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MATH 506
Presentation Notes
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The space R^∞ is complete.

Proof. Let $\{x^n\}$ be a Cauchy sequence in R^∞, where $x^n = (x_1^n, x_2^n, \dots)$. Our goal is to find an $x = (x_1, x_2, \dots) \in R^\infty$ such that $x^n \to x$. Let $\varepsilon > 0$. Because $\{x^n\}$ is Cauchy, we have $\exists N > 0$ such that $d(x^n, x^m) < \varepsilon, \forall n, m > N$. Therefore, for all i and all $n, m > N$, we have $|x_i^n - x_i^m| < \varepsilon$. Furthermore, this means, for each i we have $(x_i^1, x_i^2, \dots)$ a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, we have a limit x_i: $x_i = \lim_{n\to\infty} x_i^n$.

Now, choose $\varepsilon > 0$ and replace ε above with $\frac{\varepsilon}{2}$. Again, we can find an $N > 0$ such that $|x_k^n - x_k^m| < \frac{\varepsilon}{2}$, for all k and $n, m > N$. Now take the limit as $m \to \infty$, we have $|x_k^n - x_k| < \frac{\varepsilon}{2}$. If we take supremum, we have $\sup_{k=1}^\infty |x_k^n - x_k| \le \frac{\varepsilon}{2}$. This implies $d(x^n, x) \le \frac{\varepsilon}{2} < \varepsilon$ and thus x^n converges to x.

∎

If $P_n \rightharpoonup P$ in $(M, \mathcal{B}_M)$ and $P_n(M_0) = P(M_0) = 1$, then $(P_n)^r \rightharpoonup P^r$ in $(M_0, \mathcal{B}_{M_0})$.

Lemma 0.1. *For any open set $G \subseteq M$, $\liminf_{n\to infty} P_n(G) \ge P(G)$.*

Proof. Let G be an open set and let $H = G^c$. Consider a sequence of functions $f_m(s) = \min(1, m \cdot d(s, G))$. Then, each f_m is a bounded continuous function, $0 \le f_m \le 1$ and $f_m \to 1$. From the weak convergence of P_n, we have

$$\int f_m \, dP = \liminf_m \int f_m \, dP_n \le \liminf_n P_n(G).$$

Letting, $m \to \infty$, we get $\liminf_n P_n(G) \ge P(G)$. ∎

By Remark 4.2, we have $P(G) = P^r(G \cup M_0)$, and we will define $G_0 \coloneqq G \cup M_0$, the general open set in M_0. Since we also have $P_n(M_0) = 1$, we can also write $P_n(G)$ as $(P_n)^r(G_0)$. Therefore, we have

$$\liminf_n (P_n)^r(G_0) \ge P^r(G_0).$$

As shown before, if we take the complement above, we also get

$$\limsup_n (P_n)^r(G_0) \ge P^r(G_0).$$

Now, let us take an $A \in \mathcal{B}_{M_0}$ such that $P^r(\partial A) = 0$. Also, let $G = \mathring{A}, K = \overline{A}$, so we have $G \subseteq A \subseteq K$ and $\partial A = K \setminus G$. For every n, then, we have $P_n^r(G) \le P_n^r(A) \le P_n^r(K)$ so that

$$\liminf_n P_n^r(G) \le \liminf_n P_n^r(A) \le \limsup_n P_n^r(A) \le \limsup_n P_n^r(K).$$

From our previous result, we can deduce that $P^r(G) \leq \liminf_n P_n^r(A) \leq \limsup_n P_n^r(A) \leq P^r(K)$. Therefore, since $P^r(K \setminus G) = 0$, we have $P^r(G) = P^r(A) = P^r(K)$, and thus

$$\lim_n P_n^r(A) = P^r(A).$$

Finally, since we have $\lim_n P_n^r(A) = P^r(A)$ for all $A \in \mathcal{B}_{M_0}$ such that $P^r(\partial A) = 0$, we can conclude that for each $f \in C_b(M_0)$,

$$\int_{M_0} f \, dP_n^r \to \int_M f \, dP^r$$

which is the definition of weak convergence.