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MATH 506
Midterm 2
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Prove the following statements. You need to provide a complete proof of each problem in order to get the full credit.

1 (10 points)
A Banach space X is reflexive iff X^* is reflexive.

We will first prove this theorem:

Theorem 0.1. *Every closed subspace of a reflexive normed space is reflexive.*

Proof. Let X be a Banach space and $M \subset X$ be a closed subspace. Also, let Q be the natural mapping of X onto X^{**}. We can see that $x \in (M^{\perp})^{\perp}$ iff $Qx \in M^{\perp\perp}$. Since, M is closed, we have from before $(M^{\perp})^{\perp} = M$. So, we have $Q(M) = M^{\perp\perp}$.

Now, take $m^{**} \in M^{**}$ and $m^{\perp\perp}$ be the corresponding member of $M^{\perp\perp}$. Then there is an m such that $Qm = m^{\perp\perp}$, so for each $x^* + M^{\perp} \in M^*$,

$$(x^* + M^{\perp}, m^{**}) = m^{\perp\perp}x^* = x^*m = (m, x^* + M^{\perp}).$$

Thus, we have M reflexive. ∎

Now, we give the proof for the problem statement:

Proof. First, we assume X is reflexive. Let Q and Q_* be the natural maps from X and X^* into X^{**} and X^{***} respectively, and $x^{***} \in X^{***}$ If we take an $x^{**} \in X^{**}$ and $x = Q^{-1}x^{**}$, then

$$(x^{**}, x^{***}) = (Qx, x^{***} = (x, x^{***}Q) = (x^{***}Q, x^{**}),$$

so we have $x^{***} = Q_*(x^{***}Q)$. Since Q_* is onto X^{***}, X^* is reflexive.

Now, suppose X^* is reflexive. Then, both X^{**} and its closed subspace $Q(X)$ are reflexive, by Theorem 0.1. Therefore, X is reflexive since it is isomorphic to $Q(X)$. ∎

2 (15 points)
Let H be a Hilbert space and let $P, Q \in B(H)$ be orthogonal projections. Then the following statements are equivalent.

(a) $\mathrm{Im}(P) \subset \mathrm{Im}(Q)$;

(b) $QP = P$;

(c) $PQ = P$;

(d) $\|Px\| \leq \|Qx\|$ for all $x \in H$;

(e) $P \leq Q$.

Proof. First, assume (a). Then, for every $h \in H$, $\exists q \in H$ with $Ph = Qq$. Therefore, $QPh = QQq = Qq = Ph$. Thus, $QP = P$ and (a) $\implies$ (b).

Now, we show (b) $\Leftrightarrow$ (c). Suppose $QP = P$. Then, $Q = Q + P - QP$, so we can compute $Q^* = (Q + P - QP)^* = Q^* + P^* - P^*Q^*$, since adjoint is linear. Thus we have $Q^* + P^* - P^*Q^* = Q + P - PQ$, so we necessarily have $QP = PQ = P$. The equivalence can be shown by swapping PQ above.

If we assume (b), we have $QP = P \Leftrightarrow (I - Q)P = 0$. Also, $\mathrm{Ker}(I - P) = \mathrm{Im}(P)$. Somehow, we have (b) $\Longrightarrow$ (a)

Now, assume (e). Then, $(Px, x) \leq (Qx, x)$ which is equivalent to $\|Px\|^2 \leq \|Qx\|^2$. Therefore, (e) $\Longrightarrow$ (d). ∎

3 (15 points)
Let X be a Banach space, and $\{f_n\} \subset X^*$. Given $1 < p < \infty$, the following two statements are equivalent.

(a) If $\sum_{n=1}^{\infty} \|x_n\|^p$, where $x_n \in X$ for each n, then $\sum_{n=1}^{\infty} f_n(x_n) < \infty$;

(b) The series $\sum_{n=1}^{\infty} f_n$ satisfies $\sum_{n=1}^{\infty} \|f_n\|^q < \infty$, where q is the conjugate of p, *i.e.* $\frac{1}{p} + \frac{1}{q} = 1$.

I really do not see where to go with this. It may have something to do with ℓ^p and ℓ^q or the Hölder inequality, since there are dual exponents involved, but I'm really not sure what to make of it. Furthermore, I can start to see how we can use $\sum_{n=1}^{\infty} \|f_n\|^q < \infty$ and $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$ to show the sum of the values of the linear functionals is also finite, but I cannot see how to start the other direction.

4 (15 points)
Let H be a Hilbert space and $G \subset H$ a closed linear subspace.

(a) Any bounded linear functional on G has a unique Hanh-Banach extension on H.

(b) Given $a \in H$, $a \neq 0$, let $G = \{x \in H : (x, a) = 0\}$. If $f_0 \in G^*$ is given by $f_0(x) = (x, b)$, for some $b \in H$, find the expression of the Hahn-Banach extension $f \in H^*$ of f_0.

We begin by proving the following theorem:

Theorem 0.2. *Every Hilbert space is uniformly convex.*

Proof. To begin, we define a uniformly convex space as a normed vector space such that for every $\varepsilon > 0, \exists \delta > 0$ such that, with $\|x\| = 1$ and $\|y\| = 1$, $\|x - y\| \geq \varepsilon$ implies that $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$. Let H be a Hilbert space. Now, let $\varepsilon > 0$ and $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$. Also, let $\delta = 1 - \frac{1}{2}\sqrt{4 - \varepsilon^2}$. Then, we have $\delta > 0$, and using the parallelogram law:

$$\begin{aligned}\|x+y\| &= \|x+y\|^2 + \|x-y\|^2 - \|x-y\|^2\\ &= 2\|x\|^2 + 2\|y\|^2 - \|x-y\|^2\\ &\le 4-\varepsilon^2\\ &= 4(1-\delta)^2.\end{aligned}$$

Thus, $\left\|\frac{x+y}{2}\right\| \le 1-\delta$. So, H is uniformly convex. ■

We now give a proof of the Taylor-Foguel Theorem (1958) to address (a) above.

Theorem 0.3 (Taylor-Foguel Theorem)**.** *Let X be a normed space. For every subspace Y of X and every $g \in Y^*$, there is a unique Hahn-Banach extension of g to X iff X^* is strictly convex.*

Proof. We will only give a proof for the reverse direction; that is, if X^* is strictly convex, there exists a unique Hahn-Banach extension of g to X. So, we start by assuming X^* is strictly convex. Let Y be a subspace of X, $g \in Y^*$, and let f_1 and f_2 be two Hahn-Banach extensions of g to X. Also, we will assume g is non-zero, and, without loss of generality, assume $\|g\| = 1$. From this, we see that $\frac{f_1+f_2}{2}$ is a continuous linear extension of g to X and that $\left\|\frac{f_1+f_2}{2}\right\| = \|g\| = 1$. Since, we have $\|f_1\| = \|f_2\| = \|g\| = 1$, the strict convexity of X^* gives us $f_1 = f_2$ and thus uniqueness. ■

Proof of (a). Since a closed linear subspace of Hilbert space is also Hilbert, we have $G \subset H$ also uniformly convex. Also, for Hilbert, strict convexity follows from uniform convexity. So (a) is given by the proof of Theorem 0.3 by letting $X = H$. ■

5 (15 points)
Let X be a normed linear space satisfying the property: $\forall \{x_n\}, \{y_n\} \subset X$, we have

$$\|x_n\| = \|y_n\| = 1,\ \|x_n + y_n\| \to 2 \implies \|x_n - y_n\| \to 0.$$

If $\{z_n\} \subset X$ converges to $z \in X$ weakly, and $\|z_n\| \to \|z\|$, then $\|z_n - z\| \to 0$.

Proof. Recalling our definition of uniform convexity given in the proof to Theorem 0.2, we can see that the property given is sufficient to imply uniform convexity. Thus, assuming $\{z_n\} \to z$ weakly, $\|z_n\| \to \|z\|$, and X is uniformly convex, we prove $\|z_n - z\| \to 0$.

Suppose $z \neq 0$. Then, we define $u_j = \frac{z_j}{\|z_j\|}$ and $u = \frac{z}{\|z\|}$, and we have $u_j \rightharpoonup u$. Also, $\frac{u_j+u_k}{2} \rightharpoonup u$ as $j,k \to \infty$. Furthermore, we have

$$1 = \|u\| \le \liminf_{j,k} \left\|\frac{u_j+u_k}{2}\right\| \le \limsup_{j,k} \left\|\frac{u_j+u_k}{2}\right\| \le 1.$$

Thus, $\left\|\frac{u_j+u_k}{2}\right\| \to 1$ and, by the uniform convexity of X, we have $\|u_j - u_k\| \to 0$. Then, u_j is Cauchy and $\|u_j - u\| \to 0$. Therefore, we have

$$\|z_j - z\| = \|\|z_j\| u_j - \|z\| u\| \le \|z_j\| \|u_j - u\| + \|\|z_j\| - \|z\|\| \|u\| \to 0.$$

■

6 (30 points)
Let H be an infinite-dimensional Hilbert space with an orthonormal basis $\{e_n\}$ and let $T \in B(H)$. If the condition $\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty$ holds then T is a Hilbert-Schmidt operator.

(a) The definition of a Hilbert-Schmidt operator is independent of the choice of the orthonormal basis of H.

(b) T is Hilbert-Schmidt iff T^* is Hilbert-Schmidt.

(c) If T is Hilbert-Schmidt then it is compact.

(d) The set of Hilbert-Schmidt operators is a linear subspace of $B(H)$.

(e) Give an example of a compact operator which is not Hilbert-Schmidt.

Proof of (a). First, let $\{e_n\}$ and $\{f_n\}$ be two orthonormal bases for H. Then, by Parseval's identity, we have $\|Te_n\|^2 = \sum_{n=1}^{\infty} |(Te_n, f_n)|^2$. Also, we have $\|T^*f_n\|^2 = \sum_{n=1}^{\infty} |(e_n, T^*f_n)|^2$ Thus, we have $\sum_n \|Te_n\|^2 = \sum_n \|T^*f_n\| = \sum_n \sum_n |(Te_n, f_n)|^2$. From this we can see that the sum $\sum_{n=1}^{\infty} \|Te_n\|$ is independent of the choice of $\{e_n\}$. ■

Proof of (b). Since the definition is independent of basis and in the previous section we showed $\sum_n \|Te_n\|^2 = \sum_n \|T^*f_n\|$, then we have both directions. ■

Proof of (c). An operator is compact iff it is the limit of a sequence of finite-rank operators. Fix an $\{e_k\}$ for which T is Hilbert-Schmidt. For each integer $n \geq 1$, define $T_n \in B(H)$ by

$$T_n(x) := \sum_{k=1}^{n} (x, e_k)Te_k \in \operatorname{span}\{Te_1, \ldots, Te_n\}$$

Thus, T_n is finite-rank. Now, we have

$$\|Tx - T_nx\| \leq \sum_{k=n+1}^{\infty} |(x, e_k)| \, \|Te_k\| \leq \left(\sum_{k=n+1}^{\infty} |(x, e_k)|^2 \right)^{1/2} \left(\sum_{k=n+1}^{\infty} \|Te_k\|^2 \right)^{1/2}$$

If we fix x such that $\|x\| \leq 1$, then $\sum_{k=n+1}^{\infty} |(x, e_k)|^2 \leq \sum_{k=1}^{\infty} |(x, e_k)|^2 = \|x\|^2 \leq 1$, so

$$\|Tx - T_nx\| \leq \left(\sum_{k=n+1}^{\infty} \|Te_k\|^2 \right)^{1/2} \to 0$$

as $n \to \infty$. Therefore, T is compact. ■

Proof of (d). For some scalar c, we have $\|cT\| = |c| \, \|T\|$ and thus $\sum_{n=1}^{\infty} \|cTe_n\|^{<} \infty$, so cT is still Hilbert-Schmidt. Furthermore, for T and V both Hilbert-Schmidt, we have $normT + V \leq \|T\| + \|V\|$, so $\sum_{n=1}^{\infty} \|(T+V)e_n\| \leq \sum_{n=1}^{\infty} \|Te_n\| + \sum_{n=1}^{\infty} \|Ve_n\| < \infty$. Therefore, $T + V$ is still Hilbert-Schmidt, so the set of Hilbert-Schmidt operators is a linear subspace. ■

Example of (e). Consider the identity operator on H given by $T : x \mapsto x$ for all $x \in H$. Then we have, $\|Tx\| = \|x\|$, so $\sum_{n=1}^{\infty} \|Te_n\| = \sum_{n=1}^{\infty} 1 \not< \infty$. ■