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MATH 506
Homework 1
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Prove the following:

1 Let ν be a signed measure on $(X, \mathcal{M})$. If $\{E_j\}$ is an increasing sequence in $\mathcal{M}$, then $\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \nu(E_j)$. If $\{E_j\}$ is a decreasing sequence in $\mathcal{M}$ and $\nu(E_1)$ is finite, then $\nu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \nu(E_j)$.

Proof. We can create a sequence of disjoint sets $\{\tilde{E}_j\}$ (defined as $\tilde{E}_1 = E_1$, $\tilde{E}_k = E_k \setminus (E_1 \cup \cdots \cup E_{k-1})$. By countable additivity, then, we have $\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \nu\left(\bigcup_{j=1}^{\infty} \tilde{E}_j\right) = \sum_{j=0}^{\infty} \nu(\tilde{E}_j)$. Since a particular $\nu(E_k) = \sum_{j=0}^{k} \nu(\tilde{E}_j)$, we have $\sum_{j=0}^{\infty} \nu(\tilde{E}_j) = lim_{k\to\infty}\nu(E_k)$, which demonstrates the first proposition.

For the second part, since we have $\nu(E_1)$ finite and $\{E_j\}$ decreasing, we construct our $\{\tilde{E}_j\}$ as $\{\tilde{E}_i = E_1 \setminus E_i\}$ which is an increasing set and $\nu(\tilde{E}_i) = \nu(E_1) - \nu(E_i)$. From our last result, we have

$$\nu\left(\bigcup_{i=1}^{\infty} \tilde{E}_i\right) = \nu(E_1) - \lim_{i\to\infty} \nu(E_i). \tag{0.1}$$

Also, by definition, $\bigcup_{i=1}^{\infty} \tilde{E}_i = E_1 \setminus \bigcap_{i=1}^{\infty} E_i$, so we also know

$$\nu\left(\bigcup_{i=1}^{\infty} \tilde{E}_i\right) = \nu(E_1) - \nu(\bigcap_{i=1}^{\infty} E_i). \tag{0.2}$$

Setting 0.1 and 0.2 equal yields the second result. ∎

2 Let ν be a signed measure. Then E is ν-null iff $|\nu|(E) = 0$.

Proof. First, we assume $|\nu|(E) = 0$. Then, we must have $\nu^+(E) = \nu^-(E) = 0$, so $\nu(E) = \nu^+(E) - \nu^-(E) = 0$. Thus, E is ν-null.

Next, we assume E is ν-null. Since a subset of a null set is also null, we have positive and negative subsets P and N of E such that $\nu^+(P)$ and $\nu^-(N)$ are null. Thus, E is ν^+-null and ν^--null, so we have $|\nu|(E) = 0$. ∎

3 Let μ and ν be two signed measures. Then $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Proof. We begin by assuming $\nu \perp \mu$. Then we have a ν-null set E and a μ-null set F such that $E \cup F$ is the entire measure space. From the previous result, we then have E is also $|\nu|$-null, so $|\nu| \perp \mu$. Then, since $|\nu| = \nu^+ + \nu^-$, we must also have E ν^+-null and ν^--null, so $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. The other direction follows from the previous result as well. ∎

4 Let μ be a measure and ν be a signed measure. Then. $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$

Proof. Assume $\nu \ll \mu$. Then we have $\nu(E) = 0$ for every μ-null set E. Since E is both μ-null and ν-null, this implies it is also $|\nu|$-null, thus $|\nu|(E) = 0$ for every μ-null set as well and $|\nu| \ll \mu$. The rest then also follows from Remark 5.1.14 and 5.1.15. ∎

5 Let μ and λ be two σ-finite measures. If $\mu \ll \lambda$ and $\lambda \ll \mu$ then $\frac{d\lambda}{d\mu}\frac{d\mu}{d\lambda} = 1$ a.e. (w.r.t. either λ or μ).
Hint: Use the Radon-Nikodym theorem.

Proof. Since $\mu \ll \lambda$ and $\lambda \ll \mu$, we have, by the Radon-Nikodym Theorem $d\mu = f d\lambda$ and $d\lambda = g d\mu$, where $f = \frac{d\mu}{d\lambda}$ and thus $g = \frac{1}{f}$ a.e. Therefore, we have $fg = \frac{d\lambda}{d\mu}\frac{d\mu}{d\lambda} = 1$ a.e. ∎

6 Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $\mathcal{N}$ a sub-σ-algebra of $\mathcal{M}$, and $\nu = \mu|\mathcal{N}$. If $f \in L^1(\mu)$, there exists $g \in L^1(\nu)$ such that $\int_E f\,d\mu = \int_E g\,d\nu$ for all $E \in \mathcal{N}$; if g' is another such function then $g = g'$ ν-a. e. In probability theory, g is called the conditional expectation of f on $\mathcal{N}$.
Hint: Apply the Radon-Nikodym theorem to the signed measure $d\lambda = f d\mu$ on $(X, \mathcal{N})$.

Proof. Let $f \in L^1$. Then, we we have $\nu(E) = \int_E f\,d\mu$, $\forall E \in \mathcal{N}$ with $\nu \ll \mu$. Thus, by the Radon-Nikodym Theorem, there exists some function called the Radon-Nikodym derivate of ν w.r.t. μ, that we will call our g. ∎