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CS 575
Theory Assignment #1
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## Part I

**1** $\sum_{i=0}^{2} 3 = 9$

**2** $\prod\limits_{i=0}^{2} = 27$

**3** $\log_3 1 = 1$

**4** $\lg \lg 32 = \lg 5$

**5** $\lg^2 8 = (lg8)^2 = 9$

**6** $\lim\limits_{n\to\infty} (\frac{1}{2})^n = 0$

**7** $\lim\limits_{n\to\infty} 2^n = \infty$

**8** $\frac{d}{dx} \ln x = \frac{1}{x}$

**9** $\frac{d}{dx} e^x = e^x$

**10** $4^{\log_3 n} = n^x$. What is x?

$$
\begin{aligned}
x \ln n &= \ln\left(4^{\log_3 n}\right) \\
x &= \frac{\ln\left(4^{\log_3 n}\right)}{\ln n}
\end{aligned}
$$

**11** Prove by induction that $1 + 3 + 5 + 7 + \cdots + (2(n-1)+1) = n^2$. (Your proof should include the following: proof of base case, the induction hypothesis, what you are trying to prove, the proof. See document in Blackboard on induction proofs)

**Proposition 1.1.** $\sum\limits_{i=0}^{n-1} (2i+1) = n^2$.

*Proof.* For $n = 1$, we have $\sum_{i=0}^{0}(2i+1) = 2 \cdot 0 + 1 = 1$. So the base case is satisfied.

Now, $\forall k > 1$ we assume $\sum\limits_{i=0}^{k-1}(2i+1) = k^2$. Therefore, we aim to show $\sum\limits_{i=0}^{k}(2i+1) = (k+1)^2$. We have

$$
\begin{aligned}
\sum_{i=0}^{k}(2i+1) &= k^2 + \sum_{i=k}^{k}(2i+1) \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}
$$

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**12** Use strong induction and the identity $F_i = F_{i-1} + F_{i-2}$ to prove that the $i$th Fibonacci number satisfies

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$. (Hint: $\phi^2 = \frac{3+\sqrt{5}}{2}$ and $\hat{\phi}^2 = \frac{3-\sqrt{5}}{2}$)

*Proof.* For i=1, we have

$$
\begin{aligned}
F_1 &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}} \\
&= \frac{\sqrt{5}}{\sqrt{5}} \\
&= 1
\end{aligned}
$$

And, for i=1, we have

$$
\begin{aligned}
F_2 &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} \\
&= \frac{\sqrt{5}}{\sqrt{5}} \\
&= 1
\end{aligned}
$$

Now, let $i \geq 2$. Assume $\forall k, 1 \leq k \leq i, F_k = \frac{\phi^k - \hat{\phi}^k}{\sqrt{5}}$. We will show $F_{k+1} = \frac{\phi^{k+1} - \hat{\phi}^{k+1}}{\sqrt{5}}$. We know $F_{k+1} = F_k + F_{k-1}$. So, we have

$$
\begin{aligned}
F_{k+1} &= F_k + F_{k-1} \\
&= \frac{\phi^k - \hat{\phi}^k}{\sqrt{5}} + \frac{\phi^{k-1} - \hat{\phi}^{k-1}}{\sqrt{5}} \\
&= \frac{\left(\phi^k - \hat{\phi}^k\right) + \left(\phi^{k-1} - \hat{\phi}^{k-1}\right)}{\sqrt{5}} \\
&= \frac{\phi^k + \phi^{k-1} - \hat{\phi}^k - \hat{\phi}^{k-1}}{\sqrt{5}} \\
&= \frac{\phi^{k-1}(\phi+1) - \hat{\phi}^{k-1}(\hat{\phi}+1)}{\sqrt{5}} \\
&= \frac{\phi^{k-1}\phi^2 - \hat{\phi}^{k-1}\hat{\phi}^2}{\sqrt{5}} \\
&= \frac{\phi^{k+1} - \hat{\phi}^{k+1}}{\sqrt{5}}
\end{aligned}
$$

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**13** $\sum_{i=1}^{n} n-1 = n \cdot (n-1)$

**14** $\sum_{i=1}^{n} 3^i = 3^1 + 3^2 + \cdots + 3^n = \frac{1-3^{n+1}}{-2}$

**15** In how many different orders can $x$ students sit around a round table? Two orders are identical if one can be rotated to form the other.

There are $n!$ different arrangements, each with $n$ rotations, so $(n-1)!$ different orders.

## Part II

**1** `CountMe` has count $n^4$.

**2** `CountMe` has count $\lceil \frac{n}{2} \rceil$ for $n$ even and $\lfloor \frac{n}{2} \rfloor$ for $n$ odd.

**3** `CountMe` has count $\lceil \frac{n}{2} \rceil$ for $n$ odd and $\lfloor \frac{n}{2} \rfloor$ for $n$ even.

**4** Given the pseudo code below for bubble sort...Let $length[A] = n$. Derive the count of the number of times that the comparison $(A[j] < A[j-1])$ is executed by the algorithm. Show how you derived your answer.

If $length[A] = n$, the outer loop executes $n-1$ times. The inner loop executes $\sum_{i=2}^{n} i = \frac{n^2+n-2}{2}$ times. So, the overall instruction count is $(n-1)\left(\frac{n^2+n-2}{2}\right)$.

**5** Suppose we are comparing implementations of bubble sort and merge sort on the same machine. For inputs of size $n$, bubble sort runs in $15n^2$ steps, while merge sort runs in $150n \lg n$ steps. For which values of $n$ does bubble sort beat merge sort? (Hint: use Excel write a program, or try different values). What is the number of steps done by each for $n = 1024$?

Bubble sort beats merge sort for all values $n, 1 < n < 59$, using the following Mathematica code.

```
N[Solve[15 n^2 == 150*n*Log[2, n], n]]
```

For $n = 1024$, bubble sort takes 15,728,640 steps, while merge sort takes 1,536,000.

**6** What is the smallest value of $n$ such that an algorithm whose running time is $2^{20}n$ runs faster than an algorithm whose running time is $2^n$ on the same machine?

Once $n$ is larger than 24, $2^n$ exceeds $2^{20}n$. So the smallest such $n$ is 25.

## Extra Credit

Prove by induction that the following program computes $n!$ correctly for all integers $n \geq 1$.

*Proof.* For initialization, we need to show the program is correct when $n = 1$. Indeed, given the condition `if(n==1) return 1`, we correctly have $1! = 1$.

For maintenance, we observe that for each call where $n > 1$, the program returns $n$ multiplied by a call to the next lowest factorial, which is correct with regards to the formula for the factorial function, $n! = \prod_{i=1}^{n} i$.

Finally, for termination, we notice that, when repeatedly subtracting one from $n$, where $n > 1$, we will eventually have $n = 1$, whereby the program will terminate. ∎