Slowly rotating black hole solutions in $f(R)$ gravity: A need for enhancement of the no-hair conjecture

Alan Sunny,[*] Semin Xavier,[†] and S. Shankaranarayanan[‡]

Department of Physics, Indian Institute of Technology Bombay, Mumbai 400076, India

(Dated: March 9, 2023)

Abstract

This work tests the no-hair conjecture in $f(R)$ gravity models. No-hair conjecture asserts that all black holes in General Relativity coupled to any matter must be Kerr–Newman type. However, the conjecture fails in some cases with non-linear matter sources. Here, we address this by explicitly constructing multiple slow-rotating black hole solutions, up to second order in rotational parameter, for a class of $f(R)$ models. We analytically show that two vacuum solutions satisfy the field equations up to the second order in the rotational parameter. The uniqueness of our result stems from the fact that these are obtained directly from metric formalism without conformal transformation. We discuss the kinematical properties of these black hole solutions and compare them with slow-rotating Kerr. Specifically, we show that the circular orbits for the black holes in $f(R)$ are smaller than that of Kerr. This implies that the inner-most stable circular orbit for black holes in $f(R)$ is smaller than Kerr's; hence, the shadow radius might also be smaller. Finally, we discuss the implications of our results for future observations.

[*] alansunny21.phy@gmail.com; Equal contribution to this work.

[†] seminxavier@iitb.ac.in; Equal contribution to this work.

[‡] shanki@iitb.ac.in

The direct observation of gravitational waves (GWs) from a binary black hole (BH) merger in 2015 opened a new frontier for investigating the event horizon of the residual BH, where the curvature scale can be as large as $10^{-2}\,\mathrm{km}^{-1}$ [1–3]. However, the validity of General Relativity (GR) has yet to be rigorously investigated in these strong-gravity regimes [4–8]. Observational cosmology's biggest surprise on the largest scale is the accelerated expansion of the current Universe [9]. This can be explained by the presence of a mysterious energy source known as dark energy or by modifications to GR on the cosmological scales [10–17].

LIGO-VIRGO-KAGRA observations confirmed GR's predictions and enriched our understanding of the Universe by providing the first direct evidence of massive stellar-mass BHs and BHs colliding to form a single, larger BH. However, the existence of BH singularities indicates that GR cannot be a universal theory of space-time and needs strong-gravity modifications. A universal feature of any strong-gravity corrections to GR is introducing higher derivative Ricci scalar, Ricci tensor, and Riemann tensor terms in action [12–16, 18]. Unlike 4-dimensional GR, whose field equations contain only up to second-order derivatives, the modified theories with higher derivative Ricci/Riemann tensor gravity models include higher derivatives [19–22]. Therefore, one expects significant differences between GR and modified theories [23, 24].

Currently, two approaches are employed to distinguish GR from modified gravity theories in Electromagnetic and GW observations: First, identify modified gravity theories with the same BH solutions as GR. Then, obtain the difference in the GW signals from these two theories [25–29]. Second, obtain new BH solutions for modified gravity theories [30–33]. Then, use template matching to match GW signals and identify the deviations, if any [34, 35]. In this work, we follow the second approach. There are two key reasons for this.

First, according to Birkhoff's theorem, the Schwarzschild solution is the unique spherically symmetric vacuum solution of the Einstein field equations. Recently, two current authors obtained an infinite number of exact static spherically symmetric vacuum solutions for a class of $f(R)$ gravity [36]. It was explicitly shown that the *Birkhoff theorem* is not valid for all modified gravity theories. In GR, the zero-spin ($J \to 0$) limit of the Kerr BH uniquely leads to the Schwarzschild solution. Thus, if a large number of spherically symmetric vacuum solutions exist in $f(R)$, the rotating solution also *may not* be unique.

Second, there is no uniqueness theorem in GR for space-time describing a rotating star. Hence, it is reasonable to assume that the stationary axisymmetric BH solutions are a large family. Remarkably, all stationary, asymptotically flat, vacuum solutions of the Einstein field equations that contain a regular horizon with no singularities outside the horizon are given by a two-parameter family [37, 38]. In the case of GR, we know one such family — the Kerr — is unique [39]. Due to *no hair theorem*, the only memory of the nature, structure, and composition of any object that collapses to form a stationary BH is embodied in the mass (M) and angular momentum (a), with any residual hair rapidly radiated away during the collapse process. This raises a few questions: Is the no-hair theorem a feature of gravity or GR[1]? In other words, do modified gravity theories support other rotating BH solutions besides Kerr? If yes, which class of modified gravity theories supports and how are they different from the Kerr solution in GR? Do we need to enhance the no-hair theorem for BHs in modified gravity theories?

This work addresses some of these questions by explicitly constructing multiple slow rotating black hole (SRBH) solutions for a class of $f(R)$ gravity models. Unfortunately, modified

[1] It has been shown that the *standard* no-hair theorem fails in some cases, like Einstein–Yang–Mills theory and nonlinear electrodynamics, among others [40, 41].

theories typically lead to more complex field equations, and obtaining exact rotating BH solutions analytically is challenging. Unlike GR, whose field equations contain only up to second-order derivatives, the equations of $f(R)$ gravity models include higher derivatives. Hence, the higher-order derivative corrections, with a significant number of degrees of freedom, cannot be treated as a perturbation to GR, especially in the strong-gravity regime [8]. However small they may appear in the Newtonian limit, higher-derivative terms make the new theory drastically different from GR [19–24].

Consequently, to keep the calculations tractable, we analytically obtain SRBH solutions up to second-order in the rotational parameter ($\chi = a/M = J/M^2$, J is the angular momentum, and M is the mass). We show that this feature leads to multiple SRBH solutions in $f(R)$. The results are then compared with Kerr and SR Kerr solutions (in GR) by evaluating different physical parameters. [We use $(-,+,+,+)$ signature for the 4-D space-time metric. Greek alphabets denote the 4-D space-time coordinates. We set $G = c = 1$. Prime denotes derivative w.r.t ρ.]

Two exact slowly-rotating solutions: The modified $f(R)$ vacuum gravity action in strong-gravity is:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g}\, f(R) = \frac{1}{16\pi} \int d^4x \sqrt{-g}\, \left[c_0 + c_1 R + c_2 R^2 + \cdots\right] \tag{1}$$

To avoid instabilities and ghosts, the coefficients $c_0, c_1, c_2, \cdots$ must satisfy $\partial f/\partial R > 0, \partial^2 f/\partial R^2 > 0$ [8]. Using metric formalism, the modified Einstein's equation is:

$$\mathcal{G}_{\mu\nu} \equiv \partial_R[f(R)] R_{\mu\nu} - \frac{1}{2} f(R)\, g_{\mu\nu} - \left[\nabla_\mu \nabla_\nu - g_{\mu\nu}\, \Box\right] \partial_R[f(R)] = 0 \tag{2}$$

where $\Box = \nabla^\mu \nabla_\mu$. Taking covariant differentiation of the above equation leads to the generalized Bianchi identity [28, 42]:

$$\partial_R^2[f(R)]\, R_{\mu\nu} \nabla^\mu R = 0\,. \tag{3}$$

The generalized Bianchi identity leads to four constraints on the Ricci tensor. For GR ($f(R) = R$), $\partial_R^2[f(R)] = 0$ and the above condition is trivially satisfied. Finding a trivial solution for $R =$ constant is trivial. In such a case, the above field equations reduce to the Einstein field equations with an effective cosmological constant and an effective gravitational constant. This includes the case where $R = 0$. However, we are interested in looking for non-trivial axisymmetric BH solutions in this work.

To keep the calculations tractable, we assume $f(R)$ in the following binomial form:

$$f(R) = (\alpha_0 + \alpha_1 R)^p \tag{4}$$

where $\alpha_0, \alpha_1 > 0$ and $p > 1$. Here are some points we want to discuss about the action: First, the coefficient α_0 is dimensionless, and α_1 has a dimension of $[L]^2$. Second, in the limit, $\kappa^2 \equiv \alpha_1/\alpha_0 \ll 1$, the above action reduces to GR with a cosmological constant. However, both these coefficients have to be non-zero. Third, for the most part, we obtain solutions for $p = 2$; however, the results can be extended to any p. Substituting the above form of $f(R)$ in Eq. (2), we have:

$$G_{\mu\nu} = -\kappa^2 R \left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R\right) + \kappa^2 \left[\nabla_\beta \nabla_\gamma - g_{\beta\gamma}\, \Box\right] R + \frac{1}{4\kappa^2} g_{\mu\nu} \tag{5}$$

We aim to construct a non-trivial slowly-rotating BH solution for the above $f(R)$ gravity. In this work, we extend Hartle's approach [43, 44] in which spin corrections to the static spherically symmetric solutions are introduced perturbatively. This approach has recently been applied to obtain SRBHs in Chern-Simons and Dilaton-Gauss-Bonnet gravity [31, 32].

As shown in Appendix (A), the first order (χ) SR solution for any p is highly degenerate (see Eq. A4). Hence, in the rest of this work, we consider the following line-element that corresponds to a slowly-rotating space-time up to quadratic in the spin parameter (χ):

$$ds^2 = -e^{\mu(\rho)}\left[U(\rho) + \chi^2\,V(\rho)\cos^2(\theta)\right]d\tau^2 + \frac{1}{U(\rho)}\left[1 - \chi^2\left(\frac{V(\rho)}{U(\rho)} + \frac{\sin^2(\theta)}{\rho^2}\right)\right]d\rho^2 \tag{6}$$
$$- 2\,\chi\,V(\rho)\rho^2\sin^2(\theta)\,d\tau\,d\phi + \left[\rho^2 + \chi^2\,\cos^2(\theta)\right]d\theta^2 + \rho^2\sin^2(\theta)\,\left[1 + \frac{\chi^2}{\rho^2} + \chi^2\,V(\rho)\sin^2(\theta)\right]d\phi^2$$

where $U(\rho), V(\rho)$ and $\mu(\rho)$ are unknown functions of the new radial coordinate ρ. Demanding $g_{\phi\phi}$ is positive definite implies that $V(\rho) > 0$. Similarly we demand that $\exp[\mu(\rho)]$ is non-zero for all ρ. τ, ρ are dimensionless time (t) and radial coordinates (r): $\tau = t/M, \rho = r/M$. A word of note regarding M: We will identify one of the unknown coefficients with the black hole mass; until then, M has no physical meaning. To obtain solutions, we group terms corresponding to different orders in χ, i. e.,

$$\mathcal{G} \simeq f^{(0)}(\rho) + f^{(I)}(\rho,\theta)\,\chi + f^{(II)}(\rho,\theta)\,\chi^2 + \mathcal{O}(\chi^3) + \cdots\,, \tag{7}$$

where $f^{(0)}(\rho)$ corresponds to spherically symmetric metric. More specifically, substituting the line element (6) in Eq. (5), eliminating terms greater than χ^2, leads to:

$$\mathcal{G}^{\tau}_{\tau} \equiv T_1[U(\rho),V(\rho),\mu(\rho)] \simeq T_1^{(0)}(\rho) + T_1^{(II)}(\rho,\theta)\chi^2 = 0 \tag{8a}$$
$$\mathcal{G}^{\rho}_{\rho} \equiv T_2[U(\rho),V(\rho),\mu(\rho)] \simeq T_2^{(0)}(\rho) + T_2^{(II)}(\rho,\theta)\chi^2 = 0 \tag{8b}$$
$$\mathcal{G}^{\theta}_{\theta} \equiv T_3[U(\rho),V(\rho),\mu(\rho)] \simeq T_3^{(0)}(\rho) + T_3^{(II)}(\rho,\theta)\chi^2 = 0 \tag{8c}$$
$$\mathcal{G}^{\rho}_{\theta} \equiv T_4[U(\rho),V(\rho),\mu(\rho)] \simeq T_4^{(II)}(\rho,\theta)\,\chi^2 = 0 \tag{8d}$$
$$\mathcal{G}^{\tau}_{\phi} \equiv T_5[U(\rho),V(\rho),\mu(\rho)] \simeq T_5^{(I)}(\rho,\theta)\,\chi = 0\,. \tag{8e}$$

This is the key equation regarding which we want to mention the following points: First, T_1 and T_3 contain fourth-order derivatives of $U(\rho), \mu(\rho)$ and $V(\rho)$, while T_2 and T_4 contain third-order derivatives of $U(\rho), \mu(\rho)$ and $V(\rho)$. T_5 contains third-order derivatives of $U(\rho)$ and $\mu(\rho)$, and second-order derivative of $V(\rho)$. Explicit expressions are not illuminating, so we do not report them here. Details are in the online MAPLE files. Second, T_1, T_2 and T_3 do not contain terms in first-order in χ, i.e. $T_i^{(0)}(\rho) + T_i^{(II)}(\rho)\chi^2 + \mathcal{O}(\chi^3)$, where $i = 1, 2$ and 3. Additionally, these three components' χ independent terms depend only on $U(\rho)$ and $\mu(\rho)$. Third, T_4 only contains second-order in χ, i. e. $T_4^{(II)}(\rho)\,\chi^2 + \mathcal{O}(\chi^3)$, while T_5 only contains first-order in χ, i. e, $T_5^{(II)}(\rho)\,\chi + \mathcal{O}(\chi^3)$. Fourth, it is possible to express $\mathcal{G}^{\phi}_{\tau}$ and $\mathcal{G}^{\theta}_{\rho}$ in terms of T_1, T_2, T_3, T_4 and T_5. These are consistent with the results in Appendix (A). Lastly, since the equations of motion contain fourth-order derivatives of $U(\rho)$, $V(\rho)$, and $\mu(\rho)$, an exact solution will have four independent constants.

Since these are highly coupled non-linear differential equations, they must be simplified to obtain analytical solutions. The steps we follow to obtain the solutions are as follows:

Step 1: We eliminate the fourth-order derivatives of $U(\rho)$ from T_1 and T_3. We obtain a third-order differential equation in $U(\rho)$.

Step 2: We use the above third-order differential equation and the T_2 equation to obtain a second-order differential equation of $U(\rho)$, $V(\rho)$ and $\mu(\rho)$. This leads to the following equation:

$$\frac{W_0(\rho)}{4\,\alpha_1^2\,\rho^5\,U(\rho)\,[\Phi(\rho)-2]^2\,[\Phi(\rho)+4]^2}\,\chi^2+\frac{W_1(\rho)\,W_2(\rho)}{2\,\alpha_1^2\,\rho^3\,U(\rho)\,[\Phi(\rho)-2]\,[\Phi(\rho)+4]}=0 \qquad (9)$$

where, $\Phi(\rho)=\rho[\mu'(\rho)+[\ln U(\rho)]']$. $W_1(\rho)$ and $W_2(\rho)$ are independent of $V(\rho)$, while W_0 is dependent of $V(\rho)$. Interestingly, we get the identical equation if, in the above two steps, we eliminate $\mu(\rho)$ instead of $U(\rho)$. The above separation is crucial as it allows us to evaluate $U(\rho)$ and $\mu(\rho)$ by solving $W_1(\rho)$ or $W_2(\rho)$ without the knowledge of $V(\rho)$.

Step 3: Using $T_5^{(I)}(\rho,\theta)$, we rewrite $V(\rho)$ in-terms of $U(\rho)$ and $\mu(\rho)$. Hence, the above expression is a function of $U(\rho)$ and $\mu(\rho)$.

Step 4: Since Eq. (9) contains two unknown functions $U(\rho)$ and $\mu(\rho)$, it is necessary to set one parameter and then derive the other function. Setting $\mu(\rho)=0$ in $W_2(\rho)=0$ leads to a simplified differential equation for $U(\rho)$ which leads to analytical expression for $U(\rho)$. Substituting $U(\rho)$ in T_5, we obtain $V(\rho)$. We thus have:

$$U(\rho)=1-\frac{C_1}{\rho}+\frac{\rho^2}{12\,\kappa^2};\qquad V(\rho)=\frac{C_4}{\rho^3}\,, \qquad (10)$$

where C_1 and C_4 are integration constants. $C_4>0$ since $V(\rho)$ is positive definite. Details of the derivation are given in Appendix (B).

Step 5: Having obtained $U(\rho)$, we now substitute $U(\rho)$ in Eq. (9) and solve the resultant differential equation for $\mu(\rho)$. This leads to the following integral form for $\mu(\rho)$:

$$\frac{\mu(\rho)}{2}=\ln\left[1+N_1\left(\int\rho^{-1/2}\left(\rho^3+12\,\rho\,\kappa^2-12\,C_1\,\kappa^2\right)^{-3/2}\,d\rho\right)\right]\,, \qquad (11)$$

where N_1 is an integration constant. $N_1<0$ ensures that $\exp[\mu(\rho)]$ is positive definite for all values of ρ.

Step 6: As a consistency check we substitute the $U(\rho), V(\rho)$ and $\mu(\rho)$ in $\mathcal{G}_\phi^\phi$. The χ^2 dependent terms vanish asymptotically. See Appendix (B) for details.

This is the crucial result of this work, regarding which we would like to discuss the following points: First, to our knowledge, this is the first time a non-trivial rotating BH solution has been obtained in any $f(R)$ gravity model. Second, according to GR, Kerr (vacuum) and Kerr-Newman (with charge) are the final states of gravitational collapse. BHs with scalar or other kinds of hair, i.e., outer space-time characterized by more than three parameters, are impossible in GR. However, in $f(R)$ gravity, the generalized Bianchi identity (3) provides a non-trivial structure for the Ricci scalar as a function of ρ leading to an infinite set of static slowly rotating black hole (SRBH) solutions for $f(R)$ gravity. Besides there are two branches ($W_1(\rho)=0$ or $W_2(\rho)=0$) of SRBH solutions and a collapse of a star might lead to a BH in either one of these branches. Third, for the $f(R)$ model (4), two branches of solutions exist: $W_1(\rho)=0$ or $W_2(\rho)=0$. By setting $W_2(\rho)=0$, we have explicitly constructed two non-trivial slowly-rotating solutions. Specifically, we have obtained the solutions for two different $\mu(\rho)$ functions and have shown that both satisfy the modified field equations. Fourth, in the $\chi = 0$ limit, we can identify C_1 with the mass of the BH, and hence, we have $C_1 = 2$. However, we can not determine the exact value of C_4, except that it has to be positive definite.

Lastly, to obtain exact solutions in $f(R)$, one usually performs a conformal transformation [45–49]. Here, we have not made any approximation or performed a conformal transformation to obtain exact solutions. Our analysis and results are valid even if the conformal transformations to the Einstein frame are not well-defined. In particular, in Refs. [46, 47], the authors proved a no-hair theorem to static and spherically symmetric or stationary axisymmetric BHs in general $f(R)$ gravity. In Ref. [50], the author showed that non-trivial BHs do not exist in four-dimensional asymptotically flat, static, and spherically symmetric or stationary axisymmetric space-time by imposing the condition that $(2f(R) - R\partial_R[f(R)]) > 0$ and $(\partial_R[f(R)] - R\partial_{RR}[f(R)])/\partial_{RR}[f(R)] > 0$. Our $f(R)$ model (4) satisfies both these conditions, and we have shown that potentially there exist many forms of $U(\rho)$ and $\mu(\rho)$ that satisfies $W_2(\rho) = 0$. It is important that the author [50] did not consider the generalized Bianchi identity (3), which leads to four constraints on the Ricci tensor. As mentioned earlier, assuming Ricci flat solutions will also satisfy the generalized Bianchi identity (3). However, the generalized Bianchi identity also implies that non-trivial solutions can exist. Here we have shown that the non-trivial solution with non-zero $\mu(\rho)$ decay to unity at asymptotic infinity.

In the rest of this work, we will discuss the key properties of these solutions and compare with Kerr and SR Kerr solution.

Properties of the two solutions: First, these solutions correspond to a BH with an identical event horizon. The event horizon is determined by the condition $g^{\rho\rho} = 0$. For our metric, we have three solutions: one real and two imaginary. The horizon ρ_H is given by:

$$\rho_H = \frac{H^{2/3}\,\left(2\,\kappa^2\right)^{1/3} - 2^{5/3}\,\kappa^{4/3}}{H^{1/3}}\,; \quad H = 2\sqrt{9 + 4\,\kappa^2} + 6\,. \tag{12}$$

Note that this matches with the slowly-rotating limit of Kerr BH ($\rho_{\rm SR} = 2$) for small values of κ^2. To see this, doing a series expansion of Eq. (12) about small κ^2, we have:

$$\rho_H \approx (24)^{1/3}\,\kappa^{2/3}$$

Thus, for $\kappa^2 = 0.33$, the above solution matches the SR Kerr solution in GR. As seen from Fig. (1a), the horizon radius can be larger or smaller than the SR Kerr depending on the value of κ^2; the larger the value of κ^2, the larger the horizon radius compared to GR. This will be crucial when we discuss the kinematical properties of space-time.

In order to confirm that both the solutions indeed correspond to BH, we evaluate the Kretschmann scalar:

$$K_{\mu(\rho)=0} = \frac{1}{6\,\kappa^2} + \frac{48}{\rho^6} + \chi^2\,\mathcal{F}_0\left(\frac{1}{\rho}\right) \tag{13}$$

$$K_{\mu(\rho)\neq 0} = \frac{1}{6\,\kappa^2} + \mathcal{F}_1\left(\frac{1}{\rho}\right) + \chi^2\mathcal{F}_2\left(\frac{1}{\rho}\right) \tag{14}$$

where $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$ are functions of ρ. All these three functions diverge at the origin and vanish at infinity.

In order to confirm that these two solutions correspond to rotating BHs, we turn our attention to the ergosphere. The radius of the ergosphere ($g_{\tau\tau} = 0$) is given by:

$$\frac{1}{12\,\kappa^2}\rho^5 + \rho^3 - 2\rho^2 + C_4\,\chi^2\,\cos^2(\theta) = 0 \tag{15}$$

Note that $C_4 > 0$. Fig. (1b) is the $x - z$ plot of the ergosphere for $\chi = 0.2$ and $\kappa^2 = 1$. The red-curve corresponds to the event-horizon, while the two black curves correspond to the ergosphere. Note that the variation due to $\chi^2 \cos\theta$ is visible in the inner part of the ergoregion. As expected the event horizon is very close to the ergosphere for SRBHs.

Having confirmed that $f(R)$ model (1), an attentive reader might wonder how to distinguish this in future observations. Comparing the GW signatures of these solutions requires one to solve the perturbed modified equations, like in the case of Chern-Simons [51] and will be discussed in future work. Here we discuss the kinematical properties, including conserved quantities, that can be used in the ngEHT observations [52].

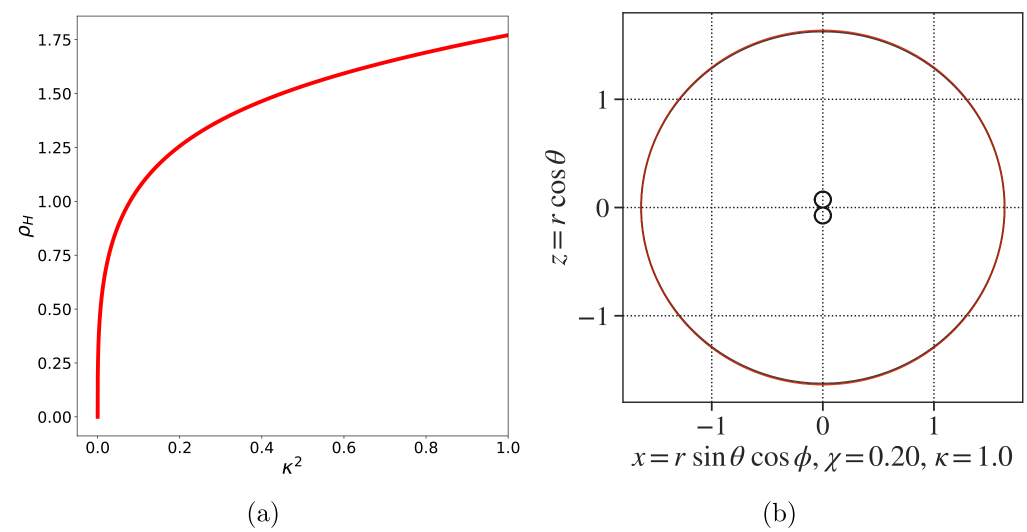


FIG. 1: (a) Plot of ρ_H as a function of κ^2 for $\chi = 0.1$ and $C_4 = 1$. (b) Plot of ergosphere for SRBH in $f(R)$ for $\chi = 0.2$ and $\kappa^2 = 1$.

Kinematical properties of the black hole space-times: Like Kerr, the SRBH space-times (6) possess two — time-like and azimuthal — Killing vectors, leading to the conservation of the specific energy (E) and the axial component of the specific angular momentum (L_z). E and L_z can be rewritten in terms of the 4-velocity (u^μ) of the time-like particle:

$$E = -\left(g_{\tau\tau}u^\tau + g_{\tau\phi}u^\phi\right); \quad L_z = g_{\phi\tau}u^\tau + g_{\phi\phi}u^\phi \,. \tag{16}$$

Since the geometry of the SR metric is axisymmetric, the orbital paths of objects about these black holes are often complex. To highlight the difference between the SRBH in $f(R)$ and GR, we consider the orbits in the equatorial plane ($\theta = \pi/2$) of the BH. Focusing on the equatorial circular orbits ($\dot{r} = 0$) around the BH satisfying the condition $u_\mu u^\mu = -1$, we

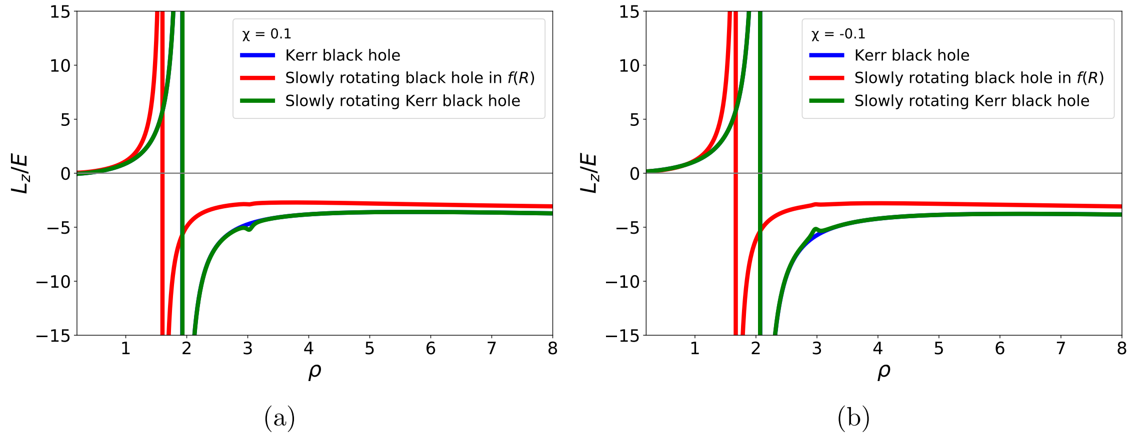


FIG. 2: Plot of L_z/E of a test particle in equatorial circular orbits as a function of ρ for $\kappa^2 = 1$ and $C_1 = 2$. We also plot for SR Kerr and Kerr solutions. (a) For $\chi = 0.1$. (b) For $\chi = -0.1$.

have:

$$E = -\frac{g_{\tau\tau} + \Omega\, g_{\tau\phi}}{\sqrt{-g_{\tau\tau} - 2\Omega\, g_{\tau\phi} - \Omega^2 g_{\phi\phi}}}; \qquad L_z = \frac{g_{\tau\phi} + \Omega\, g_{\phi\phi}}{\sqrt{-g_{\tau\tau} - 2\Omega\, g_{\tau\phi} - \Omega^2 g_{\phi\phi}}}, \tag{17}$$

where the angular velocity (Ω) is

$$\Omega = \frac{-\partial_\rho g_{\tau\phi} + \sqrt{(\partial_\rho g_{\tau\phi})^2 - \partial_\rho g_{\tau\tau}\partial_\rho g_{\phi\phi}}}{\partial_\rho g_{\phi\phi}}. \tag{18}$$

Fig. (2) contains the plot of L_z/E by evaluating up to the second order in χ for $\mu(\rho) = 0$. Like in the previous plots, we have compared the results of L_z/E for slowly-rotating $f(R)$ BH space-times with Kerr and slowly-rotating Kerr. Note that L_z/E is zero at the horizon for all the three BH space-times and diverges at the horizon. Since L_z/E is related to the impact parameter, from the figure, we infer that the impact parameter of the BHs in $f(R)$ is smaller than that of Kerr. This means that the inner-most stable circular orbit for BHs in $f(R)$ is smaller; hence, the shadow radius might also be smaller. Thus, ngEHT can potentially be used to constrain κ^2. This is currently under investigation.

Using Eq. (17), the radial equation on the equatorial plane is [37, 38, 53]:

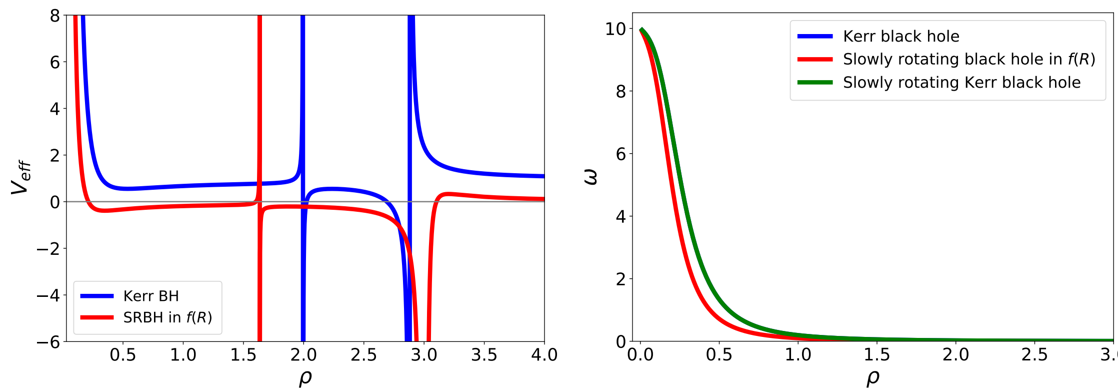
$$\frac{E^2 - 1}{2} = \frac{1}{2}\left(\frac{d\rho}{d\tau}\right)^2 + V_{\rm eff}(\rho, E, L_z) \tag{19}$$

where $V_{\rm eff}$ is the effective potential of the test particle given by:

$$V_{\rm eff} = \frac{E^2 g_{\phi\phi} + 2EL_z g_{\tau\phi} + L_z^2 g_{\tau\tau}}{g_{\tau\phi}^2 - g_{\tau\tau}g_{\phi\phi}} - 1 \tag{20}$$

Fig. (3a) contains the plot of $V_{\rm eff}$ of a test particle in the equatorial plane for $\mu(\rho) = 0$. Like in the previous plots, we have compared the results of $V_{\rm eff}$ for SRBH in $f(R)$ with Kerr and

SR Kerr. It is known that $V_{\rm eff} = 0$ corresponds to the circular orbits in the equatorial plane. These plots show that the circular orbits in the SRBH in $f(R)$ are smaller than Kerr. Note that $V_{\rm eff}$ diverges near the horizon. Again, we want to point out that for $\kappa^2 = 1$, the horizon radius of the slowly-rotating $f(R)$ black hole is larger than GR. The divergence near $\rho = 3$ corresponds to the existence of ISCO [37, 53]. This needs detailed investigation which is beyond the scope of this work.



(a) (b)

FIG. 3: (a) Plot of $V_{\rm eff}$ of a test particle in equatorial plane as a function of ρ. We set $C_1 = 2$. (b) Plot of the equatorial plane ZAMO's angular velocity as a function of ρ. For line-element (6), we have set $\kappa^2 = 1, \chi = 0.1$ and $C_4 = 1$. We also plot for SR Kerr and exact Kerr.

Let us now consider an observer moving with four-velocity u^μ in the stationary axisymmetric space-time (6). A special class of observers (ZAMO) has the property that their angular momentum vanishes. More specifically, the angular velocity ω vanishes for a ZAMO at infinity, but in the general case, it is nonzero and position-dependent. ZAMOs are defined by the condition [37, 38, 53]:

$$g_{\tau\phi}\dot{\tau} + g_{\phi\phi}\dot{\phi} = 0 \tag{21}$$

For the line-element (6), the ZAMO's angular velocity is

$$\omega = -\frac{g_{\tau\phi}}{g_{\phi\phi}} = \frac{\chi\, C_4}{\rho^3 + \chi^2\left(C_4 \sin^2(\theta) + \rho\right)} \, . \tag{22}$$

Fig. (3b) contains ZAMO's angular velocity for the SR $f(R)$ space-time and compares it with Kerr and slowly-rotating Kerr for the equatorial plane ($\theta = \pi/2$). From the figure, we infer that the ZAMO's angular velocity for an SRBH in $f(R)$ is always smaller than that of Kerr. This is intriguing because, for $\kappa^2 = 1$, the horizon radius of the slowly-rotating $f(R)$ black hole is larger than GR.

Conclusions and Discussions: For a class of $f(R)$ models, we have obtained SRBH solutions up to second-order in the rotational parameter χ. Specifically, we obtained two BH solutions with the event horizon at the same point with different asymptotic features. Besides there are two branches ($W_1(\rho) = 0$ or $W_2(\rho) = 0$) of SRBH solutions and a collapse of a star might lead to a BH in either one of these branches. Thus our analysis provide sufficient evidence for the need to enhance the no-hair theorem for modified gravity theories [39, 54]. This is currently under investigation.

In GR, the no-hair theorem demands that the only memory of the structure and composition of any object that collapses to form a stationary BH is embodied in the mass (M) and angular momentum (a), with any residual hair rapidly radiated away during the collapse process. In the case of $f(R)$, our analysis shows that this is affected by the choice of N_2. Since $f(R)$ contains fourth-order derivatives, we need four boundary (initial) conditions to fix these constants. Here, we only demanded the asymptotic properties of the line element; hence, two of the constants (C_4, N_2) are still undetermined. One possibility is to impose boundary conditions on the horizon or ergosphere. This is currently under investigation.

In Ref. [50], the author showed that non-trivial BHs do not exist in four-dimensional asymptotically flat, static, and spherically symmetric or stationary axisymmetric by imposing two conditions on $f(R)$. Although our $f(R)$ model (4) satisfies both these conditions, we have shown that multiple SRBH solutions exist. One possible reason for this discrepancy is that the generalized Bianchi identity (3) leads to two conditions $\partial_R^2[f(R)] = 0$ or $R_{\mu\nu}\nabla^\mu R = 0$. Trivial solutions like Ricci-flat space-times will automatically satisfy the above condition. However, the generalized Bianchi identity also implies that non-trivial solutions can exist. Here we have shown that the non-trivial solution with non-zero $\mu(\rho)$ decay to unity at asymptotic infinity.

We have analyzed some of the kinematical properties of the SRBHs and have shown that the horizon structure is different from that of Kerr. In addition, we have shown that the circular orbits for the SRBHs in $f(R)$ are smaller than that of Kerr. This means that the inner-most stable circular orbit for BHs in $f(R)$ is smaller; hence, the shadow radius might also be smaller. Thus, ngEHT can potentially be used to constrain κ^2. This is currently under investigation.

The next-generation GW detectors like the Einstein telescope and Cosmic explorer have higher sensitivity than the current LIGO-VIRGO-KAGRA detectors. These detectors will be highly sensitive in the quasi-normal mode (QNM) regime and can probe the QNM structure accurately in the strong gravity regime. It will be interesting to study the QNM frequencies for these non-trivial SRBH solutions and can provide strong constraints on the coupling constants [55]. This is currently under investigation.

We obtained the SRBH solutions by imposing the condition $W_2(\rho) = 0$. It will be interesting to know whether one can obtain non-singular BH solutions from the other branch $W_1(\rho) = 0$. Even the non-existence of non-singular BH solutions in this $f(R)$ model might explain the cause of the persistence of singularities in modified gravity models. This is under investigation.

ACKNOWLEDGMENTS

The authors thank K. Chandra, S. M. Chandran, A. Chowdhury, J. P. Johnson, A. Kushwaha and T. Parvez for discussions and comments on the earlier draft. The work of SS is supported by the SERB-Core Research Grant.

Appendix A: Slowly rotating black hole solution: Deformation linear in spin

Consider the following SRBH line element:

$$ds^2 = -e^{\delta(\rho)}\, A(\rho)d\tau^2 + \frac{d\rho^2}{A(\rho)} - 2\,\chi\, B(\rho)\, \sin^2(\theta)\, d\tau\, d\phi + \rho^2 d\Omega^2 \tag{A1}$$

where $A(\rho)$ and $\delta(\rho)$ are unknown functions of the reparameterized radial coordinate ρ. We use two different — modified Bianchi identity (3) and modified equations of motion (2) — approaches to obtain the solution.

1. Approach I: Bianchi Identity

As mentioned earlier, the generalized Bianchi identity (3) leads to two conditions. First, $f''(R) = 0$ leads to a trivial solution and corresponds to GR. The second condition can potentially lead to non-trivial solutions:

$$R_{\mu\nu}\nabla^{\mu}R = 0 \quad \text{or} \quad R^{\beta}_{\gamma}\nabla_{a}R = 0 \tag{A2}$$

For the SRBH line element (A1), using the fact that Ricci scalar is time-independent, i. e. $\partial_t R = 0$ and Ricci tensor components are non-zero, the above four equations can be combined to give the following expression:

$$[R_{\rho\rho} + R_{\rho\theta}]\; g^{\rho\rho}\partial_{\rho}R + [R_{\theta\rho} + R_{\theta\theta}]\; g^{\rho\rho}\partial_{\rho}R = 0\,. \tag{A3}$$

Substituting the line-element (A1) in the above expressing and ignoring all terms containing χ^q for $q > 1$, we can write the above expression as a product of two second-order differentials in $A(r)$ and $\delta(r)$:

$$2A^2(\rho)\rho^4\; \times\; \mathcal{E}_1[\delta(\rho), A(\rho)] \times \mathcal{E}_2[\delta(\rho), A(\rho)] = 0 \tag{A4}$$

where,

$$\mathcal{E}_1[\delta(\rho), A(\rho)] = \delta''(\rho) + \frac{\delta'(\rho)^2}{2} + \left[\frac{3}{2}\,\delta'(\rho) + 1 + \frac{2}{\rho}\right]\frac{A'(\rho)}{A(\rho)} \tag{A5}$$

$$\begin{aligned}\mathcal{E}_2[\delta(\rho), A(\rho)] = {}& \delta'''(\rho) + \left[\delta'(\rho) + \frac{5}{2}\frac{A'(\rho)}{A(\rho)} + \frac{2}{\rho}\right]\delta''(\rho) + \frac{1}{2}\frac{A'(\rho)}{A(\rho)}\delta'(\rho)^2 + 2\left[\frac{1}{\rho}\frac{A'(\rho)}{A(\rho)} - \frac{1}{\rho^2}\right]\delta'(\rho) \\ & + \frac{A'''(\rho)}{A(\rho)} + \left[\frac{3}{2}\delta'(\rho) + \frac{4}{\rho}\right]\frac{A''(\rho)}{A(\rho)} - \frac{2}{\rho^2}\frac{A'(\rho)}{A(\rho)} - \frac{4}{\rho^3}\left[1 - \frac{1}{A(\rho)}\right],\end{aligned} \tag{A6}$$

and prime denotes derivative w.r.t ρ. This is an interesting result regarding which we want to discuss the following: First, the above condition (A4) is valid for any $f(R)$ model and any form of stress-tensor. This is because the Bianchi identity (3) is derived by setting $\nabla_{\mu}T^{\mu\nu} = 0$. Second, the above condition is independent of $B(\rho)$. This implies that at linear-order in spin $B(\rho)$ is arbitrary. In Ref. [36], for spherically symmetric space-times, the authors showed that $\delta(\rho)$ could take infinitely many values. Here we have shown that $B(\rho)$ also can take infinite values. To know whether this is indeed the case, we proceed to obtain the condition on $B(\rho)$ using the equations of motion (2).

2. Approach II: Equations of Motion

To confirm the same, we use the equations of motion (2) for model given in Eq. (4) for $p = 2$. Substituting the line element (A1), in Eq. (5), up to leading order in χ, we obtain the following form of the modified field equations:

$$\mathcal{G}^{\tau}_{\tau} \equiv \mathcal{T}_1[A(\rho), \delta(\rho)] = 0 \tag{A7a}$$
$$\mathcal{G}^{\tau}_{\phi} \equiv \mathcal{T}_2[A(\rho), \delta(\rho), B(\rho)] = 0 \tag{A7b}$$
$$\mathcal{G}^{\rho}_{\rho} \equiv \mathcal{T}_3[A(\rho), \delta(\rho)] = 0 \tag{A7c}$$
$$\mathcal{G}^{\theta}_{\theta} \equiv \mathcal{G}^{\phi}_{\phi} \equiv \mathcal{T}_4[A(\rho), \delta(\rho)] = 0 \tag{A7d}$$
$$\mathcal{G}^{\phi}_{\tau} \equiv \mathcal{T}_5[A(\rho), \delta(\rho), B(\rho)] = 0 \tag{A7e}$$

where $\mathcal{T}_1, \mathcal{T}_3, \mathcal{T}_4$ and $\mathcal{T}_5$ are functions of $A(\rho)$ and $\delta(\rho)$ while $\mathcal{T}_2$ is a function of $A(\rho), B(\rho)$ and $\delta(\rho)$. Specifically, like in spherically symmetric case [36], $\mathcal{T}_1$ and $\mathcal{T}_4$ have up to 4^{th} order derivatives of $A(\rho)$ and $\delta(\rho)$, $\mathcal{T}_3$ have up to 3^{rd} order derivatives of $A(\rho)$ and $\delta(\rho)$, $\mathcal{T}_2$ and $\mathcal{T}_5$ have up to 3^{rd} order terms of both $A(\rho)$ and $\delta(\rho)$ and up to 2^{nd} order terms for $B(\rho)$. It is possible to rewrite $\mathcal{T}_5$ in terms of the other components.

Since the equations of motion contain up to fourth-order derivatives of $A(\rho)$ and $\delta(\rho)$, an exact solution will contain up to four independent constants. Using the procedure used in Ref. [36] for the spherically symmetric space-times, by combining $\mathcal{T}_1, \mathcal{T}_3$ and $\mathcal{T}_4$, we obtain the following expression:

$$\frac{3\, w_1(\rho)\, w_2(\rho)}{4\, \rho^3\, A^2(\rho)\ \ [\Phi(\rho) - 2]\, [\Phi(\rho) + 4]} = 0, \tag{A8}$$

where, $\Phi(\rho) = \rho(\delta'(\rho) + [\ln A(\rho)]')$,

$$w_1 = A(\rho) \left[2\, [\ln A(\rho)]'(2 - \rho) + \frac{\Phi(\rho)}{2} \left(\rho\, [\ln A(\rho)]' + 2 \right) + \rho\, \Phi'(\rho) + 2 \right] - 2 \left[1 + \frac{\alpha_0 \rho^2}{2\alpha_1} \right] \tag{A9}$$

$$\begin{aligned} w_2 =& A^2(\rho)\, \rho^2 \left(\frac{\Phi(\rho)}{\rho} - [\ln A(\rho)]' \right)^2 \left(\frac{3}{2}\rho[\ln A(\rho)]' + \Phi(\rho) + 1 \right) - 2A(\rho)\, (A(\rho) - 1)\, \left([\ln A(\rho)]' - 2 \right) \\ &+\ 62 A(\rho) \rho^2 \left(\left[\frac{\Phi(\rho)}{\rho} \right]' + ([\ln A(\rho)]')^2 \right) \left(\Phi(\rho) + \frac{4}{3} \right) - \frac{2\alpha_0\, \rho}{3\, \alpha_1} \left(2\rho^2\, A'(\rho) - A(\rho)\Phi(\rho) + \rho \right) \\ &+ \left(\frac{\Phi(\rho)}{\rho} - \frac{\rho[\ln A(\rho)]'}{2} \right) \left(3\rho \left(\rho[A'(\rho)]^2 + 2[\ln A(\rho)]' \right) + 4\, A(\rho)(A(\rho) + 1) \right) \end{aligned} \tag{A10}$$

Demanding a non-trivial solution to be satisfied for any finite value of ρ leads to the following conditions

$$w_1 = 0 \text{ or } w_2 = 0 \text{ while } \ \Phi(\rho) \neq 2,\, -4 \tag{A11}$$

Thus, we have rewritten the fourth-order differential equation as a product of two second-order non-linear differential equations. This allows for obtaining an exact slowly-rotating black hole solution. It is important to note that the above differential equations impose conditions on $A(\rho)$ and $\delta(\rho)$ only and does impose any condition on $B(\rho)$. To obtain a condition on $B(\rho)$, we eliminate the third-derivative in $A(\rho)$ or third-derivative using $\mathcal{T}_2$ and $\mathcal{T}_5$. This leads to the following equation:

$$\frac{w_1(\rho)\, w_3(\rho)}{2[\rho\, B'(\rho) - 2\, B(\rho)]\ [B(\rho)\, \Phi(\rho) - \rho\, B'(\rho)]} = 0 \tag{A12}$$

where

$$w_3 = \left[\frac{B'(\rho)}{B(\rho)} - \frac{2}{\rho}\right]\left(\left[\frac{\Phi(\rho)}{\rho}\right]' + ([\ln A(\rho)]')^2 + \left[\frac{\Phi(\rho)}{\rho} - [\ln A(\rho)]'\right]\left[\frac{\Phi(\rho)}{\rho} - [\ln A(\rho)]' + \frac{2}{A(\rho)}\right] + 2\right)$$
$$- \frac{B''(\rho)}{\rho B(\rho)}\left(\Phi(\rho) - 2\right) + \frac{2}{\rho^2}B(\rho)\left(\Phi(\rho) - \rho[\ln A(\rho)]'\right) + \frac{2}{\rho^2}\left([\ln A(\rho)]' - \frac{1}{B(\rho)}\right) \tag{A13}$$

Here again, demanding a non-trivial solution to be satisfied for any finite value of ρ leads to the following conditions

$$w_1 = 0 \text{ or } w_3 = 0 \quad \text{while} \quad B(\rho) \neq 0,\, \rho^2,\, \exp\left[\int \Phi(\rho) d(\ln\rho)\right]. \tag{A14}$$

From Eqs. (A11, A14), we infer that $\omega_1 = 0$ will satisfy all the modified Einstein's equations (A7). Rewriting $\Phi(\rho)$ as:

$$\delta'(\rho) + [\ln A(\rho)]' = \nu(\rho) \text{ and } \nu(\rho) \neq \frac{-4}{\rho} \text{ or } \frac{2}{\rho}, \tag{A15}$$

we see that given a $\nu(\rho)$, we have a relation between $A(\rho)$ and $\delta(\rho)$. More specifically, for a given $\Phi(\rho)$, we can obtain $A(\rho)$ by solving $w_1 = 0$. However, there are infinite choices for $\nu(\rho)$. For instance, solving $w_1 = 0$, leads to:

$$A(\rho) = 1 - \frac{C_1}{\rho} + \frac{C_2}{\rho^2} + \frac{\rho^2}{12\kappa^2} \tag{A16}$$

Hence, we have infinite vacuum solutions with infinite forms of $\delta(\rho)$. Moreover, since obtaining $A(\rho)$ does not constrain the form of $B(\rho)$, we have an infinite number of SRBH solutions (A1).

Appendix B: Slowly rotating black hole solution: Quadratic order in spin

This appendix details calculations of the SRBH solution at the quadratic order in the spin parameter (χ). We do this in two steps. First, we set $\mu(\rho) = 0$ and obtain the analytical expressions for $U(\rho)$ and $V(\rho)$. We then use the form of $U(\rho)$ in Eq. (8) to obtain the form of $\mu(\rho)$.

1. Obtaining $U(\rho)$ by setting $\mu(\rho) = 0$ in Eq. 9

Since Eq. (9) is highly non-linear and depends on three unknown functions, we first consider the standard form of (6) by setting $\mu(\rho) = 0$. In this case, we have:

$$W_1(\rho) = \left[3\,\rho^2 U''(\rho) - 6(U(\rho) - 1)\right]\left[\rho U'(\rho) + 2U(\rho)\right] + \frac{\alpha_0}{\alpha_1}\rho^2\left[\rho U'(\rho) - 2U(\rho)\right] \tag{B1}$$

$$W_2(\rho) = \rho^2\, U''(\rho) + 4\,\rho\, U'(\rho) + 2\, U(\rho) - \rho^2\frac{\alpha_0}{\alpha_1} - 2 \tag{B2}$$

Demanding that Eq. (9) is satisfied at each order of χ [43, 44], we have the following conditions:

$$W_1(\rho) = 0 \;\text{ or }\; W_2(\rho) = 0;\; W_0(\rho) = 0 \tag{B3}$$

Setting $W_2(\rho) = 0$, we obtain the following solution for $U(\rho)$:

$$U(\rho) = 1 - \frac{C_0}{12}\rho^2 - \frac{C_1}{\rho} + \frac{C_2}{\rho^2} \tag{B4}$$

where C_0, C_1 and C_2 are constants. Note that this is identical to Eq. (A16) we derived in the case of deformation linear in spin. Having determined $U(\rho)$, our next step is determining $V(\rho)$. For this, we can use any of the following expressions:

$$W_0(\rho) = 0 \quad \text{or} \quad T_4^{(II)}(\rho) = 0 \quad \text{or} \quad T_5^{(I)}(\rho) = 0\,. \tag{B5}$$

Setting

$$T_5 \equiv \left[V''(\rho)\,\rho + 4\,V'(\rho)\right]\left[\frac{\rho^3}{\kappa^2} - 6\,C_1 + 6\,\rho\right] = 0\,, \tag{B6}$$

and $C_2 = 0$ in Eq. (B4), we get,

$$V(\rho) = C_3 + \frac{C_4}{\rho^3}\,. \tag{B7}$$

As expected, we have four unknown constants — C_1, C_2, C_3, and C_4. In order to fix these constants, we substitute the above form of $U(\rho), V(\rho)$ and $\mu(\rho)$ in Eq. (9) and impose the condition that at asymptotic infinity ($\rho \longrightarrow \infty$), the modified Einstein tensor ($\mathcal{G}^\mu_\nu$) components should vanish. They are of the following form:

$$T_i \longrightarrow \chi^2 T_i^{(II)}\left(\frac{1}{\rho}\right) \quad \text{if} \quad C_0 = -\frac{1}{\kappa^2} \quad \text{and} \quad C_3 = 0\,.$$

We thus have:

$$U(\rho) = 1 - \frac{C_1}{\rho} + \frac{\rho^2}{12\,\kappa^2}, \qquad V(\rho) = \frac{C_4}{\rho^3}\,. \tag{B8}$$

2. Obtaining $\mu(\rho)$

Having obtained $U(\rho)$ and $V(\rho)$ (B8), we now determine the form of $\mu(\rho)$ by solving $W_2(\rho) = 0$. Substituting the above form of $U(\rho)$ (and not fixing the value of C_1), we have:

$$\mu''(\rho) + \frac{\mu'(\rho)^2}{2} + \frac{\left[5\,\rho^3 + 24\,\kappa^2\,\rho - 12\,C_1\,\kappa^2\right]}{\rho\,\left[\rho^3 + 12\,\kappa^2\,\rho - 12\,C_1\,\kappa^2\right]}\mu'(\rho) = 0 \tag{B9}$$

Solving for $\mu(\rho)$ we get a solution in the integral form:

$$\frac{\mu(\rho)}{2} = \ln\left[N_1\left(\int \rho^{-1/2}\left(\rho^3 + 12\,\rho\,\kappa^2 - 12\,C_1\,\kappa^2\right)^{-3/2}\,d\rho\right) + N_2\right] \tag{B10}$$

where N_1, N_2 are constants of integration. In the asymptotic limit, the above integral form reduces to:

$$\frac{\mu(\rho)}{2} \propto \ln\left[1 - \frac{5\,N_1}{\rho^4}\right]\,, \tag{B11}$$

where we have scaled the constant and positive definiteness of $\exp[\mu(\rho)]$ demands that $N_1 < 0$. Substituting the above integral form of $\mu(\rho)$ (B10), and $U(\rho), V(\rho)$ from (B8) all the modified Einstein tensor components vanish as

$$T_i \propto \chi^2 \, T_i^{(II)} \left(\frac{1}{\rho^2}\right); \quad T_4 \propto \chi^2 \, T_4^{(II)} \left(\frac{1}{\rho}\right); \quad T_5 \propto \chi \, T_5^{(II)} \left(\frac{1}{\rho^9}\right). \tag{B12}$$

where $i = 1, 2, 3$.

[1] B. P. Abbott *et al.* (Virgo, LIGO Scientific), Phys. Rev. Lett. **116**, 061102 (2016).
[2] R. Abbott *et al.* (LIGO Scientific, Virgo), Phys. Rev. D **103**, 122002 (2021), arXiv:2010.14529 [gr-qc].
[3] R. Abbott *et al.* (LIGO Scientific, VIRGO, KAGRA), (2021), arXiv:2112.06861 [gr-qc].
[4] D. Psaltis, Living Rev. Rel. **11**, 9 (2008), arXiv:0806.1531 [astro-ph].
[5] E. Berti, K. Yagi, and N. Yunes, Gen. Rel. Grav. **50**, 46 (2018), arXiv:1801.03208 [gr-qc].
[6] E. Berti, K. Yagi, H. Yang, and N. Yunes, Gen. Rel. Grav. **50**, 49 (2018), arXiv:1801.03587 [gr-qc].
[7] L. Barack *et al.*, Class. Quant. Grav. **36**, 143001 (2019), arXiv:1806.05195 [gr-qc].
[8] S. Shankaranarayanan and J. P. Johnson, Gen. Rel. Grav. **54**, 44 (2022), arXiv:2204.06533 [gr-qc].
[9] A. G. Riess and Others, Astron. J. **116**, 1009 (1998), arXiv:astro-ph/9805201 [astro-ph].
[10] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. **75**, 559 (2003), arXiv:astro-ph/0207347.
[11] T. Padmanabhan, Phys. Rept. **380**, 235 (2003).
[12] A. De Felice and S. Tsujikawa, Living Rev. Rel. **13**, 3 (2010), arXiv:1002.4928 [gr-qc].
[13] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010), arXiv:0805.1726 [gr-qc].
[14] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, Phys. Rept. **513**, 1 (2012).
[15] A. Joyce, L. Lombriser, and F. Schmidt, Ann. Rev. Nucl. Part. Sci. **66**, 95 (2016), arXiv:1601.06133 [astro-ph.CO].
[16] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, Phys. Rept. **692**, 1 (2017).
[17] A. G. Riess *et al.*, (2021), arXiv:2112.04510 [astro-ph.CO].
[18] R. P. Woodard, Lect. Notes Phys. **720**, 403 (2007), arXiv:astro-ph/0601672.
[19] N. H. Barth and S. M. Christensen, Phys. Rev. D **28**, 1876 (1983).
[20] J. Z. Simon, Phys. Rev. D **41**, 3720 (1990).
[21] F. Mueller-Hoissen, Annalen Phys. **48**, 543 (1991).
[22] S. W. Hawking and T. Hertog, Phys. Rev. D **65**, 103515 (2002), arXiv:hep-th/0107088.
[23] J. Z. Simon, Phys. Rev. D **43**, 3308 (1991).
[24] J. Z. Simon, Phys. Rev. D **45**, 1953 (1992).
[25] S. Bhattacharyya and S. Shankaranarayanan, Phys. Rev. D **96**, 064044 (2017).
[26] S. Bhattacharyya and S. Shankaranarayanan, Eur. Phys. J. C **78**, 737 (2018).
[27] S. Bhattacharyya and S. Shankaranarayanan, Phys. Rev. D **100**, 024022 (2019).
[28] S. Shankaranarayanan, Int. J. Mod. Phys. D **28**, 1944020 (2019), arXiv:1905.03943 [gr-qc].
[29] A. Chowdhury, S. Xavier, and S. Shankaranarayanan, (2022), arXiv:2206.06756 [gr-qc].
[30] N. Yunes and F. Pretorius, Phys. Rev. D **79**, 084043 (2009), arXiv:0902.4669 [gr-qc].
[31] K. Yagi, N. Yunes, and T. Tanaka, Phys. Rev. D **86**, 044037 (2012), [Erratum: Phys.Rev.D 89, 049902 (2014)].

[32] A. Maselli, P. Pani, L. Gualtieri, and V. Ferrari, Phys. Rev. D **92**, 083014 (2015), arXiv:1507.00680 [gr-qc].
[33] P. A. Cano and A. Ruipérez, JHEP **05**, 189 (2019), [Erratum: JHEP 03, 187 (2020)], arXiv:1901.01315 [gr-qc].
[34] M. Isi, M. Giesler, W. M. Farr, M. A. Scheel, and S. A. Teukolsky, Phys. Rev. Lett. **123**, 111102 (2019), arXiv:1905.00869 [gr-qc].
[35] V. Baibhav, E. Berti, V. Cardoso, and G. Khanna, Phys. Rev. D **97**, 044048 (2018).
[36] S. Xavier, J. Mathew, and S. Shankaranarayanan, Class. Quant. Grav. **37**, 225006 (2020).
[37] S. Chandrasekhar, *The mathematical theory of black holes* (Clarendon, Oxford, UK, 1985).
[38] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (San Francisco: W.H. Freeman and Co., 1973, 1973).
[39] P. T. Chrusciel, J. Lopes Costa, and M. Heusler, Living Rev. Rel. **15**, 7 (2012), arXiv:1205.6112 [gr-qc].
[40] C. A. R. Herdeiro and E. Radu, Int. J. Mod. Phys. D **24**, 1542014 (2015), arXiv:1504.08209 [gr-qc].
[41] J. Barrientos, P. A. González, and Y. Vásquez, Eur. Phys. J. C **76**, 677 (2016), arXiv:1603.05571 [hep-th].
[42] D. W. Tian, Gen. Rel. Grav. **48**, 110 (2016), arXiv:1507.07448 [gr-qc].
[43] J. B. Hartle, Astrophys. J. **150**, 1005 (1967).
[44] J. B. Hartle and K. S. Thorne, Astrophys. J. **153**, 807 (1968).
[45] T. P. Sotiriou and V. Faraoni, Phys. Rev. Lett. **108**, 081103 (2012), arXiv:1109.6324 [gr-qc].
[46] P. Cañate, L. G. Jaime, and M. Salgado, Class. Quant. Grav. **33**, 155005 (2016), arXiv:1509.01664 [gr-qc].
[47] J. Sultana and D. Kazanas, Gen. Rel. Grav. **50**, 137 (2018), arXiv:1810.02915 [gr-qc].
[48] M. Calzà, M. Rinaldi, and L. Sebastiani, Eur. Phys. J. C **78**, 178 (2018), arXiv:1802.00329 [gr-qc].
[49] G. G. L. Nashed and S. Nojiri, Phys. Rev. D **102**, 124022 (2020), arXiv:2012.05711 [gr-qc].
[50] P. Cañate, Classical and Quantum Gravity **35**, 025018 (2017).
[51] M. Srivastava, Y. Chen, and S. Shankaranarayanan, Phys. Rev. D **104**, 064034 (2021).
[52] P. Tiede, M. D. Johnson, D. W. Pesce, D. C. M. Palumbo, D. O. Chang, and P. Galison, Galaxies **10**, 111 (2022), arXiv:2210.13498 [astro-ph.HE].
[53] K. Bamba, S. Capozziello, S. Nojiri, and S. D. Odintsov, Astrophys. Space Sci. **342**, 155 (2012), arXiv:1205.3421 [gr-qc].
[54] A. Garcia, E. Hackmann, C. Lämmerzahl, and A. Macías, Phys. Rev. D **86**, 024037 (2012).
[55] V. Cardoso and L. Gualtieri, Class. Quant. Grav. **33**, 174001 (2016), arXiv:1607.03133 [gr-qc].