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MATH 506
Homework 5
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Prove the following statements:

1
Let H, K, and L be Hilbert spaces, let $R, S \in B(H, K)$ and let $T \in B(K, L)$. Let $\lambda, \mu \in \mathbb{R}$. Then

(a) $(\mu R + \lambda S)^* = \mu R^* + \lambda S^*$;

(b) $(TR)^* = R^* T^*$.

Proof. For (a), we have for all $x \in H$ and $y \in K$

$$\begin{aligned}(x, (\mu R + \lambda S)^* y) &= ((\mu R + \lambda S)x, y) \\ &= (\mu R x, y) + (\lambda S x, y) \\ &= (x, \mu R^* y) + (x, \lambda S^* y) \\ &= (x, (\mu R^* + \lambda S^*) y)\end{aligned}$$

Thus, for all y, we have $(\mu R + \lambda S)^* = \mu R^* + \lambda S^*$.

For (b), we have

$$(x, (TR)^* y) = ((TR)x, y) = (Rx, T^* y) = (x, R^* T^* y).$$

■

2
Let H be a Hilbert space and let $T \in B(H)$. Then

(a) $\operatorname{Ker}(T) = \operatorname{Ker}(T^* T)$.

(b) $\overline{\operatorname{Im}(T^*)} = \overline{\operatorname{Im}(T^* T)}$.

Proof. For (a), we first show that $\|T^*\| = \|TT^*\| = \|T\|^2$. From the Schwarz inequality, we have

$$\|Tx\|^2 = (Tx, Tx) = (T^* T x, x) \leq \|T^* T x\| \|x\| \leq \|T^* T x\| \|x\|^2 .$$

Then, taking the supremum over all $x, \|x\| = 1$, we have $\|T\|^2 \leq \|T^* T\|$. Applying some properties of adjoint, we then have

$$\|T\|^2 \leq \|T^* T\| \leq \|T^*\| \|T\| = \|T\|^2 .$$

Thus, $\|T^* T\| = \|T\|^2$. Replacing T with T^*, we have also $\|T^{**} T^*\| = \|TT^*\| = \|T\|^2$, since $T^{**} = T$. From this result, we have $T^* T = 0$ iff $T = 0$, thus, the kernels are equal. ■

3
Let H be a Hilbert space, and $T \in B(H)$ be a self-adjoint operator. Then for all $\mu \in V(T)$,

$$\inf \{\lambda : \lambda \in \sigma(T)\} \leq \mu \leq \sup \{\lambda : \lambda \in \sigma(T)\}.$$

Proof. For self-adjoint operators we have $\inf \sigma(T) = \inf \{(Tx, x) : \|x\| = 1\}$ and $\sup \sigma(T) = \sup \{(Tx, x) : \|x\| = 1\}$. Furthermore, for T bounded and self-adjoint, we can define $v \colon H \to \mathbb{R} : x \mapsto (Tx, x)$. Since $V(T) = v(B)$, where $B = \{x \in H : \|x\| = 1\}$ and T is bounded, we conclude $V(T)$ is bounded. Furthermore, since B is connected and $V(T)$ is the continuous of image of B, we have $V(T)$ connected and bounded. Therefore, we can express $V(T)$ as an interval at least of the form (m, M), where

$$m = \inf v(x) = \inf \sigma(T) \qquad \text{and} \qquad M = \sup v(x) = \sup \sigma(T).$$

Thus, each $m \leq \mu \leq M$ for each $\mu \in V(T)$. ∎

4
Let H be a Hilbert space. If Q is an orthogonal projection in $B(H)$ then $\mathrm{Im}(Q)$ is a closed linear subspace and $Q = P_{\mathrm{Im}(Q)}$.

Proof. Since Q is an orthogonal projection in H, we have $H = \mathrm{Ker}(Q) \oplus \mathrm{Im}(Q)$. If we take $x = Qy$ and $z \in \mathrm{Ker}(Q)$, then

$$(x, z) = (Qy, z) = (y, Qz) = 0$$

so we have $\mathrm{Im}(Q) \perp \mathrm{Ker}(Q)$. Thus, H is the orthogonal direct sum of the image and kernel of Q, so $Image(Q) = \mathrm{Ker}(Q)^{\perp}$; therefore, Q is closed. ∎

5
Let H be a Hilbert space and let $P, Q \in B(H)$ be orthogonal projections.

(a) If $PQ = QP$, then PQ is an orthogonal projection.

(b) $\mathrm{Im}(P)$ is orthogonal to $\mathrm{Im}(Q)$ iff $PQ = 0$.

Proof. First, we note that since P and Q are projections, we have $P = P^2$ and $Q = Q^2$. Also, we have P and Q are self-adjoint. For (a), then, we have $PQ = QP = Q^*P^* = (PQ)^*$; thus, PQ is also self-adjoint. We then also have $(PQ)^2 = PQPQ = PPQQ = P^2Q^2 = PQ$, so it follows that PQ is a projection. ∎

6
Let H be a Hilbert space and let $y, z \in H$. Define $T \in B(H)$ by $Tx = (x, y)z$. Then T is compact.

Proof. If we take $T = RS$, where $Rx = (x, y)$ and $Sx = z$, where S is definitely compact (since it is constant), then we have two bounded operators, one of which is compact, so their product must be compact. ∎