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The space R^∞ is complete.

Proof. Let $\{x^n\}$ be a Cauchy sequence in R^∞, where $x^n = (x_1^n, x_2^n, \dots)$. Our goal is to find an $x = (x_1, x_2, \dots) \in R^\infty$ such that $x^n \to x$. Let $\varepsilon > 0$. Because each x^n is Cauchy, we have $\exists N > 0$ such that $d(x^n, x^m) < \varepsilon, \forall n, m > N$. Therefore, for all i and all $n, m > N$, we have $|x_i^n - x_i^m| < \varepsilon$. Furthermore, this means, for each i we have $(x_i^1, x_i^2, \dots)$ a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, we have a limit x_i: $x_i = \lim_{n\to\infty} x_i^n$.

Now, choose $\varepsilon > 0$ and replace ε above with $\frac{\varepsilon}{2}$. Again, we can find an $N > 0$ such that $|x_i^n - x_i^m| < \frac{\varepsilon}{2}$, for all k and $n, m > N$. Now take the limit as $m \to \infty$, we have $|x_i^n - x_i| < \frac{\varepsilon}{2}$. If we take supremum, we have $\sup_{k=1}^\infty |x_i^n - x_i| \le \frac{\varepsilon}{2}$. This implies $d(x^n - x) \le \frac{\varepsilon}{2} < \varepsilon$ and thus x^n converges to x.

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