}=\begin{pmatrix}4+1+1&-2-2-1&2+1+1\\-2-2-1&1+4+1&-1-2-2\\2+1+2&-1-2-2&1+1+4\end{pmatrix}$$

$$=\begin{pmatrix}6&-5&5\\-5&6&-5\\5&-5&6\end{pmatrix}$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{pmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} - 6\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + 9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 22-36+18-4 & -21+30-9 & 21-30+9 \\ -21+30-9 & 22-36+18-4 & -21+30-9 \\ 21-30+9 & -21+30-9 & 22-36+18-4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In order to find A^{-1} consider $A^3 - 6A^2 + 9A - 4I = 0$

$A^{-1}(A^3 - 6A^2 + 9A - 4I) = A^{-1} \cdot 0$ [Pre multiplying by A^{-1}]

$$A^2 - 6A + 9I - 4A^{-1} = 0\ ;\ A^2 - 6A + 9I = 4A^{-1}\ ;\ A^{-1} = \frac{1}{4}[A^2 - 6A + 9I]$$

$$A^{-1} = \frac{1}{4}\left[\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{4}\begin{pmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

12. Verify that the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}$ satisfies the characteristic equation and hence compute A^{-1}.

Solution :

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}\ ;\ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\ \therefore\ \lambda I = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\ ;\ A - \lambda I = \begin{pmatrix} 1-\lambda & 2 & 1 \\ 0 & 1-\lambda & -1 \\ 3 & -1 & 1-\lambda \end{pmatrix}$$

Characteristic equation of A is $|A - \lambda I| = 0$ or $\begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 1-\lambda & -1 \\ 3 & -1 & 1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)[(1-\lambda)^2 - 1] - 2[3] + 1[-3(1-\lambda)] = 0$$

$$(1-\lambda)(1+\lambda^2 - 2\lambda - 1) - 6 - 3 + 3\lambda = 0$$

$$(1-\lambda)[\lambda^2 - 2\lambda] - 9 + 3\lambda = 0$$

$$\lambda^2 - 2\lambda - \lambda^3 + 2\lambda^2 - 9 + 3\lambda = 0$$

$$-\lambda^3 + 3\lambda^2 + \lambda - 9 = 0$$

$$\lambda^3 - 3\lambda^2 - \lambda + 9 = 0 \qquad \text{---(1)}$$

Now we shall show: square Matrix A satisfies eq. **(1)** *i.e.* To show $A^3 - 3A^2 - A + 9I = 0$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}\ ;\ A^2 = A \cdot A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+3 & 2+2-1 & 1-2+1 \\ 0+0-3 & 0+1+1 & 0-1-1 \\ 3+0+3 & 6-1-1 & 3+1+1 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{pmatrix}$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4+0+0 & 8+3+0 & 4-3+0 \\ -3+0-6 & -6+2+2 & -3-2-2 \\ 6+0+15 & 12+4-5 & 6-4+5 \end{pmatrix} = \begin{pmatrix} 4 & 11 & 1 \\ -9 & -2 & -7 \\ 21 & 11 & 7 \end{pmatrix}$$

Now $A^3 - 3A^2 - A + 9I = \begin{pmatrix} 4 & 11 & 1 \\ -9 & -2 & -7 \\ 21 & 11 & 7 \end{pmatrix} - 3\begin{pmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 4-12-1+9 & 11-9-2+0 & 1-0-1+0 \\ -9+9+0+0 & -2-6-1+9 & -7+6+1+0 \\ 21-18-3+0 & 11-12+1+0 & 7-15-1+9 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In order to find A^{-1} consider $A^3 - 3A^2 - A + 9I = 0$

$A^{-1}(A^3 - 3A^2 - A + 9I) = A^{-1}0 = 0$ [Pre multiplying by A^{-1}]

$A^{-1}A^3 - 3A^{-1}A^2 - A^{-1}A + 9A^{-1}I = 0$

$A^2 - 3A - I + 9A^{-1} = 0 \Rightarrow A^2 - 3A - I = -9A^{-1}$ $\quad \therefore \; A^{-1} = \frac{-1}{9}[A^2 - 3A - I]$

Consider $A^2 - 3A - I$

$$= \begin{pmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{pmatrix} - 3\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4-3-1 & 3-6-0 & 0-3-0 \\ -3+0+0 & 2-3-1 & -2+3+0 \\ 6-9-0 & 4+3+0 & 5-3-1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -3 & -3 \\ -3 & -2 & 1 \\ -3 & 7 & 1 \end{pmatrix} \qquad \therefore \; A^{-1} = \frac{-1}{9}[A^2 - 3A - I] = \frac{1}{9}\begin{pmatrix} 0 & 3 & 3 \\ 3 & 2 & -1 \\ 3 & -7 & -1 \end{pmatrix}$$

13. Verify that the matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$ satisfies its characteristic equation and hence find A^{-1}.

Solution:

$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$; $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad \therefore \lambda I = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$; $A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{pmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$ *i.e.* $\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$

$(1-\lambda)[(2-\lambda)(3-\lambda)] + 2(-2[2-\lambda]) = 0$; $(1-\lambda)[6 - 2\lambda - 3\lambda + \lambda^2] - 4(2-\lambda) = 0$

$(1-\lambda)[\lambda^2 - 5\lambda + 6] - 8 + 4\lambda = 0$; $\lambda^2 - 5\lambda + 6 - \lambda^3 + 5\lambda^2 - 6\lambda - 8 + 4\lambda = 0$

$-\lambda^3 + 6\lambda^2 - 7\lambda - 2 = 0$; $\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$

Now we show that A satisfies above equation. Consider $A^3 - 6A^2 + 7A + 2I$

$$= \begin{pmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 49 \end{pmatrix} - 6\begin{pmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{pmatrix} + 7\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 21-30+7+2 & 0+0+0+0 & 34-48+14+0 \\ 12-12+0+0 & 8-24+14+2 & 23-30+7+0 \\ 34-48+14+0 & 0+0+0+0 & 49-78+21+2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ Matrix A satisfies its characteristic equation

In order to find A^{-1} consider $A^3 - 6A^2 + 7A + 2I = 0$

$A^{-1}(A^3 - 6A^2 + 7A + 2I) = A^{-1}0$ [Pre multiplying by A^{-1}] $A^2 - 6A + 7I + 2A^{-1} = 0$

$$A^2 - 6A + 7I = -2A^{-1}\ ;\ A^{-1} = \frac{-1}{2}[A^2 - 6A + 7I]$$

$$A^2 - 6A + 7I = \begin{pmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 5-6+7 & 0+0+0 & 8-12+0 \\ 2+0+0 & 4-12+7 & 5-6+0 \\ 8-12+0 & 0+0+0 & 13-18+7 \end{pmatrix} = \begin{pmatrix} 6 & 0 & -4 \\ 2 & -1 & -1 \\ -4 & 0 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{-1}{2}\begin{pmatrix} 6 & 0 & -4 \\ 2 & -1 & -1 \\ -4 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{pmatrix}$$

Exercise

1. Examine the following system of vectors for linearly dependence. If dependent, find the relation between then.
   (i) $X_1 = (3, 1, -4)$, $X_2 = (2, 2, -3)$, $X_3 = (0, -4, 1)$
   (ii) $X_1 = (1, 1, 1, 3)$, $X_2 = (1, 2, 3, 4)$, $X_3 = (2, 3, 4, 7)$
   [Ans. (i) Dependent, $2X_1 - 3X_2 - X_3 = 0$, (ii) Dependent, $X_1 + X_2 - X_3 = 0$]

2. Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.
   Verify cayley Hamilton theorem. Hence find A^{-1} **[Ans. $A^{-1} = \frac{1}{20}\begin{bmatrix} 7 & -2 & -3 \\ 1 & 4 & 1 \\ -2 & 2 & 8 \end{bmatrix}$]**

3. Use cayley Hamilton theorem to find A^{-1} where $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ **[Ans. $A^{-1} = \frac{1}{8}\begin{bmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix}$]**

4. If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of the matrix $A = \begin{bmatrix} -2 & -9 & 5 \\ -5 & -10 & 7 \\ -9 & -21 & 14 \end{bmatrix}$, then find $\lambda_1 + \lambda_2 + \lambda_3$ **[Ans. 2]**

5. Find the inverse of the matrix. $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ 2 & -4 & -4 \end{bmatrix}$ using cayley - Haminton. **[Ans. $A^{-1} = \frac{1}{28}\begin{bmatrix} 12 & 4 & 6 \\ 1 & 5 & -3 \\ 5 & -3 & -1 \end{bmatrix}$]**

6. Verify cayley hamilton theorem for the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$. Hence find A^{-1}. **[Ans. $\frac{1}{5}\begin{bmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{bmatrix}$]**

7. Find characteristic equation of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$. Verify cayley hamilton theorem and hence find A^{-1}.

[Ans. $A^{-1} = \begin{bmatrix} -5/12 & 7/12 & 3/12 \\ 7/12 & -8/12 & 3/12 \\ 1/12 & 4/12 & -3/12 \end{bmatrix}$ **]**

8. Verify cayley - hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ Hence find A^{-1}. **[Ans.** $A^{-1} = \frac{1}{11} \begin{bmatrix} -2 & 5 & -1 \\ -1 & -3 & 5 \\ 7 & -1 & -2 \end{bmatrix}$ **]**

9. Find eigen vectors of the following matrix $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ **[Ans.** $\lambda = -1, 1, 4$ **]**

10. Find the eigen values of the matrix $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ **[Ans.** $\lambda = 0, 1, 1$ **]**

CHAPTER - 3

[Limits and Continuity]

3·1 [Introduction]

Let f be a function defined by $f(x) = \dfrac{x^2-4}{x-2}$. Thus $f(x)$ is defined for all x except $x = 2$.

At $x = 2$, $f(x) = \dfrac{2^2-4}{2-2} = \dfrac{0}{0}$. Thus at $x = 2, f(x)$ is not defined because denominator can never be zero.

When $x \neq 2, x-2 \neq 0 \quad \therefore f(x) = \dfrac{x^2-4}{x-2} = \dfrac{(x-2)(x+2)}{x-2} = x+2$

Now we consider the values of $f(x)$ when $x \neq 2$, but the very-very close to 2 and $x < 2$.

x	1.9	1.99	1.999	1.9999	1.99999
$f(x) = x+2$	3.9	3.99	3.999	3.9999	3.99999

It is clear from the above table that as x approaches 2 i.e. as $x \to 2$ through the values less than 2, the value of $f(x)$ approaches 4 i.e. $f(x) \to 4$.

We will express this fact by saying that left hand limit of $f(x)$ as $x \to 2$ exists and is equal to 4 and in symbols we shall write

$$\underset{x \to 2-0}{\text{Lt}}\ f(x) = 4 \qquad \text{or} \qquad \underset{x \to 2^-}{\text{Lt}}\ f(x) = 4$$

Again we consider the values of $f(x)$ when $x \neq 2$, but is very-very close to 2 and $x > 2$.

x	2.1	2.01	2.001	2.0001	2.00001
$f(x) = x+2$	4.1	4.01	4.001	4.0001	4.00001

It is clear from the table given above that as x approaches 2 i.e. as $x \to 2$ through the values greater than 2, $f(x)$ approaches 4 i.e. $f(x) \to 4$. We will express this fact by saying that right hand limit of $f(x)$ as $x \to 2$ exists and is equal to 4 and in symbols we will write.

$$\underset{x \to 2+0}{\text{Lt}}\ f(x) = 4 \qquad \text{or} \qquad \underset{x \to 2^+}{\text{Lt}}\ f(x) = 4$$

Thus we see that $f(x)$ is not defined at $x = 2$ but its left hand and right hand limits as $x \to 2$ exist and are equal.

When

$\underset{x \to 2^-}{\text{Lt}}\ f(x)$ and $\underset{x \to 2^+}{\text{Lt}}\ f(x)$ are equal to the same number l, we say that

$\underset{x \to 2}{Lt}\ f(x)$ exists and is equal to l.

Here in the example considered $\underset{x \to 2^-}{Lt}\ f(x) = \underset{x \to 2^+}{Lt}\ f(x) = 4 \quad \therefore \underset{x \to 2}{Lt}\ f(x)$ exists and is equal to 4

Definition : 1

Limit of a Function :

Let $f(x)$ we a function of x. If for every positive integer ε, however small it may be, there exist a positive number δ such that $0 < |x-a| < \delta \Rightarrow |f(x) - l| < \varepsilon$, then $f(x)$ tends to the limit l as 'x tends to a' (It is denoted as $\underset{x \to a}{Lt}\ f(x) = l$).

Definition : 2

Left Hand Limit :

Let $f(x)$ be a function of x. Function f is said to tend to a limit ℓ as x tends to 'a' (from left), if given positive integer ε, however small it may be, there exists a positive number δ such that

$a-\delta < x < a \Rightarrow |f(x)-\ell| < \varepsilon$ [It is denoted as $\underset{x\to a^-}{Lt} f(x) = \ell$]

Definition : 3

Right Hand Limit :

Let $f(x)$ be the function of x. Function f is said to tend to a limit ℓ as x tends to a (from right), it given positive integer ε, however small it may be, there exists a positive number δ such that

$a < x < a+\delta \Rightarrow |f(x)-\ell| < \varepsilon$ [it is denoted as $\underset{x\to a^+}{Lt} f(x) = \ell$]

Remark : 1

A function will have a limiting value only if its right hand limit is equal to the left hand limit.

$$\underset{x\to a}{Lt} f(x) = \ell \Leftrightarrow \underset{x\to a^+}{Lt} f(x) = \underset{x\to a^-}{Lt} f(x) = \ell$$

Remark : 2

To evaluate left hand limit $x \to a^-$. Put $x = a-h$ in $f(x)$, then $\underset{x\to a^-}{Lt} f(x) = \underset{h\to 0}{Lt} f(a-h)$

Remark 3 :

To evaluate right hand limit $x \to a^+$. Put $x = a+h$ in $f(x)$, then $\underset{x\to a^+}{Lt} f(x) = \underset{h\to 0}{Lt} f(a+h)$

Two Important Results :

$$x \to 0^+ \Rightarrow e^{-1/x} \to 0 \text{ and } x \to 0^- \Rightarrow e^{1/x} \to 0$$

Examples

1. If $f(x) = \dfrac{x^2-9}{x-3}$, find if $\underset{x\to 3}{Lt} f(x)$ exists.

Solution:

$$f(x) \frac{x^2-9}{x-3} \;;\; \underset{x\to 3^-}{Lt} f(x) = \underset{h\to 0}{Lt} f(3-h) = \underset{h\to 0}{Lt} \frac{(3-h)^2-9}{3-h-3}$$

$$\underset{h\to 0}{Lt} \frac{h^2-6h}{-h} = \underset{h\to 0}{Lt} \frac{h(h-6)}{-h} = \underset{h\to 0}{Lt} -(h-6) = 0+6 = 6$$

Also $\underset{x\to 3^+}{Lt} f(x) = \underset{h\to 0}{Lt} f(3+h) = \underset{h\to 0}{Lt} \dfrac{(3+h)^2-9}{(3+h)-3} = \underset{h\to 0}{Lt} \dfrac{h^2-6h}{h}$; $\underset{h\to 0}{Lt} \dfrac{h(h+6)}{h} = \underset{h\to 0}{Lt} (h+6) = 6$

Now as $\underset{x\to 3^-}{Lt} f(x) = \underset{x\to 3^+}{Lt} f(x)$ $\therefore \underset{x\to 3}{Lt} f(x)$ exists and is equal to 6 i.e. $\underset{x\to 3}{Lt} f(x) = 6$.

2. Show that $\underset{x\to 1}{Lt} f(x)$ does not exists where $f(x) = \begin{Bmatrix} 1+x^2 & 0 \le x \le 1 \\ 2-x & x > 1 \end{Bmatrix}$.

Solution :

$$\underset{x\to 1^-}{Lt} f(x) = \underset{x\to 1^-}{Lt} (1+x^2) = \underset{h\to 0}{Lt} (1+(1-h)^2) = \underset{h\to 0}{Lt} (1+1+h^2-2h) = \underset{h\to 0}{Lt} (h^2-2h+2) = 2$$

Also $\underset{x\to 1^+}{Lt} f(x) = \underset{x\to 1^+}{Lt} (2-x) = \underset{h\to 0}{Lt} (2-(1+h))$; $\underset{h\to 0}{Lt} (1-h) = 1$

Now $\underset{x\to 1^-}{Lt} f(x) \neq \underset{x\to 1^+}{Lt} f(x)$ $\therefore \underset{x\to 1}{Lt}$ does not exist.

3. Show that $\underset{x\to 0}{Lt}\frac{|x|}{x}, x\neq 0$ does not exist

Solution :

Let $f(x)=\frac{|x|}{x}$; $\underset{x\to 0^-}{Lt} f(x)=\underset{x\to 0^-}{Lt}\frac{|x|}{x}=\underset{h\to 0}{Lt}\frac{|0-h|}{0-h}=\underset{h\to 0}{Lt}\frac{|-h|}{-h}=\underset{h\to 0}{Lt}\frac{h}{-h}=-1$

Also $\underset{x\to 0^+}{Lt} f(x)=\underset{x\to 0^+}{Lt}\frac{|x|}{x}=\underset{h\to 0}{Lt}\frac{|0+h|}{h}=\underset{h\to 0}{Lt}\frac{h}{h}=1$. As $\underset{x\to 0^-}{Lt}\frac{|x|}{x}\neq\underset{x\to 0^+}{Lt}\frac{|x|}{x}$ $\therefore \underset{x\to 0}{Lt}\frac{|x|}{x}$ does not exi

4. For what value of p does $\underset{x\to 1}{Lt} f(x)$ exist where f is defined as $f(x)=\begin{Bmatrix}2px+3 & x<1\\ 1-px^2 & x>1\end{Bmatrix}$.

Solution :

For $\underset{x\to 1}{Lt} f(x)$ to exist we must have $\underset{x\to 1^-}{Lt} f(x)=\underset{x\to 1^+}{Lt} f(x)$.

Now $\underset{x\to 1^-}{Lt} f(x)=\underset{x\to 1^-}{Lt}(2px+3)=\underset{h\to 0}{Lt}(2p(1-h)+3)=\underset{h\to 0}{Lt}(2p-2ph+3)$

$=2p+3$ —(1)

$=\underset{x\to 1^+}{Lt} f(x)=\underset{x\to 1^+}{Lt}(1-px^2)=\underset{h\to 0}{Lt}(1-p(1+h)^2)$

$=\underset{h\to 0}{Lt}(1-p(h^2+1+2h))=\underset{h\to 0}{Lt}(1-p-ph^2-2ph)$

$=1-p$ —(2)

From (1) and (2) we get $2p+3=1-p \Rightarrow 3p=-2\Rightarrow p=-2/3$.

5. Show that $\underset{x\to 0}{Lt}\frac{e^{1/x}-1}{e^{1/x}+1}$ does not exist.

Solution :

$$f(x)=\underset{x\to 0}{\mathrm{Lt}}\frac{e^{1/x}-1}{e^{1/x}+1}$$

$$\text{Now}\quad \underset{x\to 0^-}{\mathrm{Lt}} f(x)=\underset{x\to 0^-}{\mathrm{Lt}}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)=\underset{h\to 0}{\mathrm{Lt}}\left(\frac{e^{\frac{1}{0-h}}-1}{e^{\frac{1}{0-h}}+1}\right)=\underset{h\to 0}{Lt}\left(\frac{e^{\frac{1}{-h}}-1}{e^{\frac{1}{-h}}+1}\right)=\underset{h\to 0}{Lt}\left(\frac{\frac{1}{e^{1/h}}-1}{\frac{1}{e^{1/h}}+1}\right)$$

Now as $h\to 0, \frac{1}{h}\to\infty$ i.e. $e^{\frac{1}{h}}\to\infty$ $\therefore e^{-1/h}\to 0$

$$\therefore \underset{x\to 0^-}{Lt} f(x)=\underset{h\to 0}{Lt}\left(\frac{\frac{1}{e^{1/h}}-1}{\frac{1}{e^{1/h}}+1}\right)=\left(\frac{0-1}{0+1}\right)=-1\quad ;\quad \underset{x\to 0^+}{Lt} f(x)=\underset{x\to 0^+}{Lt}\left(\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right)=\underset{h\to 0}{Lt}\left(\frac{e^{\frac{1}{0+h}}-1}{e^{\frac{1}{0+h}}+1}\right)$$

$$\underset{h\to 0}{Lt}\left(\frac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1}\right)=\underset{h\to 0}{Lt}\left(\frac{1-e^{-\frac{1}{h}}}{1+e^{-\frac{1}{h}}}\right)$$ [Dividing numerator and denominator by $e^{\frac{1}{h}}$]

$$\therefore \underset{x\to 0^+}{Lt} f(x)=\underset{h\to 0}{Lt}\left(\frac{1-e^{-\frac{1}{h}}}{1+e^{-\frac{1}{h}}}\right)=\underset{h\to 0}{Lt}\frac{\left(1-\frac{1}{e^{1/h}}\right)}{1+\frac{1}{e^{1/h}}}=\left(\frac{1-0}{1+0}\right)=1\quad\left[\underset{h\to 0}{Lt}\frac{1}{e^{1/h}}=0\right]$$

Clearly $\underset{x\to 0^-}{Lt} f(x)\neq\underset{x\to 0^+}{Lt} f(x)$

6. Evaluate $\underset{x \to 3}{Lt}[x]$.

Solution :

Note $[x]$ is a greatest integer function less than as equal to x

i.e. $[x] =$

0	$0 \le x < 1$
1	$1 \le x < 2$
2	$2 \le x < 3$
3	$3 \le x < 4$ so on

$\underset{x \to 3^-}{Lt} f(x) = \underset{h \to 0}{Lt}[3 - h] = 2$ [as $3 - h$ is slightly less than 3]

$\underset{x \to 3^+}{Lt} f(x) = \underset{h \to 0}{Lt}[3 + h] = 3$ [as $3 + h$ is slightly greater than 3] $\therefore \underset{x \to 3}{Lt}[x]$ does not exist

3·2 [Evaluation of Limits (Algebraic)]

We evaluate algebraic limit from following 3 methods

Method : 1

Direct Substitution Method :

Under this method, we directly substitute the value of 'a' in the given function $f(x)$ to get the finite number as a limit.

Examples

7. Evaluate $\underset{x \to 0^-}{Lt}\left[\frac{|x|}{x} + x^3 + 5\right]$

Solution :

$$f(x) = \frac{|x|}{x} + x^3 + 5 \quad ; \quad \underset{x \to 0^-}{Lt} f(x) = \underset{x \to 0^-}{Lt}\frac{|x|}{x} + \underset{x \to 0^-}{Lt} x^3 + 5$$

$$= \underset{x \to 0^-}{Lt}\frac{|x|}{x} + 0 + 5 = -1 + 5 = 4 \qquad \text{[See example 3]}$$

8. Evaluate $= \underset{x \to 0}{Lt}\left[9x^2 - 10x + 5\right]$.

Solution :

$$f(x) = 9x^2 - 10x + 5 \quad \therefore \underset{x \to 0}{Lt} f(x) = \underset{x \to 0}{Lt} f(x) = \underset{x \to 0}{Lt}[9x^2 - 10x + 5] = 0 + 0 + 5 = 5$$

Method 2 :

Factorisation Method :

Under this method we use the following algebraic identities to remove $\left(\frac{0}{0}\right)$ and hence evaluate the limit.

(i) $a^2 - b^2 = (a - b)(a + b)$

(ii) $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$

(iii) $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$

(iv) $a^4 - b^4 = (a^2 + b^2)(a - b)(a + b)$

9. Evaluate $\underset{x \to -5}{Lt}\frac{2x^2 + 9x - 5}{x + 5}$.

Solution :

$f(x) = \frac{2x^2 + 9x - 5}{x + 5}$ [If we put $x = -5$ in numerator and denominator we get $\left(\frac{0}{0}\right)$]

$$f(x)=\frac{2x^2+10x-x-5}{x+5}=\frac{2x(x+5)-1(x+5)}{x+5}$$

$$=\frac{(2x-1)(x+5)}{x+5}=(2x-1)\ ;\ \underset{x\to-5}{Lt} f(x)=\underset{x\to-5}{Lt}(2x-1)=2\times(-5)-1=-11$$

10. Evaluate $\underset{x\to 2}{Lt}\dfrac{x^3-8}{x-2}$.

Solution :

$$f(x)=\frac{x^3-8}{x-2}$$ [If we put $x=2$ in numerator and denominator we get $\left(\frac{0}{0}\right)$]

$$\therefore\ f(x)=\frac{x^3-(2)^3}{x-2}=\frac{(x-2)(x^2+4+2x)}{(x-2)}\qquad\therefore f(x)=x^2+4+2x\qquad\therefore\ \underset{x\to2}{Lt} f(x)=4+4+4=12$$

11. Evaluate $\underset{x\to-1/2}{Lt}\dfrac{8x^3+1}{2x+1}$.

Solution :

$$f(x)=\frac{8x^3+1}{2x+1}$$ [If we put $x=-1/2$ in numerator and denominator we get $\left(\frac{0}{0}\right)$]

$$f(x)=\frac{(2x)^3+(1)^3}{2x+1}=\frac{(2x+1)(4x^2+1-2x)}{(2x+1)}\ ;\ f(x)=4x^2+1-2x$$

$$\underset{x\to-1/2}{Lt} f(x)=\underset{x\to-1/2}{Lt}[4x^2+1-2x]=1+1+1=3$$

12. Evaluate $\underset{x\to\sqrt{2}}{Lt}\dfrac{x^4-4}{x^2+3\sqrt{2}x-8}$.

Solution :

$$f(x)=\frac{x^4-4}{x^2+3\sqrt{2}x-8}$$ [If we put $x=\sqrt{2}$ in numerator and denominator. we get $\left(\frac{0}{0}\right)$]

$$\therefore\ f(x)=\frac{x^4-4}{x^2+3\sqrt{2}x-8}=\frac{(x^2)^2-(2)^2}{x^2+3\sqrt{2}x-8}=\frac{(x^2-2)(x^2+2)}{x^2+4\sqrt{2}x-\sqrt{2}x-8}$$

$$=\frac{(x-\sqrt{2})(x+\sqrt{2})(x^2+2)}{(x+4\sqrt{2})(x-\sqrt{2})}=\frac{(x+\sqrt{2})(x^2+2)}{x+4\sqrt{2}}$$

$$\therefore\ \underset{x\to\sqrt{2}}{Lt} f(x)=\frac{(\sqrt{2}+\sqrt{2})(2+2)}{\sqrt{2}+4\sqrt{2}}=\frac{4(2\sqrt{2})}{5\sqrt{2}}=\frac{8}{5}$$

Method : 3

Rationalisation Method :

This method is used for the fractions which involves square roots, so we rationalise either numerator or denominator to evaluate the limit.

Examples

13. Evaluate $\underset{x\to 0}{Lt}\dfrac{\sqrt{1+3x}-\sqrt{1-3x}}{x}$

Solution :

$$f(x)=\frac{\sqrt{1+3x}-\sqrt{1-3x}}{x}$$ [If we put $x=0$ in numerator and denominator, we get $\left(\frac{0}{0}\right)$]

$$f(x)=\frac{\sqrt{1+3x}-\sqrt{1-3x}}{x}\times\frac{\sqrt{1+3x}+\sqrt{1-3x}}{\sqrt{1+3x}+\sqrt{1-3x}}=\frac{(\sqrt{1+3x})^2-(\sqrt{1-3x})^2}{x(\sqrt{1+3x}+\sqrt{1-3x})}=\frac{(1+3x)-(1-3x)}{x(\sqrt{1+3x}+\sqrt{1-3x})}$$

$$f(x)=\frac{6x}{x(\sqrt{1+3x}+\sqrt{1-3x})}=\frac{6}{\sqrt{1+3x}+\sqrt{1-3x}}$$

$$\underset{x\to 0}{Lt}f(x)=\underset{x\to 0}{Lt}\left[\frac{6}{\sqrt{1+3x}+\sqrt{1-3x}}\right]=\frac{6}{2}=3$$

14. Evaluate $\underset{x\to a}{Lt}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}};\quad a\neq 0$

Solution :

$$f(x)=\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$$

$$f(x)=\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\times\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}}\times\frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}}$$

$$=\frac{\left(\sqrt{(a+2x)}\right)^2-\left(\sqrt{3x}\right)^2}{\left(\sqrt{3a+x}\right)^2-\left(2\sqrt{x}\right)^2}\;\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}}=\left[\frac{a+2x-3x}{3a+x-(4x)}\right]\left[\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}}\right]$$

$$=\frac{a-x}{3(a-x)}\;\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}}=\frac{1}{3}\;\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}}$$

$$\underset{x\to a}{Lt}f(x)=\underset{x\to a}{Lt}\frac{1}{3}\left[\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}}\right]=\frac{1}{3}\left[\frac{\sqrt{4a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}\right]=\frac{1}{3}\,\frac{(2\sqrt{a}+2\sqrt{a})}{2\sqrt{3a}}=\frac{4\sqrt{a}}{6\sqrt{3a}}=\frac{2}{3\sqrt{3}}$$

Method : 4

Using Result :

Using Result $\underset{x\to a}{Lt}\dfrac{x^n-a^n}{x-a}=na^{n-1}$

Examples

15. Evaluate $\underset{x\to 2}{Lt}\dfrac{x^{10}-1024}{x^5-32}$

Solution :

Let $f(x)=\dfrac{x^{10}-1024}{x^5-32}$ [If we put $x=2$ in numerator and denominator we get $\left(\frac{0}{0}\right)$]

$$f(x)=\frac{x^{10}-(2)^{10}}{x^5-(2)^5}=\frac{\dfrac{x^{10}-(2)^{10}}{x-2}}{\dfrac{x^5-(2)^5}{x-2}}\qquad \text{Now } \lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{\dfrac{x^{10}-(2)^{10}}{x-2}}{\dfrac{x^5-(2)^5}{x-2}}$$

$$=\frac{10(2)^{10-1}}{5(2)^{5-1}}=\frac{2(2)^9}{(2)^4}=\frac{(2)^{10}}{(2)^4}=(2)^6=64$$

16. Evaluate $\lim_{x\to a}\dfrac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$

Solution :

$$f(x)=\frac{(x+2)^{5/3}-(a+2)^{5/3}}{(x+2)-(a+2)}.$$ Put $x+2=y$ and $a+2=b$. Clearly as $x\to a,\ y\to b$

$$\lim_{x\to a} f(x)=\lim_{y\to b}\frac{y^{5/3}-b^{5/3}}{y-b}=5/3(b)^{5/3-1}=5/3(b)^{2/3}=\frac{5}{3}(a+2)^{2/3}$$

17. If $\lim_{x\to a}\dfrac{x^9-a^9}{x-a}=\lim_{x\to 5}(4+x)$, find all possible values of 'a'.

Solution :

$$\lim_{x\to a}\frac{x^9-a^9}{x-a}=9(a)^{9-1}=9a^8.\qquad \text{Also } \lim_{x\to 5}(4+x)=4+5=9$$

Now $9a^8=9 \Rightarrow a^8=1 \Rightarrow a=1,-1.$

3·3 [Evaluation Of Trigonometric Limits And Inverse Limits]

In this section we shall use following standard trigonometric results to evaluate the limit.

(i) $\lim_{x\to 0}\sin x=0$ (ii) $\lim_{x\to 0}\cos x=1$ (iii) $\lim_{x\to 0}\dfrac{\sin x}{x}=1$

(iv) $\lim_{x\to 0}\dfrac{\tan x}{x}=1$ (v) $\lim_{x\to\infty}\dfrac{\sin x}{x}=0$ (vi) $\lim_{x\to\infty}\dfrac{\cos x}{x}=0$

Examples

18. Evaluate $\lim_{x\to 0}\dfrac{\sin 3x+7x}{4x+\sin 2x}$

Solution :

$$f(x)=\frac{\sin 3x+7x}{4x+\sin 2x}.$$ Dividing numerator and denominator by x, we get, $f(x)=\dfrac{\dfrac{\sin 3x}{x}+7}{4+\dfrac{\sin 2x}{x}}$

$$\lim_{x\to 0}\frac{3\dfrac{\sin 3x}{3x}+7}{4+2\dfrac{\sin 2x}{2x}}=\frac{3\lim_{x\to 0}\dfrac{\sin 3x}{3x}+7}{4+2\lim_{x\to 0}\dfrac{\sin 2x}{2x}}=\frac{3\cdot 1+7}{4+2\cdot 1}=\frac{10}{6}=\frac{5}{3}$$

19. Evaluate $\underset{x\to 0}{Lt}\left(\dfrac{\tan x-\sin x}{\sin^3 x}\right)$

Solution :

$$f(x)=\frac{\tan x-\sin x}{\sin^3 x}=\frac{\frac{\sin x}{\cos x}-\sin x}{\sin^3 x}=\frac{\sin x(1-\cos x)}{\cos x\sin^3 x}$$

$$=\frac{(1-\cos x)}{\cos x(\sin^2 x)}=\frac{1-\cos x}{\cos x(1-\cos^2 x)}=\frac{1-\cos x}{\cos x(1-\cos x)(1+\cos x)}=\frac{1}{\cos x(1+\cos x)}$$

$$\underset{x\to 0}{Lt}f(x)=\underset{x\to 0}{Lt}\left(\frac{1}{\cos x(1+\cos x)}\right)=\frac{1}{1(1+1)}=\frac{1}{2}$$

20. Evaluate $\underset{x\to 0}{Lt}\left(\dfrac{x^3\cot x}{1-\cos x}\right)$

Solution :

$$f(x)=\frac{x^3\cot x}{1-\cos x}=\frac{x^3\cot x}{1-\cos x}\times\frac{1+\cos x}{1+\cos x}=\frac{x^3\cos x(1+\cos x)}{\sin x(1-\cos^2 x)}$$

$$=\frac{x^3\cos x(1+\cos x)}{\sin x\sin^2 x}=\frac{x^3\cos x(1+\cos x)}{\sin^3 x}=\frac{x^3\cos x(1+\cos x)}{\frac{\sin^3 x}{x^3}}$$

$$\underset{x\to 0}{Lt}f(x)=\frac{\underset{x\to 0}{Lt}\cos x\ \underset{x\to 0}{Lt}(1+\cos x)}{\underset{x\to 0}{Lt}\left(\frac{\sin x}{x}\right)^3}=\frac{1(1+1)}{1}=2$$

21. Evaluate $\underset{x\to 0}{Lt}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1}x}$

Solution :

$$f(x)=\frac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1}x}$$

$$f(x)=\frac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1}x}\times\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$$

$$=\frac{(1+x)-(1-x)}{\sin^{-1}x\left(\sqrt{1+x}+\sqrt{1-x}\right)}=\frac{2x}{\sin^{-1}x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$\underset{x\to 0}{Lt}f(x)=\frac{2x}{\sin^{-1}x(\sqrt{1+0}+\sqrt{1-0})}=\frac{2x}{\sin^{-1}x}\cdot\frac{1}{2}=\frac{x}{\sin^{-1}x}$$

Put $x=\sin\theta$

Now as $x\to 0,\ \theta\to 0$ $\quad\therefore\ \underset{x\to 0}{Lt}f(x)=\underset{\theta\to 0}{Lt}\dfrac{\sin\theta}{\sin^{-1}(\sin\theta)}=\underset{\theta\to 0}{Lt}\dfrac{\sin\theta}{\theta}=1$

22. Evaluate $\underset{x\to 1}{Lt}\dfrac{\tan^{-1}(1-x)^2}{x^2-2x+1}$

Solution :

$$f(x)=\frac{\tan^{-1}(1-x)^2}{x^2-2x+1} \qquad \text{Put } (1-x)^2=\tan\theta \qquad \text{Now as } x\to 1,\ \theta\to 0$$

$$\therefore \lim_{x\to 1} f(x)=\frac{\tan^{-1}(1-x)^2}{x^2-2x+1}=\lim_{\theta\to 0}\frac{\tan^{-1}(\tan\theta)}{\tan\theta}=\lim_{\theta\to 0}\frac{\theta}{\tan\theta}=\frac{1}{\lim_{\theta\to 0}\frac{\tan\theta}{\theta}}=\frac{1}{1}=1$$

23. Evaluate $\lim_{x\to\frac{1}{\sqrt{2}}}\frac{x-\cos(\sin^{-1}x)}{1-\tan(\sin^{-1}x)}$

Solution :

$$f(x)=\frac{x-\cos(\sin^{-1}x)}{1-\tan(\sin^{-1}x)} \qquad \text{Put } x=\sin\theta \qquad \therefore \text{ As } x\to\frac{1}{\sqrt{2}},\ \theta\to\frac{\pi}{4}$$

$$\therefore \lim_{x\to\frac{1}{\sqrt{2}}}\frac{x-\cos(\sin^{-1}x)}{1-\tan(\sin^{-1}x)}=\lim_{\theta\to\pi/4}\frac{\sin\theta-\cos\theta}{1-\tan\theta}$$

$$=\lim_{\theta\to\pi/4}\frac{\sin\theta-\cos\theta}{1-\frac{\sin\theta}{\cos\theta}}=\lim_{\theta\to\pi/4}\frac{(\sin\theta-\cos\theta)\cos\theta}{(\cos\theta-\sin\theta)}=-\lim_{\theta\to\pi/4}\cos\theta=-\cos\frac{\pi}{4}=\frac{-1}{\sqrt{2}}$$

24. Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{\cot x-\cos x}{(\pi-2x)^3}$.

Solution :

$$f(x)=\frac{\cot x-\cos x}{(\pi-2x)^3} \qquad \left[\text{If we put } x=\pi/2 \text{ in numerator and denominator we get } \left(\frac{0}{0}\right)\right]$$

$$\lim_{x\to\frac{\pi}{2}}\frac{\cot x-\cos x}{(\pi-2x)^3}=\lim_{h\to 0}\frac{\cot(\pi/2+h)-\cos(\pi/2+h)}{[\pi-2(\pi/2+h)]^3} \qquad \left[\text{Put } x=\frac{\pi}{2}+h,\ x\to\frac{\pi}{2}\Rightarrow h\to 0\right]$$

$$=\lim_{h\to 0}\frac{-\tan h+\sin h}{-8h^3}=\lim_{h\to 0}\frac{-\frac{\sin h}{\cos h}+\sin h}{-8h^3}=\lim_{h\to 0}\frac{-\sin h(1-\cos h)}{\cos h(-8h^3)}$$

$$=\frac{1}{8}\lim_{h\to 0}\frac{\tan h}{h}\left(\frac{2\sin^2 h/2}{h^2}\right)=\frac{2}{8}\lim_{h\to 0}\frac{\tan h}{h}\cdot\frac{1}{4}\lim_{h\to 0}\left(\frac{\sin h/2}{h/2}\right)^2=\frac{2}{8}\cdot 1\cdot\frac{1}{4}\cdot 1=\frac{2}{32}=\frac{1}{16}$$

25. Evaluate $\lim_{x\to\infty} x\left(\tan^{-1}\frac{x+1}{x+4}-\frac{\pi}{4}\right)$

Solution :

$$f(x)=x\left(\tan^{-1}\frac{x+1}{x+4}-\tan^{-1}1\right)$$

$$f(x)=x\tan^{-1}\left(\frac{\frac{x+1}{x+4}-1}{1+\frac{x+1}{x+4}\cdot 1}\right) \qquad \left[\tan^{-1}a-\tan^{-1}b=\tan^{-1}\left(\frac{a-b}{1+ab}\right)\right]$$

$$= x\tan^{-1}\left(\frac{\frac{x+1-(x+4)}{x+4}}{\frac{2x+5}{x+4}}\right) = x\tan^{-1}\left(\frac{-3}{2x+5}\right) = x\frac{\tan^{-1}\left(\frac{-3}{2x+5}\right)}{\left(\frac{-3}{2x+5}\right)}\cdot\left(\frac{-3}{2x+5}\right)$$

$$\text{Now } \underset{x\to\infty}{Lt}\frac{\tan^{-1}\left(\frac{-3}{2x+5}\right)}{\left(\frac{-3}{2x+5}\right)};\ \underset{x\to\infty}{Lt}\left(\frac{-3x}{2x+5}\right)$$

Put $x = 1/t$ i.e. $x\to\infty \Rightarrow t\to 0 \Rightarrow \underset{t\to 0}{Lt}\frac{\tan^{-1}\left(\frac{-3t}{2+5t}\right)}{\left(\frac{-3t}{2+5t}\right)}\cdot \underset{t\to 0}{Lt}\left(\frac{-3}{2+5t}\right)$

$\Rightarrow 1.(-3/2) = -3/2$ $\qquad\left[\underset{x\to 0}{Lt}\frac{\tan^{-1}x}{x} = 1\right]$

26. Evaluate $\underset{x\to 0}{Lt}\dfrac{x(1-\sqrt{1-x^2})}{\sqrt{1-x^2}\,(\sin^{-1}x)^3}$

Solution :

$f(x) = \dfrac{x(1-\sqrt{1-x^2})}{\sqrt{1-x^2}\,(\sin^{-1}x)^3}$ Put $x = \sin\theta$ Then $1-x^2 = 1-\sin^2\theta = \cos^2\theta$

$$f(x) = \frac{\sin\theta(1-\cos\theta)}{\cos\theta(\theta)^3} = \frac{\sin\theta(2\sin^2\theta/2)}{\cos\theta(\theta)^3} = \frac{\sin\theta}{\theta}\cdot\frac{2\sin\theta/2}{2\cdot\theta/2}\cdot\frac{\sin\theta/2}{2\cdot\theta/2}\cdot\frac{1}{\cos\theta}$$

$$\underset{\theta\to 0}{Lt}f(x) = \frac{2}{4}\cdot\underset{\theta\to 0}{Lt}\frac{\sin\theta}{\theta}\cdot\underset{x\to 0}{Lt}\left(\frac{\sin\theta/2}{\theta/2}\right)^2\cdot\underset{\theta\to 0}{Lt}\frac{1}{\cos\theta} = \frac{2}{4}\cdot 1\cdot(1)^2\cdot\frac{1}{1} = \frac{2}{4} = \frac{1}{2}$$

27. Evaluate $\underset{x\to\theta}{Lt}\dfrac{\sin^2x-\sin^2\theta}{x^2-\theta^2}$.

Solution :

$$f(x) = \frac{\sin^2x-\sin^2\theta}{x^2-\theta^2} = \frac{(\sin x-\sin\theta)(\sin x+\sin\theta)}{(x-\theta)(x+\theta)}$$

$$\underset{x\to\theta}{Lt}f(x) = \underset{x\to\theta}{Lt}\left[\frac{\sin x+\sin\theta}{x+\theta}\right]\underset{x\to\theta}{Lt}\left[\frac{\sin x-\sin\theta}{x-\theta}\right]$$

$$= \underset{x\to\theta}{Lt}\left[\frac{\sin x+\sin\theta}{x+\theta}\right]\underset{x\to\theta}{Lt}\left[\frac{2\cos\left(\frac{x+\theta}{2}\right)\cdot\sin\left(\frac{x-\theta}{2}\right)}{x-\theta}\right] \qquad \left[\sin C-\sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$= \left[\frac{2\sin\theta}{2\theta}\right]\cdot\underset{x\to\theta}{Lt}\left[\frac{2\cos\left(\frac{x+\theta}{2}\right)\cdot\sin\left(\frac{x-\theta}{2}\right)}{2\cdot\left(\frac{x-\theta}{2}\right)}\right] = \frac{\sin\theta}{\theta}\underset{x\to\theta}{Lt}\frac{2\cos}{2}\left(\frac{x+\theta}{2}\right)\cdot\underset{x\to\theta}{Lt}\frac{\sin\left(\frac{x-\theta}{2}\right)}{\left(\frac{x-\theta}{2}\right)}$$

$$= \frac{\sin\theta}{\theta}\cdot\cos(\theta)\cdot 1 = \frac{\sin\theta\cdot\cos\theta}{\theta}$$

$\left[\because \underset{x\to a}{Lt}\dfrac{\sin(x-a)}{x-a} = 1\right]$

3·4 [Evaluation of Exponential and Logarithmic Limits]

In this section we shall use the following results to evaluate the limit.

(i) $\underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)^n = e$ (ii) $\underset{x\to 0}{Lt}(1+x)^{1/x} = e$ (iii) $\underset{n\to\infty}{Lt}\left(1+\frac{x}{n}\right)^n = e^x$

(iv) $\underset{x\to 0}{Lt}\left(\frac{a^x-1}{x}\right) = \log_e a$ (v) $\underset{x\to 0}{Lt}\left(\frac{e^x-1}{x}\right) = 1$ (vi) $\underset{x\to 0}{Lt}\frac{\log(1+x)}{x} = 1$

(vii) $\underset{x\to 1}{Lt}\left(\frac{\log x}{x-1}\right) = 1$

Examples

28. Evaluate $\underset{x\to 0}{Lt}\left(\frac{5^x-2^x}{\tan x}\right)$

Solution :

$$f(x) = \frac{(5^x-1)-(2^x-1)}{\tan x} = \frac{(5^x-1)-(2^x-1)}{x}\cdot\frac{x}{\tan x} = \left[\frac{(5^x-1)}{x}-\frac{(2^x-1)}{x}\right]\cdot\frac{x}{\tan x}$$

$$\Rightarrow \underset{x\to 0}{Lt} f(x) = \left[\underset{x\to 0}{Lt}\left[\frac{5^x-1}{x}\right]-\underset{x\to 0}{Lt}\left[\frac{2^x-1}{x}\right]\right]\underset{x\to 0}{Lt}\left[\frac{x}{\tan x}\right]$$

$$= [\log_e 5-\log_e 2]\cdot 1 = \log_e(5/2) \qquad [\log_e a-\log_e b = \log_e a/b]$$

29. Evaluate $\underset{x\to 0}{Lt}\left(\frac{e^x+e^{-x}-2}{x^2}\right)$

Solution :

$$f(x) = \left(\frac{e^x+e^{-x}-2}{x^2}\right) = \frac{e^x(e^x+e^{-x}-2)}{x^2\cdot e^x} = \frac{e^{2x}+1-2e^x}{x^2e^x} = \left(\frac{e^x-1}{x}\right)^2\cdot\frac{1}{e^x}$$

$$\underset{x\to 0}{Lt} f(x) = \underset{x\to 0}{Lt}\left(\frac{e^x-1}{x}\right)^2 \underset{x\to 0}{Lt}\frac{1}{e^x} = 1\cdot 1 = 1$$

30. Evaluate $\underset{x\to\infty}{Lt}\left(\frac{x-3}{x+3}\right)^{x+3}$

Solution :

$$f(x) = \left(\frac{x-3}{x+3}\right)^{x+3} = \left[1+\frac{x-3}{x+3}-1\right]^{x+3} = \left[1+\frac{(x-3)-(x+3)}{x+3}\right]^{x+3} = \left[1-\frac{6}{x+3}\right]^{x+3}$$

$$\Rightarrow \underset{x\to\infty}{Lt}\left(\frac{x-3}{x+3}\right)^{x+3} = \underset{y\to\infty}{Lt}\left(1-\frac{6}{y}\right)^y \qquad \text{[By putting } x+3=y\text{]}$$

As $[x\to\infty, y\to\infty]$ $\quad\therefore \underset{x\to\infty}{Lt} f(x) = e^{-6}$

31. Evaluate $\underset{x\to 2}{Lt}\frac{\sin(e^{x-2}-1)}{\log(x-1)}$

Solution :

Let $$f(x)=\frac{\sin(e^{x-2}-1)}{\log(x-1)}=\frac{\sin(e^{x-2}-1)}{e^{x-2}-1}\cdot\frac{e^{x-2}-1}{x-2}\,\frac{(x-2)}{\log\,(1+(x-2))}$$

$$\underset{x\to2}{Lt}\,f(x)=\underset{x-2\to0}{Lt}\left[\frac{\sin(e^{x-2}-1)}{e^{x-2}-1}\cdot\frac{(e^{x-2}-1)}{x-2}\cdot\frac{(x-2)}{\log[1+(x-2)]}\right]$$

$$=\underset{x-2\to0}{Lt}\frac{\sin(e^{x-2}-1)}{e^{x-2}-1}\;\underset{x-2\to0}{Lt}\frac{(e^{x-2}-1)}{x-2}\;\underset{x-2\to0}{Lt}\frac{(x-2)}{\log[1+(1-2)]}$$

$$=1\times1\times1=1\quad\left(\underset{x\to0}{Lt}\frac{\sin x}{x}=1,\ \underset{x\to0}{Lt}\frac{e^x-1}{x}=1\ \ \underset{x\to0}{Lt}\log\frac{1+x}{x}=1\right)$$

32. Evaluate $\underset{x\to\pi/2}{Lt}\dfrac{e^{\cos x}-1}{\cos x}$

Solution :

$$f(x)=\frac{e^{\cos x}-1}{\cos x}\qquad \text{Put }\frac{\pi}{2}-x=y\text{, when }x\to\frac{\pi}{2}\Rightarrow y\to0$$

$$\underset{x\to\pi/2}{Lt}\frac{e^{\cos x}-1}{\cos x}=\underset{y\to0}{Lt}\frac{e^{\cos\left(\frac{\pi}{2}-y\right)}-1}{\cos\left(\frac{\pi}{2}-y\right)}=\underset{y\to0}{Lt}\frac{e^{\sin y}-1}{\sin y}=1$$

33. Evaluate $\underset{x\to0}{Lt}\dfrac{e^x-1}{\sqrt{1-\cos x}}$.

Solution :

$$f(x)=\frac{e^x-1}{\sqrt{1-\cos x}}=\frac{e^x-1}{\sqrt{2\sin^2x/2}}=\frac{1}{\sqrt2}\frac{e^x-1}{\sin x/2}=\frac{1}{\sqrt2}\frac{e^x-1}{2}\frac{2\cdot\frac{x}{2}}{\sin x/2}$$

$$=\frac{2}{\sqrt2}\frac{e^x-1}{x}\frac{\frac{x}{2}}{\sin x/2}=\sqrt2\frac{e^x-1}{x}\frac{\frac{x}{2}}{\sin x/2}$$

$$\underset{x\to0}{Lt}\,f(x)=\sqrt2\,\underset{x\to0}{Lt}\frac{e^x-1}{x}\,\underset{x\to0}{Lt}\frac{x/2}{\sin x/2}=\sqrt2\cdot1\cdot1=\sqrt2$$

34. Evaluate $\underset{x\to a}{Lt}\dfrac{a^x-1}{x}=\log_e a,$ find $\dfrac{2^x-1}{(1+x)^{1/2}-1}$

Solution :

$$f(x)=\frac{2^x-1}{(1+x)^{1/2}-1}=\frac{2^x-1}{\sqrt{1+x}-1}\times\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}=\frac{2^x-1}{x}\sqrt{1+x}+1$$

$$\underset{x\to0}{Lt}\,f(x)=\underset{x\to0}{Lt}\left(\frac{2^x-1}{x}\right)\underset{x\to0}{Lt}\left(\sqrt{1+x}+1\right)=(\log_e2)\cdot2\ =2\log_e2$$

35. Evaluate $\underset{x\to0}{Lt}\left[\tan\left(\dfrac{\pi}{4}+x\right)\right]^{1/x}$

Solution :

$$f(x)=\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\frac{1}{x}}=\left(\frac{\tan \pi/4+\tan x}{1-\tan \pi/4 \tan x}\right)^{\frac{1}{x}}=\left(\frac{1+\tan x}{1-\tan x}\right)^{\frac{1}{x}}$$

$$=\left[\left(1+\frac{2\tan x}{1-\tan x}\right)^{\frac{1-\tan x}{2\tan x}}\right]^{\frac{2\tan x}{x(1-\tan x)}} \quad \text{Now } \underset{x\to 0}{Lt}\left[\left(1+\frac{2\tan x}{1-\tan x}\right)^{\frac{1-\tan x}{2\tan x}}\right]^{\frac{2\tan x}{x(1-\tan x)}}$$

$$=e^{2\underset{x\to 0}{Lt}\left(\frac{\tan x}{x}\right)\cdot\underset{x\to 0}{Lt}\left(\frac{1}{1-\tan x}\right)}=e^{2\cdot 1\left(\frac{1}{1-0}\right)}=e^{2}$$

36. Evaluate $\underset{x\to 0}{Lt}\dfrac{10^x-2^x-5^x+1}{\sin^2 x}$

Solution :

$$f(x)=\frac{10^x-2^x-5^x+1}{\sin^2 x}=\frac{2^x(5^x-1)-1(5^x-1)}{\sin^2 x}=\frac{(2^x-1)(5^x-1)}{\sin^2 x}$$

$$\therefore \underset{x\to 0}{Lt} f(x)=\underset{x\to 0}{Lt}\frac{(2^x-1)}{x}\cdot\underset{x\to 0}{Lt}\frac{(5^x-1)}{x}\cdot\left(\frac{x^2}{\sin^2 x}\right)$$

$$=\underset{x\to 0}{Lt}\frac{(2^x-1)}{x}\cdot\underset{x\to 0}{Lt}\frac{(5^x-1)}{x}\cdot\left(\frac{x}{\sin x}\right)^2=(\log_e 2)(\log_e 5)\cdot 1=(\log_e 2)(\log_e 5)$$

3·5 [Continuity]

Let $f:[ab]\to R$ and $a<c<b$.

Definition :

(i) A function $f(x)$ is said to be continuous at $x=c$ if for every $\varepsilon>0$, there exists a real number $\delta>0$ such that $|f(x)-f(c)|<\varepsilon$, whenever $|x-c|<\delta$. It is denoted as $\underset{x\to c}{Lt} f(x)=f(c)$

i.e. $\underset{x\to c^-}{Lt} f(x)=\underset{x\to c^+}{Lt} f(x)=f(c)$

(ii) A function $f(x)$ is said to be continuous in a open interval (a, b) if it is continuous at every other point in (a, b).

(iii) A function $f(x)$ is said to be continuous in closed interval (a, b), if it is continuous at every point of interval (ab) and if it is continuous at the point 'a' from the right and continuous at 'b' from left.

i.e. $\underset{x\to a^+}{Lt} f(x)=f(a)$ and $\underset{x\to b^-}{Lt} f(x)=f(b)$

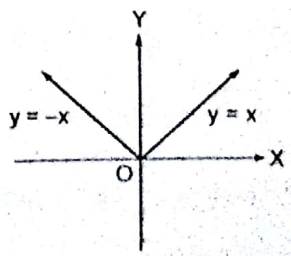
(iv) A function $f(x)$ is said to be continuous at the right end point b of $[ab]$ if $\underset{x\to b^-}{Lt} f(x)=f(b)$.

(v) A function $f(x)$ is said to be continuous at the left end point a of $[ab]$ if $\underset{x\to a^+}{Lt} f(x)=f(a)$

(vi) A function $f(x)$ is said to be discontinuous at a point $x=c$ if it is not continuous at $x=c$

3.6 [Geometrical Meaning of Continuity]

Geometrically, $f(x)$ is continuous at $x=a$ if there is no gap (break) in the graph of the function $y=f(x)$ either on the left or on the right in the neighbourhood of point $x=a$.

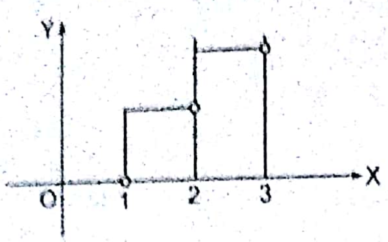


Example:

(i) $y=|x|$

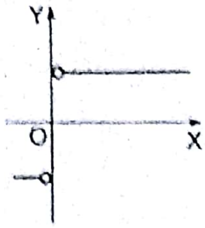
$|x|$ is continuous at $x=0$.

(ii) $y = [x]$ = integral part of x.



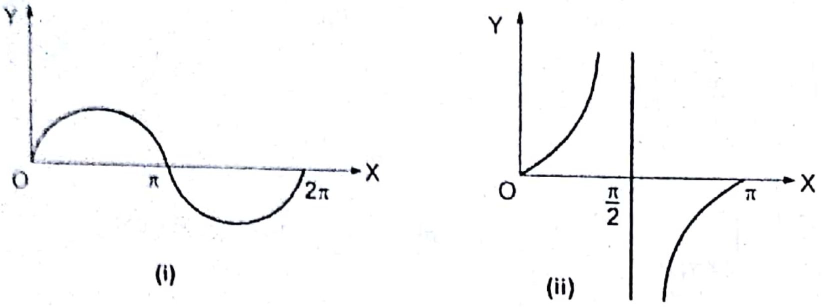
This function is discontinuous at all integral from left.

(iii) $y = 1, x > 0 = -1, x < 0 = 0, x = 0$



This function is discontinuous at $x = 0$ from both sides.

(iv)



The graph of $y = \sin x$ is continuous at all points as there is not gap in the graph. The graph of $y = \tan x$ is discontinuous at odd multiples of $\frac{\pi}{2}$ from both sides.

3·7 [Types of Discontinuities]

Consider $f:[ab] \to R$ and $a < c < b$

Definition :

(i) A function $f(x)$ is said to have a removable discontinuity at $x = c$ if $\underset{x \to c}{Lt} f(x)$ exist but is not equal to $f(c)$

i.e. $\underset{x \to c^-}{Lt} f(x) = \underset{x \to c^+}{Lt} f(x) \neq f(c)$

(ii) A function $f(x)$ is said to have a discontinuity of first kind at $x = c$ if $\underset{x \to c^-}{Lt} f(x)$ and $\underset{x \to c^+}{Lt} f(x)$ both exist but are not equal.

(iii) A function $f(x)$ is said to have a discontinuity of first kind from left at $x = c$ if $\underset{x \to c^-}{Lt} f(x)$ exist but is not equal to $f(c)$.

(iv) A function $f(x)$ is said to have a discontinuity of first kind from right at $x = c$ if $\underset{x \to c^+}{Lt} f(x)$ exist but is not equal to $f(c)$

(v) A function $f(x)$ is said to have a discontinuity of second kind from left at $x = c$ if $\underset{x \to c^-}{Lt} f(x)$ does not exist.

(vi) A function $f(x)$ is said to have a discontinuity of second kind from right at $x = c$ if $\underset{x \to c^+}{Lt} f(x)$ does not exist.

(vii) A function $f(x)$ is said to have a discontinuity of second kind at $x = c$ if $\underset{x \to c^-}{Lt} f(x)$ and $\underset{x \to c^+}{Lt} f(x)$ does not exist.

Examples

37. Let f be the function defined on R defined as $f(x)=\dfrac{e^{\frac{1}{x}}-e^{-\frac{1}{x}}}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}$, $x \neq 0$, $f(0)=1$.

Show that f is continuous from right at $x=0$ and has discontinuity of first kind from the left.

Solution :

$f(x)=\dfrac{e^{\frac{1}{x}}-e^{-\frac{1}{x}}}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}$. Now $\underset{x\to 0^-}{Lt} f(x)=\underset{x\to 0^-}{Lt}\left(\dfrac{e^{\frac{1}{x}}-e^{-\frac{1}{x}}}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}\right)=\underset{x\to 0^-}{Lt}\left[\dfrac{e^{1/x}-\dfrac{1}{e^{1/x}}}{e^{1/x}+\dfrac{1}{e^{1/x}}}\right]=\underset{x\to 0^-}{Lt}\left[\dfrac{e^{1/x}\cdot e^{1/x}-1}{e^{1/x}\cdot e^{1/x}+1}\right]$

As $x\to 0^-, e^{1/x}\to 0$ $\therefore$ we have, $\underset{x\to 0^-}{Lt}\left(\dfrac{e^{\frac{1}{x}}\cdot e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}\cdot e^{\frac{1}{x}}+1}\right)=\left(\dfrac{0.0-1}{0.0+1}\right)=-1$

Now $\underset{x\to 0^-}{Lt} f(x)=\underset{x\to 0^+}{Lt}\left(\dfrac{e^{\frac{1}{x}}-e^{-\frac{1}{x}}}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}\right)=\underset{x\to 0^+}{Lt}-\left[\dfrac{1-e^{1/x}\cdot e^{-1/x}}{1+e^{1/x}\cdot e^{1/x}}\right]$ [Dividing Nr end Dr by $e^{1/x}$]

As $x\to 0^+, e^{-1/x}\to 0$ $\therefore$ We have $\underset{x\to 0^+}{Lt}\left(\dfrac{1-e^{-\frac{1}{x}}\cdot e^{-\frac{1}{x}}}{1+e^{-1/x}e^{-1/x}}\right)=\left(\dfrac{1-0}{1+0}\right)=1$

Now $\underset{x\to 0^+}{Lt} f(x)=f(0)=1$ $\therefore$ f is continuous from right at $x=0$ and $\underset{x\to 0^-}{Lt} f(x)=-1\neq f(0)$

$\therefore$ $f(x)$ is discontinuous from left at $x=0$

38. Discuss the continuity of the function $f(x)$ defined as

$f(0)=0$;

$f(x)=1-x$ $\quad 0<x<1,\ f(1)=1$

$f(x)=2-x$ $\quad 1<x<2,\ f(2)=0$ at the point $x=0, 1, 2$. Also discuss the kind of discontinuity.

Solution :

Consider $x=0$ $\quad \underset{x\to 0^+}{Lt} f(x)=\underset{x\to 0^+}{Lt}(1-x)=\underset{h\to 0}{Lt}(1-(0+h))=\underset{h\to 0}{Lt}(1-h)=1$

also $f(0)=0$ $\quad\therefore \underset{x\to 0^+}{Lt} f(x)\neq f(0)$ $\quad\therefore$ f is discontinuous of first kind from right at $x=0$

Consider $x=1$ $\quad \underset{x\to 1^+}{Lt} f(x)=\underset{x\to 1^+}{Lt}(2-x)=\underset{h\to 0}{Lt}(2-(1+h))=\underset{h\to 0}{Lt}(1-h)=1$

$\underset{x\to 1^-}{Lt} f(x)=\underset{x\to 1^-}{Lt}(1-x)=\underset{h\to 0}{Lt}(1-(1-h))=\underset{h\to 0}{Lt}(h)=0$

also $f(1)=1$ $\quad\therefore \underset{x\to 1^-}{Lt} f(x)\neq f(1)$ $\quad\therefore$ f is discontinuous of first kind from left at $x=0$

39. Examine the function $f(x)=\dfrac{\sin^2 ax}{x^2}$, $x\neq 0$, $f(0)=1$ for continuity at $x=0$

Solution :

$f(x)=\dfrac{\sin^2 ax}{x^2}$, $x\neq 0$, $f(0)=1$; $\quad \underset{x\to 0^+}{Lt} f(x)=\underset{h\to 0}{Lt} f(0+h)=\underset{h\to 0}{Lt}\dfrac{\sin^2 a(0+h)}{(0+h)^2}$

$=\underset{h\to 0}{Lt}\dfrac{\sin^2 ah}{h^2}=\underset{h\to 0}{Lt} a^2\cdot\left(\dfrac{\sin ah}{ah}\right)^2=a^2\underset{h\to 0}{Lt}\left(\dfrac{\sin ah}{h}\right)^2=a^2\cdot 1=a^2$

$$\underset{x\to 0^-}{Lt} f(x) = \underset{h\to 0}{Lt} f(0-h) = \underset{h\to 0}{Lt} \frac{\sin^2 a(-h)}{(-h)^2} = \underset{h\to 0}{Lt} \frac{\sin(-ah)\sin(-ah)}{h^2}$$

$$= \underset{h\to 0}{Lt} \left(\frac{\sin ah}{ah}\right)^2 \cdot a^2 \ [\sin(-\theta) = -\sin\theta] = 1\cdot a^2 = a^2$$

Now $f(x)$ will be continuous at $x = 0$. Only iff $\underset{x\to 0^+}{Lt} f(x) = \underset{x\to 0^-}{Lt} f(x) = f(0) = 1$

$\therefore$ For f to be continuous we must have $a^2 = 1$ *i.e.* $a = \pm 1$

Hence $f(x)$ is discontinuous at $x = 0$ except when $a = \pm 1$

40. Examine the function given by $f(x) = \begin{cases} \dfrac{\cos x}{\pi/2 - x} & x \neq \pi/2 \\ 1 & x = \pi/2 \end{cases}$ for continuity at $x = \pi/2$.

Solution :

$$\underset{x\to \pi/2^-}{Lt} f(x) = \underset{h\to 0}{Lt} f(\pi/2 - h) = \underset{h\to 0}{Lt} \frac{\cos(\pi/2 - h)}{\pi/2 - (\pi/2 - h)} = \underset{h\to 0}{Lt} \frac{\sin h}{h} = 1$$

$$\underset{x\to \pi/2^+}{Lt} f(x) = \underset{h\to 0}{Lt} f(\pi/2 + h) = \underset{h\to 0}{Lt} \frac{\cos(\pi/2 + h)}{\pi/2 - (\pi/2 + h)} = \underset{h\to 0}{Lt} \frac{-\sin h}{-h} = \underset{h\to 0}{Lt} \frac{\sin h}{h} = 1$$

Also $f(\pi/2) = 1$ $\therefore$ f is continuous at $x = \pi/2$

41. Find the value of 'a' if $f(x) = \begin{cases} 2x-1 & x < 2 \\ a & x = 2 \\ x+1 & x > 2 \end{cases}$ is continuous at $x = 2$

Solution :

Consider the point at $x = 2$ $\underset{x\to 2^-}{Lt} f(x) = \underset{x\to 2}{Lt} (2x-1) = 2\times 2 - 1 = 3$

As f is continuous $\therefore$ $\underset{x\to 2^-}{Lt} f(x) = \underset{x\to 2}{Lt} f(x)$; $\underset{x\to 2^+}{Lt} f(x) = \underset{x\to 2}{Lt} (x+1) = 2+1 = 3$

As f is continuous $\underset{x\to 2^+}{Lt} f(x) = \underset{x\to 2}{Lt} f(x)$. Also $f(2) = a$.

Now as f is continuous at $x = 2$ $\therefore$ $\underset{x\to 2^-}{Lt} f(x) = \underset{x\to 2^+}{Lt} (x) = f(2)$; $3 = 3 = a \Rightarrow$ $a = 3$.

42. Show that $f(x) = |x-1| + |x-2|$ is continuous at $x = 1$ and $x = 2$.

Solution :

$f(x) = |x-1| + |x-2|$; $|x-1| = x-1$ *if* $x - 1 \geq 0$ *i.e* $x \geq 1$

$= -(x-1)$ *if* $x-1 < 0$ *i.e* $x < 1$; $|x-2| = x-2$ *if* $x-2 \geq 0$ *i.e* $x \geq 2$

$= -(x-2)$ *if* $x-2 < 0$ *i.e* $x < 2$

$\therefore$ $|x-1| + |x-2| = -(x-1) - (x-2) = 3 - 2x$ if $x < 1$ $= (x-1) - (x-2) = 1$ *if* $1 \leq x < 2$

$= (x-1) + (x-2) = 2x - 3$ *if* $x \geq 2$; $|x-1| + |x-2| = \begin{cases} 3-2x & x < 1 \\ 1 & 1 \leq x < 2 \\ 2x-3 & x \geq 2 \end{cases}$

Now first consider the point $x = 1$

$\underset{x\to 1^-}{Lt} f(x) = \underset{x\to 1^-}{Lt} (3-2x) = 3 - 2(1) = 1$; $\underset{x\to 1^+}{Lt} f(x) = \underset{x\to 1^+}{Lt} (1) = 1$. Also $f(1) = 1$

As $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$ $\therefore f$ is continuous at $x = 1$

Now consider the point $x = 2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (1) = 1 \; ; \; \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (2x-3) = 2\times 2 - 3 = 1$$

Also $f(2) = 2(2)-3 = 1$; As $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$ $\therefore f$ is continuous at $x = 2$

43. Examine the continuity of the function $f(x)$ defined on $[0\ 1]$

$$f(0) = 2 \; ; \; f(x) = x + \frac{1}{2} \quad 0 < x < 1/2 \; ; \quad f(1/2) = 1/2$$

$$f(x) = 3x - 1/2 \quad 1/2 < x < 1 \; ; f(1) = 1 \quad \text{At } x = 0, 1/2 \text{ and } 1$$

Also discuss the type of discontinuity at each point.

Solution :

Consider the point $x = 0$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x + 1/2) = \frac{1}{2} \quad \text{Also } f(0) = 2 \quad \therefore \lim_{x\to 0^+} f(x) \neq f(0)$$

$\therefore$ f is discontinuous of the first kind from right at $x = 0$

Consider the point $x = 1/2$

$$\lim_{x\to 1/2^-} f(x) = \lim_{x\to 1/2^-} (x + 1/2) = 1/2 + 1/2 = 1$$

$$\lim_{x\to 1/2^+} f(x) = \lim_{x\to 1/2^+} (3x - 1/2) = 3/2 + 1/2 = 2/2 = 1$$

But $f(1/2) = 1/2$ $\therefore \lim_{x\to 0^+} f(x) = f(0)$ *i.e.* $\lim_{x\to 1/2^+} f(x) = \lim_{x\to 1/2^-} f(x) \neq f(1/2)$

$\therefore$ f has a removable discontinuity at $x = 1/2$

Consider the point $x = 1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \left(3x - \frac{1}{2}\right) = 3(1) - 1/2 = 5/2 \; ; \; f(1) = 1 \quad \therefore \lim_{x\to 1^-} f(x) \neq f(1)$$

$\therefore$ f has a discontinuity of first kind from left at $x = 1$

44. Prove that the greatest integer function $[x]$ is continuous at all point except at integer points. Also discuss the type of discontinuity.

Solution :

$$\begin{aligned}[x] &= 0 && 0 \le x < 1\\ &= 1 && 1 \le x < 2\\ &= 2 && 2 \le x < 3\\ &= 4 && 3 \le x < 4\\ &\vdots\\ &= -1 && -1 \le x < 0\\ &= -2 && -2 \le x < -1\\ &\vdots\end{aligned}$$

$$[x] = \begin{bmatrix} k-1 & k-1 \le x < k \\ k & k \le x < k+1 \end{bmatrix}$$

Let k be any arbitrary integer

$$\lim_{x\to k^-} f(x) = \lim_{h\to 0} f(k-h) \; ; \; \lim_{h\to 0} [k-h] \; ; \quad \begin{aligned} &k-1 \le k-h < k\\ &\therefore [k-h] = k-1\end{aligned}$$

i.e. $\lim_{x\to k^-} f(x) = \lim_{h\to 0} (k-1) = k-1$; $\lim_{x\to k^+} f(x) = \lim_{h\to 0} f(k+h) = \lim_{h\to 0} [k+h]$

Now $k \le k+h < k+1$ $\therefore$ $[k+h]=k$ i.e. $\underset{x\to k^+}{Lt} f(x) = \underset{h\to 0}{Lt} [k+h] = \underset{h\to 0}{Lt} k = k$

i.e. $\underset{h\to k^-}{Lt} f(x) \ne \underset{h\to k^+}{Lt} f(x)$ i.e. f is discontinuous of first kind at $x = k$

As k is any arbitrary integer $\therefore$ function is discontinuous for every integer.

Now let c be any real number except integer. There exist an integer k such that $k-1 < c < k$

Now $\underset{x\to c^-}{Lt} f(x) = \underset{h\to 0}{Lt} f(c-h) = \underset{h\to 0}{Lt} [c-h]$

Now $k-1 < c-h < k$ $\therefore [c-h] = k-1$: $\underset{x\to c^-}{Lt} f(x) = \underset{h\to 0}{Lt} (k-1) = k-1$

Now $\underset{x\to c^+}{Lt} f(x) = \underset{h\to 0}{Lt} f(c+h) = \underset{h\to 0}{Lt} [c+h]$ $\therefore$ $k-1 < c+h < k$ $\therefore [c+h] = k-1$

$\underset{x\to c^+}{Lt} f(x) = \underset{h\to 0}{Lt} (k-1) = (k-1)$ Also $f(c) = k-1$ $[\because k-1 < k < c \text{ i.e.} [c] = k-1]$

i.e. $\underset{x\to c^-}{Lt} f(x) = \underset{x\to c^+}{Lt} f(x) = f(c)$ i.e. f is continuous at all point except integers

45. Prove that $f(x) = \sin\dfrac{1}{x}$ is not continuous at $x = 0$. Also name the kind of discontinuity it has

Solution :

$$f(x) = \sin\frac{1}{x} \qquad \underset{x\to 0^+}{Lt} f(x) = \underset{h\to 0}{Lt} f(0+h) = \underset{h\to 0}{Lt} \sin\left(\frac{1}{0+h}\right) = \underset{h\to 0}{Lt} \sin\left(\frac{1}{h}\right)$$

As $h \to 0$ the value of $\sin\dfrac{1}{h}$ oscillates between + 1 and – 1, passing through zero and intermediate values, an infinite number of times. Thus there is no definite number to which $\sin\dfrac{1}{h}$ tends as h tends to zero. Hence $\underset{h\to 0}{Lt} \sin\left(\dfrac{1}{h}\right)$ does not exist. Similarly $\underset{x\to 0^-}{Lt} f(x)$ does not exist. $\therefore$ f has discontinuity of second kind at $x = 0$

Remark :

Similarly $\underset{x\to 0}{Lt} \cos\left(\dfrac{1}{x}\right)$ does not exist.

46. Discuss the continuity of the function $f(x) = (x-[x])^2$, $x \in [0,4]$ at $x = 1, 2, 3$.

Solution :

We have
$$\begin{aligned} [x] &= 0 & 0 \le x < 1 \\ &= 1 & 1 \le x < 2 \\ &= 2 & 2 \le x < 3 \\ &= 3 & 3 \le x < 4 \end{aligned} \qquad \begin{aligned} \therefore f(x) = (x-[x])^2 &= (x-0)^2 & 0 \le x < 1 \\ &= (x-1)^2 & 1 \le x < 2 \\ &= (x-2)^2 & 2 \le x < 3 \\ &= (x-3)^2 & 3 \le x < 4 \end{aligned}$$

Consider the point $x = 1$
$$\underset{x\to 1^-}{Lt} f(x) = \underset{x\to 1^-}{Lt} (x)^2 = 1$$
$$\underset{x\to 1^+}{Lt} f(x) = \underset{x\to 1^+}{Lt} (x-1)^2 = (1-1)^2 = 0$$

Now $\underset{x\to 1^-}{Lt} f(x) \ne \underset{x\to 1^+}{Lt} f(x)$ $\therefore$ f has discontinuity of first kind at $x = 1$.

Consider the point $x = 2$

$\underset{x\to 2^-}{Lt} f(x) = \underset{x\to 2^-}{Lt} (x-1)^2 = (2-1)^2 = 1$; $\underset{x\to 2^+}{Lt} f(x) = \underset{x\to 2^+}{Lt} (x-2) = (2-2) = 0$

As $\underset{x\to 2^-}{Lt} f(x) \ne \underset{x\to 2^+}{Lt} f(x)$ $\therefore$ f has discontinuity of first kind at $x = 2$

Consider the point $x = 3$

$$\underset{x\to 3^-}{Lt} f(x) = \underset{x\to 3^-}{Lt} (x-2)^2 = (3-2)^2 = 1 \quad ; \quad \underset{x\to 3^+}{Lt} f(x) = \underset{x\to 3^+}{Lt} (x-3)^2 = (3-2)^2 = 0$$

$\therefore$ f has discontinuity of first kind at $x = 3$

47. Discuss the continuity of the function $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}; & \text{when } x \neq 0 \\ 0; & \text{when } x = 0 \end{cases}$ at $x = 0$. **[BCA I.P. 2011]**

Solution :

$$\underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^-}{Lt} \frac{\sin[x]}{[x]} = \underset{h\to 0}{Lt} \frac{\sin[0-h]}{[0-h]} = \underset{h\to 0}{Lt} \frac{\sin(-1)}{(-1)} = \frac{-\sin 1}{-1} = \sin 1$$

$$\underset{x\to 0^+}{Lt} f(x) = \underset{x\to 0^+}{Lt} \frac{\sin[x]}{[x]} = \underset{h\to 0}{Lt} \frac{\sin[0+h]}{[0+h]} \quad ; \quad \underset{h\to 0}{Lt} \frac{\sin 0}{0}, \text{ which does not exist}$$

Thus $\underset{x\to 0^-}{Lt} f(x) \neq \underset{x\to 0^+}{Lt} f(x)$ $\qquad \therefore$ $f(x)$ is not continuous at $x = 0$.

48. If $f(x) = \begin{cases} \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}, & x \neq 2 \\ k, & x = 2 \end{cases}$ is continuous at $x = 2$, then find the value of k.

Solution :

Since $f(x)$ is continuous at $x = 2$; $\quad \therefore \underset{x\to 2}{Lt} f(x) = f(2) \quad \Rightarrow \underset{x\to 2}{Lt} f(x) = k \quad$ ... (1)

Now, $$\underset{x\to 2}{Lt} f(x) = \underset{x\to 2}{Lt} \frac{x^3 + x^2 - 16x + 20}{(x-2)^2} = \underset{x\to 2}{Lt} \frac{(x-2)(x^2+3x-10)}{(x-2)^2} = \underset{x\to 2}{Lt} \frac{(x-2)(x-2)(x+5)}{(x-2)^2}$$

$$= \underset{x\to 2}{Lt} (x+5) = 7 \qquad \text{... (2)}$$

From (1) and (2), we get $k = 7$

49. If $[x]$ denotes the integral part of x and $f(x) = [x]\left\{\dfrac{\sin\left(\dfrac{\pi}{[x+1]}\right) + \sin \pi\,[x+1]}{1+[x]}\right\}$, then show that $f(x)$ is discontinuous at all integral points.

Solution :

We know that for all integral values $\sin n\pi = 0$. Now $[x+1]$ also gives only integral values $\therefore \sin \pi[x+1] = 0$

Also $[x+1] = [x] + 1$ $\quad \therefore f(x) = \dfrac{[x]}{1+[x]} \sin \dfrac{\pi}{[x]+1}$. $\qquad$ At $x = n$, $n \in I$, $f(x) = \dfrac{n}{1+n} \sin \dfrac{\pi}{n+1}$

For $n < x < n+1$, $n \in I$, $[x] = n$ $\quad \therefore f(x) = \dfrac{n}{1+n} \sin \dfrac{\pi}{n+1}$ $\quad \therefore \underset{x\to n^+}{Lt} f(x) = \dfrac{n}{1+n} \dfrac{\sin \pi}{n+1}$ ... (1)

For $n-1 < x < n$; $[x] = n-1$ $\quad \therefore f(x) = \dfrac{n-1}{n} \sin \dfrac{\pi}{n}$ $\quad \therefore \underset{x\to n^-}{Lt} f(x) = \dfrac{n-1}{n} \sin \dfrac{\pi}{n}$ ... (2)

From (1) and (2) $\quad \underset{x\to n^-}{Lt} f(x) \neq \underset{x\to n^+}{Lt} f(x)$ $\quad \therefore$ $f(x)$ is not continuous at all $n \in I$.

50. If $f(x) = \begin{cases} \dfrac{\sqrt{1+Px} - \sqrt{1-Px}}{x}, & -1 \leq x < 0 \\ \dfrac{2x+1}{x-1}, & 0 \leq x \leq 1 \end{cases}$ is continuous is the interval $[-1, 1]$; find P.

Solution :

As $f(x)$ is continuous in $[-1, 1]$; thus it is continuous at $x = 0$. $\quad \therefore \underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^+}{Lt} f(x) = f(0)$ ...(1)

Now, $\underset{x\to 0}{Lt} f(x) = \underset{x\to 0}{Lt} \frac{\sqrt{1+Px}-\sqrt{1-Px}}{x} = \underset{x\to 0}{Lt} \frac{\sqrt{1+Px}-\sqrt{1-Px}}{x} \times \frac{\sqrt{1+Px}+\sqrt{1-Px}}{\sqrt{1+Px}+\sqrt{1-Px}}$

$= \underset{x\to 0}{Lt} \frac{(1+Px)-(1-Px)}{x(\sqrt{1+Px}+\sqrt{1-Px})} = \underset{x\to 0}{Lt} \frac{2Px}{x(\sqrt{1+Px}+\sqrt{1-Px})}$

$= \underset{x\to 0}{Lt} \frac{2P}{(\sqrt{1+Px}+\sqrt{1-Px})} = \underset{h\to 0}{Lt} \frac{2P}{\sqrt{1+P(0-h)}+\sqrt{1-P(0-h)}} = \underset{h\to 0}{Lt} \frac{2P}{\sqrt{1-Ph}+\sqrt{1+Ph}} = \frac{2P}{2} = P$ ...(2)

Also $f(0) = \frac{2(0)+1}{0-1} = -1$ ...(3) From (1), (2) and (3), we get $P = -1$.

51. If the function $f(x) = \begin{cases} 3ax+b, & \text{for } x>1 \\ 11 & \text{for } x=1 \\ 5ax-b, & \text{for } x<1 \end{cases}$ is continuous at $x = 1$, find the values of a and b.

Solution :

$f(1) = 11$; $\underset{x\to 1^+}{Lt} f(x) = \underset{x\to 1^+}{Lt} 3ax+b = \underset{h\to 0}{Lt} 3a(1+h)+b = 3a+b$

$\underset{x\to 1^-}{Lt} f(x) = \underset{x\to 1^-}{Lt} 5ax-b = \underset{h\to 0}{Lt} 5a(1-h)-b = 5a-2b$

Since $f(x)$ is continuous at $x = 1$, we have

$\underset{x\to 1^+}{Lt} f(x) = \underset{x\to 1^-}{Lt} f(x) = f(1)$ $\therefore 3a+b = 5a-2b = 11$.

On solving $(3a+b=11)$ and $(5a-2b=11)$, we get $a=3, b=2$. Hence, $a=3, b=2$.

52. Show that the function $f(x) = \begin{cases} \frac{e^{1/x}-1}{e^{1/x}+1}, & \text{when } x\neq 0 \\ 0, & \text{when } x=0 \end{cases}$ is discontinuous at $x = 0$.

Solution :

$f(0) = 0$. Now $\underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^-}{Lt} \frac{e^{1/x}-1}{e^{1/x}+1} = \frac{0-1}{0+1} = -1 \left[\because \underset{x\to 0^-}{Lt} e^{1/x} = 0\right]$; $\underset{x\to 0^+}{Lt} f(x) = \underset{x\to 0^+}{Lt} \frac{e^{1/x}-1}{e^{1/x}+1}$

$= \underset{x\to 0^+}{Lt} \frac{1-\frac{1}{e^{1/x}}}{1+\frac{1}{e^{1/x}}} = \underset{x\to 0^+}{Lt} \frac{1-e^{-1/x}}{1+e^{-1/x}} = \frac{1-0}{1+0} = 1 \left[\because \underset{x\to 0^+}{Lt} e^{-1/x} = 0\right]$

Thus, $\underset{x\to 0^+}{Lt} f(x) \neq \underset{x\to 0^-}{Lt} f(x)$ and therefore $\underset{x\to 0}{Lt} f(x)$ does not exist. Hence $f(x)$ is discontinuous at $x = 0$.

53. If the function $f(x)$ defined by $f(x) = \begin{cases} \frac{\log(1+ax)-\log(1-bx)}{x}, & \text{if } x\neq 0 \\ k, & \text{if } x=0 \end{cases}$ is continuous at $x = 0$, find k.

Solution :

Since $f(x)$ is continuous at $x = 0$ $\therefore \underset{x\to 0}{Lt} f(x) = f(0)$

$\Rightarrow \underset{x\to 0}{Lt}\left[\frac{\log(1+ax)-\log(1-bx)}{x}\right] = k \Rightarrow \underset{x\to 0}{Lt}\left[\frac{\log(1+ax)}{x} - \frac{\log(1-bx)}{x}\right] = k$

$\Rightarrow \underset{x\to 0}{Lt} \frac{a\log(1+ax)}{ax} - (-b)\frac{\log(1-bx)}{(-bx)} = k \Rightarrow a(1)+b(1) = k \left[\text{using } \underset{x\to 0}{Lt} \frac{\log(1+x)}{x} = 1\right] \Rightarrow a-b=k$

Thus, $f(x)$ is continuous at $x = 0$, if $k = a+b$

54. A function is defined as $f(x)=\begin{cases} x^m \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

what condition should be imposed on m so that $f(x)$ may be continuous at $x = 0$?

Solution :

If $f(x)$ is continuous at $x = 0$, then $\underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^+}{Lt} f(x) = f(0) = 0$

Now $\underset{x\to 0^-}{Lt} f(x) = \underset{h\to 0}{Lt} f(0-h) = \underset{h\to 0}{Lt} (-h)^m \sin\left(-\frac{1}{h}\right) = -\underset{h\to 0}{Lt} (-h)^m \sin\left(\frac{1}{h}\right) = 0$, only when $m > 0$

and $\underset{x\to 0^+}{Lt} f(x) = \underset{h\to 0}{Lt} f(0+h) = \underset{h\to 0}{Lt} h^m \sin\left(\frac{1}{h}\right) = 0$, only where $m > 0$

Hence, $f(x)$ is continuous at $x = 0$, if $m > 0$.

55. Show that the function $f(x) = (1 + 2x)^{1/x}$ for $x \neq 0$ and $f(x) = e^2$ for $x = 0$ is continuous at $x = 0$.

Solution :

Given $\quad f(0) = e^2 \quad$ ...(i)

Also, $f(0 + h) = [1 + 2(0 + h)]^{1/0+h} = (1 + 2h)^{1/h}$

$\therefore \quad \underset{x\to 0^+}{Lt} f(x) = \lim_{h\to 0} (1 + 2h)^{1/h} = \lim_{h\to 0} [(1 + 2h)^{1/2h}]^2 = e^2, \quad \because \lim_{h\to 0} (1 + y)^{1/y} = e \quad$ ...(ii)

And $\quad f(0 - h) = [1 + 2h]^{1/(0-h)} = (1 - 2h)^{-1/h}$

$\therefore \quad \underset{x\to 0^-}{Lt} f(x) = \lim_{h\to 0} (1 - 2h)^{-1/h} = \lim_{h\to 0} [(1 - 2h)^{1/-2h}]^2 = e^2 \quad$ ...(iii)

Hence from (i), (ii) and (iii) we find that $\underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^+}{Lt} f(x) = f(0)$

Hence the given function $f(x)$ is continuous at $x = 0$.

56. Show that the function $f(x) = |x|$ is continuous for all x. Also draw the graph of the function.

Solution :

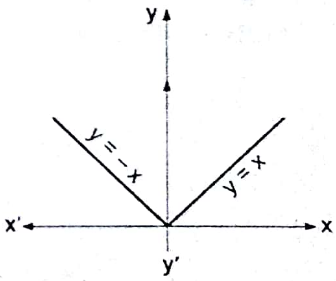
The function $f(x)$ can be defined as $f(x) = |x| = \begin{cases} x, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -x & \text{if } x < 0 \end{cases}$.

We know that a polynomial function is continuous at each point of its domain and so $f(x) = x, x > 0$ and $f(x) = -x, x < 0$ both are continuous.

We have $\quad f(0) = 0$

Now $\quad f(0 + 0) = \lim_{h\to 0} f(0 + h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} (h) = 0$

$f(0 - 0) = \lim_{h\to 0} f(0 - h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} [-(h)] = \lim_{h\to 0} (h) = 0.$



Since $f(0 + 0) = f(0) = f(0 - 0)$,

therefore, the function $f(x)$ is continuous at $x = 0$. Proved.

Graph of $f(x)$:The graph of the function consists of the following straight lines:

$y = x, x \geq 0 \;;\; y = -x, x \leq 0 \quad$ Clearly, the function is continuous at every point.

57. Discuss the continuity of the function $f(x)$ defined by $f(x) = |x - 1| + 2|x - 2| + 3|x - 3|$ in $[0, 4]$. Also draw the graph of the function.

Solution :

From the given definition of the function, we have

$$f(x) = \begin{cases} (1 - x) + 2(2 - x) + 3(3 - x) = 14 - 6x, & \text{if } 0 \leq x \leq 1, \\ (x - 1) + 2(2 - x) + 3(3 - x) = 12 - 4x, & \text{if } 1 \leq x \leq 2, \\ (x - 1) + 2(x - 2) + 3(3 - x) = 4, & \text{if } 2 \leq x \leq 3, \\ (x - 1) + 2(x - 2) + 3(x - 3) = 6x - 14, & \text{if } 3 \leq x \leq 4 \end{cases}$$

We know that a polynomial function is continuous at each point of its domain and so $f(x) = 14 - 6x$ for $0 \le x \le 1$, $f(x) = 12 - 4x$ for $1 \le x \le 2$, $f(x) = 4$ for $2 \le x \le 3$ and $f(x) = 6x - 14$ for $3 \le x \le 4$ are continuous. Now, it remains to check continuity of the function $f(x)$ at $x = 1, 2$ and 3.

Continuity at $x = 1$. We have $f(1) = 14 - 6 = 8$

$$\underset{x \to 1^-}{Lt} f(x) = \lim_{h \to 0} f(1 - h) = \lim_{h \to 0} [14 - 6(1 - h)] = \lim_{h \to 0} (8 + 6h) = 8.$$

$$\underset{x \to 1^+}{Lt} f(x) = \lim_{h \to 0} f(1 + h) = \lim_{h \to 0} [12 - 4(1 + h)] = \lim_{h \to 0} (8 - 4h) = 8.$$

Since $\underset{x \to 1^-}{Lt} f(x) = f(1) = \underset{x \to 1^+}{Lt} f(x)$, therefore, $f(x)$ is continuous at $x = 1$.

Continuity at $x = 2$. We have $f(2) = 12 - 4 \times 2 = 4$

$$\underset{x \to 2^-}{Lt} f(x) = \lim_{h \to 0} f(2 - h) = \lim_{h \to 0} [12 - 4(2 - h)] = \lim_{h \to 0} [4 + 4h] = 4$$

and $$\underset{x \to 2^+}{Lt} f(x) = \lim_{h \to 0} f(2 + h) = \lim_{h \to 0} [4] = 4.$$

Since $\underset{x \to 2^-}{Lt} f(x) = f(2) = \underset{x \to 2^+}{Lt} f(x)$, therefore, $f(x)$ is continuous at $x = 2$.

Continuity at $x = 3$. We have $f(3) = 4$

Now, $$\underset{x \to 3^-}{Lt} f(x) = \lim_{h \to 0} f(3 - h) = \lim_{h \to 0} 4 = 4.$$

and $$\underset{x \to 3^+}{Lt} f(x) = \lim_{h \to 0} f(3 + h) = \lim_{h \to 0} [6(3 + h) - 14] = 4$$

Since $\underset{x \to 3^-}{Lt} f(x) = f(3) = \underset{x \to 3^+}{Lt} f(x)$, therefore, $f(x)$ is continuous at $x = 3$.

Hence, $f(x)$ is continuous at every $x \in [0, 4]$.

Graph of the function : The graph of the function consists:

(i) the segment of the line $y = 14 - 6x$ for $0 \le x \le 1$.

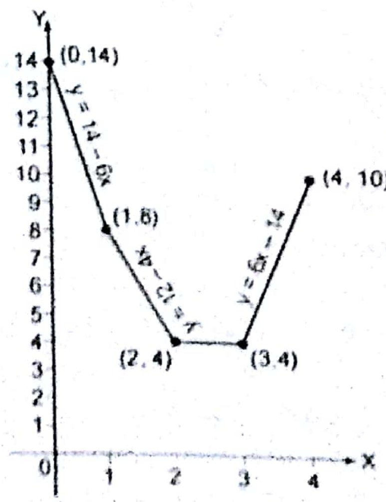
(ii) the segment of the line $y = 12 - 4x$ for $1 \le x \le 2$.

(iii) the segment of the line $y = 4$ for $2 \le x \le 3$.

(iv) the segment of the line $y = 6x - 14$ for $3 \le x \le 4$.

The graph of the function for the interval $[0, 4]$ is as given in the adjacent figure.

Clearly, the function $f(x)$ is continuous at every $x \in [0, 4]$.



58. Test the continuity of the following function: $f(x) = \frac{1}{(x-a)} \operatorname{cosec}\left(\frac{1}{x-a}\right)$, when $x \ne a$; $= 0$, when $x = a$.

Solution :

We have $f(a) = 0$. Now, Right hand limit

$$\underset{x \to 0^+}{Lt} f(x) = \lim_{h \to 0} \frac{1}{a+h-a} \operatorname{cosec}\left(\frac{1}{a+h-a}\right) = \lim_{h \to 0} \frac{1}{h} \operatorname{cosec}\left(\frac{1}{h}\right)$$

$$= \lim_{h \to 0}\left[\frac{1}{h \sin(1/h)}\right] = \frac{1}{0}, \qquad \because \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) \to 0$$

$$= \infty$$

And left hand limit $= \underset{x \to a^-}{Lt} f(x) = \lim_{h \to 0} \frac{1}{(a-h)-a} \operatorname{cosec}\left(\frac{1}{a-h-a}\right) = \lim_{h \to 0} \frac{1}{h} \operatorname{cosec}\left(\frac{1}{h}\right) = \infty$, as above.

From above, it is clear that right left hand limits of $f(x)$ as $x \to 0$ though equal to each other are not equal to $f(a)$. Hence the given function is discontinuous at $x = a$.

Miscellaneous Examples

59. Examine the continuity of $f(x) = \dfrac{e^{-1/x}}{1+e^{1/x}}$, $x \neq 0$, $f(0) = 1$ at $x = 0$

Solution :

$$f(x) = \frac{e^{-1/x}}{1+e^{1/x}}. \qquad \text{Now as } x \to 0^-, \quad e^{1/x} \to 0$$

$$\underset{x \to 0^-}{Lt} f(x) = \underset{x \to 0^-}{Lt}\left(\frac{e^{-1/x}}{1+e^{1/x}}\right) = \underset{x \to 0^-}{Lt}\left(\frac{1}{e^{1/x}(1+e^{1/x})}\right) = \frac{1}{0(1+0)} = \frac{1}{0} = \infty$$

i.e. $\underset{x \to 0^-}{Lt} f(x)$ does not exist $\qquad \therefore f$ has discontinuity of second kind from left at $x = 0$

60. Discuss the continuity of f at $x = 0$ for $f(x) = \left(\dfrac{e^{x^2}}{e^{\frac{1}{x^2}} - 1}\right)$, $x \neq 1$, $f(0) = 1$.

Solution :

Now when $x \to 0, \dfrac{1}{x^2} \to +\infty \Rightarrow e^{1/x^2} \to +\infty \Rightarrow e^{-\frac{1}{x^2}} \to 0$

$$\underset{x \to 0}{Lt} f(x) = \underset{x \to 0}{Lt}\left(\frac{e^{x^2}}{e^{1/x^2} - 1}\right) = \underset{x \to 0}{Lt}\, e^{x^2}\left(\frac{1}{e^{1/x^2} - 1}\right); \quad \underset{x \to 0}{Lt}\, e^{x^2}\, \underset{x \to 0}{Lt}\left(\frac{e^{-1/x^2}}{1 - e^{-1/x^2}}\right)$$

But $f(0) = 1 \quad \therefore \underset{x \to 0}{Lt} f(x) \neq f(0) \qquad \therefore f$ has removable discontinuity at $x = 0$

61. Let be the function defined on R as $f(x) = \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$, $x \neq 0$, $f(0) = -1$.

Show that f is continuous from left at x=0 and has discontinuity of first kind from right.

Solution :

$$f(x) = \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} \qquad \text{As } x \to 0^-, e^{1/x} \to 0 \quad \text{and} \quad x \to 0^+, e^{-1/x} \to 0$$

$$\underset{x \to 0^-}{Lt} f(x) = \underset{x \to 0^-}{Lt}\left(\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right) = \underset{x \to 0^-}{Lt}\left(\frac{e^{1/x}e^{1/x} - 1}{e^{1/x}e^{1/x} + 1}\right) = \frac{0.0 - 1}{0.0 + 1} = -1$$

$$\underset{x\to 0^+}{Lt} f(x) = \underset{x\to 0^+}{Lt}\left(\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right) = \underset{x\to 0^+}{Lt}\left(\frac{1-e^{-1/x}e^{-1/x}}{1+e^{-1/x}e^{-1/x}}\right) = \frac{1-0.0}{1+0.0} = 1 \qquad f(0) = -1$$

i.e. $\underset{x\to 0^-}{Lt} f(x) = f(0)$ But $\underset{x\to 0^+}{Lt} f(x) \neq f(0)$

$\therefore$ f is continuous from left at $x = 0$ and has discontinuity of first kind from right.

62. Evaluate $\underset{x\to\infty}{Lt} \dfrac{x^2-4}{x^2+7x+12}$

Solution :

$$f(x) = \frac{x^2-4}{x^2+7x+12}$$

If we put $x = \infty$ in numerator and denominator we get $\left(\frac{\infty}{\infty}\right)$ which is an indeterminant form

$\therefore$ we shall divide numerator and denominator by x^2 (Highest power of x)

$$\therefore f(x) = \frac{1-\frac{4}{x^2}}{1+\frac{7}{x}+\frac{12}{x^2}} \; ; \; \underset{x\to\infty}{Lt} f(x) = \frac{1-4\underset{x\to\infty}{Lt}\left(\frac{1}{x^2}\right)}{1+7\underset{x\to\infty}{Lt}\left(\frac{1}{x}\right)+12\underset{x\to\infty}{Lt}\left(\frac{1}{x^2}\right)}$$

$$\frac{1-4(0)}{1+7(0)+12(0)} = \frac{1}{1} = 1$$

63. Evaluate $\underset{x\to\infty}{Lt}\left[\sqrt{x+1}-\sqrt{x-1}\right] = 0$

Solution :

$$f(x) = \sqrt{x+1}-\sqrt{x-1} \quad i.e. \quad f(x) = \frac{\left(\sqrt{x+1}-\sqrt{x-1}\right)\left(\sqrt{x+1}+\sqrt{x-1}\right)}{\left(\sqrt{x+1}+\sqrt{x-1}\right)}$$

$$= \frac{(x+1)-(x-1)}{\sqrt{x+1}+\sqrt{x-1}} = \frac{2}{\sqrt{x+1}+\sqrt{x-1}} \; ; \; \underset{x\to\infty}{Lt} f(x) = \underset{x\to\infty}{Lt}\left(\frac{2}{\sqrt{x+1}+\sqrt{x-1}}\right) = \frac{2}{\infty} = 0$$

64. $\underset{x\to 0}{\text{Lt}} \dfrac{\sqrt{1+x^2}-\sqrt{1+x}}{x}$.

Solution :

$$\underset{x\to 0}{\text{Lt}} \frac{\sqrt{1+x^2}-\sqrt{1+x}}{x} = \underset{x\to 0}{\text{Lt}}\left(\frac{\sqrt{1+x^2}-\sqrt{1+x}}{x}\right)\left(\frac{\sqrt{1+x^2}+\sqrt{1+x}}{\sqrt{1+x^2}+\sqrt{1+x}}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{(1+x^2)-(1+x)}{x(\sqrt{1+x^2}+\sqrt{1+x})} = \underset{x\to 0}{\text{Lt}} \frac{x^2-x}{x(\sqrt{1+x^2}+\sqrt{1+x})}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x(x-1)}{x(\sqrt{1+x^2}+\sqrt{1+x})} = \underset{x\to 0}{\text{Lt}} \frac{x-1}{\sqrt{1+x^2}+\sqrt{1+x})} = \frac{-1}{\sqrt{1}+\sqrt{1}} = -\frac{1}{2}$$

65. Find $\underset{x\to 2}{\text{Lt}} \dfrac{x-2}{\sqrt{x^2-4}+\sqrt{x-2}}$.

Solution :

$$\underset{x\to 2}{\text{Lt}} \frac{(\sqrt{x-2})^2}{\sqrt{(x-2)(x+2)}+\sqrt{x-2}} = \underset{x\to 2}{\text{Lt}} \frac{(\sqrt{x-2})^2}{\sqrt{x-2}\left[\sqrt{x+2}+1\right]} = \underset{x\to 2}{\text{Lt}} \frac{(\sqrt{x-2})^2}{\left[\sqrt{x+2}+1\right]} = \frac{2-2}{\sqrt{2+2}+1} = \frac{0}{3} = 0$$

66. Find $\underset{x\to 1}{Lt}\dfrac{x^7-2x^5+1}{x^3-3x^2+2}$.

Solution :

When $x = 1$ numerator and denominator both become zero and hence $(x-1)$ is a factor of both

Now $\underset{x\to 1}{Lt}\dfrac{x^7-2x^5+1}{x^3-3x^2+2} = \underset{x\to 1}{Lt}\dfrac{x^7-x^6+x^6-x^5-x^5+x^4-x^4+x^3-x^3+x^2-x^2+x-x+1}{x^3-x^2-2x^2+2x-2x+2}$

$= \underset{x\to 1}{Lt}\dfrac{x^6(x-1)+x^5(x-1)-x^4(x-1)-x^3(x-1)-x^2(x-1)-x(x-1)-(x-1)}{x^2(x-1)-2x(x-1)-2(x-1)}$

$= \underset{x\to 1}{Lt}\dfrac{(x-1)(x^6+x^5-x^4-x^3-x^2-x-1)}{(x-1)(x^2-2x-2)}$

$= \underset{x\to 1}{Lt}\dfrac{(x^6+x^5-x^4-x^3-x^2-x-1)}{x^2-2x-2} = \dfrac{-3}{-3} = 1$

67. Evaluate : $\underset{x\to y}{Lt}\dfrac{\tan x-\tan y}{x-y}$.

Solution :

Let $x = y+h$, then as $x\to y$, $h\to 0$

Now $\underset{x\to y}{Lt}\dfrac{\tan x-\tan y}{x-y} = \underset{h\to 0}{Lt}\dfrac{\tan(y+h)-\tan y}{y+h-y}$

$= \underset{h\to 0}{Lt}\dfrac{1}{h}\left[\dfrac{\sin(y+h)}{\cos(y+h)}-\dfrac{\sin y}{\cos y}\right] = \underset{h\to 0}{Lt}\dfrac{\sin(y+h)\cos y-\cos(y+h)\sin y}{h\cos(y+h)\cos y}$

$= \underset{h\to 0}{Lt}\dfrac{\sin(y+h-y)}{h\cos(y+h)\cos y} = \underset{h\to 0}{Lt}\dfrac{\sin h}{h}\cdot\dfrac{1}{\cos(y+h)\cos y} = 1\cdot\dfrac{1}{\cos^2 y} = \sec^2 y$

68. Find $\underset{x\to \alpha}{Lt}\dfrac{x\sin\alpha-\alpha\sin x}{x-\alpha}$.

Solution :

Let $x = \alpha+h$, then as $x\to\alpha$, $h\to 0$

Now $\underset{x\to \alpha}{Lt}\dfrac{x\sin\alpha-\alpha\sin x}{x-\alpha} = \underset{h\to 0}{Lt}\dfrac{(\alpha+h)\sin\alpha-\alpha\sin(\alpha+h)}{\alpha+h-\alpha}$

$= \underset{h\to 0}{Lt}\dfrac{\alpha\sin\alpha+h\sin\alpha-\alpha\sin(\alpha+h)}{h} = \underset{h\to 0}{Lt}\dfrac{\alpha[\sin\alpha-\sin(\alpha+h)]+h\sin\alpha}{h}$

$= \underset{h\to 0}{Lt}\left[\dfrac{\alpha\cdot 2\cos\dfrac{2\alpha+h}{2}\sin\left(-\dfrac{h}{2}\right)}{h}+\dfrac{h\sin\alpha}{h}\right] = \underset{h\to 0}{Lt}\dfrac{\alpha\cdot 2\cos\dfrac{2\alpha+h}{2}\dfrac{\sin\left(-\dfrac{h}{2}\right)}{\left(-\dfrac{h}{2}\right)}\cdot\left(-\dfrac{h}{2}\right)}{h}+\sin\alpha$

$= -\alpha\cos\alpha + \sin\alpha = \sin\alpha - \alpha\cos\alpha$

69. Determine the values of a, b, c for which the function

$f(x) = \dfrac{\sin(a+1)x+\sin x}{x}$, when $x<0 = c$, when $x = 0 = \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{\frac{3}{2}}}$, when $x>0$ continuous at $x = 0$

Solution :

L.H. limit : In this case $x<0$ $\quad\therefore\quad \underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0}{Lt}\dfrac{\sin(a+1)x+\sin x}{x}$

$$= \underset{x\to 0}{Lt}\left[\frac{\sin(a+1)x}{x}+\frac{\sin x}{x}\right] = \underset{x\to 0}{Lt}\frac{\sin(a+1)x}{(a+1)x}(a+1)+\underset{x\to 0}{Lt}\frac{\sin x}{x} = a+1+1 = a+2.$$

R.H. limit : In this case $x > 0$

$$\therefore \underset{x\to 0+0}{Lt} f(x) = \underset{x\to 0}{Lt}\frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{\frac{3}{2}}} = \underset{x\to 0}{Lt}\frac{(\sqrt{x+bx^2}-\sqrt{x})(\sqrt{x+bx^2}+\sqrt{x})}{bx^{\frac{3}{2}}(\sqrt{x+bx^2}+\sqrt{x})}$$

$$= \underset{x\to 0}{Lt}\frac{x+bx^2-x}{bx^{\frac{3}{2}}(\sqrt{x+bx^2}+\sqrt{x})} = \underset{x\to 0}{Lt}\frac{bx^2}{bx^2(\sqrt{1+bx}+1)} = \underset{x\to 0}{Lt}\frac{1}{\sqrt{1+bx}+1} = \frac{1}{2}, b \neq 0$$

Given, $f(0) = c$. Since $f(x)$ is continuous at $x = 0$ $\therefore \underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^+}{Lt} f(x) = f(0)$

$$\therefore a+2 = \frac{1}{2} = c \quad \therefore a = \frac{3}{2}, \quad c = \frac{1}{2} \text{ and } b \neq 0$$

70. Let $f(x+y) = f(x)+f(y)$ for all x and y. If $f(x)$ is continuous at $x = a$, show that $f(x)$ is continuous at all x.

Solution :

Given, $f(x+y) = f(x)+f(y)$ ...(1) and $f(x)$ is continuous at $x = a$ i.e. $\underset{x\to a}{Lt} f(x) = f(a)$ ...(2)

To prove $f(x)$ is continuous at $x = c$ for all c. i.e. $\underset{x\to c}{Lt} f(x) = f(c)$

Put $x = c+h-a$, so that as $x \to c$, $h \to a$. Now $\underset{x\to c}{Lt} f(x) = \underset{h\to a}{Lt} f(c+h-a) = \underset{h\to a}{Lt} f\{(c-a)+h\}$

$$= \underset{h\to a}{Lt}\{f(c-a)+f(h)\} \quad \text{[from (1)]} = \underset{h\to a}{Lt} f(c-a)+\underset{h\to a}{Lt} f(h)$$

$$= f(c-a)+\underset{x\to a}{Lt} f(x) \quad \left[\because \underset{h\to a}{Lt} f(h) = \underset{x\to a}{Lt} f(x)\right] = f(c-a)+f(a) \quad \text{[from (2)]}$$

$= f\{(c-a)+a\} = f(c)$. Hence $f(x)$ is continuous at $x = c$ for all c.

71. The function

$$f(x) = \frac{x^2}{a}, \text{ if } 0 \le x < 1 \quad ; \quad = a, \text{ if } 1 \le x < \sqrt{2} \quad ; \quad = \frac{2b^2-4b}{x^2}, \text{ if } \sqrt{2} \le x < \infty$$

is continuous for $0 \le x < \infty$, then find the most suitable values of a and b.

Solution :

Given, $f(1) = a$; $\underset{x\to 1^-}{Lt} f(x) = \underset{x\to 1^-}{Lt}\frac{x^2}{a} = \frac{1}{a}$ and $\underset{x\to 1^+}{Lt} f(x) = \underset{x\to a}{Lt}\, a = a$

Given, $f(\sqrt{2}) = \dfrac{2b^2-4b}{(\sqrt{2})^2} = b^2-2b$; $\underset{x\to\sqrt{2}^-}{Lt} f(x) = \underset{x\to\sqrt{2}^-}{Lt}(a) = a$

and $\underset{x\to\sqrt{2}^+}{Lt} f(x) = \underset{x\to\sqrt{2}^+}{Lt}\dfrac{2b^2-4b}{x^2} = b^2-2b$ $\quad \because f(x)$ is continuous in $[0, \infty[$

$\therefore$ it is continuous at $x = 1$ and $x = \sqrt{2}$

$\therefore \underset{x\to 1^-}{Lt} f(x) = \underset{x\to 1^+}{Lt} f(1)$ or $\dfrac{1}{a} = a \quad \therefore a = \pm 1$

and $\underset{x\to\sqrt{2}^-}{Lt} f(x) = \underset{x\to\sqrt{2}^+}{Lt} f(x) = f(\sqrt{2})$ or $a = b^2-2b$ or, $b^2-2b-a = 0$

When $a = 1$, $b^2-2b-1 = 0$, $\therefore b = \dfrac{2\pm 2\sqrt{2}}{2} = 1\pm\sqrt{2}$

When $a = -1$, $b^2-2b+1 = 0$, $(b-1)^2 = 0$ or $b = 1$. So $a = 1$, $b = 1\pm\sqrt{2}$; $a = -1$, $b = 1$

23. If $f(x) = [x]^2 - [x^2]$, (where $[x]$ is the greatest integer less than or equal to x, show that $f(x)$ is continuous at $x = 1$ and discontinuous at all other integral points.

Solution :

Let $n \in I$. Given, $f(x) = [x]^2 - [x^2]$; $f(n) = [n]^2 - [n^2] = n^2 - n^2 = 0$

$$\underset{x\to n^-}{Lt} f(x) = \underset{x\to n^-}{Lt} ([x]^2 - [x^2]) = (n-1)^2 - (n^2-1) = 2 - 2n \, ;$$

$$\underset{x\to n^+}{Lt} f(x) = \underset{x\to n^+}{Lt} ([x]^2 - [x^2]) = n^2 - n^2 = 0$$

Thus $\underset{x\to n^+}{Lt} f(x) = f(0) = 0$ and $\underset{x\to n^-}{Lt} f(x) = 2(1-n) = 0$ only at $n = 1$

Hence $f(x)$ is continuous at $x = 1$ and discontinuous at all other integral points.

Exercise

Type : 1

Questions based on left hand limit, right hand limit, limit at a point.

1. Evaluate $\lim\limits_{x\to 3} \dfrac{1}{[x]}$ **[Ans : limit does not exist]**

2. Evaluate $\lim\limits_{x\to \frac{5}{2}} [x]$ **[Ans. 2]**

3. If $f(x) = \begin{cases} x & x<0 \\ 1 & x=0 \\ x^2 & x>0 \end{cases}$ find $\lim\limits_{x\to 0} f(x)$ **[Ans. 0]**

4. If $f(x) = \begin{cases} x, & x<0 \\ 0 & x=0 \\ x^2 & x>0 \end{cases}$ find $\lim\limits_{x\to 0} f(x)$ **[Ans. 1]**

5. If $f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x=0 \end{cases}$ find $\lim\limits_{x\to 0} f(x)$ **[Ans. 1]**

6. If $f(x) = \begin{cases} 1+x^2 & 0 \le x \le 1 \\ 2-x & x>1 \end{cases}$ find $\lim\limits_{x\to 1^-} f(x)$, $\lim\limits_{x\to 1^+} f(x)$ **[Ans. 2, 1]**

7. If $f(x) = \begin{cases} \dfrac{|x-4|}{x-4} & x \neq 4 \\ 0 & x = 4 \end{cases}$ find $\lim\limits_{x\to 4^-} f(x)$ **[Ans. – 1]**

8. If $f(x) = \begin{cases} \dfrac{3x}{|x|+2x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ Show that $\lim\limits_{x\to 0} f(x)$ does not exist

9. If $f(x) = \begin{cases} 4 & x \ge 3 \\ x+1 & x<3 \end{cases}$ find $\lim\limits_{x\to 3} f(x)$ **[Ans. 4]**

10. If $f(x) = \begin{cases} x^2-1 & x \le 1 \\ -x^2-1 & x>1 \end{cases}$ find $\lim\limits_{x\to 1} f(x)$ **[Ans : Does not exist]**

Type : 2

Questions based on Algebraic limit. Evaluate the following limits:

1. $\lim_{x \to 4} \dfrac{x^2 - 7x + 12}{x^2 - 3x - 4}$ [Ans. $\dfrac{1}{5}$]

2. $\lim_{x \to 2} \left[\dfrac{1}{x-2} - \dfrac{2}{x^2 - 2x}\right]$ [Ans. $\dfrac{1}{2}$]

3. $\lim_{x \to 4} \dfrac{3 - \sqrt{5+x}}{1 - \sqrt{5-x}}$ [Ans. $-\dfrac{1}{3}$]

4. $\lim_{x \to 0} \dfrac{3x+1}{x+3}$ [Ans. $\dfrac{1}{3}$]

5. $\lim_{x \to 3} \dfrac{x^3 - 7x^2 + 15x - 9}{x^4 - 5x^3 + 27x - 27}$ [Ans. $\dfrac{2}{9}$]

6. $\lim_{x \to 1} \left[\dfrac{1}{x^2 + x - 2} - \dfrac{x}{x^3 - 1}\right]$ [Ans. $-\dfrac{1}{9}$]

7. $\lim_{x \to 1} \dfrac{\sqrt{x^2 - 1} + \sqrt{x - 1}}{\sqrt{x^2 - 1}}$, $x > 1$ [Ans. $\dfrac{\sqrt{2}+1}{\sqrt{2}}$]

8. $\lim_{x \to 5} \dfrac{x^2 - 9x + 20}{x^2 - 6x + 5}$ [Ans. $\dfrac{1}{4}$]

9. $\lim_{x \to 2} \dfrac{x^3 + 3x^2 - 9x - 2}{x^3 - x - 6}$ [Ans. $\dfrac{15}{11}$]

10. $\lim_{x \to 1} \dfrac{(2x-3)\sqrt{x} - 1}{3x^2 + 3x - 6}$ [Ans. $\dfrac{-1}{18}$]

11. $\lim_{x \to a} \dfrac{x\sqrt{x} - a\sqrt{a}}{x - a}$ [Ans. $\dfrac{3}{2}\sqrt{a}$]

12. $\lim_{x \to 9} \dfrac{x^{3/2} - 27}{x - 9}$ [Ans. $\dfrac{9}{2}$]

13. $\lim_{x \to 2} \dfrac{x - 2}{\sqrt[3]{x} - \sqrt[3]{2}}$ [Ans. $3(2^{\frac{2}{3}})$]

14. $\lim_{x \to -1} \dfrac{x^3 + 1}{x + 1}$ [Ans. 3]

15. $\lim_{x \to a} \dfrac{x^{\frac{2}{7}} - a^{\frac{2}{7}}}{x - a}$ [Ans. $\dfrac{2}{7} a^{\frac{-5}{7}}$]

Type : 3

Question based on Trigonometric and Inverse limits. Evaluate the following limits:

1. $\lim_{x \to 0} \dfrac{\sin 3x}{x^2}$ [Ans. 9]

2. $\lim_{x \to 0} \dfrac{\tan mx}{\tan nx}$ [Ans. $\dfrac{m}{n}$]

3. $\lim_{x \to 0} \dfrac{\sin 2x(\cos 3x - \cos x)}{x^3}$ [Ans. -8]

4. $\lim_{x \to \frac{\pi}{4}} \dfrac{1 - \sin 2x}{1 + \cos 4x}$ [Ans. $\dfrac{1}{4}$]

5. $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{1-\tan x}{x-\frac{\pi}{4}}$ [Ans. -2]

6. $\lim\limits_{x \to \infty} x \tan\left(\dfrac{1}{x}\right)$ [Ans. 1]

7. $\lim\limits_{x \to 0} \dfrac{1-\cos ax}{x \sin 3x}$ [Ans. $\dfrac{a^2}{6}$]

8. $\lim\limits_{x \to 0} \dfrac{\tan^{-1} 4x}{\sin 3x}$ [Ans. $\dfrac{4}{3}$]

9. $\lim\limits_{x \to 1} \dfrac{1-x}{\pi - 2\sin^{-1} x}$ [Ans. 0]

10. $\lim\limits_{x \to 1} \dfrac{1-\sqrt{x}}{(\cos^{-1} x)^2}$ [Ans. $\dfrac{1}{4}$]

11. $\lim\limits_{x \to 0} \dfrac{\sec^{-1}\sqrt{1+x^2}}{x}$ [Ans. 1]

12. $\lim\limits_{x \to 0} \mathrm{Cosec}^{-1}\left(\dfrac{1}{x}\right)$ [Ans. 1]

Type : 4

Questions based on exponential and logarithmic limits. Evaluate the following limits:

1. $\lim\limits_{x \to 0} \dfrac{e^{4x}-1}{x}$ [Ans. 4]

2. $\lim\limits_{x \to 0} \dfrac{x(e^{x}-1)}{1-\cos 2x}$ [Ans. $\dfrac{1}{2}$]

3. $\lim\limits_{n \to \infty} \left(1+\dfrac{2}{n}\right)^{2n}$ [Ans. e^4]

4. $\lim\limits_{x \to 0} (1+ax)^{\frac{b}{x}}$ [Ans. e^{ab}]

5. $\lim\limits_{x \to 0} (a^{\frac{1}{x}}-1)x$ [Ans. $\log_e a$]

6. $\lim\limits_{x \to 0} \left(\dfrac{1+5x^2}{1+3x^2}\right)^{\frac{1}{x^2}}$ [Ans. e^2]

7. $\lim\limits_{x \to 0} \dfrac{\log(1+x^3)}{\sin^3 x}$ [Ans. 1]

8. $\lim\limits_{x \to 0} \dfrac{e^{\sin x}-1}{x}$ [Ans. 1]

9. $\lim\limits_{x \to 0} \dfrac{a^x-b^x}{\sin x}$ [Ans. $\log\dfrac{a}{b}$]

10. $\lim\limits_{x \to 0} \dfrac{3^{2x}-1}{2^{3x}-1}$ [Ans. $\dfrac{\log_e 9}{\log_e 8}$]

Type : 5

Questions based on continuity and dis continuity. Evaluate the following limits:

1. Show that $f(x) = \begin{cases} 5x-4 & 0 < x \le 1 \\ 4x^3-3x & 1 < x < 2 \end{cases}$ is continuous at $x = 1$

2. Find the value of constant λ so that the $f(x) = \begin{cases} \dfrac{x^2-2x-3}{x+1} & x \neq -1 \\ \lambda & x = -1 \end{cases}$ is continuous at $x = -1$.

3. Show that $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ does not exist

4. Prove that $f(x) = \begin{cases} x - |x| & x \neq 0 \\ 2 & x = 0 \end{cases}$ is discontinuous at $x = 0$

5. If $f(x) = \begin{cases} \frac{x^2}{a} - a & x < a \\ 0 & x = a \\ a - \frac{a^2}{x} & x > a \end{cases}$ Prove that $f(x)$ is continuous at $x = a$

6. If $f(x) = \begin{cases} -x & x \leq 0 \\ x & x \geq 0 \end{cases}$ Prove $f(x)$ is continuous at $x = 0$

7. If $f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{2} - x & 0 < x < \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \\ \frac{3}{2} - x & \frac{1}{2} < x < 1 \\ 1 & x = 1 \end{cases}$ Find the points of discontinuity. **[Ans.** $x = 0, \frac{1}{2}, 1$**]**

8. Show that $f(x) = |x| + |x-1|$ is continuous at $x = 0$ and $x = 1$

9. Show that $f(x) = \begin{cases} \frac{|x-a|}{x-a} & x \neq a \\ 1 & x = a \end{cases}$ is discontinuous at $x = a$

10. Show that $f(x) = \begin{cases} \frac{x}{1+e^{\frac{1}{x}}} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous at $x = 0$

11. Show that $f(x) = \begin{cases} \frac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is discontinuous at $x = 0$

12. Show that $f(x) = \left\{ \frac{x-1}{1+e^{\frac{1}{x-1}}} \right\}$, $f(1) = 0$ is continuous at $x = 1$

13. Determine the values of a and b such that $f(x) = \begin{cases} ax^2 + b & x \leq 0 \\ \frac{-3}{x^2+1} + 1 & x > 0 \end{cases}$ is continuous. **[Ans.** $a \in R$, $b = -2$**]**

14. Show that $f(x) = \begin{cases} (1+2x)^{\frac{1}{x}} & x \neq 0 \\ e^2 & x = 0 \end{cases}$ is continuous at $x = 0$

15. If $f(x) = \begin{cases} \frac{x^3 - 8}{x^2 - 4} & x \neq 2 \\ 3 & x = 2 \end{cases}$ Show that $f(x)$ is continuous at $x = 2$

Chapter - 4

[Differentiation]

4·1 [Differentiability At A Point]

Definition : (Left hand derivative) :-

Let $f(x)$ be a real valued function defined on an open interval (a, b) and let c be a point in (a, b) i.e. $c \in (a, b)$, then the L.H.D. of $f(x)$ at c is defined by the limit

$$\boxed{Lf'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}} \quad \text{provided the limit exists}$$

Definition : (Right hand derivative) :-

Let $f(x)$ be a real valued function defined on an open interval (a, b) and let $c \in (a, b)$, then the R.H.D. of $f(x)$ at c is defined by the limit

$$\boxed{Rf'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}} \quad \text{provided the limit exists}$$

Definition : (Derivative at a point) :-

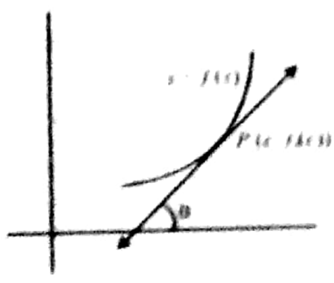
Let $f(x)$ be a real valued function defined on open interval (a, b) and $c \in (a, b)$ then the derivative of the function $f(x)$ at the point 'c' is defined as

$$\boxed{f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}}$$

Remark : A function f is derivable at a point $c \in (a, b)$ iff L.H.D. = R.H.D.

i.e. $f'(c)$ exists iff $\lim_{x \to c^-} \dfrac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \dfrac{f(x) - f(c)}{x - c}$

4·2 [Geometrical Meaning of Derivative at A Point]



Let $y = f(x)$ be a real valued function, then the derivative of the function $f(x)$ at a point P whose co-ordinates are $(c, f(c))$ represents the slope of the tangent drawn at the point P to the curve $y = f(x)$

thus $$\boxed{f'(c) = \left(\frac{dy}{dx}\right)_{(c,\, f(c))} = \text{slope of tangent at point } P = \tan\theta}$$

4·3 [Differentiability and Continuity]

Article :

If $f(x)$ is differentiable at $x = c$, then it is also continuous at $x = c$ but converse may not be true.

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Proof :

Since the function $f(x)$ is differentiable at $x = c$; thus $\quad f'(c) = \underset{x \to c}{\text{Lt}} \dfrac{f(x) - f(c)}{x - c} \quad (1)$

In order to prove that $f(x)$ is continuous at $x = c$; it is sufficient to prove that $\underset{x \to c}{\text{Lt}} f(x) = f(c)$

$$\underset{x \to c}{\text{Lt}} f(x) - f(c) + f(c) = \underset{x \to c}{\text{Lt}} [f(x) - f(c)] + f(c)$$

$$= \underset{x \to c}{\text{Lt}} \left\{ \frac{f(x) - f(c)}{(x - c)} \cdot (x - c) \right\} + f(c) = \underset{x \to c}{\text{Lt}} \frac{f(x) - f(c)}{x - c} \times \underset{x \to c}{\text{Lt}} (x - c) + f(c)$$

$= f'(c) \times 0 + f(c)$ [From (1)]

$= f(c)$ $\quad \underset{x \to c}{\text{Lt}} f(x) = f(c)$

Hence $f(x)$ is continuous at $x = c$

Converse may not be true. That is, a continuous function may not be differentiable at a point

e.g. Let $f(x) = |x|$, then $f(x)$ is continuous at $x = 0$ but it is not differentiable at $x = 0$

[See Example 1 Ahead]

Examples

1. Show that $f(x) = |x|$ is not differentiable at $x = 0$

Solution :

$$L f'(c) = \underset{x \to c^-}{\text{Lt}} \frac{f(x) - f(c)}{x - c}$$

$$L f'(0) = \underset{x \to 0^-}{\text{Lt}} \frac{f(x) - f(0)}{x - 0} = \underset{x \to 0^-}{\text{Lt}} \frac{|x| - |0|}{x} = \underset{h \to 0}{\text{Lt}} \frac{|0 - h|}{0 - h} = \underset{h \to 0}{\text{Lt}} \frac{h}{-h} = -1$$

$$R f'(0) = \underset{x \to 0^+}{\text{Lt}} \frac{f(x) - f(c)}{x - 0} = \underset{h \to 0}{\text{Lt}} \frac{f(0 + h) - f(0)}{0 + h} = \underset{h \to 0}{\text{Lt}} \frac{|0 + h| - |0|}{h} = \underset{h \to 0}{\text{Lt}} \frac{h}{h} = 1$$

$L f'(0) \neq R f'(0)$. Hence f is not differentiable at $x = 0$

2. Show that $f(x) = \begin{cases} 12x - 13, & \text{if} \quad x \le 3 \\ 2x^2 + 5, & \text{if} \quad x > 3 \end{cases}$ is differentiable at $x = 3$. Also find $f'(3)$.

Solution:

$$L f'(3) = \underset{x \to 3^-}{\text{Lt}} \frac{f(x) - f(3)}{x - 3}$$

$$= \underset{x \to 3^-}{\text{Lt}} \frac{(12x - 13) - (36 - 13)}{x - 3} = \underset{x \to 3^-}{\text{Lt}} \frac{12x - 36}{x - 3} = \underset{x \to 3^-}{\text{Lt}} \frac{12(x - 3)}{x - 3} = 12$$

$$R f'(3) = \underset{x \to 3^+}{\text{Lt}} \frac{f(x) - f(3)}{x - 3} = \underset{x \to 3^+}{\text{Lt}} \frac{(2x^2 + 5) - [2(3)^2 + 5]}{x - 3}$$

$$= \underset{x \to 3^+}{\text{Lt}} \frac{2x^2 - 18}{x - 3} = \underset{x \to 3^+}{\text{Lt}} \frac{2(x - 3)(x + 3)}{(x - 3)} = 12$$

As $L f'(3) = R f'(3)$ $\quad f(x)$ is differentiable at $x = 3$ and $f'(3) = L f'(3) = R f'(3) = 12$

3. If $f(2) = 4$ and $f'(2) = 1$, then find $\underset{x \to 2}{\text{Lt}} \dfrac{x f(2) - 2 f(x)}{x - 2}$

Solution :

$$\underset{x \to 2}{\text{Lt}} \frac{x f(2) - 2 f(x)}{x - 2} = \frac{2 f(2) - 2 f(2)}{2 - 2} = \frac{0}{0} \text{ form}$$

Thus we can apply L Hospital's rule [See chapter on indeterminate forms]

or $\underset{x \to 2}{\text{Lt}} \dfrac{x f(2) - 2 f(x)}{x - 2} = \underset{x \to 2}{\text{Lt}} \dfrac{1 . f(2) - 2 f'(x)}{1} = f(2) - 2 \ f'(2) = 4 - 2(1) = 2$

4. Check the differentiability of $f(x) = x^2|x|$ at $x = 0$

Solution :

$$f(x) = \begin{cases} -x^3 & \text{if } x < 0 \\ x^3 & \text{if } x \geq 0 \end{cases}$$

$$L f'(0) = \underset{x \to 0^-}{\text{Lt}} \frac{f(x) - f(0)}{x - 0} = \underset{x \to 0^-}{\text{Lt}} \frac{-x^3 - (0)^3}{x - 0} = \underset{x \to 0^-}{Lt} \frac{-x^3}{x} = \underset{x \to 0^-}{\text{Lt}} -x^2 = 0$$

$$R f'(0) = \underset{x \to 0^+}{\text{Lt}} \frac{f(x) - f(0)}{x - 0} = \underset{x \to 0^+}{\text{Lt}} \frac{x^3 - (0)^3}{x - 0} = \underset{x \to 0^+}{\text{Lt}} \frac{x^3}{x} = \underset{x \to 0^+}{\text{Lt}} x^2 = 0$$

As $L f'(0) = R f'(0) \Rightarrow f$ is differentiable at $x = 0$

5. For what choice of a and b is the function $f(x) = \begin{cases} x^2 & ; \; x \leq c \\ ax + b & ; \; x > c \end{cases}$ differentiable at $x = c$.

Solution :

We know that a function f differentiable at a point $x = c$ is also continuous at $x = c$

$$\therefore \underset{x \to c^-}{\text{Lt}} f(x) = \underset{x \to c^+}{\text{Lt}} f(x) \quad \text{or} \quad \underset{x \to c}{\text{Lt}} x^2 = \underset{x \to c}{\text{Lt}} (ax + b) \Rightarrow c^2 = ac + b \qquad \text{...(1)}$$

As $f(x)$ is differentiable at $x = c$ $\quad \therefore L f'(c) = R f'(c)$

$$\underset{x \to c^-}{\text{Lt}} \frac{f(x) - f(c)}{x - c} = \underset{x \to c^+}{\text{Lt}} \frac{f(x) - f(c)}{x - c} \; ; \quad \underset{x \to c}{\text{Lt}} \frac{x^2 - c^2}{x - c} = \underset{x \to c}{\text{Lt}} \frac{(ax + b) - c^2}{x - c}$$

$$\Rightarrow \underset{x \to c}{\text{Lt}} (x + c) = \underset{x \to c}{\text{Lt}} \frac{(ax + b) - (ac + b)}{x - c} \qquad \text{[From (1)]}$$

$$\Rightarrow \underset{x \to c}{\text{Lt}} (x + c) = \underset{x \to c}{\text{Lt}} \frac{a(x - c)}{x - c} \Rightarrow 2c = a \qquad \text{...(2)}$$

From (1) and (2); we get $\quad c^2 = 2c^2 + b \Rightarrow b = -c^2 \quad \therefore a = 2c$ and $b = -c^2$.

4·4 [Derivatives Of Standard Functions]

Derivatives Of Some Standard Functions Are Given Below :

(1) $\frac{d}{dx}(x^n) = nx^{n-1}$

(2) $\frac{d}{dx}(e^x) = e^x$

(3) $\frac{d}{dx}(a^x) = a^x \log_e a, \quad a > 0$

(4) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(5) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}, \; a > 0, a \neq 1$

(6) $\frac{d}{dx}(\sin x) = \cos x$

(7) $\frac{d}{dx}(\cos x) = -\sin x$

(8) $\frac{d}{dx}(\tan x) = \sec^2 x$

(9) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(10) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(11) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(12) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}, \; x \in (-1, 1)$

(13) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}, \; x \in (-1, 1)$

(14) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}, \; x \in R$

(15) $\frac{d}{dx}(\cot^{-1}x) = \frac{1}{1+x^2},\ x \in R$

(16) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}},\ x \in R -]-1,1[$

(17) $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}},\ x \in R -]-1,1[$

Remark : 1

Derivative of a constant is always zero. *i.e.* $\frac{d}{dx}(c) = 0$ where c is a constant.

Remark : 2

$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x))$ [k is a constant]

4·5 [Sum and Difference Rule]

If $f(x)$ and $g(x)$ are two functions of x, then $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$.

(This result can also be generalised to 'n' numbers of functions of x)

Examples

6. Differentiate $3+4x-7x^2-\sqrt{2}x^3$.

Solution :

Let $y = 3+4x-7x^2-\sqrt{2}x^3$; $\frac{dy}{dx} = 0+4(1)-7\frac{d}{dx}(x^2)-\sqrt{2}\frac{d}{dx}(x^3)$

$\frac{dy}{dx} = 0+4-7(2x)-\sqrt{2}(3x^2) = 4-14x-3\sqrt{2}x^2$

7. Differentiate $y = x^{\frac{2}{3}}+2x^2$.

Solution : $\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{2}{3}}\right)+2\frac{d}{dx}(x^2) = \frac{2}{3}x^{\frac{2}{3}-1}+2(2x) = \frac{2}{3}x^{-\frac{1}{3}}+4x$

8. Differentiate $(x+\frac{1}{x})^2$

Solution :

Let $y = (x+\frac{1}{x})^2 = x^2+\frac{1}{x^2}+2x.\frac{1}{x} = x^2+\frac{1}{x^2}+2$

$\frac{dy}{dx} = \frac{d}{dx}(x^2)+\frac{d}{dx}\left(\frac{1}{x^2}\right)+\frac{d}{dx}(2) = 2x-2x^{-3}+0 = 2x-2\frac{1}{x^3} = 2x-\frac{2}{x^3}$

4·6 [Product Rule]

If u and v are two functions of x, then $\frac{d}{dx}(u.v) = u.\frac{d}{dx}(v)+v\frac{d}{dx}(u)$

Examples

9. Differentiate : xe^x

Solution :

Let $y = xe^x$ $\frac{dy}{dx} = x\frac{d}{dx}(e^x)+e^x\frac{d}{dx}(x) = xe^x+e^x = e^x(x+1)$

10. Differentiate : $\sin x \log x$

Solution :

Let $y = \sin x \log x$

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin x) = \sin x \frac{1}{x} + \log x(\cos x) = \frac{1}{x}\sin x + \cos x \log x$$

11. Differentiate : $(x+1)(2x+3)^3(5x+7)^2$

Solution :

Let $y = (x+1)(2x+3)^3(5x+7)^2$

$$\frac{dy}{dx} = (x+1)(2x+3)^3\frac{d}{dx}(5x+7)^2 + (2x+3)^3(5x+7)^2\frac{d}{dx}(x+1) + (x+1).(5x+7)^2.\frac{d}{dx}(2x+3)^3$$

$$= (x+1)(2x+3)^3 \cdot 2(5x+7) \cdot 5 + (2x+3)^3(5x+7)^2 \cdot 1 + (x+1)(5x+7)^2 \cdot 3(2x+3)^2 \cdot 2$$

$$= 10(x+1)(2x+3)^3(5x+7) + (2x+3)^3(5x+7)^2 + 6(x+1)(2x+3)^2(5x+7)^2$$

4·7 [Quotient Rule]

If $f(x)$ and $g(x)$ are two functions of x, and $g(x) \neq 0$, then $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$.

Examples

12. Differentiate $\frac{(2+5x)^2}{x^3-1}$.

Solution :

Let $y = \frac{(2+5x)^2}{x^3-1}$

$$\frac{dy}{dx} = \frac{(x^3-1)\frac{d}{dx}(2+5x)^2 - (2+5x)^2\frac{d}{dx}(x^3-1)}{(x^3-1)^2}$$

$$= \frac{(x^3-1).2(2+5x).5 - (2+5x)^2 3x^2}{(x^3-1)^2} = \frac{10(x^3-1)(2+5x) - 3x^2(2+5x)^2}{(x^3-1)^2}$$

$$= \frac{(2+5x)[10(x^3-1) - 3x^2(2+5x)]}{(x^3-1)^2}$$

13. Differentiate $\frac{\sin 3x}{x-6}$.

Solution :

Let $y = \frac{\sin 3x}{x-6}$

$$\frac{dy}{dx} = \frac{(x-6)\frac{d}{dx}(\sin 3x) - \sin 3x\frac{d}{dx}(x-6)}{(x-6)^2} = \frac{(x-6)\cos 3x.3 - \sin 3x(1)}{(x-6)^2} = \frac{3\cos 3x(x-6) - \sin 3x}{(x-6)^2}$$

14. Differentiate $\frac{e^x}{\log x}$

Solution : Let $y = \frac{e^x}{\log x}$; $\frac{dy}{dx} = \frac{\log x\frac{d}{dx}(e^x) - \frac{d}{dx}(\log x)e^x}{(\log x)^2} = \frac{\log x(e^x) - e^x\frac{1}{x}}{(\log x)^2} = \frac{x\log xe^x - e^x}{x(\log x)^2} = \frac{e^x[x\log x - 1]}{x(\log x)^2}$

4·8 [Chain Rule]

If $f(x)$ and $g(x)$ are differentiable functions, then $(fog)'(x) = f'(g(x))\,g'(x)$.

Also if $z = f(y)$ and $y = g(x)$ then $\frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx}$.

Examples

15. Differentiate $\sin(2x+1)$.

Solution :

Let $y = \sin(2x+1)$

$$\frac{dy}{dx} = \cos(2x+1)\frac{d}{dx}(2x+1) = \cos(2x+1)\,.\,2 = 2\cos(2x+1).$$

16. Differentiate $e^{\tan x}$.

Solution :

Let $y = e^{\tan x}$ $\qquad \frac{dy}{dx} = e^{\tan x}\frac{d}{dx}(\tan x) = e^{\tan x}\sec^2 x$

17. Differentiate $\sin(\log\sin x)$.

Solution :

$$y = \sin(\log\sin x)$$

$$y = \cos(\log\sin x)\frac{d}{dx}(\log\sin x) = \cos(\log\sin x)\frac{1}{\sin x}\frac{d}{dx}(\sin x)$$

$$= \cos(\log\sin x)\frac{1}{\sin x}(\cos x) = \cot x\cos(\log\sin x)$$

18. Differentiate $\log\left(x+\sqrt{a^2+x^2}\right)$.

Solution :

Let $y = \log\left(x+\sqrt{a^2+x^2}\right)$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(x+\sqrt{a^2+x^2}\right)\right]$$

$$= \frac{1}{x+\sqrt{x^2+a^2}}\frac{d}{dx}\left[x+\sqrt{x^2+a^2}\right] = \frac{1}{x+\sqrt{x^2+a^2}}\left[1+\frac{1}{2}\left(x^2+a^2\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(x^2+a^2\right)\right]$$

$$= \frac{1}{x+\sqrt{x^2+a^2}}\left[1+\frac{1}{2\sqrt{x^2+a^2}}.(2x)\right] = \frac{1}{x+\sqrt{x^2+a^2}}\left[1+\frac{x}{\sqrt{x^2+a^2}}\right]$$

$$= \frac{1}{x+\sqrt{x^2+a^2}}\left[\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right] = \frac{1}{\sqrt{x^2+a^2}}$$

19. Differentiate $a^{(\sin^{-1}x)^2}$

Solution :

$y = a^{(\sin^{-1}x)^2}$ $\qquad \frac{dy}{dx} = \frac{d}{dx}\left(a^{(\sin^{-1}x)^2}\right) = a^{(\sin^{-1}x)^2}\log a.\frac{d}{dx}\left((\sin^{-1}x)^2\right)$

$$= a^{(\sin^{-1}x)^2} \log a\left(2(\sin^{-1}x)^{2-1}.\frac{d}{dx}(\sin^{-1}x)\right)$$

$$= a^{\sin^{-1}x} \log a \; 2(\sin^{-1}x)\frac{1}{\sqrt{1-x^2}} = \frac{2a^{\sin^{-1}x}\log a \sin^{-1}x}{\sqrt{1-x^2}}$$

20. If $y = \sqrt{\frac{1+e^x}{1-e^x}}$. Show that $\frac{dy}{dx} = \frac{e^x}{(1-e^x)\sqrt{1-e^{2x}}}$

Solution :

$$y = \sqrt{\frac{1+e^x}{1-e^x}} = \left(\frac{1+e^x}{1-e^x}\right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1+e^x}{1-e^x}\right)^{1/2-1}.\frac{d}{dx}\left(\frac{1+e^x}{1-e^x}\right) = \frac{1}{2}\left(\frac{1+e^x}{1-e^x}\right)^{-1/2}.\frac{d}{dx}\left(\frac{1+e^x}{1-e^x}\right)$$

$$= \frac{1}{2}\left(\frac{1+e^x}{1-e^x}\right)^{-1/2}\left[\frac{(1-e^x)\frac{d}{dx}(1+e^x)-\left[(1+e^x)\frac{d}{dx}(1-e^x)\right]}{(1-e^x)^2}\right]$$

$$= \frac{1}{2}\left(\frac{1+e^x}{1-e^x}\right)^{-1/2}\left[\frac{(1-e^x)(e^x)-[(1+e^x)(-e^x)]}{(1-e^x)^2}\right]$$

$$= \frac{1}{2}\sqrt{\frac{1-e^x}{1+e^x}}\left[\frac{e^x-e^{2x}+e^x+e^{2x}}{(1-e^x)^2}\right] = \frac{1}{2}\sqrt{\frac{1-e^x}{1+e^x}}\left[\frac{2e^x}{(1-e^x)^2}\right]$$

$$= \frac{e^x\sqrt{1-e^x}}{\sqrt{1+e^x}.(1-e^x)^2} = \frac{e^x}{\sqrt{1+e^x}.(1-e^x)^{3/2}} = \frac{e^x}{\sqrt{1+e^x}.(1-e^x)\sqrt{1-e^x}}$$

$$= \frac{e^x}{.(1-e^x)\sqrt{1-(e^x)^2}} = \frac{e^x}{.(1-e^x)\sqrt{1-e^{2x}}}$$

21. If $y = \log\left[\sqrt{x-1}-\sqrt{x+1}\right]$ Show that $\frac{dy}{dx} = \frac{-1}{2\sqrt{x^2-1}}$

Solution :

$$y = \log\left[\sqrt{x-1}-\sqrt{x+1}\right]$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left[\sqrt{x-1}-\sqrt{x+1}\right]\right] = \frac{1}{\sqrt{x-1}-\sqrt{x+1}}\frac{d}{dx}\left[\sqrt{x-1}-\sqrt{x+1}\right]$$

$$= \frac{1}{\sqrt{x-1}-\sqrt{x+1}}\frac{d}{dx}\left[(x-1)^{1/2}-(x+1)^{1/2}\right] = \frac{1}{\sqrt{x-1}-\sqrt{x+1}}\left[\frac{1}{2}(x-1)^{-1/2}-\frac{1}{2}(x+1)^{-1/2}\right]$$

$$= \frac{1}{\sqrt{x-1}-\sqrt{x+1}}\left[\frac{1}{2\sqrt{x-1}}-\frac{1}{2\sqrt{x+1}}\right] = \frac{1}{2}\frac{1}{(\sqrt{x-1}-\sqrt{x+1})}\left(\frac{1}{\sqrt{x-1}}-\frac{1}{\sqrt{x+1}}\right)$$

$$= \frac{1}{2}\frac{1}{(\sqrt{x-1}-\sqrt{x+1})}\left(\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x-1}\cdot\sqrt{x+1}}\right)$$

$$= \frac{-1}{2(\sqrt{x-1}-\sqrt{x+1})}\left(\frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x-1}.\sqrt{x+1}}\right) = \frac{-1}{2(\sqrt{x-1}\cdot\sqrt{x+1})} = \frac{-1}{2\sqrt{x^2-1}}$$

22. If $y = \frac{1}{2}\log\left(\frac{1-\cos 2x}{1+\cos 2x}\right)$. Prove that $\frac{dy}{dx} = 2\cos ec 2x$.

Solution :

$$y = \frac{1}{2}\log\left(\frac{1-\cos 2x}{1+\cos 2x}\right)$$

$$y = \frac{1}{2}\log\left(\frac{2\sin^2 x}{2\cos^2 x}\right) = \frac{1}{2}\log\left(\tan^2 x\right) = \frac{1}{2}\log(\tan x)^2 = \frac{2}{2}\log(\tan x) = \log(\tan x)$$

$$\frac{dy}{dx} = \frac{d}{dx}(\log(\tan x)) = \frac{1}{\tan x}\frac{d}{dx}(\tan x) = \frac{1}{\tan x}\left(\sec^2 x\right)$$

$$= \frac{\cos x}{\sin x.\cos^2 x} = \frac{1}{\sin x.\cos x} = \frac{2}{2\sin x.\cos x} = \frac{2}{\sin 2x} = 2\text{cosec } 2x$$

23. Differentiate : $y = \log\left(\frac{a+b\sin x}{a-b\sin x}\right)$.

Solution :

$$y = \log(a+b\sin x) - \log(a-b\sin x)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log(a+b\sin x) - \frac{d}{dx}\log(a-b\sin x)\right)$$

$$= \frac{1}{(a+b\sin x)}\frac{d}{dx}(a+b\sin x) - \left[\frac{1}{(a-b\sin x)}\frac{d}{dx}(a-b\sin x)\right]$$

$$= \frac{1}{(a+b\sin x)}(b\cos x) - \left[\frac{1}{(a-b\sin x)}(-b\cos x)\right]$$

$$= \frac{b\cos x}{a+b\sin x} + \frac{b\cos x}{a-b\sin x} = \frac{b\cos x[(a-b\sin x)+(a+b\sin x)]}{(a+b\sin x)(a-b\sin x)} = \frac{b\cos x(2a)}{a^2-(b\sin x)^2} = \frac{2ab\cos x}{a^2-b^2\sin^2 x}$$

24. If $y = \sqrt{\frac{1-x}{1+x}}$, prove that $(1-x^2)\frac{dy}{dx} + y = 0$.

Solution :

$y = \sqrt{\frac{1-x}{1+x}}$ Differentiating w.r.t. x we get

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{1/2-1}\frac{d}{dx}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2}\frac{\left[(1+x)\frac{d}{dx}(1-x)\right]-\left[(1-x)\frac{d}{dx}(1+x)\right]}{(1+x)^2}$$

$$= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2}\frac{[(1+x)(-1)]-[(1-x)(1)]}{(1+x)^2} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2}\left(\frac{-1-x-1+x}{(1+x)^2}\right)$$

$$= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2}\left(\frac{-2}{(1+x)^2}\right) = -\sqrt{\frac{1+x}{1-x}}\frac{1}{(1+x)^2}$$

Multiply $(1-x^2)$ on both sides we get

$$(1-x^2)\frac{dy}{dx} = -\sqrt{\frac{1+x}{1-x}}\frac{1}{(1+x)^2}(1-x^2)$$

$$(1-x^2)\frac{dy}{dx}=-\sqrt{\frac{1+x}{1-x}}\frac{(1-x)}{(1+x)^2}(1+x) \quad = \quad -\sqrt{\frac{1+x}{1-x}}\frac{(1-x)}{(1+x)}$$

$$=-\sqrt{\frac{(1+x)(1-x)^2}{(1-x)(1+x)^2}} \quad = \quad -\sqrt{\left(\frac{1-x}{1+x}\right)}=-y \qquad (1-x^2)\frac{dy}{dx}=-y\Rightarrow(1-x^2)\frac{dy}{dx}+y=0$$

4·9 [Derivatives Of Inverse Trigonometric Functions]

Following Trigonometric results are used to find derivatives.

(1) $\sin 2x = 2\sin x\cos x$ (2) $1-\cos 2x = 2\sin^2 x$

(3) $1+\cos 2x = 2\cos^2 x$ (4) $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$

(5) $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$ (6) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

(7) $\sin 3x = 3\sin x - 4\sin^3 x$ (8) $\cos 3x = 4\cos^3 x - 3\cos x$

(9) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$.

Remark : For $\sqrt{a^2+x^2}$; Put $x = a\tan\theta$ For $\sqrt{a^2-x^2}$; Put $x = a\sin\theta$

For $\sqrt{x^2-a^2}$; Put $x = a\sec\theta$

25. Differentiate $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}+1}{x}\right]$, $x\neq 0$

Solution :

Let $y=\tan^{-1}\left[\dfrac{\sqrt{1+x^2}+1}{x}\right]$ Put $x=\tan\varphi$ $1+x^2=1+\tan^2\varphi=\sec^2\varphi$; $\sqrt{1+x^2}=\sqrt{\sec^2\varphi}=\sec\varphi$

$\therefore\ y=\tan^{-1}\left(\dfrac{\sec\varphi+1}{\tan\varphi}\right)$; $y=\tan^{-1}\left(\dfrac{\frac{1}{\cos\varphi}+1}{\frac{\sin\varphi}{\cos\varphi}}\right)$; $y=\tan^{-1}\left(\dfrac{1+\cos\varphi}{\sin\varphi}\right)$

$$y=\tan^{-1}\left(\frac{2\cos^2\varphi\ 2}{2\sin\varphi\ 2.\cos\varphi\ 2}\right)=\tan^{-1}(\cot\varphi/2)$$

$$y=\tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{\varphi}{2}\right)\right]=\frac{\pi}{2}-\frac{\varphi}{2}=\frac{\pi}{2}-\frac{1}{2}\tan^{-1}x$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\pi}{2}-\frac{1}{2}\tan^{-1}x\right)=0-\frac{1}{2}\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{-1}{2(1+x^2)}$$

26. Differentiate $\sin^{-1}(3x-4x^3)$, $\dfrac{-1}{2}<x<\dfrac{1}{2}$ wrt. x

Solution :

Let $y=\sin^{-1}(3x-4x^3)$ Put $x=\sin\varphi$; $y=\sin^{-1}(3\sin\varphi-4\sin^3\varphi)$

$\sin^{-1}(\sin 3\varphi)=3\varphi=3\sin^{-1}x$ $\dfrac{dy}{dx}=\dfrac{3}{\sqrt{1-x^2}}$

27. Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$, $-1<x<1,\ x\neq 0$.

Solution :

Let $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$ Put $x^2 = \cos 2\varphi$ $\therefore$ $y = \tan^{-1}\left(\dfrac{\sqrt{1+\cos 2\varphi}+\sqrt{1-\cos 2\varphi}}{\sqrt{1+\cos 2\varphi}-\sqrt{1-\cos 2\varphi}}\right)$

$$y = \tan^{-1}\left(\frac{\sqrt{2\cos^2\varphi}+\sqrt{2\sin^2\varphi}}{\sqrt{2\cos^2\varphi}-\sqrt{2\sin^2\varphi}}\right) = \tan^{-1}\left(\frac{\cos\varphi+\sin\varphi}{\cos\varphi-\sin\varphi}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\varphi}{1-\tan\varphi}\right) \qquad \text{(Dividing by } \cos\varphi\text{)}$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\varphi}{1-\tan\frac{\pi}{4}.\tan\varphi}\right) \; -1<x<1 = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\varphi\right)\right] \; 0<x^2<1 = \frac{\pi}{4}+\varphi$$

Now as $\cos 2\varphi = x^2$ $\qquad 2\varphi = \cos^{-1}(x^2)$ $\qquad \varphi = \frac{1}{2}\cos^{-1}x^2$

$$\therefore \; y = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2 \; ; \; \frac{dy}{dx} = 0+\frac{1}{2}\frac{d}{dx}(\cos^{-1}x^2) = -\frac{1}{2}\frac{1}{\sqrt{1-x^4}}\frac{d}{dx}(x^2) = \frac{-2x}{2\sqrt{1-x^4}} = \frac{-x}{\sqrt{1-x^4}}$$

28. Differentiate $\tan^{-1}\left[\dfrac{\cos x}{1+\sin x}\right], 0<x<\pi$

Solution :

Let $y = \tan^{-1}\left[\dfrac{\cos x}{1+\sin x}\right]$

$$= \tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2}+x\right)}{1-\cos\left(\frac{\pi}{2}+x\right)}\right] = \tan^{-1}\left[\frac{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right] = \tan^{-1}\left[\cot\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)\right] = \frac{\pi}{4}-\frac{x}{2}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}-\frac{x}{2}\right) = 0-\frac{1}{2} = -\frac{1}{2}$$

29. Differentiate $\tan^{-1}\left(\dfrac{1-\cos x}{\sin x}\right)$ wrt x, $-\pi<x<\pi$

Solution :

Let $y = \tan^{-1}\left(\dfrac{1-\cos x}{\sin x}\right)$ $\qquad y = \tan^{-1}\left(\dfrac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}.\cos\frac{x}{2}}\right)$ $\qquad \left(\begin{array}{l} 1-\cos 2x = 2\sin^2 x \\ \therefore 1-\cos x = 2\sin^2\frac{x}{2} \\ \sin 2x = 2\sin x.\cos x \\ \sin x = 2\sin\frac{x}{2}.\cos\frac{x}{2} \end{array}\right)$

$y = \tan^{-1}\left(\tan\dfrac{x}{2}\right) = \dfrac{x}{2}$ $\qquad \left(\begin{array}{c} -\pi<x<\pi \\ -\frac{\pi}{2}<\frac{x}{2}<\frac{\pi}{2} \end{array}\right)$ $\qquad \dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x}{2}\right) = \dfrac{1}{2}$

30. Differentiate $\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $0 < x < 1$, wrt x.

Solution :

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) \qquad \text{Put } x = \tan\varphi$$

$$y = \sin^{-1}\left(\frac{1-\tan^2\varphi}{1+\tan^2\varphi}\right) \; ; \; y = \sin^{-1}(\cos 2\varphi)$$

$$y = \sin^{-1}\left(\sin\left(\frac{\pi}{2} - 2\varphi\right)\right) = \frac{\pi}{2} - 2\varphi = \frac{\pi}{2} - 2\tan^{-1}x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 2\tan^{-1}x\right) = -2\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{-2}{1+x^2}$$

31. Differentiate $\sin^{-1}\left[2x\sqrt{1-x^2}\right]$, $\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$, wrt x.

Solution :

$$\text{Let } y = \sin^{-1}\left[2x\sqrt{1-x^2}\right] \qquad \text{Put } x = \sin\varphi$$

$$\text{then } y = \sin^{-1}\left[2\sin\varphi\sqrt{1-\sin\varphi^2}\right] \; ; \; y = \sin^{-1}[2\sin\varphi.\cos\varphi] \; ; \; y = \sin^{-1}[\sin 2\varphi] = 2\varphi$$

$$y = 2\sin^{-1}x \; ; \; \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

4·10 [Implicit Functions]

If we have function $f(x, y) = 0$ and we can not express y as a functions of x, in the form $y = \varphi(x)$, in that case, y is called an implicit function of x. In such cases, we differentiate both sides of the function taking 'x' constant and derivative of $\varphi(y)$ wrt x, becomes $\frac{d\varphi}{dy}\frac{dy}{dx}$.

Illustrations : $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$; $\frac{d}{dx}(\log y) = \frac{1}{y}\frac{dy}{dx}$; $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$

Examples

32. If $4x + 3y = \log(4x - 3y)$, find $\frac{dy}{dx}$.

Solution :

We have $4x + 3y = \log(4x - 3y)$. Differentiate both sides wrt x, we have

$$4.1 + 3\frac{dy}{dx} = \frac{d}{dx}(\log(4x-3y)) \; ; \; 4 + 3\frac{dy}{dx} = \frac{1}{(4x-3y)}\frac{d}{dx}(4x-3y) \; ; \; 4 + 3\frac{dy}{dx} = \frac{1}{(4x-3y)}\left(4 - 3\frac{dy}{dx}\right)$$

$$4 + 3\frac{dy}{dx} = \frac{4}{4x-3y} - \frac{3}{(4x-3y)}\frac{dy}{dx} \; ; \; \frac{dy}{dx}\left[3 + \frac{3}{4x-3y}\right] = \frac{4}{4x-3y} - 4$$

$$\frac{dy}{dx}\left[\frac{12x-9y+3}{4x-3y}\right] = \frac{4-16x+12y}{4x-3y} \; ; \; \frac{dy}{dx}[12x-9y+3] = -16x+12y+4$$

$$\frac{dy}{dx} = \frac{-16x+12y+4}{12x-9y+3}$$

33. If $\tan(x-y) + \tan(x-y) = 1$, find $\frac{dy}{dx}$.

Solution :

We have $\tan(x-y) + \tan(x-y) = 1$. Differentiate both sides w.r.t x, we have

$$\frac{d}{dx}\left(\tan(x+y)\right) + \frac{d}{dx}\left(\tan(x-y)\right) = 0$$

$$\sec^2(x+y)\frac{d}{dx}(x+y) + \sec^2(x-y)\frac{d}{dx}(x-y) = 0$$

$$\sec^2(x+y)\left[1+\frac{dy}{dx}\right] + \sec^2(x-y)\left[1-\frac{dy}{dx}\right] = 0$$

$$\therefore \frac{dy}{dx}\left[\sec^2(x+y) - \sec^2(x-y)\right] = -\left[\sec^2(x-y) + \sec^2(x+y)\right]\ ; \quad \frac{dy}{dx} = \frac{\sec^2(x-y)+\sec^2(x+y)}{\sec^2(x-y)-\sec^2(x+y)}$$

34. If $x^2 + y^2 = t - \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$. Prove that $\frac{dy}{dx} = \frac{1}{x^3 y}$.

Solution :

We have $x^2 + y^2 = t - \frac{1}{t}$ $\quad \therefore \quad (x^2+y^2)^2 = \left(t - \frac{1}{t}\right)^2$

i.e. $x^4 + y^4 + 2x^2y^2 = t^2 + \frac{1}{t^2} - 2$ *i.e.* $t^2 + \frac{1}{t^2} + 2x^2y^2 = t^2 + \frac{1}{t^2} - 2$ $\left[x^4 + y^4 = t^2 + \frac{1}{t^2}\right]$

$$\Rightarrow 2x^2y^2 = -2 \quad \Rightarrow \quad x^2y^2 = -1 \quad \Rightarrow \quad y^2 = \frac{-1}{x^2} \quad \text{or} \quad y^2 = -x^{-2}$$

Now differentiate both sides w.r.t. x.

$$\frac{d}{dx}(y^2) = -\frac{d}{dx}(x^{-2})\ ; \quad 2y.\frac{dy}{dx} = 2x^{-3}\ ; \quad \frac{dy}{dx} = \frac{2x^{-3}}{2y} = \frac{x^{-3}}{y} = \frac{1}{x^3 y}$$

35. If $e^x + e^y = e^{x+y}$, Prove that $\frac{dy}{dx} = \frac{e^x(e^y-1)}{e^y(1-e^x)}$

Solution :

We have $e^x + e^y = e^{x+y}$. Differentiate both sides w.r.t. x, we get $\frac{d}{dx}(e^x + e^y) = \frac{d}{dx}e^{x+y}$.

$$\Rightarrow \frac{d}{dx}(e^x) + \frac{d}{dx}(e^y) = e^{x+y}\frac{d}{dx}(x+y) \quad \Rightarrow \quad e^x + e^y.\frac{dy}{dx} = e^{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow e^x + e^y\frac{dy}{dx} = e^{x+y} + e^{x+y}\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx}(e^y - e^{x+y}) = e^{x+y} - e^x$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{x+y}-e^x}{e^y - e^{x+y}} = \frac{e^x.e^y - e^x}{e^y - e^x.e^y} = \frac{e^x(e^y-1)}{e^y(1-e^x)}$$

36. If $\sin y = x\sin(a+y)$, Prove that $\Rightarrow \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

Solution :

We have $\sin y = x\sin(a+y)$. Differentiate both sides w.r.t x.

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}[x\sin(a+y)]\ ; \quad \cos y\frac{dy}{dx} = \frac{d}{dx}(x)\sin(a+y) + x\frac{d}{dx}\sin(a+y)$$

$$\cos y\frac{dy}{dx} = \sin(a+y) + x\cos(a+y).\frac{d}{dx}(a+y)$$

$$\frac{dy}{dx}[\cos y - x\cos(a+y)] = \sin(a+y)$$

$$\frac{dy}{dx}\left[\cos y - \frac{\sin y}{\sin(a+y)}\cdot\cos(a+y)\right] = \sin(a+y) \qquad \left[\because x = \frac{\sin y}{\sin(a+y)}\right]$$

$$\frac{dy}{dx}\left[\frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin(a+y)}\right] = \sin(a+y)$$

$$\frac{dy}{dx}\left[\frac{\sin(a+y-y)}{\sin(a+y)}\right] = \sin(a+y) \;;\; \frac{dy}{dx}(\sin a) = \sin^2(a+y) \;;\; \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

4·11 [Logarithmic Differentiation]

When a function is of the form $(\text{variable})^{\text{variable}}$ e.g. x^x, $x^{\sin x}$ etc. or the fraction is very complex; we use this method to find the derivative.

Remark :

If y is a function of x; then

(i) $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$ (ii) $\frac{d}{dx}(\log y) = \frac{1}{y}\frac{dy}{dx}$ (iii) $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$ etc.

Note: (i) $e^{\log x^m} = x^m$, (ii) $\log_e m^n = n\log_e m$, (iii) $\log_e \frac{m}{n} = \log_e m - \log_e n$,

(iv) $\log_e mn = \log_e m + \log_e n$ (v) $\log_e e = 1$

Examples

37. $y = x^x$ w.r.t. x.

Solution :

$y = x^x$ Taking log both the sides, we get $\log y = \log x^x = x\log x$

Differentiate both the sides, w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x\cdot 1 \quad\Rightarrow\quad \frac{dy}{dx} = y[1+\log x] = x^x[1+\log x] \qquad (y = x^x)$$

38. Differentiate $x^{\sin x}$ wrt x.

Solution:

Let $y = x^{\sin x}$. Taking log both sides $\log y = \log(x)^{\sin x} = \sin x\log x$

Differentiating both sides *w.r.t.* x

$$\frac{1}{y}\frac{dy}{dx} = \sin x\cdot\frac{d}{dx}(\log x) + \log x\cdot\frac{d}{dx}(\sin x)$$

$$= \sin x\cdot\frac{1}{x} + \log x\cdot\cos x$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x\log x\right] = x^{\sin x}\left[\frac{\sin x}{x} + \cos x\log x\right]$$

39. Differentiate $(\sin x)^{\cos x}$ w.r.t. x

Solution :

Let $y = (\sin x)^{\cos x}$ $\quad\therefore\quad \log y = \log(\sin x)^{\cos x} = \cos x\cdot\log\sin x$

Differentiating *w.r.t.* x

$$\frac{1}{y}\frac{dy}{dx} = \cos x \cdot \frac{d}{dx}(\log \sin x) + \log \sin x \cdot \frac{d}{dx}(\cos x)$$

$$\frac{1}{y}\frac{dy}{dx} = \left[\cos x \cdot \frac{1}{\sin x}\frac{d}{dx}(\sin x)\right] + \log \sin x(-\sin x)$$

$$= \left[\frac{\cos x}{\sin x} \cdot \cos x\right] + \quad (-\sin x \log \sin x)$$

$$\frac{dy}{dx} = y\left[\cot x \cos x - \sin x \log \sin x\right] = (\sin x)^{\cos x}\left[\cot x \cdot \cos x - \sin x \log \sin x\right]$$

40. Differentiate $x^{(\sin x - \cos x)} + \dfrac{x^2-1}{x^2+1}$ wrt x.

Solution :

Let $y = x^{(\sin x - \cos x)} + \dfrac{x^2-1}{x^2+1}$ $\quad \therefore\ y = u + v \quad$ or $\quad \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ ... (1)

where $u = (x)^{\sin x - \cos x}$ and $v = \dfrac{x^2-1}{x^2+1}$ $\quad$ Consider $\quad u = (x)^{\sin x - \cos x}$

$\Rightarrow \quad \log u = \log (x)^{\sin x - \cos x} = (\sin x - \cos x) \log x$

Differentiating both sides w.r.t. x

$$\frac{1}{u}\frac{du}{dx} = (\sin x - \cos x) \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(\sin x - \cos x)$$

$$= (\sin x - \cos x) \times \frac{1}{x} + (\log x)(\cos x + \sin x)$$

$$\therefore \quad \frac{du}{dx} = u\left[\frac{\sin x - \cos x}{x} + \log x(\cos x + \sin x)\right]$$

$$= x^{\sin x - \cos x}\left[\frac{\sin x - \cos x}{x} + \log x(\cos x + \sin x)\right] \quad \text{... (2)}$$

Now $\quad v = \dfrac{x^2-1}{x^2+1}$

Differentiating w.r.t. x

$$\frac{dv}{dx} = \frac{(x^2+1) \cdot \frac{d}{dx}(x^2-1) - (x^2-1) \times \frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$$

$$= \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2} \quad \text{... (3)}$$

Putting the values from (2) and (3) in (1)

$$\frac{dy}{dx} = x^{(\sin x - \cos x)}\left[\frac{\sin x - \cos x}{x} + \log x(\cos x + \sin x)\right] + \frac{4x}{(x^2+1)^2}$$

41. Differentiate $x^{\cos x} + (\cos x)^x$ wrt x

Solution :

Let $y = u + v$ $\quad \therefore\ \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ ...(1) $\quad$ where $u = x^{\cos x}$ and $v = (\cos x)^x$

Consider $\quad u = x^{\cos x}$

$\Rightarrow \quad \log u = \log (x)^{\cos x} = \cos x \cdot \log x$ Differentiating w.r.t. x

$$\frac{1}{u}\frac{du}{dx} = \cos x \times \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(\cos x) = \cos x \cdot \frac{1}{x} + \log x(-\sin x)$$

$$\therefore \quad \frac{du}{dx} = u\left[\frac{\cos x}{x} - \sin x \log x\right]$$

$$= x^{\cos x}\left[\frac{\cos x}{x} - \sin x \cdot \log x\right] \qquad ...(2)$$

Now $\quad v = (\cos x)^x \Rightarrow \log v = \log (\cos x)^x = x \log (\cos x)$

Differentiating *w.r.t.* x, we get

$$\frac{1}{v}\frac{dv}{dx} = \left[x \cdot \frac{d}{dx}(\log \cos x) + (\log \cos x) \cdot \frac{d}{dx}(x)\right]$$

$$= \left[\left\{x \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x)\right\} + (\log \cos x) \cdot (1)\right]$$

$$= \left[x \cdot \frac{1}{\cos x}(-\sin x) + (\log \cos x)\right]$$

$$\frac{dv}{dx} = v[-x \tan x + \log \cos x]$$

$$= (\cos x)^x [\log \cos x - x \tan x] \qquad ... (3)$$

Putting the values from (2) and (3) in (1)

$$\frac{dy}{dx} = (x)^{\cos x}\left[\frac{\cos x}{x} - \sin x \cdot \log x\right] + (\cos x)^x [\log \cos x - x \tan x]$$

42. If $x^y = y^x$, find $\frac{dy}{dx}$

Solution :

$x^y = y^x$ Taking log both sides we get $\log (x^y) = \log (y^x)$

$y \log x = x \log y$

Differentiating both the side w.r.t. x

$$\frac{d}{dx}(y \log x) = \frac{d}{dx}(x \log y)$$

$$y \cdot \frac{d}{dx}(\log x) + (\log x)\frac{dy}{dx} = x\frac{d}{dx}(\log y) + \log y \cdot 1$$

$$= \frac{y}{x} + \log x \frac{dy}{dx} = x\left(\frac{1}{y}\frac{dy}{dx}\right) + \log y \quad ; \quad \frac{y}{x} + \log x \frac{dy}{dx} = \frac{x}{y}\frac{dy}{dx} + \log y$$

$$\frac{dy}{dx}\left[\log x - \frac{x}{y}\right] = \log y - \frac{y}{x} \quad ; \quad \frac{dy}{dx}\left[\frac{y \log x - x}{y}\right] = \frac{x \log y - y}{x} \quad ; \quad \frac{dy}{dx} = \frac{y}{x}\left[\frac{x \log y - y}{y \log x - x}\right]$$

43. If $x^y \cdot y^x = 1$ prove that $\frac{dy}{dx} = -\frac{y}{x}\left(\frac{y + x \log y}{y \log x + x}\right)$.

Solution :

Let $x^y \cdot y^x = 1$. Taking log both sides we get

$\log (x^y \cdot y^x) = \log 1 = 0$

$\log x^y + \log y^x = 0$ [Using result $\log (ab) = \log a + \log b$]

$y \log x + x \log y = 0$

Now, differentiating both the sides wrt x

$$\frac{d}{dx}[y\log x + x\log y] = 0 \quad ; \quad \frac{d}{dx}(y\log x) + \frac{d}{dx}(x\log y) = 0$$

$$y\frac{d}{dx}(\log x) + (\log x)\frac{dy}{dx} + x\frac{d}{dx}(\log y) + \log y \cdot 1 = 0$$

$$\frac{y}{x} + \log x\frac{dy}{dx} + \frac{x}{y}\frac{dy}{dx} + \log y = 0 \quad ; \quad \frac{dy}{dx}\cdot\left[\log x + \frac{x}{y}\right] = -\frac{y}{x} - \log y$$

$$\frac{dy}{dx}\left[\frac{y\log x + x}{y}\right] = -\left[\frac{y + x\log y}{x}\right] \quad ; \quad \frac{dy}{dx} = -\frac{y}{x}\left[\frac{y + x\log y}{y\log x + x}\right]$$

44. If $x^y + y^x = (x+y)^{x+y}$, find $\frac{dy}{dx}$.

Solution :

Given $x^y + y^x = (x+y)^{x+y}$. Let $u = x^y,\ v = y^x,\ z = (x+y)^{x+y}$ then $u + v = z$

$$\therefore \quad \frac{du}{dx} + \frac{dv}{dx} = \frac{dz}{dx} \qquad \ldots(1) \qquad u = x^y \Rightarrow \log u = \log x^y = y\log x$$

Differentiating w.r.t. x

$$\frac{1}{u}\frac{du}{dx} = y\cdot\frac{d}{dx}(\log x) + \log x\cdot\frac{d}{dx}(y)$$

$$\frac{du}{dx} = u\left[\frac{y}{x} + \log x\cdot\frac{dy}{dx}\right] = x^y\left[\frac{y}{x} + \log x\cdot\frac{dy}{dx}\right]$$

$$= y\cdot x^{y-1} + (x^y\log x)\frac{dy}{dx} \qquad \ldots(2)$$

Now $\quad v = y^x \Rightarrow \log v = \log(y)^x = x\log y$. Differentiating $w.r.t.\ x$

$$\frac{1}{v}\frac{dv}{dx} = x\cdot\frac{d}{dx}(\log y) + \log y\cdot\frac{d}{dx}(x) = \left[x\cdot\frac{1}{y}\frac{dy}{dx} + \log y\right]$$

$$\frac{dv}{dx} = v\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] = y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right]$$

$$= (x\cdot y^{x-1})\frac{dy}{dx} + y^x\log y \qquad \ldots(3)$$

Also $\quad z = (x+y)^{x+y} \Rightarrow \log z = \log(x+y)^{x+y}$

$$\log z = (x+y)\cdot\log(x+y)$$

Differentiating w.r.t. x, we get

$$\frac{1}{z}\cdot\frac{dz}{dx} = (x+y)\cdot\frac{d}{dx}\log(x+y) + \log(x+y)\cdot\frac{d}{dx}(x+y)$$

$$\frac{1}{z}\cdot\frac{dz}{dx} = \left[(x+y)\cdot\frac{1}{x+y}\cdot\frac{d}{dx}(x+y) + \log(x+y)\cdot\left(1 + \frac{dy}{dx}\right)\right]$$

$$\frac{dz}{dx} = z\left[1 + \frac{dy}{dx} + \log(x+y)\cdot\left(1 + \frac{dy}{dx}\right)\right]$$

$$= (x+y)^{x+y}\left[1 + \log(x+y) + \frac{dy}{dx}[1 + \log(x+y)]\right]$$

$$= (x+y)^{x+y}[1 + \log(x+y)] + (x+y)^{x+y}[1 + \log(x+y)]\frac{dy}{dx}$$

Putting the values from (2), (3) and (4) in (1), we get

$$y \cdot x^{y-1} + (x^y \log x)\frac{dy}{dx} + x \cdot y^{x-1}\frac{dy}{dx} + y^x \log y$$

$$= (x+y)^{x+y}[1+\log(x+y)] + (x+y)^{x+y}[1+\log(x+y)]\frac{dy}{dx}$$

$$\Rightarrow (x^y \log x)\frac{dy}{dx} + xy^{x-1}\frac{dy}{dx} - (x+y)^{(x+y)}[1+\log(x+y)]\frac{dy}{dx}$$

$$= (x+y)^{x+y}[1+\log(x+y)] - yx^{y-1} - y^x \log y$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x+y)^{x+y}[1+\log(x+y)] - yx^{y-1} - y^x \log y}{(x^y \log y) + x \cdot y^{x-1} - (x+y)^{x+y}(1+\log(x+y))}$$

45. Differentiate $\dfrac{(x^2-1)(2x-1)}{\sqrt{(x-3)(4x-1)}}$ wrt x.

Solution :

Let $y = \dfrac{(x^2-1)^3(2x-1)}{\sqrt{(x-3)(4x-1)}}$. Taking log both sides we get. $\log y = \log \dfrac{(x^2-1)^3(2x-1)}{\sqrt{(x-3)(4x-1)}}$

$$\log y = \log (x^2-1)^3(2x-1) - \log\left[\sqrt{x-3}\sqrt{(4x-1)}\right] \quad \left[\log\left(\frac{a}{b}\right) = \log a - \log b\right]$$

$$\Rightarrow \log y = \log (x^2-1)^3 + \log(2x-1) - \log\left(\sqrt{x-3}\right) - \log\left(\sqrt{4x-1}\right) \quad [\log(ab) = \log a + \log b]$$

$$\Rightarrow \log y = 3\log(x^2-1) + \log(2x-1) - \frac{1}{2}\log(x-3) - \frac{1}{2}\log(4x-1) \quad (\log x^m = m\log x)$$

Differentiating both sides w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = 3\frac{d}{dx}\log(x^2-1) + \frac{d}{dx}\log(2x-1) - \frac{1}{2}\frac{d}{dx}\log(x-3) - \frac{1}{2}\frac{d}{dx}\log(4x-1)$$

$$\frac{1}{y}\frac{dy}{dx} = 3\frac{1}{(x^2-1)}\frac{d}{dx}(x^2-1) + \frac{1}{(2x-1)}\frac{d}{dx}(2x-1) - \frac{1}{2}\frac{1}{(x-3)}\frac{d}{dx}(x-3) - \frac{1}{2}\frac{1}{(4x-1)}\frac{d}{dx}(4x-1)$$

$$= \frac{3}{(x^2-1)}(2x) + \frac{1}{(2x-1)}(2) - \frac{1\times 1}{2(x-3)} - \frac{4}{2(4x-1)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{6x}{x^2-1} + \frac{2}{2x-1} - \frac{1}{2(x-3)} - \frac{2}{(4x-1)}$$

$$\frac{dy}{dx} = y\left[\frac{6x}{x^2-1} + \frac{2}{2x-1} - \frac{1}{2(x-3)} - \frac{2}{(4x-1)}\right]$$

$$\frac{dy}{dx} = \frac{(x^2-1)^3(2x-1)}{\sqrt{(x-3)(4x-1)}}\left[\frac{6x}{x^2-1} + \frac{2}{2x-1} - \frac{1}{2(x-3)} - \frac{2}{(4x-1)}\right]$$

46. Differentiate $\dfrac{e^{ax}\sec x \log x}{\sqrt{1-2x}}$

Solution :

Let $y = \dfrac{e^{ax}\sec x \log x}{\sqrt{1-2x}}$. Taking log both sides we get $\log y = \log\left(\dfrac{e^{ax}\sec x \log x}{\sqrt{1-2x}}\right)$

$$\log y = \log (e^{ax} \sec x \log x) - \log \left(\sqrt{1-2x}\right)$$

$$\log y = \log (e^{ax}) + \log \sec x + \log (\log x) - \log \left(\sqrt{1-2x}\right)$$

$$\log y = ax \log e + \log \sec x + \log (\log x) - \frac{1}{2} \log (1-2x)$$

Differentiating both sides wrt x

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(ax) + \frac{d}{dx}(\log (\sec x) + \frac{d}{dx}(\log (\log x)) - \frac{1}{2}\frac{d}{dx}(\log (1-2x))$$

$$\frac{1}{y}\frac{dy}{dx} = a \cdot 1 + \frac{1}{(\sec x)}\frac{d}{dx}(\sec x) + \frac{1}{(\log x)}\frac{d}{dx}(\log x) - \frac{1}{2}\frac{1}{(1-2x)}\frac{d}{dx}(1-2x)$$

$$\frac{1}{y}\frac{dy}{dx} = a + \frac{1}{(\sec x)}(\sec x \cdot \tan x) + \frac{1}{\log x}\left(\frac{1}{x}\right) - \frac{1}{2(1-2x)}(-2)$$

$$\frac{1}{y}\frac{dy}{dx} = a + \tan x + \frac{1}{x \log x} + \frac{1}{1-2x}$$

$$\frac{dy}{dx} = y\left[a + \tan x + \frac{1}{x \log x} + \frac{1}{1-2x}\right]$$

$$\frac{dy}{dx} = \frac{e^{ax} \sec x \log x}{\sqrt{1-2x}}\left[a + \tan x + \frac{1}{x \log x} + \frac{1}{1-2x}\right]$$

47. If $y = x^{y^{x \cdots \infty}}$ find $\frac{dy}{dx}$.

Solution :

Deleting single term from an infinte series, series will remain same. $\therefore\ y = x^y$

Now taking log both sides $\log y = \log (x^y) = y \log x$

Differentiating both sides wrt x

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(y \log x) \quad ; \quad \frac{1}{y}\frac{dy}{dx} = y\frac{d}{dx}(\log x) + \log x \frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{y}{x} + \log x \frac{dy}{dx} \quad ; \quad \frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x}$$

$$\frac{dy}{dx}\left(\frac{1 - y \log x}{y}\right) = \frac{y}{x} \quad ; \quad \frac{dy}{dx} = \frac{y^2}{x(1 - y \log x)}$$

4·12 [Parametric Functions]

Let $x = f(t)$ and $y = g(t)$ be two functions of t. (t is a variable)

Then x and y are called parametric functions and t is called parameter and value of $\frac{dy}{dx}$ is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Examples

48. If $x = at^2$ and $y = 2at$, find $\frac{dy}{dx}$.

Solution :

$$x = at^2 \ ; \quad \frac{dx}{dt} = \frac{d}{dt}(at^2) = 2at \quad ; \quad \frac{dy}{dt} = \frac{d}{dt}(2at) = 2a \quad ; \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

49. Find $\frac{dy}{dx}$ if $x = a(\varphi - \sin \varphi)$ and $y = a(1 - \cos \varphi)$.

Solution :

$$\frac{dx}{d\varphi} = a\,(1-\cos\varphi)\ :\ \frac{dy}{d\varphi} = a\sin\varphi$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\varphi}}{\frac{dx}{d\varphi}} = \frac{a\sin\varphi}{a(1-\cos\varphi)} = \frac{2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}}{2\sin^2\frac{\varphi}{2}} = \frac{\cos\frac{\varphi}{2}}{\sin\frac{\varphi}{2}} = \cot\varphi/2$$

50. If $x = \frac{3at}{1+t^2}$ and $y = \frac{3at^2}{1+t^2}$. Find $\frac{dy}{dx}$.

Solution :

$$\frac{dy}{dt} = \frac{(1+t^2)\frac{d}{dt}(3at^2) - 3at^2\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)\,(6at) - 3at^2\,(2t)}{(1+t^2)^2} = \frac{6at + 6at^3 - 6at^3}{(1+t^2)^2} = \frac{6at}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{(1+t^2)\frac{d}{dt}(3at) - 3at\frac{d}{dt}(1+t^2)}{(1+t^2)^2} = \frac{(1+t^2)(3a) - 3at(2t)}{(1+t^2)^2}$$

$$= \frac{3a + 3at^2 - 6at^2}{(1+t^2)^2} = \frac{3a - 3at^2}{(1+t^2)^2} = \frac{3a(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6at}{(1+t^2)^2} \times \frac{(1+t^2)^2}{3a(1-t^2)} = \frac{6at}{3a(1-t^2)} = \frac{2t}{1-t^2}$$

51. Find $\frac{d^2y}{dx^2}$ when $x = at^2, y = 2at$.

Solution :

Clearly $\frac{dy}{dx} = \frac{1}{t}$ **[See example 48]** Now $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{t}\right)$

$$= -\frac{1}{t^2}\cdot\frac{dt}{dx} = -\frac{1}{t^2}\cdot\frac{1}{2at} = \frac{-1}{2at^3}$$

52. If $x = a(\varphi + \sin\varphi)$, $y = a(1+\cos\varphi)$. Prove that $\frac{d^2y}{dx^2} = \frac{-a}{y^2}$.

Solution :

$x = a(\varphi + \sin\varphi)$ $\quad\therefore \frac{dx}{d\varphi} = a(1+\cos\varphi)$; $y = a(1+\cos\varphi)$ $\quad \frac{dy}{d\varphi} = -a\sin\varphi$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\varphi}}{\frac{dx}{d\varphi}} = \frac{-a\sin\varphi}{a(1+\cos\varphi)} = \frac{-\sin\varphi}{1+\cos\varphi}\ ;\ \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-\sin\varphi}{1+\cos\varphi}\right)$$

$$= -\left(\frac{\left[(1+\cos\varphi)\frac{d}{d\varphi}(\sin\varphi)\right] - \left[(\sin\varphi)\frac{d}{\varphi}(1+\cos\varphi)\right]}{(1+\cos\varphi)^2}\right)\cdot\frac{d\varphi}{dx}$$

$$= -\left(\frac{(1+\cos\varphi)\cos\varphi - (\sin\varphi)(-\sin\varphi)}{(1+\cos\varphi)^2}\right)\cdot\frac{1}{a(1+\cos\varphi)}$$

$$= -\left(\frac{\cos\varphi + \cos^2\varphi + \sin^2\varphi}{(1+\cos\varphi)^2}\right)\frac{1}{a(1+\cos\varphi)} = -\left(\frac{\cos\varphi + 1}{(1+\cos\varphi)^2}\,\frac{1}{a(1+\cos\varphi)}\right)$$

$$= -\frac{1}{a(1+\cos\varphi)^2} = \frac{-a}{a^2(1+\cos\varphi)^2} = -\frac{a}{y^2}$$

53. If $x = a(1-\cos^3\varphi)$, $y = a\sin^3\varphi$. Prove that $\frac{d^2y}{dx^2} = \frac{32}{27a}$, at $\varphi = \frac{\pi}{6}$.

Solution :

$$x = a(1-\cos^3\varphi)\ ;\quad \frac{dx}{d\varphi} = -3a\cos^2\varphi(-\sin\varphi) = 3a\cos^2\varphi.\sin\varphi\ ;\quad y = a\sin^3\varphi$$

$$\frac{dy}{d\varphi} = 3a\sin^2\varphi\cos\varphi\ ;\quad \frac{dy}{dx} = \frac{\frac{dy}{d\varphi}}{\frac{dx}{d\varphi}} = \frac{3a\sin^2\varphi\cos\varphi}{3a\cos^2\varphi\sin\varphi} = \frac{a\sin\varphi}{\cos\varphi} = \tan\varphi$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan\varphi) = \sec^2\varphi\cdot\frac{d\varphi}{dx} = \frac{\sec^2\varphi}{3a\cos^2\varphi\sin\varphi} = \frac{1}{3a\cos^4\varphi\sin\varphi} \quad \text{at } \varphi = \frac{\pi}{6}$$

$$\frac{d^2y}{dx^2} = \frac{1}{3a\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right)} = \frac{(2)^5}{3\text{x}9a} = \frac{32}{27a}$$

4.13 [Derivative of one Function with Respect to another Function]

Let $f(x)$ and $g(x)$ be two functions of x. In order to find the derivative of $f(x)$ with respect to $g(x)$, we put $u = f(x)$ and $v = g(x)$. Now, find $\frac{du}{dv} = \frac{(du/dx)}{(dv/dx)}$, which is the required derivative.

Examples

54. Differentiate $\sin^{-1}x$ w.r.t. $\tan^{-1}x$.

Solution :

Let $u = \sin^{-1}x$ and $v = \tan^{-1}x$. Then, $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ and $\frac{dv}{dx} = \frac{1}{(1+x^2)}$.

$$\frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{1}{\sqrt{1-x^2}}\times(1+x^2) = \frac{(1+x^2)}{\sqrt{1-x^2}}$$

55. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\tan^{-1}x$.

Solution :

Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \tan^{-1}x$. Now, $v = \tan^{-1}x \Rightarrow x = \tan v$. Putting $x = \tan v$, we get

$$u = \tan^{-1}\left\{\frac{\sqrt{1+\tan^2 v}-1}{\tan v}\right\} = \tan^{-1}\left(\frac{\sec v - 1}{\tan v}\right) = \tan^{-1}\left(\frac{1-\cos v}{\sin v}\right) = \tan^{-1}\left\{\frac{2\sin^2(v/2)}{2\sin(v/2)\cos(v/2)}\right\}$$

$$\tan^{-1}\left\{\tan\frac{v}{2}\right\} = \frac{v}{2} \qquad u = \frac{v}{2} \Rightarrow \frac{du}{dv} = \frac{1}{2}.$$

56. Differentiate $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ *w.r.t.* $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$.

Solution :

Let $u=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ and $v=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$. Putting $x=\tan\theta$, we get

$$u=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)=2\theta$$

$$v=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\cos^{-1}(\cos 2\theta)=2\theta \qquad \therefore\ u=v\Rightarrow\frac{du}{dv}=1.$$

57. Differentiate $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ *w.r.t.* $\sin^{-1}\left(\dfrac{2x}{1-x^2}\right)$.

Solution :

Let $u=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ and $v=\sin^{-1}\left(\dfrac{2x}{1-x^2}\right)$. Putting $x=\tan\theta$, we get

$$u=\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)=\tan^{-1}(\tan 2\theta)=2\theta,$$

$$v=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)=2\theta. \qquad \therefore\ u=v\Rightarrow\frac{du}{dv}=1.$$

58. Differentiate $\tan^{-1}\left\{\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ w.r.t. $\cos^{-1}x^2$.

Solution :

Let $u=\tan^{-1}\left\{\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ and $v=\cos^{-1}x^2$. Then, $\cos^{-1}x^2=v\Rightarrow x^2=\cos v$.

Putting $x^2=\cos v$, we get $u=\tan^{-1}\left\{\dfrac{\sqrt{1+\cos v}-\sqrt{1-\cos v}}{\sqrt{1+\cos v}+\sqrt{1-\cos v}}\right\}$

$$=\tan^{-1}\left\{\frac{\sqrt{2\cos^2(v/2)}-\sqrt{2\sin^2(v/2)}}{\sqrt{2\cos^2(v/2)}+\sqrt{2\sin^2(v/2)}}\right\}$$

$$=\tan^{-1}\left\{\frac{\cos(v/2)-\sin(v/2)}{\cos(v/2)+\sin(v/2)}\right\}=\tan^{-1}\left\{\frac{1-\tan(v/2)}{1+\tan(v/2)}\right\}$$

[dividing num. and denom. by $\cos(v/2)$].

$$=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{v}{2}\right)\right\}=\left(\frac{\pi}{4}-\frac{v}{2}\right) \qquad \therefore\ u=\left(\frac{\pi}{4}-\frac{v}{2}\right)\Rightarrow\frac{du}{dv}=\frac{-1}{2}$$

Miscellaneous Examples

59. If $y = (\sin x)^{(\sin x)^{(\sin x)^{\cdots\infty}}}$, then prove that $\frac{dy}{dx} = \frac{y^2 \cot x}{(1 - y\log\sin x)}$.

Solution :

We may write the given series as $y = (\sin x)^y$. Taking log both sides, we get $\log y = y \log \sin x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = y\cdot\cot x + (\log\sin x)\frac{dy}{dx} \Rightarrow \frac{dy}{dx}\left(\frac{1}{y} - \log\sin x\right) = y\cot x$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1-y\log\sin x}{y}\right) = y\cot x \Rightarrow \frac{dy}{dx} = \frac{y^2\cot x}{1 - y\log\sin x}$$

60. If $y = x + \cfrac{1}{x + \cfrac{1}{x + \cdots\infty}}$, prove that $\frac{dy}{dx} = \frac{y}{2y - x}$.

Solution :

We may write the given series as $y = x + \frac{1}{y} \Rightarrow y = \frac{xy+1}{y} \Rightarrow y^2 = xy + 1$.

Differentiating both sides w.r.t. x, we get

$$2y\frac{dy}{dx} = x\frac{dy}{dx} + y \Rightarrow \frac{dy}{dx}(2y - x) = y \Rightarrow \frac{dy}{dx} = \frac{y}{2y - x}.$$ Hence proved.

61. If $x = a\sin 2t(1+\cos 2t)$ and $y = b\cos 2t(1-\cos 2t)$, show that $\left(\frac{dy}{dx}\right)_{\text{at } t=\frac{\pi}{4}} = \frac{b}{a}$.

Solution :

We have $x = a\sin 2t(1+\cos 2t) \Rightarrow \frac{dx}{dt} = a\cdot[\sin 2t(-2\sin 2t) + (1+\cos 2t)(2\cos 2t)]$

$$= (2a)\cdot[-\sin^2 2t + \cos 2t + \cos^2 2t] = (2a)[\cos 4t + \cos 2t] \quad \{\because (\cos^2 2t - \sin^2 2t) = \cos 4t\}.$$

And, $y = b\cos 2t\,(1 - \cos 2t)$

$$\Rightarrow \frac{dy}{dt} = b[\cos 2t(2\sin 2t) + (1-\cos 2t)(-2\sin 2t)] = (2b)[\sin 2t\cos 2t - \sin 2t + \sin 2t\cos 2t]$$

$$= (2b)[2\sin 2t\cos 2t - \sin 2t] = (2b)[\sin 4t - \sin 2t]$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{(2b)(\sin 4t - \sin 2t)}{(2a)(\cos 4t + \cos 2t)} \Rightarrow \left(\frac{dy}{dx}\right)_{\left(t=\frac{\pi}{4}\right)} = \frac{b}{a}\cdot\frac{\left\{\sin\left(4\times\frac{\pi}{4}\right) - \sin\left(2\times\frac{\pi}{4}\right)\right\}}{\left\{\cos\left(4\times\frac{\pi}{4}\right) - \cos\left(2\times\frac{\pi}{4}\right)\right\}}$$

$$= \frac{b}{a}\cdot\frac{\left(\sin\pi - \sin\frac{\pi}{2}\right)}{\left(\cos\pi + \cos\frac{\pi}{2}\right)} = \frac{b}{a}\cdot\frac{(0-1)}{(-1+0)} = \frac{b}{a}.$$

62. If $x = a\left(\frac{1+t^2}{1-t^2}\right)$ and $y = \frac{2t}{(1-t^2)}$, find $\frac{dy}{dx}$.

Solution :

We have $x = a\left[-1 + \frac{2}{(1-t^2)}\right] = a[-1 + 2(1-t^2)^{-1}]$

$$\Rightarrow \frac{dx}{dt} = a[0 + 2(-1)(1-t^2)^{-2}(-2t)] = a \times \frac{4t}{(1-t^2)^2} = \frac{4at}{(1-t^2)^2} \quad \text{And, } y = \frac{2t}{(1-t^2)}$$

$$\Rightarrow \frac{dy}{dt} = \frac{(1-t^2)\cdot 2 - 2t(-2t)}{(1-t^2)^2} = \frac{2(1+t^2)}{(1-t^2)^2} \quad \therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \left\{\frac{2(1+t^2)}{(1-t^2)^2} \times \frac{(1-t^2)^2}{4at}\right\} = \frac{(1+t^2)}{2at}$$

63. If $x = 3\sin t - \sin 3t,\ y = 3\cos t - \cos 3t$, find $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{3}$.

Solution :

We have $x = 3\sin t - \sin 3t \Rightarrow \frac{dx}{dt} = 3\cos t - 3\cos 3t$ ... (i)

And, $y = 3\cos t - \cos 3t \Rightarrow \frac{dy}{dt} = -3\sin t + 3\sin 3t$ ... (ii)

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{-3\sin t + 3\sin 3t}{3\cos t - 3\cos 3t} = \frac{\sin 3t - \sin t}{\cos t - \cos 3t} = \frac{2\cos 2t \sin t}{2\sin 2t \sin t} = \cot 2t$$

$$\therefore \frac{d^2y}{dx^2} = -2\operatorname{cosec}^2 2t \cdot \frac{dt}{dx} = \frac{-2\operatorname{cosec}^2 2t}{(dx/dt)} = \frac{-2\operatorname{cosec}^2 2t}{3(\cos t - \cos 3t)}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(t=\frac{\pi}{3}\right)} = \frac{-2\operatorname{cosec}^2(2\pi/3)}{3\left(\cos\frac{\pi}{3} - \cos\pi\right)} = -2 \times \left(\frac{2}{\sqrt{3}}\right)^2 \cdot \frac{1}{3\left(\frac{1}{2}+1\right)} = \left(-2 \times \frac{4}{3} \times \frac{2}{9}\right) = \frac{-16}{27}.$$

64. If $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = \frac{-y}{x}$.

Solution :

We have

$$x^2 = a^{\sin^{-1}t} \text{ and } y^2 = a^{\cos^{-1}t} \Rightarrow 2x\frac{dx}{dt} = a^{\sin^{-1}t} \cdot \frac{1}{\sqrt{1-t^2}} \text{ and } 2y\frac{dy}{dt} = a^{\cos^{-1}t} \cdot \frac{(-1)}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{2y}{2x} \cdot \frac{(dy/dt)}{(dx/dt)} = \frac{-a^{\cos^{-1}t}}{\sqrt{1-t^2}} \times \frac{\sqrt{1-t^2}}{a^{\sin^{-1}t}} \Rightarrow \frac{y}{x} \cdot \frac{dy}{dx} = -\frac{a^{\cos^{-1}t}}{a^{\sin^{-1}t}} = -\frac{y^2}{x^2}$$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$ [on dividing both sides by $\frac{y}{x}$]. Hence, $\frac{dy}{dx} = \frac{-y}{x}$.

65. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

Solution :

We have, $x^y = e^{x-y} \Rightarrow y\log x = (x-y) \Rightarrow (1+\log x)y = x$

$$\Rightarrow y = \frac{x}{(1+\log x)} \quad \text{...}(i)$$

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x)\cdot\frac{d}{dx}(x) - x\cdot\frac{1}{x}(1+\log x)}{(1+\log x)^2}$$

$$= \frac{(1+\log x)\cdot 1 - x\cdot\frac{1}{x}}{(1+\log x)^2} = \frac{(1+\log x - 1)}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

66. If $x^x + x^y + y^x = a^b$, find $\dfrac{dy}{dx}$.

Solution:

Let $u = x^x$, $v = x^y$ and $w = y^x$. Then, $u + v + w = a^b$

$$\Rightarrow \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0 \quad ...(i) \; [\because a^b = \text{constant}] \qquad \text{Now,} \quad u = x^x \Rightarrow \log u = x \log x$$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(x) \quad [\text{on differentiating both sides w.r.t. } x]$$

$$\Rightarrow \frac{du}{dx} = u \cdot \left\{x \cdot \frac{1}{x} + (\log x)\cdot 1\right\} \Rightarrow \frac{du}{dx} = x^x(1 + \log x) \qquad ...(ii)$$

And, $v = x^y \Rightarrow \log v = y \log x$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = y \cdot \frac{d}{dx}(\log x) + (\log x)\cdot\frac{d}{dx}(y) \quad [\text{on differentiating both sides w.r.t. } x]$$

$$\Rightarrow \frac{dv}{dx} = v \cdot \left\{y \cdot \frac{1}{x} + (\log x)\cdot\frac{dy}{dx}\right\} \Rightarrow \frac{dv}{dx} = x^y\left\{\frac{y}{x} + (\log x)\frac{dy}{dx}\right\} \qquad ...(iii)$$

And, $w = y^x \Rightarrow \log w = x \log y$

$$\Rightarrow \frac{1}{w}\cdot\frac{dw}{dx} = x \cdot \frac{d}{dx}(\log y) + (\log y)\cdot\frac{d}{dx}(x) \quad [\text{on differentiating both sides w.r.t. } x]$$

$$\Rightarrow \frac{dw}{dx} = w \cdot \left\{x \cdot \frac{1}{y}\cdot\frac{dy}{dx} + (\log y)\cdot 1\right\} \Rightarrow \frac{dw}{dx} = y^x \cdot \left\{\frac{x}{y}\cdot\frac{dy}{dx} + (\log y)\right\} \qquad ...(iv)$$

Using (ii), (iii) and (iv) in (i), we get

$$x^x(1 + \log x) + x^y\left\{\frac{y}{x} + (\log x)\frac{dy}{dx}\right\} + y^x \cdot \left\{\frac{x}{y}\cdot\frac{dy}{dx} + (\log y)\right\} = 0$$

$$\Rightarrow \{x^x(1 + \log x) + y \cdot x^{(y-1)} + y^x(\log y)\} + \{x^y(\log x) + x\,y^{(x-1)}\}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\{x^x(1 + \log x) + y \cdot x^{(y-1)} + y^x(\log y)\}}{[x^y(\log x) + xy^{(x-1)}]}$$

67. If $y = x^{(x)^x}$, find $\dfrac{dy}{dx}$.

Solution:

Let $x^x = u$. Then, $y = x^u$. $\quad\therefore\; x\log x = \log u$ and $\log y = u \log x$ Now, $\log u = x \log x$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(x) \quad [\text{on differentiating w.r.t. } x]$$

$$\Rightarrow \frac{du}{dx} = u \cdot \left[x \cdot \frac{1}{x} + \log x \cdot 1\right] \Rightarrow \frac{du}{dx} = x^x(1 + \log x) \qquad \text{And, } \log y = u \log x$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = u \cdot \frac{d}{dx}(\log x) + (\log x)\cdot\frac{d}{dx}(u) = u \cdot \frac{1}{x} + (\log x)\cdot\frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \left[\frac{u}{x} + (\log x)\cdot\frac{du}{dx}\right] = x^{(x)^x}\left[\frac{x^x}{x} + (\log x)\{x^x(1 + \log x)\}\right]$$

$$= x^{(x)^x}\,[x^{(x-1)} + x^x(\log x) + x^x(\log x)^2]$$

Exercise

1. Differentiate the following functions w.r.t. x.

(i) $x^3 + 4x^2 + 7x + 2$ [Ans : $3x^2 + 8x + 7$]

(ii) $3 + 4x - 7x^2 - \sqrt{2}x^3 + \pi x^4 - \frac{2}{5}x^5 + \frac{4}{3}$ [Ans : $4 - 14x - 3\sqrt{2}x^2 + 4\pi x^3 - 2x^4$]

(iii) $\frac{1}{x} + \frac{3}{x^2} + \frac{2}{x^3} + \frac{7}{x^{\frac{4}{3}}}$, $x \neq 0$ [Ans : $\frac{-1}{x^2} - \frac{6}{x^3} - \frac{6}{x^4}$]

(iv) $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$, $x \neq 0$ [Ans : $1 - \frac{1}{x^2}$]

(v) $\sqrt[3]{3x^2} - \frac{1}{\sqrt{5x}}$ [Ans : $\frac{2}{\sqrt[3]{9x}} + \frac{1}{2x\sqrt{5x}}$]

2. Differentiate the following functions wrt x.

(i) $(ax + b)(cx + d)$ [Ans : $c(ax + b) + a(cx + d)$]

(ii) $(1 + 2x)(2 + 3x)^2(3 + 4x)^3$ [Ans. $2(2 + 3x)(3 + 4x)^2(72x^2 + 89 + 27)$]

(iii) $\sin x \cdot \log(\sin x)$ [Ans : $\log(\log x)\cos x + \frac{\sin x}{x \log x}$]

(iv) $(x^2 - 4x + 5)(x^3 - 2)$ [Ans : $5x^4 - 16x^3 + 15x^2 - 4x + 8$]

(v) $\frac{1}{ax^2 + bx + c}$ [Ans : $\frac{-(2ax + b)}{(ax^2 + bx + c)^2}$]

(vi) $\frac{x^2 - 3}{x + 4}$ [Ans : $\frac{x^2 + 8x + 3}{(x + 4)^2}$]

(vii) $\frac{e^x + e^{-x}}{e^x - e^{-x}}$ [Ans : $\frac{-4}{(e^x - e^{-x})^2}$]

(viii) $\frac{\sin x + x^2}{\cot 2x}$ [Ans : $2(\sin x + x^2)\sec^2 2x + (\cos x + 2x)\tan 2x$]

(ix) $e^{3x}Cos2x$ [Ans : $e^{3x}(3\cos 2x - 2\sin 2x)$]

(x) $\frac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}$ [Ans : $\frac{-8}{(e^{2x} - e^{-2x})^2}$]

3. Differentiate the following function wrt x.

(i) $x = a\cos\varphi,\ y = b\sin\varphi$ [Ans : $\frac{-b}{a}\cot\varphi$]

(ii) $x = \frac{e^t + e^{-t}}{2},\ y = \frac{e^t - e^{-t}}{2}$ [Ans : $\frac{x}{y}$]

(iii) $x = a(\cos\varphi + \varphi\sin\varphi)$, $y = a(\sin\varphi - \varphi\cos\varphi)$ [Ans : $\tan\varphi$]

(iv) If $x = a\left(t + \frac{1}{t}\right)$, and $y = a\left(t - \frac{1}{t}\right)$, $Prove\ that\ \frac{dy}{dx} = \frac{x}{y}$

(v) If $x = e^{\cos 2t}\ and\ y = e^{\sin 2t}\ Prove\ that\ \frac{dy}{dx} = \frac{-y\log x}{x\log y}$

(vi) If $x = a\sec\varphi,\ y = b\tan\varphi$, prove that $\frac{d^2y}{dx^2} = \frac{-b^4}{a^2y^3}$

(vii) If $x = a(\cos\varphi + \varphi\sin\varphi)$, $y = a(\sin\varphi - \varphi\cos\varphi)$ Prove that $\dfrac{d^2y}{dx^2} = \dfrac{\sec^3\varphi}{a\varphi}$

(viii) If $x = a(1-\cos\varphi)$, $y = a(\varphi + \sin\varphi)$. Prove that $\dfrac{d^2y}{dx^2} = \dfrac{-1}{a}$ at $\varphi = \pi/2$

(ix) Find $\dfrac{d^2y}{dx^2}$ at $\varphi = \dfrac{\pi}{4}$ when $x = a\sec^3\varphi$, $y = a\tan^3\varphi$

(x) If $y = e^x\cos x$, prove that $\dfrac{d^2y}{dx^2} = 2e^x\cos(x + \dfrac{\pi}{2})$

4 Differentiate the following functions wrt x.

(i) $\sin(\log x)$ [Ans : $\dfrac{\cos(\log x)}{x}$]

(ii) $e^{\sin^{-1}2x}$ [Ans : $\dfrac{2}{\sqrt{1-4x^2}}e^{\sin^{-1}2x}$]

(iii) $\tan(e^{\sin x})$ [Ans : $\sec^2(e^{\sin x})e^{\sin x}\cos x$]

(iv) 2^{x^3} [Ans : $3x^2.2^{x^3}\log 2$]

(v) $\cos(\log x)^2$ [Ans : $\dfrac{-2\log x\sin(\log x)^2}{x}$]

(vi) $(\sin^{-1}x^4)^4$ [Ans : $\dfrac{16x^3(\sin^{-1}x^4)^3}{\sqrt{1-x^8}}$]

(vii) If $y = (x-1)\log(x-1) - (x+1)\log(x+1)$. Prove that $\dfrac{dy}{dx} = \log\left(\dfrac{x-1}{x+1}\right)$

(viii) If $y = \sqrt{a^2 - x^2}$, Prove that $y\dfrac{dy}{dx} + x = 0$

(ix) If $y = x\sin^{-1}x + \sqrt{1-x^2}$, Prove that $\dfrac{dy}{dx} = \sin^{-1}x$

(x) If $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, Prove that $\dfrac{dy}{dx} = 1 - y^2$.

5. Differentiate the following functions wrt x.

(i) x^x [Ans : $\dfrac{dy}{dx} = x^x(1 + \log x)$]

(ii) $x^{\sqrt{x}}$ [Ans : $\dfrac{dy}{dx} = x^{\sqrt{x}}\left(\dfrac{\log x}{2\sqrt{x}} + \dfrac{1}{\sqrt{x}}\right)$]

(iii) $x^{\cos^{-1}x}$ [Ans : $x^{\cos^{-1}x}\left[\dfrac{-\log x}{\sqrt{1-x^2}} + \dfrac{\cos^{-1}x}{x}\right]$]

(iv) $\cos(x^x)$ [Ans : $-x^x\sin(x^x)\log(x+1)$]

(v) If $x^y = e^{x-y}$, Prove that $\dfrac{dy}{dx} = \dfrac{\log x}{(1+\log x)^2}$

(vi) If $y = \sin(x^x)$, Prove that $\dfrac{dy}{dx} = \cos(x^x).x^x(1 + \log x)$

(vii) If $(\cos x)^y = (\tan y)^x$. Prove that $\dfrac{dy}{dx} = \dfrac{\log\tan y + y\tan x}{\log \ldots}$

(viii) If $xy\log(x+y)=1$, Prove that $\frac{dy}{dx}=\frac{-y(x^2y+x+y)}{x(xy^2+x+y)}$

(ix) If $e^y=y^x$, Prove that $\frac{dy}{dx}=\frac{(\log y)^2}{\log y-1}$

(x) If $xy\log(x+y)=1$, Prove that $\frac{dy}{dx}=\frac{-y(x^2y+x+y)}{x(xy^2+x+y)}$

6. Differentiate the following wrt x.

(i) $\tan^{-1}\left(\frac{2x}{1-x^2}\right),\ x\in(-1,1)$ [Ans : $\frac{dy}{dx}=\frac{2}{1+x^2}$]

(ii) $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)\quad \frac{-1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$ [Ans : $\frac{dy}{dx}=\frac{3}{1+x^2}$]

(iii) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\quad 0<x<1$ [Ans : $\frac{dy}{dx}=\frac{2}{1+x^2}$]

(iv) $\sec^{-1}\left(\frac{1}{2x^2-1}\right)\quad 0<x<\frac{1}{\sqrt{2}}$ [Ans : $\frac{dy}{dx}=\frac{-2}{\sqrt{1-x^2}}$]

(v) $\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right),\ \frac{-\pi}{2}<x<\frac{\pi}{2}$ [Ans : $\frac{dy}{dx}=\frac{1}{2}$]

(vi) $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right),\ x\neq 0$ [Ans : $\frac{dy}{dx}=\frac{1}{2}\left(\frac{1}{1+x^2}\right)$]

(vii) If $y=\sin^{-1}\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right)$ and $0<x<1$, then find $\frac{dy}{dx}$ [Ans : $\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}$]

(viii) Differentiate $\tan^{-1}\left(\frac{a+x}{1-ax}\right)$ wrt x [Ans : $\frac{1}{1+x^2}$]

(ix) If $\cos^{-1}(2x)+2\cos^{-1}\sqrt{1-4x^2},\ 0<x<\frac{1}{2}$ Find $\frac{dy}{dx}$. [Ans : $\frac{2}{\sqrt{1-4x^2}}$]

(x) Differentiate $\cot^{-1}\left(\frac{1-x}{1+x}\right)$, wrt x. [Ans : $\frac{1}{1+x^2}$]

7. If $e^x+e^y=e^{x+y}$, prove that $\frac{dy}{dx}=\frac{e^x\ (e^y-1)}{e^y\ (e^x-1)}$

8. If $\sin(xy)+\frac{y}{x}=x^2-y^2$, find $\frac{dy}{dx}$ [Ans. $\frac{2x^3+y-x^2y\cos(xy)}{x\left\{x^2\cos xy+1+2xy\right\}}$]

9. If $\tan^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right)=a$, prove that $\frac{dy}{dx}=\frac{x}{y}\ \frac{(1-\tan a)}{(1+\tan a)}$

10. If $y=x\sin(a+y)$, prove that $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin(a+y)-y\cos(a+y)}$

Chapter - 5

[Indeterminate Forms]

5.1 [Introduction]

Let $f(x)$ and $g(x)$ be two functions of x: then we know that $\underset{x\to a}{\text{Lt}} \dfrac{f(x)}{g(x)} = \dfrac{\underset{x\to a}{\text{Lt}}\, f(x)}{\underset{x\to a}{\text{Lt}}\, g(x)}$ ...(1)

In case both numerator and denominator tend to zero as $x \to a$, then the form $\dfrac{0}{0}$ is called indeterminate form. It has no definite value. The other indeterminate forms are $\dfrac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$, 1^{∞}, 0^0 and ∞^0.

Remark:

If $\underset{x\to a}{\text{Lt}} \dfrac{f(x)}{g(x)}$ takes the form $\dfrac{0}{0}$ then it does not mean than $\underset{x\to a}{\text{Lt}} \dfrac{f(x)}{g(x)}$ will not exist.

For example $\underset{x\to 1}{\text{Lt}} \dfrac{x^2-1}{x-1}$ is of form $\dfrac{0}{0}$ but $\underset{x\to 1}{\text{Lt}} \dfrac{x^2-1}{x-1} = \underset{x\to 1}{\text{Lt}} \dfrac{(x-1)(x+1)}{x-1} = 2$

showing that the limit exists in this case.

5.2 [Evaluation of $\frac{0}{0}$ form (L' Hospital's Rule)]

Statement:

If $f(x)$ and $g(x)$ be two functions of x which can be expanded by Taylor's theorem in the neighbourhood of $x = a$ if $f(a) = g(a) = 0$; then $\underset{x\to a}{\text{Lt}} \dfrac{f(x)}{g(x)} = \underset{x\to a}{\text{Lt}} \dfrac{f'(x)}{g'(x)}$ provided, the latter limit exists, finite or infinite.

Remark:

In order to evaluate $\underset{x\to a}{\text{Lt}} \dfrac{f(x)}{g(x)}$ which is of form $\dfrac{0}{0}$; the differentiation of numerator and denominator should be done separately and the process is to be continued till the form $\left(\dfrac{0}{0}\right)$ persists.

5.3. [Some Standard Expansions]

(i) $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots$

(ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots, |x| < 1$

(iii) $a^x = 1 + x\log a + \dfrac{x^2}{2!}(\log a)^2 + \dfrac{x^3}{3!}(\log a)^3 + \ldots$

(iv) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$

(v) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots$

(vi) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots$

(vii) $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \ldots$

(viii) $e = 1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \ldots$

(ix) $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$

(x) $\log(1-x) = -\left(x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \ldots\right)$

(xi) $\sin^{-1} x = x + \dfrac{x^3}{6} + \dfrac{3x^5}{40} + \ldots$

(xii) $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \ldots$

Following results on limits should also be remembered

(i) $\underset{x\to 0}{\text{Lt}} \dfrac{\sin x}{x} = 1$

(ii) $\underset{x\to 0}{\text{Lt}} \dfrac{\tan x}{x} = 1$

(iii) $\underset{x\to 0}{\text{Lt}} \dfrac{\tan^{-1} x}{x} = 1$

(iv) $\underset{x\to 0}{\text{Lt}} \dfrac{\sin^{-1} x}{x} = 1$

(v) $\underset{x\to 0}{\text{Lt}} (1+x)^{1/x} = e$

(vi) $\underset{x\to \infty}{\text{Lt}} \left(1 + \dfrac{1}{x}\right)^x = e$

(vii) $\underset{x\to \infty}{\text{Lt}} \left(1 + \dfrac{a}{x}\right)^x = e^a$

(viii) $\underset{x\to 0}{\text{Lt}} \dfrac{a^x - 1}{x} = \log_e a$

(ix) $\underset{x\to 0}{\text{Lt}} \dfrac{e^x - 1}{x} = 1$

(x) $\underset{x\to 0}{\text{Lt}} \log \dfrac{(1+x)}{x} = 1$

Following values also help in evaluating limits

$$\log 1 = 0 \;;\; \log_e e = 1 \;;\; \log 0 = -\infty,\ \log \infty = \infty$$

Examples

1. Evaluate $\underset{x\to 0}{\text{Lt}} \dfrac{1-\cos x}{x^2}$.

Solution.

As $\underset{x\to 0}{\text{Lt}} \dfrac{1-\cos x}{x^2}$ is of the form $\left(\dfrac{0}{0}\right)$; thus applying L-Hospital's rule

$$\underset{x\to 0}{\text{Lt}} \frac{1-\cos x}{x^2} = \underset{x\to 0}{\text{Lt}} \frac{\sin x}{2x} = \frac{1}{2}\underset{x\to 0}{\text{Lt}} \frac{\sin x}{x} = \frac{1}{2}.1 = \frac{1}{2}$$

2. Evaluate $\underset{x\to 0}{\text{Lt}} \dfrac{x-\sin x}{x^3}$.

Solution.

$\underset{x\to 0}{\text{Lt}} \dfrac{x-\sin x}{x^3}$ [$\dfrac{0}{0}$ form, so we will apply L' Hospital's rule]

$$= \underset{x\to 0}{\text{Lt}} \frac{1-\cos x}{3x^2} \quad \left[\text{Again } \frac{0}{0} \text{ form}\right] = \underset{x\to 0}{\text{Lt}} \frac{\sin x}{6x} = \frac{1}{6}\underset{x\to 0}{\text{Lt}} \frac{\sin x}{x} = \frac{1}{6} \times 1 = \frac{1}{6}$$

3. Evaluate $\underset{x\to 0}{\text{Lt}} \dfrac{1-\cos x}{x\log(1+x)}$.

Solution.

$\underset{x\to 0}{\text{Lt}} \left[\dfrac{1-\cos x}{x\log(1+x)}\right]$ is $\dfrac{0}{0}$ form; thus we can apply L' Hospital's Rule

$$= \underset{x\to 0}{\text{Lt}} \frac{\sin x}{\log(1+x) + \frac{x}{1+x}} \quad \left[\frac{0}{0} \text{ Form}\right] = \underset{x\to 0}{\text{Lt}} \frac{\cos x}{\frac{1}{1+x} + \frac{1+x-x}{(1+x)^2}} = \underset{x\to 0}{\text{Lt}} \frac{\cos x}{\frac{1}{1+x} + \frac{1}{(1+x)^2}}$$

Now this fraction is not $\frac{0}{0}$ form; therefore we evaluate it by direct substitution

$$\underset{x\to 0}{\mathrm{Lt}} \frac{\cos x}{\frac{1}{1+x}+\frac{1}{(1+x)^2}} = \frac{1}{1+1}=\frac{1}{2}$$

4. Evaluate $\underset{x\to 0}{\mathrm{Lt}} \frac{\log(1+x^2)}{1-\cos x}$.

Solution.

$$\underset{x\to 0}{\mathrm{Lt}} \frac{\log(1+x^2)}{1-\cos x}\left[\frac{0}{0}\text{ Form}\right] = \underset{x\to 0}{\mathrm{Lt}} \frac{\frac{1}{1+x^2}\cdot 2x}{\sin x} = \underset{x\to 0}{\mathrm{Lt}} \frac{2}{1+x^2}\cdot\frac{x}{\sin x}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{2}{1+x^2}\cdot \underset{x\to 0}{\mathrm{Lt}} \frac{1}{\frac{\sin x}{x}} = 2\times 1 = 2$$

5. Evaluate $\underset{x\to b}{\mathrm{Lt}} \frac{x^b-b^x}{x^x-b^b}$.

Solution.

$\underset{x\to b}{\mathrm{Lt}} \frac{x^b-b^x}{x^x-b^b}$ is of the form $\frac{0}{0}$; thus to differentiate numerator and denominator; we require derivative of x^x.

i.e., $\frac{d}{dx}(x^x)$ which is equal to $x^x(1+\log x)$ [see logarithmic differentiation]

$$\therefore \underset{x\to b}{\mathrm{Lt}} \frac{x^b-b^x}{x^x-b^b} = \underset{x\to b}{\mathrm{Lt}} \frac{bx^{b-1}-b^x\log b}{x^x(1+\log x)-0} = \frac{b\cdot b^{b-1}-b^b\log b}{b^b(1+\log b)} = \frac{b^b(1-\log b)}{b^b(1+\log b)} = \frac{1-\log b}{1+\log b}$$

6. Evaluate $\underset{x\to 0}{\mathrm{Lt}} \frac{\log(1-x^2)}{\log\cos x}$.

Solution.

$$\underset{x\to 0}{\mathrm{Lt}} \frac{\log(1-x^2)}{\log\cos x}\left[\frac{0}{0}\text{ Form}\right] = \underset{x\to 0}{\mathrm{Lt}} \frac{\frac{-2x}{1-x^2}}{-\frac{\sin x}{\cos x}} = \underset{x\to 0}{\mathrm{Lt}} \frac{2x}{(1-x^2)\tan x}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{2}{1-x^2}\times \underset{x\to 0}{\mathrm{Lt}} \frac{1}{\frac{\tan x}{x}} \text{ [Note this step]} = 2\times 1 = 2$$

7. Evaluate $\underset{x\to 0}{\mathrm{Lt}} \frac{e^x-e^{\sin x}}{x-\sin x}\left[\frac{0}{0}\text{ Form}\right]$.

Solution.

Given limit is $\frac{0}{0}$ form; thus, we apply L' Hospital's rule

$$\text{Required limit} = \underset{x\to 0}{\mathrm{Lt}} \frac{e^x-e^{\sin x}\cdot\cos x}{1-\cos x}\left[\frac{0}{0}\text{ Form}\right] = \underset{x\to 0}{\mathrm{Lt}} \frac{e^x+e^{\sin x}\sin x-\cos^2 x\, e^{\sin x}}{\sin x}\left[\frac{0}{0}\text{ Form}\right]$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{e^x+e^{\sin x}\cos x+\sin x\cos x\cdot e^{\sin x}-\cos^2 x\cdot e^{\sin x}\cdot\cos x-e^{\sin x}(-\sin 2x)}{\cos x}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{e^x+e^{\sin x}\cos x+\sin x\cos x\, e^{\sin x}-\cos^3 x\, e^{\sin x}+e^{\sin x}\cdot\sin 2x}{\cos x} = \frac{1+1+0-1+0}{1}=1$$

8. Evaluate $\underset{x\to 0}{\mathrm{Lt}} \frac{e^x-e^{-x}-2\log(1+x)}{x\sin x}$.

Solution.

The given limit is of form $\frac{0}{0}$; therefore we apply L' Hospital's rule

$$\text{Required limit} = \underset{x\to 0}{\mathrm{Lt}} \frac{e^x+e^{-x}-\frac{2}{1+x}}{x\cos x+\sin x}\left[\frac{0}{0}\text{ Form}\right] = \underset{x\to 0}{\mathrm{Lt}} \frac{e^x-e^{-x}+\frac{2}{(1+x)^2}}{-x\sin x+\cos x+\cos x} = \frac{1-1+2}{0+1+1}=\frac{2}{2}=1$$

9. Evaluate $\underset{x\to 0}{\mathrm{Lt}} \frac{e^x-1-\log(1+x)}{x^2}$.

Solutions.

As the given limit is of the form $\frac{0}{0}$, so we apply L' Hospital's rule

$$\text{Required limit} = \underset{x\to 0}{\mathrm{Lt}} \frac{e^x-\frac{1}{1+x}}{2x}\left[\frac{0}{0}\text{ Form}\right] = \underset{x\to 0}{\mathrm{Lt}} \frac{e^x+\frac{1}{(1+x)^2}}{2} = \frac{1+1}{2}=1$$

10. Evaluate $\underset{x\to 0}{\mathrm{Lt}} \frac{a^x-b^x}{x}$.

Solution.

Given limit is of $\frac{0}{0}$ form; thus we can apply L' Hospital's rule

$$\text{Required limit} = \underset{x\to 0}{\mathrm{Lt}} \frac{a^x.\log_e a-b^x\log_e b}{1} = \log_e a-\log_e b = \frac{\log_e a}{\log_e b}$$

11. Evaluate $\underset{x\to 0}{\mathrm{Lt}} \frac{\tan x-x}{x^2\tan x}$.

Solutions.

The given limit canbe written as

$$\underset{x\to 0}{\mathrm{Lt}} \frac{\tan x-x}{x^3\cdot\frac{\tan x}{x}} \quad \text{[Note this step]} \quad = \underset{x\to 0}{\mathrm{Lt}} \frac{\tan x-x}{x^3}\left[\because \underset{x\to 0}{\mathrm{Lt}} \frac{\tan x}{x}=1\right]$$

Now this limit is of form $\frac{0}{0}$; so we apply L' Hospital's rule

$$\text{Required limit} = \underset{x\to 0}{\mathrm{Lt}} \frac{\sec^2 x-1}{3x^2}\left[\frac{0}{0}\right]$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{2\sec^2 x\tan x}{6x} = \left[\frac{0}{0}\right] = \underset{x\to 0}{\mathrm{Lt}} \frac{2\sec^4 x+2\tan x\cdot(2\sec^2 x\tan x)}{6}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{2\sec^4 x+4\sec^2 x\tan^2 x}{6} = \frac{2}{6}=\frac{1}{3}$$

12. Evaluate $\underset{x\to 0}{\mathrm{Lt}} \frac{a^x-1-x\log a}{3x^2}$.

Solution.

$$\text{Required limit} = \underset{x\to 0}{\mathrm{Lt}} \frac{a^x - 1 - x\log a}{3x^2} \quad \left[\frac{0}{0}\text{ Form}\right]$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{a^x \log a - \log a}{6x} \left[\frac{0}{0}\text{ Form}\right] = \underset{x\to 0}{\mathrm{Lt}} \frac{a^x (\log a)^2}{6} = \frac{1}{6}(\log a)^2$$

13. Evaluate $\underset{x\to 1}{\mathrm{Lt}} \dfrac{x^x - x}{1 - x + \log x}$.

Solutions:

$$\underset{x\to 1}{\mathrm{Lt}} \frac{x^x - x}{1 - x + \log x} \left[\frac{0}{0}\text{ Form}\right] = \underset{x\to 1}{\mathrm{Lt}} \frac{x^x(1+\log x) - 1}{-1 + \left(\frac{1}{x}\right)} \quad \left[\because \frac{d}{dx}(x^x) = x^x(1+\log x)\right]$$

$$= \underset{x\to 1}{\mathrm{Lt}} \frac{x^x(1/x) + x^x(1+\log x)\cdot(1+\log x)}{(-1/x^2)} = \frac{1 + 1(1+0)(1+0)}{-1} = -2$$

14. Evaluate $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\log(1+x^3)}{\sin x^3}$.

Solutions.

The given limit can be written as

$$\underset{x\to 0}{\mathrm{Lt}} \frac{\log(1+x^3)}{x^3} \times \frac{1}{\underset{x\to 0}{\mathrm{Lt}} \frac{\sin x^3}{x^3}} \quad [\text{Note this step}] = \underset{x\to 0}{\mathrm{Lt}} \frac{\log(1+x^3)}{x^3} \times 1 \quad \left[\because \underset{x\to 0}{\mathrm{Lt}} \frac{\sin x}{x} = 1\right]$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{\log(1+x^3)}{x^3} \quad \left[\frac{0}{0}\text{ form; so we apply L' Hospital's rule}\right]$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{\frac{(3x^2)}{(1+x^3)}}{3x^2} = \underset{x\to 0}{\mathrm{Lt}} \frac{1}{1+x^3} = 1$$

Exercise

Evaluate the following limits

1. $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\log(1+kx^2)}{1-\cos x}$ [Ans. $2k$]
2. $\underset{x\to 0}{\mathrm{Lt}} \dfrac{e^x - e^x\cos x}{x - \sin x}$ [Ans. 3]
3. $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\sqrt{x}\tan x}{(e^x-1)^{3/2}}$ [Ans. 1]
4. $\underset{x\to 0}{\mathrm{Lt}} \dfrac{e^x + e^{-x} + 2\cos x - 4}{x^4}$ [Ans. 1/6]
5. $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\log(1-x^2)}{\log\cos x}$ [Ans. 2]
6. $\underset{x\to 0}{\mathrm{Lt}} \dfrac{x\cos x - \log(1+x)}{x^2}$ [Ans. $\frac{1}{2}$]
7. $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$ [Ans. 4]
8. $\underset{x\to 1/2}{\mathrm{Lt}} \dfrac{\cos^2(\pi x)}{e^{2x} - 2ex}$ [Ans. $\frac{\pi^2}{2e}$]
9. $\underset{x\to 0}{\mathrm{Lt}} \dfrac{e^x\sin x - x - x^2}{x^2 + x\log(1-x)}$ [Ans. $-\frac{2}{3}$]
10. $\underset{x\to 0}{\mathrm{Lt}} \dfrac{xe^x - \log(1+x)}{x^2}$ [Ans. $\frac{3}{2}$]

5.4. [Evaluation of the form $\frac{\infty}{\infty}$]

If $\underset{x\to a}{\mathrm{Lt}} \dfrac{f(x)}{g(x)}$ is of the form $\frac{\infty}{\infty}$; then in this case also, we can apply L' Hospital's rule to find the required limit

$$\therefore \underset{x\to a}{\mathrm{Lt}} \frac{f(x)}{g(x)} = \underset{x\to a}{\mathrm{Lt}} \frac{f'(x)}{g'(x)}\text{; if the limit is of form } \frac{\infty}{\infty}$$

Examples on Form $\frac{\infty}{\infty}$

15. Evaluate $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\log\sin 3x}{\log\sin x}$.

Solution.

$$\text{We have } \underset{x\to 0}{\mathrm{Lt}} \frac{\log\sin 3x}{\log\sin x}\left[\text{Form } \frac{\infty}{\infty}\right] = \underset{x\to 0}{\mathrm{Lt}} \frac{\left(\frac{1}{\sin 3x}\right)\times(3\cos 3x)}{\left(\frac{1}{\sin x}\right)\cdot\cos x} = \underset{x\to 0}{\mathrm{Lt}} \frac{3\cot 3x}{\cot x}\left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{3\tan x}{\tan 3x}\left[\text{Form } \frac{0}{0}\right] = \underset{x\to 0}{\mathrm{Lt}} \frac{3\sec^2 x}{3\sec^2 3x} = \frac{1\times(1)^2}{(1)^2} = 1$$

16. Evaluate $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\log(\tan^2 3x)}{\log(\tan^2 x)}$.

Solution.

$$\text{Given limit } \underset{x\to 0}{\mathrm{Lt}} \frac{\log(\tan^2 3x)}{\log(\tan^2 x)} = \underset{x\to 0}{\mathrm{Lt}} \frac{2\log\tan 3x}{2\log\tan x}\left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{\left(\frac{1}{\tan 3x}\right)\cdot 3\sec^2 3x}{\left(\frac{1}{\tan x}\right)\cdot\sec^2 x} = \underset{x\to 0}{\mathrm{Lt}} \frac{3\tan x\cos^2 x}{\tan 3x\cdot\cos^2 3x}\left[\frac{0}{0}\right] = \underset{x\to 0}{\mathrm{Lt}} \frac{3\sin x\cos x}{\sin 3x\cos 3x}$$

$$= \frac{3}{2}\underset{x\to 0}{\mathrm{Lt}} \frac{\sin 2x}{\sin 3x}\times\underset{x\to 0}{\mathrm{Lt}} \frac{1}{\cos 3x} = \frac{3}{2}\left[\frac{\underset{x\to 0}{\mathrm{Lt}} \frac{\sin 2x}{2x}}{\underset{x\to 0}{\mathrm{Lt}} \frac{\sin 3x}{3x}}\times\frac{2x}{3x}\right]\times 1 \left[\because \underset{x\to 0}{\mathrm{Lt}} \cos 3x = 1\right] = \frac{3}{2}\times\frac{2}{3} = 1$$

17. Evaluate $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\log x^2}{\cot x^2}$

Solutions.

$$\underset{x\to 0}{\mathrm{Lt}} \frac{\log x^2}{\cot x^2}\left[\text{Form } \frac{\infty}{\infty}\right] = \underset{x\to 0}{\mathrm{Lt}} \frac{\frac{1}{x^2}\cdot 2x}{-2x\,\mathrm{cosec}^2 x^2} = -\underset{x\to 0}{\mathrm{Lt}} \frac{\sin x^2}{x^2} = -1 \left[\underset{y\to 0}{\mathrm{Lt}} \frac{\sin y}{y} = 1\right]$$

18. Evaluate $\underset{x\to\infty}{\mathrm{Lt}} \dfrac{x^n}{e^x}$**; where n is a natural number.**

Solution.

$$\underset{x\to\infty}{\mathrm{Lt}} \frac{x^n}{e^x}\left[\frac{\infty}{\infty}\text{ Form}\right] = \underset{x\to\infty}{\mathrm{Lt}} \frac{n\,x^{n-1}}{e^x}\left[\frac{\infty}{\infty}\text{ Form}\right] = \underset{x\to\infty}{\mathrm{Lt}} \frac{n(n-1)\,x^{n-2}}{e^x}\left[\frac{\infty}{\infty}\text{ Form}\right]$$

$$\text{After } n \text{ steps} = \underset{x\to\infty}{\mathrm{Lt}} \frac{n(n-1)(n-2)\ldots 3.2.1}{e^x} = \underset{x\to\infty}{\mathrm{Lt}} \frac{n!}{e^x} = n!\times\underset{x\to\infty}{\mathrm{Lt}} \frac{1}{e^x} = 0 \left[\because \underset{x\to\infty}{\mathrm{Lt}} \frac{1}{e^x} = 0\right]$$

19. Evaluate $\underset{x\to 0}{\text{Lt}} \log_{\tan^2 x} \tan^2 2x$.

Solution.

We know that $\log_a m = \dfrac{\log_e m}{\log_e a}$ $\quad\therefore\quad \log_{\tan^2 x} \tan^2 2x = \dfrac{\log_e \tan^2 2x}{\log_e \tan^2 x}$

Thus required limit $= \underset{x\to 0}{\text{Lt}} \dfrac{\log \tan^2 2x}{\log \tan^2 x} = \underset{x\to 0}{\text{Lt}} \dfrac{2\log \tan 2x}{2\log \tan x} \left[\dfrac{\infty}{\infty}\text{ Form}\right]$

$$= \underset{x\to 0}{\text{Lt}} \frac{\left(\frac{1}{\tan 2x}\right)\cdot 2\sec^2 2x}{\left(\frac{1}{\tan x}\right)\cdot \sec^2 x} = \underset{x\to 0}{\text{Lt}} \frac{2\tan x\cos^2 x}{\tan 2x\cos^2 2x} = \underset{x\to 0}{\text{Lt}} \frac{2\sin x\cos x}{\sin 2x\cos 2x}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\sin 2x}{\sin 2x\cdot\cos 2x} = \underset{x\to 0}{\text{Lt}} \frac{1}{\cos 2x} = \frac{1}{1} = 1$$

Exercise

Evaluate the following limits

1. $\underset{x\to 1}{\text{Lt}} \dfrac{\log(1-x^2)}{\cot \pi x}$ **[Ans. 0]**

2. $\underset{x\to \infty}{\text{Lt}} \dfrac{\log x}{x^n}$; **whee n is any natural number** **[Ans. 0]**

3. $\underset{x\to \infty}{\text{Lt}} \dfrac{\log x^2}{\cot^2 x}$ **[Ans. 0]**

4. $\underset{x\to 0}{\text{Lt}} \dfrac{\log\log(1-x^2)}{\log\log\cos x}$ **[Ans. 1]**

5. $\underset{x\to \frac{\pi}{2}}{\text{Lt}} \dfrac{\tan 5x}{\tan x}$ **[Ans. 1/5]**

6. $\underset{x\to 0}{\text{Lt}} \log_{\sin 2x} \sin x$ **[Ans. 1]**

5.5. [Evaluation of the form $(\infty - \infty)$]

When $\underset{x\to a}{\text{Lt}} f(x) = \infty$ and $\underset{x\to a}{\text{Lt}} g(x) = \infty$

then $\underset{x\to a}{\text{Lt}} [f(x) - g(x)]$ **is of the form $(\infty - \infty)$ and it can be evaluated as**

$$\underset{x\to a}{\text{Lt}}\left[\frac{1}{1/f(x)} - \frac{1}{1/g(x)}\right] = \underset{x\to a}{\text{Lt}}\left[\frac{1/g(x) - 1/f(x)}{\{1/f(x)\}\cdot\{1/g(x)\}}\right]$$

which is of form $\dfrac{0}{0}$ **and thus can be evaluated by applying L' Hospital's rule.**

Examples

20. Evaluate $\underset{x\to 0}{\text{Lt}}\left[\dfrac{1}{e^x - 1} - \dfrac{1}{x}\right]$.

Solution.

$\underset{x\to 0}{\text{Lt}}\left[\dfrac{1}{e^x-1} - \dfrac{1}{x}\right]$ [Form $\infty - \infty$] $= \underset{x\to 0}{\text{Lt}} \dfrac{x - e^x + 1}{x(e^x - 1)}$ [Form 0/0] $\therefore$ By L' Hospital's rule

$$= \underset{x\to 0}{\text{Lt}} \frac{1 - e^x}{e^x - 1 + xe^x}\left[\text{Form }\frac{0}{0}\right] = \underset{x\to 0}{\text{Lt}} \frac{-e^x}{e^x + e^x + xe^x} = \frac{-1}{1+1+0} = \frac{-1}{2}$$

21. Evaluate $\underset{x\to \frac{\pi}{2}}{\text{Lt}} (\sec x - \tan x)$.

Solution.

$\underset{x\to \frac{\pi}{2}}{\text{Lt}} (\sec x - \tan x)$ [Form $\infty - \infty$] $= \underset{x\to \frac{\pi}{2}}{\text{Lt}}\left[\dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x}\right]$

$= \underset{x\to \frac{\pi}{2}}{\text{Lt}} \dfrac{1 - \sin x}{\cos x}\left[\text{Form }\dfrac{0}{0}\right]$ $\therefore$ Applying L' Hospital's rule

Required limit $= \underset{x\to \frac{\pi}{2}}{\text{Lt}} \dfrac{-\cos x}{-\sin x} = \underset{x\to \frac{\pi}{2}}{\text{Lt}} \cot x = 0$

22. Evaluate $\underset{x\to 0}{\text{Lt}}\left[\dfrac{1}{x(1+x)} - \dfrac{\log(1+x)}{x^2}\right]$.

Solution.

The given limit is of the form $(\infty - \infty)$; we can write it as

Required limit $= \underset{x\to 0}{\text{Lt}} \dfrac{x - (1+x)\log(1+x)}{x^2(1+x)}\left[\text{Form }\dfrac{0}{0}\right]$

$$= \underset{x\to 0}{\text{Lt}} \frac{1 - (1+x)\cdot\left(\frac{1}{1+x}\right) - \log(1+x)}{2x + 3x^2} \quad \text{[By L' Hospital's Rule]}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{-\log(1+x)}{2x + 3x^2}\left[\text{Form }\frac{0}{0}\right] = \underset{x\to 0}{\text{Lt}} \frac{-\left\{\frac{1}{(1+x)}\right\}}{2 + 6x} = \frac{-1}{2}$$

23. Evaluate $\underset{x\to 0}{\text{Lt}}\left[\dfrac{1}{x} - \cot x\right]$.

Solution.

As the given limit is of the form $(\infty - \infty)$; thus we can write it as:

Required limit $= \underset{x\to 0}{\text{Lt}} \dfrac{1 - x\cot x}{x} = \underset{x\to 0}{\text{Lt}} \dfrac{\sin x - x\cos x}{x\sin x}\left[\text{Form }\dfrac{0}{0}\right]$

Now applying L' Hospital's rule

Required limit $= \underset{x\to 0}{\text{Lt}} \dfrac{\cos x + x\sin x - \cos x}{\sin x + x\cos x} = \underset{x\to 0}{\text{Lt}} \dfrac{x\sin x}{\sin x + x\cos x}\left[\text{Form }\dfrac{0}{0}\right]$

$$= \underset{x\to 0}{\text{Lt}} \frac{\sin x + x\cos x}{\cos x - x\sin x + \cos x} = \underset{x\to 0}{\text{Lt}} \frac{\sin x + x\cos x}{2\cos x - x\sin x} = \frac{0+0}{2-0} = 0$$

24. Evaluate $\underset{x\to 0}{\text{Lt}}\left(\cot^2 x - \frac{1}{x^2}\right)$

Solution.

$$\underset{x\to 0}{\text{Lt}}\left(\cot^2 x - \frac{1}{x^2}\right) \quad [\infty - \infty \text{ Form}]$$

$$= \underset{x\to 0}{\text{Lt}}\left(\frac{1}{\tan^2 x} - \frac{1}{x^2}\right) = \underset{x\to 0}{\text{Lt}}\frac{x^2-\tan^2 x}{x^2\tan^2 x} = \underset{x\to 0}{\text{Lt}}\frac{x^2-\tan^2 x}{x^2\tan^2 x} = \underset{x\to 0}{\text{Lt}}\frac{x^2-\tan^2 x}{x^4}\times\left(\frac{x}{\tan x}\right)^2$$

$$= \underset{x\to 0}{\text{Lt}}\frac{x^2-\tan^2 x}{x^4} \qquad \left[\underset{x\to 0}{\text{Lt}}\frac{x}{\tan x}=1\right] = \underset{x\to 0}{\text{Lt}}\frac{2x-2\tan x\sec^2 x}{4x^3} = \underset{x\to 0}{\text{Lt}}\frac{2x-2\tan x\,(1+\tan^2 x)}{4x^3}$$

$$= \underset{x\to 0}{\text{Lt}}\frac{x-\tan x-\tan^3 x}{2x^3}\left[\text{Form }\frac{0}{0}\right] = \underset{x\to 0}{\text{Lt}}\frac{1-\sec^2 x-3\tan^2 x\sec^2 x}{6x^2}$$

$$= \underset{x\to 0}{\text{Lt}}\frac{1-(1+\tan^2 x)-3\tan^2 x\,(1+\tan^2 x)}{6x^2} = \underset{x\to 0}{\text{Lt}}\frac{-4\tan^2 x-3\tan^4 x}{6x^2}$$

$$= \underset{x\to 0}{\text{Lt}}\frac{-4+3\tan^2 x}{6}\times\left(\frac{\tan x}{x}\right)^2 = \frac{-4+0}{6}\cdot(1)^2 = -\frac{2}{3}$$

Exercise

1. $\underset{x\to 0}{\text{Lt}}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]$ $\left[\text{Ans. }\frac{1}{2}\right]$
2. $\underset{x\to 2}{\text{Lt}}\left[\frac{1}{\log(x-1)}-\frac{1}{x-2}\right]$ $\left[\text{Ans. }\frac{1}{2}\right]$
3. $\underset{x\to 0}{\text{Lt}}(\operatorname{cosec} x-\cot x)$ [Ans. 0]
4. $\underset{x\to 0}{\text{Lt}}\dfrac{\cot x-\left(\frac{1}{x}\right)}{x}$ $\left[\text{Ans. }-\frac{1}{3}\right]$
5. $\underset{x\to 0}{\text{Lt}}\left(\frac{1}{x^2}-\operatorname{cosec}^2 x\right)$ $\left[\text{Ans. }-\frac{1}{3}\right]$

5.6. [Evaluation of the form $0\cdot\infty$]

Let $\underset{x\to a}{\text{Lt}} f(x) = 0$ and $\underset{x\to a}{\text{Lt}} g(x) = \infty$ then $\underset{x\to a}{\text{Lt}} f(x)\cdot g(x)$ is of the form $(0\,.\,\infty)$ and it can be written as

$$\underset{x\to a}{\text{Lt}}\frac{f(x)}{\dfrac{1}{g(x)}}\left[\text{Form }\frac{0}{0}\right] \quad or \quad \underset{x\to a}{\text{Lt}}\frac{g(x)}{\dfrac{1}{f(x)}}\left[\text{Form }\frac{\infty}{\infty}\right]$$

Both these forms can be evaluated by using L' Hospital's rule.

Examples

25. Evaluate $\underset{x\to 0}{\text{Lt}}\, x\log x$

Solution. $\underset{x\to 0}{\text{Lt}}\, x\log x \quad [\text{Form } 0\times\infty]$

$$= \underset{x\to 0}{\text{Lt}}\frac{\log x}{1/x}\left[\frac{\infty}{\infty}\text{ Form}\right] = \underset{x\to 0}{\text{Lt}}\frac{1/x}{-1/x^2} = \underset{x\to 0}{\text{Lt}}(-x) = 0$$

26. Evaluate $\underset{x\to\infty}{\text{Lt}}\, a^x\sin\left(\frac{b}{a^x}\right); a>1$

Solution.

$$\underset{x\to\infty}{\text{Lt}}\, a^x\sin\left(\frac{b}{a^x}\right)[\infty\times 0\text{ Form}] = \underset{x\to\infty}{\text{Lt}}\frac{\sin(b\cdot a^{-x})}{a^{-x}}\left[\text{Form }\frac{0}{0}\right]$$

$$= \underset{x\to\infty}{\text{Lt}}\frac{\cos(b\,.\,a^{-x})\,.\,b\,.\,a^{-x}(\log a)(-1)}{a^{-x}(\log a)(-1)} = \underset{x\to\infty}{\text{Lt}}\, b\cos\left(\frac{b}{a^x}\right) = b\cdot\cos 0 = b\cdot 1 = b$$

27. Evaluate $\underset{x\to 0^+}{\text{Lt}}\, x\log\tan x$.

Solution.

$\underset{x\to 0^+}{\text{Lt}}\, x\log\tan x \quad [0\times\infty\text{ Form}]$

$$\underset{x\to 0^+}{\text{Lt}}\frac{\log\tan x}{1/x}\left[\frac{\infty}{\infty}\text{ Form}\right] = \underset{x\to 0^+}{\text{Lt}}\frac{\frac{1}{\tan x}\times\sec^2 x}{-\frac{1}{x^2}} = \underset{x\to 0^+}{\text{Lt}}\frac{\frac{\cos x}{\sin x}\times\frac{1}{\cos^2 x}}{\frac{-1}{x^2}}$$

2

$$= \underset{x\to 0^+}{\text{Lt}}\frac{2\sin x\cos x}{-1/x^2} = \underset{x\to 0^+}{\text{Lt}}\frac{-2x^2}{\sin 2x} = \underset{x\to 0^+}{\text{Lt}}(-x)\times\left(\underset{x\to 0}{\text{Lt}}\frac{2x}{\sin 2x}\right) = 0\times 1 = 0$$

Exercise

Evaluate the following limits

1. $\underset{x\to 0^+}{\text{Lt}}\, x^m\log m$, where $m>0$ [Ans. 0]
2. $\underset{x\to 1}{\text{Lt}}(1-x)\tan\frac{\pi x}{2}$ [Ans. $\frac{2}{\pi}$]
3. $\underset{x\to 1}{\text{Lt}}\sin x\log x^2$ [Ans. 0]
4. $\underset{x\to 1}{\text{Lt}}\sec\frac{\pi x}{2}\log\frac{1}{2}$ [Ans. $\frac{2}{\pi}$]
5. $\underset{x\to\infty}{\text{Lt}}\, 2^x\sin\left(\frac{b}{2x}\right)$ [Ans. b]

5.7. [Evaluation of the form $0^0, 1^\infty, \infty^0$]

We have the evaluate $\underset{x\to a}{\text{Lt}}[f(x)]^{g(x)}$ where

(*i*) $\underset{x\to a}{\text{Lt}} f(x)=0;\ \underset{x\to a}{\text{Lt}} g(x)=0$ (*ii*) $\underset{x\to a}{\text{Lt}} f(x)=1;\ \underset{x\to a}{\text{Lt}} g(x)=\infty$ (*iii*) $\underset{x\to a}{\text{Lt}} f(x)=\infty;\ \underset{x\to a}{\text{Lt}} g(x)=0$

All these limits are evaluated by taking logarithm and then simplifying. This method should be clear from the following illustrations.

Examples

28. Evaluate $\underset{x\to 0}{\text{Lt}}(\sin x)^{\tan x}$.

Solution.

Let $y = \underset{x\to 0}{\text{Lt}}(\sin x)^{\tan x}$ [0^0 form]. Taking log both sides $\log y = \underset{x\to 0}{\text{Lt}}\log(\sin x)^{\tan x} = \underset{x\to 0}{\text{Lt}}\tan x\,.\log(\sin x)$

$$\log y = \operatorname*{Lt}_{x\to 0} \frac{\log(\sin x)}{\cot x} \quad \left[\frac{\infty}{\infty}\text{ Form}\right] = \operatorname*{Lt}_{x\to 0} \frac{\frac{1}{\sin x}\times\cos x}{-\operatorname{cosec}^2 x} \quad \text{[By L' Hospital's Rule]}$$

$$= \operatorname*{Lt}_{x\to 0} -(\sin x\cos x) = 0 \quad \therefore \log y = 0 \Rightarrow y = e^0 = 1 \quad \therefore \operatorname*{Lt}_{x\to 0} (\sin x)^{\tan x} = 1$$

29. Evaluate $\operatorname*{Lt}_{x\to 0^+} (\cot x)^x$

Solution.

Let $y = \operatorname*{Lt}_{x\to 0^+} (\cot x)^x$ $[\infty^0 \text{ Form}]$ $\therefore \log y = \operatorname*{Lt}_{x\to 0^+} x\log(\cot x)$

$$= \operatorname*{Lt}_{x\to 0^+} \frac{\log\cot x}{\frac{1}{x}} \quad \left[\frac{\infty}{\infty}\text{ Form}\right] = \operatorname*{Lt}_{x\to 0^+} \frac{\frac{1}{\cot x}\cdot(-\operatorname{cosec}^2 x)}{-\frac{1}{x^2}} = \operatorname*{Lt}_{x\to 0^+} \frac{x^2 \sin x}{\cos x\cdot\sin^2 x}$$

$$= \operatorname*{Lt}_{x\to 0^+} \tan x \times \left(\operatorname*{Lt}_{x\to 0^+} \frac{x^2}{\sin^2 x}\right) = 0\times 1 = 0 \quad \therefore \log y = 0 \Rightarrow y = e^0 = 1 \Rightarrow \operatorname*{Lt}_{x\to 0^+} (\cot x)^x = 1$$

30. Evaluate $\operatorname*{Lt}_{x\to 0^+} (x)^{\frac{1}{\log x}}$.

Solution.

Let $y = \operatorname*{Lt}_{x\to 0^+} (x)^{\frac{1}{\log x}}$ $[0^0 \text{ Form}]$. Taking log both sides $\log y = \operatorname*{Lt}_{x\to 0^+} \frac{1}{\log x}\times\log x = 1$

$\therefore y = e^1 \Rightarrow \operatorname*{Lt}_{x\to 0^+} x^{\frac{1}{\log x}} = e.$

31. Evaluate $\operatorname*{Lt}_{x\to\infty} (1+x)^{1/x}$

Solution.

Let $y = \operatorname*{Lt}_{x\to\infty} (1+x)^{1/x}$ $[\infty^0 \text{ Form}]$ Taking log both sides

$$\log y = \operatorname*{Lt}_{x\to\infty} \frac{1}{x}\log(1+x) = \operatorname*{Lt}_{x\to\infty} \frac{\log(1+x)}{x} \quad \left[\frac{\infty}{\infty}\text{ Form}\right] = \operatorname*{Lt}_{x\to\infty} \frac{\frac{1}{1+x}}{1}$$

$$\therefore \log y = \operatorname*{Lt}_{x\to\infty} \frac{1}{1+x} = 0 \Rightarrow y = e^0 = 1 \Rightarrow \operatorname*{Lt}_{x\to\infty} (1+x)^{1/x} = 1$$

32. Evaluate $\operatorname*{Lt}_{x\to 0^+} (\cot x)^{\sin x}$

Solution.

Let $y = \operatorname*{Lt}_{x\to 0^+} (\cot x)^{\sin x}$ $[\infty^0 \text{ Form}]$ Taking log both sides

$$\log y = \operatorname*{Lt}_{x\to 0^+} (\sin x)\log\cot x = \operatorname*{Lt}_{x\to 0^+} \frac{\log\cot x}{\operatorname{cosec} x} \quad \left[\frac{\infty}{\infty}\text{ Form}\right]$$

$$= \operatorname*{Lt}_{x\to 0^+} \frac{\frac{1}{\cot x}\times(-\operatorname{cosec}^2 x)}{-\operatorname{cosec} x\cot x} = \operatorname*{Lt}_{x\to 0^+} \frac{\operatorname{cosec} x}{\cot^2 x} = \operatorname*{Lt}_{x\to 0^+} \frac{\sin^2 x}{\sin x\cos^2 x} = \operatorname*{Lt}_{x\to 0^+} \frac{(\sin x)}{\cos^2 x} = 0$$

$$\therefore \log y = 0 \Rightarrow y = e^0 = 1 \quad \therefore \operatorname*{Lt}_{x\to 0^+} (\cot x)^{\sin x} = 1$$

33. Evaluate $\operatorname*{Lt}_{x\to 0^+} (\cot x)^{\frac{1}{\log x}}$ **[∞^0 Form}**

Solution.

Let $y = \operatorname*{Lt}_{x\to 0^+} (\cot x)^{\frac{1}{\log x}}$ $\therefore \log y = \operatorname*{Lt}_{x\to 0^+} \frac{1}{\log x}\log\cot x$

$$= \operatorname*{Lt}_{x\to 0^+} \frac{\log\cot x}{\log x} \quad \left[\frac{\infty}{\infty}\text{ Form}\right] = \operatorname*{Lt}_{x\to 0^+} \frac{\frac{1}{\cot x}\times(-\operatorname{cosec}^2 x)}{\frac{1}{x}} \quad \text{[By L' Hospital's rule]}$$

$$= \operatorname*{Lt}_{x\to 0^+} \frac{-(\sin x)(x)}{\cos x\cdot\sin^2 x} = -\operatorname*{Lt}_{x\to 0^+} \frac{1}{\cos x}\times\left[\operatorname*{Lt}_{x\to 0^+} \frac{x}{\sin x}\right] = -1$$

$$\therefore y = e^{-1} = \frac{1}{e} \Rightarrow \operatorname*{Lt}_{x\to 0^+} (\cot x)^{\frac{1}{\log x}} = \frac{1}{e}$$

34. Evaluate $\operatorname*{Lt}_{x\to 0} \left(\frac{a^x+b^x}{2}\right)^{1/x}$.

Solution.

$$\text{Let } y = \operatorname*{Lt}_{x\to 0} \left(\frac{a^x+b^x}{2}\right)^{1/x} \qquad [1^\infty \text{ Form}]$$

$$\therefore \log y = \operatorname*{Lt}_{x\to 0} \frac{1}{x}\log\left(\frac{a^x+b^x}{2}\right) \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \operatorname*{Lt}_{x\to 0} \frac{\log(a^x+b^x)-\log 2}{x}$$

$$= \operatorname*{Lt}_{x\to 0} \frac{\frac{a^x\log a + b^x\log b}{a^x+b^x} - 0}{1} = \frac{\log a+\log b}{1+1} = \frac{1}{2}\log(ab) = \log(\sqrt{ab})$$

$$\therefore \log y = \log\sqrt{ab} \Rightarrow y = \sqrt{ab} \quad \text{or} \quad \operatorname*{Lt}_{x\to 0} \left(\frac{a^x+b^x}{2}\right)^{1/x} = \sqrt{ab}$$

35. Evaluate $\operatorname*{Lt}_{x\to 0} (e^x+4x)^{1/x}$.

Solution.

Let $y = \operatorname*{Lt}_{x\to 0} (e^x+4x)^{1/x}$ $[1^\infty \text{ Form}]$ $\therefore \log y = \operatorname*{Lt}_{x\to 0} \left(\frac{1}{x}\right)\log[e^x+4x]$

$$= \operatorname*{Lt}_{x\to 0} \frac{\log(e^x+4x)}{x} \quad \left[\text{Form } \frac{0}{0}\right] = \operatorname*{Lt}_{x\to 0} \frac{\frac{1}{e^x+4x}\cdot(e^x+4)}{1} \quad \text{[By L' Hospital's rule]}$$

$$= \frac{1+4}{1+0} = 5 \quad \therefore \log y = 5 \Rightarrow y = e^5 \quad \therefore \operatorname*{Lt}_{x\to 0} (e^x+4x)^{1/x} = e^5$$

Exercise

Evaluate the following limits

1. $\underset{x \to 0}{\text{Lt}} (1 + \sin x)^{\cos x}$ [Ans. e]
2. $\underset{x \to \frac{\pi}{4}}{\text{Lt}} (\tan x)^{\tan 2x}$ [Ans. $\frac{1}{e}$]
3. $\underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$ [Ans. $e^{-1/6}$]
4. $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x^2}\right)^{\tan x}$ [Ans. 1]
5. $\underset{x \to 0}{\text{Lt}} (x)^x$ [Ans. 1]
6. $\underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right)^{1/x^2}$ [Ans. e^2]
7. $\underset{x \to \frac{\pi}{2}}{\text{Lt}} (\sec x)^{\cot x}$ [Ans. 1]
8. $\underset{x \to 0}{\text{Lt}} (\cos x)^{\frac{1}{x^2}}$ [Ans. $e^{-1/2}$]
9. $\underset{x \to 0}{\text{Lt}} (\cos x)^{\cot^2 x}$ [Ans. $e^{-1/2}$]
10. $\underset{x \to \frac{\pi}{2}}{\text{Lt}} (\sin x)^{\tan x}$ [Ans. 1]
11. $\underset{x \to 0}{\text{Lt}} \left(\frac{\tan x}{x}\right)^{1/x}$ [Ans. 1]
12. $\underset{x \to 0}{\text{Lt}} (\text{cosec}\, x)^{\frac{1}{\log x}}$ [Ans. e^{-1}]

Miscellaneous Examples

36. Find the values of a and b so that $\underset{x \to 0}{\text{Lt}} \dfrac{x(1 + a\cos x) - b\sin x}{x^3}$ may be equal to 1.

Solution.

$$\underset{x \to 0}{\text{Lt}} \frac{x(1 + a\cos x) - b\sin x}{x^3} \quad \left[\text{Form } \frac{0}{0}\right]$$

$$\text{Applying L'Hospital's rule} = \underset{x \to 0}{\text{Lt}} \frac{1 + a\cos x - ax\sin x - b\cos x}{3x^2} \quad \text{...(1)}$$

The denominator $\to 0$ as $x \to 0$; thus in order that limit may exist and be equal to 1; the numerator should be 0 which is possible when

$$1 + a - b = 0 \qquad [\because \cos 0 \to 1;\ \sin 0 \to 0] \quad \text{...(2)}$$

With the condition in (2): equation (1) has the form $\frac{0}{0}$. Thus again applying L' Hospital's rule

$$= \underset{x \to 0}{\text{Lt}} \frac{-a\sin x - a\sin x - ax\cos x + b\sin x}{6x} = \underset{x \to 0}{\text{Lt}} \frac{-ax\cos x + (b - 2a)\sin x}{6x} \quad \left[\text{Form } \frac{0}{0}\right]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{-a\cos x + ax\sin x + (b - 2a)\cos x}{6} = \frac{-a + b - 2a}{6} = \frac{b - 3a}{6}$$

Now the given limit is equal to 1 $\quad \therefore \dfrac{b - 3a}{6} = 1 \Rightarrow b - 3a = 6 \quad$ ...(3)

Adding (2) and (3) $\quad -2a = 5 \Rightarrow a = \dfrac{-5}{2} \quad \therefore \ b = a + 1 = \dfrac{-5}{2} + 1 = \dfrac{-3}{2}$

Hence $\quad a = \dfrac{-5}{2}, b = \dfrac{-3}{2}$

37. Find the values of a and b in order that $\underset{x \to 0}{\text{Lt}} \dfrac{(1 - a\cos x) + b\sin x}{x^3} = \dfrac{1}{3}$.

Similar to example 36 ; $a = \dfrac{1}{2}, b = \dfrac{-1}{3}$

38. Find the values of a, b, c so that $\underset{x \to 0}{\text{Lt}} \dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$.

Solution.

The given limit is of the form $\dfrac{a - b + c}{0}$; thus in order that it may be equal to 2; we must have

$$a - b + c = 0 \quad \text{...(1)}$$

With this condition; we apply L' Hospital's rule

$$\therefore \text{Given limit} = \underset{x \to 0}{\text{Lt}} \frac{ae^x + b\sin x - ce^{-x}}{\sin x + x\cos x} \quad \text{...(2)}$$

Now the form is $\dfrac{a - c}{0}$. $\quad \therefore$ again, we should have $a - c = 0 \quad$ ...(3)

Again applying L' Hospital's rule to (2)

$$\text{Given limit} = \underset{x \to 0}{\text{Lt}} \frac{ae^x + b\cos x + ce^{-x}}{\cos x + \cos x - x\sin x} = \frac{a + b + c}{2}.$$

As value of limit is 2 $\quad \therefore \dfrac{a + b + c}{2} = 2 \Rightarrow a + b + c = 4 \quad$ ...(4)

Solving (i), (2) and (4); we get $a = 1, b = 2, c = 1$.

39. Evaluate $\underset{x \to y}{\text{Lt}} \dfrac{x^y - y^x}{x^x - y^y}$.

Solution.

Here $x \to y$; thus we will consider x as variable and y as constant

$$\therefore \underset{x \to y}{\text{Lt}} \frac{x^y - y^x}{x^x - y^y} \quad \left[\text{Form } \frac{0}{0}\right] = \underset{x \to y}{\text{Lt}} \frac{yx^{y-1} - y^x \log y}{x^x(1 + \log x) - 0} \quad \left[\because \frac{d}{dx}(x^x) = x^x(1 + \log x)\right]$$

$$= \frac{y \cdot y^{y-1} - y^y \log y}{y^y(1 + \log y)} = \frac{y^y(1 - \log y)}{y^y(1 + \log y)} = \frac{1 - \log y}{1 + \log y}.$$

40. Evaluate $\underset{x \to 0}{\text{Lt}} \dfrac{(1 + x)^{1/x} - e}{x}$;

Solution.

As $\underset{x \to 0}{\text{Lt}} \dfrac{(1 + x)^{1/x} - e}{x} = e$; thus the given limit is of the form $\dfrac{0}{0}$.

To solve it; first we obtain expansion for $(1 + x)^{1/x}$. Let $y = (1 + x)^{1/x}$

$$\therefore \log y = \frac{1}{x}\log(1 + x) = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) = 1 - \frac{x}{2} + \frac{x^2}{3} - \dots$$

$$= 1 + z; \text{ where } z = -\frac{x}{2} + \frac{x^2}{3} - \dots \qquad \therefore y = e^{1+z} = e \cdot e^z = e\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots\right)$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \dots\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} - \dots\right)^2 + \dots\right]$$

$$= e\left[1 - \frac{x}{2} + \frac{x^2}{3} + \frac{1}{8}x^2 + \dots \text{higher terms of } x\right] = e\left[1 - \frac{x}{2} + \frac{11}{24}x^2 + \dots\right]$$

Now $\underset{x\to 0}{\text{Lt}} \dfrac{(1+x)^{1/x}-e}{x} = \underset{x\to 0}{\text{Lt}} \dfrac{e\left[1-\frac{x}{2}+\frac{11}{24}x^2+\ldots\right]-e}{x}$

$= \underset{x\to 0}{\text{Lt}}\, e\left[-\frac{x}{2}+\frac{11}{24}x^2+\ldots\right]\Big/x = \underset{x\to 0}{\text{Lt}}\, e\left[-\frac{1}{2}+\frac{11}{24}x+\ldots\right] = -\frac{1}{2}e$

41. Evaluate $\underset{x\to 0}{\text{Lt}} \dfrac{(1+x)^{1/x}-e+\frac{1}{2}ex}{x^2}$

Solution.

$$\underset{x\to 0}{\text{Lt}} \frac{(1+x)^{1/x}-e+\frac{1}{2}ex}{x^2}\left[\text{Form } \frac{0}{0}\right] = \underset{x\to 0}{\text{Lt}} \frac{e\left[1-\frac{x}{2}+\frac{11}{24}x^2+\ldots\right]-e+\frac{1}{2}ex}{x^2} \quad \text{[See ex. 40]}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{e\left[\frac{11}{24}x^2 + \text{terms containing higher powers of } x\right]}{x^2}$$

$$= \underset{x\to 0}{\text{Lt}}\, e\left[\frac{11}{24} + \text{terms containing higher powers of } x\right] = \frac{11e}{24}$$

42. Evaluate $\underset{x\to 0}{\text{Lt}} \left(\dfrac{\tan x}{x}\right)^{1/x}$.

Solutions.

Let $y = \underset{x\to 0}{\text{Lt}} \left(\dfrac{\tan x}{x}\right)^{1/x}$ [Form 1^∞]. Taking log both sides

$\log y = \underset{x\to 0}{\text{Lt}} \dfrac{1}{x}\log\left[\dfrac{\tan x}{x}\right]$ ...(1) Now $\tan x = x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\ldots$

Putting this expansion in (1)

$$\log y = \underset{x\to 0}{\text{Lt}} \frac{1}{x}\log\left[\frac{x+(x^3/3)+(2x^5/15)+\ldots}{x}\right] = \frac{1}{x}\log\left[1+\frac{x^2}{3}+\frac{2x^4}{15}+\ldots\right]$$

$$= \underset{x\to 0}{\text{Lt}} \frac{1}{x}\log(1+z);\ \text{where } z = \frac{x^2}{3}+\frac{2x^4}{15}+\ldots = \underset{x\to 0}{\text{Lt}} \frac{1}{x}\left[z-\frac{z^2}{2}+\ldots\right]$$

$$= \underset{x\to 0}{\text{Lt}} \frac{1}{x}\left[\left(\frac{x^2}{3}+\frac{2x^4}{15}+\ldots\right)-\frac{1}{2}\left(\frac{x^2}{3}+\frac{2x^4}{15}+\ldots\right)^2+\ldots\right] = \underset{x\to 0}{\text{Lt}} \frac{1}{x}\left[\frac{x^2}{3}+\left(\frac{2}{15}-\frac{1}{18}\right)x^4+\ldots\right]$$

$$= \underset{x\to 0}{\text{Lt}} \frac{1}{x}\left[\frac{x^2}{3}+\frac{7}{90}x^3+\ldots\right] = 0 \qquad \therefore\ y = e^0 = 1$$

43. Evaluate $\underset{x\to 0}{\text{Lt}} \left(\dfrac{\tan x}{x}\right)^{1/x^2}$

Solution.

Proceeding as in example (42), we get

$$\log y = \underset{x\to 0}{\text{Lt}} \frac{1}{x^2}\left[\frac{x^2}{3}+\frac{7}{90}x^4+\ldots\right] = \underset{x\to 0}{\text{Lt}}\left[\frac{1}{3}+\frac{7}{90}x^2+\ldots\right] = \frac{1}{3} \qquad \therefore\ y = e^{1/3}$$

44. Evaluate $\underset{x\to 0}{\text{Lt}} \left(\dfrac{\sin x}{x}\right)^{1/x^2}$.

Solution.

Let $y = \underset{x\to 0}{\text{Lt}} \left(\dfrac{\sin x}{x}\right)^{1/x^2}$ [1^∞ Form] $\therefore\ \log y = \underset{x\to 0}{\text{Lt}} \dfrac{1}{x^2}\log\left(\dfrac{\sin x}{x}\right)$ ...(1)

Now $\sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5}-\ldots$ Putting this expansion of $\sin x$ in (1)

$$= \underset{x\to 0}{\text{Lt}} \frac{1}{x^2}\log\left[\frac{x-(x^3/3!)+(x^5/5!)-\ldots}{x}\right] = \underset{x\to 0}{\text{Lt}} \frac{1}{x^2}\log\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\ldots\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{1}{x^2}\log\left(1-\frac{x^2}{6}+\frac{x^4}{120}-\ldots\right) = \underset{x\to 0}{\text{Lt}} \frac{1}{x^2}\log(1-z);\ \text{where } z = \frac{x^2}{6}-\frac{x^4}{120}+$$

$$= \underset{x\to 0}{\text{Lt}} \frac{1}{x^2}\left(-z-\frac{z^2}{2}-\ldots\right) = \underset{x\to 0}{\text{Lt}} \frac{1}{x^2}\left[-\left(\frac{x^2}{6}-\frac{x^4}{120}\right)-\frac{1}{2}\left(\frac{x^2}{6}-\frac{x^4}{120}+\ldots\right)-\ldots\right]$$

$$= \underset{x\to 0}{\text{Lt}} \frac{1}{x^2}\left[-\frac{x^2}{6}+\left(\frac{x^4}{120}-\frac{x^4}{72}\right)+\ldots\right] = \underset{x\to 0}{\text{Lt}} \frac{1}{x^2}\left[-\frac{x^2}{6}-\frac{x^4}{180}+\ldots\right]$$

$$= \underset{x\to 0}{\text{Lt}}\left[-\frac{1}{6}-\frac{x^2}{180}+\ldots\right] = -\frac{1}{6} \qquad \therefore\ y = e^{-1/6}$$

45. Evaluate $\underset{x\to 0}{\text{Lt}} \left(\dfrac{\sin x}{x}\right)^{1/x}$

Similar to example 44. **[Ans. 1]**

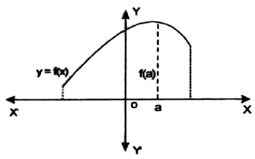
CHAPTER - 6

[Maximum and Minimum Values of Function]

6.1. [Maximum and Minimum Values of A Function]

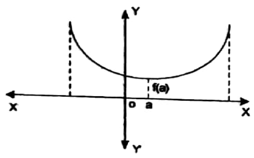
Definition:

1. Let $f(x)$ be a real function defined on interval I. $f(x)$ is said to have maximum value in I, if there exist a point a in I such that $f(x) \le f(a)$ for all $x \in$ I.



Also, $f(a)$ is called as the maximum value of $f(x)$ in the interval I and point 'a' is called a point of maximum value of f in I.

2. Let $f(x)$ be a real function defined on a interval I. Then $f(x)$ is said to have a minimum value in the interval I, if there exist a point a in I such that $f(x) \ge f(a)$ for all $x \in I$.



Also, $f(a)$ is called as the minimum value of $f(x)$ in the interval I and point 'a' is called a point of minimum value of $f(x)$ in I.

Example

1. Find the maximum and minimum values, if any of the given function:

$$f(x) = -|x-1| + 5 \text{ for all } x \in R.$$

Solution:

1 $f(x) = -|x-1| + 5$ for all $x \in R$ We know that $|x-1| \ge 0$ for all $x \in R$

$\Rightarrow -|x-1| \le 0$ for all $x \in R$ $\Rightarrow$ $-|x-1| + 5 \le 5$ for all $x \in R$ $\therefore$ 5 is the maximum value of $f(x)$

Also, $f(x) = 5 \Rightarrow -|x-1| + 5 = 5$ $\Rightarrow$ $-|x-1| = 0$ $\Rightarrow$ $|x-1| = 0$ $\Rightarrow$ $x = 1$

$\therefore$ '1' is the point of maximum value of $f(x)$

Now, $f(x)$ can be made as small as we want. $\therefore$ minimum value of $f(x)$ does not exist.

6.2. [Local Maxima and Local Minima]

Definition:

1. **Local Maximum :** A function $f(x)$ is said to attain a local maximum at $x = a$ if there exists a neighbourhood $(a - \delta, a + \delta)$ of 'a' such that $f(x) < f(a)$ for all $x \in (a - \delta, a + \delta), x \pm a$.
2. **Local Minimum :** A function $f(x)$ is said to attain a local minimum at $x = a$ if there exists a neighbourhood $(a - \delta, a + \delta)$ of 'a' such that $f(x) > f(a)$ for all $x \in (a - \delta, a + \delta), x \pm a$.

6.3. [Tests To Find Points of Local Maxima or Local Minima]

TEST 1:

First Derivative Test:

Let f be a differential function defined on an interval I and let $a \in$ I. If,

(1) $f'(a) = 0$ and $f'(x)$ changes sign from positive to negative as x increases through a i.e., $f'(x) > 0$ at every point sufficiently closed to a and to left of 'a' and $f'(x) < 0$ at every point sufficiently close to 'a' and to the right 'a' then $x = a$ is a point of local maximum value of f.

(2) $f(a) = 0$ and $f'(x)$ changes sign from negative to positive as x increases through a, i.e., $f(x) < 0$ at every point sufficiently close to a and to the left of a, and $f'(x) > 0$ at every point sufficiently close to 'a' and to the right of 'a', then $x = a$ is a point of local minimum value of f.

TEST 2:

Second Derivative Test:

Let f be a differential function defined on an interval I. and let 'c' be an interior point of I such that $f'(c) = 0$, then

(a) $x = c$ is called point of local minimum if $f'(c) > 0$

(b) $x = c$ is called point of local maximum if $f''(c) < 0$

Example

2. Find all points of local maxima and local minima, local maximum values and local minimum values for the function: $f(x) = (x-1)^3 (x+1)^2$

Solution:

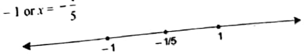
Let $f(x) = y = (x-1)^3 (x+1)^2$ then $\frac{dy}{dx} = 3(x-1)^2 (x+1)^2 + 2(x+1)(x-1)^3$

$= (x-1)^2 (x+1) [3(x+1) + 2(x-1)] = (x-1)^2 (x+1)(5x+1)$

$= (x-1)^2 (x+1)\, 5\left(x + \frac{1}{5}\right)$ ...(1)

Put $\frac{dy}{dx} = 0$ i.e., $(x-1)^2 (x+1)[5x+1] = 0 \Rightarrow (x-1)^2 = 0$ or $(x+1) = 0$ or $(5x+1) = 0$

$\Rightarrow x = 1$ or $x = -1$ or $x = -\frac{1}{5}$



For $x = 1$;

If x is sufficiently close and to the left of 1, $\frac{dy}{dx} = (+ve)(+ve)(+ve) = +ve$ and if x is sufficiently close and to right of 1 then

$\frac{dy}{dx} = (+ve)(+ve)(+ve) = +ve$ [From Equation (1)] i.e., $\frac{dy}{dx}$ does not changes its sign.

$\therefore$ $x = 1$ is neither a point of local maximum nor a point of local minimum.

For $x = -1$, If x is sufficiently close and to the left of -1.

$\frac{dy}{dx} = (+ve)(-ve)(-ve) = +ve$. If x is sifficiently close and to the right of -1.

$\frac{dy}{dx} = (+ve)(+ve)(-ve) = -ve$ $\quad \therefore x = -1$ is a point of local maximum.

And $\therefore$ local maximum value of $f(x)$ is $f(-1) = (-1-1)^3(-1+1)^2 = 0$

For $x = -\frac{1}{5}$. If x is sufficiently close and to the left of $-\frac{1}{5}$. $\frac{dy}{dx} = (+ve)(+ve)(-ve) = -ve$

If x is sufficiently close and to the right of $-\frac{1}{5}$. $\frac{dy}{dx} = (+ve)(+ve)(+ve) = +ve$

$\therefore x = -\frac{1}{5}$ is a point of local minimum $\quad$ And $\therefore$ local minimum value of $f(x)$ is

$$f\left(-\frac{1}{5}\right) = \left(-\frac{1}{5}-1\right)^3\left(-\frac{1}{5}+1\right)^3 = \left(-\frac{6}{5}\right)^3\left(\frac{4}{5}\right)^3 = \frac{(-6)^3(4)^3}{(5)^6} = -\frac{3456}{3125}$$

3. **Find the local maxima or local minima of the function $f(x) = \sin x + \cos x;\ 0 < x < \frac{\pi}{2}$.**

Solution:

Let $f(x) = y = \sin x + \cos x$.

Put $\frac{dy}{dx} = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}\ \left[\frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)\right]$.

Now for $x = \frac{\pi}{4}$. If x is sufficiently close and to the left of $\frac{\pi}{4}$ then $\cos x > \sin x \Rightarrow f'(x) > 0$.

If x is sufficiently close and to the right of $\frac{\pi}{4}$ then $\cos x < \sin x \Rightarrow f'(x) < 0$.

i.e., $x = \frac{\pi}{4}$ is a point of local maximum.

4. **Find the maximum and minimum values of y where $y = (x-1)(x-2)^2$.**

Solution:

Let $y = f(x) = (x-1)(x-2)^2$

$$\frac{dy}{dx} = \frac{d}{dx}(x-1)(x-2)^2 + (x-1)\frac{d}{dx}(x-2)^2$$

$$= (x-2)^2 + (x-1)\,2(x-2) = (x-2)[(x-2) + 2(x-1)] = (x-2)(3x-4)$$

Put $\frac{dy}{dx} = 0$ *i.e.*, $x = 2$ or $x = \frac{4}{3}$

Now $\frac{d^2y}{dx^2} = (x-2)\frac{d}{dx}(3x-4) + (3x-4)\frac{d}{dx}(x-2) = (x-2)3 + (3x-4)\cdot 1 = 6x - 10$

$$\left.\frac{d^2y}{dx^2}\right)_{x=2} = 6(2) - 10 = 2\ (+ve);\quad \left.\frac{d^2y}{dx^2}\right)_{x=4/3} = 6\left(\frac{4}{3}\right) - 10 = \frac{24}{3} - 10 = -\frac{6}{3}\quad (-ve)$$

$\therefore$ from second derivative test. $f(x)$ is minimum at $x = 2$ and $f(x)$ is maximum at $x = \frac{4}{3}$

$\therefore$ Minimum value of $y = f(2) = (2-1)(2-2)^2 = 0$

$$\text{Maximum value of } y = f\left(\frac{4}{3}\right) = \left(\frac{4}{3}-1\right)\left(\frac{4}{3}-2\right)^2 = \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)^2 = \frac{4}{27}$$

5. **Find all points of local maxima and local minima, local maximum values and local minimum values for the function $f(x) = x^4 - 62x^2 + 120x + 9$.**

Solution: Let $f(x) = y = x^4 - 62x^2 + 120x + 9$; $f'(x) = 4x^3 - 124x + 120$

$$f'(x) = 4(x^3 - 31x + 30) = 4(x-1)(x^2 + x - 30)$$

$$= 4(x-1)(x^2 + 6x - 5x - 30) = 4(x-1)(x-5)(x+6)$$

Put $f'(x) = 0 \Rightarrow x = 1$ or $x = 5$ or $x = -6$. Now $f''(x) = 12x^2 - 124$. Now from second derivative test:

For $x = 1$

$f''(x) = 12(1)^2 - 124 = 12 - 124 = -112 < 0$ $\quad \therefore x = 1$ is a point of local maximum

For $x = 5$

$f''(x) = 12(5)^2 - 124 = 176 > 0$ $\quad \therefore x = 5$ is a point of local maximum

For $x = -6$

$f''(x) = 12(-6)^2 - 124 = 432 - 124 = 308 > 0$ $\quad \therefore x = -6$ is a point of local minimum.

Now as $x = 1$ is a point of local maximum.

local maximum value $= f(1) = (1)^4 - 62(1)^2 + 120(1) + 9 = 1 - 62 + 120 + 9 = 130 - 62 = 68$

Now as $x = 5, -6$ are the points of local minimum

$\therefore$ local minimum value $= f(5) = (5)^4 - 62(5)^2 + 120(5) + 9 = 625 - 1550 + 600 + 9 = -316$

and $f(-6) = (-6)^4 - 62(-6)^2 + 120(-6) + 9 = 1296 - 2232 - 720 + 9 = -1647$

6. **If $y = \frac{px - q}{(x-1)(x-4)}$ has a turning point $T(2, -1)$, find the values of p and q and show that y is maximum at T.**

Solution:

$$y = \frac{px - q}{(x-1)(x-4)} = \frac{px - q}{x^2 - 5x + 4}$$

$$\frac{dy}{dx} = \frac{\left[(x^2 - 5x + 4)\frac{d}{dx}(px - q)\right] - \left[(px - q)\frac{d}{dx}(x^2 - 5x + 4)\right]}{(x^2 - 5x + 4)^2}$$

$$= \frac{(x^2 - 5x + 4)(p) - (px - q)(2x - 5)}{(x^2 - 5x + 4)^2} \quad \text{...(A)}$$

$$\text{Now } \left(\frac{dy}{dx}\right)_{T=(2,-1)} = \frac{(4 - 10 + 4)p - [2p - (q)][2(2) - 5]}{(4 - 10 + 4)^2}$$

$$= \frac{(-2)p - (2p - q)(-1)}{(-2)^2} = \frac{-2p + 2p - q}{4} = -\frac{q}{4}$$

Now put $\left(\frac{dy}{dx}\right)_{T=(2,T)} = 0$ *i.e.*, $-\frac{q}{4} = 0 \Rightarrow q = 0$ $\quad$ ...(*i*)

Now as $T(2, -1)$ lies on $y = \frac{px - q}{x^2 - 5x + 4}$ $\quad \therefore$ We get $-1 = \frac{2p - q}{8 - 10}$ $\quad$ *i.e.*, $-1 = \frac{2p - q}{-2}$

$\Rightarrow 2 = 2p - q$ $\quad$ ...(*ii*)

From (*i*) and (*ii*) we get $2 = 2p - 0 \Rightarrow p = 1$ *i.e.*, $p = 1$ and $q = 0$

Substituting these values in (A) we get

$$\frac{dy}{dx} = \frac{(x^2 - 5x + 4)(1) - x(2x - 5)}{(x^2 - 5x + 4)^2} = \frac{x^2 - 5x + 4 - 2x^2 + 5x}{(x^2 - 5x + 4)^2} = \frac{-x^2 + 4}{(x^2 - 5x + 4)^2}$$

$$\text{Now } \frac{d^2y}{dx^2} = \frac{\left[(x^2 - 5x + 4)^2\frac{d}{dx}(-x^2 + 4)\right] - \left[(-x^2 + 4)\frac{d}{dx}(x^2 - 5x + 4)^2\right]}{(x^2 - 5x + 4)^4}$$

$$= \frac{(x^2 - 5x + 4)^2(-2x) - [(-x^2 + 4)2(x^2 - 5x + 4)(2x - 5)]}{(x^2 - 5x + 4)^4}$$

$$= \frac{(x^2-5x+4)[(x^2-5x+4)(-2x)[(-x^2+4)2(2x-5)]}{(x^2-5x+4)^4} = \frac{[(x^2-5x+4)(-2x)-[(x^2+4)(4x-10)]]}{(x^2-5x+4)^3}$$

$$\left.\frac{d^2y}{dx^2}\right)_{(2,-1)} = \frac{((2)^2-5\times 2+4)(-4)}{((2)^2-10+4)^3} = \frac{(-2)(-4)}{(-2)^3} = \frac{8}{-8} = -1 < 0 \quad \therefore y \text{ is maximum at } T \text{ when } p = 1;\ q = 0$$

7. Show that $\sin^x\theta\cos^y\theta$ attains a maximum when $\theta = \tan^{-1}\sqrt{\dfrac{x}{y}}$.

Solution:

Let $p = \sin^x\theta\cos^y\theta$

$$\therefore \frac{dp}{d\theta} = \frac{d}{d\theta}(\sin^x\theta)(\cos^y\theta + \sin^x\theta\frac{d}{d\theta}(\cos^y\theta)$$

$$= x\sin^{x-1}\theta\frac{d}{d\theta}(\sin\theta)\cos^y\theta + \sin^x\theta\, y\cos^{y-1}\theta\frac{d}{d\theta}(\cos\theta)$$

$$= x\sin^{x-1}\theta\cos\theta\cos^y\theta + y\sin^x\theta\cos^{y-1}\theta(-\sin\theta)$$

$$= x\sin^{x-1}\theta\cos^{y+1}\theta - y\sin^{x+1}\theta\cos^{y-1}\theta = \frac{\sin^x\theta\cos^y\theta(x\cos^2\theta - y\sin^2\theta)}{\sin\theta\cos\theta}$$

$$= \sin^x\theta\cos^y\theta\left(\frac{x\cos^2\theta}{\sin\theta\cos\theta} - y\frac{\sin^2\theta}{\sin\theta\cdot\cos\theta}\right) = \sin^x\theta\cos^y\theta(x\cot\theta - y\tan\theta)$$

Put $\dfrac{dp}{d\theta} = 0$; We get $\sin^x\theta\cos^y\theta(x\cot\theta - y\tan\theta) = 0$

$\Rightarrow \sin^x\theta = 0$ or $\cos^y\theta = 0$ or $x\cot\theta - y\tan\theta = 0 \Rightarrow \theta = 0$ or $\theta = \dfrac{\pi}{2}$ or $\tan^2\theta = \dfrac{x}{y}$.

$\Rightarrow \theta = 0$ or $\theta = \dfrac{\pi}{2}$ or $\tan\theta = \sqrt{\dfrac{x}{y}}$. Now $\dfrac{dp}{d\theta} = \sin^x\theta\cos^y\theta(x\cot\theta - y\tan\theta) = p(x\cot\theta - y\tan\theta)$

$$\frac{d^2p}{d\theta^2} = p\frac{d}{d\theta}(x\cot\theta - y\tan\theta) + \frac{dp}{d\theta}(x\cot\theta - y\tan\theta)$$

$$= p\left(x\frac{d}{d\theta}(\cot\theta) - y\frac{d}{d\theta}(\tan\theta)\right) + \frac{dp}{d\theta} + (x\cot\theta - y\tan\theta)$$

$$= p(-x\operatorname{cosec}^2\theta - y\sec^2\theta) + \frac{dp}{d\theta}(x\cot\theta - y\tan\theta)$$

$$\left.\frac{d^2p}{d\theta^2}\right)_{\theta=\tan^{-1}\sqrt{\frac{x}{y}}} = \sin^x\theta\cos^y\theta(-x\operatorname{cosec}^2\theta - y\sec^2\theta) + \left.\frac{dp}{d\theta}\right)_{\theta=\tan^{-1}\sqrt{\frac{x}{y}}} \times \left[x\sqrt{\frac{p}{q}} - y\sqrt{\frac{p}{q}}\right]$$

$$= -\sin^x\theta\cos^y\theta(x\operatorname{cosec}^2\theta + y\sec^2\theta) + 0 < 0 \left[\left.\frac{dp}{d\theta}\right)_{\theta=\tan^{-1}\sqrt{\frac{x}{y}}} = 0\right]$$

$\therefore p$ is maximum when $\theta = \tan^{-1}\sqrt{\dfrac{x}{y}}$

8. Show that maximum value of $\left(\dfrac{1}{x}\right)^x$ is $e^{\frac{1}{e}}$.

Solution:

Let $y = \left(\dfrac{1}{x}\right)^x = x^{-x}$. Taking log both the sides, we get $\log y = \log(x^{-x}) = x\log x$

Differentiate both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = -\left[x\frac{d}{dx}(\log x) + \log x\,1\right] = -\left(\frac{x}{x} + \log x\right) = -(1+\log x)\ ;\ \frac{dy}{dx} = -y(1+\log x)$$

Again differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{dy}{dx}(1+\log x) + y\frac{d}{dx}(1+\log x)\right] = -\left[\frac{dy}{dx}(1+\log x) + y\left(\frac{1}{x}\right)\right] = -\frac{dy}{dx}(1+\log x) - \frac{y}{x}$$

$$= -(-y)(1+\log x)^2 - \frac{y}{x}\ ;\ \frac{d^2y}{dx^2} = y(1+\log x)^2 - \frac{y}{x}$$

$$\frac{d^2y}{dx^2} = x^{-x}(1+\log x)^2 - \frac{x^{-x}}{x} = x^{-x}(1+\log x)^2 - x^{-x-1}$$ Put $\dfrac{dy}{dx} = 0$, we get

$$-y\ (1+\log x) = 0 \Rightarrow 1+\log x = 0 \Rightarrow \log x = -1 \Rightarrow x = e^{-1} = \frac{1}{e}$$

$$\text{Now } \left.\frac{d^2y}{dx^2}\right)_{x=\frac{1}{e}} = -\left(\frac{1}{e}\right)^{-\frac{1}{e}}\left(1+\log\frac{1}{e}\right)^2 - \left(\frac{1}{e}\right)^{-\frac{1}{e}-1} = -(e^{-1})^{-\frac{1}{e}}(1+\log e^{-1})^2 - (e^{-1})^{-\left(\frac{1}{e}+1\right)}$$

$$= -(e)^{\frac{1}{e}}(1-\log e)^2 - (e^{-1})^{-\left(\frac{1}{e}+1\right)} = -e^{\frac{1}{e}}(1-1)^2 - e^{\left(\frac{1}{e}+1\right)} = 0 - e^{\left(\frac{1}{e}+1\right)} < 0$$

$\therefore x = \dfrac{1}{e}$ is a point of local maximum. $\therefore$ local maximum value $= f\left(\dfrac{1}{e}\right) = (e)^{\frac{1}{e}}\ \left(x = \dfrac{1}{e}\right)$.

9. Find the maximum value of $(\log x)/x$ in $0 < x < \infty$.

Solution.

Let $f(x) = y = (\log x)/x$

$$\frac{dy}{dx} = f'(x) = \frac{\left[x\frac{d}{dx}(\log x)\right] - [(\log x)\cdot 1]}{x^2} = \frac{x\frac{1}{x} - \log x}{x^2} = \frac{1-\log x}{x^2}$$

Put $f'(x) = 0 \Rightarrow 1 - \log x = 0 \Rightarrow \log x = 1 \Rightarrow x = e$

$$\frac{d^2y}{dx^2} = f''(x) = \frac{\left(x^2\frac{d}{dx}(1-\log x)\right) - (1-\log x)\frac{d}{dx}(x^2)}{x^4}$$

$$= \frac{x^2\left(-\frac{1}{x}\right) - (1-\log x)2x}{x^4} = \frac{-x-(1-\log x)2x}{x^4} = -\frac{1}{x^3} - \frac{2(1-\log x)}{x^3}$$

$$\text{Now } \left.\frac{d^2y}{dx^2}\right)_{x=e} = -\frac{1}{e^3} - \frac{2}{e^3}(1-\log e) = -\frac{1}{e^3} - 0\ (\log e = 1) = -\frac{1}{e^3} < 0$$

$\therefore x = e$ is a point of maxima and maximum value $= f(e) = \dfrac{\log e}{e} = \dfrac{1}{e}$ $(\log e = 1)$

10. Prove that $x^2\log(1/x)$ has a maximum value when $x = \dfrac{1}{\sqrt{e}}$.

Solution. Let $y = f(x) = x^2\log\left(\dfrac{1}{x}\right) = x^2[\log 1 - \log x] = -x^2\log x$ $[\log 1 = 0]$

$$\frac{dy}{dx} = -\left[\frac{d}{dx}(x^2)\log x + x^2\frac{d}{dx}(\log x)\right] = -\left[2x\log x + \frac{x^2}{x}\right] = -[2x\log x + x] = -x(2\log x + 1)$$

Put $\frac{dy}{dx} = 0$ we get $-x(2\log x + 1) = 0 \Rightarrow x = 0$ or $\log x = \frac{-1}{2} \Rightarrow x = 0$ or $x = e^{-1/2}$

$$\frac{d^2y}{dx^2} = -\left[(2\log x + 1)\cdot 1 + x\frac{d}{dx}(2\log x + 1)\right] = -\left[(2\log x + 1) + x\left(\frac{2}{x}\right)\right]$$
$$= -[2\log x + 1 + 2] = -2\log x - 3$$

$$\left.\frac{d^2y}{dx^2}\right|_{x = e^{-1/2}} = -2\log\left(\frac{1}{\sqrt{e}}\right) - 3 = -2\log(e^{-1/2}) - 3 = -2\left(-\frac{1}{2}\right) - 3 = 1 - 3 = -2 < 0$$

$\therefore$ y is maximum when $x = \frac{1}{\sqrt{e}}$ **Note:** For $x = 0$, $f(x)$ does not exist.

11. If $\frac{dy}{dx} = (x-a)^{2n}(x-b)^{2p+1}$, where n and p are positive integers, show that $x = a$ gives neither a maximum nor a minimum value of y, and $x = b$ gives a minimum value.

Solution.

$\frac{dy}{dx} = (x-a)^{2n}(x-b)^{2p+1}$ Put $\frac{dy}{dx} = 0$; we get $x = a$ or $x = b$

For $x = a$, since $(x-a)^{2n}$ is even, therefore for the point which are sufficiently close to 'a' and to be left of 'a' the sign of $\frac{dy}{dx}$ is same for the points which are sufficiently close to 'a' and to the right of 'a'.

For $x = b$: the points which are sufficiently close to 'b' and to the left of 'b' $\frac{dy}{dx}$ is negative because $2p + 1$ is odd. Also, for the points which are sufficiently close to 'b' and to the right of 'b' $\frac{dy}{dx}$ is positive.

Hence, $x = b$ is the point of minimum.

6.4. [Maximum and Minimum Values in Closed Interval]

Definition:

Absolute Maximum Value : Let $f(x)$ be differentiable and continuous on closed interval $[a\ b]$. Absolute maximum value is the largest value, taken from $f(a), f(p_1), f(p_2), \ldots, f(p_n), f(b)$

where $x = p_1, p_2, \ldots, p_n$ are the points for which $f'(x) = 0$.

Definition:

Absolute Minimum Value : Let $f(x)$ be differentiable and continuous on closed interval $[a\ b]$. Absolute minimum value is the smallest value taken from $f(a), f(p_1), f(p_2), \ldots, f(p_n), f(b)$

where $x = p_1, p_2, \ldots, p_n$ are the points for which $f'(x) = 0$.

12. Find the absolute maximum value and the absolute minimum value of the function.

$$f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25 \text{ in } [0, 3]$$

Solution.

$f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$; $f'(x) = 12x^3 - 24x^2 + 24x - 48$ Put $f'(x) = 0$

$12x^3 - 24x^2 + 24x - 48 = 0 \Rightarrow x^3 - 2x^2 + 2x - 4 = 0 \Rightarrow (x-2)(x^2+2) = 0$

$\Rightarrow x = 2, x = \pm\sqrt{2}\,i$ Neglect $\pm\sqrt{2}\,i$ $\therefore$ Consider the points $a = 0, 2, b = 3$

$f(0) = 25$; $f(2) = -39$; $f(3) = 3(3)^4 - 8(3)^3 + 12(3)^2 - 48(3) + 25$
$= 243 - 216 + 108 - 144 + 25 = 16$

Absolute maximum value $= 25$ at $x = 0$. Absolute minimum value $= -39$ at $x = 2$.

13. Find the absolute maximum and minimum values of $f(x) = x + \sin 2x$ in the interval $[0, 2\pi]$.

Solution.

$f(x) = x + \sin 2x$: $f'(x) = 1 + 2\cos 2x$ Put $f'(x) = 0$ $1 + 2\cos 2x = 0$

$\cos 2x = -\frac{1}{2}$; $2x = \pi - \frac{\pi}{3}, 2x = \pi + \frac{\pi}{3}$ $[0 \le x \le 2\pi]$ $2x = \frac{2\pi}{3}, \frac{4\pi}{3}$ $[0 \le 2x \le 4\pi]$

i.e., $x = \frac{\pi}{3}, \frac{2\pi}{3}$. Consider the points $a = 0, \frac{\pi}{3}, \frac{2\pi}{3}, b = 2\pi$; $f(0) = 0 + 0 = 0$

$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$; $f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

$f(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi$. Absolute maximum value $= 2\pi$ at $x = 2\pi$

Absolute minimum value $= 0$ at $x = 0$

14. If $f(x) = x^4 - 62x^2 + ax + 9$ attains its maximum value in the interval $[0, 2]$ at $x = 1$, find the value of a.

Solution.

$$f(x) = x^4 - 62x^2 + ax + 9$$
$$f'(x) = 4x^3 - 124 + a$$

Now as $f(x)$ attains maximum value at $x = 1$ i.e., $f'(1) = 0$; $4 - 124 + a = 0$; $a = 120$

6.5. [Applications of Maxima and Minima]

15. Find two positive numbers whose product is 64 having minimum sum.

Solution.

Consider two positive numbers x and y. Its given that $xy = 64$. i.e., $y = \frac{64}{x}$ ...(*i*)

Let S be the sum of given two numbers i.e., $S = x + y$

or $S(x) = x + \frac{64}{x}$ from (*i*) $S'(x) = 1 - \frac{64}{x^2}$.

Now put $S'(x) = 0$ is order to minimize the sum

$$S'(x) = 0 \Rightarrow 1 - \frac{64}{x^2} = 0 \Rightarrow x^2 = 64 \Rightarrow x = \pm 8$$

Neglect $x = -8$ (as x and y are positive number). Now consider $S''(x) = \frac{128}{x^3}$

$S''(x) = \frac{128}{(8)^3} = \frac{128}{512} > 0$ $\therefore$ S is minimum at $x = 8$ $y = \frac{64}{8}$ $\therefore y = 8$

$\therefore$ Two positive numbers are $x = 8, y = 8$

16. Show tht the area of a rectangle of given perimeter is maximum when rectangle is a square.

Solution.

Let x be the length and y be the breadth of the rectangle.

Let P be the given perimeter i.e., $P = 2(x + y) \Rightarrow \frac{P}{2} = x + y \Rightarrow y = \frac{P}{2} - x$

Now we have to maximize the area of the rectangle. Let A be the area of the rectangle.

i.e., $A = xy$ i.e., $A(x) = x\left(\frac{P}{2} - x\right) = \frac{xP}{2} - x^2$. Now, $\frac{dA}{dx} = \frac{P}{2} - 2x$

Put $\frac{dA}{dx} = 0$ i.e., $\frac{P}{2} - 2x = 0$ i.e., $P = 4x$

Now when $P = 4x$ then $P = 2(x + y) \Rightarrow 4x = 2x + 2y \Rightarrow 2x = 2y \Rightarrow x = y$

Now consider $\left.\dfrac{d^2A}{dx^2}\right)_{x=y} = (-2) < 0 \quad \therefore\ A$ is maximum when $x = y$

i.e., length is equal to the breadth *i.e.*, rectangle is a square.

17. Prove that the perimeter of a right angled triangle of given hypotenuse is maximum when the triangle is isosceles.

Solution.

Let x, y, h be the two sides and hypotenuse of the right angled triangle respectively.

i.e., $h = \sqrt{x^2 + y^2}$ *i.e.*, $h^2 = x^2 + y^2$ or $y = \sqrt{h^2 - x^2}$

We have to maximize the perimeter of the triangle.

Let P be the perimeter of the triangle. *i.e.*, $P = h + x + y$ or $P(x) = h + x + \sqrt{h^2 - x^2}$

$$\frac{dP}{dx} = 0 + 1 + \frac{d}{dx}\left(\sqrt{h^2 - x^2}\right) = 0 + 1 + \frac{(-2x)}{2\sqrt{h^2 - x^2}} \quad (h \text{ is a constant})$$

$$\frac{dP}{dx} = 1 - \frac{x}{\sqrt{h^2 - x^2}}. \quad \text{Put } \frac{dP}{dx} = 0 \text{ i.e., } 1 - \frac{x}{\sqrt{h^2 - x^2}} = 0$$

$$\Rightarrow x = \sqrt{h^2 - x^2} \Rightarrow x^2 = h^2 - x^2 \Rightarrow 2x^2 = h^2 \Rightarrow x^2 = \frac{h^2}{2} \Rightarrow x = \frac{h}{\sqrt{2}}. \text{ Now consider}$$

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left(1 - \frac{x}{\sqrt{h^2 - x^2}}\right) = 0 - \frac{d}{dx}\left(\frac{x}{\sqrt{h^2 - x^2}}\right) = -\left[\frac{\left(\sqrt{h^2 - x^2}\right)(1) - x\dfrac{d}{dx}\left(\sqrt{h^2 - x^2}\right)}{(h^2 - x^2)}\right]$$

$$= -\left[\frac{\sqrt{h^2 - x^2}\cdot 1 - \dfrac{x}{2}\dfrac{\frac{d}{dx}(-x^2)}{\sqrt{h^2 - x^2}}}{(h^2 - x^2)}\right] = -\left[\frac{\sqrt{h^2 - x^2} - \dfrac{x}{2}\dfrac{(-2x)}{\sqrt{h^2 - x^2}}}{(h^2 - x^2)}\right]$$

$$= -\left[\frac{(h^2 - x^2) + x^2}{(h^2 - x^2)^{3/2}}\right] = -\frac{h^2}{(h^2 - x^2)^{3/2}}$$

$$\left.\frac{d^2P}{dx^2}\right)_{x=\frac{h}{\sqrt{2}}} = -\frac{h^2}{\left(h^2 - \frac{h^2}{2}\right)^{3/2}} = -\frac{h^2}{\left(\frac{h^2}{2}\right)^{3/2}} = -\frac{h^2}{\dfrac{h^3}{(2)^{3/2}}} - \frac{(2)^{3/2}}{h} < 0$$

Now when $x = \dfrac{h}{\sqrt{2}}$, $y = \sqrt{h^2 - \dfrac{h^2}{2}} = \dfrac{h}{\sqrt{2}}$

i.e., $x = y$ *i.e.*, two sides of the triangle are equal *i.e.*, hypotenuse is maximum when triangles is isosceles.

18. Buses are to be charted for excursion. The price per ticket is ` 30 for the first 200 tickets with 10 paise rebate for every ticket per passenger in excess of 200. What number of passengers will produce the maximum gross income for the company?

Solution.

Let x be the number of passengers in excess of 200

Rebate per ticket Rs. $\left(x \times \dfrac{1}{10}\right)$ $\quad \therefore$ Price per ticket = Rs. $\left(30 - \dfrac{x}{10}\right)$

Let y be the gross income, then $y = (200 + x) \times \left(30 - \dfrac{x}{10}\right) = 6000 + 10x - \dfrac{x^2}{10}$

Now $\dfrac{dy}{dx} = 10 - \dfrac{2x}{10}$ *i.e.*, $10 - \dfrac{x}{5}$. Put $\dfrac{dy}{dx} = 0$ *i.e.*, $10 - \dfrac{x}{5} = 0$ *i.e.*, $x = 50$

Now $\dfrac{dy^2}{dx^2} = -\dfrac{2}{10} = -\dfrac{1}{5} < 0$

$\therefore\ x = 50$ for maximum gross income. and total number of passenger should be $= 200 + 50 = 250$.

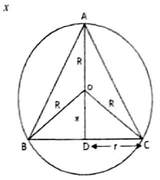
19. Find the volume of largest cone. that can be inscribed in a sphere of radius R.

Solution.

Clearly $OC = BO = AO = R$ (radius of sphere). Let $OD = x$ and let $DC = r$ (base radius)

We have to maximize the volume of the sphere. Let V be the volume of sphere.

i.e., $V = \dfrac{1}{3}\pi r^2 h$ where h is the height of the cone. *i.e.*, $h = R + x$



Consider $\Delta\, ODC$. Clearly $R^2 = x^2 + r^2$ *i.e.*, $r^2 = R^2 - x^2$

$$\therefore \quad V = \frac{1}{3}\pi(R^2 - x^2)(R + x) = \frac{1}{3}\pi(R^3 + R^2x - x^2R - x^3)$$

$$\frac{dV}{dx} = \frac{\pi}{3}[0 + R^2 - 2xR - 3x^2] \quad [R \text{ is constant}]$$

$$= \frac{\pi}{3}[R^2 - 2xR - 3x^2]$$

Put $\dfrac{dV}{dx} = 0$ *i.e.*, $R^2 - 2xR - 3x^2 = 0$

i.e., $R^2 - 3xR + xR - 3x^2 = 0$ *i.e.*, $R(R - 3x) + x(R - 3x) = 0$ *i.e.*, $(R + x)(R - 3x) = 0$

i.e., $R = -x$ or $R = 3x$ *i.e.*, $x = \dfrac{R}{3}$ or $x = -R$ *i.e.*, $x = \dfrac{R}{3}$ (Neglect $x = -R$ as $x + R \neq 0$)

Now consider $\dfrac{d^2V}{dx^2} = \dfrac{\pi}{3}[-2R - 6x] = -\dfrac{2\pi}{3}(R + 3x)$

$$\left.\frac{d^2V}{dx^2}\right)_{x=\frac{R}{3}} = \frac{2\pi}{3}\left(R + \frac{3R}{3}\right) = -\frac{2\pi}{3}(R + R) = -\frac{4}{3}R\pi < 0$$

$\therefore$ when $x = \dfrac{R}{3}$, V is maximum. Now maximum volume of the cone

$$= \frac{1}{3}\pi\left(R^2 - \frac{R^2}{9}\right)\left(R + \frac{R}{3}\right) = \frac{1}{3}\pi\left(\frac{8R^2}{9}\right)\left(\frac{4R}{3}\right) = \frac{32\pi R^3}{81}$$

$$= \frac{8}{27}\left[\frac{4}{3}\pi R^3\right] = \frac{8}{27} \text{ [Volume of sphere]}$$

20. Show that the volume of the greatest cylinder which can be considered in a cone of height h and semivertical angle α is $\dfrac{4}{27}\pi h^3 \tan^2\alpha$.

Solution.

Consider the given cone ABC, with height 'h' and semivertical angle 'α'.

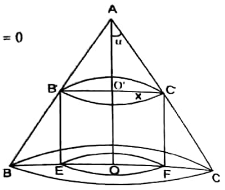
Cylinder whose radius of base $= x$ is inscribed in the cone.

Now clearly OO' = height of the cylinder *i.e.*, $OO' = AO - AO'$ ...(*i*)

Now $AO = h$ (height of the cone) also $\cot\alpha = \dfrac{AO'}{x}$ *i.e.*, $AO' = x\cot\alpha$

Substituting in (*i*), we get $OO' = h - x\cot\alpha$

We have to maximize the volume of the cylinder. Let V be the volume of the cylinder.



i.e., $V = \pi x^2 (OO') = \pi x^2 (h - x \cot \alpha) = \pi x^2 h - \pi x^3 \cot \alpha$

Now $\frac{dV}{dx} = 2\pi x h - 3\pi x^2 \cot \alpha$. Put $\frac{dV}{dx} = 0$ *i.e.*, $2\pi x h - 3\pi x^2 \cot \alpha = 0$

i.e., $2\pi x h = 3\pi x^2 \cot \alpha$ *i.e.*, $2h = 3x \cot \alpha$

i.e. $x = \frac{2h}{3 \cot \alpha} = \frac{2}{3} h \tan \alpha$ Now $\frac{d^2V}{dx^2} = 2\pi h - 6\pi x \cot \alpha$;

$$\left.\frac{d^2V}{dx^2}\right)_{x = \frac{2}{3}h \tan \alpha} = 2\pi h - 6\pi \left(\frac{2}{3} h \tan \alpha \cdot \cot \alpha\right)$$

$$= 2\pi h - 4\pi h (\tan \alpha \cdot \cot \alpha) = -2\pi h < 0$$

$\therefore$ V is maximum when $\alpha = \frac{2}{3} h \tan \alpha$; Maximum volume of cylinder is given by

$$V = \pi\left(\frac{2}{3} h \tan \alpha\right)^2 \left(h - \frac{2h}{3} \tan \alpha \cdot \cot \alpha\right) = \pi\left(\frac{4}{9} h^2 \tan^2 \alpha\right)\left(\frac{h}{3}\right) = \frac{4}{27} h^3 \tan^2 \alpha$$

21. Show that the semivertical angle of the cone of maximum volume and of given slant height is $\tan^{-1} \sqrt{2}$.

Solution.

Let ABC be the cone whose slant height is l (given) and semivertical angle be θ.

Now consider ΔAOC $\sin \theta = \frac{OC}{l}$ *i.e.*, $OC = \sin \theta \, l$

i.e., radius of base of cone $= OC = l \sin \theta$ *i.e.*, height of the cone $= AO = \cos \theta \, l$

We have to maximise the radius of the cone. Let V be the volume of the cone.

$$\therefore \quad V = \frac{1}{3}\pi (OC)^2 AO = \frac{1}{3}\pi (l \sin \theta)^2 l \cos \theta = \frac{1}{3}\pi l^3 \sin^2 \theta \cdot \cos \theta$$

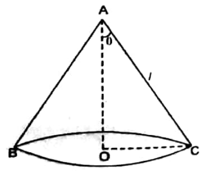
$$\frac{dV}{d\theta} = \frac{1}{3}\pi l^3 \frac{d}{d\theta}(\sin^2 \theta \cdot \cos \theta) = \frac{1}{3}\pi l^3 \left[\sin^2 \theta \frac{d}{d\theta}(\cos \theta) + \cos \theta \frac{d}{d\theta}(\sin^2 \theta)\right]$$

$$= \frac{1}{3}\pi l^3 [\sin^2 \theta (-\sin \theta) + \cos \theta \cdot 2 \sin \theta \cdot \cos \theta] = \frac{1}{3}\pi l^3 [-\sin^3 \theta + 2 \sin \theta \cos^2 \theta]$$

Put $\frac{dV}{d\theta} = 0$ *i.e.*, $-\sin^3 \theta + 2 \sin \theta \cdot \cos^2 \theta = 0$ *i.e.*, $\sin \theta (2 \cos^2 \theta - \sin^2 \theta) = 0$

i.e., $\sin \theta (2 \cos^2 \theta - \sin^2 \theta) = 0$; $\sin \theta = 0$ or $\tan^2 \theta = 2$

$\Rightarrow$ $\theta = 0$ or $\tan \theta = \sqrt{2}$ neglect $\theta = 0$



Now consider $\frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}\left[\frac{d}{d\theta}(-\sin^3 \theta) + \frac{d}{d\theta}(2 \sin \theta \cos^2 \theta)\right]$

$$= \frac{\pi l^3}{3}[-3 \sin^2 \theta \cos \theta + 2 \cos^3 \theta - 4 \cos \theta \sin^2 \theta]$$

$$= \frac{\pi l^3}{3}[-3 \sin^2 \theta \cdot \cos \theta - 4 \sin^2 \theta \cdot \cos \theta + 2 \cos^3 \theta]$$

$$= \frac{\pi}{3} l^3 [2 \cos^3 \theta - 7 \sin^2 \theta \cdot \cos \theta]$$

$$= \frac{\pi}{3} l^3 \cos^3 \theta \left[2 - 7\frac{\sin^2 \theta}{\cos^2 \theta}\right] = \frac{\pi}{3} l^3 \cos^3 \theta [2 - 7 \tan^2 \theta]$$

Clearly $\left.\frac{d^2V}{d\theta^2}\right)_{\tan \theta = \sqrt{2}} = -ve$ [$\cos \theta$ is $+ve$ since θ is acute. $2 - 7(2) = 2 - 14 = -12 < 0$]

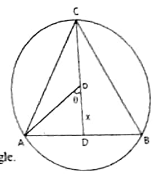
$\therefore$ Volume V is maximum for $\tan \theta = \sqrt{2}$ *i.e.*, $\theta = \tan^{-1} \sqrt{2}$

22. Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.

Solution.

Let O be the centre of given circle and let R be the radius. Let ABC be the triangle inscribed in *a* given circle. Let $OD = x$

Area will be maximum when C is at maximum distance from AB. *i.e.*, when C lies on diameter $\perp$ to AB *i.e.*, $CD \perp AB$ $\therefore$ ΔABC is isosceles. Consider the right angled triangle ADO.



We have $AO^2 = OD^2 + AD^2$

i.e., $R^2 = x^2 + AD^2$ *i.e.*, $AD^2 = R^2 - x^2$ *i.e.*, $AD = \sqrt{R^2 - x^2}$.

Now, $AB = 2AD = 2\sqrt{R^2 - x^2}$

Also, $CD = CO + x = R + x$ [CO = Radius of the circle]

We have to maximize the area of the triangle. Let A be the area of the triangle.

$$\therefore \quad A = \frac{1}{2} \times AD \times CD = \frac{1}{2} \times 2\sqrt{R^2 - x^2}\,(R + x) = \sqrt{R^2 - x^2}\,(R + x)$$

$$\frac{dA}{dx} = \sqrt{R^2 - x^2} \frac{d}{dx}(R + x) + (R + x)\frac{d}{dx}\sqrt{R^2 - x^2}$$

$$= \sqrt{R^2 - x^2}\,(-1) + \frac{(R + x)(-2x)}{2\sqrt{R^2 - x^2}} = \sqrt{R^2 - x^2} - \frac{x(R + x)}{\sqrt{R^2 - x^2}} = \frac{R^2 - 2x^2 - xR}{\sqrt{R^2 - x^2}}$$

Put $\frac{dA}{dx} = 0$ *i.e.*, $R^2 - 2x^2 - xR = 0$ *i.e.*, $R^2 - xR - 2x^2 = 0$ *i.e.*, $R^2 - 2xR + xR - 2x^2 = 0$

i.e., $R(R - 2x) + x(R - 2x) = 0$ *i.e.*, $(R + x)(R - 2x) = 0$ *i.e.*, $R - 2x = 0$ $(R + x \neq 0)$ *i.e.*, $x = \frac{R}{2}$

$$\frac{d^2A}{dx^2} = \frac{\sqrt{R^2 - x^2} \frac{d}{dx}(R^2 - Rx - 2x^2) - (R^2 - Rx - 2x^2)\frac{d}{dx}\sqrt{R^2 - x^2}}{(R^2 - x^2)}$$

$$= \frac{\sqrt{R^2 - x^2}\,(-R - 4x) - (R^2 - Rx - 2x^2)\frac{1(-2x)}{2\sqrt{R^2 - x^2}}}{(R^2 - x^2)}$$

$$= \frac{\sqrt{R^2 - x^2}\,(-R - 4x)}{(R^2 - x^2)} - \frac{(R^2 - Rx - 2x^2)(-x)}{\sqrt{R^2 - x^2} \cdot (R^2 - x^2)}$$

$$= \frac{-(R + 4x)}{\sqrt{R^2 - x^2}} - \frac{(R^2 - Rx - 2x^2)(-x)}{(R^2 - x^2)^{3/2}}$$

$$\left.\frac{d^2A}{dx^2}\right)_{x = \frac{R}{2}} = -\frac{(R + 2R)}{\sqrt{R^2 - \frac{R^2}{4}}} - \frac{\left(R^2 - \frac{R^2}{2} - \frac{2R^2}{4}\right)\left(-\frac{R}{2}\right)}{\left(R^2 - \frac{R^2}{4}\right)^{3/2}} = -\frac{3R}{\sqrt{\frac{3R^2}{4}}} - 0 = \frac{-6R}{\sqrt{3}R} = -2\sqrt{3} < 0$$

$\therefore$ A is maximum when $x = \frac{R}{2}$. Now, $AD = \sqrt{R^2 - x^2} = \sqrt{R^2 - \frac{R^2}{4}} = \frac{\sqrt{3}R}{2}$

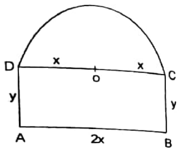
Consider ΔAOD, $\tan\theta = \dfrac{AD}{OD} = \dfrac{\frac{\sqrt{3}}{2}R}{\frac{R}{2}} = \sqrt{3}$ i.e.. $\theta = 60^\circ$ $\therefore \angle ACB = \theta = 60^\circ = \angle C$

Now ΔABC is isosceles

$\therefore AC = CB$ i.e., $\angle A = \angle B$ i.e.. $\angle A = \angle B = \angle C = 60^\circ$ $\therefore \Delta ACB$ is equilateral.

23. A window is in the form of a rectangle surmounted by a semicircle. If the perimeter is 30 metres, find the dimensions so that greatest possible amount of light may be admitted.

Solution.



Let x be radius of semicircle. $\therefore$ We have $AB = 2x$

Let $AD = BC = y$

Given: Perimeter of window = 30 m i.e., $2(x+y) + \pi \cdot x = 30$

$\Rightarrow 2y = 30 - \pi x - 2x$ We have to maximize the area of the window.

i.e.. A = area of rectangle + area of semicircle $= 2x \cdot y + \dfrac{\pi}{2}x^2$

$$A(x) = x(30 - 2x - \pi x) + \frac{1}{2}\pi x^2 = 30x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2 = 30x - 2x^2 - \frac{\pi x^2}{2}$$

Put $\dfrac{dA}{dx} = 0$ $\quad \dfrac{dA}{dx} = 30 - 4x - \pi x$ $\quad \dfrac{dA}{dx} = 0 \Rightarrow -x(4+\pi) = -30 \Rightarrow x = \dfrac{30}{4+\pi}$

Now, $\dfrac{d^2A}{dx^2} = -4 - \pi = -ve$ $\quad \therefore A$ is maximum when $x = \dfrac{30}{4+\pi}$

Now $2y = 30 - \pi x - 2x = 30 - \dfrac{\pi \cdot 30}{4+\pi} - \dfrac{2(30)}{4+\pi} = 30 + \dfrac{-30\pi - 60}{4+\pi} = \dfrac{30(4+\pi) - 30\pi - 60}{4+\pi}$

$2y = \dfrac{120 + 30\pi - 30\pi - 60}{4+\pi}$; $y = \dfrac{30}{4+\pi}$ i.e., $x = y = \dfrac{30}{4+\pi}$ m

24. A wire of given length is cut into the portions which are divided into shapes of a circle and a square respectively. Show that the sum of the areas of two circle and the square will be least when the side of the square is equal to the diameter of the circle.

Solution.

- Let 'l' be the lengths of the given wire.

Now this wire is cut into portions of (1) circle (let x be its radius), and (2) square (let y be its side)

Now clearly l = circumference of circle + circumference square. $l = 2\pi x + 4y$ ...(*i*)

We have to minimise the sum of areas of circle and square.

Let A be the sum of areas of circle and square

$$A = \pi x^2 + y^2 \quad [\pi x^2 = \text{area of circle } y^2 = \text{area of square}] = \pi x^2 + \left[\frac{l - 2\pi x}{4}\right]^2 \qquad \text{from } (i)$$

$$\frac{dA}{dx} = 2\pi x + 2\left[\frac{l-2\pi x}{4}\right]\left[-\frac{2\pi}{4}\right] = 2\pi x - \frac{4\pi}{16}(l - 2\pi x) = 2\pi x - \frac{\pi}{4}(l - 2\pi x) = 2\pi x - \frac{\pi l}{4} + \frac{\pi^2 x}{2}$$

Put $\dfrac{dA}{dx} = 0$ i.e., $2\pi x + \dfrac{\pi^2 x}{2} = \dfrac{\pi l}{4}$; $\dfrac{x(4\pi + \pi^2)}{2} = \dfrac{\pi l}{4}$; $x(4\pi + \pi^2) = \dfrac{\pi l}{2}$

$x = \dfrac{\pi l}{2\pi(4+\pi)} = \dfrac{l}{2(4+\pi)}$. Now, $\dfrac{d^2A}{dx^2} = 2\pi + \dfrac{\pi^2}{2} = +ve$. Now $4y = l - 2\pi x$

$4y = l - \dfrac{2\pi l}{2(4+\pi)} = l - \dfrac{\pi l}{(4+\pi)} = \dfrac{4l + l\pi - \pi l}{4+\pi}$; $y = \dfrac{4l}{4(4+\pi)} = \dfrac{l}{4+\pi} = \dfrac{2l}{2(4+\pi)} = 2x$

Now A is minimum when $y = 2x$ i.e.. side of square = diameter of circle.

Exercise

1. Find maximum and minimum values of the function if any:
   (a) $f(x) = x^3 + 1 \ \forall x \in R$
   (b) $f(x) = \sin 2x + 5 \ \forall x \in R$ **[Ans. no minimum and maximum value]**
   [Ans. Max = 6, Min = 4]
2. Find the points of local maxima and local minima for the function $f(x) = (x-2)^4 (x+1)^3$
   [Ans. Point of local maxima = 2/7, Point of local minima : x=2]
3. Find the points of local maxima and local minima for the functions $f(x) = \sin 2x \ ; 0 < x < \pi$
   [Ans. Point of local maxima = $\dfrac{\pi}{4}$, Point of local minma = $\dfrac{3\pi}{4}$]
4. For the following functions, find points of local maxima, local minima, local maximum and local minimum values
   (1) $f(x) = x^3 - 6x^2 + 9x + 15$ **[Ans. Point of local maxima = 1, Local max value = 19, Point of local minima = 3, Local min value = 15]**
   (2) $f(x) = (x+1)(x+2)^{1/3}\ x \geq -2$ **[Ans. Point of local minima = $\dfrac{-7}{4}$, Local min value = $\dfrac{-3}{4^{4/3}}$]**
5. Find the absolute maximum value and absolute minimum value for $f(x) = (x-2)\sqrt{x-1}$ in $[1, 9]$
   [Ans. Absolute max value = 14, Absolute min value = $\dfrac{-3}{4^{4/3}}$]
6. Show that of all the rectangles of given area, the square has smallest perimeter.
7. Prove that the area of right-angled triangle of given hypotenuse is maximum when triangle is isosceles.
8. Find the value of the largest cylinder that can be inscribed in the sphere of radius r cm **[Ans. $\dfrac{4\pi r^3}{3\sqrt{3}}$]**
9. Show that the semi-vertical angle of a right circular cone of given surface area and maximum volume is $\sin^{-1}(1/3)$.
10. Find the area of the greatest isosceles triangle that can be inscribed in given eclipse having the vertex coincident with one end of the major axis. **[Ans. $\dfrac{3\sqrt{3}ab}{4}$]**
11. Find the shortest distance of the point (0, c) from the parabola $y = x^2, 0 \leq c \leq 5$. **[Ans. $\dfrac{\sqrt{4c-1}}{2}$]**
12. Find the point on the curve $y^2 = 4ax$ which is nearest to the point (2, -8). **[Ans. (4, -4)]**
13. The combined resistance R of two resistors R_1 and R_2 ($R_1, R_2 > 0$) is given by $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$. If $R_1 + R_2 = a$ (constant), show that the maximum resistance R is obtained by choosing $R_1 = R_2$.
14. An open box with a sqaure base is to be made out of given quantity of card-board of area a^2 square units. Show that maximum volume of box is $\dfrac{a^3}{6\sqrt{3}}$ cubic units.
15. Show that the height of the cylinder of maximum volume that can be inscribed in sphere of radius a is $\dfrac{2a}{\sqrt{3}}$.

CHAPTER - 7

[Successive Differentiation]

7.1. [Definition]

Let $y = f(x)$ [*i.e.*, y is a function of x]. Then

1. $y_1 = \frac{dy}{dx}$ is called its first derivative

2. $y_2 = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ is called its second derivative

3. $y_3 = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$ is called its third derivative. Continuing in the same manner we get

$$y_n = \frac{d}{dx}\left(\frac{d^{n-1}}{dx^{n-1}}\right) = \frac{d^ny}{dx^n},\ n\text{th derivative of } y = f(x)$$

Examples

1. Find the second derivative of $y = \frac{\log x}{x}$.

Solution.

Differentiating $y = \frac{\log x}{x}$ w.r.t. x we get first derivative $y = \frac{dy}{dx} = \frac{x\frac{d}{dx}(\log x) - (\log x)1}{(x)^2}$

First derivative $y_1 = \frac{dy}{dx} = \frac{x\frac{d}{dx}(\log x) - (\log x)\frac{d}{dx}(x)}{(x)^2}$; $y_1 = \frac{x\frac{1}{x} - \log x}{x^2}$; $y_1 = \frac{1 - \log x}{x^2}$

Now in order to find second derivative y_2, differentiate y_1 w.r.t. x.

$$y_2 = \frac{d^2y}{dx^2} = \frac{x^2\frac{d}{dx}(1 - \log x) - [(1 - \log x)\frac{d}{dx}(x^2)]}{x^4}$$

$$y_2 = \frac{x^2\left(-\frac{1}{x}\right) - [(1 - \log x)(2x)]}{x^4} = \frac{-x - [2x - \log x\, 2x]}{x^4} = \frac{-3x + 2x\log x}{x^4} = \frac{-3 + 2\log x}{x^3}$$

2. If $y = a\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$.

Solution.

$$y_1 = \frac{d}{dx}[a\sin(\log x)] = a\cos(\log x)\frac{d}{dx}(\log x) = \frac{a\cos(\log x)}{x}$$

$$y_2 = \frac{d}{dx}(y_1) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{a\cos(\log x)}{x}\right) = a\left[\frac{x\frac{d}{dx}[\cos(\log x)] - [\cos(\log x)]\frac{d}{dx}(x)}{x^2}\right]$$

$$= a\left[\frac{x(-\sin(\log x))\frac{d}{dx}(\log x) - \cos(\log x)}{x^2}\right] = a\left[\frac{x(-\sin(\log x))\frac{1}{x} - \cos(\log x)}{x^2}\right]$$

$$= a\left[\frac{-\sin(\log x) - \cos(\log x)}{x^2}\right]$$

Now $x^2y_2 + xy_1 + y$

$$= x^2\left[\frac{a(-\sin(\log x) - \cos(\log x))}{x^2}\right] + x\frac{a\cos(\log x)}{x} + y$$

$= -a\sin(\log x) - a\cos(\log x) + a\cos(\log x) + a\sin(\log x)$ $\quad [\because y = a\sin(\log x)]$

$= -a\sin(\log x) - a\cos(\log x) + a\cos(\log x) + a\sin(\log x) = 0$

7.2 [Standard Results of the *n*th Derivatives of Standard Function]

1. The nth derivative of $y = (ax + b)^m$

Proof : $y = (ax + b)^m$. Differentiate w.r.t. x we get $y_1 = m(ax + b)^{m-1}a = ma(ax + b)^{m-1}$

Again differentiate w.r.t. x we get

$$y_2 = ma(m - 1)(ax + b)^{m-2}a = ma^2(m - 1)(ax + b)^{m-2} = m(m - 1)a^2(ax + b)^{m-2}$$

Continuing in the same manner, we get

$$y_n = m(m - 1)(m - 2)\ldots(m - (n - 1))a^m(ax + b)^{m-n} = \frac{m!}{(m - n)!}a^n(ax + b)^{m-n}$$

Remark 1:

If $m = n$

$$y_n = n(n - 1)(n - 2)\ldots(n - n + 1)a^n(ax + b)^{n-n}$$

$$= \frac{[n(n - 1)(n - 2)\ldots 3.2.1.]}{(n - n)!}a^n(ax + b)^0 = \frac{n!}{0!}a^n \cdot 1 = a^n n! \quad [\because 0! = 1]$$

Remark 2:

If m is a positive integer $< n$, then $y_n = 0$

Remark 3:

If $m = -1$; $y = \frac{1}{ax + b}$ differentiate w.r.t. x we get $y_1 = -\frac{a}{(ax + b)^2} = \frac{(-1)(a)}{(ax + b)^2}$

Again differentiate w.r.t. x, we get $y_2 = \frac{(-1)(-2)a^2 \cdot a}{(ax + b)^3} = \frac{(-1)^2\, 2!\, .a^2}{(ax + b)^3}$

In general $y_n = \frac{(-1)^n n! a^n}{(ax + b)^{n+1}}$,

2. The nth derivative of $(ax + b)$ and $\sin(ax + b)$

Proof: Let $y = \cos(ax + b)$ differentiate w.r.t. x we get

$$y_1 = -a\sin(ax + b) = a\cos\left(\frac{\pi}{2} + ax + b\right) \quad \left[\because \cos\left(\frac{\pi}{2} + \theta\right) = \sin\theta\right]$$

Now again differentiating w.r.t. x we get

$$y_2 = -a^2\sin\left(\frac{\pi}{2} + ax + b\right) = a^2\cos\left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right) = a^2\cos(ax + b + \pi) = a^2\cos\left(ax + b + \frac{2}{2}\pi\right)$$

Now again differentiate w.r.t. x we get

$$y_3 = -a^3\sin\left(ax + b + \frac{2}{2}\pi\right) = a^3\cos\left(ax + b + \frac{2}{2}\pi + \frac{\pi}{2}\right) = a^3\cos\left(ax + b + \frac{3\pi}{2}\right)$$

In general, $y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$. Now consider $y = \sin(ax + b)$

Differentiate it w.r.t. x we get $y_1 = a\cos(ax + b)$

$$= a\sin\left(ax + b + \frac{\pi}{2}\right) \qquad \left[\because \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta\right]$$

Now again differentiate w.r.t. x, we get

$$y_2 = a^2\cos\left(ax + b + \frac{\pi}{2}\right) = a^2\sin\left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= a^2\sin(ax + b + \pi) = a^2\sin\left(ax + b + \frac{2\pi}{2}\right)$$

Now again differentiating it w.r.t. x, we get

$$y_3 = a^3\cos\left(ax + b + \frac{2\pi}{2}\right) = a^3\sin\left(ax + b + \frac{2\pi}{2} + \frac{\pi}{2}\right) = a^3\sin\left(ax + b + \frac{3\pi}{2}\right)$$

$\therefore$ In general we have, $y_n = a^n\sin\left(ax + b + \frac{n\pi}{2}\right)$

3. The *nth* derivative of $e^{ax}\sin(bx + c)$

Proof: $y = e^{ax}\sin(bx + c)$

Differentiate it w.r.t. x, are get $y_1 = e^{ax}\cos(bx + c)\,b + \sin(bx + c)\,ae^{ax}$

$$= e^{ax}[a\sin(bx + c) + b\cos(bx + c)]$$

Put $a = r\cos\theta$, $b = r\sin\theta$, we get $r = (a^2 + b^2)^{1/2}$; $\theta = \tan^{-1}(b/a)$

$$\therefore \quad y_1 = e^{ax}[r\cos\theta\sin(bx + c) + r\sin\theta\cos(bx + c)]$$

$$= re^{ax}[\sin(bx + c)\cos\theta + \sin\theta\cos(bx + c)]$$

$$= re^{ax}\sin(bx + c + \theta) \qquad [\sin(A + B) = \sin A\cos B + \cos A\sin B]$$

Again differentiating it w.r.t. x we get

$$y_2 = re^{ax}[\cos(bx + c + \theta)\cdot b] + ra\,e^{ax}\sin(bx + c + \theta)$$

$$= re^{ax}[a\sin(bx + c + \theta) + b\cos(bx + c + \theta)]$$

$$= re^{ax}[r\cos\theta\sin(bx + c + \theta) + r\sin\theta\cos(bx + c + \theta)]$$

$$= r^2e^{ax}[\cos\theta\sin(bx + c + \theta) + \sin\theta\cos(bx + c + \theta)]$$

$$= r^2e^{ax}\sin(bx + c + \theta + \theta) = r^2e^{ax}\sin(bx + c + 2\theta)$$

$\therefore$ In general we get $y_n = r^n e^{ax}\sin(bx + c + n\theta)$

4. The *n*th derivative of $\log(ax + b)$

Proof: $y = \log(ax + b)$

Differentiating it w.r.t. x we get $\quad y_1 = \dfrac{a}{(ax + b)} = \dfrac{(-1)^{1-1}\,0!\,a^1}{(ax + b)^1}$

Again differentiating it w.r.t. x we get $\quad y_2 = -\dfrac{a^2}{(ax + b)^2} = \dfrac{(-1)^{2-1}(2-1)!}{(ax + b)^2}a^2$

Again differnetiating it w.r.t. x; we get $\quad y_3 = \dfrac{-(-2)a^3}{(ax + b)^3} = \dfrac{(-1)^{3-1}(3-1)!\,a^3}{(ax + b)^3}$

In general we can write $\quad y_n = \dfrac{(-1)^{n-1}(n-1)!\,a^n}{(ax + b)^n}$

5. The *n*th *derivatie* of e^{ax}

Proof: $y = e^{ax}$: $y_1 = ae^{ax}$

Again differentiating it w.r.t. x we get $\quad y_2 = a\dfrac{d}{dx}(e^{ax}) = a\cdot ae^{ax} = a^2e^{ax}$

In general we can write $\quad y_n = a^n e^{ax}$

Remark 1: If $a = 1$; $\dfrac{d^n}{dx^n}(e^x) = e^x$

Remark 2: If $y = a^x = e^{x\log a}$. Then $\dfrac{d^n}{dx^n}(a^x) = (\log_e a)^n\cdot a^x$

Examples

3. **Find n^{th} differential coefficients of $\sin ax\cos bx$**

Solution.

Let $y = \sin ax\cos bx$

$$= \frac{1}{2}[2\sin ax\cos bx] = \frac{1}{2}[\sin(a + b)x + \sin(a - b)x]$$

Now $y_n = \dfrac{d^n}{dx^n}(\sin ax\cos bx)$

$$= \frac{1}{2}\left[\frac{d^n}{dx^n}(\sin(a + b)x) + \frac{d^n}{dx^n}(\sin(a - b)x)\right]$$

$$= \frac{1}{2}\left[\frac{d^n}{dx^n}[\sin((a + b)x + 0)\right] + \frac{d^n}{dx^n}[\sin(a - b)x + 0]$$

$$= \frac{1}{2}\left[(a + b)^n\sin\left((a + b)x + \frac{n\pi}{2}\right) + (a - b)^n\sin((a - b)x + \frac{n\pi}{2})\right]$$

4. **Find n^{th} derivative of $\dfrac{x^2 + 4x + 1}{x^3 + 2x^2 - x - 2}$.**

Solution.

Let $y = \dfrac{x^2 + 4x + 1}{x^3 + 2x^2 - x - 2} = \dfrac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 2)}$

$$\therefore \quad \frac{x^2 + 4x + 1}{(x - 1)(x - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 2}$$

$\therefore \quad x^2 + 4x + 1 = A(x + 1)(x + 2) + B(x - 1)(x + 2) + C(x - 1)(x + 1)$

Put $x = 1$ $\qquad 6 = A(2)(3) \Rightarrow 6 = A(6) \Rightarrow A = 1$

Put $x = -1$ $\qquad -2 = B(-2)(1) \Rightarrow B = 1$

Put $x = -2$ $\qquad -3 = C(-3)(-1) \Rightarrow C = -1$ *i.e.*, $A = 1; B = 1; C = -1$

$$\therefore \quad y = \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 2)} = \frac{1}{x - 1} + \frac{1}{x + 1} - \frac{1}{x + 2}$$

$$y_n = \frac{d^n}{dx^n}\left(\frac{1}{x - 1}\right) + \frac{d^n}{dx^n}\left(\frac{1}{x + 1}\right) + \frac{d^n}{dx^n}\left(-\frac{1}{x + 2}\right) = \frac{(-1)^n n!(1)^n}{(x - 1)^{n+1}} + \frac{(-1)^n n!(1)^n}{(x + 1)^{n+1}} - \frac{(-1)^n n!(1)^n}{(x + 2)^{n+1}}$$

$$(-1)^n n! = \left[\frac{1}{(x - 1)^{n+1}} + \frac{1}{(x + 1)^{n+1}} - \frac{1}{(x + 2)^{n+1}}\right]$$

5. **Find n^{th} derivative of $\dfrac{x^4}{(x-1)(x-2)}$.**

Solution.

Let $y = \dfrac{x^4}{(x-1)(x-2)}$. Dividing numerator by denominator we get $y = x^2 + 3x + 7 + \dfrac{15x-14}{(x-1)(x-2)}$

Consider $\dfrac{15x-14}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$: $15x - 14 = A(x-2) + B(x-1)$

Put $x = 2$ $\quad 16 = B(1) \Rightarrow B = 16$

Put $x = 1$ $\quad 1 = A(-1) + 0 \Rightarrow A = -1$

$$\frac{15x-14}{(x-1)(x-2)} = -\frac{1}{x-1} + \frac{16}{x-2} \qquad \therefore y = x^2 + 3x + 7 - \frac{1}{x-1} + \frac{16}{x-2}$$

$$= (x+0)^2 + 3(x+0)^1 + 7 - \frac{1}{(x-1)} + \frac{16}{(x-2)}$$

$$y_n = 0 + 0 + 0 - \frac{(-1)^n n!(1)^n}{(x-1)^{n+1}} + \frac{16(-1)^n n!(1)^n}{(x-2)^{n+1}} \quad (\text{if } n > 2)$$

$$= \frac{16(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x-1)^{n+1}} = (-1)^n n!\left[\frac{16}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right]$$

6. **Find n^{th} derivative of $e^x \sin^4 x$**

Solution.

Let $y = e^x \sin^4 x = e^x((\sin^2 x)^2) = e^x\left(\frac{1}{2}\cdot 2\sin^2 x\right)^2$

$$= \frac{1}{4}e^x(1-\cos 2x)^2 = \frac{1}{4}e^x[1 + \cos^2 2x - 2\cos 2x]$$

$$= \frac{e^x}{4}\left[1 + \left(\frac{1}{2}\cdot 2\cos^2 2x\right) - 2\cos 2x\right]$$

$$= \frac{e^x}{4}\left[1 - 2\cos 2x + \frac{1}{2}[1 + \cos 4x]\right] = \frac{e^x}{4}\left[1 - 2\cos 2x + \frac{1}{2} + \frac{\cos 4x}{2}\right]$$

$$= e^x\left[\frac{1}{4} - \frac{2}{4}\cos 2x + \frac{1}{8} + \frac{\cos 4x}{8}\right] = e^x\left[\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{\cos 4x}{8}\right]$$

$$\therefore \quad y_n = \frac{3}{8}e^x - \frac{1}{2}\left[((1)^2 + (2)^2)^{n/2} e^x \cos\left(2x + 0 + n\tan^{-1}\left(\frac{2}{1}\right)\right)\right]$$

$$+ \frac{1}{8}\left[((1)^2 + (4)^2)^{n/2} e^x \cos\left(4x + 0 + n\tan^{-1}\left(\frac{4}{1}\right)\right)\right]$$

$$y_n = \frac{3}{8}e^x - \frac{1}{2}\left[(5)^{n/2} e^x \cos\left(2x + n\tan^{-1}2\right)\right] + \frac{1}{8}\left[(17)^{n/2} e^x \cos(4x + n\tan^{-1}(4))\right]$$

7. **Find n^{th} derivative of $\tan^{-1}\left[\dfrac{2x}{(1-x^2)}\right]$.**

Solution.

Let $y = \tan^{-1}\left[\dfrac{2x}{(1-x^2)}\right] = 2\tan^{-1} x$

Clearly $y_1 = \dfrac{2}{1+x^2} = \dfrac{2}{(x-i)(x+i)} = \dfrac{1}{i}\left[\dfrac{1}{x-i} - \dfrac{1}{x+i}\right]$ (from partial fraction)

Now differentiate $(n-1)$ times w.r.t. x we get

$$y_n = \frac{1}{i}\left[\frac{(-1)^{n-1}(n-1)!(1)^n}{(x-i)^{n-1+1}} - \frac{(-1)^{n-1}(n-1)!(1)^n}{(x+i)^{n-1+1}}\right] = \frac{1}{i}(-1)^{n-1}(n-1)!\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right]$$

Put $x = r\cos\phi$ and $1 = r\sin\phi$

$$\therefore \quad y_n = \frac{(-1)^{n-1}(n-1)!}{i}[r^{-n}(\cos\phi - i\sin\phi)^{-n} - r^{-n}(\cos\phi + i\sin\phi)^{-n}]$$

$$= \frac{(-1)^{n-1}(n-1)!}{i} r^{-n}[(\cos n\phi + i\sin n\phi) - (\cos n\phi - i\sin n\phi)]$$

$$= \frac{(-1)^{n-1}(n-1)!}{i} r^{-n}[2i\sin n\phi]$$

$$= \frac{2(-1)^{n-1}(n-1)!}{i}\left(\frac{1}{\sin\phi}\right)^{-n}\sin n\phi \qquad \left[\because r = \frac{1}{\sin\phi}\right]$$

$$= \frac{2(-1)^{n-1}(n-1)!}{i}\sin^n\phi\sin n\phi \text{ where } \phi = \tan^{-1}\left(\frac{1}{x}\right) \qquad \left[\frac{\sin\phi}{\cos\phi} = \frac{1}{x};\ \tan\phi = \frac{1}{x}\right]$$

7.3. [Leibnitz Theorem]

Statement:

If u and v are any two functions of x such that all their desired differential coefficients exist, then the nth differential coefficients of their product is given by

$$D^n(uv) = D^n u\, v + {}^nC_1 D^{n-1}u\, Dv + {}^nC_2 D^{n-2}u\, D^2v + {}^nC_3 D^{n-3}u D^3v + \ldots\ldots + {}^nC_r D^{n-r}u D^r v + \ldots\ldots + uD^n v$$

Proof : We will prove this result using mathematical induction.

Step 1 : Now clearly (1) is true for $n = 1$ as for $n = 1$ L.H.S. $D(uv) = (Du)v + u(Dv)$ = RHS.

Step 2 : Assume that (1) is true for $n = k$, i.e.,

$$D^k(uv) = (D^k u)v + {}^kC_1 D u^{k-1} Dv + {}^kC_2 D^{k-2}u D^2v + \ldots\ldots$$
$$+ {}^kC_r D^{k-r}u D^r v + {}^kC_{r+1} D^{k-r-1}u D^{r+1}v + \ldots + uD^k v \qquad \ldots(ii)$$

Now differentiating both sides of (ii) w.r.t. x, we get

$$D^{k+1}(uv) = [(D^{k+1}u)\cdot v + D^k u\, Dv] + [{}^kC_1 D^k u\, Dv + {}^kC_1 D^{k-1}u\, D^2 v]$$
$$+ [{}^kC_2 D^{k-1}u\, D^2v + {}^kC_2 D^{k-2}u\, D^3v] + \ldots\ldots + [{}^kC_r D^{k-r+1}u\cdot D^r v + {}^kC_r D^{k-r}u\, D^{r+1}v]$$
$$+ [{}^kC_{r+1} D^{k-r}u\,.\,D^{r+1}v + {}^kC_{r+1} D^{k-r-1}u\cdot D^{r+2}v] + \ldots\ldots + (Du\, D^k v + uD^{k+1}v]$$

Re-arranging the terms, we get

$$D^{k+1}(uv) = (D^{k+1}u)\,.\,v + (1 + {}^kC_1)(D^k u\,.\,Dv) + ({}^kC_1 + {}^kC_2) D^{k-1}u\, D^2v + \ldots\ldots$$
$$+ ({}^kC_r + {}^kC_{r+1})(D^{k-r}u\, D^{r+1}v) \ldots uD^{k+1}\,.\,v \qquad \ldots(iii)$$

Also we know that ${}^kC_r + {}^kC_{r+1} = {}^{k+1}C_{r+1}$. Hence $1 + {}^kC_1 = {}^kC_0 + {}^kC_1 = {}^{k+1}C_1$

${}^kC_1 + {}^kC_2 = {}^{k+1}C_2 \ldots$ so on

$\therefore$ Substituting these relations in (iii), we get

$$D^{k+1}(uv) = (D^{k+1}u)\,v + {}^{k+1}C_1 (D^k u)\, D(v) + {}^{k+1}C_2 (D^{k-1}u)(D^2v) + \ldots\ldots$$
$$+ {}^{k+1}C_{r+1} D^{k-r}(u)\, D^{r+1}(v) + \ldots\, uD^{k+1}(v)$$

Hence from mathematical induction we get, that result is true for all positive integral values of n.

Examples

8. Find y_n, if $y = x^2 e^x \cos x$.

Solution.

Let $u = e^x \cos x,\ v = x^2$ $\quad \therefore\ y = uv \quad$ ...(i)

Now differentiate (i) n times and applying Leibnitz's theorem such we get

$$y_n = D^n(u)\,v + {}^nC_1 D^{n-1}(u)\,D(v) + {}^nC_2 D^{n-2}(u) D^2(v) \quad \text{(other terms become zero)}$$

$$= D^n(e^x \cos x)(x^2) + nD^{n-1}(e^x \cos x)(2x) + \frac{n(n-1)}{2!} D^{n-2}(e^x \cos x)(2) \quad \text{...(ii)}$$

Now consider $D^n(e^x \cos x) = r^n e^x \sin(x + n\theta)$ where $\theta = \tan^{-1}(1) = \frac{\pi}{4}$ and $r = \sqrt{1+1} = \sqrt{2}$

$$\therefore D^n(e^x \cos x) = (2)^{n/2} e^x \cos\left(x + \frac{n\pi}{4}\right)$$

Substituting these values in (ii) we get

$$y_n = (2)^{n/2} e^x \cos\left(x + \frac{n\pi}{4}\right) x^2 + n\,[2^{(n-1)/2} e^x \cos(x + (n-1)\frac{\pi}{4})(2x)]$$

$$+\left[\frac{n(n-1)}{2!} 2^{(n-2)/2} e^x \cos\left(x + (n-2)\frac{\pi}{4}\right)\cdot 2\right]$$

$$= e^x\left[2^{n/2} x^2 \cos\left(x + \frac{n\pi}{4}\right) + 2\,nx + 2^{(n-1)/2} \cos\left(x + (n-1)\frac{\pi}{4}\right) + n(n-1)\,2^{(n-2)/2} \cos\left(x + (n-2)\frac{\pi}{4}\right)\right]$$

9. Find n^{th} derivative of $e^x (2x+3)^3$.

Solution.

Let $u = e^x;\ v = (2x+3)^3$ $\quad \therefore\ y = uv \quad$ ...(i)

Now differentiate (i) w.r.t. x n times and apply Leibnitz's theorem we get

$$y_n = D^n(u)\,v + {}^nC_1 D^{n-1}(u)\,D(v) + {}^nC_2 D^{n-2}(u)\,D^2(v) + {}^nC_3 D^{n-3}(u)\,D^3(v)$$

$$y_n = D^n(e^x)(2x+3)^3 + nD^{n-1}(e^x)\,D(2x+3)^3 +$$

$$\frac{n(n-1)}{2!} D^{n-2}(e^x)\,D^2(2x+3)^3 + \frac{n(n-1)(n-2)}{3!} D^{n-3}(e^x) + D^3(2x+3)^3$$

[other terms become zero]

$$= e^x(2x+3)^3 + n(e^x)\cdot 3(2x+3)^2\cdot 2 + \frac{n(n-1)}{2!} e^x \cdot 6 \times 2(2x+3)\,2 + \frac{n(n-1)(n-2)}{3!} e^x\,24 \times 2$$

$$= e^x(2x+3)^3 + ne^x \cdot 6(2x+3)^2 + \frac{n(n-1)}{2}\cdot 24e^x(2x+3) + \frac{n(n-1)(n-2)}{3!} e^x \cdot 48$$

$$= e^x(2x+3)^3 + 6ne^x(2x+3)^2 + 12n(n-1)e^x(2x+3) + 8e^x n(n-1)(n-2)$$

$$= e^x\,[(2x+3)^3 + 6n(2x+3)^2 + 12n(n-1)(2x+3) + 8n(n-1)(n-2)]$$

10. If $y = e^{m\sin^{-1}x}$, show that $(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - (n^2+m^2)\,y_n = 0$.

Solution.

$$y = e^{m\sin^{-1}x} \ ;\ y_1 = e^{m\sin^{-1}x}\,\frac{d}{dx}(m\sin^{-1}x) \ ;\ y_1 = \frac{me^{m\sin^{-1}x}}{\sqrt{1-x^2}} = \frac{my}{\sqrt{1-x^2}} \quad \text{...(i)}$$

Squaring both the sides we get $y_1^2 = \dfrac{(my)^2}{1-x^2}$ $\quad$ i.e., $\ y_1^2(1-x^2) = m^2 y^2 \quad$ ...(ii)

Now, differentiate (ii) w.r.t. x, we get $(1-x^2)\dfrac{d}{dx}(y_1^2) + y_1^2\dfrac{d}{dx}(1-x^2) = m^2\dfrac{d}{dx}(y^2)$

$$(1-x^2)\left(2y_1\frac{d}{dx}(y_1)\right) + y_1^2(-2x) = m^2\,2y\frac{dy}{dx} \ ;\ (1-x^2)(2y_1 y_2) + y_1^2(-2x) = m^2\,2y\,y_1$$

$$(1-x^2)\,2y_1 y_2 - 2xy_1^2 = m^2\,2yy_1 \ ;\ 2y_1\,[(1-x^2)\,y_2 - xy_1] = 2y_1(m^2 y)$$

$$(1-x^2)\,y_2 - xy_1 = m^2 y \quad \text{...(iii)} \quad \text{(by cancelling } 2y_1)$$

Now again Differentiate (iii) w.r.t. x n times and using Leibnitz's theorem, we get

$$D^n((1-x^2)\,y_2) - D^n(xy_1) - m^2 D^n y = 0$$

$$D^n(y_2)(1-x^2) + {}^nC_1 D^{n-1}(y_2)\,D(1-x^2) + {}^nC_2 D^{n-2}(y_2)\,D^2(1-x^2)$$

$$-[D^n(y_1)(x) + {}^nC_1 D^{n-1}(y_1)\,D(x)] - m^2 D^n(y) = 0 \quad \text{(other terms become zero)}$$

$$y_{n+2}(1-x^2) + n\,y_{n+1}(-2x) + \frac{n(n-1)}{2!} y_n(-2) - (y_{n+1}x + ny_n) - m^2 y_n = 0$$

$$(1-x^2)\,y_{n+2} + y_{n+1}[-2xn - x] + y_n[-n(n-1) - m^2 - n] = 0$$

$$(1-x^2)\,y_{n+2} + y_{n+1}(-x)(2n+1) + y_n[-n^2 - m^2] = 0$$

$$(1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} - (n^2+m^2)\,y_n = 0$$

11. If $y = \sin(m\sin^{-1}x)$, then prove that $(1-x^2)\,y_{n+2} = (2n+1)\,x\,y_{n+1} + (n^2-m^2)\,y_n$.

Solution.

$$y = (\sin(m\sin^{-1}x) \ ;\ y_1 = \cos(m\sin^{-1}x)\,\frac{d}{dx}(m\sin^{-1}x)$$

$$y_1 = \cos(m\sin^{-1}x)\,\frac{m}{\sqrt{1-x^2}} \quad \text{...(i)}$$

Squaring both the sides and cross-multipliying we get

$$y_1^2(1-x^2) = m^2\cos^2(m\sin^{-1}x)$$

$$y_1^2(1-x^2) = m^2(1-\sin^2(m\sin^{-1}x))$$

$$y_1^2(1-x^2) = m^2(1-y^2)$$

$$y_1^2(1-x^2) + m^2 y^2 = m^2 \quad \text{...(ii)}$$

Again differentiating w.r.t. x we get

$$\frac{d}{dx}(y_1^2)(1-x^2) + \frac{d}{dx}(1-x^2)(y_1^2) + m^2\frac{d}{dx}(y^2) = 0$$

$$2y_1\frac{d}{dx}(y_1)(1-x^2) + (-2x)\,y_1^2 + m^2\,2y\frac{dy}{dx} = 0$$

$$2y_1 y_2(1-x^2) - 2xy_1^2 + m^2\,2y\,y_1 = 0$$

$$2y_1\,[y_2(1-x^2) - xy_1 + m^2 y] = 0$$

$$y_2(1-x^2) - xy_1 + m^2 y = 0 \quad \text{(by cancelling } 2y_1) \quad \text{...(iii)}$$

Differentiating (iii) n times w.r.t. x, and using Leibnitz's theorem we get

$$D^n(y_2(1-x^2)) - D^n(xy_1) + m^2 D^n(y_1) = 0$$

$$D^n(y_2)(1-x^2) + {}^nC_1 D^{n-1}(y_2)\,D(1-x^2) + {}^nC_2 D^{n-2}(y_2)\,D^2(1-x^2)$$

$$-[D^n(y_1)\,x + {}^nC_1 D^{n-1}(y_1)\,D(x)] + m^2 D^n(y) = 0$$

$$y_{n+2}(1-x^2) + n\,y_{n+1}(-2x) + \frac{n(n-1)}{2!} y_n(-2) - [y_{n+1}x + ny_n] + m^2 y_n = 0$$

$$(1-x^2)\,y_{n+2} - xy_{n+1}\,\{2n+1\} + y_n\,[m^2 - n^2] = 0$$

$$\text{i.e., } (1-x^2)\,y_{n+2} = xy_{n+1}(2n+1) + y_n(n^2-m^2)$$

12. If $y = a\cos(\log x) + h\sin(\log x)$; show that $x^2y_2 + xy_1 + y = 0$ $x^2y_2 + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.

Solution.

$$y = a\cos(\log x) + b\sin(\log x)\ ;\ y_1 = -a\sin(\log x)\frac{1}{x} + b\cos(\log x)\times\frac{1}{x}$$

$$y_1 = \frac{1}{x}[b\cos(\log x) - a\sin(\log x)]$$

$$xy_1 = b\cos(\log x) - a\sin(\log x) \quad ...(i)$$

Differentiate (*i*) w.r.t. x we get

$$x\frac{d}{dx}(y_1) + y_1\cdot 1 = b\frac{d}{dx}(\cos(\log x)) - a\frac{d}{dx}(\sin(\log x))$$

$$xy_2 + y_1 = -b\sin(\log x)\frac{1}{x} - a\cos(\log x)\frac{1}{x} = -\frac{1}{x}(a\cos(\log x) + b\sin(\log x)]$$

$$= -\frac{y}{x} \qquad [\because y = a\cos(\log x) + b\sin(\log x)]$$

i.e., $x^2y_2 + xy_1 + y = 0 \quad ...(ii)$

Now differentiating (*ii*) 'n' times w.r.t. x and applying Leibnitz's theorem, we get

$$D^n(x^2y_2) + D^n(xy_1) + D^n(y) = 0$$

$$D^n(y_2)x^2 + {}^nC_1D^{n-1}(y_2)D(x^2) + {}^nC_2D^{n-2}(y_2)D^2(x^2) + D^n(y_1)x + {}^nC_1D^{n-1}(y_1)D(x) + D^n(y) = 0$$

$$y_{n+2}x^2 + ny_{n+1}(2x) + \frac{n(n-1)}{2!}y_n\cdot 2 + y_{n+1}(x) + n\,y_n\cdot 1 + y_n = 0$$

$$y_{n+2}x^2 + y_{n+1}(2xn + x) + y_n(n^2 - n + n + 1) = 0$$

$$y_{n+2}x^2 + y_{n+1}x(2n+1) + y_n(n^2+1) = 0$$

or $x^2y_{n+2} + x(2n+1)y_{n+1} + (n^2+1)y_n = 0$

13. If $y = (x^2-1)^n$, prove that $(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$

Hence if $P_n = \frac{d^n}{dx^n}(x^2-1)^n$, show that $\frac{d}{dx}\left\{(1-x)^2\frac{dP_n}{dx}\right\} + n(n+1)P_n = 0$

Solution.

Let $y = (x^2-1)^n$

Differentiate w.r.t x, we get

$$y_1 = n(x^2-1)^{n-1}\cdot 2x = 2xn(x^2-1)^{n-1} \quad i.e.\ \ y_1 = \frac{2xn(x^2-1)^n}{(x^2-1)}$$

i.e., $(x^2-1)y_1 = 2xn(x^2-1)^n$ *i.e.*, $(x^2-1)y_1 = 2xny \Rightarrow (x^2-1)y_1 - 2xny = 0 \quad ...(i)$

Now differentiating (*i*) $(n+1)$ times and from Leibnitz's theorem we get

$$D^{n+1}((x^2-1)y_1) - 2nD^{n+1}(xy) = 0$$

$$D^{n+1}(y_1)(x^2-1) + {}^{n+1}C_1D^n(y_1)D(x^2-1) + {}^{n+1}C_2D^{n-1}(y_1)D^2(x^2-1) - 2n[D^{n+1}(y)x + {}^{n+1}C_1D^n(y)D(x)] = 0 \quad \text{(other term will be 0)}$$

i.e., $y_{n+2}(x^2-1) + (n+1)y_{n+1}(2x) + \frac{n(n+1)}{2!}y_n\cdot 2 - 2n\times[y_{n+1}x + (n+1)y_n\cdot 1] = 0$

i.e., $y_{n+2}(x^2-1) + 2(n+1)xy_{n+1} + n(n+1)y_n - 2nxy_{n+1} - 2n(n+1)y_n = 0$

$$y_{n+2}(x^2-1) + y_{n+1}[2nx + 2x - 2nx] + y_n[n^2 + n - 2n^2 - 2n] = 0$$

$$y_{n+2}(x^2-1) + y_{n+1}(2x) + y_n(-n^2-n) = 0$$

$$y_{n+2}(x^2-1) + 2xy_{n+1} - n(n+1)y_n = 0$$

(*ii*) is the required result $\quad ...(ii)$

i.e., $(x^2-1)D^2(y_n) + 2xD(y_n) - n(n+1)y_n = 0$

Put $y_n = \frac{d^n}{dx^n}(x^2-1)^n = P_n$ in (*ii*); we get

$$(x^2-1)D^2(P_n) + 2xD(P_n) - n(n+1)P_n = 0$$

$$\Rightarrow \quad -(1-x^2)D^2(P_n) + 2xD(P_n) - n(n+1)P_n = 0$$

$$-\frac{d}{dx}[(1-x^2)D(P_n)] - n(n+1)P_n = 0$$

$$\frac{d}{dx}[(1-x^2)\frac{d}{dx}(P_n)] + n(n+1)P_n = 0$$

14. If $y^{1/m} + y^{-1/m} = 2x$, prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.

Solution.

Let $t = y^{1/m}$ $\quad\therefore\quad y^{1/m} + y^{-1/m} = 2x$ becomes $t + \frac{1}{t} = 2x$

i.e. $t^2 + 1 - 2xt = 0 \Rightarrow t^2 - 2xt + 1 = 0\ :\ t = \frac{2x \pm \sqrt{4x^2-4}}{2} = \frac{2x \pm 2\sqrt{(x^2-1)}}{2}$

$\Rightarrow\ t = x \pm \sqrt{x^2-1}$ *i.e.*, $y^{1/m} = \left[x \pm \sqrt{x^2-1}\right] \Rightarrow y = \left[x \pm \sqrt{x^2-1}\right]^m \quad ...(i)$

Now differentiating both sides w.r.t. x, we get

$$y_1 = m\left[x \pm \sqrt{x^2-1}\right]^{m-1}\cdot\left[1 \pm \frac{x}{\sqrt{x^2-1}}\right]$$

$$y_1 = m\left[x \pm \sqrt{x^2-1}\right]^{m-1}\cdot\frac{\left[\sqrt{x^2-1} \pm x\right]}{\sqrt{x^2-1}}$$

$$y_1 = \pm\frac{my}{\sqrt{x^2-1}} \qquad \text{[using eqn. (i)]}$$

Squaring both the sides wrt x, we get

$$y_1^2 = \frac{m^2y^2}{(x^2-1)} \Rightarrow (x^2-1)y_1^2 = m^2y^2 \quad ...(ii)$$

Now differentiate again w.r.t. x we get

$$(x^2-1)\frac{d}{dx}(y_1^2) + y_1^2\frac{d}{dx}(x^2-1) = m^2\frac{d}{dx}(y^2)$$

$\Rightarrow\ (x^2-1)(2y_1y_2) + y_1^2(2x) = m^2(2y\cdot(y_1)) \Rightarrow 2y_1[(x^2-1)y_2 + y_1x] = 2y_1(ym^2)$

$\Rightarrow\ (x^2-1)y_2 + y_1x = ym^2 \quad$ (cancelling out $2y_1$)

Differentiating, w.r.t. 'x' n times, we get

$$D^n[(x^2-1)y_2] + D^n(y_1x) = m^2D^n(y)$$

$$D^n(y_2)(x^2-1) + {}^nC_1D^{n-1}(y_2)D(x^2-1) + {}^nC_2D^{n-2}(y_2)D^2(x^2-1) + D^n(y_1)x + {}^nC_1D^{n-1}(y_1)D(x) = m^2D^n(y)$$

$$y_{n+2}(x^2-1)+n\,y_{n+1}(2x)+\frac{n(n-1)}{2}y_n(2)+y_{n+1}x+n\,y_n\cdot 1=m^2y_n$$

$$y_{n+2}(x^2-1)+y_{n+1}(2xn+x)+y_n(n^2-n+n-m^2)=0$$

$$y_{n+2}(x^2-1)+xy_{n+1}(2n+1)+y_n(n^2-m^2)=0$$

or $\quad (x^2-1)\,y_{n+2}+(2n+1)\,xy_{n+1}+(n^2-m^2)\,y_n=0$

15. If $y=e^{\tan^{-1}x}$, prove that $(1+x^2)\,y_{n+2}+[2(n+1)x-1]\,y_{n+1}+n(n+1)\,y_n=0$.

Solution.

$y=e^{\tan^{-1}x}$ Differentiating both sides w.r.t. x, we get $\; y_1=e^{\tan^{-1}x}\,\dfrac{1}{(1+x^2)}=\dfrac{y}{(1+x^2)}$

$\therefore \quad y_1(1+x^2)=y\Rightarrow y_1(1+x^2)-y=0 \qquad ...(i)$

Now differentiating (i) w.r.t. x $(n+1)$ times and using Leibneitz's rule, we get

$$D^{n+1}[y_1(1+x^2)]+{}^{n+1}C_1D^n(y_1)D(1+x^2)+{}^{n+1}C_2D^{n-1}Cy\cdot D^2(1+x^2)-D^{n+1}(y)=0$$

$$y_{n+2}(1+x^2)+(n+1)\,y_{n+1}(2x)+\frac{(n+1)(n)}{2}\,y_n(2)-y_{n+1}=0$$

$$y_{n+2}(1+x^2)+y_{n+1}[2(n+1)x-1]+y_n(n)(n+1)=0$$

$\Rightarrow \quad (1+x^2)\,y_{n+2}+[2(n+1)x-1]\,y_{n+1}+n(n+1)\,y_n=0$

16. If $y=\sin(m\sin^{-1}x)$, find $(y_n)=0$

Solution.

$$y=\sin(m\sin^{-1}x) \qquad ...(i)$$

Differentiate w.r.t. x, we get

$$y_1=\cos(m\sin^{-1}x)\frac{m}{\sqrt{1-x^2}} \qquad ...(ii)$$

Squaring both sides we get $\quad y_1^2=\dfrac{m^2}{(1-x^2)}\cos^2(m\sin^{-1}x)$

$$y_1^2=\frac{m^2}{(1-x^2)}[1-\sin^2(m\sin^{-1}x)]$$

$$y_1^2=\frac{m^2}{(1-x^2)}(1-y^2)$$

$\therefore \quad (1-x^2)\,y_1^2+y^2m^2-m^2=0 \qquad ...(iii)$

Differentiate both sides w.r.t. x, we get

$$(1-x^2)\frac{d}{dx}(y_1^2)+y_1^2\frac{d}{dx}(1-x^2)+m^2\frac{d}{dx}(y^2)-0=0$$

$$(1-x^2)(2y_1y_2)+y_1^2(-2x)+m^2(2y\,y_1)=0$$

$$2y_1[(1-x^2)\,y_2-xy_1+ym^2]=0$$

$(1-x^2)\,y_2-xy_1+ym^2=0 \qquad$ (cancelling $2y_1$) $\qquad ...(iv)$

Now differentiate (iv) w.r.t. x 'n' times and being Leibnitz's rule, we get

$$D^n(1-x^2)\,y_2-D^n(xy_1)+m^2D^n(y)=0$$

$$D^n(y_2)(1-x^2)+{}^nC_1D^{n-1}(y_2)D(1-x^2)+{}^nC_2D^{n-2}(y_2)D^2(1-x^2)$$
$$-[D^n(y_1)x+{}^nC_1D^{n-1}(y_1)D(x)]+m^2D^n(y)=0$$

$$y_{n+2}(1-x^2)+ny_{n+1}(-2x)+\frac{n(n-1)}{2}y_n(-2)-[y_{n+1}x+ny_n\cdot 1]+m^2y_n=0$$

$$y_{n+2}(1-x^2)-2xny_{n+1}-(n^2-n)\,y_n-y_{n+1}x-ny_n+m^2y_n=0$$

$$y_n+2(1-x^2)+y_{n+1}[-2xn-x]+y_n[-n^2+n-n+m^2]=0$$

$$y_{n+2}(1-x^2)-xy_{n+1}(2n+1)+y_n(m^2-n^2)=0$$

or $\quad y_{n+2}(1-x^2)-(2n+1)\,xy_{n+1}+y_n(m^2-n^2)=0$

Now put $x=0$ in (i), (ii) and (iv) and (iv) we get $\qquad ...(v)$

$$(y)_0=\sin(m\sin^{-1}0)=\sin(m\times 0)=\sin 0=0$$

$$(y_1)_0=[\cos(m\sin^{-1}0)]\frac{m}{\sqrt{1-0^2}}=\cos(0)\,m=m$$

$$(y_{n+2})_0=(n^2-m^2)(y_n)_0=0 \qquad ...(vi)$$

Clearly (vi) is a reduction formula. Putting $n-2$ in place of n. in (vi) we get

$$(y_n)_0=[(n-2)^2-m^2](y_{n-2})_0=0=(y_n)_0=(n-2)^2-m^2(y_{n-2})_0=0 \qquad ...(vii)$$

Now put $n-4$ instead of n in (vi). we get

$$(y_{n-4+2})=[(n-4)^2-m^2](y_{n-4})_0 \qquad ...(viii)$$

Substitute $(viii)$ in (vii)

$$(y_n)_0=[(n-2)^2-m^2][(n-4)^2-m^2](y_{n-4})_0=0$$

Case I : n is odd, Put $n=1, 3, 5, 7, \ldots$ in (vi) we get

$$(y_3)_0=(1-m^2)(y_1)_0=(1-m^2)\cdot m$$

$$(y_5)_0=(9-m^2)(y_3)_0=(9-m^2)(1-m_2)m$$

$$(y_7)_0=(25-m^2)(y_5)_0=(25-m^2)(9-m^2)(1-m^2)\,m \qquad \text{...... so on}$$

i.e. when n is odd

$$(y_n)_0=\{(n-2)^2-m^2\}\{(n-4)^2-m^2\}\ldots$$
$$(3^3-m^2)(1^2-m^2)m$$

Case II : n is even: Put $n=2, 4, 6\ldots$ in (vi), we get

$$(y_4)_0=(2^2-m^2)[y_2)_0=(4-m^2)(0)=0$$

$$(y_6)_0=(4^2-m^2)(y_4)_0=0\ldots$$

$\therefore$ in general we can write when n is even $(y_n)_0=0$

17. If $y=\cos(m\sin^{-1}x)$ find $(y_n)_0$.

Solution.

$$y=\cos(m\sin^{-1}x) \qquad ...(i)$$

Differentiating wrt x, we get

$$y_1=-\sin(m\sin^{-1}x)\frac{m}{\sqrt{1-x^2}} \quad i.e. \quad \left(\sqrt{1-x^2}\right)y_1=-m\sin(m\sin^{-1}x) \qquad ...(ii)$$

Squaring both sides of (i) wrt x we get

$$(1-x^2)y_1^2=m^2\sin^2(m\sin^{-1}x)$$

$$(1-x^2)y_1^2=m^2(1-\cos^2(m\sin^{-1}x))$$

$$(1-x^2)y_1^2=m^2(1-y^2) \qquad ...(iii)$$

i.e. $\quad (1-x^2)y_1^2+m^2y^2-m^2=0$

Now again differentiating (ii) w.r.t x we get

$$\left(1-x^2\right)\frac{d}{dx}(y_1^2)+y_1^2\frac{d}{dx}\left(1-x^2\right)+m^2\frac{d}{dx}(y^2)-0=0$$

$$(1-x^2)(2y_1 \cdot y_2)+y_1^2(-2x)+m^2(2yy_1)=0$$

$$2y_1(y_2(1-x^2)-xy_1+ym^2)=0$$

or $$(1-x^2)y_2-xy_1+m^2y=0 \qquad ...(iv) \quad \text{(cancelling } 2y_1\text{)}$$

Now differentiate (*iii*) n times wrt x, we get

$$D^n((1-x^2)y_2)-D^n(xy_1)+m^2D^n(y)=0$$

$$D^n(y_2)(1-x^2)+{}^nC_1D^{n-1}(y_2)D(1-x^2)+{}^nC_2D^{n-2}(y_2)D^2(1-x^2)$$

$$-[D^n(y_1)x+{}^nC_1D^{n-1}(y_1)D(x)]+m^2D^n(y)=0$$

$$y_{n+2}(1-x^2)+n\,y_{n+1}(-2x)+n\frac{(n-1)}{2}y_n(-2)$$

$$-[y_{n+1}+ny_n\cdot 1]+m^2y_n= 0$$

$$y_{n+2}(1-x^2)-2xn\,y_{n+1}-n(n-1)y_n-xy_{n+1}-ny_n+m^2y_n=0.$$

$$y_{n+2}(1-x^2)+y_{n+1}\left[-2xn-x\right]+y_n[-n^2+n-n+m^2]=0$$

$$y_{n+2}(1-x^2)-xy_{n+1}\left[2n+1\right]+y_n[m^2-n^2]=0 \qquad ...(v)$$

Now put $x=0$ in (*i*), (*ii*), (*iv*), (*v*) we get

$$(y)_0=\cos(m\sin^{-1}0)=\cos(0)=1$$

$$(y_1)_0=-m\sin(m\sin^{-1}0)=-m(\sin 0)=0 \quad \text{i.e.} \quad (y_1)_0=0$$

$$(1-0)(y_2)_0-0y_1+m^2(y)_0=0 \Rightarrow (y_2)_0+m^2(1)=0 \Rightarrow (y_2)_0=-m^2$$

$$(1-0)(y_{n+2})_0-(2n+1).0(y_{n+1})_0-(n^2-m^2)(y_n)_0=0$$

$$(y_{n+2})_0=(n^2-m^2)(y_n)_0 \qquad ...(vi)$$

Now clearly (*vi*) is a reduction formulae. Putting n - 2 in place of n in (*vi*) we get

$$(y_n)_0=\left((n-2)^2-m^2\right)(y_{n-2})_0 \qquad ...(vii)$$

Now put n-4 in place of n in (*vi*) we get

$$(y_{n-2})_0=\left((n-4)^2-m^2\right)(y_{n-4})_0 \qquad ...(viii)$$

Substituting (*vii*) in (*viii*) we get

$$(y_n)_0=\left((n-2)^2-m^2\right)\left((n-4)^2-m^2\right)(y_{n-4})_0$$

Case I. n is odd put n = 1, 3, 5, 7, in (*vi*) we get

$$(y_3)_0=(1^2-m^2)(y_1)_0=(1-m^2)(y_1)_0$$

$$(y_5)_0=(3^2-m^2)(y_3)_0=(3^2-m^2)(1-m^2)(y_1)_0$$

$$(y_7)_0=(5^2-m^2)(y_5)_0=(5^2-m^2)(3^2-m^2)(1-m^2)(y_1)_0 \qquad \text{...... so on}$$

In general we can write

$$(y_n)_0=\left((n-2)^2-m^2\right)\left((n-4)^2-m^2\right)\text{..........}(3^2-m^2)(1-m^2)(y_1)_0=0 \qquad [\because (y_1)_0=0]$$

Case II. n is even put n=2, 4, 6, in (*vi*)

$$(y_4)_0=(2^2-m^2)(y_2)_0$$

$$(y_6)_0=(4^2-m^2)(y_4)_0=(4^2-m^2)(2^2-m^2)(y_2)_0$$

$$(y_8)_0=(6^2-m^2)(y_6)_0=(6^2-m^2)(4^2-m^2)(2^2-m^2)(y_2)_0 \qquad \text{...... so on}$$

In general we can say

$$(y_n)_0=\left((n-2)^2-m^2\right)\left((n-4)^2-m^2\right)\text{..........}(4^2-m^2)(2^2-m^2)(y_2)_0$$

$$=\left((n-2)^2-m^2\right)\left((n-4)^2-m^2\right)\text{..........}(4^2-m^2)(2^2-m^2)(1-m^2)(-m^2)$$

$$=\left((n-2)^2-m^2\right)\left((n-4)^2-m^2\right)\text{..........}(4^2-m^2)(2^2-m^2)m^2$$

18. If $y=e^{a\sin^{-1}x}$, Find y_n at $x=0$.

Solution:

Let $$y=e^{a\sin^{-1}x} \qquad ...(i)$$

Differentiate (1) wrt x we get

$$y_1=e^{a\sin^{-1}x}\frac{d}{dx}\left(e^{a\sin^{-1}x}\right)=e^{a\sin^{-1}x}\frac{a}{\sqrt{1-x^2}}$$

$$y_1=\frac{ay}{\sqrt{1-x^2}} \qquad ...(ii)$$

$$\Rightarrow y_1\left(\sqrt{1-x^2}\right)=ay \Rightarrow y_1^2(1-x^2)=a^2y^2 \qquad ...(iii)$$

Now Differentiate (*ii*) wrt x we get

$$(1-x^2)\frac{d}{dx}\left(y_1^2\right)+y_1^2\frac{d}{dx}(1-x^2)=a^2\frac{d}{dx}(y^2)$$

$$(1-x^2)(2y_1y_2)+y_1^2(-2x)=a^2(2yy_1)$$

$$2y_1\left[(1-x^2)y_2\right]+2y_1(-y_1x)=2y_1(ya^2)$$

$$(1-x^2)y_2-(y_1x)=ya^2 \qquad \text{(cancelling } 2y_1\text{)}$$

$$\Rightarrow (1-x^2)y_2-y_1x-ya^2=0 \qquad ...(iv)$$

Now differentiate (*iii*) n times wrt x and using Leibnitz's theorem, we get

$$D^n\left(y_2(1-x^2)\right)-D^n(xy_1)-a^2D^n(y)=0$$

$$D^n(y_2)(1-x^2)+{}^nC_1D^{n-1}(y_2)D(1-x^2)+{}^nC_2D^{n-2}(y_2)D^2(1-x^2)$$

$$-\left[D^n(y_1)x+{}^nC_1D^{n-1}(y_1)D(x)\right]-a^2D^n(y)=0$$

$$y_{n+2}(1-x^2)+ny_{n+1}(-2x)+\frac{n(n-1)}{2}y_n(-2)$$

$$-\left[y_{n+1}x+ny_n.1\right]-a^2y_n=0$$

$$y_{n+2}(1-x^2)+y_{n+1}(-2xn-x)+y_n[-n^2+n-n-a^2]=0$$

$$y_{n+2}(1-x^2)-xy_{n+1}(2n+1)+y_n[-n^2-a^2]=0 \qquad ...(v)$$

$$y_{n+2}(1-x^2)-x(2n+1)y_{n+1}-y_n[n^2+a^2]=0$$

Put $x=0$ in (*i*), (*ii*), (*iv*), (*v*) we get

$$(y)_0=e^{a\sin^{-1}0}=e^{a.0}=e^0=1$$

$$(y_1)_0=\frac{a(y_0)}{\sqrt{1-0^2}}=a.1=a$$

$$(1-0)(y_2)_0 - 0.(y_1)_0 - a^2(y)_0 = 0$$

$$(y_2)_0 - 0.(y_1)_0 - a^2.1 = 0 \Rightarrow (y_2)_0 = a^2$$

$$(y_{n+2})_0(1-0^2) - 0 - (y_n)_0(n^2+a^2) = 0$$

$$\Rightarrow (y_{n+2})_0(1) = (y_n)_0(n^2+a^2) \qquad ...(vi)$$

Case I.- n is odd, put n = 1, 3, 5, ...in (vi), we get

$$(y_3)_0 = (y_1)_0(1^2+a^2) = (1^2+a^2).a$$

$$(y_5)_0 = (y_3)_0(3^2+a^2) = (3^2+a^2)(1^2+a^2).a$$

$$(y_7)_0 = (y_5)_0(5^2+a^2) = (5^2+a^2)(3^2+a^2)(1^2+a^2).a \quad \text{...... so on}$$

In general we can write

$$(y_n)_0 = \left[(n-2)^2+a^2\right]\left[(n-4)^2+a^2\right]............(3^2+a^2)(1^2+a^2).a$$

Case II.- n is even, put n=2, 4, 6, 8, ...in (vi) we get

$$(y_4)_0 = (y_2)_0(2^2+a^2) = (2^2+a^2).(a^2)$$

$$(y_6)_0 = (y_4)_0(4^2+a^2) = (4^2+a^2)(2^2+a^2).(a^2)$$

$$(y_8)_0 = (y_6)_0(6^2+a^2) = (6^2+a^2)(4^2+a^2)(2^2+a^2).(a^2) \quad \text{...... so on}$$

$\therefore$ In general we can say

$$(y_n)_0 = \left[(n-2)^2+a^2\right]\left[(n-4)^2+a^2\right]............(4^2+a^2)(2^2+a^2)(a^2)$$

Exercise

Find the n^{th} differentiate coefficients of following functions

1. $\sin^3 x$ [**Ans.** $\frac{3}{4}\sin\left(x+\frac{n\pi}{2}\right)-\frac{1}{4}3^n \sin\left(x+\frac{n\pi}{2}\right)$]

2. $\cos^4 x$ [**Ans.** $\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$]

3. $e^{ax}\cos^2 x\text{anx}$ [**Ans.** $\frac{1}{4}\left[\left(a^2+1\right)^{n/2} e^{ax}\sin\left(x+n\tan^{-1}(1/a)\right)\right]+\left[\left(a^2+9\right)^{n/2} e^{ax}\sin\left(3x+n\tan^{-1}(3/a)\right)\right]$]

4. $\sin x \cos 3x$ [**Ans.** $\frac{3}{4}\left[\left(2^2+1^2\right)^{1/2}\right]^n e^{2x}\sin\left[x+n\tan^{-1}(1/2)\right]-\frac{1}{4}\left[\left(2^2+3^2\right)^{1/2}\right]^n e^{ax}\sin\left[2x+n\tan^{-1}(3/2)\right]$]

5. $\sin^2 x \sin 2x$ [**Ans.** $2^{n-1}\sin\left(2x+\frac{n\pi}{2}\right)-4^{n-1}\sin\left(4x+\frac{n\pi}{2}\right)$]

6. $\frac{x^2}{(x-a)(x-b)}$ [**Ans.** $\frac{(-1)^n n!}{(a-b)}\left[\frac{a^2}{(x-a)^{n+1}}-\frac{b^2}{(x-b)^{n+1}}\right]$]

7. $\frac{x}{1+3x+2x^2}$ [**Ans.** $(-1)^n n!\left[\frac{1}{(x+1)^{n+1}}-\frac{2^n}{(2x+1)^{n+1}}\right]$]

8. $\tan^{-1}(x/a)$ [**Ans.** $(-1)^n(n-1)!a^{-n}\sin^n\phi \sin\phi\left[\phi=\tan^{-1}(a/x)\right]$]

9. Find n^{th} derivative of $x^3\log x$ [**Ans.** $6(-1)^n(n-4)!x^{3-n}$]

10. Find y_n for $y=x^2\tan^{-1}x$

[**Ans.** $(-1)^{n-1}(n-3)!\,[(n-1)(n-2)x^2\sin^n\varphi\sin\varphi$ $-2nx(n-2)\sin^{n-1}\varphi\sin(n-1)\phi+n(n-1)\sin^{n-2}\phi\sin(n-2)\phi]\,[\phi=\tan^{-1}(1/x)]$]

11. Prove that $\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)=\frac{(-1)^n(n)!}{x^{n+1}}\left[\log x-1-1/2-1/3......-1/n\right]$

12. If $y=\tan^{-1}x$ find $(y_7)_0$ and $(y_8)_0$

[**Ans.** $(y_7)_0=-720,\ (y_8)_0=0$]

13. If $x+y=1$, prove that: $\frac{d^n}{dx^n}(x^n y^n)=n!\left[y^n-({}^nC_1)^2y^{n-1}x+({}^nC_2)^2y^{n-2}x^2+.....(-1)^n x^n\right]$

14. If $y=e^{m\cos^{-1}x}$, find $(y_n)_0$ [**Ans.** $\left[(n-2)^2+m^2\right].......(2^2+m^2)(m^2e^{m\pi/2})$]

15. If $y=\left[\log\left\{x+\sqrt{1+x^2}\right\}\right]^2$, prove that $(y_{n+2})_0=-n^2(y_n)_0$, hence find $(y_n)_0$

[**Ans.** $(-1)^{(n-2)/2}(n-2)^2(n-4)^2(n-6)^2.....4^2 2^2 2$]

CHAPTER - 8

[Rolle's Theorem, Mean Value Theorem] Expansion of Functions

8·1 [Rolle's Theorem]

Statement :

If $f(x)$ is a real valued function such that

(i) $f(x)$ is continuous in the closed interval $[a, b]$

(ii) f is derivable in open interval (a, b)

(iii) $f(a) = f(b)$; then there exists atleast one value of x; say c where $a < c < b$ such that $f'(c) = 0$

Proof :

Case I : If f is a constant function then $f'(x) = 0 \; \forall \; x \in [a, b] \Rightarrow f'(c) = 0 \; \forall \; c \in (a, b)$

$\therefore$ the result is true in this case.

Case II :

If f is not a constant function; then as f is continuous in $[a, b]$; thus it is bounded and attains its supremum and infimum in $[a, b]$; As f is not constant; thus one of its bounds must be different from $f(a)$ or $f(b)$. Let this bound be attained at $x = c$ (say) where $c \in (a, b)$

Without loss of generality; we assume that infimum of f is attained at $x = c$

$\Rightarrow f(c + h) \le f(c)$; where $c + h$ is any point in the *nbd.* of c, h may be +ve or –ve.

If $h > 0$; then $\dfrac{f(c+h) - f(c)}{h} \le 0 \Rightarrow \underset{h \to 0^+}{\text{Lt}} \dfrac{f(c+h) - f(c)}{h} \le 0$

$\Rightarrow R f'(c) \le 0$ ...(1)

Again if $h < 0$; then $\dfrac{f(c+h) - f(c)}{h} \ge 0 \Rightarrow \underset{h \to 0^-}{\text{Lt}} \dfrac{f(c+h) - f(c)}{h} \ge 0 \Rightarrow L f'(c) \ge 0$ ...(2)

As $f(x)$ is derivable in (a, b)

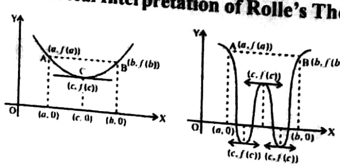
$\Rightarrow f'(x)$ exists for all $x \in (a, b) \Rightarrow f'(c)$ exists $\therefore R f'(c) = L f'(c)$ ...(3)

From (1), (2) and (3)

$f'(c) = 0$ [$\because$ 0 is the common value of $L f'(c)$ and $R f'(c)$] $\Rightarrow f'(c) = 0$ for same $c \in (a, b)$

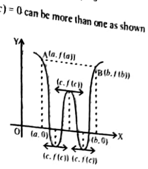
Hence the theorem is proved

8·2 [Geometrical Interpretation of Rolle's Theorem]



Let $f(x)$ be a real valued function defined on $[a, b]$ such that the curve $y = f(x)$ is a continuous curve between points $A(a, f(a))$ and $B(b, f(b))$ Now if $f(a) = f(b)$; then there exists atleast one point $(c, f(c))$ lying between A and B on the curve $y = f(x)$ where tangent is parallel to x-axis *i.e.* $f'(c) = 0$

Note : Number of such points where $f'(c) = 0$ can be more than one as shown below



8·3 [Algebraic Interpretation of Rolle's Theorem

Let $f(x)$ be a polynomial function with a and b as two of its roots; then $f(a) = f(b) = 0$. Now $f(x)$ being a polynomial function is continuous and derivable, thus $f(x)$ satisfies all conditions of Rolle's theorem; therefore there exists $c \in (a, b)$ such that $f'(c) = 0$; where $c \in (a, b)$. So, we can say $x = c$ is a root of its derivative $f'(x)$.

Examples

1. **Verify Rolle's theorem for the following functions on the indicated intervals.**

(i) $f(x) = 2x^2 - 5x + 3$ on $[1, 3/2]$ (ii) $f(x) = e^x \cos x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iii) $f(x) = \sin 2x$ on $\left[0, \frac{\pi}{2}\right]$ (iv) $f(x) = 4 \sin x$ on $[0, \pi]$

(v) $f(x) = x^2 + 5x + 6$ on $[-3, -2]$

Solution :

(i) Here $f(x)$ is a polynomial function; which is continuous and derivable on R;

Also $f(a) = f(1) = 2(1)^2 - 5(1) + 3 = 0$

$f(b) = f(3) = 2(3/2)^2 - 5(3/2) + 3 = 0$

$\therefore f(a) = f(b)$: thus by Rolle's theorem; there exists atleast a point $c \in (1, 3/2)$ such that $f'(c) = 0$

Now $f'(x) = 4x - 5$ $\therefore f'(c) = 0 \Rightarrow 4c - 5 = 0 \Rightarrow c = \frac{5}{4} \in (1, 3/2)$

Hence Rolle's theorem is verified

(ii) $f(x) = e^x \cos x$

Here $f(x)$ is a product of exponential function and cosine function which are both continuous and derivable on R and thus their product is also continuous and derivable on R

Now $f\left(-\frac{\pi}{2}\right) = e^{-\pi/2} \cos\left(-\frac{\pi}{2}\right) = 0$; $f\left(\frac{\pi}{2}\right) = e^{\pi/2} \cos\left(\frac{\pi}{2}\right) = 0$ $\therefore f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = 0$

$\Rightarrow$ there exist same $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ so that $f'(c) = 0$.

Now $f'(x) = e^x(-\sin x) + \cos x \, e^x = e^x(\cos x - \sin x)$

$f'(c) = 0 \Rightarrow e^c(\cos c - \sin c) = 0 \Rightarrow \cos c - \sin c = 0$ [$\because e^c \ne 0$]

$\Rightarrow \tan c = 1 \Rightarrow c = \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ Hence Rolle's theorem is verified.

(iii) $f(x) = \sin 2x$ on $\left(0, \frac{\pi}{2}\right)$ sine function is continuous and derivable on R $f(0) = f\left(\frac{\pi}{2}\right) = 0$

thus all conditions of Rolle's theorem are verified so there exists some $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$

Now $f'(x) = 2\cos 2x$; $f'(c) = 0 \Rightarrow 2\cos 2c = 0$; $\cos 2c = 0 \Rightarrow 2c = \frac{\pi}{2} \Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

Hence Rolle's theorem is verified.

(iv) $f(x) = 4^{\sin x}$ on $[0, \pi]$

$f(x)$ being a composite function of 4^x and $\sin x$ is continuous and derivable on R.

Also $f(0) = 4^{\sin(0)} = 4^0 = 1$; $f(\pi) = 4^{\sin\pi} = 4^0 = 1 \Rightarrow f(0) = f(\pi)$

Thus all conditions of Rolle's theorem are verified; thus there exists some

$c \in (0, \pi)$ such that $f'(c) = 0$. Now $f'(x) = 4^{\sin x} . \cos x$ $f'(c) = 0 \Rightarrow 4^{\sin c} \cos c = 0$

$\Rightarrow \cos c = 0$ $[\because 4^{\sin c} \neq 0]$ $\Rightarrow c = \frac{\pi}{2} \in (0, \pi)$ Hence Rolle's theorem is verified

(v) $f(x) = x^2 + 5x + 6$ on $[-3, -2]$

$f(x)$ being a polynomial function is continuous and derivable on R $f(-3) = f(-2) = 0$

$\therefore$ there exists same $c \in (-3, -2)$ such that $f'(c) = 0$. Now $f'(x) = 2x + 5$

$$f'(c) = 0 \Rightarrow 2c + 5 = 0 \Rightarrow c = -\frac{5}{2} = -2.5 \in (-3, -2).$$ Hence Rolle's theorem is verified.

2. Show that $f(x) = (x-a)^m (x-b)^n$, m, n being integer satisfies Rolle's theorem in $[a, b]$. Find the value of c.

Solution:

$f(x)$ being the product of two polynomial functions is continuous and derivable on R. Also $f(a) = f(b) = 0$.

Thus all conditions of Rolle's theorem are verified; so there must exist some $c \in (a, b)$ such that $f'(c) = 0$.

Now $f'(x) = (x-a)^m n(x-b)^{n-1} + m(x-a)^{m-1}(x-b)^n = (x-a)^{m-1}(x-b)^{n-1}[x(m+n) - (mb+na)]$

$$\therefore f'(c) = 0 \Rightarrow (c-a)^{m-1}(c-b)^{n-1}[c(m+n) - (mb+na)] \Rightarrow c = a, c = b; c = \frac{mb+na}{m+n}$$

As $c \in (a, b)$; thus $c = \frac{mb+na}{m+n}$ is the required value.

3. Verify Rolle's theorem for the function $f(x) = \log\left\{\frac{x^2+ab}{x(a+b)}\right\}$ on $[a, b]$ where $0 < a < b$.

Solution :

$$f(x) = \log\frac{x^2+ab}{x(a+b)} = \log(x^2+ab) - \log x - \log(a+b) \quad ....(1)$$

Now logarithmic function is derivable and continuous on $(0, \infty)$

Also $f(a) = \log\left[\frac{a^2+ab}{a(a+b)}\right] = \log\left[\frac{a(a+b)}{a(a+b)}\right] = \log 1 = 0$ similarly $f(b) = 0$

Thus all conditions of Rolle's theorem are verified; so there exists some $c \in (a, b)$ such that $f'(c) = 0$.

Now $f'(x) = \frac{1}{x^2+ab}(2x) - \frac{1}{x}$ $\therefore f'(c) = 0 \Rightarrow \frac{2c}{c^2+ab} - \frac{1}{c} = 0$

$\Rightarrow 2c^2 = c^2 + ab \Rightarrow c^2 = ab \Rightarrow c = \sqrt{ab} \in (a, b)$. Hence Rolle's theorem is verified.

4. Verify Rolle's theorem on $f(x) = \frac{|x|}{x}$ on $[-1, 1]$.

Solution :

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ +x & \text{if } x \geq 0 \end{cases} \qquad \therefore \frac{|x|}{x} = \begin{cases} -\frac{x}{x} = -1 & \text{if } x < 0 \\ \frac{x}{x} = 1 & \text{if } x \geq 0 \end{cases}$$

Thus $f(x)$ being a constant function is continuous and derivable on its domain possibly except $x = 0$.

We check; First continuity on $x = 0$ and then derivability at $x = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1) = -1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1) = 1$$

$\therefore$ L.H.L. $\neq$ R.H.L $\Rightarrow f(x)$ is not continuous at $x = 0$. Thus Rolle's theorem can not be applied

5. Verify Rolle's theorem for $f(x) = |x|$ on $[-2, 2]$

Solution :

$$|x| = \begin{cases} -x & \text{if } -2 \leq x < 0 \\ +x & \text{if } 0 \leq x < 2 \end{cases}$$

Since a polynomial function is continuous and differentiable on R; there for $f(x)$ is continuous and derivable on $x < 0$ and $x > 0$ except possibly at $x = 0$.

(i) Continuity at $x = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (-x) = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (x) = 0 \ ; \ f(0) = 0$$

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) \Rightarrow f \text{ is continuous at } x = 0$$

(ii) Derivability at $x = 0$

$$L f'(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x-0}{x-0} = -1$$

$$R f'(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x-0}{x-0} = 1$$

$\therefore L f'(0) \neq R f'(0)$

$\Rightarrow f(x)$ is not differentiable at $x = 0$. Thus Rolle's Theorem cannot be applied

6. At what point on the curve $f(x) = e^{1-x^2}$ is tangent parallel to x-axis if $x \in [-1, 1]$

Solution :

$f(x)$ is continuous and derivable on $[-1, 1]$

$f(-1) = f(1) = 0$ thus all conditions of Rolle's theorem are verified; so there exists some

$c \in (a, b)$ such $f'(c) = 0$ or $c \in (a, b)$ at which tangent is parallel to x-axis

$f'(x) = e^{(1-x^2)}(-2x)$;

$f'(c) = 0 \Rightarrow e^{1-c^2}(-2c) = 0$

$c = 0 \in (-1, 1)$

Now at $x = 0$; $f(0) = e^{1-0} = e$ $\therefore$ Required point is $(0, e)$

Exercise

Verify Rolle's Theorem for the following functions:

1. $f(x) = x^3 - 6x^2 + 11x - 6$ on $[1, 3]$ **[Ans : applicable; $c = 2 \pm \frac{1}{\sqrt{3}}$]**
2. $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ **[Ans : applicable $c = 0$]**
3. $f(x) = x(x + 3)e^{-x/2}$ on $[-3, 0]$ **[Ans : applicable; $c = -2$]**
4. It is given that for the function f given by $f(x) = x^3 + bx^2 + ax; x \in [1, 3]$ Rolle's theorem holds with $c = 2 + \frac{1}{\sqrt{3}}$. Find the values of a and b. **[Ans : $a = 11, b = -6$]**
5. At what points on the following curves; is the tangent parallel to x-axis ?
   (i) $y = x^2$ on $[-2, 2]$
   (ii) $y = 12(x + 1)(x - 2)$ on $[-1, 2]$
   (iii) $y = 16 - x^2$ on $[-1, 1]$ **[Ans. (i) (0, 0); (ii) $\left(\frac{1}{2}, -27\right)$; (iii) (0, 16)]**

Discuss the applicability of Rolle's theorem on following functions

6. $f(x) = 3 + (x - 2)^{2/3}$ on $[1, 3]$ **[Ans : Not applicable]**
7. $f(x) = [x]$ for $-2 \le x \le 2$; where $[x]$ = greatest integer; **[Ans : Not applicable]**

8·4 [Lagrange's Mean Value Theorem]

Statement :-

Let $f(x)$ be a real valued function defination $[a, b]$ such that

(i) $f(x)$ is continuous on $[a, b]$

(ii) $f(x)$ is differentiable on $]a, b[$ then there exists a real number $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Proof :-

Consider a function.

$$\phi(x) = f(x) + Ax; \text{ where } A \text{ is a constant choosen such that } \phi(a) = \phi(b)$$

Now

(i) $\phi(x)$ being a sum of two continuous functions $f(x)$ and Ax is also continuous on $[a, b]$ [$\because Ax$ is a polynomial function]

(ii) $f(x)$ is derivable on (a, b) and Ax being a polynomial function is derivable on R; thus $\phi(x)$ is also derivable on (a, b).

(iii) A has been choosen such that $\phi(a) = \phi(b)$

Thus $\phi(x)$ satisfies all the conditions of Rulle's theorem; thus there exists same point $c \in (a, b)$ such that $\phi'(c) = 0$

Now $\phi'(x) = f'(x) + A$ ...(1)

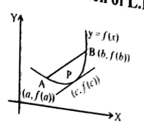
Now $\phi(a) = \phi(b) \Rightarrow f(a) + Aa = f(b) + Ab \Rightarrow A(a - b) = f(b) - f(a)$

$\Rightarrow A = \frac{f(b) - f(a)}{a - b} = -\frac{[f(b) - f(a)]}{b - a}$ Putting this value in (1) $\phi'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$

Now $\phi'(c) = 0 \Rightarrow f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$.

Hence the theorem.

8·5 [Geometrical Interpretation of L.M.V. Theorem]



Let $y = f(x)$ be a continuous curve and is derivable on (a, b); then there exists a point $c \in (a, b)$ such that the slope of the chord joining $A(a, f(a))$ and $B(b, f(b))$ is equal to the slope of the tangent at $P(c, f(c))$

$\therefore$ Geometrically $\boxed{\text{Slope of chord } AB = \text{Slope of tangent at } (c, f(c))}$

8·6 [Another Form of Lagrange's Mean Value Theorem]

Statement :

If a function $f(x)$ is

(i) Continuous on $[a, b]$

(ii) differentiable is the open interval $(a, a + h)$, then there exists atleast one number θ lying between 0 and 1 such that

$f(a + h) = f(a) + h\, f'(a + \theta h)$. Proof is omitted.

Examples

7. **Verify L.M.V. theorem for the following functions on the indicated intervals**

(i) $f(x) = x^3 - 2x^2 - x + 3$ on $[0, 1]$ (ii) $f(x) = \tan^{-1} x$ on $[0, 1]$

(iii) $f(x) = \frac{1}{x}$ on $[-1, 1]$ (iv) $f(x) = \sin x - \sin 2x - x$ on $[0, \pi]$

(v) $f(x) = x(x + 4)^2$ on $[0, 4]$

Solution :

(i) $f(x)$ being a polynomial function is continuous and derivable on R thus it is continuous on $[0, 1]$ and derivable on $(0, 1)$. So both conditions of L.M.V. theorem are satisfied; thus there must exist some $c \in (0, 1)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad ...(1)$$

Now $f'(x) = 3x^2 - 4x - 1 \Rightarrow f'(c) = 3c^2 - 4c - 1$

$f(a) = f(0) = 3; f(b) = f(1) = 1 - 2 - 1 + 3 = 1$

Putting these values in (1)

$$3c^2 - 4c - 1 = \frac{1 - 3}{1 - 0}$$

$$3c^2 - 4c - 1 = -2 \Rightarrow 3c^2 - 4c + 1 = 0$$

$$3c^2 - 3c - c + 1 = 0 \Rightarrow 3c(c - 1) - 1(c - 1) = 0$$

$$(3c - 1)(c - 1) = 0 \Rightarrow c = \frac{1}{3}, 1$$

Now $\frac{1}{3} \in (0, 1)$; thus $c = \frac{1}{3}$ is the required value.

Hence L.M.V. theorem is verified.

(ii) $f(x) = \tan^{-1} x$

For $0 \le x \le 1$; $\tan^{-1} x$ is well defined; thus $f(x)$ is continuous on $[0, 1]$

$f'(x) = \frac{1}{1+x^2}$; which exists for all values between 0 and 1; thus f is derivable on $(0, 1)$

So, both the conditions of L.M.V. theorem are satisfied; thus there must exist some $c \in (0, 1)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a} \qquad ...(1)$$

Now $f'(c) = \frac{1}{1+c^2}$; $f(b) = f(1) = \tan^{-1} 1 = \frac{\pi}{4}$

$f(a) = f(0) = \tan^{-1}(0) = 0$

Putting these values in (1)

$$\frac{1}{1+c^2} = \frac{\frac{\pi}{4} - 0}{1-0} \Rightarrow \frac{1}{1+c^2} = \frac{\pi}{4} \Rightarrow 1 + c^2 = \frac{4}{\pi} \Rightarrow c = \sqrt{\frac{4}{\pi} - 1} \in (0, 1)$$

Hence L.M.V. theorem is verified.

(iii) $f(x) = \frac{1}{x}$ on $[-1, 1]$

$f(x)$ is not defined at $x = 0 \Rightarrow f$ is not continuous at $x = 0$

$\therefore$ L.M.V. theorem can not be applied.

(iv) $f(x) = \sin x - \sin 2x - x \; [0, \pi]$

$f(x)$ is continuous as it is a sum of three continuous functions and is also derivable on $(0, \pi)$

So, both the conditions of L.M.V; theorem are satisfied; thus there must exist some $c \in (0, \pi)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a} \qquad ...(1)$$

Now $f'(x) = \cos x - 2\cos 2x - 1 = \cos x - 2(2\cos^2 x - 1) = \cos x - 4\cos^2 x + 1$

$\therefore \; f'(c) = \cos c - 4\cos^2 c + 1$

$f(b) = f(\pi) = \sin \pi - \sin 2\pi - \pi = -\pi$

$f(a) = f(0) = \sin 0 - \sin 0 - 0 = 0$

Putting these values in (1); we get

$$\cos c - 4\cos^2 c + 1 = \frac{-\pi - 0}{\pi - 0}; \; \cos c - 4\cos^2 c + 1 = -1$$

$$\Rightarrow \cos c - 4\cos^2 c + 2 = 0 \Rightarrow 4\cos^2 c - \cos c - 2 = 0$$

$$\cos c = \frac{1 \pm \sqrt{(1)^2 - 4(4)(-2)}}{2 \times 4}; \; \cos c = \frac{1 \pm \sqrt{33}}{8} \quad \therefore c = \cos^{-1} \frac{1+\sqrt{33}}{8} \quad [\because c \in (0, \pi)]$$

Hence L.M.V. theorem is verified.

(v) $f(x) = x(x+4)^2$ on $[0, 4]$

$f(x)$ being a polynomial function is continuous and derivable on R thus it is continuous on $[0, 4]$ and derivable on $(0, 4)$; thus both the conditions of L.M.V. theorem are verified; so there must exist some $c \in (0, 4)$ such that.

$$f'(c) = \frac{f(b) - f(a)}{b-a} \qquad ...(1)$$

Now $f'(x) = x.2(x+4) + (x+4)^2 = (x+4)[2x + x + 4] = (x+4)(3x+4)$

$= 3x^2 + 4x + 12x + 16 = 3x^2 + 16x + 16 \qquad \therefore \; f'(c) = 3c^2 + 16c + 16$

$f(b) = f(4) = 4(4+4)^2 = 256$

$f(a) = f(0) = 0$

Putting these values in (1), we get

$$3c^2 + 16c + 16 = \frac{256 - 0}{4 - 0}$$

$$3c^2 + 16c + 16 = 64$$

$$3c^2 + 16c - 48 = 0$$

$$c = \frac{-16 \pm \sqrt{16^2 - 4(3)(-48)}}{2 \times 3} = \frac{-16 \pm \sqrt{256 + 576}}{6} = \frac{-16 \pm \sqrt{832}}{6}$$

or $c = \frac{-16 \pm 8\sqrt{13}}{6}$ or $c = \frac{-8 + 4\sqrt{13}}{3}$ $\left[\because \frac{-8 - 4\sqrt{13}}{3} \notin (0, 4)\right]$

Hence L.M.V. theorem is verified.

8. Using L.M.V. theorem, find a point on the curve $y = \sqrt{x-2}$ defined on [2, 3], where tangent is parallel to the chord joining the end points of the curve.

Solution :

$f(x) = \sqrt{x-2}$

Now $f(x)$ exists if $x - 2 \ge 0$ *i.e.* $x \ge 2$ $\qquad \therefore f$ is continuous on $[2, 3]$

Also $f'(x) = \frac{1}{2\sqrt{x-2}}$ exists for all values on $(2, 3)$ $\qquad \therefore f$ is differentiable on $(2, 3)$

so both conditions of L.M.V. theorem are verified; thus there must exist same $c \in (2, 3)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a} \qquad —(1)$$

Now $f'(c) = \frac{1}{2\sqrt{c-2}}$; $f(b) = f(3) = \sqrt{3-2} = 1$

$f(a) = f(2) = \sqrt{2-2} = 0$

Putting these values in (1); we get

$$\frac{1}{2\sqrt{c-2}} = \frac{1-0}{3-2} \Rightarrow \frac{1}{2\sqrt{1-2}} = 1 \Rightarrow \sqrt{c-2} = \frac{1}{2} \Rightarrow c - 2 = \frac{1}{4} \Rightarrow c = \frac{9}{4} \in (2, 3)$$

Putting $x = \frac{9}{4}$ in $f(x)$; $f\left(\frac{9}{4}\right) = \sqrt{\frac{9}{4} - 2} = \frac{1}{2}$

$\therefore \; (c, f(c))$ *i.e.* $\left(\frac{9}{4}, \frac{1}{2}\right)$ is the required point.

9. If $f(x)$ is a quadratic polynomial and a, b are any two numbers, show that $\frac{a+b}{2}$ is the only value of c which satisfies L.M.V. theorem.

Solution :

Let $f(x) = px^2 + qx + r$

As $f(x)$ is a polynomial function, thus it is continuous and derivable on R so it is continuous on $[a, b]$ and derivable on (a, b). Thus by L.M.V. theorem, there exists some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a} \qquad ...(1)$$

$f'(x) = 2px + q \Rightarrow f'(c) = 2pc + q$

$f(b) = pb^2 + qb + r$, $f(a) = pa^2 + qa + r$

Putting these values in (1); we get

$$2pc + q = \frac{(pb^2 + qb + r) - (pa^2 + qa + r)}{b-a}$$

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$$2pc + q = \frac{p(b^2-a^2)+q(b-a)}{b-a} \Rightarrow 2pc + q = p(b+a) + q \Rightarrow 2pc = p(b+a)$$

$\Rightarrow \quad c = \frac{b+a}{2}$ is the only value of c which satisfies L.M.V. theorem in (a, b)

10. By using L.M.V. theorem, prove that $|\cos x - \cos y| \le |x-y|$ for all $x, y \in R$.

Solution :

Consider the function $f(t) = \cos t$; then

(i) f is continuous in (x, y) (ii) f is derivable in (x, y).

$\therefore$ by L.M.V; theorem, there exists atleast one real number $c \in (x, y)$ such that

$$f'(c) = \frac{f(y)-f(x)}{y-x} \Rightarrow -\sin c = \frac{\cos y - \cos x}{y-x}$$

Now $|\sin c| \le 1 \ \forall c \in R \quad \therefore \left|\frac{\cos y - \cos x}{y-x}\right| \le 1$

$$\Rightarrow \left|\frac{\cos x - \cos y}{x-y}\right| \le 1 \Rightarrow \frac{|\cos x - \cos y|}{|x-y|} \le 1 \Rightarrow |\cos x - \cos y| \le |x-y| \ \forall \ x, y \in R$$

Hence proved.

11. Using L.M.V. Theorem prove that (i) $\sin x < x$ for $x > 0$; (ii) $\tan x > x$ for all $x \in \left(0, \frac{\pi}{2}\right)$.

Solution :

(i) Let x be any point in $(0, \infty)$ such that $[0, x] \subset (0, \infty)$

Let $f(x) = x - \sin x$; then f is continuous on $[0, x]$ and derivable on $(0, x)$ thus by L.M.V. Theorem; there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x)-f(0)}{x-0} \Rightarrow 1 - \cos c = \frac{x - \sin x}{x}$$

Now $|\cos x| \le 1 \quad \therefore 1 - \cos c > 0$

$$\Rightarrow \frac{x-\sin x}{x} > 0 \Rightarrow x - \sin x > 0 \Rightarrow x > \sin x \quad \text{or} \quad \sin x < x \text{ for } x > 0$$

(ii) Let x be any point in the interval $\left(0, \frac{\pi}{2}\right)$ such that $[0,x] \subset \left[0, \frac{\pi}{2}\right]$ consider $f(x) = \tan x - x \Rightarrow f'(x) = \sec^2 x - 1$

then f is continuous on $[0, x]$ and derivable on $(0, x)$

so there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x)-f(0)}{x-0}$$

$$\sec^2 c - 1 = \frac{\tan x - x - 0}{x} \qquad ...(1)$$

Now $\sec^2 c > 1$ for $c \in \left(0, \frac{\pi}{2}\right) \quad \therefore \sec^2 c - 1 > 0$

Thus from (1) $\frac{\tan x - x}{x} > 0 \Rightarrow \tan x > x$ for all $x \in \left(0, \frac{\pi}{2}\right)$

Exercise

Verify Lagrange's mean value theorem for the following functions in the indicated intervals.

1. $f(x) = |x|$ on $[-1, 1]$

[Ans. Not applicable; f is not derivable at $x = 0$]

2. $f(x) = \frac{1}{4x-1}; 1 \le x \le 4$

[Ans. Not applicable; f is not continuous at $x = 4$]

3. $f(x) = x - 2\sin x$ on $[-\pi, \pi]$

[Ans. $c = \pm\frac{\pi}{2}$]

4. $f(x) = 2\sin x + \sin 2x$ on $[0, \pi]$

[Ans. $c = \frac{\pi}{3}$]

5. $f(x) = \log_e x$ on $[1, 2]$

[Ans. $c = \log_2 e$]

6. $f(x) = (x-3)(x-6)(x-9)$

[Ans. $c = 8\cdot8, 4\cdot8$]

7. $f(x) = (x-1)(x-2)(x-3)$

[Ans. $c = 2 \pm \frac{2}{\sqrt{3}}$]

8. Find the point on the parabola $y = (x-3)^2$, where the tangent is parallel to the chord joining (4, 0) and (5, 1).

[Ans. $\frac{9}{2}, \frac{1}{4}$]

9. Find a point on the curve $y = x^3 - 3x$, where the tangent to the curve is parallel to the chord joining (1, − 2) and (2, 2).

[Ans. $\left(\sqrt{\frac{7}{3}}, \frac{-2}{3}\sqrt{\frac{7}{3}}\right), \left(-\sqrt{\frac{7}{3}}, \frac{2}{3}\sqrt{\frac{7}{3}}\right)$]

10. Verify L.M.V. theorem for the function $f(x) = \begin{cases} 2 + x^2 & \text{if } x \le 1 \\ 3x & \text{if } x > 1 \end{cases}$ on $[-1, 2]$.

[Ans. $c = \pm(\sqrt{5}/3)$; Hint: Check continuity and derivability on $x = 1$]

8·7 [Cauchy's Mean Value Theorem Or Generalised Mean Value Theorem]

Statement :

If $f(x)$ and $\phi(x)$ are continuous in the closed interval $[a, b]$ and derivable at each point in the open interval (a, b); then there exists a point $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(c)}{\phi'(x)} \ (a < c < b)$$

Provided that $\phi(a) \ne \phi(b)$, and $\phi'(x)$ and $f'(x)$ do not vanish simultaneously in (a, b).

Proof is omitted.

Examples

12. If in Cauchy's Mean value theorem; $f(x) = e^x$; $\phi(x) = e^{-x}$; show that 'c' is the arithmetic mean between a and b.

Solution :

As $f(x)$ and $\phi(x)$ satisfy both the conditions of Cauchy's mean value theorem; therefore

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(c)}{\phi'(c)} \qquad ...(1)$$

Now $f(x) = e^x \Rightarrow f(b) = e^b; f(a) = e^a$. Also $f'(x) = e^x \Rightarrow f'(c) = e^c$

And $\phi(b) = e^{-x} \Rightarrow \phi(b) = e^{-b}$; $\phi(a) = e^{-a}$. Also $\phi'(x) = -e^{-x} \Rightarrow \phi'(c) = -e^{-c}$

Putting these values in (1); we get

$$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}} \ ; \ \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = -e^{2c} \Rightarrow \frac{e^b - e^a}{-(e^b - e^a)} \times e^{a+b} = -e^{2c}$$

$\Rightarrow e^{a+b} = e^{2c} \Rightarrow a + b = 2c \Rightarrow c = \frac{a+b}{2}$. Hence proved

13. **If $f(x) = \sqrt{x}$; $\phi(x) = \frac{1}{\sqrt{x}}$ in cauchy's mean value theorem; then prove that 'c' is the geometric mean between a and b.**

Solution :

As per hypothesis: conditions of cauchy's mean value theorem are satisfied: thus

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(c)}{\phi'(c)} \quad ...(1)$$

Now $\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = -\sqrt{ab}$; $f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(c) = \frac{1}{2\sqrt{c}}$

$$\phi'(x) = -\frac{1}{2(x)^{3/2}} \Rightarrow \phi'(c) = -\frac{1}{2(c)^{3/2}} \Rightarrow \frac{f'(c)}{\phi'(c)} = -\frac{c^{3/2}}{c^{1/2}} = -c$$

Putting these values in (1), we get $-\sqrt{ab} = -c \Rightarrow c = \sqrt{ab}$

$\therefore$ c is geometric mean between a and b.

14. **Verify cauchy's mean value theorem for functions x^2 and x^3 in the interval [1, 3]**

Solution :

Let $f(x) = x^2$; $\phi(x) = x^3$ · Now

(i) both $f(x)$ and $\phi(x)$ are continuous in [1, 3] as these are polynomial functions

(ii) both $f(x)$ and $\phi(x)$ are differentiable in (1, 3)

(iii) $\phi'(x) = 3x^2 \neq 0$ for any point in (1, 3)

Thus all conditions of cauchy's mean value theorem are satisfied

$$\therefore \quad \frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(c)}{\phi'(c)} \quad ....(1)$$

Now $\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f(3)-f(1)}{\phi(3)-\phi(1)} = \frac{9-1}{27-1} = \frac{8}{26} = \frac{4}{13}$. Also $\frac{f'(c)}{\phi'(c)} = \frac{2c}{3c^2} = \frac{2}{3c}$

Putting these values in (1); we get

$\frac{4}{13} = \frac{2}{3c} \Rightarrow c = \frac{2\times 13}{4\times 3} = \frac{13}{6}$ which lies in open interval (1, 3). This verifies the theorem.

Exercise

1 If $f(x) = \frac{1}{x^2}$; $\phi(x) = \frac{1}{x}$ in cauchy's mean value theorem: then prove that 'c' is the geometric mean between a and b.

2. Verify cauchy's mean value theorem for the functions x^2 and x^3 in [1, 2] **[Ans. $c = 14/9$]**

8·8 [Expansion Of Functions]

In this section; we shall use the result of the following theorems

(1) Taylor's Theorem :

Let $f(x + h)$ be a function of h (x being independent) which can be expanded in powers of h and the expansion be differentiable any number of times; then

$$\boxed{f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}\,f''(x) + ... + \frac{h^n}{n!}\,f^n(x) + ...} \quad ....(1)$$

Another form : putting $h = (x - a)$ in (1), we get

$$\boxed{f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2!}\,f''(a) + \frac{(x-a)^3}{3!}\,f'''(a) + ...} \quad ...(2)$$

(2) Maclaruin's Theorem :

Putting $a = 0$ and $h = x$ in (1), we get

$$\boxed{f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}\,f''(0) + \frac{x^3}{3!}\,f'''(0) + ... + \frac{x^n}{n!}\,f^n(0) + ...} \quad ...(3)$$

Examples

Type : 1 Based on Taylor's Theorem

15. **Expand $\log_e \sin(x + h)$ in powers of h by Taylor's theorem.**

Solution :

Let $f(x + h) = \log_e \sin(x + h)$ $\quad \therefore f(x) = \log_e \sin x$

Now $f'(x) = \cot x$; $f''(x) = -\operatorname{cosec}^2 x$; $f'''(x) = -2\operatorname{cosec} x \cot x$

By Taylor's theorem

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}\,f''(x) + ... + \frac{h^n}{n!}\,f^n(x) + ...$$

$$\therefore \quad \log_e \sin(x+h) = \log_e \sin x + h(\cot x) + \frac{h^2}{2!}(-\operatorname{cosec}^2 x) + \frac{h^3}{3!}(2\cot x \operatorname{cosec}^2 x) + ...$$

$$= \log_e \sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\operatorname{cosec}^2 x \cot x + ...$$

16. **Expand $\cos x$ in powers of $\left(x - \frac{\pi}{2}\right)$.**

Solution :

Let $f(x) = \cos x$; then $f\left(\frac{\pi}{2}\right) = 0$

$f'(x) = -\sin x$; $f'\left(\frac{\pi}{2}\right) = -1$

$f''(x) = -\cos x$; $f''\left(\frac{\pi}{2}\right) = 0$

$f'''(x) = \sin x$; $f'''\left(\frac{\pi}{2}\right) = 1$, and so on

Now $f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2!}\,f''(a) + \frac{(x-a)^3}{3!}\,f'''(x) + ...$

$$\therefore \quad \sin x = f\left(\frac{\pi}{2}\right) + \left(x-\frac{\pi}{2}\right) f'\left(\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} f''\left(\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!} f'''\left(\frac{\pi}{2}\right)$$

$$= 0 + \left(x-\frac{\pi}{2}\right)(-1) + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!}(0) + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!}(1) + ... = -\left(x-\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!} + ...$$

17. **Expand $\tan^{-1} x$ in power of $\left(x - \frac{\pi}{4}\right)$.**

Solution :

Let $f(x) = \tan^{-1} x$; then $f\left(\frac{\pi}{4}\right) = \tan^{-1}\frac{\pi}{4}$

$$f'(x) = \frac{1}{1+x^2}\ ;\ f'\left(\frac{\pi}{4}\right) = \frac{1}{1+\left(\frac{\pi}{4}\right)^2} = \frac{1}{1+\frac{\pi^2}{16}}$$

$$f''(x) = \frac{-2x}{(1+x^2)^2}\ ;\ f''\left(\frac{\pi}{4}\right) = \frac{-2\left(\frac{\pi}{4}\right)}{\left(1+\frac{\pi^2}{16}\right)^2} = -\frac{\pi}{2\left(1+\frac{\pi^2}{16}\right)^2}$$

Now $f(x) = f(a) + (x-a)\, f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots\ldots$

$$\therefore \tan^{-1} x = \tan^{-1}\left(\frac{\pi}{4}\right) + \left(x-\frac{\pi}{4}\right)\left(\frac{1}{1+\frac{\pi^2}{16}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2 \cdot \left[-\frac{\pi}{2\left(1+\frac{\pi^2}{16}\right)^2}\right]}{2!} + \ldots\ldots$$

$$= \tan^{-1}\frac{\pi}{4} + \frac{\left(x-\frac{\pi}{4}\right)}{1+\frac{\pi^2}{16}} - \frac{\pi\left(x-\frac{\pi}{4}\right)^2}{4\left(1+\frac{\pi^2}{16}\right)^2} + \ldots\ldots$$

18. Expand $\log_e \sin x$ in powers of $(x-2)$.

Solution :

Let $f(x) = \log_e \sin x$; then $f(2) = \log_e \sin 2$

$f'(x) = \cot x\ ;\ f'(2) = \cot(2)$

$f''(x) = -\operatorname{cosec}^2 x;\ f''(2) = -\operatorname{cosec}^2(2)$

$f'''(x) = +2\operatorname{cosec}^2 x \cot x;\ f'''(2) = 2\operatorname{cosec}^2(2)\cot(2)$

Now $f(x) = f(a) + (x-a)\, f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots\ldots$

$\therefore\ \log_e \sin x = \log_e \sin 2 + (x-2)\cot(2) - \frac{(x-2)^2}{2}\operatorname{cosec}^2(2) + \frac{(x-2)^3}{3}\operatorname{cosec}^2(2)\cot(2) + \ldots\ldots$

19. Expand $2x^3 + 7x^2 + x - 1$ in power of $(x-2)$.

Solution :

Let $f(x) = 2x^3 + 7x^2 + x - 1 : f(2) = 2(2)^3 + 7(2)^2 + 2 - 1 = 45$

$f'(x) = 6x^2 + 14x + 1;\ f'(2) = 6(2)^2 + 14(2) + 1 = 53$

$f''(x) = 12x + 14;\ f''(2) = 12(2) + 14 = 38$

$f'''(x) = 12;\ f'''(2) = 12$

$f^{iv}(x) = 0; f^{iv}(2) = 0$

Now $f(x) = f(a) + (x-a)\, f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a)$

or $2x^3 + 7x^2 + x - 1 = 45 + (x-2)\,53 + \frac{38}{2}(x-2)^2 + \frac{12}{3!}(x-2)^3$

$= 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3.$

Type II : Based on Maclaurin's Theorem

20. Expand $\cos x$ with the help of Maclaurin's Theorem.

Solution :

Let $f(x) = \cos x$; then $f(0) = 1$

$f'(x) = -\sin x;\ f'(0) = 0$

$f''(x) = -\cos x;\ f''(0) = -1$

$f'''(x) = \sin x;\ f'''(0) = 0$

$f^{iv}(x) = \cos x; f^{iv}(0) = 1$

Now, by Maclaurin's Theorem

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

$$= 1 + 0 + \frac{x^2}{2}(-1) + 0 + \frac{x^4}{4!} + \ldots\ldots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots$$

21. Obtain Maclaurin's series expansion of e^x.

Solution :

Let $f(x) = e^x$; then $f'(x) = e^x;\ f''(x) = e^x, \ldots\ldots, f^n(x) = e^x, \ldots\ldots \Rightarrow f'(0) = 1;\ f''(0) = 1, \ldots\ldots$

Now by Maclaurin's series expansion

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots \quad \therefore\ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$$

22. Obtain Maclaurin's series expansion of $\log(1+x)$.

Solution :

Let $f(x) = \log(1+x) \Rightarrow f(0) = \log 1 = 0$ then $f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}\ \forall n \in N$ and $\forall x > -1$

[See Chapter on Successive Differentiation]

$\therefore\quad f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1$

$f'''(x) = \frac{2!}{(1+x)^3} \Rightarrow f'''(0) = 2!$

$f^{iv}(x) = \frac{3!}{(1+x)^4} \Rightarrow f^{iv}(0) = 3!$ and so on

Now by Maclaurin's series expansion

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) - \frac{x^4}{4!} f^{iv}(0) + \ldots\ldots$$

$$= 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots, -1 < x < 1$$

23. Obtain Maclaurin's series expansion of $(1+x)^m$; $-1 < x < 1$.

Solution :

Let $f(x) = (1+x)^m$

Case I :

When m is a positive integer, then

$f'(x) = m(1+x)^m;\ f''(x) = m(m-1)(1+x)^m$

$$f^m(x) = m(m-1)(m-2)\ldots\ldots 3 \cdot 2 \cdot 1 = m! \qquad \therefore\ f^n(x) = 0 \text{ for } n > m$$

Thus by Maclaurin's series expansion

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^m}{m!} f^m(x) = 1 + mx + \frac{m(m-1)}{2!} x^2 + \ldots\ldots + x^m$$

Case II :

When m is not a +ve integer; then

$$f'(x) = m(1+x)^m;\ f''(x) = m(m-1)(1+x)^{m-2}$$

and $f^n(x) = m(m-1)(m-2)\ldots\ldots(m-n+1)(1+x)^{m-n}$

Now by Maclaurin's series expansion

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots = 1 + mx + \frac{m(m-1)}{2!} x^2 + \frac{m(m-1)(m-2)}{3!} x^3 + \ldots\ldots;\ |x| < 1$$

Miscellaneous Examples

Note : If the function $f(x)$ is denoted by y, then Maclaurin's Infinite Series may be written in the form

$$y = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \ldots\ldots$$

where $y(0), y_1(0), y_2(0), y_3(0), \ldots\ldots$ denote the values of $y, y_1, y_2, y_3, \ldots\ldots$ respectively for $x = 0$.

1. **Use Maclaurin's formula to show that $e^x \sec x = 1 + x + \frac{2x^2}{2!} + \frac{4x^3}{3!} + \ldots$.**

Solution :

Let $y = e^x \sec x$, then $(y)_0 = 1$

$y_1 = e^x \sec x + e^x \sec x \tan x = y + y \tan x$ so that $(y_1)_0 = 1$

$y_2 = y_1 + y_1 \tan x + y \sec^2 x$ so that $(y_2)_0 = 1 + 0 + 1 = 2$

$y_3 = y_2 + y_2 \tan x + 2y_1 \sec^2 x + 2y \sec^2 x \tan x$ so that

$(y_3)_0 = 2 + 2 = 4$, and so on

Substituting these values in Maclaurin's theorem, we get $e^x \sec x = 1 + x + \frac{2x^2}{2!} + \frac{4x^3}{3!} + \ldots$

2. **Expand the following functions by Maclaurin's theorem : (i) $e^{\sin x}$ (ii) $\log(1 + \sin x)$.**

Solution :

(i) Let $y = e^{\sin x}$, then $(y)_0 = e^{\sin 0} = e^0 = 1$

$y_1 = e^{\sin x} \cos x = y \cos x$ so that $(y_1)_0 = (y)_0 \cos 0 = 1 \times 1 = 1$

$y_2 = y_1 \cos x - y \sin x$ so that $(y_2)_0 = 1 \times 1 - 1 \times 0 = 1$

$y_3 = y_2 \cos x - y_1 \sin x - y_1 \sin x - y \cos x$

$= y_2 \cos x - 2y_1 \sin x - y \cos x$ so that $(y_3)_0 = 1 - 0 - 1 = 0$

$y_4 = y_3 \cos x - 3y_2 \sin x - 3y_1 \cos x + y \sin x$ so that $(y_4)_0 = -3$ and so on

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \ldots$$

$$\therefore\ e^{\sin x} = 1 + x \cdot 1 + \frac{x^2}{2!} \cdot 1 + \frac{x^3}{3!} \cdot 0 + \frac{x^4}{4!} \cdot (-3) + \ldots = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots$$

(ii) Let $y = \log(1 + \sin x)$ then $(y)_0 = 0$

$$y_1 = \frac{\cos x}{1 + \sin x} \text{ so that } (y_1)_0 = 1$$

$$y_2 = \frac{-\sin x(1 + \sin x) - \cos^2 x}{(1 + \sin x)^2} = \frac{-(1 + \sin x)}{(1 + \sin x)^2} = -\frac{1}{1 + \sin x} \text{ so that } (y_2)_0 = -1$$

$$y_3 = \frac{\cos x}{(1 + \sin x)^2} = \frac{\cos x}{1 + \sin x} \cdot \frac{1}{1 + \sin x} = -y_1 y_2$$

so that $(y_3)_0 = -1 \cdot (-1) = 1$

$y_4 = -y_1 y_3 - y_2^2$ so that $(y_4)_0 = -1 \cdot 1 - (-1)^2 = -1 - 1 = -2$

$y_5 = -y_1 y_4 - y_2 y_3 - 2y_2 y_3 = -y_1 y_4 - 3y_2 y_3$ so that

$(y_5)_0 = -1 \cdot (-2) - 3 \cdot (-1) \cdot 1 = 2 + 3 = 5$, and so on

Substituting these values in Maclaurin's theorem, we get

$$\log(1 + \sin x) = 0 + x \cdot 1 + (x^2/2!) \cdot (-1) + (x^3/3!) \cdot 1 + (x^4/4!) \cdot (-2) + (x^5/5!) \cdot 5 + \ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} + \ldots$$

3. **Apply Maclaurin's theorem to find the expansion in ascending powers of x of $\log_e(1 + e^x)$ to the term containing x^4.**

Solution :

Let $y = \log_e(1 + e^x)$ then $(y)_0 = \log_e(1 + e^0) = \log_e 2$

$$y_1 = \frac{e^x}{1 + e^x} = \frac{(1 + e^x) - 1}{1 + e^x} = 1 - \frac{1}{1 + e^x} \text{ so that } (y_1)_0 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$y_2 = 0 + \frac{e^x}{(1 + e^x)^2} = \frac{e^x}{(1 + e^x)} \cdot \frac{1}{1 + e^x} = y_1(1 - y_1) = y_1 - y_1^2$$

$$\text{so that } (y_2)_0 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$y_3 = y_2 - 2y_1 y_2 \text{ so that } (y_3)_0 = \frac{1}{4} - 2 \cdot \frac{1}{2} \cdot \frac{1}{4} 0$$

$$y_4 = y_3 - 2y_2^2 - 2y_1 y_3 \text{ so that } (y_4)_0 = 0 - 2 \cdot \left(\frac{1}{4}\right)^2 - 0 = -1/8 \quad \text{and so on}$$

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \ldots \qquad \therefore \log(1 + e^x)$$

$$= \log 2 + x \cdot \frac{1}{2} + \frac{x^2}{2!} \cdot \frac{1}{4} + \frac{x^3}{3!} 0 + \frac{x^4}{4!} \cdot (-1/8) + \ldots = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \ldots$$

4. **Show that**

(i) $e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} - \frac{2^2 x^5}{5!} + \frac{2^3 x^7}{7!} + \ldots + 2^{n/2} \cos \frac{n\pi}{4} \cdot \frac{x^n}{n!} + \ldots$

(ii) $e^x \sin x = x + x^2 - \frac{2}{3!} x^3 - \frac{2^2}{5!} x^5 - \ldots + \sin\left(\frac{1}{4} n\pi\right) \frac{2^{n/2}}{n!} x^n + \ldots$

Solution :

(i) Let $y = e^x \cos x$; then $(y)_0 = e^0 \cos 0 = 1$

$y_1 = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x)$

so that $(y_1)_0 = 1(1 - 0) = 1$

$y_2 = e^x(\cos x - \sin x) + e^x(-\sin x - \cos x) = -2e^x \sin x$

so that $(y_2)_0 = 0$

$y_3 = -2e^x \sin x - 2e^x \cos x = -2e^x(\sin x + \cos x)$

so that $(y_3)_0 = -2$

$y_4 = -2e^x(\sin x + \cos x) - 2e^x(\cos x - \sin x)$

$= -4e^x \cos x = -2^2 y$ so that $(y_4)_0 = -2^2$

$y_5 = -2^2 y_1$ so that $(y_5)_0 = -2^2$

$y_6 = -2^2 y_2$ so that $(y_6)_0 = 0$

$y_7 = -2^2 y_3$ so that $(y_7)_0 = 2^3$, and so on

In general

$$y_n = (1+1)^{n/2} \cos(x + n\tan^{-1} 1) = 2^{n/2} \cos(x + n\pi/4) \text{ so that } (y_n)_0 = 2^{n/2} \cos\left(\frac{1}{4} n\pi\right)$$

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \ldots + \frac{x^n}{n!}(y_n)_0 + \ldots$$

$$= 1 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot (-2) + \frac{x^4}{4!} \cdot (-2^2) + \frac{x^5}{5!} \cdot (-2^2)$$

$$+ \frac{x^6}{6!} \cdot 0 + \frac{x^7}{7!} \cdot 2^3 + \ldots + \frac{x^n}{n!} 2^{n/2} \cos\left(\frac{1}{4} n\pi\right) + \ldots$$

$$= 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} - \frac{2^2 x^5}{5!} + \frac{2^3 x^7}{7!} + \ldots + 2^{n/2} \cos\left(\frac{1}{4} n\pi\right) \frac{x^n}{n!} + \ldots$$

(ii) Proceed as in part (i).

5. **Apply Maclaurin's theorem to prove that :**

(i) $e^{ax} \sin bx = bx + abx^2 + \frac{3a^2 b - b^3}{n!} x^3 + \ldots + \frac{(a^2+b^2)^{n/2}}{n!} x^n \sin\{n \tan^{-1}(b/a)\} + \ldots$

(ii) $e^{ax} \cos bx = 1 + ax + \frac{a^2 - b^2}{2} x^2 + \frac{a(a^2 - 3b^2)}{3!} x^3 + \ldots + \frac{(a^2+b^2)^{n/2}}{n!} x^n \cos\{n \tan^{-1}(b/a)\} + \ldots$

Hence deduce that $e^{x\cos\alpha} \cos(x \sin\alpha) = 1 + x\cos\alpha + \frac{x^2}{2!}\cos 2\alpha + \frac{x^3}{3!}\cos 3\alpha + \ldots$

Solution :

(i) Let $y = e^{ax} \sin bx$. Then $(y)_0 = e^0 \sin 0 = 0$

$y_1 = ae^{ax} \sin bx + be^{ax} \cos bx = ay + be^{ax} \cos bx$

so that $(y_1)_0 = b$

$y_2 = ay_1 + abe^{ax} \cos bx - b^2 e^{ax} \sin bx = ay_1 - b^2 y + abe^{ax} \cos bx$ so that $(y_2)_0 = ab - 0 + ab = 2ab$

$y_3 = ay_2 - b^2 y_1 + a^2 be^{ax} \cos bx - ab^2 e^{ax} \sin bx = ay_2 - b^2 y_1 - ab^2 y + a^2 be^{ax} \cos bx$

so that $(y_3)_0 = 2a^2 b - b^3 + a^2 b = 3a^2 b - b^3$, and so on

In general $y_n = (a^2+b^2)^{n/2} \sin\{bx + n\tan^{-1}(b/a)\}$ so that $(y_n)_0 = (a^2+b^2)^{n/2} \sin\{n\tan^{-1}(b/a)\}$

Now by Maclaurin's theorem, we have

$$y = (y)_0 + \frac{x}{1!} \cdot (y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots + \frac{x^n}{n!}(y_n)_0 + \ldots$$

$$= 0 + \frac{x}{1!} \cdot b + \frac{x^2}{2!} \cdot (2ab) + \frac{x^3}{3!}(3a^2 b - b^3) + \ldots + \frac{x^n}{n!}(a^2+b^2)^{n/2} \sin\{n\tan^{-1}(b/a)\} + \ldots$$

$$= bx + abx^2 + \frac{3a^2 b - b^3}{3!} x^3 + \ldots + \frac{(a^2+b^2)^{n/2}}{n!} x^n \sin\{n\tan^{-1}(b/a)\} + \ldots$$

(ii) Let $y = e^{ax} \cos bx$. Then $(y)_0 = e^0 \cos 0 = 1$

$y_1 = ae^{ax} \cos bx - be^{ax} \sin bx = ay - be^{ax} \sin bx$ so that $(y_1)_0 = a$

$y_2 = ay_1 - abe^{ax} \sin bx - b^2 e^{ax} \cos bx = ay_1 - b^2 y - abe^{ax} \sin bx$

so that $(y_2)_0 = a^2 - b^2$

$y_3 = ay_2 - b^2 y_1 - a^2 be^{ax} \sin bx - ab^2 e^{ax} \cos bx = ay_2 - b^2 y_1 - ab^2 y - a^2 be^{ax} \sin bx$

so that $(y_3)_0 = a(a^2 - b^2) - b^2 a - ab^2 = a(a^2 - 3b^2)$, and so on

In general, $y_n = (a^2+b^2)^{n/2} \cos\{bx + n\tan^{-1}(b/a)\}$ so that $(y_n)_0 = (a^2+b^2)^{n/2} \cos\{n\tan^{-1}(b/a)\}$

Substituting these values in Maclaurin's theorem, we get

$$e^{ax} \cos bx = 1 + ax + \frac{a^2 - b^2}{2!} x^2 + \frac{a(a^2 - 3b^2)}{3!} x^3 + \ldots + \frac{(a^2+b^2)^{n/2}}{n!} x^n \cos\{n\tan^{-1}(b/a)\} + \ldots$$

Deduction: Putting $a = \cos\alpha$ and $b = \sin\alpha$, we get

$(y_n)_0 = (\cos^2\alpha + \sin^2\alpha)^{n/2} \cos(n\tan^{-1}\tan\alpha) = \cos n\alpha$ so that

$(y_1)_0 = \cos\alpha$, $(y_2)_0 = \cos 2\alpha$, $(y_3)_0 = \cos 3\alpha$, etc.

$\therefore\ e^{x\cos\alpha} \cos(x\sin\alpha) = 1 + x\cos\alpha + (x^2/2!)\cos 2\alpha + (x^3/3!)\cos 3\alpha + \ldots$

6. **Expand** $e^{a\sin^{-1}x}$ **by Maclaurin's theorem and find the general term. Hence show that**

$$e^\theta = 1 + \sin\theta + \frac{1}{2!}\sin^2\theta + \frac{2}{3!}\sin^3\theta + \ldots$$

Solution :

Let $y = e^{a\sin^{-1}x}$

we get $y(0) = 1$, $y_1(0) = a$, $y_2(0) = a^2$ [Please see chapter on Successive Differentiation]

and $y_{n+2}(0) = (n^2 + a^2) y_n(0)$. ...(1)

Putting $n = 1, 2, 3, 4, \ldots$ in (1), we get

$y_3(0) = (1^2 + a^2) y_1(0) - (1^2 + a^2) a$, $y_4(0) = (2^2 + a^2) y_2(0)$

$= (2^2 + a^2) a^2$, $y_5(0) = (3^2 + a^2) y_3(0) = (3^2 + a^2)(1^2 + a^2) a$

$y_6(0) = (4^2 + a^2) y_4(0) = (4^2 + a^2)(2^2 + a^2) a^2$, etc.

In general, $y_n(0) = \begin{cases} a\ (1^2 + a^2)(3^2 + a^2)\ldots[(n-2)^2 + a^2] \text{ if } n \text{ is odd} \\ xa^2\ (2^2 + a^2)(4^2 + a^2)\ldots[(n-2)^2 + a^2] \text{ if } n \text{ is even} \end{cases}$

Substituting these values in Maclaurin's expansion

$$y + y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \ldots + \frac{x^n}{n!} y_n(0) + \ldots,$$

we get $e^{a\sin^{-1}x} = 1 + ax + \frac{a^2}{2!} x^2 + \frac{a(1^2 + a^2)}{3!} x^3 + \frac{a^2(2^2 + a^2)}{4!} x^4 + \ldots$ ...(2)

The general term is $(x^n/n!) y_n(0)$, where $y_n(0)$ is as given above.

Now putting $x = \sin\theta$ and $a = 1$ in (2), we get

$$e^\theta = 1 + \sin\theta + \frac{1}{2!}\sin^2\theta + \frac{2}{3!}\sin^3\theta + \ldots$$

Exercise

Using Taylor's series, prove that :

1. $e^{x+h} = e^x\left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots\ldots\right)$

2. $\frac{1}{x+h} = \frac{1}{x}\left(1 - \frac{h}{x} + \frac{h^2}{x^2} - \frac{h^3}{x^3} + \ldots\ldots\right)$

3. $\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \ldots\ldots$

4. $\sin^{-1}(x+h) = \sin^{-1}x + \frac{h}{\sqrt{1-x^2}} + \frac{x}{(1-x^2)^{3/2}} \cdot \frac{h^2}{2!} + \ldots\ldots$

5. $\tan(x+h) = \tan x + h\sec^2 x + h^2 \sec^2 x \tan x + \frac{h^3}{3}\sec^2 x(1 + 3\tan^2 x) + \ldots$

Using Taylor's series, expand :

6. e^x in powers of $(x-2)$. [Ans. $e^2\left[1+(x-2)+\frac{(x-2)^2}{2!}+\frac{(x-2)^3}{3!}+\ldots\ldots\right]$]

7. $\sin x$ in powers of $\left(x-\frac{\pi}{2}\right)$. [Ans. $1-\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}-\ldots\ldots$]

8. $\tan^{-1}x$ in powers of $(x-1)$ [Ans. $\frac{\pi}{4}+\frac{1}{2}(x-1)-\frac{1}{4}(x-1)2+\frac{1}{12}(x-1)3+\ldots\ldots$]

9. $2+x^2-3x^5+7x^6$ in powers of $(x-1)$.
[Ans. $7+29(x-1)+76(x-1)^2+110(x-1)^3+90(x-1)^4+39(x-1)^5+7(x-1)^6$]

Using Maclaurin's series prove that :

10. $e^{\sin x}=1+x+\frac{x^2}{2}-\frac{x^4}{8}+\ldots\ldots$

11. $\frac{x}{\sin x}=1+\frac{x^2}{6}+\frac{7x^4}{360}+\ldots\ldots$

12. $\log\sec x=\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\ldots\ldots$

13. $\log(1+\sin x)=x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\ldots\ldots$

14. $\frac{e^x}{1+e^x}=\frac{1}{2}+\frac{x}{4}-\frac{x^3}{48}+\ldots\ldots$

15. $e^{ax}\sin bx=bx+abx^2+\frac{b(3a^2-b^2)}{3!}x^3+\ldots\ldots$

16. $e^x\cos x=1+x-\frac{2x^3}{3!}-\frac{2^2x^4}{4!}\ldots\ldots$

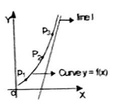
CHAPTER - 9

[Assymptotes]

9.1. [Definition of Assymptote]

A straight line l is called an assymptote of of a curve if only if the perpendicular distance of a point $p(x, y)$ on the curve from the straight line tends to zero as p moves to infinity along that curve.

Geometrically if $y=f(x)$ is a curve as shown in diagram and p_1, p_2 and p_3 are three points on it. We see that distance between the points and the straight line l keeps on reducing as we move along the curve and we can say that the distance will get reduced to zero as the point $p(x, y)$ will move along the curve at infinity. Thus the line l is assymptote to the curve.

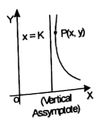


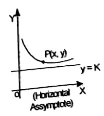
Remark:
A curve of degree n can not have more than n assymptotes real or imaginary.

9.2. [Types of Assymptotes]

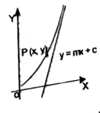
There are two types of assymptotes. (*i*) Rectangular Assymptotes (*ii*) Oblique Assymptotes.

Rectangular Assymptote:
If an assymptote is either parallel to x-axis or parallel to y-axis, then it is called a rectangular assymptote. An assymptote parallel to x-axis is called horizontal assymptote and an assymptote parallel to y-axis is called vertical assymptote.





Oblique Assymptote
Any assymptote which is neither parallel to x-axis or y-axis is called an oblique assymptote.



9.3. [Method For Finding Assymptote Parallel To Axes]

Rule:
(1) To find an assymptote parallel to x-axis equate to zero the real linear factors in the coefficient of highest power of x in the equation of the curve.
(2) To find an assymptote parallel to y-axis equate to zero the real linear factors in the coefficient of highest powers of y in the equation of the given curve.

Examples

1. Find the assymptotes parallel to axes of the curve $x^2y^2 + y^2 = 2$

Solution.

The equation of the given curve is $x^2y^2 + y^2 - 2 = 0$

The coefficient of highest power of x is y^2

$\therefore\quad y^2 = 0 \Rightarrow y = 0$ is the only assymptote parallel to x-axis

Now the coefficient of highest power of y is $(x^2 + 1)$

$\therefore\quad x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x$ has no real linear factors

Thus the given curve has no assymptote parallel to y-axis.

2. Find the assymptotes (if any) parallel to co-ordinate axes of the curve $4x^2 + 9y^2 = x^2y^2$

Solution.

The equation of the given curve is $4x^2 + 9y^2 - x^2y^2 = 0$

The terms of highest power of x is x^2 which has coefficient $(4 - y^2)$

Equating it to zero: $4 - y^2 = 0 \Rightarrow y = \pm 2$

$\therefore\quad y = 2, y = -2$ are two assymptotes parallel to x-axis

The coefficient of highest power of y is $(9 - x^2)$

Equality it to zero; $9 - x^2 = 0 \Rightarrow x = \pm 3$

$\therefore\quad x = 3, x = -3$ are assymptotes parallel to y-axis.

Exercise

1. Find the assymptotes (if any) parallel to coordinate axes of the following curves:

1. $3xy + 5x - 4y - 3 = 0$ **[Ans.** $3y + 5 = 0; 3x - y = 0$**]**
2. $x^2y^3 + x^2 + 3y^2 - 9xy + 8x - 25 = 0$ **[Ans.** $y + 1 = 0; x = 0$**]**
3. $(x^2 + y^2)x - ay^2 = 0$ **[Ans.** No Assymptote**]**
4. $x^2y^2 - a^2(x^2 + y^2) = 0$ **[Ans.** $y = \pm a, x = \pm a$**]**

9.4. [Method of Finding Oblique Assymptotes of a Curve]

The following procedure should be adapted to find oblique assymptotes of a given curve.

Let the equation of the curve be $\phi_n(x, y) + \phi_{n-1}(x, y) + ... + \phi_1(x, y) + c = 0$ ...(1)

Where $\phi_n(x, y)$ denotes the term of highest degree of curve.

Step 1:

Put $x = 1, y = m$ in $\phi_n(x, y), \phi_{n-1}(x, y)$ ... and all terms in (1)

Step 2:

Find all real roots of $\phi_n(m) = 0$

Step 3:

It m is a non-repeated root; then the corresponding value of c is given by

$$c\phi_n'(m) + \phi_{n-1}(m) = 0; \text{ provided } \phi_n'(m) \neq 0$$

If $\phi_n'(m) = 0$; then there is no assymptote to the curve corresponding to this value of m.

Step 4:

If m is a repeated root occuring twice; then the two values of care given by

$$\frac{c^2}{2!}\phi''_n(m) + \frac{c_1}{1!}\phi'_{n-1}(m) + \phi_{n-2}(m) = 0, \text{ provided } \phi''_n(m_{1)} \neq 0$$

Step 5:

The assymptote of the curve is $y = mx + c$.

Remark:

In order to find all assymptotes of a curve; first find assymptotes parallel to co-ordinate axes and then find oblique assymptotes.

Examples

3. Find all the assymptotes of the curve $x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0$.

Solution.

(a) In the curve the highest degree of x is 3 and its coefficient is 1; thus $1 = 0$; which is absurd; thus the curve has no assymptote parallel to x-axis.

Also the highest degree of y is 3 and its coefficient is 1; thus the curve has no assymptote parallel to y-axis.

(b) Now to find oblique assymptotes.

Here $\phi_3(x, y) = x^3 - x^2y + y^3 - xy^2$; $\phi_2(x, y) = 2x^2 - 4y^2 + 2xy$; $\phi_1(x, y) = x + y$ ...(1)

Let the assymptotes be given by $y = mx + c$

Step 1:

Putting $x = 1$ and $y = m$ in all terms of (1), we get

$$\phi_3(m) = m^3 - m^2 - m + 1; \phi_2(m) = 2 - 4m^2 + 2m; \phi_1(m) = 1 + m$$

Step 2:

The values of m are obtained by $\phi_3(m) = 0$

$$\therefore\quad m^3 - m^2 - m + 1 = 0 \Rightarrow (m^2 - 1)(m - 1) = 0 \Rightarrow m = 1, 1, -1$$

Here $m = -1$ is a non-repeated root and $m = 1$ is a repeated root

Step 3:

Fom $= -1$ (non-repeated root), the corresponding value of c is given by

$c\phi'_3(m) + \phi_2(m) = 0$; $\phi_3(m) = m^3 - m^2 - m + 1 \Rightarrow \phi'_3 = 3m^2 - 2m - 1$

or $\quad c(3m^2 - 2m - 1) + (2 + 2m - 4m^2) = 0$

or $\quad c(3m^2 - 2m - 1) = 4m^2 - 2m - 2$ or $c = \dfrac{4m^2 - 2m - 2}{3m^2 - 2m - 1}$

For $m = -1$ $\quad c = \dfrac{4 + 2 - 2}{3 + 2 - 1} = \dfrac{4}{4} = 1$ Thus for $m = -1$; $c = 1$

Now the assymptote is $y = mx + c$ i.e., $y = -x + 1$ or $y + x = 1$

Step 4:

For $m = 1$ (repeated root); the value of c is given by

$$\frac{c^2}{2}\phi''_3(m) + c\phi'_2(m + \phi_1(m) = 0 \quad ...(2)$$

Now
$$\phi'_3(m) = 3m^2 - 2m - 1 \Rightarrow \phi''_3(m) = 6m - 2$$
$$\phi_2(m) = 2 + 2m - 4m^2 \Rightarrow \phi'_2(m) = 2 - 8m$$
$$\phi_1(m) = 1 + m$$

Putting these values is (2); we get $\dfrac{c^2}{2}(6m - 2) + c(-8m + 2) + 1 + m = 0$

For $m = 1$; the above equation becomes

$$\frac{c^2}{2}(4) + c(-6) + 2 = 0 \Rightarrow c^2 - 3c + 1 = 0 \Rightarrow c = \frac{3 \pm \sqrt{5}}{2} = \frac{3 + \sqrt{5}}{2}, \frac{3 - \sqrt{5}}{2}$$

For $m = 1, c = \frac{3+\sqrt{5}}{2}$; the assymptote is $y = mx + c$

or $y = x + \frac{3+\sqrt{5}}{2}$ or $2y = 2x + (3+\sqrt{5})$

For $m = 1; c = \frac{3-\sqrt{5}}{2}$; the assymptote is

$y = mx + c$

or $2y = 2x + (3 - \sqrt{5})$

$\therefore$ the three assymptotes of the given curve as

$$x + y = 1; 2(y - x) = (3 + \sqrt{5}); 2(y - x) = 3 - \sqrt{5}$$

4. Find all assymptotes of the curve $x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 - 1 = 0$

Solution.

The equation of the given curve is $x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 - 1 = 0$ ...(1)

(*a*) Assymptotes parallel to axis

The coefficient of highest power of x in (1) is 1: thus $1 = 0$; which is absurd

$\therefore$ there is no assymptote parallel to x-axis

The coefficient of highest power of y in (1) is -2; thus $-2 = 0$, not possible;

thus there is no assymptote parallel to y-axis.

(*b*) Oblique Assymptotes: $\phi_3(x, y) = x^3 + 2x^2y - xy^2 - 2y^3$; $\phi_2(x, y) = xy - y^2$, $\phi_0(x, y) = -1$

Step 1:

Putting $x = 1$ and $y = m$ is ϕ_3 and ϕ_2, we get

$$\phi_3(m) = 1 + 2m - m^2 - 2m^3; \phi_2(m) = m - m^2; \phi_0(m) = -1$$

Step 2:

The values of m are obtained by $\phi_3(m) = 0$

$\Rightarrow 1 + 2m - m^2 - 2m^3 = 0$ or $2m^3 + m^2 - 2m - 1 = 0$

$\Rightarrow m^2(2m + 1) - 1(2m + 1) = 0 \Rightarrow (m^2 - 1)(2m + 1) = 0 \Rightarrow m = 1, -1, -\frac{1}{2}$.

Thus all the roots of m are non-repeating

Step 3:

The values of c will be obtained from the equation $c\phi'_3(m) + \phi_2(m) = 0$

Now $\phi'_3(m) = 2 - 2m - 6m^2$; $\phi_2(m) = m - m^2$

$\therefore c(2 - 2m - 6m^2) + (m - m^2) = 0 \Rightarrow c = \frac{m^2 - m}{2 - 2m - 6m^2}$

For $m = 1; c = \frac{1-1}{2-2-6} = 0 \Rightarrow$ Assymptote is $y = mx + c \Rightarrow y = x$

Form $m = -1; c = \frac{1+1}{2+2-6} = \frac{2}{-2} = -1$

$\therefore$ Assymptote is $y = mx + c \Rightarrow y = (-1)x + (-1) \Rightarrow y + x + 1 = 0$

Form $m = -\frac{1}{2}; c = \frac{\left(-\frac{1}{2}\right)^2 + \frac{1}{2}}{2 - 2\left(-\frac{1}{2}\right) - 6\left(-\frac{1}{2}\right)^2} = \frac{\frac{3}{4}}{\frac{3}{2}} = \frac{1}{2}$

$\therefore$ Assymptote is $y = \frac{-1}{2}x + \frac{1}{2}$ or $x + 2y = 1$

Hence assymptotes of the given curve are $x - y = 0; x + y + 1 = 0; x + 2y = 1$.

Find the assymptotes of the curve $y^3 + x^2y + 2xy^2 - y + 1 = 0$

olution.

(*a*) Assymptotes parallel to co-ordinate axes:

The highest power of y is y^3 whose coefficient is 1; thus $1 = 0$; not possible $\Rightarrow$ No assymptote parallel to y-axis.

The coefficient of highest power of x (*i.e.*, x^2) is y; thus $y = 0$ is the required assymptote parallel to x-axis.

(*b*) To find oblique Assymptotes $\phi_3(x, y) = y^3 + x^2y + 2xy^2$; $\phi_2(x, y) = 0$; $\phi_1(x, y) = -y$; $\phi_0(x, y) = 1$

Step 1:

Putting $x = 1$, $y = m$ in above terms, we get

$\phi_3(m) = m^3 + m + 2m^2$; $\phi_2(m) = 0$; $\phi_1(m) = -m$, $\phi_0(m) = 1$

$\therefore \phi'_3(m) = 3m^2 + 4m + 1$

Step 2:

The values of m are obtained by $\phi_3(m) = 0$

i.e., $m^3 + 2m^2 + m = 0 \Rightarrow m(m^2 + 2m + 1) = 0 \Rightarrow m = 0, -1, -1$

Step 3:

For $m = 0$ (non-repeating value); the value of c is given by

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{0}{3m^2 + 4m + 1} = 0 \quad \therefore y = 0 \text{ is the assymptote}$$

Note:

For $m = 0$; you are not required to find assymptote as it is already found in assymptotes parallel to co-ordinate axes.

Step 4:

For $m = -1, -1$ (repeating value)

$$\frac{c^2}{2!}\phi''_3(m) + \frac{c}{1!}\phi''_2(m) + \phi_1(m) = 0$$

$$\frac{c^2}{2}(6m + 4) + c(0) - m = 0 \quad [\because \phi'_3(m) = 3m^2 + 4m + 1 \Rightarrow \phi''_3(m) = 6m + 4 \text{ and } \phi_1(m) = -m]$$

$\Rightarrow (3m + 2)c^2 - m = 0$

Putting $m = -1$ in this equation, we get

$$-c^2 + 1 = 0 \Rightarrow c^2 = 1 \Rightarrow c = \pm 1$$

$\therefore$ Assymptotes for $m = -1; c = 1$ is $y = -x + 1 \Rightarrow y + x - 1 = 0$

and assymptotes for $m = -1, c = -1$ is $y = -x - 1 \Rightarrow y + x + 1 = 0$

Thus the assymptotes of the given curve are

$$y = 0, \ y + x - 1 = 0, \text{ and } y + x + 1 = 0$$

Find all assymptotes of the curve

$3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0$

tion.

(*a*) There are no assymptotes parallel to coordinate axes.

(*b*) $\phi_3(x, y) = 3x^3 + 2x^2y - 7xy^2 + 2y^3$; $\phi_2(x, y) = -14xy + 7y^2$

$\phi_1(x, y) = 4x + 5y$

Step 1:

Put $x = 1$, $y = m$ in above terms

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$\phi_3(m) = 3 + 2m - 7m^2 + 2m^3;\ \phi_2(m) = -14m + 7m^2$

$\phi_1(m) = 4 + 5m$

$\phi'_3(m) = 2 - 14m + 6m^2$

Step 2:

The values of m are obtained by $\phi_3(m) = 0$

$3 + 2m - 7m^2 + 2m^3 = 0 \Rightarrow (m-1)(2m^2 - 5m - 3) = 0$

$\Rightarrow (m-1)(2m+1)(m-3) = 0 \Rightarrow m = 1, 3, -\frac{1}{2}$

Step 3:

The value of c is given by

$$c = \frac{\phi_2(m)}{\phi'_3(m)} = \frac{14m - 7m^2}{2 - 14m + 6m^2}$$

For $m = 1: c = \frac{14-7}{2-14+6} = -\frac{7}{6}$

For $m = 3: c = -1$

For $m = -\frac{1}{2}; c = -\frac{5}{6}$

Hence assymptotes are

$m = 1: c = -\frac{7}{6} \Rightarrow y = x - \frac{7}{6} \Rightarrow 6y - 6x + 7 = 0$

$m = 3; c = -1 \Rightarrow y = 3x - 1$

$m = -\frac{1}{2}; c = -\frac{5}{6} \Rightarrow y = -\frac{x}{2} - \frac{5}{6} \Rightarrow 6y + 3x + 5 = 0$

7. Find all the asympotes of

$$y^4 - 2xy^3 + 2x^3y - x^4 - 3x^3 + 3x^2y + 3xy^2 - 3y^3 - 2x^2 + 2y^2 - 1 = 0.$$

[BCA I.P. 2011]

Solution:

Let $\phi(x, y) = (y^4 - 2xy^3 + 2x^3y - x^4) + (-3x^3 + 3x^2y + 3xy^2 - 3y^3) - 2x^2 + 2y^2 - 1 = 0$

Since the coefficients of x^4 and y^4 are constants. Therefore, there is no asymptote parallel to the co-ordinate axes.

Now putting $x = 1, y = m$

$\phi(m) = (m^4 - 2m^3 + 2m - 1) + (-3 + 3m + 3m^2 - 3m^3) + (-3 + 2m^2) - 1 = 0$

Here $\phi_4(m) = m^4 - 2m^3 + 2m - 1$

$\phi_3(m) = -3 + 3m + 3m^2 - 3m^3$

$\phi_2(m) = -2 + 2m^2$

$\phi_1(m) = 0$

Putting $\phi_4(m) = 0$

$\Rightarrow m^4 - 2m^3 + 2m - 1 = 0$

$\Rightarrow (m^4 - 1) - 2m(m^2 - 1) = 0$

$\Rightarrow (m^2 - 1)(m-1)^2 = 0$

$\Rightarrow m = 1, 1, 1, -1$

Now $c = \frac{-\phi_3(m)}{\phi'_4(m)} = \frac{-(-3 + 3m + 3m^2 - 3m^3)}{4m^3 - 6m^2 + 2}$

when $m = -1$, $c = \frac{-(-3-3+3+3)}{-4-6+2} = 0$

$\therefore$ Asymptote is $y = -x$

Since $m = 1$ repeat thrice, therefore, to obtain values of c, we use

$$\frac{c^3}{3!}\phi_4'''(m) + \frac{c^2}{2!}\phi_3''(m) + c\cdot\phi_2'(m) + \phi_1(m) = 0$$

[see note I]

i.e., $\frac{c^3}{6}(24m - 12) + \frac{c^2}{2}(6 - 18m) + c(4m) + 0 = 0$

Putting $m = 1$, we get $\Rightarrow 2c^3 - 6c^2 + 4c = 0 \Rightarrow c = 0, 1, 2$

Thus asymptotes are $y = x,\ y = x + 1,\ y = x + 2.$

Hence, the required asymptotes are

$$y + x = 0,\quad y - x = 0,\quad y - x = 1,\quad y - x = 2.$$

8. Find all the asymptotoes to the curve

$$(x - y)^2 (x - 2y)(x - 3y) - 2a(x^3 - y^3) - 2a^2(x - 2y)(x + y) = 0$$

Solution:

The equation of the curve is

$$(x - y)^2 (x - 2y)(x - 3y) - 2a(x^3 - y^3) - 2a^2(x - 2y)(x + y) = 0 \quad ...(1)$$

Asymptotes parallel to the co-ordinate axes:

Since the coefficients of x^4 and y^4 are constant, therefore there are no asymptotes parallel to the co-ordinate axes.

Oblique Asymptotes:

Putting $x = 1, y = m$ in the fourth, third and second degree terms of (1), one by one, we get

$\phi_4(m) = (1 - m)^2 (1 - 2m)(1 - 3m)$

$\phi_4(m) = -2a(1 - m^3)$

$\phi_2(m) = -2a^2(1 - 2m)(1 + m)$

Slopes of the asymptotes are roots of equation $\phi_3(m) = 0$

i.e., $(1 - m)^2 (1 - 2m)(1 - 3m) = 0 \Rightarrow m = 1, 1, \frac{1}{2}, \frac{1}{3}$

Now $\phi_4'(m) = -2(1 - m)(1 - 2m)(1 - 3m) - 2(1 - m)^2(1 - 3m) - 3(1 - m)^2(1 - 2m)$

$\phi_4''(m) = -2[-(1 - 2m)(1 - 3m) - 2(1 - m)(1 - 3m) - 3(1 - m)(1 - 2m)]$

$-2[-2(1 - m)(1 - 3m) - 3(1 - m)^2] - 3[-2(1 - 2m) - 2(1 - m)]$

and $\phi_3'(m) = 6am^2$

When $m = 1, \phi_4''(m) = 4,\ \phi_3'(m) = 6a,\ \phi_2(m) = 4a^2$

Thus c is given by $\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0$

i.e., $\frac{c^2}{2}\cdot 4 + c\cdot 6a + 4a^2 = 0$

or $c^2 + 3ac + 2a^2 = 0$

or $(c + a)(c + 2a) = 0 \Rightarrow c = -a, -2a.$

Thus the two corresponding parallel asymptotes are $y = x - a$ and $y = x - 2a$.

When $m = \frac{1}{2}, \phi_2'(m) = -2\left(\frac{1}{4}\right)\left(-\frac{1}{2}\right) = \frac{1}{4}$

$$\phi_3(m) = -2a\left(1-\frac{1}{8}\right)=\frac{7a}{4}$$

$$c = -\frac{\phi_3(m)}{\phi_4'(m)} = \frac{4}{1/4} = 7a$$

$\therefore$ The corresponding asymptote is $y = \frac{1}{2}x + 7a \Rightarrow x - 2y + 14a = 0$

When $\quad m = \frac{1}{3}, \quad \phi_4'(m) = -3\left(\frac{4}{9}\right)\left(\frac{1}{3}\right) = -\frac{4}{9}$

$$\phi_3(m) = -2a\left(\frac{26}{27}\right) = -\frac{52}{27}a$$

$$\therefore \quad c = -\frac{\phi_3(m)}{\phi_4'(m)} = \frac{\frac{52}{27}a}{-\frac{4}{9}} = -\frac{52}{27}a \times \frac{9}{4} = -\frac{13}{3}a$$

Thus the corresponding asymptote is

$$y = \frac{1}{3}x - \frac{13}{13}a \Rightarrow x - 3y - 13a = 0$$

Hence the asymptote to the curve are

$$y = x - a,\ y = x - 2a,\ x - 2y + 14a = 0,\ x - 3y - 13a = 0$$

Exercise

Find all the assymptotes of the following curves:

1. $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$ [Ans. $x - y = 0; x + 2y + 1 = 0, x + 2y - 1 = 0$]
2. $x^3 - x^2y - xy^2 + y^3 + 2x^3 - 4y^2 + 2xy + x + y + 1 = 0$ [Ans. $x + y = 1; x - y + \frac{3+\sqrt{5}}{2} = 0, x - y + \frac{3-\sqrt{5}}{2} = 0$]
3. $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$ [Ans. $x + 2y = 0; x + y - 1 = 0; x - y + 1 = 0$]
4. $x^3 + 6x^2y + 11xy^2 + 6y^3 + 3x^2 + 12xy + 11y^2 + 2x + 3y + 5 = 0$ [Ans. $x + y + 1 = 0; x + 2y + 1 = 0; x + 3y + 1 = 0$]
5. $3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0$ [Ans. $6x - 6y - 7 = 0, 6x - 2y - 3 = 0, 3x + 6y + 5 = 0$]
6. $x^3 - 4x^2y + 5xy^2 - 2y^3 + 3x^2 - 4xy + 2y^2 - 3x + 2y - 1 = 0$ [Ans. $x - 2y + 6 = 0$]
7. $3x^3 + 17x^2y + 21xy^2 - 9y^3 - 2ax^2 + 12axy - 18ay^2 - 3a^2x + a^2y = 0$ [Ans. $3x - y - 2a = 0; x + 3y + a = 0; x + 3y - a = 0$]

9.5 [Assymptotes of a Polar Curve]

Assymptotes of a Polar Curve : To find the assymptotes of a polar curve, we follow the following procedure.

(1) Express the equation of the given curve by $\frac{1}{r} = f(\theta)$

(2) Let α be a root of the equation $f(\theta) = 0$

(3) (i) $r\sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$ is an assymptote of the curve, provided $f'(\alpha) \neq 0$

(ii) If $f'(\alpha) = 0$, then the curve has no assymptote.

Example

Find the assymptotes of the curve $r(\pi+\theta) = ae^{\theta}$. [BCA IP 2009]

Solution.

Step 1. Here $r = \frac{ae^{\theta}}{\pi+\theta} \Rightarrow \frac{1}{r} = \frac{\pi+\theta}{ae^{\theta}} = \frac{1}{a}(\pi+\theta)e^{-\theta} = f(\theta)$

Step 2. Now $f(\theta) = 0 \Rightarrow \frac{1}{a}(\pi+\theta)e^{-\theta} = 0 \Rightarrow \pi+\theta = 0 \Rightarrow \theta = -\pi \left[\because \frac{1}{a} \neq 0 \text{ and } e^{-\theta} \neq 0\right]$

Now $f'(\theta) = \frac{1}{a}\left[e^{-\theta} + (\pi+\theta)(-e^{-\theta})\right] = \frac{e^{-\theta}}{a}[1-(\pi+\theta)]$

$\therefore \quad f'(\alpha) = f'(-\pi) = \frac{e^{\pi}}{a}[1-(\pi-\pi)] = \frac{e^{\pi}}{a}$

Step 3. The assymptotes are given by

$$r\sin(\theta-\alpha) = \frac{1}{f'(\alpha)} \Rightarrow r\sin(\theta+\pi) = \frac{a}{e^{\pi}} \Rightarrow -r\sin\theta e^{\pi} = a$$

$\Rightarrow r\sin\theta e^{\pi} + a = 0$ is the required assymptote.

Find all the assymptotes of the curve $r\sin n\theta = a$.

Solution.

Step 1. Here $\frac{1}{r} = \frac{\sin n\theta}{a} = f(\theta)$; where $f(\theta) = \frac{1}{a}\sin(n\theta)$

Step 2. $f(\theta) = 0 \Rightarrow \sin n\theta = 0 \Rightarrow n\theta = k\pi$, k is any integer $\Rightarrow \theta = \frac{k\pi}{n}$

Thus if α is root of equation $f(\theta) = 0 \Rightarrow \alpha = \frac{k\pi}{n}$. Here $f'(\theta) = \frac{n\cos n\theta}{a}$

$$\Rightarrow f'(\alpha) = f'\left(\frac{k\pi}{n}\right) = \frac{n\cos k\pi}{a}$$

Step 3. The assymptote is given by $r\sin(\theta-\alpha) = \frac{1}{f'(\alpha)}$

$$\Rightarrow r\sin\left(\theta - \frac{k\pi}{n}\right) = \frac{a}{n\cos k\pi},\ k \text{ is any integer}$$

Find the assymptote of the curve $r\log\theta = a$.

Solution.

Step 1. Here $r = \frac{a}{\log\theta} \Rightarrow \frac{1}{r} = \frac{1}{a}\log\theta = f(\theta)$

Step 2. $f(\theta) = 0 \Rightarrow \log\theta = 0 \Rightarrow \theta = 1 \therefore \alpha = 1$, Now $f'(\theta) = \frac{1}{a\theta} \Rightarrow f'(1) = \frac{1}{a}$

Step 3. The assymptote is $r\sin(\theta-\alpha) = \frac{1}{f'(\alpha)} \Rightarrow r\sin(\theta - 1) = a$ is the required assymptote.

Exercise

Find the assymptotes of the following curves:

1. $r = \frac{1}{1-2\sin\theta}$ [Ans: $r\sin\left(\theta - \frac{\pi}{6}\right) + \frac{1}{\sqrt{3}} = 0$]
2. $r\sin\theta = ae^{\theta}$ [Ans : $r\sin\theta = ae^{n\pi}$]
3. $r = a\tan\theta$ [Ans: $r\cos\theta = \pm a$]

Miscellaneous Examples

12. Show that the parabola $x^2 = 4ay$ has no assymptotes.

Solution.

The equation of parabola is $x^2 - 4ay = 0$

(a) There is no assymptote parallel to co-ordinate axis.

(b) Oblique Assymptote.

$\phi_2(x, y) = x^2$; $\phi_1(x, y) = -4ay$

Putting $x = 1$; $y = m$

$\phi_2(m) = 1$; $\phi_1(m) = -4am$

Now value of m are obtained by $\phi_2(m) = 0 \Rightarrow 1 = 0$; which is absurd.

Thus there are no oblique assymptotes.

$\therefore$ The given curve has no assymptotes.

13. Find all assymptotes of the curve $(x + y)^2 (x + 2y + 2) = x + 9y + 2$.

Solution.

The equation of the given curve is

$$(x + y)^2 (x + 2y) + 2(x + y)^2 - (x + 9y) - 2 = 0$$

$\therefore$ $\phi_3(x, y) = (x + y)^2 (x + 2y)$; $\phi_2(x, y) = 2(x + y)^2$; $\phi_1(x, y) = -(x + 9y)$

Putting $x = 1$, $y = m$, we get

$\phi_3(m) = (1 + m)^2 (1 + 2m)$, $\phi_2(m) = 2(1 + m)^2$, $\phi_1(m) = -(1 + 9m)$

$\therefore$ $\phi'_3(m) = 2(1 + m)(1 + 2m) + 2(1 + m)^2$

Now $\phi_3(m) = 0 \Rightarrow (1 + m)^2 (1 + 2m) = 0 \Rightarrow m = \frac{1}{2}, -1, -1$

$$\text{For } m = -\frac{1}{2};\ c = \frac{-\phi_2(m)}{\phi'_3(m)} = \frac{-2(1+m)^2}{2(1+m)(1+2m) + 2(1+m)^2}$$

$$= \frac{-2\left(1 - \frac{1}{2}\right)^2}{2\left(1 - \frac{1}{2}\right)(1 - 1) + 2\left(1 - \frac{1}{2}\right)^2} = -1$$

$\therefore$ the corresponding assymptote is $y = -\frac{1}{2}x - 1$

or $\quad 2y + x + 2 = 0 \quad$ ...(1)

For $m = -1$ (Repeating root); c is determined by equation

$$\frac{c^2}{2!}\phi''_3(m) + \frac{c}{1!}\phi'_2(m) + \phi_1(m) = 0 \quad \text{...(2)}$$

Now $\phi''_3(m) = 2(1 + 2m) + 4(1 + m) + 4(1 + m) = 4 + 4m + 8 + 8m = 12 + 12m$

$\phi'_2(m) = 4(1 + m) = 4 + 4m$; $\phi_1(m) = -(1 + 9m)$

Putting these values in (2); we get

$$\frac{c^2}{2}[12 + 12m] + c[4 + 4m] - (1 + 9m) = 0$$

Putting $m = -1$, we get

$$-c^2 + 8 = 0 \Rightarrow c = \pm 2\sqrt{2}$$

$\therefore$ Corresponding assymptote are $y = -x + 2\sqrt{2}$

and $\quad y = -x - 2\sqrt{2} \quad$ ...(3)

From (1) and (3); we find that the required assymptote are

$$2y + x + 2 = 0 \text{ and } y + x = 2\sqrt{2};\ y + x = -2\sqrt{2}$$

Find the assymptotes of the curve $x^2y^2 = a^2 (x^2 + y^2)$

lutions.

e given curve is of degree 4; thus it can not have more than four assymptotes.

Equating to zero the coefficient of highest power of x (i.e., of x^2)

$$y^2 - a^2 = 0 \Rightarrow y = \pm a$$

Equating to zero the coefficient of highest power of y (i.e., y^2)

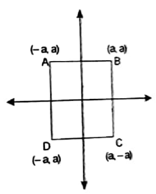
$$x^2 - a^2 = 0 \Rightarrow x = \pm a$$

Thus all the four assymptotes of the curve are $x = \pm a$, $y = \pm a$.

Show that the assymptotes of the curve $x^2 y^2 - a^2 (x^2 + y^2) - a^3 (x + y) + a^4 = 0$ form a square through two of whose angular points the curve passes.

lution.

Equating to zero the coefficient of highest power of x (i.e., of x^2); we get the assymptotes parallel to x-axis as $y^2 - a^2 = 0$ i.e., $y = \pm a$.



Similarly the assymptotes parallel to y-axis are $x = \pm a$. Now all these four assymptotes form a square as shown in diagram.

The angular points $(a, -a)$ and $(-a, a)$ satisfy the equation of curve; thus the curve passes through these two points.

CHAPTER - 10

[Integration]

10.1 [Introduction]

Integration is the reverse process of differentiation. If $f(x)$, $g(x)$ are two functions of x such that

$\frac{d}{dx}[f(x)] = g(x)$; then we say that $f(x)$ is integral of $g(x)$.

We write it as $\int g(x)dx = f(x)$

e.g. (i) $\frac{d}{dx}(\sin x) = \cos x$, thus integral of $\cos x$ is $\sin x$ i.e. $\int \cos x dx = \sin x$

(ii) $\frac{d}{dx}(x^2) = 2x$; thus integral of $2x$ is x^2 i.e. $\int 2x dx = x^2$

10.2 [Indefinite Integral]

From derivatives; we know that

$\frac{d}{dx}(\sin x) = \cos x$

and $\frac{d}{dx}(\sin x + 3) = \cos x$; but in integration

$\int \cos x dx = \sin x$; $\int \cos x dx = \sin x + 3$ as by definiton.

Thus integral of a function can be infinite. This type of integral is called indefinite integral. Thus the formula is modified and is written as

$\int \cos x dx = \sin x + c$; where c=constant of integration.

10.3 [Standard Formulae of Integration]

Based on the definition, following are formulae of some standard functions in integration.

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c; n \neq -1$ (2) $\int a^x dx = \frac{a^x}{\log_e a} + c$

(3) $\int \frac{1}{x} dx = \log_e x + c$ (4) $\int e^x dx = e^x + c$

(5) $\int 1 dx = x + c$ (6) $\int \cos x dx = \sin x + c$

(7) $\int \sin x dx = -\cos x + c$ (8) $\int \sec^2 x dx = \tan x + c$

(9) $\int \operatorname{cosec}^2 x dx = -\cot x + c$ (10) $\int \sec x \tan x dx = \sec x + c$

(11) $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$ (12) $\int \tan x dx = -\log|\cos x| + c$ or $\log|\sec x| + c$

$\int \operatorname{cosec} x dx = \log\left|\tan\frac{x}{2}\right| + c$ or $\log|\operatorname{cosec} x - \cot x| + c$

$\int \sec x dx = \log|\sec x + \tan x| + c$ or $\log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

$\int \cot x dx = \log|\sin x| + c$ (16) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$ (18) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$ (20) $\int \frac{dx}{\sqrt{a^2 + x^2}} = \log\left|x + \sqrt{a^2 + x^2}\right| + c$

$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + c$ (22) $\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

$\int \sqrt{a^2 + x^2} dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log\left|x + \sqrt{a^2 + x^2}\right| + c$

$\int \sqrt{x^2 - a^2} dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$

:

$\int cf(x)dx = c\int f(x)dx$ [where c is a constant]

$\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$ [This can also be generated to 'n' number of functions.]

Examples

$\int\left(x^3 + 5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}\right)dx$.

:

$\int x^3 dx + 5\int x^2 dx - 4\int 1.dx + 7\int \frac{1}{x}dx + 2\int \frac{1}{\sqrt{x}}dx$

$= \frac{x^4}{4} + \frac{5x^3}{3} - 4x + 7\int x^{-1}dx + 2\int x^{-1/2}dx = \frac{x^4}{4} + \frac{5x^3}{3} - 4x - 7\log x + 2\frac{x^{-1/2+1}}{-1/2+1} + c$

$= \frac{x^4}{4} + \frac{5x^3}{3} - 4x - 7\log x + 4\sqrt{x} + c$

$\int(3\sin x - 2\cos x + 4\sec^2 x - 5\operatorname{cosec}^2 x)dx$.

:

$3\int \sin dx - 2\int \cos x dx + 4\int \sec^2 x dx - 5\int \operatorname{cosec}^2 x dx$

$= -3\cos x - 2\sin x + 4\tan x - 5(-\cot x) + c = -3\cos x - 2\sin x + 4\tan x + 5\cot x + c$

$\int\left(\frac{(1+x)^3}{\sqrt{x}}\right)dx$.

:

$\int\left(\frac{(1+x)^3}{\sqrt{x}}\right)dx = \int \frac{1 + x^3 + 3x + 3x^2}{\sqrt{x}}dx$

$$= \int \frac{1}{\sqrt{x}}dx + \int \frac{x^3}{\sqrt{x}}dx + 3\int \frac{x}{\sqrt{x}}dx + 3\int \frac{x^2}{\sqrt{x}}dx = \int x^{-1/2}dx + \int x^{5/2}dx + 3\int x^{1/2}dx + 3\int x^{3/2}dx$$

$$= \frac{2x^{1/2}}{1} + \frac{2x^{7/2}}{7} + 3.\frac{2x^{3/2}}{3} + 3.\frac{2x^{5/2}}{5} + c = 2\sqrt{x} + \frac{2}{7}x^{7/2} + 2x^{3/2} + \frac{6}{5}x^{5/2} + c$$

4. $\int \frac{2+3\cos x}{\sin^2 x}dx$.

Solution:

$$= 2\int \operatorname{cosec}^2 x\, dx + 3\int \frac{\cos x}{\sin^2 x}dx = -2\cot x + 3\int \operatorname{cosec} x.\cot x$$

$$= -2\cot x + 3(-\operatorname{cosec} x) + c$$

5. $\int \frac{2+3\cos x}{\sin^2 x}dx$.

Solution:

$$\int \frac{2\sin^2 x}{2\cos^2 x}dx = \int \tan^2 x dx = \int (\sec^2 x - 1)dx = \int \sec^2 x dx - \int dx = \tan x - x + c$$

6. $\int \frac{1}{1-\sin x}dx$.

Solution:

$$\int \frac{1}{1-\sin x} \times \frac{1+\sin x}{1+\sin x}dx = \int \frac{1+\sin x}{1-\sin^2 x}dx$$

$$= \int \frac{1+\sin x}{\cos^2 x}dx = \int \sec^2 x\, dx + \int \tan x \sec x dx = \tan x + \sec x + c$$

7. $\int (2x-3)\,(3+2x)\,(1-2x)dx$.

Solution:

$$\int -(3-2x)(3+2x)(1-2x)dx = -\int ((3)^2 - (2x)^2)\,(1-2x)dx$$

$$= -\int (9-4x^2)(1-2x)dx = -\int (9-18x-4x^2+8x^3)dx$$

$$= -\int 9dx + 18\int xdx + 4\int x^2 dx - 8\int x^3 dx = -9x + 18\frac{x^2}{2} + 4\frac{x^3}{3} - 8\frac{x^4}{4} + c$$

$$= -9x + 9x^2 + \frac{4}{3}x^3 - 2x^4 + c$$

10.4 [Integration By Substitution]

Substition method is used to evaluate integrals of the form $\int f(\phi(x))\phi'(x)dx$ ...(1)

We substitute $\phi(x) = t$ and $\phi'(x)dx = dt$. so (1) reduces to integral $\int f(t)dt$ and after solving this integral we substitute back the value of t.

Examples

8. $\int \frac{\sec^2 x}{3+\tan x}dx$.

Solution:

Put $\tan x = t$

$$\frac{dt}{dx} = \sec^2 x \Rightarrow dt = \sec^2 x\, dx \quad ; \int \frac{\sec^2 x}{3+\tan x}dx = \int \frac{dt}{3+t} = \log|3+t| + c = \log|3+\tan x| + c$$

$\int \frac{1-\sin x}{x+\cos x}dx$.

Solution:

Put $x + \cos x = t$

$$\frac{dt}{dx} = 1 - \sin x \Rightarrow dt = (1-\sin x)dx$$

$$\int \frac{1-\sin x}{x+\cos x}dx = \int \frac{dt}{t} = \log|t| + c = \log|x+\cos x| + c$$

$\int \frac{e^{2x}}{e^{2x}-2}dx$.

Solution:

Put $e^{2x} - 2 = t$

$$\frac{dt}{dx} = \frac{d}{dx}(e^{2x}) - \frac{d}{dx}(2) = 2e^{2x}$$

$$\frac{dt}{dx} = 2e^{2x} \Rightarrow \frac{dt}{2} = e^{2x}dx$$

$$\int \frac{1}{2}\frac{dt}{t} = \frac{1}{2}\log|t| + c = \frac{1}{2}\log|e^{2x}-2| + c$$

$\int \frac{\cos 2x}{(\sin x+\cos x)^2}dx$.

Solution:

$$I = \int \frac{\cos^2 x - \sin^2 x}{(\sin x+\cos x)^2}dx$$

$$= \int \frac{(\cos x-\sin x)(\cos x+\sin x)}{(\sin x+\cos x)^2}dx = \int \frac{\cos x-\sin x}{\sin x+\cos x}dx$$

Put $\sin x + \cos x = t$; $\cos x - \sin x = \frac{dt}{dx}$

$$\int \frac{\cos 2x}{(\sin x+\cos x)^2}dx = \int \frac{dt}{t} = \log|t| + c = \log|\sin x+\cos x| + c$$

$\int \frac{\sin 2x}{a^2\sin^2 x + b^2\cos^2 x}dx$.

Solution:

Put $a^2\sin^2 x + b^2\cos^2 x = t$

$$(2a^2\sin^2 x\cos x - 2b^2\cos x\sin x)dx = dt$$

$$dx = \frac{dt}{2\sin x\cos x(a^2-b^2)} = \frac{dt}{\sin 2x(a^2-b^2)}$$

$$\int \frac{\sin 2x}{a^2\sin^2 x+b^2\cos^2 x}dx = \int \frac{\sin 2x}{t.\sin 2x(a^2-b^2)}.dt = \int \frac{dt}{t(a^2-b^2)}$$

$$= \frac{1}{(a^2-b^2)}\log|t| + c = \frac{1}{(a^2-b^2)}\log|a^2\sin^2 x + b^2\cos^2 x| + c$$

10.5 [Integration by Parts]

If $f(x)$ and $g(x)$ are two functions of x, then let $f(x)$ be first function and $g(x)$ be second function.

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left(\frac{d}{dx}(f(x))\int g(x)dx\right)dx$$

First function × (Integral of second function) – Integral of [(Differentiation of first function) (Integral of second function)]

Remark : We choose the first function which comes first in the word I L A T E

I = Inverse trigonometric Functions
L = Logarithmic Functions
A = Algebraic Functions
T = Trigonometric Functions
E = Exponential Functions

Examples

13. $\int x^2 \sin x\, dx.$

Solution : $I = \int \underset{\text{I}}{x^2} \underset{\text{II}}{\sin x}\, dx$

$$= x^2 \left\{\int \sin x\, dx\right\} - \int\left\{\frac{d}{dx}(x^2)\int \sin x\, dx\right\} dx$$

$$= -x^2 \cos x - \int (2x)(-\cos x)\, dx = -x^2 \cos x + 2\int \underset{\text{I}}{x} \underset{\text{II}}{\cos x}\, dx$$

$$= -x^2 \cos x + 2\left[x\left\{\int \cos x\, dx\right\} - \int\left\{\frac{d}{dx}(x)\int \cos x\, dx\right\}dx\right]$$

$$= -x^2 \cos x + 2\left[x \sin x - \int 1\cdot \sin x\, dx\right] = -x^2 \cos x + 2x \sin x - 2(-\cos x) + c$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + c$$

14. $\int (\log x)^2\, dx.$

Solution:

$$I = \int \underset{\text{I}}{(\log x)^2} \underset{\text{II}}{1}\, dx$$

$$= (\log x)^2 \int 1 \cdot dx - \int\left\{\frac{d}{dx}(\log x)^2 \int 1\cdot dx\right\} dx = (\log x)^2 x - \int 2\log x \cdot \frac{1}{x}\cdot x\, dx$$

$$= (\log x)^2 x - 2\int \log x\, dx = (\log x)^2 x - 2\int \underset{\text{I}}{\log x}\cdot \underset{\text{II}}{1}\, dx$$

$$= (\log x)^2 x - 2\left[\log x \int 1\cdot dx - \int\left\{\frac{d}{dx}(\log x)\int 1\cdot dx\right\}\right] dx$$

$$= (\log x)^2 x - 2\left[\log x\, x - \int\left(\frac{1}{x}\right)(x)\, dx\right] = (\log x)^2 x - 2\left[\log x \cdot x - \int 1\cdot dx\right]$$

$$= (\log x)^2 x - 2[\log x \cdot x - x] + c$$

$$= (\log x)^2 x - 2\log x \cdot x + 2x + c$$

$\int \tan^{-1} x\, dx.$

tion:

$$I = \int \underset{\text{I}}{(\tan^{-1} x)} \underset{\text{II}}{(1)}\, dx$$

$$= \tan^{-1} x \int 1\cdot dx - \int\left\{\int \frac{d}{dx}(\tan^{-1} x)\int 1\cdot dx\right\} dx = \tan^{-1} x \cdot x - \int \frac{1}{1+x^2}\cdot x \cdot dx$$

$$= \tan^{-1} x \cdot x - \frac{1}{2}\int \frac{2x}{1+x^2}\, dx$$

Put $1 + x^2 = t \Rightarrow 2x\, dx = dt = \tan^{-1} x \cdot x - \frac{1}{2}\int \frac{dt}{t} = \tan^{-1} x \cdot x - \frac{1}{2}\log|t| + c$

$$= \tan^{-1} x \cdot x - \frac{1}{2}\log|1 + x^2| + c$$

$\int \sin(\log x)\cdot dx.$

ion:

$$= \int \underset{\text{I}}{\sin(\log x)}\cdot \underset{\text{II}}{1}\cdot dx = \sin(\log x)\cdot x - \int\left[\frac{d}{dx}(\sin(\log x))\cdot \int 1\cdot dx\right]dx$$

$$= \sin(\log x)\cdot x - \int\left[\cos(\log x)\cdot \frac{d}{dx}(\log x)\int 1\cdot dx\right]dx$$

$$= \sin(\log x)\cdot x - \int\left[\cos(\log x)\cdot \frac{1}{x}\cdot x\right]dx = \sin(\log x)\cdot x - \int[\cos(\log x)\cdot 1]\, dx$$

$$= \sin(\log x)x - \int[\underset{\text{I}}{\cos(\log x)}\cdot \underset{\text{II}}{1}]\, dx$$

$$= \sin(\log x)x - \left[\cos(\log x)\cdot x - \int\left\{\frac{d}{dx}(\cos \log x)\int 1\cdot dx\right\}dx\right]$$

$$= \sin(\log x)x - \left[\cos(\log x)x - \int\left\{-\sin \log x \frac{1}{x}\cdot x\right\}dx\right]$$

$$= \sin(\log x)x - \left[\cos(\log x)\, x \cdot + \int \sin \log x\, dx\right]$$

$$= \sin(\log x)x - (\cos(\log x)x) - \int \sin(\log x)\cdot x\, dx$$

$$= \sin(\log x)x - \cos(\log x)x - I$$

$$2I = \sin(\log x)x - \cos(\log x)x$$

$$I = \frac{\sin(\log x)x - \cos(\log x)x}{2}$$

$\int e^{ax} \cos(bx + c)\, dx.$

ution:

$$I = \int \underset{\text{II}}{e^{ax}} \underset{\text{I}}{\cos(bx + c)}\, dx$$

$$= \cos(bx + c)\int e^{ax}\, dx - \int\left[\frac{d}{dx}(\cos bx + c)\int e^{ax} dx\right]dx$$

$$= \cos(bx + c)\frac{e^{ax}}{a} - \int -\sin(bx + c)\cdot b\frac{e^{ax}}{a}\, dx$$

$$= \cos(bx+c)\,\frac{e^{ax}}{a} + \frac{b}{a}\int \sin(bx+c)\,e^{ax}\,dx$$

$$= \cos(bx+c)\,\frac{e^{ax}}{a} + \frac{b}{a}\left[\sin(bx+c)\int e^{ax}dx - \int\left[\frac{d}{dx}(\sin(bx+c)\int e^{ax}dx\right]dx\right]$$

$$= \cos(bx+c)\,\frac{e^{ax}}{a} + \frac{b}{a}\left[\sin(bx+c)\frac{e^{ax}}{a} - \int b\cos(bx+c)\frac{e^{ax}}{a}\,dx\right.$$

$$I = \cos(bx+c)\,\frac{e^{ax}}{a} + \frac{b}{a}\left[\sin(bx+c)\frac{e^{ax}}{a} - \frac{b}{a}I\right]$$

$$I = \cos(bx+c)\,\frac{e^{ax}}{a} + \frac{b}{a}\,[\sin(bx+c)\cdot - \frac{b}{a}\cdot I]$$

$$I = \cos(bx+c)\,\frac{e^{ax}}{a} + \frac{b}{a}\sin(bx+c) - \frac{b^2}{a^2}I$$

$$I + \frac{b^2}{a^2}I = \cos(bx+c)\,\frac{e^{ax}}{a} + \frac{b}{a}\sin(bx+c)$$

$$\frac{(a^2+b^2)}{a^2}I = \cos(bx+c)\,\frac{e^{ax}}{a} + \frac{b}{a}\sin(bx+c)$$

$$I = \frac{a^2}{a^2+b^2}\left[\cos(bx+c)\,e^{ax} + \frac{b}{a}\sin(bx+c)\right] + \text{constant}$$

18. $\int e^x\left(\frac{1+\sin x\cos}{\cos^2 x}\right)dx.$

Solution:

$$I = \int e^x\left(\frac{1+\sin x\cos}{\cos^2 x}\right)dx$$

$$= \int e^x\left(\frac{1}{\cos^2 x}\right)dx + \int e^x\,\frac{\sin x\cos dx}{\cos^2 x} = \int e^x\left(\frac{1}{\cos^2 x}\right)dx + \int e^x\tan x\,dx$$

$$= \int e^x\sec^2 x\,dx + \int \underset{\text{II}}{e^x}\,\underset{\text{I}}{\tan x}\,dx$$

$$= \tan x\int e^x - \int\left[\frac{d}{dx}(\tan x)\int e^x dx\right]dx + \int e^x\sec^2 x\,dx$$

$$= \tan x\,e^x - \int \sec^2 x\,e^x\,dx + \int e^x\sec^2 x\,dx$$

$= \tan x\,e^x$ + constant.

19. $\int \sec^3 x\,dx.$

Solution:

$$I = \int \underset{\text{I}}{\sec x}\,\underset{\text{II}}{(\sec^2 x)}\,dx$$

$$= \sec x\int \sec^2 x\,dx - \int\left[\frac{d}{dx}(\sec x)\int \sec^2 x\,dx\right]dx$$

$$= \sec x\tan x - \int(\sec x\tan x)\tan x\,dx = \sec x\tan x - \int \sec x\,\tan^2 x\,dx$$

$$= \sec x\tan x - \int \sec x\,(\sec^2 x - 1)\,dx$$

$$I = \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$I = \sec x\tan x - I - \log|\sec x + \tan x| + \text{constant}$$

$$2I = \sec x\tan x - \log|\sec x + \tan x| + \text{constant}$$

$$I = \frac{1}{2}\,[\sec x\tan x - \log|\sec x + \tan x| +]\ \text{constant.}$$

20. $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx.$

Solution:

$$I = \int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

$$= \int e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx = \int e^x\left(\frac{1}{2}\text{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right)dx$$

$$= -\int e^x\cot\frac{x}{2}\,dx + \int e^x\,\frac{1}{2}\,\text{cosec}^2\frac{x}{2}\,dx$$

$$= -\left[\cot\frac{x}{2}\int e^x dx - \int\left[\frac{d}{dx}\left(\cot\frac{x}{2}\right)\int e^x\,dx\right]dx + \frac{1}{2}\int e^x\,\text{cosec}^2\frac{x}{2}\,dx\right.$$

$$= -\left[\cot\frac{x}{2}e^x - \int -\text{cosec}^2\frac{x}{2}\,\frac{1}{2}\,e^x\,dx\right] + \frac{1}{2}\int e^x\,\text{cosec}^2\frac{x}{2}\,dx$$

$$= -\cot\frac{x}{2}\,e^x - \frac{1}{2}\int \text{cosec}^2\frac{x}{2}\,e^x\,dx + \frac{1}{2}\int \text{cosec}^2\frac{x}{2}\,e^x\,dx$$

$= -\cot\frac{x}{2}\,e^x$ + constant.

21. $\int \frac{e^x(x-3)}{(x-1)^3}\,dx.$

Solution:

$$I = \int \frac{e^x(x-3)}{(x-1)^3}\,dx$$

$$= \int \frac{e^x[x-1-2]}{(x-1)^3}\,dx = \int\left[e^x\frac{(x-1)}{(x-1)^3} - \frac{2e^x}{(x-1)^3}\right]dx$$

$$= \int \underset{\text{II}}{e^x}\,\underset{\text{I}}{\frac{1}{(x-1)^2}}\,dx - \int \frac{2e^x}{(x-1)^3}\,dx$$

$$= \frac{1}{(x-1)^2}\int e^x\,dx - \int\left[\frac{d}{dx}\left(\frac{1}{(x-1)^2}\right)\int e^x\,dx\right]dx - \int \frac{2e^x}{(x-1)^3}\,dx$$

$$= \frac{1}{(x-1)^2}\,e^x - \int \frac{d}{dx}\,((x-1)^{-2})\,e^x\,dx - \int \frac{2e^x}{(x-1)^3}\,dx$$

$$= \frac{1e^x}{(x-1)^2} - \int \frac{(-2)1}{(x-1)^3}\,e^x\,dx - \int \frac{2e^x}{(x-1)^3}\,dx$$

$$= \frac{e^x}{(x-1)^2} + \int \frac{2e^x}{(x-1)^3}\,bx - \int \frac{2e^x}{(x-1)^3}\,dx$$

$= \frac{e^x}{(x-1)^2}$ + constant

10.6 [Integration using partial fractions]

Let $f(x)$ and $g(x)$ be two polynomials, then $\frac{f(x)}{g(x)}$ is called algebriac rational functions,

If degree $f(x)$<degree $g(x)$, then $\frac{f(x)}{g(x)}$ is called proper rational function.

Any proper rational function can be expressed as a sum of rational functions, where each rational number has a simple factor of $g(x)$. These rational fractions are called **partial fractions** and this process of expressing $\frac{f(x)}{g(x)}$ into sum of rational funcions is called partial functions.

Following are the rules to integrate using partial fractions.

(1) If $g(x)=(x-a_1)(x-a_2)\ldots\ldots(x-a_n)$ then $\frac{f(x)}{g(x)}=\frac{A_1}{(x-a_1)}+\frac{A_2}{(x-a_2)}\ldots\ldots\ldots\frac{A_n}{(x-a_n)}$

where $A_1, A_2, \ldots\ldots\ldots A_n$ are constant which are obtained by equating numerator of LHS and RHS by putting values $x=a_1, a_2 \ldots\ldots\ldots a_n$.

(2) If $g(x)=(x-a)^k.(x-a_1)(x-a_2)\ldots\ldots\ldots(x-a_n)$ then $\frac{f(x)}{g(x)}$

$$=\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}\ldots\ldots\ldots+\frac{A_k}{(x-a)^k}+\frac{B_1}{x-a_1}+\frac{B_2}{x-a_2}+\ldots\ldots\ldots\frac{B_n}{x-a_n}$$

where $A_1, A_2, \ldots\ldots\ldots A_k, B_1, B_2, \ldots\ldots\ldots B_n$ are constants obtained by equating numerator both sides and by equating like terms both sides

(3) If $g(x)$ has factor quadratic, (irreducible), say ax^2+bx+c, then partial fraction for ax^2+bx+c is taken as

$$\frac{Bx+C}{ax^2+bx+c}$$

Example:

If $g(x)=(x-p)(ax^2+bx+c)$ then $\frac{f(x)}{g(x)}=\frac{A}{x-p}+\frac{BX+C}{ax^2+bx+c}$.

Where $A, B,$ and C are constants which are obtained by equating like terms both sides.

Remark:

If $g(x)$ has factors quadratic (irreducible) which are repeating, then partial fraction for

$(ax^2+bx+c)^k$, is taken as $\frac{BX+C}{ax^2+bx+c}+\frac{DX+E}{(ax^2+bx+c)^2}\ldots\ldots\ldots\frac{PX+Q}{(ax^2+bx+c)^k}$

where B, C, D, E, ……… P, Q are constants obtained by equating like terms both sides.

Example:

If $g(x)=(x-p)^2(ax^2+bx+c)^2$ then $\frac{f(x)}{g(x)}=\frac{A}{(x-p)}+\frac{B}{(x-p)^2}+\frac{(CX+D)}{(ax^2+bx+c)}+\frac{(EX+F)}{(ax^2+bx+c)^2}$

Examples

22. $\int\frac{x-1}{(x+1)(x-2)}dx.$

Solution:

$I=\int\frac{x-1}{(x+1)(x-2)}dx$ Let $\frac{x-1}{(x+1)(x-2)}=\frac{A}{x+1}+\frac{B}{x-2}$; $(x-1)=A(x-2)+B(x+1)$

Put $x=2$ $(2-1)=B(3)\Rightarrow 1=B(3)$ $\therefore B=\frac{1}{3}$ Put $x=-1$

$(-1-1)=A(-1-2)+0 \; -2=A(-3)+0$ $A=\frac{2}{3}$

$\therefore \frac{x-1}{(x+1)(x-2)}=\frac{2}{3(x+1)}+\frac{1}{3(x-2)}$ $\therefore \int\frac{x-1}{(x+1)(x-2)}dx=\frac{2}{3}\int\frac{1}{x+1}dx+\frac{1}{3}\int\frac{dx}{x-2}$

$=\frac{2}{3}\log|x+1|+\frac{1}{3}\log|x-2|+\text{constant}$

$\int\frac{x^2+1}{(x-1)^2(x+3)}dx.$

…ution:

$I=\int\frac{x^2+1}{(x-1)^2(x+3)}dx$; $\frac{x^2+1}{(x-1)^2(x+3)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x+3)}$

i.e. $x^2+1=A(x-1)(x+3)+B(x+3)+c(x-1)^2$

Put $x=1$

$1+1=B(4)$; $2=B(4)$ $\therefore B=\frac{2}{4}=\frac{1}{2}$

Put $x=-3$

$(-3)^2+1=C(-3-1)^2$; $10=C(-4)^2$; $10=C(16)$; $C=\frac{10}{16}=\frac{5}{8}$

Compare coefficient of x^2 $1=A+C$; $A=1-C=$ i.e. $1-\frac{5}{8}=\frac{3}{8}$

$\therefore \frac{x^2+1}{(x-1)^2(x+3)}\;\frac{3}{8}\left(\frac{1}{(x-1)}\right)+\frac{1}{2(x-1)^2}+\frac{5}{8}\frac{1}{(x+3)}$

$\therefore \int\frac{(x^2+1)\,dx}{(x-1)^2(x+1)}=\frac{3}{8}\log|x-1|+\frac{1}{2}\int(x-1)^{-2}+\frac{5}{8}\log|x+3|+\text{constant}$

$=\frac{3}{8}\log(x-1)-\frac{1}{2}\frac{1}{(x-1)}+\frac{5}{8}\log|x+3|+\text{constant}$

$\int\frac{2x}{(x^2+1)(x^2+3)}dx.$

…tion:

$I=\int\frac{2x}{(x^2+1)(x^2+3)}dx$

Put $x^2=t$; $2x\,dx=dt$

$\therefore \int\frac{dt}{(t+1)(t+3)}$; consider $\frac{1}{(t+1)(t+3)}=\frac{A}{t+1}+\frac{B}{t+3}$; $1=A(t+3)+B(t+1)$

Put $t=-1$: $1=A(-1+3)+B(0)$; $1=A(2)\Rightarrow A=\frac{1}{2}$

Put $t=-3$: $1=A(0)+B(-3+1)$; $1=B(-2)$ $\therefore B=-\frac{1}{2}$

$\therefore \frac{1}{(t+1)(t+3)}=\frac{1}{2(t+1)}-\frac{1}{2(t+3)}$ $\therefore \int\frac{dt}{(t+1)(t+3)}=\frac{1}{2}\int\frac{dt}{(t+1)}-\frac{1}{2}\int\frac{dt}{(t+3)}$

$=\frac{1}{2}\log|t+1|-\frac{1}{2}\log(t+3)$ *i.e.* $I=\frac{1}{2}\log|x^2+1|-\frac{1}{2}\log(x^2+3)+\text{constant}$

25. $\int \frac{x^2}{(x^2+1)(3x^2+4)}\,dx.$

Solution:

$I = \int \frac{x^2}{(x^2+1)(3x^2+4)}\,dx$ Let $x^2 = y$ $\therefore \frac{x^2}{(x^2+1)(3x^2+4)} = \frac{y}{(y+1)(3y+4)}$

Let $\frac{y}{(y+1)(3y+4)} = \frac{A}{y+1} + \frac{B}{3y+4}$ $\therefore y = A(3y+4) + B(y+1)$

Put $y = -1$

$-1 = A(-3+4) + B(0)$; $-1 = A(1) + 0$ $\therefore A = -1$

Put $y = -\frac{4}{3}$

$-\frac{4}{3} = A(0) + B\left(-\frac{4}{3}+1\right)$; $-\frac{4}{3} = B\left(-\frac{1}{3}\right)$ $\therefore$ $B = 4$

$\therefore \frac{y}{(y+1)(3y+4)} = -\frac{1}{(y+1)} + \frac{4}{(3y+4)}$ Replace y by x^2

$\therefore \frac{x^2}{(x^2+1)(3x^2+4)} = \frac{-1}{(x^2+1)} + \frac{4}{(3x^2+4)}$

$\therefore \int \frac{x^2\,dx}{(x^2+1)(3x^2+4)} = \int \frac{-1}{(x^2+1)}\,dx + \int \frac{4\,dx}{(3x^2+4)}$

$= -1\int \frac{dx}{(x^2+1)} + 4\int \frac{dx}{(3x^2+4)} = -\tan^{-1}x + \frac{4}{3}\int \frac{dx}{\left(x^2+\frac{4}{3}\right)}$

$= -\tan^{-1}x + \frac{4}{3}\int \frac{dx}{x^2+\left(\frac{2}{\sqrt{3}}\right)^2} = -\tan^{-1}x + \frac{4}{3}\times\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{\sqrt{3}x}{2}\right) + \text{constant}$

$= -\tan^{-1}x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}x}{2}\right) + \text{constant}$

26. $\int \frac{\sin x}{\sin 4x}\,dx.$

Solution:

$I = \int \frac{\sin x\,dx}{2\sin 2x\cos 2x} = \int \frac{\sin x\,dx}{2(2\sin x\cos x)\cos 2x}$

[Using the result $\sin 2x = 2\sin x\cos x$, $\sin 4x = 2\sin 2x\cos 2x$]

$= \frac{1}{4}\int \frac{dx}{(\cos x)(\cos 2x)} = \frac{1}{4}\int \frac{\cos x\,dx}{(\cos^2 x)(\cos 2x)}$

$= \frac{1}{4}\int \frac{\cos x\,dx}{(1-\sin^2 x)(1-2\sin^2 x)}$ [Using the result $\cos^2 x + \sin^2 x = 1$, $1 - \cos 2x = 2\sin^2 x$]

Put $\sin x = t$ $\Rightarrow$ $\cos x\,dx = dt$; $I = \frac{1}{4}\int \frac{dt}{(1-t^2)(1-2t^2)}$

Put $t^2 = y$ Now $\frac{1}{(1-y)(1-2y)} = \frac{A}{1-y} + \frac{B}{1-2y}$ $\therefore$ $1 = A(1-2y) + B(1-y)$

Put $y = 1$

$1 = A(1-2) + B(0)$; $1 = A(-1) + 0$; $A = -1$

Put $y = \frac{1}{2}$: $1 = A(0) + B\left(1-\frac{1}{2}\right)$; $1 = 0 + B\left(\frac{1}{2}\right)$ $\Rightarrow$ $B = 2$

$\therefore \frac{1}{(1-y)(1-2y)} = \frac{-1}{(1-y)} + \frac{2}{1-2y}$ $\therefore \frac{1}{(1-t^2)(1-2t^2)} = \frac{-1}{1-t^2} + \frac{2}{1-2t^2}$

$I = \frac{1}{4}\int \frac{dt}{(1-t^2)(1-2t^2)} = \frac{1}{4}\left[-\int \frac{dt}{1-t^2} + 2\int \frac{dt}{1-2t^2}\right]$

$= \frac{1}{4}\left[\frac{-1}{2}\log\left|\frac{1+t}{1-t}\right| + 2\int \frac{dt}{1-(\sqrt{2}t)^2}\right]$ [See standard results]

$= \frac{1}{4}\left[\frac{-1}{2}\log\left|\frac{1+t}{1-t}\right| + \frac{2}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}t}{1-\sqrt{2}t}\right|\right] + \text{constant}$

$= \frac{1}{4}\left[\frac{-1}{2}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{2}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right|\right] + \text{constant}$

$= \frac{-1}{8}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + \text{constant}$

$\int \frac{\cos x}{(1-\sin x)^3(2+\sin x)}\,dx.$

tion:

$I = \int \frac{\cos x}{(1-\sin x)^3(2+\sin x)}\,dx$ Put $\sin x = t$ $-\cos x\,dx = dt$

$I = \int \frac{-dt}{(1-t)^3(2+t)}$ Consider $\frac{1}{(1-t)^3(2+t)} = \frac{A}{(1-t)} + \frac{B}{(1-t)^2} + \frac{C}{(1-t)^3} + \frac{D}{(2+t)}$

$1 = A(1-t)^2(2+t) + B(1-t)(2+t) + C(2+t) + D(1-t)^3$ —(1)

Put $t = 1$

$1 = A(0) + B(0) + C(2+1) + D(0)$; $1 = C(3) \Rightarrow C = \frac{1}{3}$

Put $t = -2$

$1 = A(0) + B(0) + C(0) + D[1-(-2)]^3$; $1 = 0 + D(3)^3$; $D = \frac{1}{27}$

Compare the constant term both the sides

$1 = A(2) + B(2) + 2C + D$; $1 = 2A + 2B + 2\times\frac{1}{3} + \frac{1}{27}$

$1 = 2A + 2B + \frac{2}{3} + \frac{1}{27}$; $1 = 2A + 2B + \frac{19}{27}$

$\frac{8}{27} = 2A + 2B$ $\Rightarrow$ $\frac{4}{27} = A + B$ ...(ii)

Compare the coefficent of both the side

$0 = A(1-4) + B(1-2) + C + D(-3)$; $0 = A(-3) + B(-1) + \frac{1}{3} - \frac{3}{27}$

$0 = -3A - B + \frac{1}{3} - \frac{1}{9}$; $\frac{1}{9} - \frac{1}{3} = -(3A+B)$; $-\frac{2}{9} = -(3A+B)$ i.e. $3A + B = \frac{2}{9}$ ...(iii)

Solving equation (ii) and (iii) we get $A = \frac{1}{27}$ and $B = \frac{1}{9}$

$$\therefore \frac{1}{(1-t)^3\,(2+t)} = \frac{1}{27\,(1-t)} + \frac{1}{9\,(1-t)^2} + \frac{1}{3\,(1-t)^3} + \frac{1}{27\,(2+t)}$$

$$\therefore I = \int \frac{dt}{(1-t)^3\,(2+t)} = -\left[\frac{1}{27}\int\frac{dt}{(1-t)} + \frac{1}{9}\int\frac{dt}{(1-t^2)^2} + \frac{1}{3}\int\frac{dt}{(1-t)^3} + \frac{1}{27}\int\frac{dt}{2+t}\right]$$

$$= -\left[\frac{1}{27}\log|1-t| - \frac{1}{9}(1-t)^{-1} + \frac{1}{3\,(-2)}\,(1-t)^2 + \frac{1}{27}\log|2+4|\right]$$

$$= -\frac{1}{27}\log|1-t| + \frac{1}{9\,(1-t)} + \frac{1}{6}\,\frac{1}{(1-t)^2} - \frac{1}{27}\log|2+t| + \text{constant}$$

$$= \frac{-1}{27}\log|1-\sin x| + \frac{1}{9\,(1-\sin x)} + \frac{1}{6\,(1-\sin x)^2} - \frac{1}{27}\log|2+\sin x| + \text{constant}$$

10.7 [Special Integrals]

The standard formulae of Integration (16) to (24) are used to calculate special integrals

Examples

28. $\int \frac{dx}{16+25x^2}$.

Solution:

$$I = \int \frac{dx}{16+25x^2} = \frac{1}{25}\int\frac{dx}{\frac{16}{25}+x^2} = \frac{1}{25}\int\frac{dx}{\left(\frac{4}{5}\right)^2+x^2} = \frac{1}{25}\cdot\frac{1}{\frac{4}{5}}\tan^{-1}\left(\frac{x}{\frac{4}{5}}\right) + \text{constant}$$

$$= \frac{1}{20}\tan^{-1}\left(\frac{5x}{4}\right) + \text{constant}$$

29. $\int \frac{dx}{1+2(x+2)^2}$.

Solution:

$$I = \int \frac{dx}{1+2(x+2)^2} \qquad \text{Put } x+2 = t \Rightarrow dx = dt$$

$$I = \int\frac{dt}{1+2t^2} = \frac{1}{2}\int\frac{dt}{\frac{1}{2}+t^2} = \frac{1}{2}\int\frac{dt}{\left(\frac{1}{\sqrt{2}}\right)^2+t^2}$$

$$= \frac{1}{2}\cdot\frac{1}{\frac{1}{\sqrt{2}}}\tan^{-1}\left(\frac{x}{\frac{1}{\sqrt{2}}}\right) + c = \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{x}\right) + c = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}}{x}\right) + c$$

30. $\int \frac{3x-1}{x^2-1}dx$.

Solution:

$$I = \int\frac{3x-1}{x^2-1}dx = \underbrace{\int\frac{3xdx}{x^2-1}}_{I_1} - \underbrace{\int\frac{dx}{x^2-1}}_{I_2} \; ; \qquad \text{Consider } I_1 = \int\frac{3xdx}{x^2-1} = 3\int\frac{xdx}{x^2-1}$$

Put $x^2 - 1 = t$; $2xdx = dt$

$$I_1 = 3\int\frac{dt}{2t} = \frac{3}{2}\int\frac{dt}{t} = \frac{3}{2}\log|t| = \frac{3}{2}\log|x^2-1|$$

Consider $I_2 = \int\frac{dx}{x^2-1} = \frac{1}{2.1}\left[\log\left|\frac{x-1}{x+1}\right|\right]$ $\quad \therefore I = \frac{3}{2}\log|x^2-1| - \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + c$

$\int \frac{x^2-1}{x^4+1}dx$.

Solution:

$$I = \int\left(\frac{x^2-1}{x^4+1}\right)dx \qquad \text{Divide the numerator and denominator by } x^2$$

$$I = \int\left(\frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\right)dx \; ; \quad I = \int\left(\frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}\right)dx$$

Put $x + \frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$

$$I = \int\frac{dt}{t^2-2} = \int\frac{dt}{t^2-\left(\sqrt{2}\right)^2} = \frac{1}{2\sqrt{2}}\log\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + c = \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + c$$

$$I = \frac{1}{2\sqrt{2}}\log\left|\frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x}\right| + c.$$

$\int \frac{dx}{32-2x^2}$.

Solution:

$$I = \int\frac{dx}{32-2x^2} = \frac{1}{2}\int\frac{dx}{\left(\frac{32}{2}-x^2\right)} = \frac{1}{2}\int\frac{dx}{(16-x^2)}$$

$$I = \frac{1}{2}\int\frac{dx}{(4)^2-(x)^2} = \frac{1}{2}\cdot\frac{1}{2\times4}\log\left|\frac{4+x}{4-x}\right| + c$$

$$I = \frac{1}{16}\log\left|\frac{4+x}{4-x}\right| + c$$

$\int \frac{dx}{a^2-b^2x^2}$.

Solution:

$$I = \int\frac{dx}{a^2-b^2x^2} = \frac{1}{b^2}\int\frac{dx}{\left(\frac{a}{b}\right)^2-x^2} = \frac{1}{b^2}\cdot\frac{b}{2a}\log\left|\frac{\frac{a}{b}+x}{\frac{a}{b}-x}\right| + c = \frac{1}{2ab}\log\left|\frac{a+bx}{a-bx}\right| + c$$

$\int \frac{dx}{\sqrt{2ax-x^2}}$.

Solution:

$$I = \int\frac{dx}{\sqrt{2ax-x^2}} = \int\frac{dx}{\sqrt{a^2-(x^2-2ax+a^2)}} = \int\frac{dx}{\sqrt{a^2-(x-a)^2}}$$

Put $x - a = t \Rightarrow dx = dt$ $\quad \therefore I = \int\frac{dx}{\sqrt{a^2-t^2}} = \sin^{-1}\left(\frac{t}{a}\right) + c = \sin^{-1}\left(\frac{x-a}{a}\right) + c$

35. $\int \frac{\sin\varphi}{\sqrt{4\cos^2\varphi - 1}} d\varphi$.

Solution:

$$I = \int \frac{\sin\varphi}{\sqrt{4\cos^2\varphi - 1}} d\varphi \qquad \text{Put } \cos\varphi = t \ ; \quad -\sin\varphi \, d\varphi = dt$$

$$I = -\int \frac{dt}{\sqrt{4t^2 - 1}} = -\int \frac{dt}{\sqrt{4\left(t^2 - \frac{1}{4}\right)}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{2}\right)^2}} = -\frac{1}{2}\log\left|t + \sqrt{t^2 - \left(\frac{1}{2}\right)^2}\right| + c$$

$$= -\frac{1}{2}\log\left|t + \frac{1}{2}\sqrt{4t^2 - 1}\right| + c = -\frac{1}{2}\log\left|\cos\varphi + \frac{1}{2}\sqrt{4\cos^2\varphi - 1}\right| + c$$

36. $\int \frac{dx}{x^{\frac{2}{3}}\sqrt{x^{\frac{2}{3}} - 4}}$.

Solution:

$$I = \int \frac{dx}{x^{\frac{2}{3}}\sqrt{\left(x^{\frac{1}{3}}\right)^2 - (2)^2}}$$

$$\text{Put } x^{\frac{1}{3}} = t \Rightarrow \frac{1}{3}x^{\frac{-2}{3}}dx = dt \Rightarrow \frac{1}{x^{\frac{2}{3}}}dx = 3dt \qquad \therefore I = \int \frac{3dt}{\sqrt{t^2 - (2)^2}} = 3\int \frac{dt}{\sqrt{t^2 - (2)^2}}$$

$$= 3\log\left|t + \sqrt{t^2 - (2)^2}\right| + c = 3\log\left|t + \sqrt{t^2 - 4}\right| + c$$

$$= 3\log\left|x^{\frac{1}{3}} + \sqrt{\left(x^{\frac{1}{3}}\right)^2 - 4}\right| + c = 3\log\left|x^{\frac{1}{3}} + \sqrt{x^{\frac{2}{3}} - 4}\right| + c$$

37. $\int \sqrt{\frac{\sin(x - \alpha)}{\sin(x + \alpha)}} dx$.

Solution:

$$I = \int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}} dx = \int \sqrt{\frac{\sin(x-\alpha)\sin(x-\alpha)}{\sin(x+\alpha)\sin(x-\alpha)}} dx$$

$$= \int \frac{\sin(x-\alpha)}{\sqrt{\sin^2 x - \sin^2\alpha}} dx \qquad \left[\because \sin(a+b)\sin(a-b) = \sin^2 a - \sin^2 b\right]$$

$$= \int \frac{\sin x\cos\alpha - \cos x\sin\alpha}{\sqrt{\sin^2 x - \sin^2\alpha}} dx$$

$$= \cos\alpha \underbrace{\int \frac{\sin x}{\sqrt{\sin^2 x - \sin^2\alpha}} dx}_{I_1} - \sin\alpha \underbrace{\int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2\alpha}} dx}_{I_2}$$

$$\text{Consider } I_1 = \int \frac{\sin x dx}{\sqrt{\sin^2 x - \sin^2\alpha}} = \int \frac{\sin x dx}{\sqrt{1 - \cos^2 x - (1 - \cos^2\alpha)}}$$

$$= \int \frac{\sin x dx}{\sqrt{\cos^2\alpha - \cos^2 x}} \qquad \left[\begin{array}{l} Put \cos x = t \\ -\sin x dx = dt \end{array}\right]$$

$$\therefore I_1 = -\int \frac{dt}{\sqrt{\cos^2\alpha - t^2}} = -\sin^{-1}\left(\frac{t}{\cos\alpha}\right) = -\sin^{-1}\left(\frac{\cos x}{\cos\alpha}\right)$$

$$\text{Consider } I_2 = \int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2\alpha}} dx \qquad \left[Put \sin x = t \Rightarrow \cos x dx = dt\right]$$

$$\therefore I_2 = \int \frac{dt}{\sqrt{t^2 - \sin^2\alpha}} = \log\left|t + \sqrt{t^2 - \sin^2\alpha}\right| = \log\left|\sin x + \sqrt{\sin^2 x - \sin^2\alpha}\right|$$

$$\therefore I = \cos\alpha\left(-\sin^{-1}\left(\frac{\cos x}{\cos\alpha}\right)\right) - \sin\alpha\log\left|\sin x + \sqrt{\sin^2 x - \sin^2\alpha}\right|$$

$\int \frac{\sec^2 x}{\sqrt{16 + \tan^2 x}} dx$.

...on:

$$I = \int \frac{\sec^2 x}{\sqrt{16 + \tan^2 x}} dx \qquad \text{Put } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\therefore I = \int \frac{dt}{\sqrt{16 + t^2}} = \int \frac{dt}{\sqrt{(4)^2 + t^2}} = \log\left|t + \sqrt{4^2 + t^2}\right| + c = \log\left|\tan x + \sqrt{16 + (\tan x)^2}\right| + c$$

$\int \sqrt{(2ax - x^2)} dx$.

...ion:

$$I = \int\left(\sqrt{a^2 - (x-a)^2}\right)dx$$

Put $x - a = t \Rightarrow dx = dt$

$$\therefore I = \int\left(\sqrt{a^2 - t^2}\right) = \frac{t}{2}\sqrt{a^2 - t^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{t}{a}\right) + c$$

$$= \frac{(x-a)}{2}\sqrt{a^2 - (x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + c$$

$$\frac{(x-a)}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + c$$

$\int e^x \sqrt{(e^{2x} + 1)} dx$.

...tion:

$$I = \int e^x \sqrt{(e^{2x} + 1)} dx$$

Put $e^x = t \Rightarrow e^x dx = dt$

$$I = \int\left(\sqrt{t^2 + 1}\right)dt = \int\left(\sqrt{t^2 + (1)^2}\right)dt = \frac{t}{2}\sqrt{t^2 + 1} + \frac{(1)^2}{2}\log\left|t + \sqrt{t^2 + a^2}\right| + c$$

$$= \frac{e^x}{2}\sqrt{e^{2x} + 1} + \frac{1}{2}\log\left|e^x + \sqrt{e^{2x} + a^2}\right| + c$$

Evaluate $\int \sqrt{3x^2 - 9} dx$.

...tion:

$$I = \int \sqrt{3x^2 - 9} dx = \sqrt{3(x^2 - 3)} dx = \sqrt{3}\sqrt{x^2 - (\sqrt{3})^2} dx$$

$$= \sqrt{3}\left[\frac{x}{2}\sqrt{x^2-(\sqrt{3})^2} - \frac{(\sqrt{3})^2}{2}\log\left|x+\sqrt{x^2-(\sqrt{3})^2}\right|\right]+c$$

$$= \sqrt{3}\left[\frac{x}{2}\sqrt{x^2-3} - \frac{3}{2}\log\left|x+\sqrt{x^2-3}\right|\right]+c$$

10.8 [Applications of Special Integrals]

Type I : $\int \frac{dx}{ax^2+bx^2+c}$ ie $\int \frac{dx}{quadratic}$

Note : Make coefficient of x^2 unity and complete the square in denominator by adding and subtracting square of half of coefficient of x

42. $\int \frac{dx}{x^2+2x+10}$.

Solution:

$$I = \int \frac{dx}{x^2+2x+10} = \int \frac{dx}{x^2+2x+(1)^2-(1)^2+10}$$

$$= \int \frac{dx}{(x+1)^2-1+10} = \int \frac{dx}{(x+1)^2+9} = \int \frac{dx}{(x+1)^2+(3)^2} = \frac{1}{3}\left[\tan^{-1}\left(\frac{x+1}{3}\right)\right]+c$$

43. $\int \frac{dx}{9x^2+6x+10}$.

Solution:

$$I = \frac{dx}{9x^2+6x+10} = \frac{1}{9}\int \frac{dx}{x^2+\frac{6}{9}x+\frac{10}{9}}$$

$$= \frac{1}{9}\int \frac{dx}{x^2+\frac{2}{3}x+\frac{10}{9}} = \frac{1}{9}\int \frac{dx}{x^2+\frac{2}{3}x+\left(\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^2+\frac{10}{9}}$$

$$= \frac{1}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2+1} = \frac{1}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2+(1)^2} = \frac{1}{9}\tan^{-1}\left(\frac{x+\frac{1}{3}}{1}\right)+c = \frac{1}{9}\tan^{-1}\left(\frac{3x+1}{3}\right)+c$$

Type II : $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \int \frac{dx}{\sqrt{quadratic}}$

Note : Make coefficient of x^2 unity and complete the square in denominator by adding and subtracting square of half of coefficient of x.

Examples

44. $\int \frac{dx}{\sqrt{3x^2-x-2}}$.

Solution:

$$I = \int \frac{dx}{\sqrt{3x^2-x-2}} \quad ; \quad I = \int \frac{dx}{\sqrt{3\left(x^2-\frac{x}{3}-\frac{2}{3}\right)}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{x^2-\frac{x}{3}+\left(\frac{1}{6}\right)^2-\left(\frac{1}{6}\right)^2-\frac{2}{3}}}$$

$$= \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(x-\frac{1}{6}\right)^2-\frac{1}{36}-\frac{2}{3}}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(x-\frac{1}{6}\right)^2-\frac{25}{36}}}$$

$$= \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(x-\frac{1}{6}\right)^2-\left(\frac{5}{6}\right)^2}} = \frac{1}{\sqrt{3}}\log\left|x-\frac{1}{6}+\sqrt{\left(x-\frac{1}{6}\right)^2-\left(\frac{5}{6}\right)^2}\right|$$

$$= \frac{1}{\sqrt{3}}\log\left|x-\frac{1}{6}+\sqrt{3x^2-x-2}\right|+c$$

$\int \frac{dx}{\sqrt{5x-6-x^2}}$.

Solution:

$$I = \int \frac{dx}{\sqrt{5x-6-x^2}} = \int \frac{dx}{\sqrt{-(x^2-5x+6)}}$$

$$I = \int \frac{dx}{\sqrt{-\left(x^2-5x+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+6\right)}}$$

$$I = \int \frac{dx}{\sqrt{-\left[\left(x-\frac{5}{2}\right)^2-\frac{25}{4}+6\right]}}$$

$$I = \int \frac{dx}{\sqrt{-\left[\left(x-\frac{5}{2}\right)^2-\frac{1}{4}\right]}} = \int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{5}{2}\right)^2}} = \sin^{-1}\left(\frac{x-\frac{5}{2}}{\frac{1}{2}}\right)+c = \sin^{-1}(2x-5)+c$$

Type III : $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ or $\int \frac{linear}{\sqrt{quadratic}}$

Note : Put $px+q = A\frac{d}{dx}(ax^2+bx+c)+B$ and find values of A and B by equating coefficients of like terms on both sides.

$\int \frac{6x+5}{\sqrt{6+x-2x^2}}dx$.

Solution:

$I = \int \frac{6x+5}{\sqrt{-2x^2+x+6}}dx$ Put $6x+5 = A\frac{d}{dx}(-2x^2+x+6)+B$

$6x+5 = A(-4x+1)+B \Rightarrow 6x+5 = -4Ax+A+B$ By equating coefficients of like terms we get

$6 = -4A$ and $5 = A+B \Rightarrow A = \frac{-6}{4} = \frac{-3}{2}$ and $5 = \frac{-3}{2}+B \Rightarrow B = \frac{13}{2}$

$$\therefore \int \frac{6x+5}{\sqrt{-2x^2+x+6}} = \int\left(\frac{-\frac{3}{2}(-4x+1)+\frac{13}{2}}{\sqrt{-2x^2+x+6}}\right)dx \quad -\frac{3}{2}\int \frac{(1-4x)}{\sqrt{-2x^2+x+6}}dx+\frac{13}{2}\int \frac{dx}{\sqrt{-2x^2+x+6}}$$

$$I = -\tfrac{3}{2}I_1 + \tfrac{13}{2}I_2$$

Consider $I_1 = \int \frac{(1-4x)}{\sqrt{-2x^2+x+6}}dx$ Put $-2x^2+x+6 = t$; $(-4x+1)\,dx = dt$

$\therefore I_1 = \int \frac{dx}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{-2x^2+x+6}$ Consider $I_2 = \int \frac{dx}{\sqrt{-2x^2+x+6}}$

$$= \int \frac{dx}{\sqrt{-2\left(x^2-\frac{x}{2}-\frac{6}{2}\right)}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{-\left(x^2-\frac{x}{2}+\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2-\frac{6}{2}\right)}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{-\left(x-\frac{1}{4}\right)^2-\frac{1}{16}-\frac{6}{2}}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{-\left(x-\frac{1}{4}\right)^2-\frac{49}{16}}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{-\left[\left(x-\frac{1}{4}\right)^2-\left(\frac{7}{4}\right)^2\right]}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(\frac{7}{4}\right)^2-\left(x-\frac{1}{4}\right)^2}} = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x-\frac{1}{4}}{\frac{7}{4}}\right) = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)$$

$$I = -\frac{3}{2}(2\sqrt{-2x^2+x+6})+\frac{13}{2}\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)+c = -3\sqrt{-2x^2+x+6}+\frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)+c$$

47. Evaluate $\int\sqrt{\frac{1+x}{x}}dx$.

Solution:

$I = \int\sqrt{\frac{1+x}{x}}dx$; $I = \int\left(\sqrt{\frac{1+x}{x}} \times \sqrt{\frac{1+x}{1+x}}\right)dx$; $I = \int\left(\frac{(1+x)}{\sqrt{x(1+x)}}\right)dx = \int\left(\frac{(1+x)}{\sqrt{x^2+x}}\right)dx$

$1+x = A\frac{d}{dx}(x^2+x)+B$; $1+x = A(2x+1)+B$ $\therefore 1+x = 2Ax+A+B$

Now equating coefficients of like terms both sides

$1 = 2A \Rightarrow A = \frac{1}{2}$; $1 = A+B \Rightarrow B = \frac{1}{2}$ $[\because A = \frac{1}{2}]$

$\therefore I = \int\frac{\frac{1}{2}(2x+1)+\frac{1}{2}}{\sqrt{x^2+x}}dx = \frac{1}{2}\int\frac{(2x+1)dx}{\sqrt{x^2+x}}+\frac{1}{2}\int\frac{dx}{\sqrt{x^2+x}}$ $\therefore I = \frac{1}{2}I_1+\frac{1}{2}I_2$

Consider $I_1 = \int\frac{(2x+1)}{\sqrt{x^2+x}}dx$. Put $x^2+x = t \Rightarrow (2x+1)dx = dt$

Consider $I_1 = \int\frac{dt}{\sqrt{t}} = 2\sqrt{t}+2 = 2\sqrt{x^2+x}$

$$I_2 = \int\frac{dx}{\sqrt{x^2+x}} = \int\frac{dx}{\sqrt{x^2+x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}} = \int\frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2-\frac{1}{(2)^2}}}$$

$$= \log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right| = \log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|$$

$$I = \frac{1}{2}2\sqrt{x^2+x}+\frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|+c = \sqrt{x^2+x}+\frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|+c$$

V: $\int\frac{px+q}{ax^2+bx+c}dx$ or $\int\frac{linear}{quadratic}dx$

Evaluate $\int\left(\frac{2x+5}{x^2-x-2}\right)dx$.

:

$$I = \int\left(\frac{2x+5}{x^2-x-2}\right)dx = \int\left(\frac{2x+5}{x^2-x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2-2}\right)dx$$

$$= \int\frac{(2x+5)}{\left(x-\frac{1}{2}\right)^2-\frac{9}{4}}dx = \int\frac{(2x+5)}{\left(x-\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2}dx$$

Put $x-\frac{1}{2} = t \Rightarrow dx = dt$

$\therefore I = \int\frac{2(t+\frac{1}{2})+5}{(t)^2-\left(\frac{3}{2}\right)^2} = \int\frac{2t+6}{t^2-\left(\frac{3}{2}\right)^2}dt$ $\therefore I = 2\int\frac{t}{(t)^2-\left(\frac{3}{2}\right)^2}dt+6\int\frac{dt}{t^2-\left(\frac{3}{2}\right)^2}$; $I = 2I_1+6I_2$

Consider $I_1 = \int\frac{t}{(t)^2-\left(\frac{3}{2}\right)^2}dt$ Put $t^2-\frac{9}{4} = y$; $2t\,dt = dy$

$$I_1 = \frac{1}{2}\int\frac{dy}{y} = \frac{1}{2}\log|y| = \frac{1}{2}\log\left|t^2-\frac{9}{4}\right|$$

Consider $I_2 = \int\frac{dt}{t^2-\left(\frac{3}{2}\right)^2} = \frac{1}{2\times\frac{3}{2}}\log\left|\frac{t-\frac{3}{2}}{t+\frac{3}{2}}\right| = \frac{1}{3}\log\left|\frac{t-\frac{3}{2}}{t+\frac{3}{2}}\right|$

$$\therefore I = 2\frac{1}{2}\log\left|t^2-\frac{9}{4}\right|+\frac{6}{3}\log\left|\frac{t-\frac{3}{2}}{t+\frac{3}{2}}\right|+c = \log\left|t^2-\frac{9}{4}\right|+2\log\left|\frac{t-\frac{3}{2}}{t+\frac{3}{2}}\right|+c$$

$$= \log\left|\left(x-\frac{1}{2}\right)^2-\frac{9}{4}\right|+2\log\left|\frac{\left(x-\frac{1}{2}\right)-\frac{3}{2}}{\left(x-\frac{1}{2}\right)+\frac{3}{2}}\right|+c$$

10.9 [Integrations of Algebriac Functions]

1: Consider the integral $\int\frac{\phi(x)}{A\sqrt{B}}dx$ where A and B are linear functions of x. To evaluate this type of integral, we put $B = t^2$

2: Consider the integral $\int\frac{\phi(x)}{A\sqrt{B}}$ where A is quadratic and B is linear. To evaluate this type of integral we pur $B=t^2$.

3: Consider the integral $\int\frac{\phi(x)}{A\sqrt{B}}$ where A is linear and B is quadratic. To evaluate this type of integral we put $A=\frac{1}{t}$.

4: Consider the integral $\int\frac{\phi(x)}{A\sqrt{B}}dx$ where A and B are both quadratic (irreducible), then put $x=\frac{1}{t}$.

Examples

49. $\int \frac{1}{(x-1)\sqrt{2x+3}}\, dx.$

Solution:

$$I = \int \frac{dx}{(x-1)\sqrt{2x+3}} \qquad \text{Put } 2x+3 = t^2 \Rightarrow \sqrt{2x+3} = t \;\therefore\; 2\,dx = 2\,t\,dt$$

$$I = \int \frac{1}{\left(\frac{t^2-3}{2}-1\right)t}\, t\,dt = \int \frac{2\,dt}{(t^2-3-2)} = 2\int \frac{dt}{t^2-\left(\sqrt{5}\right)^2} = \frac{2}{2\sqrt{5}} \log\left|\frac{t-\sqrt{5}}{t+\sqrt{5}}\right| + \text{constant}$$

$$= \frac{1}{\sqrt{5}} \log\left|\frac{\sqrt{2x+3}-\sqrt{5}}{\sqrt{2x+3}+\sqrt{5}}\right| + \text{constant}$$

50. $\int \frac{dx}{(x^2+1)\sqrt{x}}.$

Solution:

$$I = \int \frac{dx}{(x^2+1)\sqrt{x}} \quad \text{Put } x = t^2 \Rightarrow x^2 = t^4$$

$$I = \int \frac{2t\,dt}{(t^4+1)\,t} = \int \frac{2\,dt}{t^4+1} = 2\int \frac{dt}{t^4+1} = 2\,I_1 \qquad \text{where } I_1 = \int \frac{dt}{t^4+1}$$

$$I_1 = \int \frac{\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt = \frac{1}{2}\int \frac{\frac{2}{t^2}}{t^2+\frac{1}{t^2}}\,dt = \frac{1}{2}\left[\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt - \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt\right]$$

$$= \frac{1}{2}\left[\int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}\,dt - \int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-2}\,dt\right]$$

Put $t - \frac{1}{t} = p$ in first integral and $t + \frac{1}{t} = q$ in second integral $\quad \therefore \left(1+\frac{1}{t^2}\right) dt = dp$

and $\left(1-\frac{1}{t^2}\right) dt = dq \qquad \therefore \; I_1 = \frac{1}{2}\left[\int \frac{dp}{(p)^2+2} - \int \frac{dq}{q^2-2}\right]\; ; \; \frac{1}{2}\left[\int \frac{dp}{p^2+\left(\sqrt{2}\right)^2} - \int \frac{dq}{q^2-\left(\sqrt{2}\right)^2}\right]$

$$\left[\frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{p}{\sqrt{2}}\right) - \frac{1}{2\cdot 2\sqrt{2}} \log\left|\frac{q-\sqrt{2}}{q+\sqrt{2}}\right|\right] + \text{constant}$$

$$\frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right) - \frac{1}{4\sqrt{2}} \log\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right| + \text{constant}$$

$$\frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{t^2-1}{t\sqrt{2}}\right) - \frac{1}{4\sqrt{2}} \log\left|\frac{t^2+1-\sqrt{2}t}{t^2+1+\sqrt{2}t}\right| + \text{constant}$$

$$I_1 = \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x-1}{\sqrt{x}\,\sqrt{2}}\right) - \frac{1}{4\sqrt{2}} \log\left|\frac{x+1-\sqrt{2}\cdot\sqrt{x}}{x+1+\sqrt{2}\,\sqrt{x}}\right| + \text{constant}$$

$$I_1 = \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x-1}{\sqrt{2x}}\right) - \frac{1}{4\sqrt{2}} \log\left|\frac{x+1-\sqrt{2x}}{x+1+\sqrt{2x}}\right| + \text{constant} \quad ; \quad I = 2\,[I_1]$$

$$\therefore 2\left[\frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x-1}{\sqrt{2x}}\right) - \frac{1}{4\sqrt{2}} \log\left|\frac{x+1-\sqrt{2x}}{x+1+\sqrt{2x}}\right|\right] + \text{constant}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x-1}{\sqrt{2x}}\right) - \frac{1}{2\sqrt{2}} \log\left|\frac{x+1-\sqrt{2x}}{x+1+\sqrt{2x}}\right| + \text{constant}$$

$\int \frac{x+2}{\left(x^2+3x+3\right)\sqrt{x+1}}\, dx.$

ution:

$$I = \int \frac{x+2}{\left(x^2+3x+3\right)\sqrt{x+1}}\, dx \qquad \text{Put } x+1 = t^2 \; ; \; dx = 2t\,dt$$

$$I = \int \frac{(t^2+1)\,2t\,dt}{(t^2-1)^2+3\,(t^2-1)+3\,t}$$

$$I = \int \frac{2\,(t^2+1)}{\left[t^4+1-2t^2+3t^2-3+3\right]}$$

$$I = 2\int \frac{t^2+1}{t^4+t^2+1}\, dt$$

$$= 2\int \left(\frac{1+\frac{1}{t^2}}{t^2+1+\frac{1}{t^2}}\right) dt = 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}\, dt = 2\int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2+1}\, dt$$

$$= 2\int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+3}\, dt = 2\int \left[\frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+\left(\sqrt{3}\right)^2}\right] dt$$

Put $t - \frac{1}{t} = p \qquad \therefore \left(1+\frac{1}{t^2}\right) dt = dp$

$$I = 2\int \frac{dp}{p^2-\left(\sqrt{3}\right)^2}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{p}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{3}}\right) + \text{constant}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t^2-1}{t\sqrt{3}}\right) + \text{constant}$$

$$I = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x+1-1}{\sqrt{x+1}\cdot\sqrt{3}}\right) + \text{constant}$$

$$I = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3x+1}}\right) + \text{constant}$$

52. $\int \frac{x}{(x^2+2x+2)\sqrt{x+1}}\, dx.$

Solution:

$$I = \int \frac{x}{(x^2+2x+2)\sqrt{x+1}}\, dx \qquad \text{Put } x+1 = t^2 \;;\; dx = 2t\, dt$$

$$\therefore I = \int \frac{(t^2-1)\, 2t\, dt}{\left((t^2-1)^2 + 2(t^2-1)+2\right) t}$$

$$I = \int \frac{2(t^2-1)\, dt}{(t^4+1-2t^2+2t^2-2+2)}$$

$$I = \int \frac{2(t^2-1)}{t^4+1}\, dt = 2\int \left(\frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\right) dt = 2\int \frac{\left(1-\frac{1}{t^2}\right)}{\left(t+\frac{1}{t}\right)^2-2}\, dt$$

$$\text{Put } t+\frac{1}{t} = v \qquad \left(1-\frac{1}{t^2}\right) dt = dv \qquad \therefore I = 2\int \frac{dv}{v^2-2} = 2\int \frac{dv}{(v)^2-\left(\sqrt{2}\right)^2}$$

$$I = \frac{2}{2\sqrt{2}} \log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + \text{constant} \quad ; \quad \frac{1}{\sqrt{2}} \log\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right| + \text{constant}$$

$$\frac{1}{\sqrt{2}} \log\left|\frac{t^2+1-\sqrt{2}t}{t^2+1+\sqrt{2}t}\right| + \text{constant} \quad ; \quad \frac{1}{\sqrt{2}} \log\left|\frac{x+1+1-\sqrt{2}\sqrt{x+1}}{x+1+1+\sqrt{2}\sqrt{x+1}}\right| + \text{constant}$$

$$\frac{1}{\sqrt{2}} \log\left|\frac{x+2-\sqrt{2(x+1)}}{x+2+\sqrt{2(x+1)}}\right| + \text{constant}$$

53. $\int \frac{dx}{(x-1)\sqrt{x^2+4}}.$

Solution:

$$I = \int \frac{dx}{(x-1)\sqrt{x^2+4}} \qquad \text{Put } x-1 = \frac{1}{t} \Rightarrow t = \frac{1}{x-1} \;;\; dx = \frac{-1}{t^2}\, dt$$

$$\therefore I = \int \frac{-\frac{1}{t^2}\, dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}+1\right)^2+4}} \quad ; \quad I = \int \frac{-1}{t} \frac{dt}{\sqrt{\frac{1}{t^2}+1+\frac{2}{t}+4}}$$

$$= -\int \frac{dt}{\sqrt{5t^2+2t+1}} = -\frac{1}{\sqrt{5}}\int \frac{dt}{\sqrt{t^2+\frac{2t}{5}+\frac{1}{5}}} - \frac{1}{\sqrt{5}}\int \frac{dt}{\sqrt{t^2+\frac{2}{5}t+\left(\frac{1}{5}\right)^2+\frac{1}{5}-\left(\frac{1}{5}\right)^2}} - \frac{1}{\sqrt{5}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{5}\right)^2+\frac{4}{25}}}$$

$$\frac{-1}{\sqrt{5}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{5}\right)^2+\left(\frac{2}{5}\right)^2}} = \frac{-1}{\sqrt{5}} \log\left|\left(t+\frac{1}{5}\right)+\sqrt{\left(t+\frac{1}{5}\right)^2+\left(\frac{2}{5}\right)^2}\right| + \text{constant}$$

$$\frac{-1}{\sqrt{5}} \log\left|\left(t+\frac{1}{5}\right)+\sqrt{t^2+\frac{2}{5}t+\frac{1}{5}}\right| + \text{constant}$$

$$= \frac{-1}{\sqrt{5}} \log\left|\left(\frac{1}{x-1}+\frac{1}{5}\right)+\sqrt{\left(\frac{1}{x-1}\right)^2+\frac{2}{5}\frac{1}{(x-1)}+\frac{1}{5}}\right| + \text{constant}$$

$$= \frac{-1}{\sqrt{5}} \log\left|\left(\frac{1}{x-1}+\frac{1}{5}\right)+\sqrt{\frac{1}{(x-1)^2}+\frac{2}{5(x-1)}+\frac{1}{5}}\right| + \text{constant}$$

$$= \frac{-1}{\sqrt{5}} \log\left|\left(\frac{1}{x-1}+\frac{1}{5}\right)+\sqrt{\frac{5+2(x-1)+(x-1)^2}{(x-1)^2 5}}\right| + \text{constant}$$

$$= \frac{-1}{\sqrt{5}} \log\left|\left(\frac{1}{x-1}+\frac{1}{5}\right)+\sqrt{\frac{x^2+1-2x+2x-2+5}{(x-1)^2\, 5}}\right| + \text{constant}$$

$$= \frac{-1}{\sqrt{5}} \log\left|\left(\frac{1}{(x-1)}+\frac{1}{5}\right)+\sqrt{\frac{x^2+4}{(x-1)^2\, 5}}\right| + \text{constant}$$

$\int \frac{1}{(x-1)\sqrt{x^2+1}}\, dx.$

Solution:

$$I = \int \frac{dx}{(x-1)\sqrt{x^2+1}} \qquad \text{Put } x-1 = \frac{1}{t} \Rightarrow t = \frac{1}{x-1} \;;\; dx = \frac{-1}{t^2}$$

$$I = \int \frac{-1}{t^2} \frac{dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}+1\right)^2+1}}$$

$$= -\int \frac{dt}{\sqrt{1+t^2+2t+t^2}} = -\int \frac{dt}{\sqrt{2t^2+2t+1}} = \frac{-1}{\sqrt{2}}\int \frac{dt}{\sqrt{t^2+t+\frac{1}{2}}}$$

$$= \frac{-1}{\sqrt{2}}\int \frac{dt}{\sqrt{t^2+t+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+\frac{1}{2}}} = \frac{-1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{1}{4}}} = \frac{-1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}$$

$$= -\frac{1}{\sqrt{2}} \log\left|\left(t+\frac{1}{2}\right)+\sqrt{\left(t+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\right| + \text{constant}$$

$$-\frac{1}{\sqrt{2}} \log\left|\left(t+\frac{1}{2}\right)+\sqrt{t^2+t+\frac{1}{2}}\right| + \text{constant}$$

$$= -\frac{1}{\sqrt{2}} \log\left|\frac{1}{x-1}+\frac{1}{2}+\sqrt{\left(\frac{1}{x-1}\right)^2+\frac{1}{x-1}+\frac{1}{2}}\right| + \text{constant}$$

$$= -\frac{1}{\sqrt{2}} \log\left|\frac{1}{x-1}+\frac{1}{2}+\sqrt{\frac{2+2(x-1)+(x-1)^2}{(x-1)^2}}\right| + \text{constant}$$

$$= -\frac{1}{\sqrt{2}} \log\left|\frac{1}{x-1}+\frac{1}{2}+\sqrt{\frac{x^2+1}{(x-1)^2}}\right| + \text{constant}$$

$$= -\frac{1}{2} \log\left|\frac{1}{x-1}+\frac{1}{2}+\frac{\sqrt{x^2+1}}{(x-1)}\right| + \text{constant}$$

55. $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$.

Solution:

$$I = \int \frac{dx}{(x+1)\sqrt{x^2+x+1}} \qquad \text{Put } x+1 = \frac{1}{t} \Rightarrow t = \frac{1}{x+1} \qquad \therefore dx = \frac{-1}{t^2}$$

$$I = \int \frac{-dt}{t^2 \frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2 + \left(\frac{1}{t}-1\right)+1}} = -\int \frac{dt}{t\sqrt{\frac{1}{t^2}+1-\frac{2}{t}+\frac{1}{t}-1+1}} = -\int \frac{dt}{t \frac{1}{t}\sqrt{1+t^2-2t+t}}$$

$$= -\int \frac{dt}{\sqrt{t^2-t+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}} = -\int \frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}} = -\int \frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= -\left[\log\left|\left(t-\frac{1}{2}\right)+\sqrt{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|\right] + \text{constant}$$

$$= -\left[\log\left|\left(t-\frac{1}{2}\right)+\sqrt{t^2-t+1}\right|\right] + \text{constant}$$

$$= -\left[\log\left|\left(\frac{1}{x+1}-\frac{1}{2}\right)+\sqrt{\left(\frac{1}{x+1}\right)^2-\frac{1}{x+1}+1}\right|\right] + \text{constant}$$

$$= -\left[\log\left|\frac{1}{x+1}-\frac{1}{2}+\sqrt{\frac{1-(x+1)+(x+1)^2}{(x+1)^2}}\right|\right] + \text{constant}$$

$$= -\left[\log\left|\frac{1}{x+1}-\frac{1}{2}+\frac{\sqrt{1-x-1+x^2+1+2x}}{(x+1)}\right|\right] + \text{constant}$$

$$= -\left[\log\left|\frac{1}{x+1}-\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{(x+1)}\right|\right] + \text{constant} - \left[\log\left|\frac{1}{x+1}-\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{(x+1)}\right|\right] + \text{constant}$$

56. $\int \frac{dx}{x^2\sqrt{1+x^2}}$.

Solution:

$$I = \int \frac{dx}{x^2\sqrt{1+x^2}} \qquad \text{Put } x = \frac{1}{t} \qquad dx = \frac{-dt}{t^2}$$

$$I = \int \frac{-dt}{t^2 \frac{1}{t}\sqrt{1+\frac{1}{t}^2}} = -\int \frac{t}{\sqrt{t^2+1}} = -\frac{1}{2}\int \frac{2t}{\sqrt{t^2+1}}$$

Put $t^2+1 = u$; $2t\,dt = du$

$$I = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}2\sqrt{u}+c = -\sqrt{t^2+1}+c = -\sqrt{\left(\frac{1}{x}\right)^2+1}+c$$

$$= -\sqrt{\frac{x^2+1}{x^2}}+c \qquad -\frac{\sqrt{x^2+1}}{x}+c$$

$\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$.

:

Put $x = \frac{1}{t}$; $dx = \frac{-dt}{t^2}$

$$I = \int \frac{-dt}{t^2\left(1+\frac{1}{t^2}\right)\sqrt{1-\left(\frac{1}{t}\right)^2}} = -\int \frac{t\,dt}{t^2\frac{(t^2+1)}{t^2}\sqrt{t^2-1}} = -\frac{t\,dt}{(t^2+1)\sqrt{t^2-1}}$$

Put $t^2-1 = u^2$; $2t\,dt = 2u\,du$

$$I = -\int \frac{u\,du}{u(u^2+2)} = -\int \frac{du}{u^2+2} = -\int \frac{du}{u^2+(\sqrt{2})^2}$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+c = -\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{t^2-1}}{\sqrt{2}}+c$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{\left(\frac{1}{x}\right)^2-1}}{\sqrt{2}}+c = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x\sqrt{2}}\right)+c$$

Exercise

(Questions based on the method of By Parts)

ate the following integrals:

$\int x\sec^2 x\,dx$ [Ans : $x\tan x + \log|\cos x| + c$]

$\int \frac{(\log x)}{x^2}\,dx$ [Ans : $\frac{-1}{x}(1+\log x)+c$]

$\int (\sin(\log x)+\cos(\log x))\,dx$ [Ans : $x\sin(\log x)+c$]

$\int (x\sin x\cos 2x)\,dx$ [Ans : $\frac{1}{2}\left[\frac{-x\cos 3x}{3}+\frac{\sin 3x}{9}+x\cos x-\sin x\right]+c$]

$\int e^x(\tan x+\log\sec x)\,dx$ [Ans : $e^x\log\sec x+c$)

$\int e^{-x}\cos x\,dx$ [Ans : $\frac{e^{-x}}{2}(\sin x-\cos x)+c$]

$\int \frac{1}{x^3}\sin(\log x)\,dx$ [Ans : $\frac{-1}{5x^2}[\cos(\log x)+2\sin(\log x)]+c$]

$\int x^n \log x\,dx$ [Ans : $\frac{x^{n+1}}{n+1}\log x-\frac{x^{n+1}}{(n+1)^2}+c$]

$\int \operatorname{cosec}^3 x\,dx$ [Ans : $\int\frac{-1}{2}\operatorname{cosec}x\cot x+\frac{1}{2}\log\left|\tan\frac{x}{2}\right|+c$]

$\int \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$ [Ans : $2x\tan^{-1}x-\log|(1+x^2)|+c$]

(Questions based on the method of Partial Fraction)

Evaluate following integrals :

1. $\int \frac{2x-1}{(x-1)(x+2)(x-3)}dx$ [Ans : $\frac{-1}{6}\log|x-1|-\frac{1}{3}\log|x+2|+\frac{1}{2}\log|x-3|+c$]

2. $\int \frac{x^2}{(x+1)(x+2)(x+3)}dx$ [Ans : $\frac{1}{2}\log|x+1|-4\log|x+2|+\frac{9}{2}\log|x+3|+c$]

3. $\int \frac{\sin\varphi\cos\varphi}{\cos^2\varphi-\cos\varphi-2}d\varphi$ [Ans : $\frac{2}{3}\log|\cos\varphi-2|\frac{-1}{3}\log|1+\cos\varphi|+c$]

4. $\int \frac{dx}{1+3e^x+2e^{2x}}$ [Ans : $x-2\log(2e^x+1)+\log(e^x+1)+c$]

5. $\int \frac{x^2}{16-x^6}dx$ [Ans : $\frac{1}{24}\log\left(\frac{4+x^3}{4-x^3}\right)+c$]

6. $\int \frac{x^3-x-2}{1-x^2}dx$ [Ans : $\log\left|\frac{x-1}{x+1}\right|-\frac{x^2}{2}+c$]

7. $\int \frac{x^2+1}{x^2-1}dx$ [Ans : $x+\log|x-1|-\log|x+1|+c$]

8. $\int \frac{x^4}{x^4-1}dx$ [Ans : $x+\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+c$]

9. $\int \frac{\sec^2\varphi\, d\varphi}{\tan^3\varphi+4\tan\varphi}$ [Ans : $\frac{1}{4}\log\left|\frac{\tan\varphi}{\sqrt{\tan^2\varphi+4}}\right|+c$]

10. $\int \frac{dx}{\sin x+\tan x}$ [Ans : $\log|\operatorname{cosec}x-\cot x|+\frac{1}{4}\log\left|\frac{1+\cos x}{1-\cos x}\right|-\frac{1}{2(1+\cos x)}+c$]

(Questions based on Special Integrals)

Evaluate following integrals:

1. $\int \frac{dx}{(x^2+36)}$ [Ans : $\frac{1}{6}\tan^{-1}\frac{x}{6}+c$]

2. $\int \frac{dx}{-16x^2+25}$ [Ans : $\frac{1}{40}\log\left|\frac{5+4y}{5-4y}\right|+c$]

3. $\int \frac{dx}{a^2-b^2x^2}$ [Ans : $\frac{1}{2ab}\log\left|\frac{a+bx}{a-bx}\right|+c$]

4. $\int \frac{dx}{\sqrt{x^2-2}}$ [Ans : $\log\left|x+\sqrt{x^2-2}\right|+c$]

5. $\int \frac{dx}{\sqrt{15-8x^2}}$ [Ans : $\frac{1}{2\sqrt{2}}\sin^{-1}\left(\sqrt{\frac{8}{15}}x\right)+c$]

6. $\int \frac{dx}{\sqrt{a^2-b^2x^2}}$ [Ans : $\frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right)+c$]

7. $\int \frac{dx}{1+2(x+2)^2}$ [Ans : $\frac{1}{\sqrt{2}}\tan^{-1}\left[(x+2)\sqrt{2}\right]+c$]

$\int \frac{\cos x}{1+\sin^2 x}dx$ [Ans : $\tan^{-1}(\sin x)+c$]

$\int \frac{\sin 8x}{\sqrt{9+\sin^4 4x}}dx$ [Ans : $\frac{1}{4}\log\left|\sin^2 4x+\sqrt{9+\sin^4 4x}\right|+c$]

$\int \frac{x\,dx}{x^4-a^4}$ [Ans : $\frac{1}{4a^2}\log\left|\frac{x^2-a^2}{x^2+a^2}\right|+c$]

(Questions based on Algebriac Integral)

...e the following integrals:

$\int \frac{dx}{(x+2)\sqrt{x+1}}$ [Ans :- $2\tan^{-1}\sqrt{x+1}+c$]

$\int \frac{dx}{(x-2)\sqrt{x+2}}$ [Ans :- $\frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right|+c$]

$\int \frac{x+1}{(x-1)\sqrt{x+2}}dx$ [Ans :- $2\sqrt{x+2}+\frac{2}{\sqrt{3}}\log\left|\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right|+c$]

$\int \frac{dx}{(x+1)\sqrt{x^2-1}}$ [Ans :- $\sqrt{\frac{x-1}{x+1}}+c$]

$\int \frac{dx}{x^2\sqrt{x+1}}$ [Ans :- $-\frac{\sqrt{x+1}}{x}-\frac{1}{2}\log\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+c$]

$\int \frac{dx}{(x+2)\sqrt{x^2+6x+7}}$ [Ans :- $\sin^{-1}\left[\frac{x+1}{\sqrt{2}(x+2)}\right]+c$]

$\int \frac{dx}{(3+4x^2)\sqrt{4-3x^2}}$ [Ans :- $\frac{-1}{5\sqrt{3}}\tan^{-1}\left[\frac{\sqrt{12-9x^2}}{5x}\right]+c$]

$\int \frac{dx}{(x^2-1)\sqrt{x^2+1}}$ [Ans :- $\frac{-1}{2\sqrt{2}}\log\left[\frac{x\sqrt{2}+\sqrt{x^2+1}}{x\sqrt{2}-\sqrt{x^2+1}}\right]+c$]

$\int \frac{\sqrt{1+x^2}}{1-x^2}dx$ [Ans :- $-\log\left[x+\sqrt{1+x^2}\right]+\frac{1}{\sqrt{2}}\log\left|\frac{x\sqrt{2}+\sqrt{x^2+1}}{x\sqrt{2}-\sqrt{x^2+1}}\right|$]

$\int \frac{x}{(x^2+4)\sqrt{x^2+9}}dx$ [Ans :- $\frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{x^2+9}-\sqrt{5}}{\sqrt{x^2+9}+\sqrt{5}}\right|+c$]

CHAPTER - 11

[Definite Integrals]

11.1 [Definite Integral As a Limit of Sum]

Let $f(x)$ be a continuous function defined on the interval [a, b]; where b>a . If the interval [a, b] can be divided into n equal parts; each of width h so that $h=\frac{b-a}{n}$; then

$$\int_a^b f(x) = \mathop{Lt}_{n\to\infty} \frac{b-a}{n}[f(a)+f(a+h)+f(a+2h)------+f(a+(n-1)h)]$$

$$\mathop{Lt}_{h\to 0} h[f(a)+f(a+h)+------+f(a+(n-1)h)]$$

$$\left[\because As\, n\to\infty, \frac{b-a}{n}\to 0 \Rightarrow h\to 0\right]$$

is called the definite integral of $f(x)$, between the limits a and b or integral as a limit of sum.

thus $\int_a^b f(x) = \mathop{Lt}_{h\to 0} h[f(x)+f(a+h)+-----+f(a+(n-1)h]$

Following results shall be very useful in evaluation of integral as a limit of sum.

(i) $1+2+3+------+(n-1) = n\frac{(n-1)}{2}$

(ii) $1^2+2^2+3^2+------+(n-1)^2 = \frac{1}{6}(n-1)(n)(2n-1)$

(iii) $1^3+2^3+3^3+------+(n-1)^3 = \left\{\frac{n(n-1)}{2}\right\}^2$

(iv) $a+ar+ar^2+------ar^{n-1} = \frac{a(r^n-1)}{r-1}$

(v) $Cos a + Cos(a+h) + Cos(a+2h) + -----+Cos[a+(n-1)h]\ \frac{\cos\left\{a+\left(\frac{n-1}{2}\right)h\right\}\sin\left(\frac{nh}{2}\right)}{\sin(h/2)}.$

(vi) $Sin a + \sin(a+h) + \sin(a+2h) + ------+ Sin[a+(n-1)h] = \frac{Sin\left\{a+\left(\frac{n-1}{2}\right)h\right\}\cdot Sin\left(\frac{nh}{2}\right)}{Sin(h/2)}$

(vii) $\mathop{Lt}_{x\to 1} \frac{e^x-1}{x} = 1$

Examples

Evaluate the following integrals as limit of sums

(i) $\int_1^4 x dx$ (ii) $\int_1^3 (2x+3)dx$ (iii) $\int_1^2 (x^2+1)dx$

(vi) $\int_1^2 e^x \cdot dx$ (v) $\int_0^{\pi/2} Sin x\, dx$ (vi) $\int_a^b Cos x\, dx$

(vii) $\int_1^2 (x^3+1)dx$

...tion

Here $f(x)=x$; a = 1 : b = 4 ; thus $h=\frac{b-a}{n}=\frac{4-1}{n}=\frac{3}{n}$

$$\int_a^b f(x)dx = \mathop{Lt}_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+-----+f(a+(n-1)h)]$$

$$\therefore \int_1^4 x\,dx = \mathop{Lt}_{h\to 0} h[a+(a+h)+(a+2h)+-----+(a+(n-1)h)]$$

$$= \mathop{Lt}_{h\to 0} h[(a+a+---n\,times)+h(1+2+---+(n-1)]$$

$$= \mathop{Lt}_{h\to 0} h\left[na+\frac{(n)(n-1)}{2}\right] = \mathop{Lt}_{h\to 0}\left[nha+\frac{(nh-h)(nh)}{2}\right] = \left[3\times 1+\left(\frac{(3-0)(3)}{2}\right)\right]$$

$[\because nh = b-a = 3\, and\, a = 1]$ or $\int_1^4 x\,dx = \left[3+\frac{9}{2}\right] = \frac{15}{2}$

$f(x) = 2x+3; a=1, b=3 \Rightarrow nh = b-a = 3-1 = 2$

Now $\int_a^b (x)dx = \mathop{Lt}_{h\to 0} h[f(a)+f(a+h)+---+f(a+(n-1)h)]$

$$\therefore \int_1^3 (2x+3)dx = \mathop{Lt}_{h\to 0} h[\{2(1)+3\}+\{2(1+h)+3\}+\{2(1+2h)+3\}+......+\{2(1+(n-1)h+3\}]$$

$$\mathop{Lt}_{h\to 0} h[(5+5+---+5)+2h(1+2+---+(n-1)]$$

$$\mathop{Lt}_{h\to 0} h\left[5n+2h\frac{n(n-1)}{2}\right]$$

$$\mathop{Lt}_{h\to 0}[5nh+nh(nh-h)] \qquad [\because nh=2]$$

$= 5(2)+2(2-0)$

$= 10+4 = 14$

$f(x) = x^2+1; a=1, b=2 \Rightarrow nh = b-a = 2-1 = 1$

Now $\int_a^b (x)dx = \mathop{Lt}_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+---+f(a+(n-1)h)]$

$$\therefore \int_1^2 (x^2+1)dx = \underset{h\to 0}{Lt}\, h\left[\{(1)^2+1\}+\{(1+h)^2+1\}+\{(1+2h)^2+1\}\right.$$

$$\left.+ \ldots\ldots + \{(1+(n-1)h)^2+1\}\right]$$

$$\underset{h\to 0}{Lt}\, h\left[(2+2+---n \text{ times}) + h^2\{1^2+2^2+---+(n-1)^2\}\right.$$

$$\left.+ 2h\,\{1+2+\ldots\ldots+(n-1)\}\right]$$

$$\underset{h\to 0}{Lt}\, h\left[2n + h^2\frac{(n)(n-1)(2n-1)}{6} + 2h\frac{(n-1)(n)}{2}\right]$$

$$\left[2nh + \frac{(nh)(nh-h)(2nh-h)}{6} + (nh-h)(nh)\right]$$

$$= \left[2(1) + \frac{(1)(1-0)(2-0)}{6} + (1-0)(1)\right] = 2+\frac{1}{3}+1 = \frac{10}{3}$$

(iv) $f(x) = e^x;\ a = 1,\ b = 2 \Rightarrow nh = b - a = 2 - 1 = 1$

Now $\int_a^b (x)dx = \underset{h\to 0}{Lt}\, h[f(a)+f(a+h)+---+f(a+(n-1)h)]$

$$\underset{h\to 0}{Lt}\, h\left[(e^a + e^{a+h} + e^{a+2h} + --- + e^a\cdot e^{(n-1)h}\right]$$

$$\underset{h\to 0}{Lt}\, h\left[(e^a + e^a\cdot e^h + e^a\cdot e^{2h} + --- + e^{a+(n-1)h}\right]$$

$$\underset{h\to 0}{Lt}\, h\left[(e^a\{1 + e^h + e^{2h} + --- + e^{(n-1)h}\}\right] \qquad [\text{G.P. with } a = 1;\ r = e^h]$$

$$= \underset{h\to 0}{Lt}\, h\left[(e^a\cdot 1\left\{\frac{(e^h)^n-1}{e^h-1}\right\}\right] = e^a\cdot \underset{h\to 0}{Lt}\, h\frac{(e^{nh}-1)}{e^h-1}$$

$$= e^a\cdot \underset{h\to 0}{Lt}(e^{nh}-1)\times\frac{1}{\underset{h\to 0}{Lt}\dfrac{(e^h-1)}{h}}$$

$$= e^1\cdot(e^1-1)\times\frac{1}{1} = e(e-1) \qquad [\because a = 1;\ nh = 1]$$

(v) $f(x) = Sin\,x;\ a = 0; b = \frac{\pi}{2} \Rightarrow nh = b - a = \frac{\pi}{2}$

Now $\int_a^b (x)dx = \underset{h\to 0}{Lt}\, h[f(a)+f(a+h)+f(a+2h)---+f(a+(n-1)h)]$

$$\therefore \int_a^b Sin\,x\,dx = \underset{h\to 0}{Lt}\, h[Sin\,a + Sin(a+h) + Sin(a+2h) --- + Sin(a+(n-1)h)]$$

$$= \underset{h\to 0}{Lt}\, h\frac{Sin\left\{a+\left(\frac{n-1}{2}\right)h\right\}\cdot Sin\frac{nh}{2}}{\sin(h/2)}$$

$$= \underset{h\to 0}{Lt}\left[2Sin\left\{a+\frac{nh-h}{2}\right\}\right]\times \underset{h\to 0}{Lt}\, Sin\left(\frac{nh}{2}\right)\times \underset{h\to 0}{Lt}\frac{h/2}{Sin(h/2)}$$

$$= 2Sin\left(a+\frac{b-a}{2}\right)\cdot Sin\left(\frac{b-a}{2}\right)\times 1 \qquad [\because nh = b-a]$$

$$= 2Sin\left(\frac{b+a}{2}\right)\cdot Sin\left(\frac{b-a}{2}\right) = \cos\left(\frac{b+a-b+a}{2}\right) - \cos\left(\frac{b+a+b-a}{2}\right)$$

$$= \cos a - \cos b \qquad [\because 2\,SinA\,SinB = Cos(A-B) - Cos(A+B)]$$

$$\int_0^{\pi/2} Sin\,x\,dx = Cos\,0 - Cos\frac{\pi}{2}$$

$$= 1 - 0 = 1$$

Please do yourself [Ans: $Sin\,b - Sin\,a$]

$f(x) = x^3+1,\ a = 1, b = 2 \Rightarrow nh = b - a = 2 - 1 = 1$

As $\int_a^b (x)dx = \underset{h\to 0}{Lt}\, h[f(a)+f(a+h)+---+f(a+(n-1)h)]$

$$\therefore \int_1^2 (x^3+1)dx = \underset{h\to 0}{Lt}\, h\left[(1^3+1)+\{(1+h)^3+1\}+(1+2h)^3\right.$$

$$\left.+\ldots\ldots+\{(1+(n-1)h)^3+1\}\right]$$

$$\underset{h\to 0}{Lt}\, h\left[\begin{array}{l}(2+2+---n\text{ times}) + h^3\{1^3+2^3+---+(n-1)^3\}\\ +3h^2\{1^2+2^2+---(n-1)^2\} + 3h\{1+2+---+(n-1)\}\end{array}\right]$$

$$= \underset{h\to 0}{Lt}\, h\left[2n + h^3\frac{n^2(n-1)^2}{4} + 3h^2\frac{(n)(n-1)(2n-1)}{6} + 3h\frac{(n)(n-1)}{2}\right]$$

$$= \underset{h\to 0}{Lt}\left[2nh + \frac{(nh)^2(nh-h)^2}{4} + \frac{(nh)(nh-h)(2nh-h)}{2} + \frac{3}{2}(nh)(nh-h)\right]$$

$$= \left[2.1 + \frac{(1)^2(1-0)}{4} + \frac{(1)(1-0)(2.1-0)}{2} + \frac{3}{2}(1)(1-0)\right]$$

$$= 2+\frac{1}{4}+1+\frac{3}{2}$$

$$= 3+\frac{1}{4}+\frac{3}{2} = \frac{19}{4}$$

Exercise

Evaluate the following integrals as limit of Sum

(1) $\int_3^5 (2-x)dx$ [Ans : -4]

(2) $\int_1^2 x^2 dx$ [Ans : $-\frac{7}{3}$]

(3) $\int_0^2 (x^2+3)dx$ [Ans: 3]

(4) $\int_1^3 (2x^2+5x)dx$ [Ans : $\frac{26}{3}$]

(5) $\int_0^2 e^{-x} dx$ [Ans : $1-\frac{1}{e^2}$]

(6) $\int_0^2 e^{3x+1} dx$ [Ans : $\frac{e(e^6-1)}{3}$]

(7) $\int_0^{\pi/2} Cos\,x\,dx$ [Ans: 1]

11.2 [Definite Integrals]

Definition:

Let $\frac{d}{dx}[\varphi(x)] = f(x)$ Then, definite integral of f(x) over [a b] is given by $\int_a^b f(x)$

where $\int_a^b f(x) = \varphi(b) - \varphi(a)$

Examples

2. **Evaluate** $\int_4^5 x^4 dx$.

Solution:

$$\int_4^5 x^4 dx = \left[\frac{x^{4+1}}{4+1}\right]_4^5 = \left[\frac{x^5}{5}\right]_4^5 = \frac{1}{5}[(5)^5-(4)^5] = \frac{1}{5}[3125-1024] = \frac{1}{5}[2101] = \frac{2101}{5}$$

3. **Evaluate** $\int_0^{\pi/2} \sin^2 x dx$.

Solution:

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}}\left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2}\int_0^{\frac{\pi}{2}}(1-\cos 2x)dx$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} dx - \int_0^{\frac{\pi}{2}} \cos 2x dx\right] = \frac{1}{2}\left[[x]_0^{\frac{\pi}{2}} - \left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}-0\right) - \frac{1}{2}[\sin\pi - \sin 0]\right] = \frac{1}{2}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{4}$$

4. **Evaluate** $\int_2^3 \frac{x}{x^2+1} dx$.

Solution:

$\int_2^3 \frac{x}{x^2+1} dx = \frac{1}{2}\int_2^3 \frac{2x}{x^2+1} dx$ Now Put $x^2+1 = t$ $\therefore$ 2xdx = dt when $x = 3 \Rightarrow t = 10$

$x = 2 \Rightarrow t = 5$ $\therefore$ $\frac{1}{2}\int_5^{10}\frac{dt}{t} = \frac{1}{2}[\log t]_5^{10} = \frac{1}{2}[\log 10 - \log 5] = \frac{1}{2}\log\left(\frac{10}{5}\right) = \frac{1}{2}\log 2$

Evaluate $\int_0^{\frac{\pi}{2}} x \sin x dx$.

Solution:

$[x(-\cos x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 1.(-\cos x)dx$ [Using by parts]

$$= [-x\cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x dx = \left[-\frac{\pi}{2}\cos\frac{\pi}{2} - (-0.\cos 0)\right] + [(\sin x)]_0^{\frac{\pi}{2}} = 0 + \left[\sin\frac{\pi}{2} - \sin 0\right] = 1$$

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$.

Solution:

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$ Put cos x = t [when $x = \frac{\pi}{2}, t = 0$ $-\sin x\, dx = dt$ when $x = 0, t = 1$]

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx = -\int_1^0 \frac{dt}{1+t^2} = -[\tan^{-1} t]_1^0 = -[\tan^{-1} 0 - \tan^{-1} 1] = -\left[0-\frac{\pi}{4}\right] = \frac{\pi}{4}$$

11.3 [Properties of Definite Integral]

(I) $\int_a^b f(x)dx = \int_a^b f(t)dt$ [integration is independent of the change of variable]

(II) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(III) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, a < c < b

(IV) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(V) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ [f is a continuous function defined on [0,a]]

(VI) $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function,} \\ 0 & \text{if } f(x) \text{ is odd function} \end{cases}$

(VII) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)$ if $f(2a-x) = f(x)$

Examples

Evaluate $\int_1^4 f(x)dx$ **where** $f(x) = \begin{pmatrix} 4x+3, & \text{if } 1 \le x \le 2 \\ 3x+5, & \text{if } 2 \le x \le 4 \end{pmatrix}$.

Solution:

$\int_1^4 f(x)dx = \int_1^2 f(x)dx + \int_2^4 f(x)dx$ [See property III]

$$\Rightarrow \int_1^2 (4x+3)dx + \int_2^4 (3x+5)dx = \left[4\frac{x^2}{2}+3x\right]_1^2 + \left[3\frac{x^2}{2}+5x\right]_2^4$$

$$= \left(\frac{4}{2}.(4)+6\right)-\left(\frac{4}{2}+3\right)+\left[\frac{3}{2}.(16)+20-\left(\frac{3}{2}.(4)\right)+10\right] = \frac{16}{2}+6-(2+3)+[24+20-(6+10)]$$

$$= 8+6-5+44-6-10 = 37$$

8. Evaluate $\int_{-5}^{5} |x-2|dx$.

Solution:

$$|x-2| = \begin{pmatrix} x-2, \text{ when } x-2 \ge 0 \text{ ie } x \ge 2 \\ -(x-2) \text{ when } x-2<0 \text{ ie } x<2 \end{pmatrix} \quad \therefore \int_{-5}^{5}|x-2|dx = \int_{-5}^{2}|x-2|dx + \int_{2}^{5}|x-2|dx$$

$$= \int_{-5}^{2} -(x-2)dx + \int_{2}^{5}(x-2)dx = \int_{-5}^{2}(-x+2)dx + \int_{2}^{5}(x-2)dx$$

$$= \left[\frac{-x^2}{2}+2x\right]_{-5}^{2} + \left[\frac{x^2}{2}-2x\right]_{2}^{5}$$

$$= \left[\frac{-4}{2}+4-\left(\frac{-25}{2}+(-10)\right)\right]+\left[\frac{25}{2}-10-\left(\frac{4}{2}-4\right)\right]$$

$$= \left[-2+4+\frac{25}{2}+10\right]+\left[\frac{25}{2}-10-2+4\right] = 29$$

9. Evaluate $\int_1^3 \frac{\sqrt{x}}{\sqrt{4-x}+\sqrt{x}}dx$.

Solution:

$$I = \int_1^3 \frac{\sqrt{x}}{\sqrt{4-x}+\sqrt{x}}dx \quad ...(i)$$

$$= \int_1^3 \frac{\sqrt{4-x}}{\sqrt{4-(4-x)}+\sqrt{4-x}}dx \quad \text{[see property iv]}$$

$$I = \int_1^3 \frac{\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}}dx \quad ...(ii) \qquad \text{Adding (i) and (ii)}$$

$$2I = \int_1^3 \frac{\sqrt{x}+\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}}dx = \int_1^3 dx = [x]_1^3 = (3)-1 = 2 \Rightarrow I = 1$$

10. Evaluate $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}dx$.

Solution:

$$I = \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}dx \quad ...(i)$$

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx \quad \text{[see property v)} \qquad I = 0$$

$$I = \int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}dx \quad ...(ii)$$

Adding (i) and (ii) we get

$$2I = \int_0^{\pi/2}\left(\frac{\sin x+\cos x}{\cos x+\sin x}\right)dx = \int_0^{\pi/2}dx = [x]_0^{\pi/2} = \pi/2$$

$$2I = \pi/2 \qquad \therefore I = \pi/4$$

Evaluate $\int_0^{\pi/2} \sin 2x.\log\tan x dx$.

tion:

$$I = \int_0^{\pi/2} \sin 2x.\log\tan x dx \quad ...(i)$$

$$I = \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2}-x\right).\log\tan\left(\frac{\pi}{2}-x\right)dx$$

$$I = \int_0^{\pi/2} \sin(2x).\log\cot x dx \quad ...(ii)$$

Add (i) and (ii), we get

$$2I = \int_0^{\pi/2} \sin 2x(\log\cot x+\log\tan x)dx$$

$$2I = \int_0^{\pi/2} \sin 2x(\log\cot x.\tan x)dx$$

$$2I = \int_0^{\pi/2} \sin 2x\log 1.dx = 0 \quad [\because \log 1 = 0]$$

Evaluate $\int_{-\pi/2}^{\pi/2} \sin^2 x dx$.

tion:

Let $f(x) = \sin^2 x$

Now $f(-x) = (\sin(-x))^2 = (-\sin x)^2 = \sin^2 x$ i.e $f(x) = f(-x)$

Hence f is an even funtion

$$\therefore \int_{\pi/2}^{\pi/2} f(x)dx = 2\int_0^{\pi/2} f(x)dx \quad \text{[using property VI]} \quad \text{i.e.} = 2\int_0^{\pi/2}\sin^2 x dx$$

$$= 2\int_0^{\pi/2}\left(\frac{1-\cos 2x}{2}\right)dx = 2\left[\frac{1}{2}\int_0^{\pi/2}dx - \frac{1}{2}\int_0^{\pi/2}\cos 2x dx\right] = 2\left[\frac{1}{2}[x]_0^{\pi/2} - \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/2}\right]$$

$$= 2\left[\frac{1}{2}\left(\frac{\pi}{2}\right)-\frac{1}{2}\left[\frac{\sin\pi}{2}-\frac{0}{2}\right]\right] = 2\left(\frac{\pi}{4}\right)-0 = \frac{\pi}{2}$$

13. Evaluate $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x dx$.

Solution:

Let $f(x) = x^3 \sin^4 x$

$f(-x) = (-x)^3 (\sin(-x))^4 = -x^3 \sin^4 x = -f(x)$ i.e. $f(x) = -f(x)$

Hence f is a odd function $\therefore \int_{-\pi/4}^{\pi/4} x^3 \sin^4 x dx = 0$ [using property VI]

Exercise

1. $\int_0^1 x(1-x)^5 dx$ [Ans. $\frac{1}{42}$]
2. $\int_0^{\pi/4} \tan^3 x dx$ [Ans. $\frac{1}{2}(1-\log 2)$]
3. $\int_0^{\pi/4} \log(1+\tan x)dx$ [Ans. $\frac{\pi}{8} \log 2$]
4. $\int_0^{\pi/2} \frac{\sin^2 x}{1+\sin x \cos x} dx$ [Ans. $\frac{\pi}{3\sqrt{3}}$]
5. $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} dx$ [Ans. $a\pi$].

CHAPTER - 12

[Reduction Formulae]

...duction :

A reduction formula for an integral is a formula that represents the integral into one or two other integrals in which ...nd is of the same type but of lower degree or lower order.

From the reduction formula of an integral; we can easily calculate the integrals which otherwise are tedious to ...ate.

12·1 [Reduction Formulae of $\sin^n x$; $\cos^n x$]

...cle : Find the reduction formula for

(i) $\int \sin^n x \, dx$ (ii) $\int \cos^n x \, dx$ and use them to evaluvate

(iii) $\int_0^{\pi/2} \sin^n x \, dx$ and $\int_0^{\pi/2} \cos^n x \, dx$ [where n is a +ve integer]

...of:

Let $I_n = \int \sin^n x \, dx = \int \sin^{n-1} x \cdot \sin x \, dx$

Integrating by parts by taking $\sin^{n-1} x$ as first function and $\sin x$ as second function;

$$I_n = -\sin^{n-1} x (\cos x) - \int (n-1) \sin^{n-2} x \cdot \cos x (-\cos x) dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1-\sin^2 x) dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

or $I_n + (n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$

or $(1+n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$

or $\boxed{I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}}$

Which is the required reduction formula for $\int \sin^n x \, dx$

(ii) $\int \cos^n x \, dx = I_n$

$\therefore I_n = \int \cos^{n-1} x \cos x \, dx$

Integrating by parts taking $\cos^{n-1} x$ as first function and $\cos x$ as second function

$$= \cos^{n-1} x (\sin x) - \int (n-1) \cos^{n-2} x (-\sin x) \times (\sin x) dx$$

$$= \cos^{n-1} x (\sin x) + (n-1) \int \cos^{n-2} x \sin^2 x \, dx$$

$$= \cos^{n-1} x (\sin x) + (n-1) \int \cos^{n-2} x (1-\cos^2 x) dx$$

$$I_n = \cos^{n-1} x\,(\sin x) + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

or $I_n + (n-1)I_n = \cos^{n-1} x\,(\sin x) + (n-1)I_{n-2}$

or $\{1 + (n-1)\}\,I_n = \cos^{n-1} x\,(\sin x) + (n-1)I_{n-2}$

or $\boxed{I_n = \dfrac{\cos^{n-1} x\,(\sin x)}{n} + \dfrac{n-1}{n} I_{n-2}}$

(iii) To evaluavate $\int_0^{\pi/2} \sin^n x$ or $\int_0^{\pi/2} \cos^n x\,dx$ [Walli's Formula]

Proof:

Let $I_n = \int \sin^n x\,dx$

then $I_n = -\dfrac{1}{n}\sin^{n-1} x\cos x + \dfrac{n-1}{n} I_{n-2}$ [From part (i)]

$$\therefore \int_0^{\pi/2} \sin^n x\,dx = -\frac{1}{n}\left[\sin^{n-1} x\cos x\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx = 0 + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$\therefore I_n = \dfrac{n-1}{n} I_{n-2}$ [Where $I_n = \int_0^{\pi/2} \sin^n x\,dx$]

Changing n to $n-2, n-4, n-6$ in successive steps

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4} \; ; \quad I_{n-4} = \frac{n-5}{n-4} I_{n-6} \quad \text{etc.}$$

$$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} I_{n-6} \quad \text{etc.}$$

As n is a positive integer the end term depends upon its value. Following two cases arise.

Case I :

In n is an even positive integer then

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots\cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \int_0^{\pi/2} 1\,dx$$

i.e. $\boxed{\int_0^{\pi/2} \sin^n x\,dx = \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdots\cdots \dfrac{5}{6} \cdot \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2} \text{ if } n \text{ is even}}$

Case II :

If n is odd positive integer; then

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{4}{5} \cdot \frac{2}{3} \cdot \int_0^{\pi/2} \sin x\,dx$$

Now $\int_0^{\pi/2} \sin x\,dx = -\left[\cos x\right]_0^{\pi/2} = 1$

$\therefore \boxed{\int_0^{\pi/2} \sin^n x\,dx = \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdots\cdots \dfrac{4}{5} \cdot \dfrac{2}{3} \cdot 1, \text{ if } n \text{ is odd}}$

(iv) $\int_0^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \cos^n(\pi/2 - x)\,dx = \int_0^{\pi/2} \sin^n x\,dx$

$\Rightarrow$ Reduction formulae for both are same.

Examples

Evaluvate $\int_0^{\pi/2} \sin^4 x\,dx$.

ion:

$\int_0^{\pi/2} \sin^4 x\,dx$; Here $n = 4$ which is even $\therefore \int_0^{\pi/2} \sin^4 x\,dx = \dfrac{4-1}{4} \cdot \dfrac{4-3}{4-2} \cdot \dfrac{\pi}{2} = \dfrac{3\pi}{16}$

Evaluvate $\int_0^{\pi/2} \cos^8 x\,dx$.

tion :

$$\int_0^{\pi/2} \cos^8 x\,dx = \frac{8-1}{8} \cdot \frac{8-3}{8-2} \cdot \frac{8-5}{8-4} \cdot \frac{8-7}{8-2} \cdot \frac{\pi}{2} = \frac{7\times5\times3\times1}{8\times6\times4\times2} \cdot \frac{\pi}{2} = \frac{105\pi}{768}$$

Evaluvate $\int_0^1 \dfrac{x^5}{2\sqrt{1-x^2}}\,dx$.

tion :

Put $x = \sin\theta$, $dx = \cos\theta\,d\theta$; then the given integral becomes

$$\frac{1}{2}\int_0^{\pi/2} \frac{\sin^5\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \frac{1}{2}\left[\int_0^{\pi/2} \sin^5\theta\,d\theta\right] = \frac{1}{2}\left[\frac{4}{5}\cdot\frac{2}{3}\cdot 1\right] = \frac{4}{15}$$

Evaluvate $\int_0^{\infty} \dfrac{dx}{(1+x^2)^4}$.

tion :

Put $x = \tan\theta$; $dx = \sec^2\theta\,d\theta$;

when $x \to 0$; $\theta \to 0$ and when $x \to \infty \Rightarrow \theta \to \dfrac{\pi}{2}$

$\therefore$ Given integral becomes

$$\int_0^{\pi/2} \frac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)^4} = \int_0^{\pi/2} \frac{\sec^2\theta}{\sec^8\theta}\,d\theta = \int_0^{\pi/2} \cos^6\theta\,d\theta = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}$$

Exercise

aluvate :

1. $\int_0^{\pi/2} \cos^7 dx$ [Ans. $\dfrac{16}{35}$]
2. $\int_0^{\pi/2} \cos^9 x\,dx$ [Ans. $\dfrac{128}{315}$]
3. $\int_0^{\pi/6} \cos^3 3x\,dx$ [Ans. $\dfrac{2}{9}$] [Hint : Put $3x = t$]
4. $\int \sin^3 x\,dx$ [Ans. $-\dfrac{1}{3}\sin^2 x\cos x - \dfrac{2}{3}\cos x + c$]

12·2 [Reduction Formulae of $\tan^n x$, $\cot^n x$, $\sec^n x$, $\operatorname{cosec}^n x$]

Article:

Obtain Reduction Formulae for

(i) $\int \tan^n x\,dx$ (ii) $\int \cot^n x\,dx$ (iii) $\int \sec^n x\,dx$ (iv) $\int \operatorname{cosec}^n x\,dx$

Proof :-

(i) Let $I_n = \int \tan^n x\,dx = \int \tan^{n-2} x \tan^2 x\,dx = \int \tan^{n-2} x (\sec^2 x - 1)\,dx$

or $I_n = \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$

Let $\tan x = t$ for first integral so that $dt = \sec^2 x\,dx$

$$\therefore\ I_n = \int t^{n-2} dt - \int \tan^{n-2} x\,dx = \frac{t^{n-1}}{n-1} - I_{n-2} \quad \text{or} \quad I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2};$$

Which is the required reduction formulae for $\int \tan^n x\,dx$

(ii) Please try your self. Similar to (i)

$$\int \cot^n x\,dx = \frac{\cot^{n-1} x}{n-1} - I_{n-2} \quad \text{or} \quad I_n = \frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

(iii) $\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$

Integrating by parts taking $\sec^{n-2} x$ as first function and $\sec^2 x$ as second function

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-3} x \cdot \sec x \cdot \tan x \cdot (\tan x)\,dx$$

$$= \sec^{n-2} x \cdot \tan x - (n-2) \int \sec^{n-2} x \tan^2 x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \cdot (\sec^2 x - 1)\,dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x\,dx + (n-2) \int \sec^{n-2} x\,dx$$

or $[1 + (n-2)] \int \sec^n x\,dx = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x\,dx$

$$\therefore\ \int \sec^n x\,dx = \frac{1}{n-1} (\sec^{n-2} x \tan x) + \frac{n-2}{n-1} \int \sec^{n-2} x\,dx$$

Which is the required reduction formula for $\int \sec^n x\,dx$

We can also write the above formula as

$$I_n = \frac{1}{n-1} (\sec^{n-2} x \tan x) + \frac{n-2}{n-1} I_{n-2}$$

(iv) Please do yourself similar to (iii)

$$\int \operatorname{cosec}^n x\,dx = -\frac{1}{n-1} (\operatorname{cosec}^{n-2} x \cot x) + \frac{n-2}{n-1} \int \operatorname{cosec}^{n-2} x\,dx$$

Examples

5. **Evaluvate $\int \tan^5 x\,dx$.**

Solution:

Let $I_n = \int \tan^n x\,dx$; then we know

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2} \qquad \ldots(1) \qquad \text{Taking } n = 5 \text{ in (1)}$$

$$I_5 = \frac{\tan^4 x}{4} - I_3 \qquad \ldots(2) \qquad \text{Taking } n = 3 \text{ in (1)}$$

$$I_3 = \frac{\tan^2 x}{2} - I_1 \qquad \ldots(3)$$

Now $I_1 = \int \tan x\,dx = \log \sec x$; Putting this value is (3)

$$I_3 = \frac{\tan^2 x}{2} - \log \sec x \qquad \therefore \text{ Putting in (2)}$$

$$I_5 = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log \sec x \quad \text{or} \quad \int \tan^5 x = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \log \sec x$$

Evaluvate $\int \sec^5 x\,dx$.

tion :

Let $I_n = \int \sec^5 x\,dx$ then $I_n = \frac{1}{n-1} \sec^{n-2} x \cdot \tan x + \frac{n-2}{n-1} I_{n-2}$ ...(1)

Putting $n = 5$ in (1)

$$I_5 = \frac{1}{4} \sec^3 x \tan x + \frac{3}{4} I_3 \qquad \ldots(2)$$

Putting $n = 3$ in (1)

$$I_3 = \frac{1}{2} \sec x \tan x + \frac{1}{2} I_1 = \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x\,dx$$

$$= \frac{1}{2} \sec x \tan x + \frac{1}{2} \log \left| \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \right|$$

Putting this value in (2); we get

$$I_5 = \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \log \left| \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \right|$$

$$\text{or} \quad \int \sec^5 x = \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \log \left| \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \right|$$

If $I_n = \int_0^{\pi/4} \tan^n x\,dx$; then prove that $I_n + I_{n-1} = \frac{1}{n-1}$; n being a positive integer > 1. Hence evaluvate I_5.

olution:

$$I_n = \int_0^{\pi/4} \tan^n x\,dx = \int_0^{\pi/4} \tan^{n-2} x \cdot \tan^2 x\,dx = \int_0^{\pi/4} \tan^{n-2} x (\sec^2 x - 1)\,dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x\,dx - \int_0^{\pi/4} \tan^{n-2} x\,dx$$

Taking $\tan x = t$ in Ist integral so that $\sec^2 x\,dx = dt$.

When $x \to 0$ then $t \to 0$ and as $x \to \frac{\pi}{4}$ then $t \to 1$

$$\therefore\ I_n = \int_0^1 t^{n-2} dt - I_{n-2} \quad \text{or} \quad I_n + I_{n-2} = \left[\frac{t^{n-1}}{n-1}\right]_0^1 \quad \boxed{\text{or} \quad I_n + I_{n-2} = \frac{1}{n-1}} \qquad \ldots(1)$$

Hence first part is proved. Putting $n = 5, 3$ in (1)

$$I_5 + I_3 = \frac{1}{4} \quad \ldots(2) \qquad I_3 + I_1 = \frac{1}{2} \quad \ldots(3)$$

$$\text{Now} \quad I_1 = \int_0^{\pi/4} \tan x\,dx = [\log \sec x]_0^{\pi/4} = \left[\log \sec \frac{\pi}{4} - \log \sec 0\right] = \log \sqrt{2} - \log 1 = \frac{1}{2} \log 2 \quad [\because \log 1 = 0]$$

From (3) $I_3 = \frac{1}{2} - \frac{1}{2}\log 2$. Putting this value in (2); we get

$$I_5 = \frac{1}{4} - I_3 = \frac{1}{4} - \frac{1}{2} + \frac{1}{2}\log 2 \qquad \therefore \; I_5 = \frac{1}{2}\log 2 - \frac{1}{4}$$

Exercise

1. Evaluate $\int \operatorname{cosec}^6 x\, dx$ [Ans. $-\frac{1}{5}\operatorname{cosec}^4 x \cdot \cot x - \frac{4}{15}\operatorname{cosec}^2 x \cot x - \frac{8}{15}\cot x$]

2. Evaluate $\int_0^a (a^2 + x^2)^{5/2}\, dx$ [Ans. $\frac{a^6}{48}(67\sqrt{2} + 15 \log \tan 3\pi/8)$] [Hint : Put $x = a \tan\theta$]

3. Prove that $\int_0^{\pi/4} \sec^3 x\, dx = \frac{1}{2}\left[\sqrt{2} + \log \tan \frac{3\pi}{8}\right]$

4. Prove that $\int \cot^5 x\, dx = -\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \log \sin x$

5. Prove that $\int \operatorname{cosec}^5 x\, dx = -\frac{1}{4}\operatorname{cosec}^3 x \cot x - \frac{3}{8}\operatorname{cosec} x \cot x + \frac{3}{8}\log \tan \frac{x}{2}$

12·3 [Reduction Formulae of $x \sin^n x$, $x \cos^n x$, $\cos^m x \cos^n x$, $\cos^m x \sin^n x$, $\sin^m x \cos^n x$]

Article:

Obtain a reduction formula for (i) $\int x \sin^n x\, dx$ (ii) $\int x \cos^n x\, dx$.

Proof:-

(i) Let $I_n = \int x \sin^n x\, dx = \int (x \sin^{n-1} x) \cdot \sin x\, dx$

$$= (x \sin^{n-1} x) \cdot (-\cos x) - \int \left[x \cdot \{(n-1)\sin^{n-2} x \cos x\} + \sin^{n-1} x \cdot 1\right](-\cos x)\, dx \quad \text{[Integrating by parts]}$$

$$= -x \sin^{n-1} x \cos x + (n-1)\int x \sin^{n-2} x \cdot \cos^2 x\, dx + \int \sin^{n-1} x \cos x\, dx$$

$$= -x \sin^{n-1} x \cos x + (n-1)\int x \sin^{n-2} x \cdot (1 - \sin^2 x)\, dx + \int \sin^{n-1} x \cos x\, dx$$

$$= -x \sin^{n-1} x \cos x + (n-1)\int x \sin^{n-2} x\, dx - (n-1)\int x \sin^n x\, dx + \frac{\sin^n x}{n}$$

$\Rightarrow$ $I_n = -x \sin^{n-1} x \cos + (n-1) I_{n-2} - (n-1) I_n + \frac{\sin^n x}{n}$

$\Rightarrow$ $[1 + (n-1)] I_n = -x \sin^{n-1} x \cos x + \frac{\sin^n x}{n} + (n-1) I_{n-2}$

$\Rightarrow$ $n \cdot I_n = -x \sin^{n-1} x \cos x + \frac{\sin^n x}{n} + (n-1) I_{n-2}$

$\Rightarrow$ $\boxed{I_n = -\frac{x \sin^{n-1} x \cos x}{n} + \frac{\sin^n x}{n^2} + \frac{n-1}{n} I_{n-2}}$

(ii) Let $I_n = \int x \cos^n x\, dx = \int (x \cos^{n-1} x) \cdot \cos x\, dx$

$$= (x \cos^{n-1} x) \cdot (\sin x) - \int \left[x\{(n-1)\cos^{n-2} x(-\sin x)\} + \cos^{n-1} x \cdot 1\right](\sin x)\, dx \quad \text{[Integrating by parts]}$$

$$= x \cos^{n-1} x \sin x + (n-1)\int x \cos^{n-1} x \sin^2 x\, dx - \int \cos^{n-1} x \sin x\, dx$$

$$= x \cos^{n-1} x \sin x + (n-1)\int x \cos^{n-2} x (1 - \cos^2 x)\, dx + \int (\cos x)^{n-1}(-\sin x)\, dx$$

$$= x \cos^{n-1} x \sin x + (n-1)\int x \cos^{n-2} x\, dx - (n-1)\int x \cos^n x\, dx + \frac{(\cos x)^n}{n}$$

$\therefore$ $I_n = x \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n + \frac{1}{n}\cos^n x$

$\Rightarrow$ $[1 + (n-1)] I_n = x \cos^{n-1} x \sin x + (n-1) I_{n-2} + \frac{\cos^n x}{n}$

$\Rightarrow$ $n \cdot I_n = x \cos^{n-1} x \sin x + \frac{\cos^n x}{n} + (n-1) I_{n-2}$

$\Rightarrow$ $\boxed{I_n = \frac{1}{n} x \cos^{n-1} x \sin x + \frac{1}{n^2}\cos^n x + \frac{n-1}{n} I_{n-2}}$ which is the required reduction formula.

Article:

Obtain a reduction formula for (i) $\int \cos^m x \sin nx\, dx$ (ii) $\int \cos^m x \cos nx\, dx$.

Proof :-

(i) Let $I_{m,n} = \int \cos^m x \sin nx\, dx$; Integrating by parts

$$= \cos^m x\left(-\frac{\cos nx}{n}\right) - \int m \cos^{m-1} x(-\sin x) \cdot \left(-\frac{\cos nx}{n}\right) dx$$

$$\therefore \; I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x (\cos nx \sin x)\, dx \qquad \text{...(1)}$$

Now $\sin (n-1)x = \sin nx \cos x - \cos nx \sin x \Rightarrow \cos nx \sin x = \sin nx \cos x - \sin (n-1)x$

Substituting value of $\cos nx \sin x$ in (1), we get,

$$I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x [\sin nx \cos x - \sin (n-1)x]\, dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^m x \sin nx\, dx + \frac{m}{n}\int \cos^{m-1} x \sin (n-1)x\, dx$$

$$\therefore \; I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

$\Rightarrow$ $\left(1 + \frac{m}{n}\right) I_{m,n} = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1}$ $\Rightarrow$ $(m+n) I_{m,n} = -\cos^m x \cos nx + m I_{m-1,n-1}$

$\Rightarrow$ $I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$

(ii) Let $I_{m,n} = \int \cos^m x \cos nx\, dx = \cos^m x\left(\frac{\sin nx}{n}\right) - \int m \cos^{m-1} x(-\sin x) \cdot \left(\frac{\sin nx}{n}\right) dx$ [Integrating by parts]

$$I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x (\sin nx \sin x)\, dx \qquad \text{...(1)}$$

Now $\cos (n-1)x = \cos nx \cos x + \sin nx \sin x \Rightarrow \sin nx \sin x = \cos (n-1)x - \cos nx \cos x$

Substituting value of sin nx sin x in (1), we get,

$$I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x\,[\cos(n-1)x - \cos nx \cos x]\,dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \cos(n-1)x\,dx - \frac{m}{n}\int \cos^m x \cos nx\,dx$$

$$\Rightarrow I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,n-1} - \frac{m}{n} I_{m,n} \quad \Rightarrow \left(1+\frac{m}{n}\right) I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow (m+n) I_{m,n} = \cos^m x \sin nx + m\, I_{m-1,n-1} \quad \Rightarrow \boxed{I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}}$$

which is the required reduction formula.

Article :-

Obtain Reduction Formula for $\int \sin^m x \cos^n x\,dx$

Proof :

Let $I_{m,n} = \int \sin^m x \cos^n x\,dx = \int \sin^m x \cos^{n-1} x \cos x\,dx = \int \cos^{n-1} x\,(\sin^m x \cos x)\,dx$

$$= \cos^{n-1} x \cdot \frac{\sin^{m+1} x}{m+1} - \int (n-1)\cos^{n-2} x\,(-\sin x)\cdot \frac{\sin^{m+1} x}{m+1}\,dx \qquad \text{[Integrating by parts]}$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \sin^{m+2} x\,dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \cdot \sin^m x \cdot \sin^2 x\,dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \cdot \sin^m x\,(1-\cos^2 x)\,dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \cdot \sin^m x\,dx - \frac{n-1}{m+1}\int \cos^n x \sin^m x\,dx$$

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$$

$$\Rightarrow \left(1+\frac{n-1}{m+1}\right) I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$\Rightarrow (m+n) I_{m,n} = \sin^{m+1} x \cos^{n-1} x + (n-1) I_{m,n-2} \quad \Rightarrow I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$\Rightarrow \boxed{\int \sin^m x \cos^n x\,dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x\,dx}$$

which is the required reduction formula.

Examples

8. If $U_n = \int_0^{\pi/2} x^n \sin x\,dx$ and $n > 1$ prove that $U_n + n(n-1)\,U_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$ Hence evaluvate.

(a) U_5 (b) $\int_0^{\pi/2} x^4 \sin x\,dx$

...tion :

$$U_n = \int_0^{\pi/2} x^n \sin x\,dx = \left[x^n(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} n\,x^{n-1}(-\cos x)\,dx$$

$$= -\left[x^n \cos x\right]_0^{\pi/2} + n\int_0^{\pi/2} x^{n-1}\cos x\,dx = -\left[\left(\frac{\pi}{2}\right)^n \cos\frac{\pi}{2} - 0\right] + n\int_0^{\pi/2} x^{n-1}\cos x\,dx$$

$$= n\int_0^{\pi/2} x^{n-1}\cos x\,dx = n\left\{\left[x^{n-1}\sin x\right]_0^{\pi/2} - \int_0^{\pi/2}(n-1)\,x^{n-2}\sin x\,dx\right\}$$

$$= n\left[x^{n-1}\sin x\right]_0^{\pi/2} - n(n-1)\int_0^{\pi/2} x^{n-2}\sin x\,dx = n\left[\left(\frac{\pi}{2}\right)^{n-1}\sin\frac{\pi}{2} - 0\right] - n(n-1)\,U_{n-2}$$

or $U_n + n(n-1)\,U_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$ ...(1)

(a) Taking $n = 5, 3$ in (1)

$U_5 + 5(5-1)\,U_3 = 5\left(\frac{\pi}{2}\right)^{5-1}$ or $U_5 + 20U_3 = \frac{5\pi^4}{16}$ ...(2)

and $U_3 + 6U_1 = \frac{3}{4}\pi^2$ ...(3)

Now $U_1 = \int_0^{\pi/2} x \sin x\,dx$

$$= \left[x(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} 1\cdot(-\cos x)\,dx = -\left[x\cos x\right]_0^{\pi/2} + \int_0^{\pi/2}\cos x\,dx$$

$$= -\left[\frac{\pi}{2}\cos\frac{\pi}{2} - 0\right] + \left[\sin x\right]_0^{\pi/2} = 0 + \left[\sin\frac{\pi}{2} - \sin 0\right] = 1$$

Putting in (3)

$$U_3 = \frac{3\pi^2}{4} - 6U_1 = \frac{3\pi^2}{4} - 6$$

Putting in (2)

$$U_5 = \frac{5\pi^4}{16} - 20U_3 = \frac{5\pi^4}{16} - 20\left(\frac{3\pi^2}{4} - 6\right) = \frac{5\pi^4}{16} - 15\pi^2 + 120$$

(b) Putting $n = 4, 2$ in (1), we get

$U_4 + 12U_2 = \frac{\pi^3}{2}$ ...(4) $\qquad U_2 + 2U_0 = \pi$ ...(5)

Now $U_0 = \int_0^{\pi/2} \sin x\,dx = \left[-\cos x\right]_0^{\pi/2} = \left[-\cos\frac{\pi}{2} + \cos 0\right] = 1$

$\therefore$ From (5) $U_2 = \pi - 2U_0 = \pi - 2$

From (4) $U_4 = \frac{\pi^3}{2} - 12U_2 = \frac{\pi^3}{2} - 12(\pi - 2) = \frac{\pi^3}{2} - 12\pi + 24.$

Evaluvate $\int x^2 \sin 2x \cdot dx.$

...lution:

From Reduction formula of $\int x^m \sin nx\,dx$; we have [See article ahead]

$$\int x^m \sin nx\,dx = -\frac{x^m \cos nx}{n} + \frac{m}{n^2}x^{m-1}\sin nx - \frac{m(m-1)}{n^2}\int x^{m-2}\sin nx\,dx$$

Taking $m = 2;\ n = 2$

$$\int x^2 \sin 2x\,dx = -\frac{x^2\cos 2x}{2} + \frac{2x\sin 2x}{4} - \frac{2}{4}\int \sin 2x\,dx = -\frac{x^2}{2}\cos 2x + \frac{x\sin 2x}{2} + \frac{1}{4}\cos 2x.$$

10. **If $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx\,dx$; show that $I_{m,n} = \frac{m}{m+n} I_{m-1,n-1}$. Hence evaluate $\int_0^{\pi/2} \cos^5 x \cos 3x\,dx$.**

Solution:

$$I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx\,dx = \left[\cos^m x\cdot\frac{\sin nx}{n}\right]_0^{\pi/2} - \int_0^{\pi/2} m\cos^{m-1} x\,(-\sin x)\cdot\frac{\sin nx}{n}\,dx$$

$$= 0 - 0 + \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x\,(\sin nx \sin x)\,dx$$

$$I_{m,n} = \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x\,(\sin nx \sin x)\,dx \qquad \text{...(1)}$$

Now $\cos(n-1)x = \cos nx\cos x + \sin nx \sin x \Rightarrow \sin nx \sin x = \cos(n-1)x - \cos nx\cos x$

Substituting value of $\sin nx \sin x$ in (1), we get.

$$I_{m,n} = \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x\,[\cos(n-1)x - \cos nx\cos x]\,dx$$

$$= \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x\cos(n-1)x\,dx - \frac{m}{n}\int_0^{\pi/2} \cos^m x\cos nx\,dx \qquad \therefore I_{m,n} = \frac{m}{n} I_{m-1,n-1} - \frac{m}{n} I_{m,n}$$

$$\Rightarrow \left(1+\frac{m}{n}\right) I_{m,n} = \frac{m}{n} I_{m-1,n-1} \Rightarrow (m+n)\,I_{m,n} = m\cdot I_{m-1,n-1} \qquad \text{...(2)}$$

$$\Rightarrow I_{m,n} = \frac{m}{m+n}\cdot I_{m-1,n-1}$$

Put $m = 5, n = 3$ in (2), $\quad I_{5,3} = \frac{5}{5+3} I_{4,2}$ ...(3)

Put $m = 4, n = 2$ in (2), $\quad I_{4,2} = \frac{4}{4+2} I_{3,1}$ ...(4)

Put $m = 3, n = 1$ in (2) $\quad I_{3,1} = \frac{3}{3+1} I_{2,0}$ ...(5)

Now $I_{2,0} = \int_0^{\pi/2} \cos^2 x\cos(0\,x)\,dx = \int_0^{\pi/2}\cos^2 x\,dx$

$$= \frac{1}{2}\int_0^{\pi/2}(1+\cos 2x)\,dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}+\frac{1}{2}\sin\pi\right) - \left(0+\frac{1}{2}\sin 0\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2}+0\right] - \left(0+\frac{1}{2}\times 0\right) = \frac{\pi}{4}$$

$\therefore$ From (5), $I_{3,1} = \frac{3}{4}\times\frac{\pi}{4} = \frac{3\pi}{16}$; From (4), $I_{4,2} = \frac{4}{6}\times\frac{3\pi}{16} = \frac{\pi}{8}$; From (3), $I_{5,3} = \frac{5}{8}\times\frac{\pi}{8} = \frac{5\pi}{64}$.

11. **If $U_n = \int_0^{\pi/2} x \sin^n x\,dx$ then prove that $U_n = \frac{n-1}{n} U_{n-2} + \frac{1}{n^2}$. Deduce that $U_5 = \frac{149}{225}$.**

Solution:

From Reduction Formula of $\int x \sin^n x\,dx$; we have

$$\int x \sin^n x\,dx = -\frac{x\sin^{n-1}x\cos x}{n} + \frac{\sin^n x}{n^2} + \frac{n-1}{n}\int x\sin^{n-2}x\,dx$$

$$\therefore \int_0^{\pi/2} x\sin^n x\,dx = -\left|\frac{x\sin^{n-1}x\cos x}{n}\right|_0^{\pi/2} + \frac{1}{n^2}\left|\sin^n x\right|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} x\cdot\sin^{n-2}x\,dx$$

or $U_n = 0 + \frac{1}{n^2}[1-0] + \frac{n-1}{n}U_{n-2}$ $\qquad \therefore U_n = \frac{n-1}{n}U_{n-2} + \frac{1}{n^2}$ ...(1)

Now putting $n = 5, 3$ in (1) $\quad U_5 = \frac{4}{5}U_3 + \frac{1}{25}$ ...(2)

$$U_3 = \frac{2}{3}U_1 + \frac{1}{9} \qquad \text{...(3)}$$

Now $U_1 = \int_0^{\pi/2} x\sin x\,dx = -\left|x\cos x\right|_0^{\pi/2} + \int_0^{\pi/2}\cos x\,dx = 0 + \left|\sin x\right|_0^{\pi/2} = 1 \qquad \therefore U_1 = 1$

Putting in (3); we get, $\quad U_3 = \frac{2}{3} + \frac{1}{9} = \frac{7}{9}$

Putting in (2), we get, $\quad U_5 = \frac{4}{5}\left(\frac{7}{9}\right) + \frac{1}{25} = \frac{28}{45} + \frac{1}{25} = \frac{149}{225}$

If $I_{m,n} = \int_0^{\pi/2} \sin^m x\cos^n x\,dx$; then prove that $I_{m,n} = \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdot\frac{m-5}{m+n-4}\cdots\cdots\frac{2}{3+n}\cdot\frac{1}{1+n}$.

where m is an odd positive integer and n is a +ve integer, even or odd.

Solution:

From reduction formula of $\int \sin^m x\cos^n x\,dx$; we have

$$\int \sin^m x\cos^n x\,dx = -\frac{\cos^{n+1}x\sin^{m-1}x}{m+n} + \frac{m-1}{m+n}\int\sin^{m-2}x\cos^n x\,dx$$

$$\therefore \int_0^{\pi/2}\sin^m x\cos^n x\,dx = -\left|\frac{\cos^{n+1}x\sin^{m-1}x}{m+n}\right|_0^{\pi/2} + \frac{m-1}{m+n}\int_0^{\pi/2}\sin^{m-2}x\cos^n x\,dx$$

$$\therefore I_{m,n} = 0 + \frac{m-1}{m+n}I_{m-2,n} = \frac{m-1}{m+n}I_{m-2,n}$$

Replacing m by $m-2, m-4, \ldots\ldots 3$; we get

$$I_{m-2,n} = \frac{m-3}{m+n-2}I_{m-4,n}$$

$$I_{m-4,n} = \frac{m-5}{m+n-4}I_{m-6,n}$$

$$\vdots \qquad \vdots$$

$$\vdots \qquad \vdots$$

$$I_{3,n} = \frac{2}{3+n}I_{1,n} \qquad \text{if } m \text{ is odd}$$

Now $I_{1,n} = \int_0^{\pi/2}\sin x\cos^n x\,dx = -\left|\frac{\cos^{n+1}x}{n+1}\right|_0^{\pi/2} = \frac{1}{n+1}$

$\therefore$ From these relations, we get

$$I_{m,n} = \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdot\frac{m-5}{m+n-4}\cdots\cdots\frac{2}{3+n}\cdot\frac{1}{1+n}$$

where m is any odd positive integer and n is any positive integer.

12·4 [Reduction Formulae of $e^{ax}\sin^n x$]

Article:-

Obtain a reduction formula for $\int e^{ax}\sin^n x\,dx$.

Proof :-

Let $I_n = \int e^{ax}\sin^n dx = \int \sin^n x \cdot e^{ax}\,dx$ [Taking $\sin^n x$ as first function]

$$= \sin^n x\cdot\left(\frac{e^{ax}}{a}\right) - \int (n\sin^{n-1}x\cos x)\cdot\left(\frac{e^{ax}}{a}\right)dx$$

$$= \frac{1}{a}e^{ax}\sin^n x - \frac{n}{a}\int(\sin^{n-1}x\cos x)\,e^{ax}\,dx$$

$$= \frac{1}{a}e^{ax}\sin^n x - \frac{n}{a}\left[(\sin^{n-1}x\cos x)\cdot\left(\frac{e^{ax}}{a}\right)\right.$$

$$\left. - \int\{(n-1)\sin^{n-2}x\cos x\cdot\cos x + \sin^{n-1}x\cdot(-\sin x)\}\left(\frac{e^{ax}}{a}\right)dx\right]$$

$$= \frac{1}{a}e^{ax}\sin^n x - \frac{n}{a}\left[\frac{e^{ax}\sin^{n-1}x\cos x}{a} - \int\{(n-1)\sin^{n-2}x\cos^2 x - \sin^n x\}\frac{e^{ax}}{a}dx\right]$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\left[\frac{e^{ax}\sin^{n-1}x\cos x}{a} - \int\{(n-1)\sin^{n-2}x(1-\sin^2 x) - \sin^n x\}\frac{e^{ax}}{a}dx\right]$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\left[\frac{e^{ax}\sin^{n-1}x\cos x}{a} - \int\{(n-1)\sin^{n-2}x - n\sin^n x\}\frac{e^{ax}}{a}dx\right]$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\left[\frac{e^{ax}\sin^{n-1}x\cos x}{a} - \frac{n-1}{a}\int e^{ax}\sin^{n-2}dx + \frac{n}{a}\int e^{ax}\sin^n x\,dx\right]$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{n}{a^2}e^{ax}\sin^{n-1}x\cos x + \frac{n(n-1)}{a^2}\int e^{ax}\sin^{n-2}x\,dx - \frac{n^2}{a^2}\int e^{ax}\sin^n x\,dx$$

$$I_n = \frac{e^{ax}\sin^n x}{a} - \frac{n}{a^2}e^{ax}\sin^{n-1}x\cos x + \frac{n(n-1)}{a^2}I_{n-2} - \frac{n^2}{a^2}I_n$$

$$\Rightarrow \left(1+\frac{n^2}{a^2}\right)I_n = \frac{a\,e^{ax}\sin^n x - n\,e^{ax}\sin^{n-1}x\cos x}{a^2} + \frac{n(n-1)}{a^2}I_{n-2}$$

$$\Rightarrow (a^2+n^2)I_n = e^{ax}\sin^{n-1}x\,(a\sin x - n\cos x) + n(n-1)I_{n-2}$$

$$\Rightarrow I_n = \frac{e^{ax}\sin^{n-1}x\,(a\sin x - n\cos x)}{a^2+n^2} + \frac{n(n-1)}{a^2+n^2}I_{n-2}$$ which is the required reduction formula.

Miscellaneous Examples

1. **Obtain Reduction Formula of $\int x^m \sin nx\,dx$.**

Solution:

Let $I_{m,n} = \int x^m \sin nx\,dx$

$$= x^m\left(-\frac{\cos nx}{n}\right) - \int m x^{m-1}\cdot\left(-\frac{\cos nx}{n}\right)dx$$ [Integrating by parts]

$$= -\frac{x^m\cos nx}{n} + \frac{m}{n}\int x^{m-1}\cos nx\,dx$$

$$= -\frac{x^m\cos nx}{n} + \frac{m}{n}\left[x^{m-1}\left(\frac{\sin nx}{n}\right) - \int(m-1)x^{m-2}\cdot\left(\frac{\sin nx}{n}dx\right)\right]$$ [Again integrating by parts]

$$= -\frac{x^m\cos nx}{n} + \frac{m}{n^2}x^{m-1}\sin nx - \frac{m(m-1)}{n^2}\int x^{m-2}\sin nx\,dx$$

$$\therefore I_{m,n} = -\frac{x^m\cos nx}{n} + \frac{m}{n^2}x^{m-1}\sin nx - \frac{m(m-1)}{n^2}I_{m-2,n}$$ which is the required reduction formula.

Obtain Reduction Formula of $\int x^m \cos nx\,dx$.

n:

Let $I_{m,n} = \int x^m \cos nx\,dx$

$$= x^m\left(\frac{\sin nx}{n}\right) - \int m x^{m-1}\cdot\left(\frac{\sin nx}{n}\right)dx$$ [Integrating by parts]

$$= \frac{x^m\sin nx}{n} - \frac{m}{n}\int x^{m-1}\cdot\sin nx\,dx$$

$$= \frac{x^m\sin nx}{n} - \frac{m}{n}\left[x^{m-1}\left(-\frac{\cos nx}{n}\right) - \int(m-1)x^{m-2}\cdot\left(-\frac{\cos nx}{n}\right)dx\right]$$ [Again integrating by parts]

$$= \frac{x^m\sin nx}{n} + \frac{m}{n^2}x^{m-1}\cos nx - \frac{m(m-1)}{n^2}\int x^{m-2}\cos nx\,dx$$

$$\therefore I_{m,n} = \frac{x^m\sin nx}{n} + \frac{m}{n^2}x^{m-1}\cos nx - \frac{m(m-1)}{n^2}I_{m-2,n}$$ which is the required reduction formula.

If $I_{m,n} = \int_0^{\pi/2}\cos^m x\cos nx\,dx$, show that $I_{m,n} = \frac{m}{m+n}I_{m-1,n-1}$. Hence evaluate $\int_0^{\pi/2}\cos^5 x\cos 3x\,dx$.

ion:

$$I_{m,n} = \int_0^{\pi/2}\cos^m x\cos nx\,dx = \left[\cos^m x\cdot\frac{\sin nx}{n}\right]_0^{\pi/2} - \int_0^{\pi/2} m\cos^{m-1}x(-\sin x)\cdot\frac{\sin nx}{n}dx$$

$$= 0 - 0 + \frac{m}{n}\int_0^{\pi/2}\cos^{m-1}x\,(\sin nx\sin x)\,dx$$

$$I_{m,n} = \frac{m}{n}\int_0^{\pi/2}\cos^{m-1}x\,(\sin nx\sin x)\,dx \quad ...(1)$$

Now $\cos(n-1)x = \cos nx\cos x + \sin nx\sin x \Rightarrow \sin nx\sin x = \cos(n-1)x - \cos nx\cos x$

Substituting value of $\sin nx\sin x$ in (1), we get

$$I_{m,n} = \frac{m}{n}\int_0^{\pi/2}\cos^{m-1}x\,[\cos(n-1)x - \cos nx\cos x]\,dx = \frac{m}{n}\int_0^{\pi/2}\cos^{m-1}x\cos(n-1)x\,dx - \frac{m}{n}\int_0^{\pi/2}\cos^m x\cos nx\,dx$$

$$\therefore I_{m,n} = \frac{m}{n}I_{m-1,n-1} - \frac{m}{n}I_{m,n} \Rightarrow \left(1+\frac{m}{n}\right)I_{m,n} = \frac{m}{n}I_{m-1,n-1}$$

$$\Rightarrow I_{m,n} = \frac{m}{m+n}\cdot I_{m-1,n-1}$$

$$\Rightarrow (m+n)I_{m,n} = m\cdot I_{m-1,n-1} \quad ...(2)$$

Put $m = 5$, $n = 3$ in (2),

$$I_{5,3} = \frac{5}{5+3}I_{4,2} \quad ...(3)$$

Put $m = 4$, $n = 2$ in (2),

$$I_{4,2} = \frac{4}{4+2}I_{3,1} \quad ...(4)$$

Put $m = 3$, $n = 1$ in (2)

$I_{3,1} = \frac{3}{3+1} I_{2,0}$ ...(5)

Now $I_{2,0} = \int_0^{\frac{\pi}{2}} \cos^2 x \cos(0\,x)\,dx = \int_0^{\frac{\pi}{2}} \cos^2 x\,dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2x)\,dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - \left(0 + \frac{1}{2}\sin 0\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} + 0\right] - \left(0 + \frac{1}{2}\times 0\right) = \frac{\pi}{4}$ $\quad\therefore$ from (5), $I_{3,1} = \frac{3}{4}\times\frac{\pi}{4} = \frac{3\pi}{16}$

From (4), $I_{4,2} = \frac{4}{6}\times\frac{3\pi}{16} = \frac{\pi}{8}$. From (3), $I_{5,3} = \frac{5}{8}\times\frac{\pi}{8} = \frac{5\pi}{64}$.

4. Write down the values of : (i) $\int_0^{\frac{\pi}{2}} \sin^8\theta\,d\theta$ (ii) $\int_0^{\frac{\pi}{2}} \sin^7\theta\,d\theta$.

Solution:

(i) $\int_0^{\frac{\pi}{2}} \sin^8\theta\,d\theta$ $\quad\left[\text{Compare with } \int_0^{\frac{\pi}{2}} \sin^n\theta\,d\theta, \text{ here } n = 8\right]$

$= \frac{7\cdot5\cdot3\cdot1}{8\cdot6\cdot4\cdot2}\cdot\frac{\pi}{2}$ $\quad\left[\frac{\pi}{2}\ (\because n = 8, a + ve \text{ even integer})\right]$

$\left[\frac{(n-1)\times\text{go on diminishing by 2}}{n\times\text{go on diminishing by 2}}\cdot\frac{\pi}{2} \text{ only if } n \text{ is a + ve even integer}\right]$ $\quad\therefore \int_0^{\frac{\pi}{2}} \sin^8\theta\,d\theta = \frac{35}{256}\pi$

(ii) $\int_0^{\frac{\pi}{2}} \sin^7\theta\,d\theta$ $\quad\left[\text{Compare with } \int_0^{\frac{\pi}{2}} \sin^n\theta\,d\theta, \text{ here } n = 7\right]$

$\frac{6\cdot4\cdot2}{7\cdot5\cdot3}$ $\quad\left[\text{no } \frac{\pi}{2}\ (\because n = 7, a + ve \text{ even integer})\right]$

$\left[\frac{(n-1)\times\text{go on diminishing by 2}}{n\times\text{go on diminishing by 2}}\cdot\frac{\pi}{2} \text{ only if } n \text{ is a + ve even integer}\right]$ $\quad\therefore \int_0^{\frac{\pi}{2}} \sin^7\theta\,d\theta = \frac{16}{35}$.

Exercise

1 If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$, prove that $I_n + I_{n-1} = \frac{1}{n-1}$, n being a positive integer > 1. Hence evaluate I_5.

[Ans. $-\frac{1}{4} + \frac{1}{2}\log 2$]

2 Obtain a reduction formula for $\int \cot^n x\,dx$, n being a +ve integer. Hence evaluate

(i) $\int \cot^3 x\,dx$ (ii) $\int \cot^4 x\,dx$ (iii) $\int \cot^5 x\,dx$

[Ans. (i) $-\frac{1}{2}\cot^2 x - \log|\sin x|$, (ii) $-\frac{1}{3}\cot^3 x + \cot x + x$, (iii) $-\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \log|\sin x|$]

3 Obtain a reduction formula for $\int \sec^{2n+1} x\,dx$. hence evaluate $\int \sec^5 x\,dx$.

[Ans. $\frac{1}{4}\sec^3 x\tan x + \frac{3}{8}\sec x\tan x + \frac{3}{8}\log|\sec x + \tan x|$]

Obtain a reduction formula for $\int e^{ax}\cos^n x\,dx$. Hence evaluate $\int e^{ax}\cos^4 x\,dx$.

[Ans. $\int e^{ax}\cos^n x\,dx = \frac{e^{ax}\cos^{n-1} x\,(a\cos x + n\sin x)}{a^2+n^2} + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\cos^{n-2} x\,dx$;

$e^{ax}\left[\frac{\cos^3 x\,(a\cos x + 4\sin x)}{a^2+4^2} + \frac{4\cdot3\cos x\,(a\cos x + 2\sin x)}{(a^2-4^2)(a^2+2^2)} + \frac{4\cdot3\cdot2\cdot1}{(a^2+4^2)(a^2+2^2)\,a}\right]$

Find a reduction formula for $\int_0^{\pi} e^{-x}\sin^n x\,dx$. [Ans. $\int_0^{\pi} e^{-x}\sin^n x\,dx = \frac{n(n-1)}{n^2+1}\int_0^{\pi} e^{-x}\sin^{n-2} x\,dx$]

If $U_n = \int_0^{\frac{\pi}{2}} \theta\sin^n\theta\,d\theta$, and $n > 1$, prove that $U_n = \frac{n-1}{n}U_{n-2} + \frac{1}{n^2}$ Hence Find U_5. [Ans. $\frac{149}{225}$]

Obtain a reduction formula for $\int x^n\sin x\,dx$ and apply it to evaluate $\int_0^{\frac{\pi}{2}} x^3\sin x\,dx$.

[Ans. $\int x^n\sin x\,dx = -x^n\cos x + n\,x^{n-1}\sin x - n(n-1)\int x^{n-2}\sin x\,dx$; $\frac{3\pi^2}{4} - 6$]

Find a reduction formula for $\int x^n\cos x\,dx$ and hence evaluate (i) $\int x^3\cos x\,dx$ (ii) $\int x^4\cos x\,dx$.

[Ans. $\int x^n\cos x\,dx = x^n\sin x + n\,x^{n-1}\cos x - n(n-1)\int x^{n-2}\cos x\,dx$

(i) $x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x$

(ii) $x^4\sin x + 4x^3\cos x - 12x^2\sin x - 24\,x\cos x + 24\sin x$]

Write down the values of : (i) $\int_0^{\frac{\pi}{2}} \cos^3 x\sin^2 x\,dx$ (ii) $\int_0^{\frac{\pi}{2}} \sin^8 x\cos^4 x\,dx$ (iii) $\int_0^{\frac{\pi}{2}} \sin^6 x\cos^5 x\,dx$.

[Ans. (i) $\frac{2}{15}$, (ii) $\frac{7\pi}{2048}$, (iii) $\frac{8}{693}$]

CHAPTER - 13

[Beta and Gamma Functions]

13.1 Eulerian Integrals

The first and second kind Eulerian integrals are known as beta and gamma functions respectively which are defined as:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx\,;\ n, m > 0 \qquad ...(1)$$

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx;\ n > 0 \qquad ...(2)$$

13.2 Properties of Beta Functions

(*i*) $B(m, n) = B(n, m)$ (symmetry)

(*ii*) $B(m, n) = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$

(Evaluation of beta function)

(*iii*) $\dfrac{B(m,n+1)}{n} = \dfrac{B(m+1,n)}{m} = \dfrac{B(m,n)}{m+n}$

(*iv*) $B(m, n) = B(m+1, n) + B(m, n+1)$

(*v*) $B(m, n) = \displaystyle\int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}}; m > 0, n > 0$

(Alternative form)

(*vi*) $B(m, n) = \displaystyle\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

Proof:

(*i*) We have

$$B(m, n) = \int_0^1 (1-x)^{m-1}x^{n-1}dx;\quad m, n > 0 \qquad ...(1)$$

Using property of definite integral $\int_0^a f(dx) = \int_0^a f(a-x)dx$, we get

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$B(m, n) = B(n, m) \qquad \text{(From (1))} \qquad ...(2)$$

(*ii*) We have $B(m, n) = \displaystyle\int_0^1 (1-x)^{m-1}x^{n-1}dx$ ...(1)

Integration by parts, we get

Let m is +ve integer $B(m, n) = \left[(1-x)^{m-1}\dfrac{x^n}{n}\right]_0^1 + \displaystyle\int_0^1 (m-1)(1-x)^{m-2}\frac{x^n}{n}dx$

$$= \frac{m-1}{n}\int_0^1 (1-x)^{m-1-1}x^{n+1-1}dx$$

$$B(m, n) = \frac{m-1}{n}B(m-1, n+1) \qquad ...(2)$$

Replacing m by $m-1$ and n by $n+1$, we get

$$B(m-1, n+1) = \frac{m-2}{n+1}B(m-2, n+2) \qquad ...(3)$$

Using Eq. (3) in Eq. (2), we get $B(m, n) = \dfrac{(m-1)(m-2)}{(n)(n+1)}B(m-2, n+2)$

Similarly,

$$B(m, n) = \frac{(m-1)(m-2)(m-3)...1}{n(n+1)(n+2)...(m+n-2)}B(1, m+n-1) = \frac{(m-1)!}{n(n+1)(n+2)...(m+n-2)}\int_0^1 x^{m+n-2}dx$$

$$= \frac{(m-1)!}{n(n+1)(n+2)...(m+n-2)(m+n-1)} \qquad ...(4)$$

If n is +ve integer then use of Eq. (1) in Eq. (4), we get

$$B(m, n) = \frac{(n-1)!}{m(m+1)...(m+n-1)} = B(n, m) \qquad ...(5)$$

If both m, n are +ve integers, then

$$B(m, n) = \frac{(m-1)!}{(m+n-1)(m+n-2)...(n+2)(n+1)n}$$

$$B(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} \qquad ...(6)$$

(*iii*) We know that $B(m, n) = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$. Therefore $B(m, n+1) = \dfrac{(m-1)!n!}{(m+n)!}$

$$B(m, n+1) = \frac{(m-1)!n(n-1)!}{(m+n)(m+n-1)!} = \frac{n}{m+n}B(m,n) \qquad ...(1)$$

Now $B(m+1, n) = \dfrac{m!(n-1)!}{(m+n)!}$

$$B(m, n+1) = \frac{m(m-1)!(n-1)!}{(m+n)(m+n-1)!} = \frac{m}{m+n}B(m,n) \qquad ...(2)$$

From Eqs. (1) and (2), we get $\dfrac{B(m,n+1)}{n} = \dfrac{B(m+1,n)}{m} = \dfrac{B(m,n)}{m+n}$

(*iv*) Adding Eqs. (1) and (2) of (*iii*), we get

$$B(m, n+1) + B(m+1, n) = \frac{n+m}{m+n}B(m,n)$$

$$B(m, n+1) + B(m+1, n) = B(m, n)$$

(*v*) We know that $B(m, n) = \displaystyle\int_0^1 (1-x)^{m-1}x^{n-1}dx$

Let $x = \dfrac{1}{1+y}$ or $y = \dfrac{1}{x} - 1$. Then $dx = -\dfrac{dy}{(1+y)^2}$

Using in Eq. (1), we get $B(m, n) = \displaystyle\int_\infty^0 \left(\frac{y}{1+y}\right)^{m-1}\cdot\left(\frac{1}{1+y}\right)^{n-1}\left(-\frac{dy}{(1+y)^2}\right)$ ...(2)

$$B(m, n) = \int_0^\infty \frac{y^{m-1}dy}{(1+y)^{m+n}} = \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}}$$

(vi) From (v), we have $B(m, n) = \int_0^\infty \frac{y^{m-1}dy}{(1+y)^{m+n}} = \int_0^1 \frac{y^{m-1}dy}{(1+y)^{m+n}} + \int_1^\infty \frac{y^{m-1}dy}{(1+y)^{m+n}}$

$$= \int_0^1 \frac{x^{n-1}dx}{(1+x)^{m+n}} + \int_1^0 \frac{x^{-1(m-1)}}{\left(1+\frac{1}{x}\right)^{m+n}}\left(-\frac{dx}{x^2}\right)\left(\because y = \frac{1}{x} \text{ is 2nd integral}\right)$$

$$= \int_0^1 \frac{x^{m-1}dx}{(1+x)^{m+n}} + \int_0^1 \frac{x^{n-1}dx}{(1+x)^{m+n}}$$

$$B(m, n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$$

13.3 Properties of Gamma Functions

(i) $\Gamma(n+1) = n\Gamma n;\ n > 0$

(ii) $\Gamma n = (n-1)!;\ n = +\text{ve integer}$

(iii) $\int_0^\infty e^{-\lambda x} x^{n-1}dx = \frac{\Gamma n}{\lambda^n}$

(iv) $\int_0^\infty e^{-(x)^n} dx = \Gamma(n+1)$

Proof:

(i) We have $\Gamma(n+1) = \int_0^\infty e^{-x}x^{n+1-1}dx$; $\Gamma(n+1) = \int_0^\infty e^{-x}x^n dx$

Integration by parts gives $\Gamma(n+1) = \left(-e^{-x}x^n\right)_0^\infty + \int_0^\infty n e^{-x}x^{n-1}dx$

$$\Gamma(n+1) = 0 + n\Gamma n \qquad \because x \to \infty \frac{x^n}{e^x} = 0\ ;\ \Gamma(n+1) = n\Gamma n$$

(ii) From (i); $\Gamma n = (n-1)\Gamma(n-1)$ ...(1)

Replacing n by $n-1$, we get $\Gamma(n-1) = (n-2)\Gamma(n-2)$

Using in Eq. (1), we get $\Gamma n = (n-1)(n-2)\Gamma(n-2)$;

$\Gamma n = (n-1)(n-2)\Gamma(n-2)$

$\vdots$

$\Gamma n = (n-1)(n-2)\ldots1\Gamma 1$ ...(2)

where $\Gamma 1 = \int_0^\infty e^{-x}x^{1-1}dx = 1$. Hence $\Gamma n = (n-1)!$

(iii) We have $\int_0^\infty e^{-\lambda x}x^{n-1}dx = \int_0^\infty e^{-y}\left(\frac{y}{\lambda}\right)^{n-1}\frac{dy}{\lambda}$ $(\because y = \lambda x)$

$$= \int_0^\infty \frac{e^{-y}y^{n-1}dy}{\lambda^n}\ ;\ \int_0^\infty e^{-\lambda x}x^{n-1}dx = \frac{\Gamma n}{\lambda^n}$$

(iv) We have $\int_0^\infty e^{-(x)^{\frac{1}{n}}} dx = \int_0^\infty e^{-y} n y^{n-1}dy$ $(\because x = y^n)$ $= n\int_0^\infty e^{-y}y^{n-1}dy$

$$\int_0^\infty e^{-(x)^{1/n}} dx = n\Gamma n = \Gamma(n+1)$$

13.4 [Relation between Beta and Gamma Functions]

o prove that $B(m, n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$; $m, n > 0$

We have $\Gamma n = \int_0^\infty e^{-t}t^{n-1}dt$ ...(1)

$B(n, m) = \int_0^1 (1-t)^{n-1}t^{m-1}dt = \int_0^\infty \frac{t^{n-1}dt}{(1+t)^{m+n}}$ ...(2)

utting $t = xz$, i.e., $dt = z\,dx$ in Eq. (1), we get

$$\Gamma n = \int_0^\infty e^{-xz}(xz)^{n-1}z\,dx$$

$$\Gamma n = \int_0^\infty z^n \cdot e^{-xz}x^{n-1}dx \qquad ...(3)$$

ultiplying by $e^{-z}z^{m-1}$ and integrating w.r.t. from 0 to ∞, we get

$$\Gamma n\int_0^\infty e^{-z} z^{m-1}dz = \int_0^\infty\left(\int_0^\infty z^n e^{-xz}x^{n-1}dx\right)e^{-z}z^{m-1}dz$$

Using Eq. (1), we get $\Gamma n\Gamma m = \int_0^\infty\left(\int_0^\infty z^n e^{-xz}x^{n-1}dx\right)e^{-z}z^{m-1}dz$

Interchanging order of integration, we get

$$\Gamma n\Gamma m = \int_0^\infty\left(\int_0^\infty e^{-(1+x)z}z^{m+n-1}dz\right)x^{n-1}dx = \int_0^\infty \frac{\Gamma(m+n)x^{n-1}}{(1+x)^{m+n}}dx \quad \text{(using (3))}$$

$$\Gamma n\Gamma m = \Gamma(m+n)B(m,n) \quad \text{(using (2))}$$

$$B(m,n) = \frac{\Gamma m\Gamma n}{\Gamma(m+n)} \qquad ...(4)$$

13.5 [To Evaluate $\int_0^{\pi/2} \sin^p x\cos^q x\,dx$]

lution : Put $sin^2 x = t \Rightarrow 2 sinx cosx dx = dt;$

ther as $x \to 0 \Rightarrow t \to 0$ and as $x \to \frac{\pi}{2} \Rightarrow t \to 1$

$$\int_0^{\pi/2} \sin^p x\cos^q x = \int_0^{\frac{\pi}{2}} (\sin^2 x)^{\frac{(p-1)}{2}}.(1-\sin^2 x)^{\frac{(q-1)}{2}} \sin x\cos x\,dx$$

$$= \frac{1}{2}\int_0^1 (t)^{\frac{(p-1)}{2}}.(1-t)^{\frac{(q-1)}{2}}dt = \frac{1}{2}B\left(\frac{p-1}{2}+1, \frac{q-1}{2}+1\right) = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\int_0^{\pi/2} \sin^p x\cos^q x\,dx = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \qquad ...(1)$$

Corollary 1 : Putting p = q = 0; we have $\dfrac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{2\Gamma(1)} = \int_0^{\pi/2} dx = \dfrac{\pi}{2} \Rightarrow \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi \quad \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Note :- Thus to prove $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, first prove result (1) and then follow as above.

Corollary 2 : Putting p = 0; q = n; we get $\int_0^{\pi/2} \cos^n x dx = \dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)} = \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \dfrac{\sqrt{\pi}}{2}$

Similarly $\int_0^{\pi/2} \sin^n x dx = \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \dfrac{\sqrt{\pi}}{2}$

Examples

1. **Evaluate** $\int_0^\infty \dfrac{x^8(1-x^6)}{(1+x)^{24}} dx$.

Solution:

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}} = \int_0^\infty \frac{x^8}{(1+x)^{24}} dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}} dx = \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}} - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}$$

$= B(9, 15) - B(15, 9)$ [see 14.2 result (ii)]

$= B(9, 15) - B(9, 15)$ [$\because B(m,n) = B(n,m)$]

$= 0$

2. **Evaluate** $\int_0^\infty \dfrac{x^4(1+x^5)}{(1+x)^{15}} dx$.

Solution:

$$\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}} dx = \int_0^\infty \frac{x^4}{(1+x)^{15}} dx + \int_0^\infty \frac{x^9}{(1+x)^{15}} dx = \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}} + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}}$$

$$= B(5, 10) + B(10, 5) = B(10, 5) + B(10, 5) = 2B(10, 5) = 2.\frac{\Gamma(5).\Gamma(10)}{\Gamma(5+10)} = 2.\frac{\Gamma(5).\Gamma(10)}{\Gamma(15)}$$

$$= \frac{2\times 4!\times 9!}{14!} = \frac{2\times 4\times 3\times 2\times 1\times 9!}{14\times 13\times 12\times 11\times 10\times 9!} = \frac{1}{5005}$$

3. **Evaluate the following**

(i) $\Gamma(7)$ **(ii)** $\dfrac{\Gamma(3).\Gamma(2)}{\Gamma(5)}$ **(iii)** $\dfrac{\Gamma(4).\Gamma(11)}{\Gamma(15)}$ **(iv)** B (3, 6) **(v)** B (11, 4).

Solution:

(i) As $\Gamma(n+1) = n!$ $\quad \therefore \Gamma(7) = \Gamma(6+1) = 6! = 6.5.4.3.2.1 = 720$

(ii) $\dfrac{\Gamma(3).\Gamma(2)}{\Gamma(5)} = \dfrac{2!\times 1!}{4!} = \dfrac{2\times 1}{4\times 3\times 2\times 1} = \dfrac{1}{12}$

(iii) $\dfrac{\Gamma(4).\Gamma(11)}{\Gamma(15)} = \dfrac{3!\times 10!}{14!} = \dfrac{3\times 2\times 1\times 10!}{14\times 13\times 12\times 11\times 10!} = \dfrac{1}{4004}$

(iv) $B(m, n) = \dfrac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$ $\quad \therefore B(3,6) = \dfrac{\Gamma(3).\Gamma(6)}{\Gamma(3+6)} = \dfrac{\Gamma(3).\Gamma(6)}{\Gamma(9)} = \dfrac{2\times 5!}{8!} = \dfrac{2\times 5!}{8\times 7\times 6\times 5!} = \dfrac{1}{112}$

(v) $B(11,4) = \dfrac{\Gamma(11).\Gamma(4)}{\Gamma(11+4)} = \dfrac{\Gamma(11).\Gamma(4)}{\Gamma(15)} = \dfrac{10!\times 4!}{14!} = \dfrac{10!\times 4\times 3\times 2\times 1}{14\times 13\times 12\times 11\times 10!} = \dfrac{1}{1001}$

Express $\int_0^1 \dfrac{xdx}{\sqrt{1-x^7}}$ **in terms of Beta Function.**

Solution:

Let $x^7 = t$ i.e. $x = t^{1/7} \Rightarrow dx = \frac{1}{7} t^{-6/7} dt$

$$\therefore \int_0^1 \frac{xdx}{\sqrt{1-x^7}} = \frac{1}{7}\int_0^1 \frac{(t)^{1/7}(t)^{-6/7}}{\sqrt{1-t}} dt = \frac{1}{7}\int_0^1 (t)^{-5/7}(1-t)^{-1/2} dt = \frac{1}{7}B\left(\frac{-5}{7}+1, \frac{-1}{2}+1\right) = \frac{1}{7}B\left(\frac{2}{7}, \frac{1}{2}\right)$$

Express $\int_0^2 x^4.(8-x^3)^{-1/3} dx$ **in terms of (i) Beta Function (ii) Gamma Function.**

Solution:

Let $x^3 = 8t \Rightarrow x = 2t^{1/3} \Rightarrow dx = \frac{2}{3}(t)^{-2/3} dt$

[**Note :** In order to express given integral in Beta Function, we should have $(1-x)^n$ form; thus the substitution is of above form]. Also when $x \to 0; t \to 0$ and when $x \to 2; t \to 1$

$$\therefore \int_0^2 x^4.(8-x^3)^{-1/3} = \int_0^1 (2)^4 t^{4/3}.(8)^{-1/3}.(1-t)^{-1/3}.\frac{2}{3}t^{-2/3} dt = \frac{16}{3}\int_0^1 t^{2/3}(1-t)^{-1/3} dt$$

$$= \frac{16}{3}B\left(\frac{2}{3}+1, \frac{-1}{3}+1\right) = \frac{16}{3}B\left(\frac{5}{3}, \frac{2}{3}\right)$$

For Gamma Function; we use the formula

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \frac{16}{3}\frac{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{5}{3}+\frac{2}{3}\right)} = \frac{16}{3}\frac{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{7}{3}\right)} = \frac{16}{3}\times\frac{\frac{2}{3}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\frac{5}{3}\cdot\frac{2}{3}\Gamma\left(\frac{2}{3}\right)} = \frac{16}{3}\times\frac{3}{5}\Gamma\left(\frac{2}{3}\right) = \frac{16}{5}\Gamma\left(\frac{2}{3}\right)$$

Express $\int_0^1 x^m(1-x^n)^p dx$ **in terms of Beta function and hence Evaluate** $\int_0^1 x^7(1-x^5)^3 dx$.

Solution:

Put $x^n = t \Rightarrow x = (t)^{1/n} \Rightarrow dx = \frac{1}{n}(t)^{\frac{(1-n)}{n}} dt$ $\quad \therefore \int_0^1 x^m(1-x^n)^p dx = \frac{1}{n}\int_0^1 (t)^{\frac{m}{n}}(1-t)^p (t)^{\frac{(1-n)}{n}} dt$

$$= \frac{1}{n}\int_0^1 (t)^{\frac{m+1}{n}-1}(1-t)^p dt \Rightarrow \int_0^1 x^m(1-x^n)^p dx = \frac{1}{n}B\left(\frac{m+1}{n}, p+1\right) \quad ...(1)$$

For evaluation of Second integral, we take m = 7, n = 5, p = 3, in (1)

$$\boxed{\therefore \int_0^1 x^7(1-x^5)^3 dx = \frac{1}{5}B\left(\frac{7+1}{5}, 3+1\right) = \frac{1}{5}B\left(\frac{8}{5}, 4\right)}$$

Show that $\int_0^{\pi/2} \dfrac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x}dx = \pi$.

Solution:

The integral is $\int_0^{\pi/2} \sin^{-1/2} x\, dx \times \int_0^{\pi/2} \sin^{1/2} x\, dx = \int_0^{\pi/2} \sin^{-1/2} x \cos^0 x dx \times \int_0^{\pi/2} \sin^{1/2} x \cos^0 x dx$

$$= \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)} \times \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{4}\right)} \quad \left[\text{using} \int_0^{\pi/2} \sin^p \varphi \cos^q \varphi d\varphi = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}\right]$$

$$= \frac{\Gamma\left(\frac{1}{4}\right)\times\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{4.\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} \quad [\because \Gamma(n+1) = (n)\Gamma(n)] \quad = \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \left(\sqrt{\pi}\right)^2 = \pi \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

8. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^n}}$.

Solution:

Put $x^n = \sin^2\varphi \Rightarrow x = \sin^{\frac{2}{n}}\varphi$; $x \to 0 \Rightarrow \varphi \to 0,\ x \to 1 \Rightarrow \varphi \to \frac{\pi}{2}$

$$\therefore dx = \frac{2}{n}\sin^{\left(\frac{2}{n}-1\right)}\varphi\cos\varphi d\varphi \qquad \therefore \int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{2}{n}\int_0^{\frac{\pi}{2}} \frac{\sin^{\left(\frac{2}{n}-1\right)}\varphi\cos\varphi d\varphi}{\cos\varphi}$$

$$= \frac{2}{n}\int_0^{\frac{\pi}{2}} \sin^{\left(\frac{2}{n}-1\right)}\varphi\cos^0\varphi d\varphi = \frac{2}{n}.\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{n}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{n+2}{2n}\right)}$$

9. Evaluate $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$.

Solution:

$$\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

Putting $x = \frac{1}{y}; dx = \left(\frac{-1}{y^2}\right)dy$ in second integral, we get [Also $x \to 0 \Rightarrow y \to \infty$; $x \to 1 \Rightarrow y \to 1$]

$$\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_\infty^1 \frac{\left(\frac{1}{y}\right)^{n-1}}{\left(1+\frac{1}{y}\right)^{m+n}}\left(-\frac{1}{y^2}\right)dy$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy \qquad \left[\because \int_a^b f(x)dx = -\int_b^a f(x)dx\right]$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx \qquad \left[\because \int_a^b f(x)dx = \int_a^b f(t)dt\right]$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

10. Show that $\int_0^1 \frac{x^2dx}{(1-x^4)^{\frac{1}{2}}} \times \int_0^1 \frac{dx}{(1+x^4)^{\frac{1}{2}}} = \frac{\pi}{4\sqrt{2}}$.

Solution:

Put $x^2 = \sin\varphi; 2xdx = \cos\varphi d\varphi$ in first integral; then $\int_0^1 \frac{x^2dx}{(1-x^4)^{\frac{1}{2}}} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\varphi}.\cos\varphi}{\cos\varphi} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sqrt{\sin\varphi}d\varphi$ ...(1)

For second integral, Put $x^2 = \tan\phi$; $x \to 0 \Rightarrow \phi \to 0,\ x \to 1, \Rightarrow \phi \to \frac{\pi}{4}$

$$\therefore 2xdx = \sec^2\phi d\phi \qquad \therefore \int_0^1 \frac{dx}{(1+x^4)^{\frac{1}{2}}} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\sec^2\phi d\phi}{\sqrt{\tan\phi}\sec\phi} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d\phi}{\sqrt{\sin\phi\cos\phi}} = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{d\phi}{\sqrt{\sin 2\phi}}$$

Let $2\phi = t \Rightarrow d\phi = \frac{t}{2}$; Also $t \to 0$ when $\phi \to 0$ and $t \to \frac{\pi}{2}$ when $\phi \to \frac{\pi}{4}$ $= \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}} t\,dt$ ...(2)

From (1) and (2)

$$\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{4\sqrt{2}}\left[\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\varphi\cos^0\varphi dq \times \int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}} t\cos^0 t dt\right] = \frac{1}{4\sqrt{2}}\pi \text{ [see example 7]} = \frac{\pi}{4\sqrt{2}}$$

Prove that $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\cdot\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)}$ **and hence establish that** $\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi$.

ion:

We know that $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$\therefore \quad B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}+n+\frac{1}{2}\right)} = \frac{\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2}{\Gamma(2n+1)} \qquad ...(1)$$

Now $B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \int_0^1 x^{n-\frac{1}{2}}(1-x)^{n-\frac{1}{2}}dx$. Put $x = \sin^2 t \Rightarrow dx = 2\sin t\cos t\,dt$

Also $x \to 0 \Rightarrow t \to 0; x \to 1 \Rightarrow t \to \frac{\pi}{2}$

$$B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2n-1} t\,(1-\sin^2 t)^{n-1/2}\cos t\sin t\,dt = 2\int_0^{\pi/2} \sin^{2n-1} t\,\cos^{2n-1} t\cos t\sin t\,dt$$

$$= 2\int_0^{\pi/2} \sin^{2n} t\,\cos^{2n} t\,dt = \frac{2}{2^{2n}}\int_0^{\pi/2} (2\sin t\cos t)^{2n}\,dt = \frac{1}{2^{2n-1}}\int_0^{\pi/2} (\sin 2t)^{2n}\,dt$$

Let $2t = z \Rightarrow dt = \frac{dz}{2}$ when $t \to 0; z \to 0; t \to \frac{\pi}{2} \Rightarrow z \to \pi$

$$\therefore \quad B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{1}{2^{2n}}\int_0^\pi \sin^{2n} z\,dz$$

$$= \frac{1}{2^{2n}}\cdot 2\int_0^{\pi/2} \sin^{2n} z\,dz \qquad \left[\because \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(2a-x) = f(x)\right]$$

$$= \frac{1}{2^{2n-1}}\int_0^{\pi/2} \sin^{2n} z\cos^0 z\,dz$$

$$= \frac{1}{2^{2n-1}} \times \frac{\Gamma\left(\frac{2n+1}{2}\right)\times\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2n+2}{2}\right)} \qquad \left[\text{using} \int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}\right]$$

$$= \frac{1}{2^{2n-1}} \times \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right] \qquad = \frac{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{2^{2n}\,\Gamma(n+1)}$$

Putting this value in (1), we get

$$\frac{\sqrt{\pi}}{2^{2n}}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}=\frac{\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2}{\Gamma(2n+1)} \quad \text{or} \quad \Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)}.$$

Hence proved.

Deduction:

Taking $n=\frac{1}{4}$; we get

$$\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+1\right)}{2^{1/2}\,\Gamma\left(\frac{1}{4}+1\right)} \quad \text{or} \quad \Gamma\left(\frac{3}{4}\right)=\frac{\sqrt{\pi}}{\sqrt{2}}\frac{\Gamma(3/2)}{\Gamma(5/4)}=\frac{\sqrt{\pi}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\sqrt{2}\cdot\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}$$

$$\therefore \Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{3}{4}\right)=\sqrt{2}\,\pi.$$

12. **Prove that** $\int_0^\infty x^3 e^{-x^3}\,dx=\frac{1}{9}\Gamma\left(\frac{1}{3}\right)$.

Solution :

Put $x^3=t$ so that $3x^2\,dx=dt$ $\quad\therefore \int_0^\infty x^3 e^{-x^3}\,dx=\frac{1}{3}\int_0^\infty x\,(3x^2)\,e^{-x^3}\,dx$

$$=\frac{1}{3}\int_0^\infty t^{1/3}e^{-t}\,dt=\frac{1}{3}\Gamma\left(\frac{1}{3}+1\right) \qquad \text{[By definition of Gamma function]}$$

$$=\frac{1}{3}\Gamma\left(\frac{4}{3}\right)=\frac{1}{3}\cdot\frac{1}{3}\Gamma\left(\frac{1}{3}\right) \qquad [\because \Gamma(n+1)=n\,\Gamma(n)]$$

$$=\frac{1}{9}\Gamma\left(\frac{1}{3}\right).$$

13. **Prove that** $\int_0^\infty e^{-ax}x^{n-1}\,dx=\frac{\Gamma(n)}{a^n}$; **where a, n are positive.**

Solution:

Put $ax=z\Rightarrow dx=\frac{dz}{a}$ $\quad\therefore \int_0^\infty e^{-ax}x^{n-1}\,dx=\int_0^\infty e^{-z}\left(\frac{z}{a}\right)^{n-1}\cdot\frac{dz}{a}=\frac{1}{a^n}\int_0^\infty e^{-z}z^{n-1}\,dz=\frac{\Gamma(n)}{a^n}$.

Hence proved.

14. **Prove that** $\int_0^\infty e^{-x^2}x^{2n-1}\,dx=\frac{1}{2}\Gamma(n)$.

Solution :

Put $x^2=t\Rightarrow 2x\,dx=dt$

$$\therefore \int_0^\infty e^{-x^2}x^{2n-1}\,dx=\frac{1}{2}\int_0^\infty e^{-x^2}x^{2n-2}(2x)\,dx=\frac{1}{2}\int_0^\infty e^{-t}(t)^{\frac{2n-2}{2}}\,dt=\frac{1}{2}\int_0^\infty e^{-t}t^{n-1}\,dt$$

$$=\frac{1}{2}\Gamma(n).$$ Hence proved.

15. **Prove that** $\int_0^\infty x^{2n-1}e^{-ax^2}\,dx=\frac{\Gamma(n)}{2a^n}$.

Solution:

Put $ax^2=t\Rightarrow 2ax\,dx=dt$ $\quad\therefore \int_0^\infty x^{2n-1}e^{-ax^2}\,dx=\frac{1}{2a}\int_0^\infty x^{2n-2}e^{-ax^2}(2ax\,dx)$

$$=\frac{1}{2a}\int_0^\infty\left(\frac{t}{a}\right)^{\frac{2n-2}{2}}e^{-t}\,dt=\frac{1}{2a}\int_0^\infty\frac{(t)^{n-1}}{a^{n-1}}e^{-t}\,dt=\frac{1}{2a^n}\int_0^\infty e^{-t}t^{n-1}\,dt=\frac{\Gamma(n)}{2a^n}.$$ Hence proved.

Prove that : (i) $\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy=\Gamma(n)$ **(ii)** $\int_0^1 x^4\left(\log\frac{1}{x}\right)^2dx=\frac{6}{125}$.

ion :

(i) Put $e^{-x}=y\Rightarrow -e^{-x}\,dx=dy$

Also $e^{-x}=y\Rightarrow -x=\log y\Rightarrow x=-\log y=\log(y^{-1})=\log\frac{1}{y}$

Also $y\to 0\Rightarrow x\to\infty$; $y\to 1$; $x\to 0$

$$\therefore \int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy=-\int_\infty^0 x^{n-1}e^{-x}\,dx=\int_0^\infty x^{n-1}e^{-x}\,dx=\Gamma(n).$$ Hence proved.

(ii) Put $\log\frac{1}{x}=t\Rightarrow\log x^{-1}=t\Rightarrow\log x=-t\Rightarrow x=e^{-t}$ $\quad\therefore dx=-e^{-t}\,dt$

when $x\to 0$; $t\to\infty$; $x\to 1$; $t\to 0$

$$\therefore \int_0^1 x^4\left(\log\frac{1}{x}\right)^2dx=\int_\infty^0(e^{-t})^4\,t^2(-e^{-t})\,dt=\int_0^\infty(e^{-t})^5\,t^2\,dt=\int_0^\infty e^{-5t}t^2\,dt$$

Put $5t=u\Rightarrow 5\,dt=du\Rightarrow dt=\frac{du}{5}$

$$\therefore \int_0^\infty e^{-5t}t^2\,dt=\int_0^\infty e^{-u}\left(\frac{u}{5}\right)^2\frac{du}{5}=\frac{1}{5^3}\int_0^\infty e^{-u}u^{3-1}\,du=\frac{1}{125}\Gamma(3)=\frac{3\times2\times1}{125}=\frac{6}{125}.$$

Evaluvate $\int_0^\infty\frac{x^c}{c^x}\,dx$.

ion :

ut $c^x=e^t$; taking log both sides $x\log_e c=t\log_e e\Rightarrow x=\frac{t}{\log c}$ $\quad[\because \log_e e=1]\Rightarrow dx=\frac{dt}{\log c}$

$$\therefore \int_0^\infty\frac{x^c}{c^x}\,dx=\int_0^\infty\left(\frac{t}{\log c}\right)^c\cdot e^{-t}\frac{dt}{\log c}=\frac{1}{(\log c)^{c+1}}\int_0^\infty e^{-t}t^c\,dt=\frac{\Gamma(c+1)}{(\log c)^{c+1}}.$$

amma function for Negative Non-Integer : If x is a negative non-integer, then we define $\Gamma(x)=\frac{\Gamma(x+1)}{x}$.

Evaluvate the following : (i) $\Gamma\left(-\frac{1}{2}\right)$ **(ii)** $\Gamma\left(-\frac{3}{2}\right)$ **(iii)** $\Gamma\left(-\frac{15}{2}\right)$.

ion :

(i) $\Gamma\left(-\frac{1}{2}\right)=-2\,\Gamma\left(-\frac{1}{2}+1\right)=-2\,\Gamma\left(\frac{1}{2}\right)=-2\sqrt{\pi}$

(ii) $\Gamma\left(-\frac{3}{2}\right)=-\frac{2}{3}\Gamma\left(-\frac{3}{2}+1\right)=-\frac{2}{3}\Gamma\left(-\frac{1}{2}\right)=\left(-\frac{2}{3}\right)(-2)\,\Gamma\left(-\frac{1}{2}+1\right)=\frac{4}{3}\Gamma\left(\frac{1}{2}\right)=\frac{4}{3}\sqrt{\pi}$.

(iii) $\Gamma\left(-\frac{15}{2}\right)=-\frac{2}{15}\Gamma\left(-\frac{15}{2}+1\right)=\frac{-2}{15}\Gamma\left(-\frac{13}{2}\right)=\left(\frac{-2}{15}\right)\left(\frac{-2}{13}\right)\Gamma\left(-\frac{13}{2}+1\right)=\frac{2^2}{15\times13}\Gamma\left(-\frac{11}{2}\right)$

$$=\frac{2^2}{15\times13}\times\left(-\frac{2}{11}\right)\Gamma\left(-\frac{9}{2}\right)=-\frac{2^3}{15\times13\times11}\Gamma\left(-\frac{9}{2}\right)$$

Continuing like this, we get $\Gamma\left(-\frac{15}{2}\right) = \frac{2^8 \times \Gamma\left(\frac{1}{2}\right)}{15\times13\times11\times9\times7\times5\times3}$

or $\Gamma\left(-\frac{15}{2}\right) = \frac{2^8\sqrt{\pi}}{15\times13\times11\times9\times7\times5\times3}$ $\left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$

Exercise

1. Prove that $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx = \frac{\sqrt{\pi}}{3}$.
2. Prove that $\int_0^\infty e^{-x^2} x^{2n-1}\, dx = \frac{1}{2}\Gamma(n)$
3. Prove that $\int_0^1 x^m (\log x)^n\, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$
4. Prove that $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\cot\theta} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right) \times \Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sqrt{2}}$
5. Prove that $\int_0^{\pi/2} \sin^3 x \cos^{5/2} x\, dx = \frac{8}{77}$
6. Prove that $\int_0^\infty x^4 e^{-x^2}\, dx = \frac{3\sqrt{\pi}}{8}$
7. Prove that $\int_0^\infty e^{-h^2x^2}\, dx = \frac{\sqrt{\pi}}{2h}$
8. Prove that $\int_0^1 x^3 (1-x)^{4/3}\, dx = \frac{243}{7280}$
9. Prove that $\int_0^1 x^5 (1-x^3)^3\, dx = \frac{1}{60}$
10. Prove that $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$
11. Prove that $\int_0^1 \frac{x\, dx}{\sqrt{1-x^5}} = \frac{1}{5} B\left(\frac{2}{5}, \frac{1}{2}\right)$
12. Prove that $\int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{4\sqrt{2}} B\left(\frac{1}{4}, \frac{1}{2}\right)$ [Hint : Put $x^2 = \tan^2\theta$]
13. Prove that : (i) $\int_0^{\pi/2} \sin^2\theta \cos^6\theta\, d\theta = \frac{5\pi}{256}$ (ii) $\int_0^{\pi/2} \sin^6\theta\, d\theta = \frac{5\pi}{32}$.
14. Prove that $\int_0^1 x^{n-1} \left[\log_e \frac{1}{x}\right]^{m-1} dx = \frac{1}{n^m}\Gamma(m)$
15. Prove That $\int_0^1 x^m (\log x)^n\, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$.

EXAMINATION PAPER
[Dec - 2006 (I.P.)]

Time : 3 Hours ***Maximum Marks : 75***

Note : Attempt five questions in all Q.1. is Compulsory. In addition, attempt one question from each unit.

Q. 1. (a) For what values of λ. $\begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 4-\lambda & \sqrt{3} \\ 0 & \sqrt{3} & 6-\lambda \end{bmatrix} = 0$

(b) Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

(c) Evaluate $\underset{x\to 0}{Lt}\left[\frac{1}{\sin x} - \frac{1}{x}\right]$

(d) Discuss the continuity of the function $f(x) = \begin{cases} \frac{x-|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$

(e) Find nth derivative of $\cos(ax+b)$

(f) Integrate $\int x^2 e^{x^3} \cos(e^{x^3})\, dx$

(g) Find the value of $(3\hat{i} - 5\hat{j} + 2\hat{k}) \times (9\hat{i} - \hat{j} - 4\hat{k})$

(h) State Liebnitz theorem

(i) Define Gamma and Beta function

(j) Write the reduction formula for $\int \sin^n dx$. (2·5 × 10 = 25)

UNIT - I

Q. 2. (a) Evaluate the determinant $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$ in a closed form. (5·5)

(b) Verify Caley Hamilton theorem for the matrix A and find its inverse $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ (7)

Q. 3. (a) Find the characteristic roots and the characteristic vectors of the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$. (6)

(b) Solve the following equations $x + z = 7$, $2x + y = 7$, $3x + 2y - z = 17$ using the Cramer's rule. (6·5)

UNIT - II

Q.4. (a) Find the indicated limits

(i) $\lim_{x \to \infty} x\left(\tan^{-1}\frac{x+1}{x+4} = \frac{\pi}{4}\right)$ **(2·5)**

(ii) $\lim_{x \to 0} \dfrac{\sin^{-1}x - 2x}{\sin^{-1}x - \left(2\sin\left(\frac{1}{2}\sin^{-1}x\right)\right)\left(3 - 4\sin^2\left(\frac{1}{2}\sin^{-1}x\right)\right)}$ **(3)**

(b) Determine a, b, c if the function $f(x) = \begin{cases} \dfrac{\sin(z+1)x + \sin x}{x}, & \text{if } x < 0 \\ c, & \text{if } x = 0 \\ \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{\frac{3}{2}}}, & \text{if } x > 0 \end{cases}$ is continuous at $x = 0$. **(7)**

Q.5. (a) Discuss the continuity of $f(x) = \begin{cases} \dfrac{[x]-1}{x-1}, & x \neq 1 \\ -1, & x = 1 \end{cases}$. **(6·5)**

(b) Find the following limits (i) $\lim_{x \to 1}\left[\dfrac{\log x}{\cos\left(\frac{\pi x}{2}\right)}\right]$ (ii) $\lim_{x \to \infty} \dfrac{(x^3 - 2x^2 + 5)}{(3x^3 + 7)}$ **(3 + 3)**

UNIT - III

Q.6. (a) Verify Rolle's theorem for $f(x) = \dfrac{\sin x}{e^x}$ in $(0, \pi)$. **(3·5)**

(b) Find the nth derivative of $e^x (2x + 3)^3$ **(4)**

(c) Trace the curve, $x = a\cos^3 t$, $y = a\sin^3 t$ **(5)**

Q.7. (a) Expand $e^{\sin x}$ by using Maclaurin's theorem: upto the term containing x^4. **(6·5)**

(b) Find the maximum value of $\sin x\,(1 + \cos x)$ and the point of maxima. **(6)**

UNIT = IV

Q.8. (a) Find the reduction formula for $\int e^{ax}\sin^n x\,dx$ and hence evaluate $\int e^x \sin^3 x\,dx$. **(6·5)**

(b) Evaluate the integral $\int_0^a \dfrac{x^7}{\sqrt{a^2 - x^2}}\,dx$. **(6)**

Q.9. (a) (i) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ **(3·5)**

(ii) Express $\int_0^2 x^3 (8 - x^3)^{-1/3}\,dx$ in terms of Beta function. **(3·5)**

(b) Find the volume of the tetrahedron having vertices

$(-\hat{j} - \hat{k};\ 4\hat{i} + 5\hat{j} + s\hat{k} : 3\hat{i} + 9\hat{j} + 4\hat{k})$ and $4(-\hat{i} + \hat{j} + \hat{k})$.

Also find the value of s for which these four points are coplanar **(5·5)**

EXAMINATION PAPER
[Dec - 2007 (I.P.)]

Time : 3 Hours *Maximum Marks : 75*

Note : Q.1. is compulsory. In the remaining paper, attempt one question from each unit.

Q.1. (a) What is the difference between minor and co-factor ?

(b) State Cramers Rule.

(c) What is mean value theorem ?

(d) Evaluate the limit $\lim_{x \to 0} \dfrac{(\log x)}{\cot x}$.

(e) Write the various types of discontinuities.

(f) State Liebnitz theorem.

(g) Write the reduction formula for $\int x^n e^{ax}\,dx$.

(h) State the condition under which two vectors are parallel.

(i) Define Gamma and Beta functions

(j) Write the partial fractions of $\dfrac{1}{(x+1)(2x+1)}$. **(2·5 × 10 = 25)**

UNIT - I

Q.2. (a) Show that $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x - y)(y - z)(z - x)(x + y + z)$ **(5)**

(b) If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, find A^{-1}, and show that $A^3 = I$ the unit matrix. **(6 + 1·5)**

Q.3. (a) Solve $x + 2y + z = 7$, $x + 3z = 11$, $2x - 3y = 1$, using Cramers rule. **(5)**

(b) Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. **(5 + 2·5)**

UNIT - II

Q.4. (a) (i) Find the limit of the following $\lim_{x \to \sqrt{2}} \dfrac{\sqrt{3+2x} - (\sqrt{2}+1)}{x^2 - 2}$ **(4)**

(ii) $\lim_{x \to 0} \dfrac{\sqrt{4+x} - \sqrt{4-x}}{\sin^{-1}x}$ **(4)**

(b) Find the values of a and b, so that $f(x) = \begin{cases} x^2 + ax + b & \text{if } 0 \le x < 2 \\ 3x + 2 & \text{if } 2 \le x \le 4 \\ 2ax + 5b & \text{if } 4 < x \le 8 \end{cases}$ is continuous in $[0, 8]$. **(4·5)**

Q.5. (a) Examine the continuity of the following

(i) $f(x) = \begin{cases} 3x-2, & x \le 0 \\ x+1, & x > 0 \end{cases}$ at $x = 0$. **(3)**

(ii) Show that $|x| + |(x-1)|$ is a continuous function. **(3)**

(b) Evaluate the limit of (i) $\lim\limits_{x \to 0} \dfrac{\tan^{-1} 2x}{\sin 3x}$ (ii) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x^2} - \sqrt{1+x}}{\sqrt{1+x^3} - \sqrt{1-x}}$. **(3 + 3·5)**

UNIT - III

Q.6. (a) (i) Find $\dfrac{d^2 y}{dx^2}$, if $x = \dfrac{2at^2}{1+t}$, $y = \dfrac{3at}{1+t}$. **(3)**

(ii) Find $\dfrac{dy}{dx}$, if $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$. **(3)**

(b) Show that the rectangle of maximum area that can be inscribed in a circle of radius r is a square of side $\sqrt{2}\, r$. **(6·5)**

Q.7. (a) Sketch the curve $y = (x-1)(x-2)(x-3)$. **(6·5)**

(b) Find a point on $y = x^2 - 6x + 1$ where the tangent is parallel to the chord joining $(1, -4)$ and $(3, -8)$. **(6)**

UNIT - IV

Q.8. (a) Find the integral of the following : (i) $\int x \cos^{-1} x \, dx$, (ii) $\int \dfrac{x^2}{(1+x^3)(2+x^3)} \, dx$. **(3 + 3)**

(b) (i) Evaluate teh following as beta function- $\int_0^1 \dfrac{x^4}{\sqrt{1-x^2}} \, dx$. **(3·5)**

(ii) Prove that $[\vec{b} \times \vec{c}, \vec{c} \times \vec{a}, \vec{a} \times \vec{b}] = [\vec{a}, \vec{b}, \vec{c}]^2$. **(3)**

Q.9. (a) (i) Integrate as limit of sum, of the following $\int_0^4 e^x \, dx$. **(3·5)**

(ii) Integrate the following- $I = \int \dfrac{x^2 \, dx}{x^2 + 6x + 12}$. **(3)**

(b) (i) Using vector analysis show that diagonals of a rhombus are perpendicular. **(3)**

(ii) Using reduction formulae find, $\int \sin^7 x \, dx$. **(3)**

EXAMINATION PAPER
[Dec - 2008 (I.P.)]

Time : 3 Hours ***Maximum Marks : 75***

Note : Q.1. is compulsory. In the remaining paper, attempt one question from each unit.

Q.1. (a) Define minor and cofactors in a square matrix.

(b) Evaluate the cofactor of 'a' in the determinant $\begin{vmatrix} 3 & -4 & -3 \\ 2 & 7 & a \\ 5 & -9 & 2 \end{vmatrix}$.

(c) Evaluate 3A–4B where $A = \begin{bmatrix} 3 & -4 & 6 \\ 5 & 1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 0 & 3 \end{bmatrix}$.

(d) Find the limit of the following $\lim\limits_{x \to 0} \dfrac{\sin 2x + \sin 3x}{2x + \sin 3x}$

(e) State Leibnitz's theorem.

(f) Define Beta and Gamma functions.

(g) Evaluate $\int_0^{\pi/2} \cos^6 x \, dx$

(h) What is the difference between scalar and vector product of two vectors?

(i) Differentiate the following w.r.t. x. $y = x^x$.

(j) Write partial fractions of $\dfrac{x-1}{(x+1)(x-2)}$. **(2·5 × 10 = 25)**

UNIT - I

Q.2. (a) If $\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = 0$, in which a, b, c are different, show that $abc = 1$. **(6)**

(b) Determine the rank of the following matrix, $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$. **(6.5)**

Q.3. (a) Verify the Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and find its inverse. Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10$ as a linear polynomial in A. **(5)**

(b) Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$. **(5 + 2.5 = 7.5)**

UNIT - II

Q.4. (a) (i) Evaluate $\lim_{x \to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3}$ (ii) If $\lim_{x \to -a} \frac{x^9 + a^9}{x + a} = 9$, find value of a. (3+3.5)

(b) Discuss the continuity of the function $f(x) = \begin{cases} 2-x, & x < 2 \\ 2+x, & x \geq 2 \end{cases}$ at $x = 2$ (6)

Q.5. (a) Show that $f(x) = \begin{cases} 5x-2, & \text{when } 0 < x \leq 1 \\ 4x^3 - 3x, & \text{when } 1 < x < 2 \end{cases}$ is discontinuous at $x = 1$. (5)

(b) Find the value of a so that the function $f(x) = \begin{cases} \frac{\sin^2 ax}{x^2}, & x \neq 0 \\ 1 & x = 0 \end{cases}$, may be continuous in R. (7.5)

UNIT - III

Q.6 (a) (i) If $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$ show that $\frac{dy}{dx} = y$. (3)

(ii) $y = e^x \sin x + x^x \cos x$, w.r.t. x. (3)

(b) Find the altitude and the semi vertical angle of a cone of least volume which can be circumscribed to a sphere of radius a. (6.5)

Q.7. (a) If $y = (\sin^{-1} x)^2$, show that $(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$. (5)

(b) Find the asymptotes of the curve $y^3 - 2xy^2 - x^2 y + 2x^3 + 3y^2 - 7xy + 2x^2 + 2y + 2x + 1 = 0$. (7.5)

UNIT - IV

Q.8. (a) Evaluate $\int_0^{\pi/2} \log \sin x \, dx$. (6)

(b) (i) Evaluate $\int_0^{\pi/4} \cos^4 3\theta \sin^3 6\theta \, d\theta$. (3.5)

(ii) Show that $\beta(p, q) = \int_0^{\infty} \frac{y^{q-1}}{(1+y)^{p+q}} dy = \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}} dx$. (3)

Q.9. (a) (i) In a triangle ABC, DEF are the midpoints of the sides BC, CA, AB, prove that

$$\Delta DEF = \Delta FCE = \frac{1}{2} \Delta ABC,$$ using vector analysis. (3.5)

(ii) Calculate area of triangle whose vertices are A (1, 0, −1), B (2, 1, 5) and C (0, 1, 2). (3)

(b) (i) Show that the points $-6\hat{i} + 3\hat{j} + 2\hat{k}, 3\hat{i} - 2\hat{j} + 4\hat{k}, 5\hat{i} + 7\hat{j} + 3\hat{k}$ and $-13\hat{i} + 17\hat{j} - \hat{k}$ are coplanar. (3)

(ii) Using vector analysis find volume of the tetrahedron formed by the points, (1, 1, 1), (2, 1, 3), (3, 2, 2) and (3, 3, 4) (3)

EXAMINATION PAPER
[Dec - 2009 (I.P.)]

Time : 3 Hours **Maximum Marks : 75**

Note : Q.1. is compulsory. Attempt one question from each unit.

Q.1. (a) Define Hermitian and Skew Hermitian matrix.

(b) Find minor and co factors of each element in the determinant $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(c) Show that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

(d) Find the volume of the parallelopiped whose edges are represented by

$$A = 2i - 3j + 4k, B = i + 2j - k, C = 3i - j + 2k.$$

(e) State Mean Value Theorem.

(f) If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then show that $A^2 - 5A - 2I = 0$, where I is the 2×2, identity matrix.

(g) Evaluate $\int_0^{\pi/2} \sin^8 x \cos^7 x \, dx$.

(h) If $x = t \log t$, $y = (\log t) / t$. Find dy/dx when $t = 1$.

(i) Evaluate $\int \frac{(x+2)^3}{x^6} dx$.

(j) State Cayley-Hamilton theorem. (2·5 × 10 = 25)

UNIT - I

Q.2. (a) Solve the following system of equations using Cramer's rule.

$$x + y + z = 1$$
$$ax + by + cz = k$$
$$a^2 x + b^2 y + c^2 z = k^2$$

(b) If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ verify Cayley Hamilon theorem for A and find its inverse. (6+6.5 = 12.5)

Q.3. (a) Define linear dependence and independence of vectors. Examine for linear dependence [1, 0, 2, 1], [3, 1, 2, 1], [4, 6, 2, −4], [−6, 0, −3, −4] and find the relation between them, if possible. (6.5+6 = 12.5)

(b) Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

UNIT-II

Q.4 (a) (i) Find $\lim_{x\to 0} \frac{e^{1/x}-1}{e^{1/x}+1}$, if it exist. (ii) Find $\lim_{x\to 0} \frac{e^{x}-1}{\log(1+x)}$, if it exist.

(b) A function is defined in (0, 3) in the following way:-

$f(x) = x^2$ when $0 < x < 1$, $f(x) = x$ when $1 \le x < 2$, $f(x) = (1/4)x^3$ when $2 \le x < 3$ **(6.5+6 = 12.5)**

Show that $f(x)$ is continuous at $x = 1$ and $x = 2$. **(6+6.5 = 12.5)**

Q.5 (a) Show that $f(x) = |x| + |x-1|$ is continuous at $x = 0$ and $x = 1$.

(b) Show that the function f defined as $f(x) = \begin{cases} \frac{e^{1/x}}{e^{1/x}+1} & when\ x \ne 0 \\ 0 & when\ x = 0 \end{cases}$ is not continuous at $x = 0$.

UNIT-III

(6+6.5 = 12.5)

Q.6 (a) If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

(b) If $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$, prove that $\frac{dy}{dx} + \frac{x}{y} = 0$.

(6+6.5 = 12.5)

Q.7 (a) Trace the curve $r^2 = a^2 \cos 2\theta$.

(b) Find all the asymptotes of the curve $r(\pi+\theta) = ae^{\theta}$.

UNIT-IV

Q.8 (a) If $I_n = \int_0^{\pi/2} \tan^n x\,dx$, show that for $n > 1$, $I_n + I_{n-1} = \frac{1}{n-1}$. Deduce the value of I_5. **(6+6.5 = 12.5)**

(b) (*i*) Evaluate $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$, (*ii*) $\int \left[\log\log x + \frac{1}{(\log x)^2}\right] dx$

Q.9 (a) If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, then show that (*i*) $\nabla\left(\frac{1}{|\bar{r}|}\right) = -\frac{\bar{r}}{|\bar{r}|^3}$ (*ii*) $\nabla|\bar{r}|^n = n|\bar{r}|^{n-2}\,\bar{r}$ **(3 + 3.5)**

(b) (*i*) Find the tangent plane and normal line to the cone $z = \sqrt{x^2+y^2}$ at the point $A\ (1, 1, \sqrt{2})$.

(*ii*) Find the directional derivative of $\phi = xy + yz + zx$ at (1, 2, 0) in the direction of $\bar{i} + 2\bar{j} + 2\bar{k}$. **(3+3)**

ABOUT THE BOOK

This Book "B.C.A. Mathematics" has been written for the students of B.C.A. (I.P. University) appearing for first semester. The author has tried to present the text material in a comprehensive and Iucid manner. A distinct feature of the book is the large number of typical solved examples. The students of other universities with similar course will also find book very useful. The book been written strictly in accordance with the latest syllabi.

ABOUT THE AUTHOR

R.Chadha is a visiting faculty to many reputed institutes affiliated to G.G.S. I.P. University for B.C.A. / M.C.A. classes. He has more than 12 years of experience for teaching Advanced mathematics to B.A. / M.C.A. / B.tech / M.Sc. (Mathematics) students. He is The Director of Landmark Institute. He is also the author of B.C.A. Mathematics (Semester-II, III and IV). His area of interest is to provide quality coaching to students aiming for I.I.T.-J.E.E. and M.C.A Entrance Examinations.

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