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MATH 506
Homework 4
March 27, 2014

1
Let X be a real linear space, p a sublinear functional on X, Y a subspace of X, and f_0 a linear functional on Y such that $f_0 \leq p(x)$ for all $x \in Y$. Then there exists a linear functional f on X such that $f(x) \leq p(x)$ for all $x \in X$ and $f|_Y = f_0$.

- Here for all $x, y \in X$ and $a \geq 0$, $p\colon X \to \mathbb{R}$ satisfies

$$p(x+y) \leq p(x) + p(y) \qquad \text{and} \qquad p(ax) = ap(x).$$

Proof. Let E be the set of all linear extensions g of f_0 with $g(x) \leq p(x)$, with $E \neq \emptyset$, since $f_0 \in E$. If we define an ordering on E where $g \leq h$ means h is an extension of g, then for each $C \subset E$, we can define $\hat{g}(x) = g(x)$ if $x \in \mathscr{D}(g)$ for $g \in C$, which is a linear functional with $\mathscr{D}(\hat{g}) = \bigcup_{g \in C} \mathscr{D}(g)$. Then, we have $g \leq \hat{g}$ for all $g \in C$, and since C was arbitrary, Zorn's lemma allows us to choose an maximal element of E, which we will call f. By definition of E, we also have $f(x) \leq p(x)$.

Now, we must show that $\mathscr{D}(f) = X$. Suppose this is not true. Then, we can pick a $y_1 \in X \setminus \mathscr{D}(f)$ and consider the subspace Y_1, where any $x \in Y_1$ can be written $x = y + \alpha y_1$ and define a functional g_1 on Y_1 by $g_1(y + \alpha y_1) = f(y) + \alpha c$, where $c \in \mathbb{R}$. By this then, we have $f(x) \leq g_1(x)$, thus contradicting the maximality of f.

(Note: this is the Hahn-Banach Theorem as appearing in the book.) ∎

2
Let X be a real linear space and $A, B \subset X$ be nonempty, disjoint, convex subsets. If A is also open, then there exists $f \in X^*$ and $a \in \mathbb{R}$ such that

$$f(x) < a < f(y), \qquad x \in A, y \in B.$$

Hint: for $C = z_0 + A - B := \{z_0 + x - y : x \in A, y \in B\}$, define Minkowski functional of C by $p_C(x) = \inf\left\{\alpha > 0 : \alpha^{-1}x \in C\right\}$, for all $x \in X$. The result is a sublinear functional on X with $C = \{x : p_C(x) < 1\}$, and that there exists $c > 0$ such that $0 \leq p_C(x) \geq c\,\|x\|$. You do not have to prove these two facts.

3
Explain that the completeness of X in Theorem 6.5.11 (principle of uniform boundedness) cannot be removed.
Hint: For example, consider $X = \left\{x = (x_n) \in \ell^2 : x_n \neq 0 \text{ for only finite many } n\right\}$.

4
Let $T\colon X \to Y$ be a bounded linear operator, where X and Y are Banach spaces. If T is bijective, then there are two constants $a, b > 0$ such that $a\,\|x\| \leq \|Tx\| \leq b\,\|x\|$, where $\|x\|$ and $\|Tx\|$ stand for the norms on X and Y respectively.

5

Let $X = C\,[0,1]$ with sup-norm and Y be the subspace of functions $x \in X$ which have a continuous derivative. Define $T\colon Y \to X$ by $Tx = x'$, where the prime denotes differentiation. Then T is closed in the sense that its graph is closed.

6

Suppose that $S = \{s_\alpha : \alpha \in A\}$ is a set of points in X such that $\overline{\operatorname{span}}\,\{S\} = X$. If $\{f_n\}$ is a bounded sequence in X^* and $\{f_n(s_\alpha)\}$ converges for all $\alpha \in A$, then there exists $f \in X^*$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$.

7

Let X be a finite-dimensional space, then for sequences $\{x_n\} \subset X$ and $\{f_n^*\} \subset X^*$, if there exist $x \in X$ and $f \in X^*$ such that $x_n \rightharpoonup x$ and $f_n \overset{*}{\rightharpoonup} f$, then we have $x_n \to x$ and $f_n \to f$.