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MATH 506
Homework 2
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Prove the following statements:

1 Let (X,d) be a metric space. Then (X,d) is complete iff every nest of closed balls has a nonempty intersection.

Proof. We will first assume (X,d) is complete. Let $\{B_n\}$ be such a nest, where B_n is the ball with center at x_n and radius r_n. We have $\lim_{n\to\infty} r_n = 0$; therefore, the sequence $\{x_n\}$ is Cauchy, and its limit lies inside every B_n, so their intersection is nonempty.

If we have such a nest as described above with $\{x_n\}$ a Cauchy sequence.

And then I'm not sure where to really go from here. ∎

2 Let X be a linear space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. Let d_1 and d_2 be the metrics defined by $d_i(x,y) = \|x-y\|_i$, $i = 1,2$. Suppose that there exists $K > 0$ such that $\|x\|_1 \leq K\|x\|_2$ for all $x \in X$. Let $\{x_n\}$ be a sequence in X.

2(a) If $\{x_n\}$ converges to x in the metric space (X,d_2), then $\{x_n\}$ converges to x in the metric space (X,d_1).

Proof. Suppose $\{x_n\}$ converges to x in (X,d_2). Then, $\lim_{n\to\infty}\|x_n - x\|_2 = 0$. We also have $\lim_{n\to\infty}\|x_n - x\|_1 \leq K\cdot 0$, so $\{x_n\}$ converges to x in (X,d_1). ∎

2(b) If $\{x_n\}$ is Cauchy in the metric space (X,d_2), $\{x_n\}$ is Cauchy in the metric space (X,d_1).

Proof. Suppose $\{x_n\}$ is Cauchy in w.r.t. $\|\cdot\|_2$ and let $\varepsilon > 0$. Then, $\exists N > 0$ such that $\|x_n - x_m\|_2 \leq K\varepsilon$ when $n,m > N$. Therefore, $\|x_n - x_m\|_1 \leq \varepsilon$ whenever $n,m > N$, so $\{x_n\}$ is Cauchy in (X,d_1). ∎

3 If X is a normed linear space and S is a linear subspace of X then $\overline{S}$ is a linear subspace of X.

Proof. For $\overline{S}$ to be a linear subspace, we must show that it is addition and scalar multiplication.

Consider $a,b \in \overline{S}$. Then we consider three cases.

Case 1. $a,b \in S$.

Then, $a+b \in S$ since S is a linear subspace already, so clearly, $a+b \in \overline{S}$.

Case 2. $a \in \overline{S}\setminus S$, $b \in S$.

By definition of closure, then, $\exists a_0 \neq a \in S$ such that $\|a_0 - a\| < \varepsilon$. We have, though, $a_0 + b \neq a + b \in S$ and $\|a_0 + b - a - b\| = \|a_0 - a\| < \varepsilon$, so the distance between $a+b \in S$ and $a_0 + b$ is arbitrarily small, so we can conclude $a_0 + b \in \overline{S}$.

Case 3. $a \in \overline{S}\setminus S$, $b \in \overline{S}\setminus S$

This case is essentially the same as the previous case, where the distance between some $a_0 + b_0 \neq a + b \in S$ can be made arbitrarily small as well.

Now, if we have some $x \in \overline{S}$ and some scalar α, we consider two more cases:

Case 4. $x \in S$

That $\alpha x \in \overline{S}$ follows from the fact that S is also a linear subspace.

Case 5. $x \in \overline{S}\setminus S$

Again, a similar argument can be made to show that the distance between some $\alpha x_0 \neq \alpha x$ can be made arbitrarily small, so $\alpha x \in \overline{S}$.

Therefore, we have shown $\overline{S}$ is a linear subspace of X. ∎

4 Let $\mathcal{P}$ be the linear space of polynomials with real coefficients defined on $[0,1]$. Then we can define the sup-norm $\|p\|_\infty = \sup\{|p(x)| : x \in [0,1]\}$ and L^1-norm $\|p\|_1 = \int_0^1 |p(x)|\, dx$. Show that these two norms are not equivalent on $\mathcal{P}$.
(Hint: consider $p_n(x) = x^n$.)

Proof. Suppose the two norms are equivalent. Then $\exists c_1, c_2$ such that $c_1 \|p\|_1 \leq \|p\|_\infty \leq c_2 \|p\|_1$. For $p_n(x) = x^n$, we have

$$\|p_n\|_\infty = \max_{x\in[0,1]} |x|^n = 1$$
$$\|p_n\|_1 = \int_0^1 |x|^n\, dx = \frac{1}{n+1}.$$

Then, for $c_2 > 0$ for all n, we have

$$\frac{\|p_n\|_\infty}{\|p_n\|_1} = n + 1 \leq c_2.$$

This cannot hold for all n. Therefore, the norms are not equivalent. ∎

5 If $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ are two normed linear spaces, then

$$\|(x,y)\| = \max\{\|x\|_1, \|y\|_2\}, \qquad x \in X, y \in Y$$

is a norm on the product space $X \times Y$.

Proof. We must show the norm as defined meets the four criteria:

1. $\|x\| \geq 0$,
2. $\|x\| = 0$ iff $x = 0$,
3. $\|\alpha x\| = |\alpha|\|x\|$,
4. $\|x + y\| \leq \|x\| + \|y\|$.

That (1) holds follows from the fact that both $\|x\|_1$ and $\|y\|_2$ are also nonnegative, so their max must be as well. Next, $\max\{\|x\|_1, \|y\|_2\}$ can only equal zero when $x = 0$ and $y = 0$, so the second property holds. If, we treat scalar multiplication and addition component-wise, the other properties hold as well. ∎

6 The linear space $L^1(M, \mu)$ is complete in its metric $d(f,g) = \int |f - g|\, d\mu$. You can use the following procedure to prove.

(a) For any Cauchy sequence $\{f_n\} \subset L^1$, find a subsequence $\{f_{n_k}\}$ such that $\left\|f_{n_{k+1}} - f_{n_k}\right\|_{L^1} \leq 2^{-k}$.

(b) Define

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} \left(f_{n_{k+1}}(x) - f_{n_k}(x)\right),$$

then $f \in L^1$, $f_{n_k}(x) \to f(x)$ a.e. and $f_{n_k}(x) \to f(x)$ in L^1.

(c) Show that $\|f_n - f\|_{L^1} \to 0$ as $n \to \infty$.

Proof. Suppose $\{f_n\}$ is Cauchy in L^1 and that we pick out a subsequence $\{f_{n_k}\}$ with $\left\|f_{n_{k+1}} - f_{n_k}\right\|_{L^1} \leq 2^{-k}$. Then, we define the infinite series $f(x)$ by

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} \left(f_{n_{k+1}}(x) - f_{n_k}(x)\right),$$

so we have $f \in L^1$, $f_{n_k}(x) \to f(x)$ a.e. and $f_{n_k}(x) \to f(x)$ in L^1. Since infinite series $f(x)$ is dominated by $\sum 2^{-k} \leq 1$, we can say, by the Dominated Convergence Theorem, that $\|f_n - f\|_{L^1} \to 0$. ∎