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MATH 506
Midterm: Exam 1
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Prove the following statements. You need to provide a complete proof of each problem in order to get the full credit.

1 (10 points)
Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be two Banach spaces. If $Z = X \times Y$ (Cartesian product) with the norm given by $\|(x, y)\| = \|x\|_1 + \|y\|_2$, then $(Z, \|\cdot\|)$ is a Banach space.

2 (10 points)
Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Then for $1 \leq p < \infty$, $L^p(X, \mu)$ is a Banach space with the norm $\|f\|_p = \left(\int |f|^p \, d\mu\right)^{1/p}$.

Proof. We must show that $L^p(X, \mu)$ is complete in the given norm.

First, let $\{f_n\}$ be Cauchy in L^p with respect to $\|\cdot\|_p$. It thus suffices to show that $\sum \|f_n\|_p < \infty$ and f_n converges to some $f \in L^p$. ∎

3 (20 points)
Let $\{H_n\}_{n=1}^{\infty}$ be a sequence of Hilbert spaces and let $H = \left\{\{x_n\} : x_n \in H_n, \sum \|x_n\|^2 < \infty\right\}$. For $\{x_n\}, \{y_n\} \in H$, $a, b \in \mathbb{R}$, define $a\{x_n\} + b\{y_n\} = \{ax_n + by_n\}$ and $(\{x_n\}, \{y_n\}) = \sum (x_n, y_n)$. Then H is a Hilbert space.

4 (10 points)
Let X be an inner product space and let $A \subset X$. then $A^{\perp} = \overline{A}^{\perp}$.

5 (10 points)
Suppose that H is a separable Hilbert space and $Y \subset H$ is a closed linear subspace. Then these is an orthonormal basis form H consisting only of elements of Y and $Y^{\perp}$.

6 (15 points)
Let Y be a closed linear subspace of a Hilbert space H. If $Y \neq H$, then $Y^{\perp} \neq \{0\}$. Is this always true if Y is not closed?

7 (10 points)
Let $T \colon C[0, 1] \to \mathbb{R}$ is the linear transformation defined by

$$T(f) = \int_0^1 f(x) \, dx.$$

Suppose that $C[0, 1]$ is equipped with the sup-norm.

7a
T is continuous.

7b
Find $\|T\|$.

8 (15 points)
Let ℓ^2 be the set of real sequences $x = (x_1, x_2, \dots)$ such that $\sum_n |x_n|^2 < \infty$.

8a
Let T be the linear transformation defined by

$$T\left(x_1, x_2, x_3, x_4, \dots\right) = \left(0, 4x_1, x_2, 4x_3, x_4, \dots\right).$$

Then $T\colon \ell^2 \to \ell^2$ is continuous.

8b
Find $\|T\|$.

Find T^2 and $\left\|T^2\right\|$.