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MATH 506
Homework 3
February 27, 2014

Prove the following statements:

1 Let X be a real inner product space with the inner product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$.

1(a) For all $x, y, z \in X$, $\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\left\|z - \frac{1}{2}(x+y)\right\|^2$

This looks something like the law of cosines, and I think we can use this, since x, y, and z describe a "triangle".

1(b) The condition $\|x\| = \|y\|$ implies $(x+y, x-y) = 0$. What is the geometric interpretation if $X = \mathbb{R}^2$.

In $\mathbb{R}^2$, this implies that x and y are vectors with the same length and thus describe two sides of a rhombus.

2 Let X be an inner product space and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences in X, with $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$. Then

$$\lim_{n\to\infty}(x_n, y_n) = (x, y).$$

Proof. The inner product is bilinear and thus continuous in both arguments. We can thus write

$$\lim_{n\to\infty}(x_n, y_n) = (\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = (x, y)$$

. ∎

3 If Y is a closed linear subspace of a Hilbert space H then $Y^{\perp\perp} = Y$.

Proof. In general, we know first that $Y \subset Y^{\perp\perp}$. We need to show $Y^{\perp\perp} \subset Y$. ∎

4 Let X and Y be linear subspaces of a Hilbert space H. Define $X + Y = \{x + y \in H : x \in X, y \in Y\}$. Then $(X+Y)^\perp = X^\perp \cap Y^\perp$.

Proof. We have $(X+Y)^\perp = \{x' + y' \in H : (x'+y', x+y) = 0, \forall x \in X, y \in Y, x+y \in H\}$. By bilinearity and nonnegativity of the inner product, this means $(x', x) = (y', y) = 0$, which implies $x' \in X^\perp$ and $y' \in Y^\perp$, so $x' + y' \in X^\perp \cap Y^\perp$. ∎

5 Let H be a Hilbert space and let $\{e_n\}$ be an orthonormal basis in H. Prove that the Parseval relation

$$(x, y) = \sum_{n=1}^{\infty}(x, e_n)(e_n, y)$$

holds for all $x, y \in H$.

Proof. Since $\{e_n\}$ is an orthonormal basis, we can write

$$x = \sum_{n=1}^{\infty} x_n e_n \qquad \text{and} \qquad y = \sum_{m=1}^{\infty} y_m e_m$$

for all $x, y \in H$. Then, we have

$$\begin{aligned}(x, y) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} x_n y_m (e_n, e_m) \\ &= \sum_{n=1}^{\infty} x_n y_n \\ &= \sum_{n=1}^{\infty} (x, e_n)(e_n, y)\end{aligned}$$

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6 Let H be a Hilbert space and let $\{e_n\}$ be an orthonormal sequence in H. The following conditions are equivalent:

(a) $\{e_n : n \in \mathbb{N}\}^{\perp} = \{0\}$;

(b) $\overline{\text{span}}\,\{e_n : n \in \mathbb{N}\} = H$;

(c) $\|x\|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2$ for all $x \in H$;

(d) $x = \sum_{n=1}^{\infty} (x, e_n) e_n$ for all $x \in H$.

Hint: It suffices to prove (b) $\Longrightarrow$ (a) and (c) $\Longrightarrow$ (a).

Proof. Since the closure of the span of e_n is the whole space that (b) $\Longrightarrow$ (a) follows from the fact that $\{0\}^{\perp} = H$. ■