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MATH 506
Homework 4
March 27, 2014

**1**
Let $X$ be a real linear space, $p$ a sublinear functional on $X$, $Y$ a subspace of $X$, and $f_0$ a linear functional on $Y$ such that $f_0 \leq p(x)$ for all $x \in Y$. Then there exists a linear functional $f$ on $X$ such that $f(x) \leq p(x)$ for all $x \in X$ and $f|_Y = f_0$.

- Here for all $x, y \in X$ and $a \geq 0$, $p\colon X \to \mathbb{R}$ satisfies

$$p(x+y) \leq p(x) + p(y) \qquad \text{and} \qquad p(ax) = ap(x).$$

*Proof.* Let $E$ be the set of all linear extensions $g$ of $f_0$ with $g(x) \leq p(x)$, with $E \neq \emptyset$, since $f_0 \in E$. If we define an ordering on $E$ where $g \leq h$ means $h$ is an extension of $g$, then for each $C \subset E$, we can define $\hat{g}(x) = g(x)$ if $x \in \mathscr{D}(g)$ for $g \in C$, which is a linear functional with $\mathscr{D}(\hat{g}) = \bigcup_{g \in C} \mathscr{D}(g)$. Then, we have $g \leq \hat{g}$ for all $g \in C$, and since $C$ was arbitrary, Zorn's lemma allows us to choose an maximal element of $E$, which we will call $f$. By definition of $E$, we also have $f(x) \leq p(x)$.

Now, we must show that $\mathscr{D}(f) = X$. Suppose this is not true. Then, we can pick a $y_1 \in X \setminus \mathscr{D}(f)$ and consider the subspace $Y_1$, where any $x \in Y_1$ can be written $x = y + \alpha y_1$ and define a functional $g_1$ on $Y_1$ by $g_1(y + \alpha y_1) = f(y) + \alpha c$, where $c \in \mathbb{R}$. By this then, we have $f(x) \leq g_1(x)$, thus contradicting the maximality of $f$.

(Note: this is the Hahn-Banach Theorem as appearing in the book.) ∎

**2**
Let $X$ be a real linear space and $A, B \subset X$ be nonempty, disjoint, convex subsets. If $A$ is also open, then there exists $f \in X^*$ and $a \in \mathbb{R}$ such that

$$f(x) < a < f(y), \qquad x \in A, y \in B.$$

Hint: For $C = z_0 + A - B := \{z_0 + x - y : x \in A, y \in B\}$, define Minkowski functional of $C$ by $p_C(x) = \inf\left\{\alpha > 0 : \alpha^{-1}x \in C\right\}$, for all $x \in X$. The result follows from Problem 1 and the facts that $p_C$ is a sublinear functional on $X$ with $C = \{x : p_C(x) < 1\}$, and that there exists $c > 0$ such that $0 \leq p_C(x) \leq c\,\|x\|$. You do not have to prove these two facts.

*Proof.* Assuming we have such a $p_C$, we must find an $f_0$ such that $f_0(x) \leq p_C(x), \forall x \in M \subset X$. We'll define $f_0(az_0) := a$. Since $z_0 \notin C$, we have $a^{-1}z_0 \notin C$ if $a \in (0,1)$. So we have $p_C(z_0) \geq 1$ so $f_0(az_0) \leq p_C(az_0)$ if $a \geq 0$. Otherwise, $a < 0 \leq p(az_0)$. So $f_0(x) \leq p_C(x)$ holds, so by Problem 1, we can extend $f_0$ to an $f \in X^*$.

Now, let $x \in A, y \in B$ and $z_0 + x - y \in C$. Since $C$ is an open set, $\exists \varepsilon > 0$ such that $(1+\varepsilon)(z_0 + x - y) \in C$. Thus, it follows that

$$p_C\left(z_0 + x - y\right) \leq \frac{1}{1+\varepsilon}$$

Thus, we have $f(x) - f(y) + 1 = f(z_0 + x - y) \leq \frac{1}{1+\varepsilon}$, so $f(x) - f(y) < 0$. Let $a = \sup\{f(x) : s \in A\}$, then we have

$$f(x) \leq a \leq f(y)$$

for all $x \in A, y \in B$. ■

**3**
Explain that the completeness of $X$ in Theorem 6.5.11 (principle of uniform boundedness) cannot be removed.
Hint: For example, consider $X = \left\{x = (x_n) \in \ell^2 : x_n \neq 0 \text{ for only finite many } n\right\}$.

If we had $X$ not Banach, we may have a sequence $\{x_n\} \in X$ that is bounded in norm but that does not converge to an $x\ inX$.

Consider the set $X = \left\{x = (x_n) \in \ell^2 : x_n \neq 0 \text{ for only finite many } n\right\}$. Let $x_n$ be the sequence such that the $n$th element is $\frac{1}{n^2}$ with all others nonzero. Then, $\sum_{n=1}^{\infty} \|x_n\| < \infty$, but $\sum x_n$ does not converge to a sequence in $X$.

**4**
Let $T\colon X \to Y$ be a bounded linear operator, where $X$ and $Y$ are Banach spaces. If $T$ is bijective, then there are two constants $a, b > 0$ such that $a\|x\| \leq \|Tx\| \leq b\|x\|$, where $\|x\|$ and $\|Tx\|$ stand for the norms on $X$ and $Y$ respectively.

*Proof.* By definition, since $T$ is bounded, there exists $c \in \mathbb{R}$ such that $\|Tx\| \leq c\|x\|$. We will call this $b$. Furthermore, as a consequence of the open mapping theorem, if $T$ is bijective, there exists a $T^{-1}$ that is also continuous and bounded. Call the bound on $T^{-1}$ $a$ and the desired result follows. ■

**5**
Let $X = C[0,1]$ with sup-norm and $Y$ be the subspace of functions $x \in X$ which have a continuous derivative. Define $T\colon Y \to X$ by $Tx = x'$, where the prime denotes differentiation. Then $T$ is closed in the sense that its graph is closed.

*Proof.* We can use an application of the Closed Graph Theorem here to show $T$ is closed. Let $\{x_n\} \in \mathscr{D}(x)$ be such that $\{x_n\}$ and $\{Tx_n\}$ converge to

$$x_n \to x \qquad \text{and} \qquad Tx_n = x_n' \to y.$$

Since convergence on the norm of $C[0,1]$ is uniform, from $x_n' \to y$, we have

$$\int_0^t y(t)\,dt = \int_0^t \lim_{n\to\infty} x_n'(t)\,dt = \lim_{n\to\infty} \int_0^t x_n'(t)\,dt = x(\tau) - x(0),$$

which gives

$$x(\tau) = x(0) + \int_0^t y(t)\,dt.$$

This means $x \in \mathscr{D}(x)$ and $x' = y$, so from our closed linear operator theorem, we conclude $T$ is closed. ■

**6**
Suppose that $S = \{s_\alpha : \alpha \in A\}$ is a set of points in $X$ such that $\overline{\text{span}}\,\{S\} = X$. If $\{f_n\}$ is a bounded sequence in $X^*$ and $\{f_n(s_\alpha)\}$ converges for all $\alpha \in A$, then there exists $f \in X^*$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$.

*Proof.* We have as a corollary to the Uniform Boundedness principle, that if a sequence of bounded operators converges pointwise for all $x \in X$, the limits define an operator $T$. Taking $\{f_n\}$ as our operators and $f \in X^*$ our limit, since $S$ is dense in $X$, the existence of $f$ follows. ∎

**7**
Let $X$ be a finite-dimensional space, then for sequences $\{x_n\} \subset X$ and $\{f_n^*\} \subset X^*$, if there exist $x \in X$ and $f \in X^*$ such that $x_n \rightharpoonup x$ and $f_n \overset{*}{\rightharpoonup} f$, then we have $x_n \to x$ and $f_n \to f$.

*Proof.* From the proof of Theorem 4.8-4, we have that strong convergence and weak convergence are equivalent is $\dim X < \infty$. Thus, if we have $x \in X$ and $f \in X^*$ such that $x_n \rightharpoonup x$ and $f_n \overset{*}{\rightharpoonup} f$, we also have $x_n \to x$ and $f_n \to f$. ∎