

Prove the following statements. You need to provide a complete proof of each problem in order to get the full credit.

**1** (10 points)

Let  $(X, \|\cdot\|_1)$  and  $(Y, \|\cdot\|_2)$  be two Banach spaces. If  $Z = X \times Y$  (Cartesian product) with the norm given by  $\|(x, y)\| = \|x\|_1 + \|y\|_2$ , then  $(Z, \|\cdot\|)$  is a Banach space.

**2** (10 points)

Let  $(X, \mathcal{M}, \mu)$  be a  $\sigma$ -finite measure space. Then for  $1 \leq p < \infty$ ,  $L^p(X, \mu)$  is a Banach space with the norm  $\|f\|_p = (\int |f|^p d\mu)^{1/p}$ .

*Proof.* We must show that  $L^p(X, \mu)$  is complete in the given norm.

First, let  $\{f_n\}$  be Cauchy in  $L^p$  with respect to  $\|\cdot\|_p$ , with  $\sum_{n=1}^{\infty} \|f_n\|_p = M < \infty$ . Define  $g_n = \sum_{k=1}^n f_k$ . By the Minkowski inequality, then, we have

$$\|g_n\|_p \leq \sum_{k=1}^n \|f_k\|_p \leq M,$$

therefore

$$\int g_n^p \leq M^p.$$

Since  $\{g_n\}$  is increasing it must converge to some  $g$  such that  $g^p$  is measurable and finite a.e. Thus our sequence  $\{f_n\}$  converges so for  $1 \leq p < \infty$ ,  $L^p(X, \mu)$  is a Banach space. ■

**3** (20 points)

Let  $\{H_n\}_{n=1}^{\infty}$  be a sequence of Hilbert spaces and let  $H = \{\{x_n\} : x_n \in H_n, \sum \|x_n\|^2 < \infty\}$ . For  $\{x_n\}, \{y_n\} \in H$ ,  $a, b \in \mathbb{R}$ , define  $a\{x_n\} + b\{y_n\} = \{ax_n + by_n\}$  and  $(\{x_n\}, \{y_n\}) = \sum (x_n, y_n)$ . Then  $H$  is a Hilbert space.

*Proof.* First, we know the given function is an inner product because it relies on the sum of inner products from the given sequence of Hilbert spaces. We need just to show the space is complete. ■

**4** (10 points)

Let  $X$  be an inner product space and let  $A \subset X$ . then  $A^{\perp} = \overline{A}^{\perp}$ .

**5** (10 points)

Suppose that  $H$  is a separable Hilbert space and  $Y \subset H$  is a closed linear subspace. Then there is an orthonormal basis for  $H$  consisting only of elements of  $Y$  and  $Y^{\perp}$ .

Since  $Y$  is a closed linear subspace of  $H$ , we know that  $H$  can be described by  $H = Y \oplus Y^\perp$  (this comes from the direct sum theorem). That is, every  $h \in H$  can be written as  $h = y + y'$  for some  $y \in Y$  and  $y' \in Y^\perp$ . We can then find an orthonormal basis  $\{e_n\}$  for  $Y$  and  $\{e_m\}$  for  $Y^\perp$  and we have  $\{e_n\} \cup \{e_m\}$  as an orthonormal basis for  $H$ .

**6** (15 points)

Let  $Y$  be a closed linear subspace of a Hilbert space  $H$ . If  $Y \neq H$ , then  $Y^\perp \neq \{0\}$ . Is this always true if  $Y$  is not closed?

*Proof.* Since  $Y$  is a closed linear subspace of  $H$ , every  $h \in H$  can be expressed as the sum of an element of  $Y$  and an element of  $Y^\perp$ . Since  $Y \neq H$ ,  $H \setminus Y$  is not empty. Thus, each  $x \in H \setminus Y$  can be written as  $x = y + y'$  for  $y \in Y$ ,  $y' \in Y^\perp$ , where  $y'$  is necessarily not 0, otherwise we would have  $x = y \in Y$ . ■

**7** (10 points)

Let  $T: C[0, 1] \rightarrow \mathbb{R}$  be the linear transformation defined by

$$T(f) = \int_0^1 f(x) dx.$$

Suppose that  $C[0, 1]$  is equipped with the sup-norm.

**7a**

$T$  is continuous.

*Proof.*

$$|T(f)| = \left| \int_0^1 f(x) dx \right| \leq \sup |f(x)| = \|f\|$$

. Thus  $T$  is bounded and thus continuous. ■

**7b**

Find  $\|T\|$ .

**8** (15 points)

Let  $\ell^2$  be the set of real sequences  $x = (x_1, x_2, \dots)$  such that  $\sum_n |x_n|^2 < \infty$ .

**8a**

Let  $T$  be the linear transformation defined by

$$T(x_1, x_2, x_3, x_4, \dots) = (0, 4x_1, x_2, 4x_3, x_4, \dots).$$

Then  $T: \ell^2 \rightarrow \ell^2$  is continuous.

*Proof.* Let  $c \in \ell^\infty$  be defined by  $(0, 4, 1, 4, 1, \dots)$ . For every  $x \in \ell^2$ , we have

$$\|Tx\|^2 = \sum_n |c_i x_i|^2 = (\sup c_i)^2 \sum_n |x_i|^2 = 16 \|x\|^2.$$

Thus, every  $Tx$  is bounded and therefore continuous. ■

**8b**

Find  $\|T\|$ .

Again, we can consider  $Tx = cx$  with  $c$  defined as above. So  $\|T\| = \|c\|_\infty = 4$ .

Find  $T^2$  and  $\|T^2\|$ .