Radio Lobe notes

July 2021

1 Introduction:

- What do we want to simulate?
  $\implies$ Jet termination and formation of radio lobe.

- What do we know about the physical parameters there?
  $\implies$ Jet velocity is non relativistic there. So the Jet velocity can be approximated as,

$$v_j = MC_{s_j} \tag{1}$$

  Where, M is the mach number in the Jet and C_{s_j} is the sound speed of the Jet.

- How to know the sound speed in the jet?
  $\implies$ We know from linear perturbation theory of fluid equations that,

$$C_s = \left(\frac{\partial P}{\partial \rho}\right)^{\frac{1}{2}}. \tag{2}$$

  If we take $P \propto \rho^\gamma$ inside the jet, then

$$C_s = \sqrt{\frac{\gamma P}{\rho}}. \tag{3}$$

  So, $C_{s_j} = \sqrt{\frac{\gamma_j P_j}{\rho_j}}$; γ_j is the ratio of specific heat inside the jet and usually taken to be equal to $\frac{5}{3}$, P_j is the pressure and ρ_j is the density of jet respectively.

- How to estimate P_j and ρ_j ?

  $\implies$ The quantities could be estimated in the following way,

  - ρ_j is estimated from the density contrast, η, defined as,

$$\eta = \frac{\rho_j}{\rho_A}. \tag{4}$$

    Where, ρ_A is the ambient density. For the case where ambient density is not a constant (for instance King's profile), then instead of taking ρ_A, one takes ρ_c, the core density.
  - P_j is taken as a constant whose value is considered to be

$$P_j = \frac{1}{\gamma_j} \tag{5}$$

    in Massaglia et al. (2016).
    One reason to take pressure to be uniform could be given as follows, due to the high collimation of the jet upto very large distance, the pressure needs to be constant else it would generate pressure force ($\propto \vec{\nabla} P$) which would further collimate the flow, **though this reason is not so good for me as the jet collimation happens due to pressure in the cocoon not because of the ambient pressure.**

- How to calculate sound speed in Jet ?
  $\implies$ The sound speed inside the jet, C_{s_j} could be written as, [following Eq. (4)]

$$\begin{aligned} C_{s_j} &= \sqrt{\frac{\gamma_j P_j}{\rho_j}} \\ &= \sqrt{\frac{\gamma_j P_j}{\eta \rho_c}} \end{aligned} \tag{6}$$

If ideal gas law is taken for the ambient medium then,

$$C_{s_j} = \sqrt{\frac{r_j P_j}{\eta \frac{P_c \mu}{RT_c}}} \tag{7}$$

Where, μ is the mean molecular mass and R is the universal gas constant. Now,

$$\begin{aligned} &C_{s_j} \\ &= \sqrt{\frac{\gamma_j R}{\mu} T_c^{\frac{1}{2}} \left(\frac{P_j}{P_c}\right)^{1/2} \eta^{-1/2}}. \end{aligned} \tag{8}$$

- How to calculate the velocity of the jet ?
$\implies$ The velocity in the jet could be written as, [following Eq. (1) and (8)]

$$\begin{aligned} v_j &= M C_{s_j} \\ &= M \sqrt{\frac{r_j R}{\mu} T_c^{\frac{1}{2}} \left(\frac{P_j}{P_c}\right)^{1/2} \eta^{-1/2}}. \end{aligned} \tag{9}$$

Again if μ_j is some distance (here jet radius), the time taken by the sound speed to cross this distance is,

$$\begin{aligned} \frac{\gamma_j}{C_{s_j}} &= \tau \\ &= r_j \sqrt{\frac{\mu}{r_j R} T_c^{-1/2} \left(\frac{P_c}{P_j}\right)^{1/2} \eta^{1/2}}. \end{aligned} \tag{10}$$

- How to calculate Jet kinetic power?
$\implies$ From fluid equation we know,

$$\frac{\partial e}{\partial t} + \nabla f(e) = 0. \tag{11}$$

Where, e is the energy density and $f(e)$ is the corresponding flux. Now volume integrating the equation gives,

$$\begin{aligned} &\int \frac{\partial e}{\partial t} dV + \int \nabla f(e) dV \\ &= 0 \\ &\implies \frac{\partial E}{\partial t} + \int f(e) d\vec{s} \\ &= 0 \end{aligned} \tag{12}$$

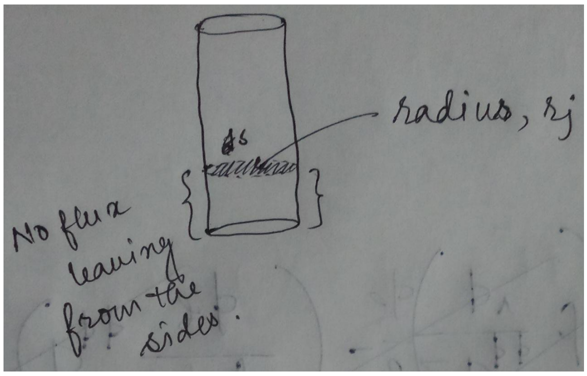

Figure 1: Geometry of the jet

Here, $\frac{\partial E}{\partial t}$ = Kinetic Power ($\mathcal{L}_{kin}$). Integrating the second term of the equation following $d\vec{s}$ to be a differential surface area of the cylinder and imposing no-flux leaving the side area of the cylinder (see Fig. (1) for example) (O'Neill & Jones, 2010),

$$\begin{aligned} \mathcal{L}_{kin} &= \int_0^{r_j} \left(\frac{1}{2}\rho v^2\right) v dV \\ &= \frac{1}{2}\int_0^{r_j} \rho v^3 dA \\ &= \frac{1}{2}\rho_j v_j^3 \pi r_j^2 \\ &= \frac{\pi}{2}\rho_j v_j^3 r_j^2, \end{aligned} \tag{13}$$

where we have taken the kinetic energy flux density as a product of kinetic energy density and velocity, $f(e) = \frac{1}{2}\rho v^2 \vec{v}$ and r_j is the radius of the jet. Now putting the values of ρ_j from Eq. (4) and v_j from Eq. (9), we get the kinetic power as,

$$\begin{aligned} &\mathcal{L}_{kin} \\ &= \pi r_j^2 \eta \rho_c M^3 \left(\frac{r_j R}{\mu}\right)^{\frac{3}{2}} T_c^{\frac{3}{2}} \left(\frac{P_j}{P_c}\right)^{\frac{3}{2}} \eta^{-3/2} \\ &= \pi \left(\frac{r_j R}{\mu}\right)^{\frac{3}{2}} r_j^2 \eta^{-1/2} \rho_c M^3 T_c^{\frac{3}{2}} \left(\frac{P_j}{P_c}\right)^{\frac{3}{2}}. \end{aligned} \tag{14}$$

which in scaled unit takes the form, (Massaglia et al., 2016)

$$\mathcal{L}_{kin} = 1.1 \times 10^{42} \left(\frac{r_j}{100\text{pc}}\right)^2 \left(\frac{\rho_c}{1\text{cm}^{-3}}\right) \left(\frac{T_c}{0.2\text{Kev}}\right)^{\frac{3}{2}} \left(\frac{M}{4}\right)^3 \left(\frac{\eta}{0.01}\right)^{-1/2} \left(\frac{P_j}{P_c}\right)^{\frac{3}{2}}. \tag{15}$$

As an example for $r_j = 100$ pc ; $\eta = 0.001$; $M = 20$ the kinetic power becomes,

$$\mathcal{L}_{kin} = 434.81 \times 10^{42} \left(\frac{\rho_c}{1\text{cm}^{-3}}\right) \left(\frac{T_c}{0.2\text{Kev}}\right)^{\frac{3}{2}} \left(\frac{P_j}{P_c}\right)^{\frac{3}{2}} = 4.3 \times 10^{44} \left(\frac{\rho_c}{1\text{cm}^{-3}}\right) \left(\frac{T_c}{0.2\text{Kev}}\right)^{\frac{3}{2}} \left(\frac{P_j}{P_c}\right)^{\frac{3}{2}}. \tag{16}$$

If we take the pressure balance ($P_j = P_c$, though do not know why? might be cause of mechanical equilibrium), then T_c can be computed following ideal gas equation of state,

$$PV = nRT \implies \frac{P}{\rho} = nRT \implies \frac{P}{\rho} \propto T. \tag{17}$$

Now making P to constant as P_c, T would be $T \propto \frac{1}{\rho}$. For King's profile ρ would have a functional dependence on 'r' and so does T.

2 Simulation of Radio lobe:

Two regions are needed to be simulated.

1. Ambient.
2. Jet nozzle.

2.1 Ambient:

This will set the initial condition in the ambient medium. For our case the cluster temperature stays constant (0.2 Kev) and because of that following Ideal gas law we know,$P = \rho n K_B T$ or P varies as ρ. For cluster ambience ρ is modelled by King's profile,

$$\rho = \frac{\rho_c}{\left(1 + \left(\frac{r}{r_c}\right)^2\right)^{\frac{3\beta}{2}}}. \tag{18}$$

with β and ρ_c and r_c being the free parameters varies from cluster to cluster.

So P also varies in the same way with r. So if some other force is not given, the cluster following this pressure will explode. For that we provide a gravitational acceleration in order to compensate the pressure. The gravitational force can be calculated following the hydro static equilibrium,

$$-\nabla P = -\rho g \tag{19}$$

For king's profile the form of gravitational potential is given in Krause, M. (2005). Following the temperature in the cluster medium we could derive the speed of sound there, which is,

$$C_s^2 = \sqrt{\frac{r K_B T}{\mu m_p}}. \tag{20}$$

2.2 Jet nozzle:

Now the Jet flow needed to be injected once the medium is specified. We take a value of the initial Mach number of the Jet, M.

Then we inject the jet material from lower z boundary following,

$$v_j = MC_s. \tag{21}$$

Where, C_s is the ambient sound speed. Also the density inside the jet is calculated following,

$$\rho_j = \eta\rho_c. \tag{22}$$

η is known as the density contrast, for us the value is 0.1. The pressure inside the jet is given as,

$$\begin{aligned} P_j &= P_c \\ &= \rho_c C_s^2. \end{aligned} \tag{23}$$

Where, C_s^2 being the ambient sound speed. This initialization of pressure ensures that the pressure inside the nozzle is same as that of the ambient and the jet formed by this pressure initialization is known as pressure-matched AGN jet.

The sound speed at the nozzle can be calculated as,

$$\begin{aligned} C_{s_j}^2 &= \frac{P_j}{\rho_j} \\ &= \frac{\cancel{\rho_c} C_s^2}{\eta \cancel{\rho_c}} \\ &= \frac{C_s^2}{\eta}. \end{aligned} \tag{24}$$

If we have jet materials flowing along $\hat{z}$ then the current density could be written as $\vec{J} = \rho\vec{v} = \rho v_z \hat{z}$.

From Biot-savart law we know,

$$B = \frac{\mu_0}{4\pi}\int_L \frac{I\vec{dl} \times \vec{r'}}{|r'|^3}. \tag{25}$$

For current density this becomes,

$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J} \times \vec{r'}}{|r'|^3} dV. \tag{26}$$

In cylindrical polar coordinate general radius vector take the form, $\vec{r'} = R'\hat{R} + z'\hat{z}$. Therefore,

$$\begin{aligned} \vec{J} \times \vec{r'} &= \rho v_z \hat{z} \times (R'\hat{R} + z'\hat{z}) \\ &= \rho v_z R'(\hat{z} \times \hat{R}) + \rho v_z z'(\hat{z} \times \hat{z}) \\ &= \rho v_z R'(\hat{z} \times \hat{R}) \\ &= \rho v_z R'\hat{\phi}. \end{aligned} \tag{27}$$

As $\hat{z} \times \hat{z} = 0$. So, B will only have ϕ component and,

$$B_\phi = \frac{\mu_0}{4\pi}\int \frac{\rho v_z R'}{|r'|^3} dV. \tag{28}$$

We can calculate B_ϕ following a very simple procedure which does not require to solve the above integration. Following Ampere's law, $\nabla \times \vec{B} = 4\pi\vec{J}$, as $\vec{J}$ only have component of $(\nabla \times B)$ along $\hat{z}$ and equal them, we get,

$$\frac{1}{R}\left(\frac{\partial}{\partial R}(RB_\phi) - \frac{\partial}{\partial \phi}(B_R)\right) = 4\pi j_z \tag{29}$$

$$\implies \frac{1}{R}\left(\frac{\partial}{\partial R}(RB_\phi)\right) = 4\pi j_z. \tag{30}$$

As B only has ϕ component so $\frac{\partial}{\partial\phi}(B_R) = 0$. So,

$$\frac{1}{R}\left(\frac{\partial}{\partial R}(RB_\phi)\right) = \begin{cases} 4\pi\rho v_z & R \leq R_j. \\ 0 & \text{else where} \end{cases} \tag{31}$$

where R_j is the jet radius. Integration leads,

$$RB_\phi = \begin{cases} 4\pi\rho v_z \frac{R^2}{2} & ; R \leq R_j. \\ D & ; R > R_j \end{cases} \tag{32}$$

where D is a constant in R. Finally we have,

$$B_\phi = \begin{cases} 4\pi\rho v_z \frac{R}{2} & ; R \leq R_j. \\ \frac{D}{R} & ; R > R_j. \end{cases} \tag{33}$$

So analogous to this, the magnetic field for the AGN jet could be written as (Lind et al., 1989),

$$B_\phi = \begin{cases} B_m \frac{R}{R_m} & ; R \le R_m \\ B_m \frac{R_m}{R} & ; R_m \le R \le R_j \\ 0 & ; R \ge R_j. \end{cases} \quad (34)$$

Now $\vec{B}$ is directed along $\hat{\phi}$, current density $\vec{J} = \rho v_z \hat{z}$ along $\hat{z}$, leads to a force $(\vec{J} \times \vec{B})$ along $-\hat{R}$. So magnetostatic condition leads to,

$$\begin{aligned} \nabla P &= (\vec{J} \times \vec{B}) \\ &= -(\nabla \times \vec{B} \times \vec{B}) \end{aligned} \quad (35)$$

$$\begin{aligned} \frac{\partial P}{\partial R} &= -\frac{1}{R}\left(\frac{\partial}{\partial R}(R B_\phi)\right) B_\phi \\ &= -\frac{1}{4\pi}\left(\frac{B_\phi^2}{R} + \cancel{R}\frac{\partial B_\phi}{\partial R}\frac{B_\phi}{\cancel{R}}\right) \\ &= -\left(\frac{B_\phi^2}{4\pi R} + \frac{\partial}{\partial R}\left(\frac{B_\phi^2}{8\pi}\right)\right) \end{aligned} \quad (36)$$

Integrating with respect to R we get,

$$\int_0^R \frac{\partial P}{\partial R} dR = -\int_0^R \left(\frac{B_\phi^2}{4\pi R'} + \frac{\partial}{\partial R'}\left(\frac{B_\phi^2}{8\pi}\right)\right) dR' \quad (37)$$

$$\begin{aligned} P(R) - P(0) = &-\int_0^R \frac{B_\phi^2}{4\pi R'} dR' \\ &- \left(\frac{B_\phi^2}{8\pi}\right)_R + \left(\frac{B_\phi^2}{8\pi}\right)_0 \end{aligned} \quad (38)$$

As the value of B_ϕ is zero on the axis ($R = 0$), we get (following Eq. 34),

$$P(R) = P(0) - \left(\frac{B_\phi^2}{8\pi}\right)_R - \int_0^R \frac{B_\phi^2}{4\pi R'} dR' \quad (39)$$

If there is no magnetic field in the ambient medium then the pressure is only due to gas, equating the external pressure with Eq. (39) at the jet radius we get,

$$P_e = P(0) - \int_0^{R_j} \frac{B_\phi^2}{4\pi R'} dR' \quad (40)$$

So the pressure can be written as,

$$\begin{aligned} P(R) = P_e &+ \int_0^{R_j} \frac{B_\phi^2}{4\pi R'} dR' \\ &- \left(\frac{B_\phi^2}{8\pi}\right)_R - \int_0^R \frac{B_\phi^2}{4\pi R'} dR' \end{aligned} \quad (41)$$

Following the magnetic field given in Eq. (34) one can write the expression for the pressure. To do the first lets evaluate the following integral,

$$\begin{aligned} \int_0^{R_j} \frac{B_\phi^2}{4\pi R'} dR' &= \int_0^{R_m} \frac{B_\phi^2}{4\pi R'} dR' + \int_{R_m}^{R_j} \frac{B_\phi^2}{4\pi R'} dR' \\ &= \int_0^{R_m} \frac{B_m^2 R'^2}{4\pi R' R_m^2} dR' \\ &\quad + \int_{R_m}^{R_j} \frac{B_m^2 R_m^2}{4\pi R'^3} dR' \\ &= \frac{B_m^2 R_m^2}{8\pi R_m^2} + \frac{B_m^2 R_m^2}{4\pi}\left(\frac{R^{-3+1}}{-3+1}\right)_{R_m}^{R_j} \\ &= \frac{B_m^2}{8\pi} - \frac{B_m^2}{8\pi}\left(\frac{R_m^2}{R_j^2} - 1\right) \\ &= \frac{B_m^2}{8\pi}\left(2 - \frac{R_m^2}{R_j^2}\right) \end{aligned} \quad (42)$$

Now, when $R < R_m$,

$$\begin{aligned} P(R) &= P_e + \frac{B_m^2}{8\pi}\left(2 - \frac{R_m^2}{R_j^2}\right) - \left(\frac{B_m}{R_m}\right)^2 \frac{R^2}{8\pi} \\ &\quad - \int_0^R \frac{B_m^2 R'^2}{4\pi R' R_m^2} dR' \\ &= P_e + \frac{B_m^2}{8\pi}\left(2 - \frac{R_m^2}{R_j^2}\right) \\ &\quad - \left(\frac{B_m}{R_m}\right)^2 \frac{R^2}{8\pi} - \left(\frac{B_m}{R_m}\right)^2 \frac{R^2}{8\pi} \\ &= P_e + \frac{B_m^2}{8\pi}\left(2 - \frac{R_m^2}{R_j^2}\right) - \frac{B_m^2}{4\pi}\frac{R^2}{R_m^2} \\ &= P_e - \frac{B_m^2}{8\pi}\frac{R_m^2}{R_j^2} + \frac{B_m^2}{4\pi}\left(1 - \frac{R^2}{R_m^2}\right) \\ &= \left\{\underbrace{1 - \overbrace{\frac{B_m^2}{8\pi P_e}}^{\text{let } \frac{1}{\beta}}\frac{R_m^2}{R_j^2}}_{\text{let } \alpha} + \underbrace{\frac{B_m^2}{4\pi P_e}}_{\frac{2}{\beta}}\left(1 - \frac{R^2}{R_m^2}\right)\right\} P_e \\ &= \left(\alpha + \frac{2}{\beta}\left(1 - \frac{R^2}{R_m^2}\right)\right) P_e, \end{aligned} \quad (43)$$

when $R_m \leq R \leq R_j$,

$$
\begin{aligned}
P(R) &= P_e + \frac{B_m^2}{8\pi}\left(2 - \frac{R_m^2}{R_j^2}\right) - \frac{B_m^2 R_m^2}{8\pi R^2} \\
&\quad - \underbrace{\int_0^{R_m} \frac{B_m^2 R'^2}{4\pi R' R_m^2} dR'}_{\frac{B_m^2}{8\pi}} - \int_{R_m}^{R} \frac{B_m^2 R_m^2}{4\pi R'^3} dR' \\
&= P_e + \frac{B_m^2}{8\pi}\left(2 - \frac{R_m^2}{R_j^2}\right) - \frac{B_m^2 R_m^2}{8\pi R^2} \\
&\quad - \frac{B_m^2}{8\pi} - \frac{B_m^2 R_m^2}{4\pi}\left(\frac{R^{-3+1}}{-3+1}\right)_{R_m}^{R} \\
&= P_e + \frac{B_m^2}{8\pi}\left(2 - \frac{R_m^2}{R_j^2}\right) - \frac{B_m^2 R_m^2}{8\pi R^2} \\
&\quad - \frac{B_m^2}{8\pi} + \frac{B_m^2 R_m^2}{8\pi}\left(\frac{1}{R^2} - \frac{1}{R_m^2}\right) \\
&= P_e + \cancel{\frac{B_m^2}{4\pi}} - \frac{B_m^2}{8\pi}\frac{R_m^2}{R_j^2} - \cancel{\frac{B_m^2 R_m^2}{8\pi R^2}} \\
&\quad - \cancel{\frac{B_m^2}{8\pi}} + \cancel{\frac{B_m^2 R_m^2}{8\pi R^2}} - \cancel{\frac{B_m^2 R_m^2}{8\pi R_m^2}} \\
&= \left(1 - \frac{B_m^2}{8\pi P_e}\frac{R_m^2}{R_j^2}\right) P_e \\
&= \alpha P_e
\end{aligned}
\tag{44}
$$

References

Krause, M. 2005, A&A, 431, 45

Lind K. R., Payne D. G., Meier D. L., Blandford R. D., 1989, ApJ, 344, 89

Massaglia S., Bodo G., Rossi P., Capetti S., Mignone A., 2016, A&A, 596, A12

O'Neill S. M., Jones T. W., 2010, The Astrophysical Journal, 710, 180